Groups Geom. Dyn. (Online first)
DOI 10.4171/GGD/796

Groups of type FP via graphical small cancellation

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Abstract. We construct an uncountable family of groups of type FP. In contrast to every previous construction of non-finitely presented groups of type FP, we do not use Morse theory on cubical complexes; instead, we use Gromov's graphical small cancellation theory.

1. Introduction

The first examples of non-finitely presented groups of type FP were constructed in the 1990s by Bestvina and Brady, using Morse theory on CAT(0) cubical complexes [2]. Brady also used similar techniques to construct finitely presented subgroups of hyperbolic groups that are not themselves hyperbolic because they are not FP_3 [5]. With the benefit of hindsight, examples due to Stallings and Bieri of groups that are FP_n but that are not FP_{n+1} can be viewed as special cases of the Bestvina–Brady construction [33] and [3, pp. 37–40]. In [11], Bux and Gonzalez pointed out the close connection between the Bestvina–Brady construction and Brown's criterion for finiteness properties [8]. The computations of finiteness properties that Brown made in [8] using his new criterion can also be rephrased in terms of Morse theory. Since then, Morse theory on polyhedral complexes has been a vital tool in computing the finiteness properties of many families of groups—for some notable recent examples, see [10, 12]. The Bestvina–Brady argument has been extended in a number of ways; in particular, the second named author has constructed continuously many isomorphism types of groups of type FP [24]. Nevertheless, it is remarkable that until now, every construction of a non-finitely presented group of type FP has relied on the same Morse-theoretic techniques that were used by Bestvina–Brady.

There are of course a number of ways to make 'new groups of type FP from old': notable examples include the Davis trick, which has been used to produce non-finitely presented Poincaré duality groups [13, 14, 24], the method used by Skipper–Witzel–Zaremsky to construct simple groups with given homological finiteness properties [32] and the proof that every countable group embeds in a group of type FP_2 [23]. Nevertheless, these constructions require a pre-existing family of groups of type FP, and so they rely ultimately on the Morse theoretic techniques of Bestvina–Brady.

Mathematics Subject Classification 2020: 20J06 (primary); 20F06, 20F65, 57M07 (secondary).
Keywords: homological finiteness conditions, graphical small cancellation, groups of type FP.

Here, we give a new construction for non-finitely presented groups of type FP, which relies on Gromov's graphical small cancellation theory [17, Sec. 2] instead of the techniques used by Bestvina–Brady. Gromov introduced graphical small cancellation as a method to embed certain families of graphs, in particular, an expanding family, as subgraphs of a Cayley graph. Our main idea is to use graphical small cancellation to construct families of surjective group homomorphisms with acyclic kernels.

Our method naturally constructs uncountable families of groups, but we claim that it is as simple as the method of [2], even if one only wants to establish the existence of some non-finitely presented group of type FP. Apart from graphical small cancellation, we use only classical tools from combinatorial and homological group theory. Some of the families of groups that we construct are isomorphic to families from [24], but some are rather different.

The remainder of this introduction consists of a sketch of the construction and the statements of our main results. For the definitions and background results that we assume concerning homological finiteness properties and graphical small cancellation, see Section 2.

The Bestvina–Brady construction takes as its input a finite flag simplicial complex L, which should be acyclic but not contractible in order to produce a group BB_L that is FP but not finitely presented. For constructing continuously many generalized Bestvina–Brady groups $G_L(S)$ as in [24], one also requires L to be aspherical. Our construction takes as input a finite CW-complex K that we call a *spectacular complex*, together with an infinite set Z of non-zero integers. In the definition, we refer to the 1- and 2-cells of the complex as 'edges' and 'polygons', respectively.

Definition 1.1. A *spectacular complex* K is a finite 2-dimensional CW-complex K with the following properties.

(1) The 1-skeleton K^1 of K is a simplicial graph.

(2) The attaching map for each polygon P of K is an embedding of the boundary circle ∂P into K^1.

(3) Every edge path between distinct vertices of K^1 of valence at least three has length at least 5.

(4) The girth g of the graph K^1 is at least 13.

(5) The perimeter l_P of each polygon P satisfies $l_P > 2g$.

(6) The polygons of K satisfy a $C'(1/6)$ condition: for each pair $P \neq Q$ of polygons of K, each component of the intersection $\partial P \cap \partial Q$ of their boundaries contains strictly fewer than $\min\{l_P/6, l_Q/6\}$ edges.

(7) K is acyclic.

We emphasize that a spectacular complex is required to be 2-dimensional: a finite tree (with no polygons) is not spectacular. The word spectacular is intended to be an acronym: Simplicial 1-skeleton, Polygons Embed, $C'(1/6)$, Two-dimensional, ACyclic,

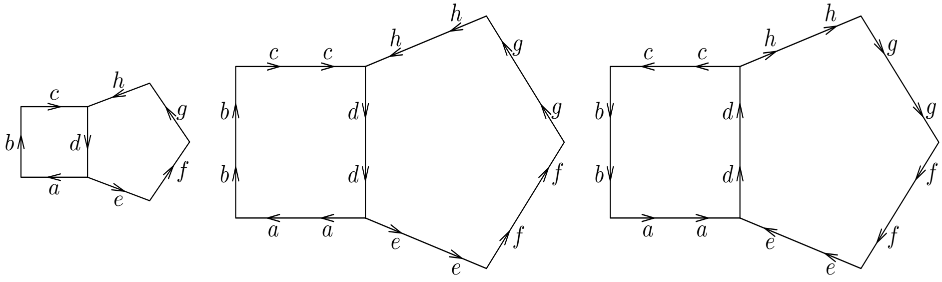

Figure 1. A graphical relator and its degree 2 and -2 subdivisions.

with lower bounds on maximal Unbranching paths, Lengths or perimeters of polygons And Rotundity, where 'rotundity' is used as a replacement for 'girth'. The existence of a spectacular complex K will be established in Section 7. For now we suppose that we are given a 2-complex K as above and an infinite set $Z \subseteq \mathbb{Z} - \{0\}$. We consider K and Z to be fixed throughout and they will usually be omitted from the notation.

For each subset $S \subseteq Z$, we define a group $H(S) = H(K, Z, S)$ whose generators are the directed edges of K, where the two orientations of the same edge are mutually inverse group elements. The relators between these generators depend on Z and S as well as K, in the following way. For each $n \in Z - S$ and each polygon P of K, the word spelt around the 'degree n subdivision of ∂P' is a relator. For each $n \in S$ and each simple cycle C in the graph K^1, the word spelt around the 'degree n subdivision of C' is a relator.

The group $H(S)$ is better understood via its graphical presentation. Recall that a graphical presentation arises from a labelled graph. The group presented by a graphical presentation has as generators the edge labels and as relators the words that represent the labellings of cycles in the graph. Each component of the labelled graph is called a graphical relator. For $H(S)$, the set of labels used is the directed edges of K, and the graphical relators are certain subdivisions of K^1 and of the boundaries of polygons P of K, with the canonical choice of labelling. In more detail, for each $n \in Z - S$ and each polygon P of K, the 'degree n subdivision of ∂P' is a graphical relator, and for each $n \in S$, the 'degree n subdivision of K^1' is a graphical relator. The degree n subdivision of a labelled graph is defined in Section 2, but see also the example in Figure 1.

For P, a polygon of K, we define $H_P(S)$ to be the subgroup of $H(S)$ generated by the directed edges of P.

Theorem 1.2. *For each $S \subseteq Z$, the graphical presentation for $H(S)$ given above satisfies the graphical small cancellation condition $C'(1/6)$. For each $S \subsetneq T \subseteq Z$, the natural bijection between generating sets extends to a surjective group homomorphism $H(S) \to H(T)$, whose kernel $K_{S,T}$ is a non-trivial acyclic group. For each polygon P, the intersection $K_{S,T} \cap H_P(S)$ is the trivial group and so the natural map restricts to an isomorphism $H_P(S) \cong H_P(T)$.*

None of the groups $H(S)$ are of type FP; however, they form the main building block in our construction of such groups.

In Proposition 3.3, as part of the proof of Theorem 1.2, it will be shown that the isomorphism type of the polygon subgroup H_P depends only on Z and on the perimeter of the polygon P. If P has perimeter $l = l_P$ with $a_1, \dots, a_l$ being the directed edge path around the boundary of P, then H_P has the following presentation:

$$H_P = \langle a_1, \dots, a_l \colon a_1^n a_2^n \cdots a_l^n = 1,\ n \in Z \rangle.$$

The final ingredient that we need is an embedding of H_P into a group G_P, where G_P is of type F. The existence of such an embedding puts some constraints on the set Z. We give explicit constructions for G_P in the cases $Z = \mathbb{Z} - \{0\}$ and $Z = \{k^n \colon n \geq 0\}$ for any integer k with $|k| > 1$. Using Sapir's version of the Higman embedding theorem [30], we show that H_P embeds in a group of type F if and only if Z is recursively enumerable. However, we emphasize that our construction requires only one example of such a Z and so does not require Sapir's landmark result.

Given such an embedding, we define a group $G(S)$ as the fundamental group of a star-shaped graph of groups. The underlying graph has one central vertex and arms of length one indexed by the polygons of K. The central vertex group is $H(S)$, with $H_P = H_P(S)$ on the edge indexed by P and G_P on the outer vertex indexed by P.

Proposition 1.3. *Let the groups $G(S)$ for $S \subseteq Z \subseteq \mathbb{Z}$ be defined as described above, for any spectacular complex K, any recursively enumerable $Z \subseteq \mathbb{Z}$, each polygon P of K and any fixed embedding of the subgroup H_P into a group G_P of type F.*

In each such case, the group $G(\emptyset)$ is of type F. For each $S \subsetneq T \subseteq Z$, there is a surjective group homomorphism $G(S) \to G(T)$ with non-trivial acyclic kernel.

Corollary 1.4. *Fix a spectacular complex K, a subset Z of $\mathbb{Z}$ and embeddings of the groups H_P into groups G_P of type F as in the statement of Proposition 1.3.*

For each such set of choices, there are continuously many isomorphism types of the groups $G(S)$ for varying $S \subseteq Z$. Each group $G(S)$ is of type FP. $G(S)$ is finitely presented if and only if S is finite, and $G(S)$ embeds as a subgroup of a finitely presented group if and only if S is recursively enumerable. Provided that each G_P has geometric dimension two, $G(\emptyset)$ also has geometric dimension two and each $G(S)$ has cohomological dimension two.

Corollary 1.4 should be compared with Theorems 1.2 and 1.3 from [24], in the special case when the flag complex L used there is acyclic and aspherical.

Although the proofs are rather different, there is an overlap between the groups $G(S)$ obtained from the new construction and the generalized Bestvina–Brady groups of [24]. In the case when $Z = \mathbb{Z} - \{0\}$, each H_P is isomorphic to a Bestvina–Brady group and we may take for G_P the corresponding right-angled Artin group. For these choices, the group $G(S)$ is naturally isomorphic to the generalized Bestvina–Brady group $G_L(S \cup \{0\})$ of [24], where L is the flag triangulation of K obtained by viewing each polygon as a cone on its boundary. In particular, with these choices, $G(\mathbb{Z} - \{0\})$ is isomorphic to the Bestvina–Brady group BB_L.

In contrast, in the case when $Z = \{k^n : n \geq 0\}$ for $|k| > 1$, our choice for G_P leads to a group $G(S)$ in which each generator for $H(S)$ is conjugate to its own kth power. Hence, any semisimple action of $G(S)$ on a CAT(0) space will have $H(S)$ in its kernel, indicating that these groups are very different to generalized Bestvina–Brady groups.

In the next section, we give some background material concerning finiteness properties, graphical presentations and graphical small cancellation. Most of this section is well-known material, but we give some foundational results concerning graphical presentations for which we have been unable to find a reference. In Section 3, we use graphical small cancellation methods to prove Theorem 1.2. In Section 4, we use standard methods from graphs of groups to prove Proposition 1.3 and deduce Corollary 1.4. In Section 5, we discuss embeddings of the polygon groups H_P into 2-dimensional groups G_P of type F. In Section 7, we establish the existence of a 2-complex K with the required properties, using some background material concerning projective linear groups, which is described in Section 6. The ordering of the material reflects the history of our work: in particular, we had a rough version of the main theorem long before we had established the existence of a spectacular complex K. Finally, Section 8 discusses some questions that remain open.

2. Background

We begin with some remarks concerning 2-dimensional CW-complexes whose 1-skeleta are simplicial graphs and such that the attaching maps for 2-cells are embeddings, i.e., complexes that satisfy conditions 1 and 2 from the definition of a spectacular complex. Two types of subdivision of such complexes will be of interest. Firstly, for $m > 1$, there is a subdivision in which each 1-cell e is subdivided into m subintervals, with $m - 1$ new 0-cells at their intersections. In this subdivision, the attaching maps for the 2-cells are unchanged although the length of each 2-cell is multiplied by m. For $m \geq 5$, conditions 3 and 4 in the definition of a spectacular complex will always hold for this subdivision. Secondly, if we replace each 2-cell by the cone on its boundary, we obtain a simplicial complex that is homeomorphic to the original complex, with one new vertex for each old 2-cell. We call this simplicial complex the *conical subdivision* of the original complex.

Although we do not need this result, we digress to explain the connection between the fundamental group of a spectacular 2-complex and a standard small cancellation group.

Proposition 2.1. *Let K be a 2-complex satisfying conditions 1, 2 and 6 from the definition of a spectacular complex and having n vertices. Let G be the group with inverse pairs of generators given by the edges of K and relators given by the words along the polygons of K. Then, G is a classical $C'(1/6)$ small cancellation group and there is an isomorphism $\pi_1(K) * F \cong G$, where F is a free group of rank $n - 1$.*

Proof. Let T be the cone on the 0-skeleton K^0 of K, so that T is a star with n arms, and let X be the 2-complex defined as the union of K and T, identifying K^0. It is easy to see that X is homotopy equivalent to the 1-point union of K and the complete bipartite graph $K(2,n)$, and so $\pi_1(X) \cong \pi_1(K) * F$. To complete the proof, it suffices to show that $\pi_1(X)$ is isomorphic to the group G described in the statement. Since T contains every vertex of X, it is a maximal tree in X. Using this maximal tree as our starting point, we obtain a presentation of $\pi_1(X)$ with generators the directed edges of X not contained in T and with relators corresponding to the 2-cells of X. This presentation is exactly the one given for the group G, and this presentation is $C'(1/6)$. ∎

An *Eilenberg–Mac Lane space* for a group G is a connected CW-complex whose fundamental group is isomorphic to G and whose universal cover is contractible. Any two such spaces are based homotopy equivalent.

A group G is *of type* F if G admits an Eilenberg–Mac Lane space with finitely many cells. A space is *acyclic* if it has the same homology as a point. A group G is *of type* FH if there is a free G-CW-complex that is acyclic and has only finitely many orbits of cells. A group G is *of type* FL if the trivial module $\mathbb{Z}$ for its group algebra $\mathbb{Z}G$ admits a finite resolution by finitely generated free $\mathbb{Z}G$-modules. Finally, a group G is *of type* FP if $\mathbb{Z}$ admits a finite resolution by finitely generated projective $\mathbb{Z}G$-modules. From the definitions, it is easy to see that

$$F \implies FH \implies FL \implies FP,$$

and it may be shown that any finitely presented group of type FL is of type F. For further details concerning these properties, see [3,7] and [2, Sec. 1]. We require one general result concerning finiteness properties. A group is said to be *acyclic* if its Eilenberg–Mac Lane space is acyclic.

Proposition 2.2. *Suppose that G is of type F and that N is an acyclic normal subgroup of G. The group G/N is of type FH, and the cohomological dimension of G/N is bounded above by the geometric dimension of G.*

Proof. Let X be the universal covering space of an Eilenberg–Mac Lane space for G, and consider the quotient X/N. This is an Eilenberg–Mac Lane space for N, equipped with a free cellular action of G/N. Since N is acyclic, X/N is acyclic. The G/N-orbits of cells in X/N correspond to the G-orbits of cells in X, and so if G is of type F, then G/N is of type FH. If X has dimension n, then so does X/N, and the dimension of X/N is an upper bound for the cohomological dimension of G/N. ∎

A *graph of groups* indexed by a graph Γ consists of groups G_v and G_e for each vertex v and edge e of Γ, together with two injective group homomorphisms $G_e \to G_v$ from the edge group G_e to the vertex groups corresponding to the ends of e. A *graph of based spaces* is defined similarly. If each vertex and edge space in a graph of spaces is an Eilenberg–Mac Lane space and the induced maps on fundamental groups are all injective, then the homotopy colimit (in the category of unbased spaces) of the graph of spaces is also an Eilenberg–Mac Lane space, whose fundamental group is by definition the *fundamental group of the graph of groups*. See [19, Ch. 1.B] for a treatment of this topic,

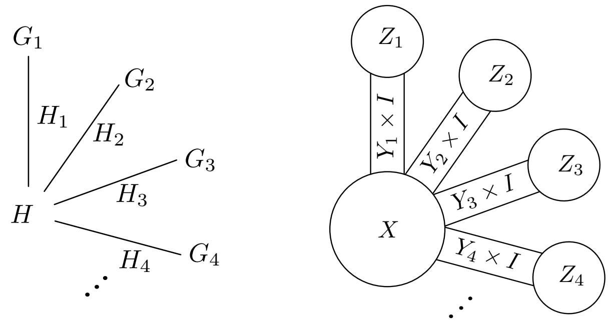

Figure 2. A star-shaped graph of groups and its Eilenberg–Mac Lane space.

and either [26, App.] or [16, Sec. 4] for a treatment that mentions homotopy colimits. We highlight two special cases of the Eilenberg–Mac Lane space for the fundamental group of a graph of groups that will appear in our work. See also Figure 2.

Proposition 2.3. *Explicit models for the Eilenberg–Mac Lane spaces for certain graphs of groups may be constructed as described below.*

(1) *Suppose that X is an Eilenberg–Mac Lane space for the group G and that $f : X \to X$ is a based map that induces an injective homomorphism $\phi : G \to G$. Then, the mapping torus of f is an Eilenberg–Mac Lane space for the ascending HNN-extension $\langle G, t : tgt^{-1} = \phi(g),\ g \in G\rangle$.*

(2) *Fix an indexing set I and suppose that X, Y_i and Z_i are Eilenberg–Mac Lane spaces for groups H, H_i and G_i, respectively. Suppose also that $f_i : Y_i \to X$ and $g_i : Y_i \to Z_i$ are based maps that induce injective group homomorphisms $\phi_i : H_i \to H$ and $\psi_i : H_i \to G_i$. In this case, the star-shaped graph of spaces given as the following identification space*

$$\frac{\left(X \sqcup \bigsqcup_{i \in I} \left(Y_i \times [0,1]\right) \sqcup \bigsqcup_{i \in I} Z_i\right)}{(y_i, 0) \sim f_i(y_i),\ (y_i, 1) \sim g_i(y_i)}$$

is an Eilenberg–Mac Lane space for the corresponding star-shaped graph of groups.

Next, we describe *graphical small cancellation theory* as in [17, Sec. 2], [18, 29]. The theory subsumes the classical small cancellation theory [27, Ch. V], which is the case in which the graph Γ considered below is a disjoint union of cycles. Note that we consider only the version that yields torsion-free groups, which corresponds in the classical case to excluding the possibility that a relator is a proper power. This is the only case discussed by Ollivier [29]. Gruber discusses the more general case but uses terms like '$C'(1/6)$' and '$C(7)$' in the case that yields only torsion-free groups and terms like 'Gr$'(1/6)$', 'Gr(7)' for the more general case [18].

Before introducing the graphical small cancellation conditions, we start by discussing graphical presentations of groups and the associated graphical presentation complexes.

A *labelling* of a graph Γ consists of a set L of labels, together with a fixed-point free involution $\tau : L \to L$, and a labelling function ϕ from the set of directed edges of Γ to L so that the label on the opposite edge to e is $\tau \circ \phi(e)$. We emphasize that when Γ is viewed as a topological space, each pair of opposite directed edges corresponds to a single 1-cell of Γ with its two orientations, rather than two distinct 1-cells. A labelling is said to be *reduced* if the graph Γ contains no vertices of valence 0 or 1 and, for every vertex v, the labelling function ϕ is injective on the outward-pointing edges incident on v. If a labelling is reduced, then any word in the elements of L will describe at most one edge path starting at any vertex v of Γ. A reduced word in L is a finite sequence of elements of L that contains no subword $(l, \tau(l))$. Conversely, if $\phi : \Gamma \to L$ is a reduced labelling, then any directed edge path in Γ can be described by its initial vertex and a word in the elements of L.

A *graphical presentation* is a 4-tuple (Γ, L, τ, ϕ), where Γ is a graph and ϕ is a reduced labelling of Γ by L. The group $G(\Gamma, L, \tau, \phi)$ presented by a graphical presentation is the group given by the following ordinary presentation. The generators are the elements of L, subject to the following relations: for each $l \in L$, $\tau(l)$ is the inverse of l, and for every simple cycle in Γ, the word in L obtained by going around that cycle is equal to the identity. This definition is used in [29, Thm. 1].

A reduced labelling of a graph Γ by L can be viewed as defining an immersion from Γ to the rose R_L, which is the 1-vertex graph with directed edges in bijective correspondence with the elements of L. We remind the reader that R_L has $|L|/2$ distinct 1-cells. The fundamental group of R_L is of course a free group with representatives of the τ-orbits in L as free generators. The group $G(\Gamma, L, \tau, \phi)$ can be viewed as the quotient of the fundamental group of R_L by the normal subgroup generated by the images of the fundamental groups of the components of Γ under this immersion. This is the definition of $G(\Gamma, L, \tau, \phi)$ used in [18, Def. 1.1]. It is immediate that the two definitions describe the same group, essentially because the fundamental group of a graph can always be generated as a normal subgroup by simple cycles.

The presentation 2-complex associated with a graphical presentation in both [18, 29] relies on a choice of free basis for the fundamental group of each component of Γ. We prefer a different complex that requires no such choice, but we shall show that our complex is homotopy equivalent to those used in [18, 29].

A *graphical* 2*-cell* of the graphical presentation (Γ, L, τ, ϕ) is the cone on a component of Γ. We emphasize that this definition does not appear in [18, 29]. In the case when Γ is a simplicial graph, each graphical 2-cell can be given the structure of a 2-dimensional simplicial complex with one new vertex at the cone point, and with new edges and triangles in bijective correspondence with the vertices and edges (respectively) of the relevant component of Γ. This is the natural generalisation of the conical subdivision of a standard 2-cell. The *boundary* of a graphical 2-cell is the base of the cone, a component of Γ.

The *graphical presentation* 2*-complex* associated with a graphical presentation (Γ, L, τ, ϕ) is the 2-dimensional space obtained by attaching the graphical 2-cells to the rose R_L, using the immersions induced by ϕ to identify the boundary of each graphical 2-cell with its image in R_L. As was mentioned in the introduction, a *graphical relator* is just a component of the labelled graph Γ. Just as in the classical case, the graphical presentation 2-complex is obtained by attaching graphical 2-cells corresponding to the graphical relators to a rose R_L.

The *graphical* 1-*skeleton* of the graphical presentation 2-complex is just the rose R_L. We emphasize that this will not usually be the whole 1-skeleton for any CW-structure on the graphical presentation 2-complex.

To show that our graphical presentation complex is naturally homotopy equivalent to those used in [18, 29], we will use the following proposition.

Proposition 2.4. *Let* (Γ, L, τ, ϕ) *be a labelled graph and for each component* Γ_i *of* Γ, *suppose that* X_i *and* Y_i *are contractible CW-complexes containing* Γ_i *as a subcomplex. Let* X *(resp.* Y*) be constructed by attaching each* X_i *(resp.* Y_i*) to* R_L *using the map* $\Gamma_i \to R_L$ *induced by* ϕ. *The complexes* X *and* Y *are homotopy equivalent relative to* R_L.

To prove this proposition, we require one lemma. Recall that a *CW-pair* (X, A) is a pair consisting of a CW-complex X and a subcomplex A.

Lemma 2.5. *Suppose that* (X, A) *and* (Y, A) *are CW-pairs, and that* Y *is contractible. Then, there is a map* $f : X \to Y$ *extending the identity map on* A, *and any two such maps are homotopic relative to* A.

If X *is also contractible, then* X *and* Y *are homotopy equivalent relative to* A.

Proof. As an inductive hypothesis, assume that f is defined on $X^n \cup A$, the union of A and the n-skeleton of X. (The induction can be started with $n = -1$ and the convention that the -1-skeleton of X is empty.) It suffices to show that f can be extended to each $(n+1)$-cell σ of X that is not contained in A. Since Y is contractible, any map from an n-sphere to Y extends to a map from the $(n+1)$-ball to Y. Applying this statement to the map $f : \partial\sigma \to Y$ proves the claim.

Given two maps $f, f' : X \to Y$ that extend the identity map on A, construct a homotopy $H : X \times I \to Y$ by a similar inductive process. Define $H(x,0) = f(x)$ and $H(x,1) = f'(x)$ for all $x \in X$ and define $H(a,t) = a$ for all $t \in [0,1]$. Now suppose as an inductive hypothesis that H is defined on $X \times \{0,1\} \cup A \times [0,1] \cup X^n \times [0,1]$. It suffices to show that H can be extended to $\sigma \times [0,1]$ for each $(n+1)$-cell σ of X. But H is already defined on the $(n+1)$-sphere $\partial(\sigma \times [0,1]) = \sigma \times \{0,1\} \cup \partial\sigma \times [0,1]$, and since Y is contractible, this map can be extended to the $(n+2)$-ball $\sigma \times [0,1]$.

For the second part, let $f : X \to Y$ and $g : Y \to X$ be maps extending the identity on A. Then, $g \circ f : X \to X$ and the identity on X both extend the identity on A, and so they are homotopic relative to A. Similarly, $f \circ g$ and 1_Y are self-maps of Y that extend the identity on A so they are homotopic relative to A. ∎

Proof of Proposition 2.4. By the lemma, for each component Γ_i of Γ, there are maps $f_i : X_i \to Y_i$ and $g_i : Y_i \to X_i$ extending the identity on Γ_i. There are also homotopies $g_i \circ f_i \sim 1_{X_i}$ and $f_i \circ g_i \sim 1_{Y_i}$ that fix Γ_i throughout. These maps and homotopies can be combined to give $f : X \to Y$ and $g : Y \to X$ extending the identity map on R_L and homotopies $g \circ f \sim 1_X$, $f \circ g \sim 1_Y$ that fix R_L throughout. In more detail, the map f is defined to be the identity on R_L and to be f_i on $X_i - \Gamma_i$. The definitions of g and the homotopies are similar. ∎

We are now ready to compare our graphical 2-complex with the ones used in [18, 29]. Both constructions when applied to a graphical presentation (Γ, L, τ, ϕ) start with the rose R_L. For each component Γ_i of Γ, Ollivier and Gruber choose a family of (based) cycles $C_{i,j}$ that freely generate the fundamental group $\pi_1(\Gamma_i)$. As subgraphs of Γ, these cycles are labelled via ϕ, and these maps are used as the attaching maps for 2-cells. Unlike our construction, this has the advantage that only ordinary 2-cells are used, so they do not need to distinguish between the 1-skeleton and the graphical 1-skeleton. On the other hand, our construction involves no choices.

Corollary 2.6. *For any graphical presentation (Γ, L, τ, ϕ), our graphical presentation complex and the one used in [18, 29] are homotopy equivalent relative to R_L.*

Proof. Both constructions contain the rose R_L. As described above, Gruber and Ollivier attach 2-cells to R_L indexed by cycles $C_{i,j}$, where Γ_i is a component of Γ and for fixed i the cycles $C_{i,j}$ form a free basis for $\pi_1(\Gamma_i)$. The same space may be obtained by a 2-stage process: first, form a space Y_i by attaching 2-cells to Γ_i along the cycles $C_{i,j}$, and then, use ϕ to identify $\Gamma_i \subseteq Y_i$ with its image in R_L. The fact that the cycles $C_{i,j}$ for fixed i form a free basis for $\pi_1(\Gamma_i)$ implies that Y_i is contractible. Now define X_i to be the cone on Γ_i. The hypotheses of Proposition 2.4 are satisfied with these choices of X_i and Y_i, and the claimed result follows, since the X in Proposition 2.4 is our graphical presentation 2-complex and the Y is the one from [18, 29]. ∎

The *graphical Cayley complex* associated with a graphical presentation (Γ, L, τ, ϕ) is the universal covering space of its graphical presentation complex. The group

$$G = G(\Gamma, L, \tau, \phi)$$

presented by the graphical presentation acts freely on the graphical Cayley complex. The inverse image of the rose R_L in the universal covering space is by definition the *graphical 1-skeleton* of the graphical Cayley complex. The graphical 1-skeleton contains one free orbit of vertices and its edges are labelled by L in such a way that the action of G preserves the labels. Thus, the graphical 1-skeleton is just the ordinary Cayley graph of G with respect to generators that are the (inverse pairs of) elements of L. The graphical Cayley complex may be constructed by attaching free G-orbits of graphical 2-cells to the graphical 1-skeleton. In more detail, one free G-orbit of copies of the cone on Γ_i is attached for each component Γ_i of Γ.

Having discussed graphical presentations and their graphical presentation complexes, we move on to discuss the graphical small cancellation condition.

A *piece* is a reduced word in L that defines edge paths that start at least at two distinct vertices of Γ. A piece is said to *belong to* each component of Γ that contains at least one of these vertices. The *length* of a piece is the number of letters in the word, or equivalently the length of the corresponding edge paths.

The *girth* of a graph Γ_i is the minimal length of any cycle in Γ_i. A graphical presentation satisfies the *small cancellation condition* $C'(1/6)$ if the length of any piece is strictly less than $1/6$ of the girth of each component to which the piece belongs. We are now ready to state the main theorem of graphical small cancellation theory.

Theorem 2.7. *If the graphical presentation* (Γ, L, τ, ϕ) *satisfies* $C'(1/6)$*, then the associated graphical Cayley complex X is contractible, and the attaching map for each graphical* 2*-cell is an injection from a component of* Γ *to the graphical* 1*-skeleton of* X*. Moreover, the Cayley graph* X^1 *has the 'Dehn property': any closed loop in* X^1 *contains strictly more than half of some cycle of* Γ *as a subpath.*

Proof. By Corollary 2.6, this statement will be true for our graphical presentation 2-complex if and only if it is true for the complex used in [18, 29]. We shall describe how to deduce the above result from theorems in these two sources. A version of this theorem is more readily seen in [29], but the version of parts of the theorem that can be found in [18] works with weaker hypotheses. Note also that both [18, 29] make the statement that the graphical presentation complex is aspherical, rather than the equivalent statement that the graphical Cayley complex is contractible.

Theorem 1 of Ollivier's article [29] implies the above result, except for two points. Firstly, because Ollivier is interested in hyperbolicity, Theorem 1 of [29] includes the condition that Γ should be finite. This condition is removed in Remark 21 of [29], at the expense of also removing the conclusion that $G(\Gamma, L, \tau, \phi)$ is hyperbolic. Secondly, Theorem 1 of [29] includes the extra hypothesis that Γ should be 'non-filamentous'; i.e., every edge of Γ should be contained in a non-trivial simple cycle. This hypothesis is an artefact of the proof and is not needed; in Remark 3 of [29], it is asserted that this hypothesis is used only to ensure that components of Γ inject into the graphical Cayley complex. This part of the theorem is proved in [18] without this hypothesis.

Theorem 2.18 of Gruber's article [18] proves that the graphical Cayley complex is contractible with our $C'(1/6)$ condition replaced by the much weaker hypothesis $C(6)$ (no cycle is a union of six or fewer pieces). Lemma 4.1 of [18] shows that the components of Γ inject into the graphical 1-skeleton (which is just the Cayley graph of G with respect to the generating set L). Again, Lemma 4.1 of [18] uses the small cancellation property Gr(6), which is even weaker than $C(6)$: see the discussion of these various conditions in and below Definitions 1.2 and 1.3 of [18]. Unfortunately, for our purposes, [18] does not discuss the Dehn property at all, but as stated above, this part of the theorem is proved in [29]. ∎

Remark 2.8. The reader who wishes to use only results concerning graphical small cancellation that are stated in [29] should note that all of the graphs that we will use in the graphical presentation for $H(S)$ will be non-filamentous in the sense of Ollivier provided that the 1-skeleton K^1 of K has no cut points. This extra hypothesis on K does hold for the examples that we construct below.

Remark 2.9. We use the phrase 'Dehn property' rather than 'Dehn algorithm' because we are interested in the case when Γ has infinitely many components although each component will be a finite graph. In this case, the Dehn property by itself does not suffice to produce an algorithm to decide whether a path in the Cayley graph is closed, i.e., to solve the word problem in the group. The other ingredient needed is an algorithm to list all of the cycles in Γ of at most a given length. (This issue arises already in the classical case.)

Before introducing the graphical presentation for the groups $H(S)$ that is central to our discussion, we discuss a well-known example in which graphical small cancellation can be applied while classical small cancellation does not apply directly.

Example 2.10. Let T be a torus with a small open disc removed. The fundamental group of T is free of rank two, and with respect to a natural choice of generating set the bounding curve of the disc represents the commutator $[a, b]$ of free generators. Now consider the space obtained by taking two copies of T and identifying their boundary curves. The van Kampen theorem gives a presentation for the fundamental group of this space, which is of course a closed orientable surface of genus two:

$$\langle a, b, c, d : [a, b] = [c, d] \rangle.$$

In the relator $[a, b][c, d]^{-1}$, the only pieces are single letters, and so this group presentation satisfies the $C'(1/6)$ condition (and even the $C'(1/7)$ condition). The presentation 2-complex obtained by attaching the octagonal 2-cell to the rose with petals a, b, c, d is homeomorphic to the given closed surface. This surface admits a Riemannian metric of constant curvature -1, and for any such metric, its universal covering space (which is the Cayley 2-complex for the given presentation) is identified with the hyperbolic plane $\mathbb{H}$. By choosing the metric as symmetrically as possible, one identifies the Cayley complex with a tesselation of $\mathbb{H}$ by regular hyperbolic octagons with interior angles $\pi/4$. (Note that eight of these octagons will meet at each vertex of the tesselation.) With respect to this metric, the circle where the two copies of T were joined together represents a long diagonal of the octagon. In the conical subdivision of the Cayley 2-complex, each regular octagon is replaced by eight isosceles triangles with angles $\pi/4, \pi/8, \pi/8$. Each of the two copies of T is made by identifying some of the sides of four of these triangles.

Now repeat the above, but instead of identifying the bounding circles of two copies of T, identify the bounding circles of *three* copies. The fundamental group G of the resulting space X has the presentation

$$G = \langle a, b, c, d, e, f : [a, b] = [c, d] = [e, f] \rangle.$$

The classical small cancellation conditions fail dismally: there are three natural choices of octagonal relators, but any two of them intersect in a piece of length 4. On the other hand, the symmetrical Riemannian metrics defined on each union of two copies of T coincide on their intersections, yielding a locally CAT(-1)-metric on X, so the fundamental group is hyperbolic.

The solution is to take one graphical relator. Take a graph Γ consisting of two vertices of valence three, joined by paths consisting of four edges; equivalently, this is the barycentric subdivision of the complete bipartite graph $K(3,2)$. If x, y are the two vertices of valence three, label the three paths from x to y by the three commutators $[a,b]$, $[c,d]$ and $[e,f]$. In this single graphical relator, the pieces once again consist of single letters, and so the graphical $C'(1/6)$ condition holds. For this group, Ollivier's graphical presentation complex is obtained by attaching two of the possible octagons to the 6-petalled rose, whereas our graphical presentation complex is obtained by attaching the cone on Γ to the 6-petalled rose. In this case, our graphical presentation 2-complex is homeomorphic to X, and even isometric to X if we view the cone on Γ as made from three halves of the regular octagon. The graphical Cayley complex is obtained from the Cayley graph of the group with respect to the generators a,b,c,d,e,f by attaching a free G-orbit of cones on Γ.

Since Γ is a simplicial graph, the conical subdivision of the graphical Cayley complex is a simplicial complex. It has two free orbits of vertices. Let v be a lift of the vertex that covers the 0-cell of the rose, and let u be a lift of the cone vertex in the cone on Γ. The link of the vertex u is identified with Γ, so we may fix our choices in such a way that v is adjacent to u and furthermore the edge $\{u,v\}$ corresponds to the vertex $x \in \Gamma$. There are 10 other vertices adjacent to u: these vertices are av, abv, $aba^{-1}v$, $[a,b]v$, cv, cdv, $cdc^{-1}v$, ev, efv, $efe^{-1}v$. The group elements arising here are the elements represented by paths in Γ from x to another vertex. The link of the vertex v is larger: it contains twelve other vertices in the same orbit (the vertices gv where g is either a generator or the inverse of a generator) together with eleven vertices in the orbit of u. Each 2-simplex of the conical subdivision contains one vertex in the orbit of u and two vertices in the orbit of v. Realizing each 2-simplex as a hyperbolic isosceles triangle with angles $\pi/4, \pi/8, \pi/8$ gives a G-equivariant CAT(-1)-metric on the (conical subdivision of the) graphical Cayley complex since Gromov's link condition [6, Ch. II.5.24] is easily verified. The locally CAT(-1)-metric that this induces on the quotient space is isometric to the locally CAT(-1)-metric on X mentioned above.

Any graph Γ admits a *tautological labelling*, in which the set L is just the set of directed edges of Γ, the function τ sends an edge to itself with the opposite orientation and the function ϕ is the identity map.

Given a labelling of a graph Γ and n a non-zero integer, the *degree n subdivision* of Γ is the labelled graph obtained by subdividing each edge of Γ into $|n|$ parts. If $n > 0$, then the label attached to each of the n new directed edges contained in the directed edge e is $\phi(e)$, whereas if $n < 0$, the label attached to each new directed edge contained in e is $\tau \circ \phi(e)$. It may be helpful to imagine that the graph is rescaled by a factor of $|n|$, so that the edges of the degree n subdivision are 'the same length' as the edges of the original graph. See Figure 1 for an example of a labelled graph and two of its subdivisions.

We are now ready to define the graphical presentation for the groups $H(S)$.

Definition 2.11. Take a spectacular complex K, and take the tautological labelling of its 1-skeleton K^1. The generators for $H(S) = H(K,Z,S)$ are the directed edges of K. For each $n \in Z - S$ and for each polygon P of K, the degree n subdivision of the tautological labelling on ∂P is a graphical relator. For each $n \in S$, the degree n subdivision of the tautological labelling on K^1 is a graphical relator.

3. Using graphical small cancellation

In this section, we prove Theorem 1.2. First, we need to establish that graphical small cancellation can be applied.

Proposition 3.1. *The given graphical presentation for $H(S)$ satisfies the small cancellation condition $C'(1/6)$.*

Proof. This is a simple check. The shortest cycle in the degree m subdivision of K^1 is of length $g|m| \geq 13|m|$, and the shortest cycle in the degree m subdivision of the polygon boundary ∂P is of length $l_P|m| > 26|m|$. Now suppose that $m \neq n$ are non-zero integers with $|m| < |n|$. If $mn > 0$, then the longest pieces contained in both a degree m subdivision and a degree n subdivision are of the form $a^m b^m$, where either (a, b) or (b, a) is a pair of consecutive edges in K^1. If on the other hand $mn < 0$, then the longest pieces are of the form a^m.

Between the degree m subdivision of either K^1 or a polygon boundary ∂P and itself, the longest pieces are of the form $a^{|m|-1}$, of length $|m| - 1$. The longest pieces between the degree m subdivisions of distinct polygon boundaries ∂P and ∂Q are potentially much longer, of length $|m|$ times the length of a piece of $\partial P \cap \partial Q$. But since the polygons of K satisfy the $C'(1/6)$ condition, it follows that each piece of the intersection of the degree m subdivisions of ∂P and ∂Q has length strictly less than both $|m|l_P/6$ and $|m|l_Q/6$. ■

For $S \subseteq Z$, let $E(S)$ denote the standard graphical Cayley 2-complex for $H(S)$. By Proposition 3.1 and Theorem 2.7, we see that $E(S)$ is contractible, with a free action of $H(S)$, and hence that $E(S)/H(S)$ is an Eilenberg–Mac Lane space for $H(S)$. Similarly, $E(T)/H(T)$ is an Eilenberg–Mac Lane space for $H(T)$.

For the proof of Theorem 1.2, we will need to compare, for $S \subseteq T \subseteq Z$, two free $H(T)$-complexes: the standard complex $E(T)$ and the quotient $E(S)/K_{S,T}$. (We remind the reader that $K_{S,T}$ is by definition the kernel of the natural surjective homomorphism $H(S) \to H(T)$; for any complex E with an $H(S)$-action, the quotient $E/K_{S,T}$ admits an $H(T)$-action.)

Since we know that $E(S)/H(S)$ is an Eilenberg–Mac Lane space for $H(S)$, the general theory of covering spaces tells us that $E(S)/K_{S,T}$ is an Eilenberg–Mac Lane space for $K_{S,T}$ which admits a free cellular action of $H(T)$. Define the graphical 1-skeleton of $E(S)/K_{S,T}$ to be the image of the graphical 1-skeleton of $E(S)$, and define a graphical 2-cell of $E(S)/K_{S,T}$ to be the image of a graphical 2-cell of $E(S)$. With this definition, the graphical 1-skeleton of each of $E(T)$ and $E(S)/K_{S,T}$ is the Cayley graph of $H(T)$ with respect to the same generating set. Thus, these two free $H(T)$-spaces have the same graphical 1-skeleta. They also share many of the same graphical 2-cells. The difference between them is that for each $n \in T - S$, $E(S)/K_{S,T}$ contains a free orbit of polygonal 2-cells attached along the degree n subdivision of ∂P for each polygon P of K, whereas $E(T)$ contains a free orbit of graphical 2-cells attached along the degree n subdivision of the graph K^1 itself. It may be seen that the attaching map for each graphical 2-cell is injective. For the graphical 2-cells of $E(T)$, this follows directly from the small cancellation properties of the presentation for $H(T)$. For the 2-cells that belong only to $E(S)/K_{S,T}$, i.e., the cells attached along the degree n subdivision of ∂P for $n \in T - S$, this follows from the fact that ∂P maps injectively into K^1, and the degree n subdivision of K^1 is already known to map injectively into the graphical 1-skeleton.

There is a 3-dimensional $H(T)$-complex F that contains both $E(S)/K_{S,T}$ and $E(T)$ as subcomplexes. The fundamental theorem of graphical small cancellation tells us that inside $E(T)$, there is a free $H(T)$-orbit of copies of cones on the degree n subdivision of K^1, for each $n \in T - S$. These copies of the degree n subdivision of K^1 are of course also present inside $E(S)/K_{S,T}$ since the graphical 1-skeleta are equal. However, in $E(S)/K_{S,T}$, instead of having a single cone vertex attached to each such copy, there is a specific subcomplex K_0 homeomorphic to K, which has this copy of the degree n subdivision of K^1 as its 1-skeleton, and in which each polygon P of K is attached to the degree n subdivision of ∂P. To make F from $E(S)/K_{S,T}$, attach a free $E(T)$-orbit of cones on K to the $E(T)$-orbit of K_0. The inclusion of K^1 into K induces an embedding of the cone $C(K^1)$ on K^1 into the cone $C(K)$, and hence an $H(T)$-equivariant inclusion $E(T) \to F$.

The reader may prefer to understand the construction in terms of the conical subdivisions of $E(T)$, F and $E(S)/K_{S,T}$, which are simplicial complexes with free $H(T)$-actions. The graphical 1-skeleton of $E(T)$ and of $E(S)/K_{S,T}$ is the cover of the rose R_L: it has one free orbit of vertices, with say v as an orbit representative. The directed edges between vertices in this orbit are labelled by elements of L, so that each vertex in this orbit is adjacent to $|L|$ other vertices in this orbit. For each $n \in S$, both $E(S)/K_{S,T}$ and $E(T)$ have another orbit of vertices with orbit representative u_n. The link of each vertex in the orbit of u_n contains only vertices in the orbit of v and is a copy of the degree n subdivision of K^1. For each $n \in Z - T$ and each polygon P of K, both $E(S)/K_{S,T}$ and $E(T)$ have an orbit of vertices with orbit representative $x_{n,P}$. The link of each vertex in this orbit again contains only vertices in the orbit of v and is a copy of the degree n subdivision of ∂P. The difference between $E(S)/K_{S,T}$ and $E(T)$ arises for the vertices corresponding to $n \in T - S$. For $n \in T - S$, $E(T)$ contains an orbit of vertices of type u_n, with a link of a copy of the degree n subdivision of K^1, whereas $E(S)/K_{S,T}$ instead contains orbits of vertices of type $x_{n,P}$ for each polygon P of K, whose links are copies of the degree n subdivision of ∂P. In these terms, the conical subdivision of the complex F is obtained by taking *both* of these types of vertices: for $n \in T - S$, the complex F contains the vertices $x_{n,P}$ for each polygon P, together with the vertices u_n. The link of each u_n vertex here contains exactly one vertex in the orbit of $x_{n,P}$, as well as a number of v-vertices. The link of each vertex in the u_n orbit for $n \in T - S$ is thus a 2-complex isomorphic to the conical subdivision of the complex obtained from K by taking the degree n subdivision of its 1-skeleton.

Proposition 3.2. *With notation as above, the inclusion $E(S)/K_{S,T} \to F$ is an $H(T)$-equivariant homology isomorphism and the inclusion $E(T) \to F$ is an $H(T)$-equivariant homotopy equivalence, whose image is an $H(T)$-equivariant deformation retract of F.*

Proof. The first claim follows because attaching a cone to an acyclic subspace does not change homology. For the second claim, note that for a polygon P, the cone $C(\partial P)$ on ∂P is a deformation retraction of the cone $C(P)$. Putting these deformation retractions together, it follows that the cone $C(K^1)$ on K^1 is a deformation retraction of the cone $C(K)$ on K. Applying this retraction simultaneously over all such cones appearing in F, it follows that $E(T)$ is an $H(T)$-equivariant deformation retraction of F as claimed. ∎

Proposition 3.3. *For P, a polygon of K with directed bounding cycle $a_1, \dots, a_l$, let H'_P be the group defined by the presentation*

$$H'_P = \langle a_1, \dots, a_l : a_1^n \cdots a_l^n,\ n \in Z \rangle.$$

The inclusion of generating sets induces an injective homomorphism $H'_P \to H(S)$ for each $S \subseteq Z$. In particular, this map gives an isomorphism from H'_P to $H_P(S)$, the subgroup of $H(S)$ generated by the edges of P.

Proof. The relators in the given presentation for H'_P all hold between the corresponding generators of $H(S)$, so there is a homomorphism $H'_P \to H(S)$ as claimed. It remains to show that this homomorphism is injective. Since $H(S)$ maps surjectively onto $H(Z)$, it clearly suffices to show the claim in the case $S = Z$. The given presentations for H'_P and $H(S)$ satisfy the $C'(1/6)$ property, so have the Dehn property. To spell this out in greater detail, we recall that a word in the generators of a group (and their inverses) is said to be *reduced* if it contains no subword of the form aa^{-1} or $a^{-1}a$. The Dehn property for the two given presentations is the following: any reduced word in the generators of H'_P that is equal to the identity in H'_P contains strictly more than half of one of the defining relators as a subword, and any reduced word in the generators of $H(S)$ that is equal to the identity in $H(S)$ contains strictly more than half of the word spelt around a simple cycle in one of the defining graphical relators as a subword.

Say that a word in the generators for H'_P is *H'_P-reduced* if it is reduced and does not contain more than half of any defining relator as a subword. By induction on word length, the Dehn property shows that any word in the generators for H'_P represents the same element of H'_P as some H'_P-reduced word. Similarly, say that a word in the generators for $H(Z)$ is *$H(Z)$-reduced* if it is reduced and does not contain more than half of any simple cycle in any of the labelled graphs used to define $H(Z)$ as a subword. Once again, applying the Dehn property and induction, one sees that every word in the generators for $H(Z)$ represents the same group element as some $H(Z)$-reduced word.

It suffices to show that the image in $H(Z)$ of any non-trivial H'_P-reduced word is not the identity. Not every H'_P-reduced word will be $H(Z)$-reduced, which complicates the argument considerably; however, we claim that every non-trivial H'_P-reduced word is equal as an element of $H(Z)$ to some non-trivial $H(Z)$-reduced word.

For the remainder of this proof, we temporarily redefine a *piece* to be a reduced word that defines a path in at least *one* of the (graphical) relators for the given presentation for $H(Z)$. Our justification for this term is that we will be considering pieces that arise as subwords of our given H'_P-reduced word, so the pieces that we will consider will still appear twice, once in a relator and once in our given word.

Suppose that a pair a, b of distinct directed edges of K have exactly one vertex in common. After possibly replacing each of the edges by its opposite, we may suppose that the terminal vertex of a is equal to the initial vertex of b. In this case, the word $a^l b^m$ with $l, m \neq 0$ is a piece of the degree n subdivision of K^1 if and only if $lm > 0$, $mn > 0$ and $|n| \geq |l|$, $|n| \geq |m|$. This word is a piece of the degree n subdivision in a unique way since the division between the a-edges and the b-edges can only appear at one vertex.

Moving on to words in three letters, if a, b, c is a directed edge path in K^1, and $l, m, n \in \mathbb{Z} - \{0\}$, then $a^l b^m c^n$ can only appear as a piece of the degree m subdivision of K^1, and will do so precisely when all of the following hold: $lm > 0$; $mn > 0$; $|l| \leq |m|$; $|n| \leq |m|$. Although we made the above statements for the subdivision of K^1, if the directed edges a, b, c are contained in P, then the same statements hold for pieces of the subdivision of P.

Given an H'_P-reduced word w, we consider how it breaks up into maximal pieces. For example, if $w = ua^n v$ where the final letter of u and the initial letter of v have no vertex in common with a (when viewed as directed edges of K), then a^n is a maximal piece. If a piece contains a subword $a^l b^m c^n$, where a, b, c is a directed edge path in P with $l, m, n \neq 0$, this piece can occur only in the degree m subdivision of P; in this case, we call it a *piece of degree m*. Short pieces, by which we mean pieces that consist of either a power of a single letter or a product of two powers of single letters, do not have a well-defined degree. The maximal pieces of w may intersect non-trivially. If $m, n > 0$ with $m \neq n$, the intersection of a maximal piece of degree m and a maximal piece of degree n may have length up to $2\min\{m, n\}$, with the worst case represented by $\ldots x^m y^m z^m a^n b^n c^n \ldots$, where x, y, z, a, b, c are consecutive directed edges (i.e., they form a directed edge path). If $m > 0$, the intersection of two distinct maximal pieces of degree m can be at most length $m - 1$, with the worst case represented by $\ldots x^m y^m z^m a^{m+1} b^m c^m \ldots$ or by $\ldots x^m y^m z^m a^{m-1} b^m c^m \ldots$. These considerations show that it is possible for a maximal piece of well-defined degree to be entirely covered by its neighbours, but that the longest such maximal pieces consist of powers of at most four letters: if $m > 0$ and $l, n > m$, then the word $\ldots x^l y^l z^m a^m b^n c^n \ldots$ contains the maximal piece $y^m z^m a^m b^m$ which is entirely covered by the neighbouring pieces of degrees l and n. In general, the length of the intersection is not the important feature: what is crucial to our argument is that the intersection of maximal pieces of distinct degrees consists of at most two powers of letters, and that the intersection of two distinct maximal pieces of the same degree consists of a power of a single letter.

By definition, if w is an H'_P-reduced word, then w contains no degree m pieces of length greater than $(m/2)l_P$. However, such a word will not necessarily be $H(Z)$-reduced: a piece of degree m that is H'_P-reduced may be further shortened using a 'shortcut' across the polygon P that contains edges from K^1 that are not in P. Such a shortcut will necessarily go between distinct vertices of K^1 of valence at least three. If there is a maximal degree m piece in w that is H'_P-reduced but not $H(Z)$-reduced, we replace it by its $H(Z)$-reduction. Since this reduction involves a path between two vertices of K^1 of valence at least three that is not contained in P, condition 3 of Definition 1.1 implies that the $H(Z)$-reduction of this piece contains a subword of the form $r^m s^m t^m u^m v^m$, where r, s, t, u, v are consecutive directed edges of K^1 that are not contained in P. In particular, this piece can only appear in the degree m subdivision of K^1, it cannot be entirely covered by its two neighbours, and it has well-defined endpoints (vertices of the degree m subdivision of K^1) that are equal to those of the piece that it replaced. Provided that any two pieces that are long enough to support shortcuts are separated by smaller pieces, this shows that there can be no further cancellation once each piece with a shortcut has been reduced in this way.

The remaining difficulty is the case when two longer pieces both admitting shortcuts are adjacent, including the case when they overlap. If these pieces have the same degree, m say, then they are not compatible, in the sense that when they are both fitted into the degree m subdivision of K^1, the intersection of their images there will not be precisely equal to their intersection in w; if this was not the case, then they would combine to form a single piece of degree m. But now the same property also holds for their $H(Z)$-reductions, and so their reductions cannot be combined into a longer piece. Note also that the intersection of two such pieces consists of at most a power of a single letter. It remains to consider the case when two long pieces corresponding to different m and n are adjacent. In this case, it is possible for their endpoints to match up and some cancellation may take place at their boundaries. Viewing the terminal vertex of the first piece as a vertex of the degree m subdivision of $P \subseteq K^1$ and the initial vertex of the second piece as a vertex of the degree n subdivision of $P \subseteq K^1$, the only potential problem is when these two vertices correspond to a single vertex of K^1 of valence at least three, as opposed to corresponding to vertices of the subdivision that appear somewhere in the middle of an edge of K^1. Again, no reduction will occur unless the same edge of $K^1 - P$ appears with opposite sign at these ends. For example, the first piece might end $r^m s^m t^m u^m v^m$ and the second piece might begin $v^{-n} u^{-n} t^{-n} s^{-n} r^{-n}$, with $m, n > 0$. Even in this case, since $m \neq n$ at most a power of one letter is lost, either from the end of the first piece (if $m < n$) or from the beginning of the second piece (if $m > n$), and so the first piece still has degree m and the second piece has degree n. Thus, no two adjacent long pieces can combine to form a longer piece, and the reduction (in the free group) of the word that they spell together is $H(Z)$-reduced and non-trivial. ■

Remark 3.4. The reader may prefer an alternative account of the above proof in terms of the simplicial complexes that are the conical subdivisions of the graphical Cayley complexes for H'_P and for $H(Z)$. The graphical 1-skeleton of this complex is the Cayley graph of the group H'_P (resp. $H(Z)$) with the directed edges of P (resp. K) as generators. For each $n \in Z$ there is an orbit of 2-cells (resp. graphical 2-cells) attached, with boundary being the degree n subdivision of ∂P (resp. the degree n subdivision of K^1). In the conical subdivision, this gives rise to an extra orbit of vertices for each $n \in Z$. Let v' (resp. v) be an orbit representative of the vertices in the graphical 1-skeleton, and for $n \in Z$, let u'_n (resp. u_n) be an orbit representative of the vertices at the centre of the degree n subdivision of ∂P (resp. at the cone point in the cone on the degree n subdivision of K^1).

In this simplicial complex, a word is represented by an edge path that stays in the graphical 1-skeleton (i.e., that only passes through vertices in the orbit of v' (resp. v)). A reduced word is such an edge path that never reverses its direction. A piece (in the sense of the above proof) is a connected subpath that is contained in the link of one of the cone vertices (i.e., the vertices in the orbits u'_n (resp. u_n)). A piece consisting of just a power of a single letter is contained in the links of many different cone vertices. A piece consisting of a product of two powers of letters is also contained in the links of many different cone vertices but is contained in the link of at most one vertex in each orbit. A piece of degree n is contained in the link of a single vertex in the orbit of u'_n (resp. u_n) and in the link of no other vertex. A reduced word is H'_P-reduced (resp. $H(Z)$-reduced) if each of its pieces of degree n consists of a shortest path in the link of its cone vertex between its two end points.

The Dehn property tells us that any non-trivial edge path in the graphical 1-skeleton that is reduced and H'_P-reduced (resp. reduced and $H(Z)$-reduced) has distinct end points.

The homomorphism $H'_P \to H(Z)$ gives a local embedding of the simplicial complex for H'_P into the simplicial complex for $H(Z)$, and the content of Proposition 3.3 is that this map is actually an embedding. To prove this, we start with an edge path that is non-trivial, reduced and H'_P-reduced and consider its image in the (subdivided) graphical Cayley complex for $H(Z)$. Since there may be shortcuts in K^1 between the ends of a shortest path in P, the image need not be $H(Z)$-reduced. However, the only places where it can fail to be $H(Z)$-reduced are pieces that are sufficiently long that they have a well-defined degree. For each such piece of degree n, we start by replacing the piece by a shortest path in the degree n subdivision of K^1 between its end points. If any two pieces of well-defined degree are separated by shorter pieces, then this replacement is already reduced and $H(Z)$-reduced. If on the other hand there are pieces of well-defined degree that are adjacent in the path, it is possible that the $H(Z)$-reductions of these pieces intersect in such a way as to introduce back-tracking, so that the new path is no longer reduced. Removing the backtracking cannot reduce the number of powers of letters involved to fewer than three (since at least five letters were involved originally and at most one letter can be lost at each end of each piece), so removing backtracking cannot change a piece of degree n to a piece that is too short to have a well-defined degree. Hence, the new path is non-trivial, reduced and $H(Z)$-reduced, which establishes that H'_P embeds in $H(Z)$.

Proof of Theorem 1.2. Since the generating sets of $H(S)$ and $H(T)$ are identified with each other, the homomorphism $H(S) \to H(T)$ is surjective. To see that its kernel is non-trivial, let $(a_1, \ldots, a_g)$ be a directed loop in the graph K^1 of length equal to the girth of K^1, let n be an element of $T - S$ and consider the element $h := a_1^n a_2^n \cdots a_g^n$ of $H(S)$.

This element is contained in $K_{S,T}$ and we claim that it is not the identity element in $H(S)$. By Proposition 3.1, the given graphical presentation for $H(S)$ is $C'(1/6)$. If h represents the identity element, then each path in the Cayley graph that follows the word defining h is a closed loop. In this case, by the main theorem of graphical small cancellation, Theorem 2.7, this loop must contain more than half of some cycle in one of the relator graphs. By condition (5) of Definition 1.1, the perimeter of each polygon is more than twice the girth g and so h cannot contain more than half of the degree n subdivision of ∂P for P, any polygon of K. The given word for h also contains only pieces consisting of less than $1/6$ of the length of any cycle in a defining relator of $H(S)$ of degree $m \neq n$ since any such piece is of the form $a_i^l a_{i+1}^l$ for l equal to the minimum of $|m|$ and $|n|$, and $g \geq 13$.

It remains to show that the kernel $K_{S,T}$ is acyclic. Since $E(S)$ is contractible and $K_{S,T}$ acts freely cellularly on $E(S)$, it follows that $E(S)/K_{S,T}$ is an Eilenberg–Mac Lane space for $K_{S,T}$. By Proposition 3.2, $E(S)/K_{S,T}$ has the same homology as F, which is contractible because it has the same homotopy type as $E(T)$. ∎

4. Graphs of groups

In this section, we prove Proposition 1.3 and Corollary 1.4. We start by proving two general results that we will use.

Proposition 4.1. *Let $f : X \to Y$ be a map of Eilenberg–Mac Lane spaces, and suppose that the induced map $f_* : G \to Q$ of fundamental groups is surjective, with kernel N. Let $\pi : \widetilde{Y} \to Y$ be the universal covering of Y, and let P be the pullback of this covering along f. Then, P is an Eilenberg–Mac Lane space for N.*

Proof. Recall that the pullback P is defined by

$$P = \{(x, y) \in X \times \widetilde{Y} : f(x) = \pi(y)\}.$$

For any $f : X \to Y$, this is a covering space of X that is regular and has Q as a group of deck transformations, where the action of $q \in Q$ is defined by $q(x, y) = (x, qy)$ and the covering map is the map $(x, y) \mapsto x$. The action of Q is transitive on each fibre of this map. In the case when $f_* : G \to Q$ is *surjective*, it may be shown that P is connected. To see this, if $\pi(y) = \pi(y') = f(x)$, then there is a loop β in Y that lifts to a path in $\widetilde{Y}$ from y to y'. Now let γ be a loop in X based at x so that $f \circ \gamma$ is in the same homotopy class as β. The loop γ and a lift to $\widetilde{Y}$ of the loop $f \circ \gamma$ together define a path in P from (x, y) to (x, y'). Since P is connected and Q acts transitively on each fibre of the map $P \to X$, it follows that in this case, Q is the whole group of deck transformations.

Up to isomorphism, there is only one connected regular covering of X with Q as its group of deck transformations: the space $\widetilde{X}/N$, where $\widetilde{X}$ is the universal covering of X. This space is an Eilenberg–Mac Lane space for N and we have seen that it is homeomorphic to P. ∎

Proposition 4.2. *For $i \in \{1, \dots, m\}$, let H_i be a subgroup of a group H and let N be a normal subgroup of H so that $N \cap H_i$ is trivial for each i. Let G_i be another group that contains H_i as a subgroup. Let G be the fundamental group of the star-shaped graph of groups with m arms where the central vertex group is H, the edge groups are $H_1, \dots, H_m$ and the outer vertex groups are $G_1, \dots, G_m$. Let $\overline{G}$ be the fundamental group of a similar graph of groups in which the central vertex group is replaced by $\overline{H} = H/N$. The quotient map $H \to H/N$ together with the identity maps on the edge groups and outer vertex groups induces a surjective group homomorphism $G \to \overline{G}$ and the kernel of this homomorphism is isomorphic to a free product of copies of N, where the copies are indexed by the cosets of $\overline{H}$ in $\overline{G}$.*

Proof. We use the previous proposition to construct an Eilenberg–Mac Lane space for the kernel of the map $G \to \overline{G}$ from which the stated result will be apparent. Let $\overline{X}$ be an Eilenberg–Mac Lane space for $\overline{H}$, let X be an Eilenberg–Mac Lane space for H, and let $f : X \to \overline{X}$ be a based map that realizes the quotient map $H \to \overline{H}$. Let Y_i be an Eilenberg–Mac Lane space for H_i, let Z_i be an Eilenberg–Mac Lane space for G_i and let $f_i : Y_i \to X$ and $g_i : Y_i \to Z_i$ be based maps so that $f_{i*} : H_i \to H$ and $g_{i*} : H_i \to G_i$ are the inclusions. Define $\overline{f_i} = f \circ f_i$ and define $\overline{g_i} = g_i$.

We can use the spaces X, Y_i, Z_i together with the maps f_i, g_i to make a star-shaped graph of spaces and we can use the spaces $\overline{X}$, Y_i, Z_i together with the maps $\overline{f_i}$, $\overline{g_i}$ to make a second star-shaped graph of spaces. By part 2 of Proposition 2.3, these spaces are Eilenberg–Mac Lane spaces for G and $\overline{G}$, respectively. Moreover, we may define a map of graphs of spaces by taking the map $f : X \to \overline{X}$ on the central vertex space and the identity map on each edge space Y_i and on each outer vertex space Z_i. This gives us an explicit map of Eilenberg–Mac Lane spaces inducing the surjection $G \to \overline{G}$ on fundamental groups. By Proposition 4.1, the pullback of the universal covering space for the space for $\overline{G}$ along this map is an Eilenberg–Mac Lane space for the kernel.

The universal covering space of the graph of spaces that is an Eilenberg–Mac Lane space for $\overline{G}$ is well understood: it can be viewed as another star-shaped graph of spaces, where each of the spaces arising is a disjoint union of copies of the universal cover of the original space, so that for example over the edge $I \times Y_i$ lies a disjoint union of copies of $I \times \widetilde{Y_i}$. This space can also be understood as a graph of spaces in a different way, with each vertex and edge space being a single component of the union described above. In this way, the universal covering is described as a graph of contractible spaces. Since it is by definition simply connected, the graph underlying this graph of spaces is a tree: this is the Scott–Wall approach to constructing the Bass–Serre tree for $\overline{G}$ as a graph of groups [19, Ch. 1.B] or [31].

The pullback space can also be understood as a graph of groups whose underlying graph is equal to the Bass–Serre tree for $\overline{G}$ expressed as a star-shaped graph of groups. In this case, every edge space is contractible, and the vertex spaces that correspond to leaf vertices of the star-shaped graph are also contractible, while the vertex spaces over vertices that map to the central vertex of the star are Eilenberg–Mac Lane spaces for N.

Thus, the kernel of the group homomorphism is the free product of copies of N indexed by the vertices of the Bass–Serre tree for $\overline{G}$ that map to the central vertex of the star, or equivalently indexed by the cosets $\overline{G}/\overline{H}$. ■

Proposition 4.3. *For each $S \subseteq T \subseteq Z$, the kernel of the homomorphism $G(S) \to G(T)$ is acyclic.*

Proof. Both $G(S)$ and $G(T)$ are constructed as star-shaped graphs of groups, with edges indexed by the polygons of K. The leaf vertex groups are G_P in each case and the edge groups are H_P in each case. The only difference is that the central vertex group is $H(S)$ for constructing $G(S)$ and $H(T)$ for constructing $G(T)$. Thus, the hypotheses of Proposition 4.2 are satisfied, and we deduce that the kernel of the surjection $G(S) \to G(T)$ is isomorphic to the free product of copies of $K_{S,T}$ and so is itself acyclic. ■

Proposition 4.4. *There is a finite Eilenberg–Mac Lane space for the group $G(\emptyset)$ as in Proposition 1.3. If each G_P has a finite 2-dimensional Eilenberg–Mac Lane space, then so does $G(\emptyset)$.*

Proof. Let X be the presentation 2-complex for $H(\emptyset)$, which is built from standard cells and has finite 1-skeleton. For each polygon P, let Y_P be the presentation 2-complex for H_P, and let Z_P be a finite Eilenberg–Mac Lane space for G_P. Since the generators and relations for each H_P are a subset of those of $H(\emptyset)$, the inclusion $H_P \to H(\emptyset)$ is induced by an isomorphism between Y_P and a subcomplex of X. Note also that each 2-cell of X is contained in exactly one of these subcomplexes.

If $f_P : Y_P \to Z_P$ is a map that induces the embedding $H_P \to G_P$, then an Eilenberg–Mac Lane space W for $G(\emptyset)$ can be built by taking a copy of the mapping cylinder of $f_P : Y_P \to Z_P$ for each P, and identifying the copy of Y_P with its image in X. If Y_P^1 denotes the finite 1-skeleton of Y_P, then the mapping cylinder of the restriction $f_P|_{Y_P^1}$ is a deformation retract of the mapping cylinder for f_P and is a finite complex. Since each 2-cell of X belongs to a unique polygon P, these deformation retractions can be combined. This shows that the finite complex W' obtained from X^1 and the mapping cylinders of the maps $f_P|_{Y_P^1}$ by identifying each copy of Y_P^1 with its image in X^1 is a deformation retract of W. If each Z_P is 2-dimensional, then so is W'. ■

Proof of Corollary 1.4. Recall from [24, Sec. 15] the set-valued invariant $\mathcal{R}(\mathbf{g}, G) \subseteq \mathbb{Z}$ for a group G and a sequence $\mathbf{g} = (g_1, \dots, g_l)$ of elements of G, defined by

$$\mathcal{R}(\mathbf{g}, G) = \{n \in \mathbb{Z} : g_1^n g_2^n \cdots g_l^n = 1\}.$$

In [24, Prop. 15.2], three properties of this invariant were established: for a fixed isomorphism type of countable group G, the invariant takes only countably many values as $\mathbf{g}$ varies; if $H \geq G$, then $\mathcal{R}(\mathbf{g}, G) = \mathcal{R}(\mathbf{g}, H)$; if G is finitely presented, then $\mathcal{R}(\mathbf{g}, G)$ is recursively enumerable.

Let $a_1, \dots, a_g$ be a directed loop in K of length equal to the girth of K^1. Viewing this loop as a sequence of elements of $H(S)$, the Dehn property implies that for any $S \subseteq Z$,

$a_1^n \cdots a_g^n = 1$ in $H(S)$ if and only if $n \in S$. Hence, for any $S \subseteq Z$, one has

$$\mathcal{R}\big((a_1, \ldots, a_g), G(S)\big) = \mathcal{R}\big((a_1, \ldots, a_g), H(S)\big) = S \cup \{0\}.$$

From this together with the known properties of the invariant $\mathcal{R}$, it follows that there are continuously many isomorphism types of groups $G(S)$, that $G(S)$ is finitely presented if and only if S is finite and that $G(S)$ can embed in a finitely presented group only when S is recursively enumerable. For the converse, note that since we know that H_P embeds in a finitely presented group (see the remark at the end of the next section), it follows that Z is recursively enumerable. Now if S and Z are both recursively enumerable, $G(S)$ is recursively presented and so by the Higman embedding theorem [20, 27], $G(S)$ does embed in some finitely presented group.

We know already that $G(\emptyset)$ has geometric dimension two and is of type F; since the kernel of the map $G(\emptyset) \to G(S)$ is acyclic, it follows from Proposition 2.2 that $G(S)$ is of type FH and has cohomological dimension two. ∎

5. Embedding polygon subgroups

In this section, we construct embeddings of the polygon subgroup H_P, whose isomorphism type depends only on the perimeter l of P and on $Z \subseteq \mathbb{Z} - \{0\}$, into groups G_P of type F. Moreover, each group G_P will have geometric dimension two.

The first case that we deal with is the case $Z = \mathbb{Z} - \{0\}$. Before starting this case, we recall that the right-angled Artin group A_L associated with a flag simplicial complex L is the group whose generators are the vertices of L, subject to the relations that the two vertices which are incident on each edge commute. The right-angled Artin group associated with a finite flag complex L is of type F and has cohomological dimension one more than the dimension of L [14, 21]. The Bestvina–Brady group BB_L is defined to be the kernel of the map $A_L \to \mathbb{Z}$ that sends each of the generators to $1 \in \mathbb{Z}$. Provided that L is connected, there is a generating set for BB_L that corresponds to the directed edges of L, where the edge from vertex x to y corresponds to the element xy^{-1} in A_L. A presentation for BB_L in terms of these generators is given in [15]. In the case when $Z = \mathbb{Z} - \{0\}$, the given presentation for H_P is equal to this presentation for $\mathrm{BB}_{\partial P}$. Hence, we may take for G_P the right-angled Artin group $A_{\partial P}$. However, it may be more helpful to use the natural isomorphism between the right-angled Artin group A_L and the Bestvina–Brady group $\mathrm{BB}_{C(L)}$ for the cone on L: if c is the cone vertex, this isomorphism takes the vertex generator x to the generator xc^{-1} corresponding to the edge from x to c.

Proposition 5.1. *Let K be a spectacular 2-complex, let $Z = \mathbb{Z} - \{0\}$ and let L be the flag complex obtained from K by replacing each polygon with the cone on its boundary. With the embedding $H_P \to G_P$ as described above, there is an isomorphism, for each $S \subseteq \mathbb{Z} - \{0\}$, from $G(S)$ to the generalized Bestvina–Brady group $G_L(S \cup \{0\})$ in the sense of* [24, Def. 1.1].

Proof. The generating set for $H(S)$ consists of the directed edges of K^1, which is a subcomplex of L, the generating set for each H_P is identified with the edges of ∂P and the generating set for G_P is identified with the edges from vertices of ∂P to the cone vertex c_P. This gives a generating set for the group $G(S)$ consisting of directed edges of L. Since each edge of L is either in the image of K^1 or incident on some cone vertex, this generating set for $G(S)$ consists of all of the directed edges of L. But the generating set for the presentation for $G_L(S \cup \{0\})$ given in [24, Def. 1.1] is also the directed edges of L. Hence, there are natural mutually inverse bijections between the generators of $G(S)$ and the generators of $G_L(S \cup \{0\})$.

It remains to show that these bijections of generating sets send the relators of each group to valid relations in the other group, where the relators taken for $G(S)$ are those implicit in its description as a graph of groups. To do this, one employs the sort of reasoning that was used in the proof of [15, Prop. 2]. The relations of the given presentation for $G(S)$ are as follows:

(1) for each triangle in L with directed boundary (a, b, c), the relators abc and $a^{-1}b^{-1}c^{-1}$;

(2) for each polygon P of K with directed boundary $(e_1, \ldots, e_l)$ and each $n \in \mathbb{Z} - S$, the relator $e_1^n e_2^n \cdots e_l^n$;

(3) for each directed cycle $(e_1, \ldots, e_l)$ in K^1 and each $n \in S$, the relator $e_1^n e_2^n \cdots e_l^n$.

On the other hand, the relations in the given presentation for $G_L(S \cup \{0\})$ are as follows:

(1) for each triangle in L with directed boundary (a, b, c), the relators abc and $a^{-1}b^{-1}c^{-1}$;

(2) for each directed cycle $(e_1, \ldots, e_l)$ in L and each $n \in S$, the relator $e_1^n e_2^n \cdots e_l^n$.

The relators that are not common to the two presentations are the relators of the second type in the presentation for $G(S)$ and the relators of the second type in the presentation for $G_L(S \cup \{0\})$ which correspond to cycles in L not contained in K^1.

The relators of the second type in the presentation for $G(S)$ associated with the boundary of a given polygon P can be deduced from the triangle relators for the triangles that form the conical subdivision of P, as in the proof of [15, Prop. 2].

To deduce the relators of the second type in the presentation for $G_L(S \cup \{0\})$ from the relators of the presentation for $G(S)$, one again reasons as in the proof of [15, Prop. 2]. Implicit in that proof is the statement that if $(a_1, \ldots, a_l)$ and $(b_1, \ldots, b_m)$ are directed cycles that are (unbased) homotopic to each other in L, then for any n, the relation $a_1^n \cdots a_l^n$ is a consequence of the relator $b_1^n \cdots b_m^n$ together with the triangle relators. This suffices since any edge cycle in L is homotopic to an edge cycle that is contained in K^1. ■

Next, we consider the case $Z = \{k^n : n \geq 0\}$ for some $k \in \mathbb{Z}$ with $|k| > 1$. In this case, the group H_P has an injective but non-surjective self-homomorphism $\phi = \phi_P$ defined by $\phi(a_i) := a_i^k$ for each of the edge generators a_i. In this case, the natural choice for G_P is the ascending HNN-extension

$$G_P = \langle a_1, \ldots, a_l, t : a_i^t = a_i^k,\ a_1 a_2 \cdots a_l = 1 \rangle,$$

in which conjugation by the stable letter $t = t_P$ acts by applying the homomorphism ϕ.

Proposition 5.2. *The presentation 2-complex for the finite presentation for G_P given above is aspherical.*

Proof. Let $Y = Y_P$ be the presentation 2-complex for the small cancellation presentation for H_P, and let $f : Y \to Y$ be the based cellular map that induces

$$\phi : H_P \to H_P$$

on fundamental groups. The natural choice of Eilenberg–Mac Lane space for G_P is the mapping torus $M = M_f$ of f. Since f sends relators to relators, there is an easy way to put a CW-structure on M, with finite 1-skeleton. Each i-cell of Y contributes one i-cell and one $(i+1)$-cell to M. Hence, the cells of M are one 0-cell, $l+1$ 1-cells labelled by the generators $a_1, \dots, a_l, t$, one family of l trapezoidal 2-cells (coming from the 1-cells of Y) whose boundaries are the words $ta_it^{-1}a_i^{-k}$, an infinite family of 2-cells and an infinite family of 3-cells. For $n \geq 0$, denote by e_n the 2-cell that corresponds to the relator $a_1^{k^n}a_2^{k^n}\cdots a_l^{k^n}$, and for $n \geq 1$, let E_n be the 3-cell coming from the 2-cell e_{n-1}, so that the boundary of E_n consists of e_{n-1}, $-e_n$, and k^{n-1} copies of each of the trapezoidal 2-cells. For $n \geq 0$, let M_n be the subcomplex of M that contains the 1-skeleton, the trapezoidal 2-cells, the 2-cells $e_0, \dots, e_n$ and the 3-cells $E_1, \dots, E_n$. There is a deformation retraction of E_n onto $\partial E_n - \mathrm{Int}(e_n)$, and combining this with the identity map on the rest of M_{n-1} defines a deformation retraction of M_n onto M_{n-1}. Applying these retractions successively so that the nth of the retractions happens during the interval $[1/2^n, 1/2^{n-1}]$ gives a deformation retraction of M onto M_0. Since M_0 is the presentation 2-complex described in the statement, this implies that M_0 is aspherical. ∎

Sapir has given an aspherical version of the Higman embedding theorem, stating that any finitely generated group with an aspherical recursive presentation can be embedded into a group with a finite aspherical presentation [30]. This gives a characterisation of which polygon groups embed into groups of type F.

Proposition 5.3. *Suppose that P is a polygon of perimeter at least 13, and let H_P be the corresponding polygon group, which depends on $Z \subseteq \mathbb{Z} - \{0\}$ as well as on P. The following statements are equivalent:*

- *The set Z is recursively enumerable.*
- *H_P embeds in a finitely presented group.*
- *H_P embeds in a group admitting a finite 2-dimensional Eilenberg–Mac Lane space.*

Proof. Since the perimeter of P is at least 13, the defining presentation for H_P satisfies the $C'(1/6)$ small cancellation condition. Given this, if $a_1, \dots, a_l$ are the directed edges making up the bounding cycle of P, it follows from the Dehn property that $a_1^n a_2^n \cdots a_l^n = 1$ in H_P if and only if $n \in Z \cup \{0\}$. Since any finitely generated subgroup of a finitely presented group is recursively presented, it follows that if H_P embeds into a finitely presented group, then Z must be recursively enumerable.

In general, the $C'(1/6)$ condition implies that the presentation 2-complex for H_P is aspherical. If Z is recursively enumerable, then H_P is recursively presented and Sapir's theorem [30] implies that H_P embeds in a group admitting a finite 2-dimensional Eilenberg–Mac Lane space. Such a group is a fortiori finitely presented. ∎

6. The projective line and its symmetries

Our construction of a spectacular 2-complex will involve the combinatorics of the 2-dimensional projective linear group over a finite field, viewed as a group of permutations of the projective line. For the benefit of the reader, we summarize those properties that we shall use. Recall that a permutation group on a set X is said to be *k-transitive* if it acts transitively on the ordered k-tuples of elements of X, and *strictly k-transitive* if in addition the stabilizer of an ordered k-tuple is trivial.

Theorem 6.1. *Let F be any field and let $G = \mathrm{PGL}(2, F)$ be the 2-dimensional projective general linear group over F. The action of G on the projective line over F is strictly 3-transitive.*

Proof. Let $\mathbf{e}_1, \mathbf{e}_2$ be the standard basis for the vector space F^2, viewed as a space of column vectors, and let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad \neq bc,$$

be an element of $\mathrm{GL}(2, F)$. Then, $M\mathbf{e}_1 = a\mathbf{e}_1 + c\mathbf{e}_2$, and since a, c can be arbitrary provided that they are not both zero, $\mathrm{GL}(2, F)$ acts transitively on the non-zero vectors. The stabilizer of the line $\langle \mathbf{e}_1 \rangle$ is the subgroup consisting of those M with $c = 0$. For such M, $ad \neq 0$ and b is arbitrary (independent of choice of a, d). For such M, $M\mathbf{e}_2 = b\mathbf{e}_1 + d\mathbf{e}_2$, an arbitrary vector in $F^2 - \langle \mathbf{e}_1 \rangle$. It follows that $\mathrm{GL}(2, F)$ acts 2-transitively on the lines in F^2. The intersection of the stabilizers of $\langle \mathbf{e}_1 \rangle$ and $\langle \mathbf{e}_2 \rangle$ is those M with $b = c = 0$; this implies that $ad \neq 0$. For such M, we have that

$$M(\mathbf{e}_1 + \mathbf{e}_2) = a\mathbf{e}_1 + d\mathbf{e}_2,$$

an arbitrary element of $F^2 - (\langle \mathbf{e}_1 \rangle \cup \langle \mathbf{e}_2 \rangle)$. Hence, $\mathrm{GL}(2, F)$ acts 3-transitively on the lines in F^2. The vector $a\mathbf{e}_1 + d\mathbf{e}_2$ is in the line $\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$ if and only if $a = d$. Thus, the intersection of the stabilizers (in $\mathrm{GL}(2, F)$) of the lines $\langle \mathbf{e}_1 \rangle$, $\langle \mathbf{e}_2 \rangle$ and $\langle \mathbf{e}_1 + \mathbf{e}_2 \rangle$ is the scalar matrices aI. Since these form the kernel of the map $\mathrm{GL}(2, F) \to G$, it follows that G acts strictly 3-transitively as claimed. ∎

Theorem 6.2. *Fix a prime power $q = p^k$, let $G = \mathrm{PGL}(2, q)$ be the 2-dimensional projective general linear group over the field with q elements and let $\mathbb{P}^1(q)$ denote the projective line over the field of q elements. The following statements hold:*

- *The orders of elements of G are p, the prime dividing q, and every factor of $q + \varepsilon$ for $\varepsilon = \pm 1$.*

- *Let g be an element of G of order $d \geq 3$ with d dividing $q + \varepsilon = q \pm 1$.*
  - *The centralizer of g is cyclic of order $q + \varepsilon$.*
  - *The normalizer of the subgroup generated by g is dihedral of order $2(q + \varepsilon)$.*
  - *In its permutation action on $\mathbb{P}^1(q)$, g fixes $1 - \varepsilon$ points and permutes all other points in $(q + \varepsilon)/d$ cycles of length d.*
- *Any element of order p acts on $\mathbb{P}^1(q)$ with one fixed point and q/p cycles of length p.*
- *When q is odd, there are two cycle types of elements of order two; for each $\varepsilon = \pm 1$, there are elements of order two whose centralizers are dihedral of order $2(q + \varepsilon)$. These elements fix $1 - \varepsilon$ points of $\mathbb{P}^1(q)$ and have $(q + \varepsilon)/2$ cycles of length two.*

Proof. We start with some general remarks concerning normalizers inside permutation groups. If g is an element of a permutation group G on a set X, and m is any integer, then the set of points fixed by g^m will contain the set of points fixed by g. If g and g^m generate the same cyclic subgroup of G, then there is an integer m' so that $g = (g^m)^{m'}$, and so by symmetry g and g^m must fix the same points. It follows that any $h \in G$ that normalizes the subgroup generated by g must preserve the set of g-fixed points.

Now consider the case of interest, when $G = \mathrm{PGL}(2, q)$ acting as a group of permutations of $\mathbb{P}^1(q)$. By Theorem 6.1, any non-identity element of G fixes at most two points. Lifting to the general linear group, it follows that any non-scalar matrix in $\mathrm{GL}(2, q)$ can fix at most two lines in $\mathbb{F}_q^2$.

There are three possible cases for the characteristic polynomial of an element of $\mathrm{GL}(2, q)$: it may factor as the square of a linear polynomial $(t - \lambda)^2$, it may factor as the product of two distinct linear polynomials $(t - \lambda)(t - \mu)$ or it may be a quadratic polynomial that has no roots in $\mathbb{F}_q$. Any eigenvalues (i.e., roots of the characteristic polynomial) must be non-zero.

A matrix whose characteristic polynomial is $(t - \lambda)^2$ is either a scalar matrix $\mathrm{Diag}(\lambda, \lambda)$ or it is conjugate to an upper triangular matrix with both diagonal entries equal to λ. A scalar matrix is in the kernel of the map $\mathrm{GL}(2, q) \to G$. The set of all upper triangular matrices with 1's on their diagonal forms a subgroup of $\mathrm{GL}(2, q)$ isomorphic to the additive group of $\mathbb{F}_q$; in particular, each of these except the identity matrix has order p and maps to a non-identity element of G which must also have order p. An upper triangular matrix with both diagonal entries equal to λ is the product of a matrix as considered above with a scalar matrix. Hence, its image in G is also an element of order p. To see that the cycle type of an element of order p acting on $\mathbb{P}^1(q)$ is as claimed, note that

$$\left|\mathbb{P}^1(q)\right| = q + 1$$

is congruent to 1 modulo p, and so the number of fixed points under the action of an element of order p must also be congruent to 1 modulo p and cannot be larger than 2 by strict 3-transitivity.

A matrix whose characteristic polynomial is $(t - \lambda)(t - \mu)$ for $\lambda \neq \mu$ is conjugate to the diagonal matrix $\mathrm{Diag}(\lambda, \mu)$ via the change of basis that sends $\mathbf{e}_1$ to an eigenvector for λ and $\mathbf{e}_2$ to an eigenvector for μ. Let g be the image of $\mathrm{Diag}(\lambda, \mu)$ inside G. Since $\lambda \neq \mu$, g is not the identity element and so g fixes just two points of $\mathbb{P}^1(q)$: the lines $\langle \mathbf{e}_1 \rangle$ and $\langle \mathbf{e}_2 \rangle$. Any element of G that normalizes the subgroup generated by g must either fix or exchange these two lines. The matrices that preserve the lines $\langle \mathbf{e}_1 \rangle$ and $\langle \mathbf{e}_2 \rangle$ are precisely the diagonal matrices, each of which centralizes $\mathrm{Diag}(\lambda, \mu)$, and the matrices that exchange these two lines are precisely the antidiagonal matrices. Each antidiagonal matrix conjugates $\mathrm{Diag}(\lambda, \mu)$ to $\mathrm{Diag}(\mu, \lambda)$. Since the product of $\mathrm{Diag}(\lambda, \mu)$ and $\mathrm{Diag}(\mu, \lambda)$ is a scalar matrix, any $h \in G$ that exchanges the two lines must conjugate g to g^{-1}. Thus, the normalizer in G of the subgroup generated by g is the image in G of the group of diagonal and antidiagonal matrices in $\mathrm{GL}(2, q)$. This is a dihedral group of order $2(q-1)$. The cyclic subgroup of order $q-1$ fixes the same two points of $\mathbb{P}^1(q)$ as g and centralizes g. The other elements of this dihedral group swap the two points fixed by g and send g to its inverse.

In this case, it remains only to consider the cycle type for the action of g. Since the pointwise stabilizer in G of the two points fixed by g is a cyclic group of order $q-1$ containing g, it follows that g is a power of an element of this order. If g has order $d > 1$, then it is the $(q-1)/d$th power of some element of order $q-1$. To determine the cycle type of g, it suffices to show that this element of order $q-1$ acts on $\mathbb{P}^1(q)$ via a single $(q-1)$-cycle. A matrix in $\mathrm{GL}(2, q)$ that maps to this element is conjugate to $\mathrm{Diag}(a, d)$ where a/d is a generator for the multiplicative group of $\mathbb{F}_q$. For $0 \leq k \leq q-2$, the vectors $a^k \mathbf{e}_1 + d^k \mathbf{e}_2$ all lie in distinct lines, showing that such an element acts as a $(q-1)$-cycle.

An irreducible quadratic polynomial over $\mathbb{F}_q$ has both of its roots in the field $\mathbb{F}_{q^2}$ of q^2 elements. We may view $\mathrm{GL}(2, q)$ as a subgroup of $\mathrm{GL}(2, q^2)$, and we may view G as a subgroup of $\mathrm{PGL}(2, q^2)$. If M is a matrix in $\mathrm{GL}(2, q)$ whose characteristic polynomial is irreducible over $\mathbb{F}_q$, then over $\mathbb{F}_{q^2}$, we see that M is conjugate to a diagonal matrix $\mathrm{Diag}(\lambda, \mu)$ for some $\mu \neq \lambda \in \mathbb{F}_{q^2}$. The study of the properties of diagonalizable matrices in the previous two paragraphs therefore applies to M as an element of $\mathrm{GL}(2, q^2)$. If g is the image of the matrix M in $\mathrm{PGL}(2, q^2)$, then we see that the normalizer in $\mathrm{PGL}(2, q^2)$ of this element is dihedral of order $2(q^2-1)$, and that the elements of this subgroup that fix the two points of $\mathbb{P}^1(q^2)$ that are fixed by g form a cyclic subgroup of order q^2-1, while the elements that swap these two points conjugate g to g^{-1}. Arguing as in the previous paragraph, we see that a generator for this cyclic group acts as a single (q^2-1)-cycle on the complement of the two fixed points. Hence, the points of $\mathbb{P}^1(q^2)$ that are not fixed by g lie in g-orbits of length equal to the order of g. The action of g on $\mathbb{P}^1(q^2)$ preserves the subset $\mathbb{P}^1(q) \subseteq \mathbb{P}^1(q^2)$. Since the two points of $\mathbb{P}^1(q^2)$ that are fixed by g are not contained in $\mathbb{P}^1(q)$, the orbits for g on $\mathbb{P}^1(q)$ are all of length equal to the order of g. Any matrix in $\mathrm{GL}(2, q)$ that fixes the same two points of $\mathbb{P}^1(q^2)$ as g must also have an irreducible quadratic as its characteristic polynomial (since its eigenvectors in $\mathbb{F}_{q^2}^2$ do not lie in $\mathbb{F}_q^2$).

The roots of a quadratic that is irreducible over $\mathbb{F}_q$ lie in a single orbit for the Galois group of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. This Galois group is a group of order two generated by the Frobenius map $\lambda \mapsto \lambda^q$. It follows that any matrix in $\mathrm{GL}(2, q)$ whose characteristic polynomial is an irreducible quadratic is conjugate over $\mathbb{F}_{q^2}$ to a matrix of the form $\mathrm{Diag}(\lambda, \lambda^q)$ for some $\lambda \in \mathbb{F}_{q^2} - \mathbb{F}_q$. Since $\lambda^{q^2} = \lambda$, the $(q+1)$st power of $\mathrm{Diag}(\lambda, \lambda^q)$ is the scalar matrix $\mathrm{Diag}(\lambda^{q+1}, \lambda^{q+1})$. Thus, any element of $\mathrm{GL}(2, q)$ whose characteristic polynomial is an irreducible quadratic gives rise to an element of $G = \mathrm{PGL}(2, q)$ of order dividing $q + 1$. Combining this information with that given in the previous paragraph, we see that if g is an element of $\mathrm{PGL}(2, q)$ that fixes no point of $\mathbb{P}^1(q)$, then the set of all elements of $\mathrm{PGL}(2, q)$ that fix the same points in $\mathbb{P}^1(q^2)$ as g is a cyclic group of order dividing $q + 1$, and that this cyclic group has index at most two in the set of all elements of $\mathrm{PGL}(2, q)$ that fix this *set* of two points. Furthermore, if g has order d, then g acts on $\mathbb{P}^1(q)$ as $(q + 1)/d$ disjoint d-cycles.

So far we have an upper bound for the normalizer of a group element g corresponding to a matrix M whose characteristic polynomial is an irreducible quadratic, but we also need a lower bound. That is, we need to construct a cyclic group of order $q + 1$ that centralizes g and a dihedral group of order $2(q + 1)$ that normalizes the subgroup generated by g. Let $f(x) \in \mathbb{F}_q[x]$ be the characteristic polynomial of such an M. Define a (unital) ring homomorphism $\widetilde{\psi}_M : \mathbb{F}_q[x] \to M_2(\mathbb{F}_q)$ by $\widetilde{\psi}_M(x) = M$ and extending $\mathbb{F}_q$-linearly. By definition, the kernel of $\widetilde{\psi}_M$ is the ideal of $\mathbb{F}_q[x]$ generated by the minimal polynomial of M. Since the characteristic polynomial $f(x)$ is an irreducible quadratic, it is equal to the minimal polynomial of M and the factor ring $E = \mathbb{F}_q[x]/(f(x))$ is isomorphic to the field $\mathbb{F}_{q^2}$. Moreover, $\widetilde{\psi}_M$ induces an injective $\mathbb{F}_q$-linear homomorphism

$$\psi_M : E \to M_2(\mathbb{F}_q).$$

Fix a non-zero vector $\mathbf{v}$ in $\mathbb{F}_q^2$ and define an $\mathbb{F}_q$-vector space homomorphism $\theta_{\mathbf{v}} : E \to \mathbb{F}_q^2$ by $\theta_{\mathbf{v}}(1) = \mathbf{v}$ and $\theta_{\mathbf{v}}(x) = M\mathbf{v}$ where we have abused notation slightly by writing x for the image of x inside E. Since E is being used as both a module and a ring, we use the notation E^1 for E viewed as a 1-dimensional vector space over E and $M_1(E)$ for E viewed as the endomorphism ring of E^1. Since M has no eigenvalues in $\mathbb{F}_q$, $\mathbf{v}$ and $M\mathbf{v}$ are linearly independent and so the map $\theta_{\mathbf{v}} : E^1 \to \mathbb{F}_q^2$ is an isomorphism of vector spaces over $\mathbb{F}_q$. Moreover, it is compatible with $\psi_M : M_1(E) \to M_2(\mathbb{F}_q)$ in the sense that for all $\lambda, \mu \in E$, $\theta_{\mathbf{v}}(\lambda\mu) = \psi_M(\lambda)\theta_{\mathbf{v}}(\mu)$. In particular, specializing to the case $\lambda = x$ gives that $\theta_{\mathbf{v}}(x\mu) = M\theta_{\mathbf{v}}(\mu)$ for any μ, and so $\theta_{\mathbf{v}} x \theta_{\mathbf{v}}^{-1} = M$. To avoid any confusion, we emphasize that this equation represents an identity between $\mathbb{F}_q$-linear automorphisms of $\mathbb{F}_q^2$, and that the symbol x in this equation represents the map $\lambda \mapsto x\lambda$ for all $\lambda \in E^1$ which is E-linear and hence a fortiori $\mathbb{F}_q$-linear.

The multiplicative group $\mathrm{GL}(1, E) \leq M_1(E)$ is a cyclic group of order $q^2 - 1$ containing x. The Galois group of E as an extension of $\mathbb{F}_q$ is a cyclic group of order two, generated by the Frobenius map $\phi_q : \lambda \mapsto \lambda^q$. This map is an $\mathbb{F}_q$-linear automorphism of E of order two which is not E-linear, so it normalizes $\mathrm{GL}(1, E)$ but is not contained in $\mathrm{GL}(1, E)$. Since $\phi_q(\lambda\mu) = \lambda^q\mu^q$, it follows that conjugation by ϕ_q acts on $\mathrm{GL}(1, E)$ as the group automorphism $\lambda \mapsto \lambda^q$. Let H be the group of $\mathbb{F}_q$-linear automorphisms of E^1 generated by $\mathrm{GL}(1, E)$ and ϕ_q. Thus, H is a group of $\mathbb{F}_q$-linear automorphisms of E that has order $2(q^2 - 1)$, which normalizes the subgroup generated by x, and contains an index two cyclic subgroup $\mathrm{GL}(1, E)$ that contains x. Now $\theta_{\mathbf{v}} : E^1 \to \mathbb{F}_q^2$ is an isomorphism of $\mathbb{F}_q$-vector spaces such that $\theta_{\mathbf{v}} x \theta_{\mathbf{v}}^{-1} = M \in M_2(\mathbb{F}_q)$. Hence, the group

$$H' = \theta_{\mathbf{v}} H \theta_{\mathbf{v}}^{-1} \leq \mathrm{GL}(2, q)$$

normalizes the subgroup generated by the matrix M and M is contained in a cyclic subgroup C of H' of order $q^2 - 1$. Since M is a power of a generator for C, we see that each generator of C has the same fixed points in $\mathbb{P}^1(q^2)$ as M. These points do not lie in $\mathbb{P}^1(q)$ and so each generator of C is irreducible over $\mathbb{F}_q$. It follows by an argument given earlier that the $(q+1)$st power of any generator of C is a scalar matrix and so lies in the kernel of the map $\mathrm{GL}(2, q) \to G$. Hence, the image of C in G is cyclic of order at most $q + 1$. Since this kernel of the map $\mathrm{GL}(2, q) \to G$ is cyclic of order $q - 1$, the image of C is cyclic of order exactly $q + 1$ and the image of H' in G is a dihedral group of order $2(q + 1)$, with g (the image of M) contained in a cyclic subgroup of order $q + 1$ (the image of C).

The statements concerning an element g of order two follow from the above cases. In particular, the case when $q = 2^k$ is even corresponds to an element g fixing one point of $\mathbb{P}^1(q)$. When q is odd, the element g must fix either 0 or 2 points of $\mathbb{P}^1(q)$, with the other points permuted in 2-cycles. Write $1 - \varepsilon$ with $\varepsilon = \pm 1$ for the number of fixed points for g. In each case, g is a power of an element of order $q + \varepsilon$, g is centralized (or equivalently normalized) by a dihedral subgroup of order $2(q + \varepsilon)$ and the cycle type of g is as claimed. ■

7. A spectacular 2-complex

We construct a 2-complex K having the required properties in stages. Initially, we ignore the requirements of large girth (or rotundity) and acyclicity and construct a 2-complex K_1 with perfect fundamental group and the required small cancellation property. By passing to a subcomplex, we obtain an acyclic subcomplex K_2. Finally, by subdividing the 1-skeleton of K_2, with a corresponding increase in the number of sides of each polygon, we obtain K.

Fix a prime power q, and fix $d \geq 3$ so that d divides $q + \varepsilon$ for some $\varepsilon \in \{\pm 1\}$. Now let $G = \mathrm{PGL}(2, q)$ be the 2-dimensional projective general linear group over the field with q elements. There is a natural action of G on the projective line, a set of $q + 1$ points. The 1-skeleton K_1^1 is the complete graph whose vertex set is the projective line. By construction, G acts on K_1^1, and the action is triply-transitive on the vertex set K_1^0. Note also that $|G| = q(q^2 - 1)$ is equal to the number of ordered triples of distinct elements of K_1^0.

Let g be an element of G of order d. Our complex will depend on the pair (d, q) and on the conjugacy class of the element g. By Theorem 6.2, the centralizer in G of g is cyclic of order $q + \varepsilon$, and the normalizer of the subgroup generated by g is dihedral of order $2(q + \varepsilon)$. It follows that the conjugacy class of g contains $q(q - \varepsilon)$ elements and that it is closed under taking inverses. The element g is a power of an element of order $q+\varepsilon$. From this it follows that the permutation action of g on K_1^0 has $1-\varepsilon$ fixed points and $(q+\varepsilon)/d$ cycles of length d. We use each of these cycles of length d to describe the attaching map for a d-gon. If x is an element of the conjugacy class of g and the vertices $v_0, \dots, v_{d-1}, v_d$ are such that $v_i = x^i(v_0)$, then there is a d-gon whose boundary is the edge loop of length d in K_1^1 consisting of the edges $(v_0, v_1), (v_1, v_2), \dots, (v_{d-1}, v_d)$. This edge loop passes through the vertices $v_0, \dots, v_{d-1}$ consecutively. Note that the vertex orbits for x^{-1} give rise to the same polygons as the vertex orbits for x, with the opposite orientation. Each pair of the form x, x^{-1} of elements of the conjugacy class of g gives a recipe for attaching $(q+\varepsilon)/d$ distinct d-sided polygons to K_1^1. The complex K_1 is thus defined by attaching $|G|/2d$ distinct d-gons to K_1^1.

Complexes of this form in the case $d = q+1$ were studied by Aschbacher and Segev in [1] although the complex that they describe is actually the conical subdivision of our complex, in which each d-gon is replaced by the cone consisting of d triangles. We encourage the reader to experiment with these complexes for small values of q. There are four such complexes with $q \le 4$: when $(d,q) = (3,2)$, the complex is a triangle; when $(d,q) = (4,3)$, the complex is a projective plane made from three squares that can be understood as the quotient of the boundary of a cube by its antipodal map; when $(d,q) = (3,4)$, the complex is the 2-skeleton of a 4-simplex; when $(d,q) = (5,4)$, the complex is the 2-skeleton of the Poincaré homology sphere [34, pp. 34–36], which is an acyclic space with fundamental group of order 120.

Proposition 7.1. *Each piece of the intersection of two polygons of K_1 contains at most one edge.*

Proof. Throughout this proof, the term *triple* will mean an ordered triple of pairwise distinct vertices of K_1. Say that a triple (u,v,w) is *contained* in a polygon P if v is a vertex of P and the edges $\{u,v\}$ and $\{v,w\}$ are both contained in P. Clearly each d-gon contains exactly $2d$ such triples. It will suffice to show that no triple is contained in more than one polygon of K_1 because the vertices contained in a piece of length 2 would form such a triple.

To do this, consider the set Π of pairs consisting of a polygon P of K_1 and a triple contained in the polygon P. We shall count the number of elements in Π in two different ways, firstly by summing over the polygons and secondly by summing over the triples. Before starting, note that the group $G = \mathrm{PGL}(2,q)$ acts on K_1 and hence also acts on Π. Since each polygon contains exactly $2d$ triples and there are $q(q^2-1)/2d$ polygons in K_1, we see that $|\Pi| = q(q^2-1)$, and so Π, the set of triples and G all have the same number of elements. Since the action of G on the set of triples is free and transitive, it follows that each triple must be contained in the same number of polygons. Hence, $|\Pi|$ is equal to the number of triples times the number of polygons containing any given fixed triple. Since $|\Pi|$ is equal to the number of triples, it follows that each triple is contained in exactly one polygon. In particular, no triple is contained in more than one polygon. ■

In the case when $d = q + 1$, the Euler characteristic of K_1 is equal to 1. Aschbacher and Segev showed that many of these complexes are rationally acyclic [1]. The only Aschbacher–Segev complexes that are known to be (integrally) acyclic are the two cases already mentioned above: $(d, q) = (3, 2)$ and $(d, q) = (5, 4)$, i.e., the triangle and the 2-skeleton of the Poincaré homology sphere. We cannot use these complexes because we need $d \geq 7$ together with Proposition 7.1 to ensure the $C'(1/6)$ condition (condition 5 in Definition 1.1).

For $d < q + 1$, the complex K_1 has too many 2-cells to be acyclic, but frequently $H_1(K_1)$ is trivial. Consider the case $(d, q) = (7, 8)$. In this case, K_1 has 36 2-cells, and a calculation shows that $H_1(K_1) = 0$ and that $H_2(K_1)$ is free abelian of rank 8. (There are three conjugacy classes of elements of order 7 in $G = \mathrm{PGL}(2, 8)$, but the action of the Galois group of the field of 8 elements by outer automorphisms of G permutes the three classes, so there is only one isomorphism type of complex K_1.) For this K_1, there are many ways to remove eight 2-cells to leave an acyclic 2-complex: randomly removing 2-cells to reduce the rank of H_2 often finds such a complex. The simplest way to produce such a complex that we have found is to fix one of the vertices v_0 of K_1, and to discard the eight 2-cells that are *not* incident on v_0.

This 2-complex K_2 has all of the required properties except that its polygons have too few sides and its girth is 3; by subdividing each edge into five, one obtains a spectacular complex K with girth 15 whose 28 2-cells are 35-gons.

8. Closing remarks

The groups $G(S)$ for $S \neq \emptyset$ are known to have cohomological dimension 2, but for most of them, we have been unable to construct a 2-dimensional Eilenberg–Mac Lane space.

In contrast to the Bestvina–Brady construction, graphical small cancellation is purely 2-dimensional, and so our methods cannot be used to construct (for example) groups that are finitely presented but not of type F.

We have been unable to find a spectacular complex K admitting an action of a non-trivial finite group Q so that the fixed point set K^Q is empty. Such a pair might give an alternative construction for the main examples in [25].

It is easy to see that the groups $H(S)$ fall into uncountably many quasi-isometry classes using Bowditch's argument [4]. It is less clear what happens with the groups $G(S)$, except in the case when $G(S)$ is isomorphic to a generalized Bestvina–Brady group $G_L(S)$, which was covered in [22]. Since this article was first submitted, the first named author has resolved the case when $Z = \{k^n : n \geq 0\}$ and G_P is as described in Proposition 5.2, in [9]. The question of whether the groups $G(S)$ form uncountably many quasi-isomorphism classes for arbitrary recursively enumerable $Z \subseteq \mathbb{Z}$ and arbitrary choices of embeddings $H_p \to G_P$ remains open in full generality. The methods introduced in [28] may prove useful in resolving this but they do not seem to apply directly. Fix a finite generating set for $G(\emptyset)$, and use the image of this set under the surjective homomorphism $G(\emptyset) \to G(S)$

as a set of generators for $G(S)$. In the language of [28], the map

$$\psi : S \mapsto G(S)$$

defines an injective function from the set of subsets of Z to the space of marked groups. To apply the main theorem of [28], it would suffice to show that the image of ψ is closed, which in turn would follow if one could show that ψ is continuous for the Tychonoff (or product) topology on the powerset of Z.

Condition (7) in the definition of a spectacular complex may be weakened, to give a weaker conclusion. For example, if K is only assumed to be 1-acyclic (or equivalently to have perfect fundamental group), then each $G(S)$ will be of type FP_2. Similarly, if K is only assumed to be rationally acyclic, then each $G(S)$ will be of type $FP(\mathbb{Q})$. Of course in these cases one could also weaken the hypotheses on G_P.

It is hard to see how conditions (1)–(6) of Definition 1.1 could be weakened. Condition (3) is used only in the proof of Proposition 3.3 and could possibly be weakened at the expense of a more complicated proof for this proposition. Condition (5) is used only to establish that the kernel $K_{S,T}$ is non-trivial and could be replaced by the more complicated condition: 'there is a closed path in K^1 whose intersection with the boundary of each polygon of K is less than half the length of that polygon' without changing the proof. Note that a single 13-gon satisfies all of the conditions except condition (5), so this condition cannot be omitted altogether.

The constants in the definition of a spectacular complex are chosen to ensure the $C'(1/6)$ graphical small cancellation condition in the associated group presentations. We have been unable to find analogous complexes that give rise to presentations satisfying the $C'(1/n)$ condition for arbitrarily large n. The technique used in Section 7 will produce suitable analogues of K_1, provided that d is chosen to be at least $n + 1$. However, for large values of d, we have been unable to find a set of polygons to remove to leave an acyclic subcomplex.

Acknowledgements. This work was done while the first named author was working on his PhD under the supervision of the second named author. Further properties of the groups $G(S)$ and other methods for constructing spectacular complexes K can be found in the PhD thesis of the first named author [9]. The main motivation for this work was the observation that the relations in the presentations for the groups $G_L(S)$ given in [24, Def. 1.1] consist of a large family of 'long' relations together with a finite number of 'short' relations. Another motivation (which predates the work in [24]) was a conversation between the second named author and Martin Bridson, in which Martin Bridson pointed out that there ought to be other constructions of non-finitely presented groups of type FP apart from that of Bestvina–Brady. The authors gratefully acknowledge this inspiration. The authors also gratefully acknowledge helpful comments on this work by Tim Riley. Finally, the authors thank the anonymous referee, whose numerous comments on four earlier versions have greatly improved the exposition and led to the correction of some minor errors.

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Received 23 September 2020.

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