

On some parametric elliptic systems at resonance

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Abstract. In this paper, we study the resonance for some parametric elliptic systems in which the nonlinearities involve convection and convolution of the solution. By applying a topological method based on fixed point theory, we establish the existence of at least one weak solution without using Landesman–Lazer-type conditions. Furthermore, we deal with a gradient-type system at resonance and prove the existence of a solution by using a variational approach.

1. Introduction

In recent years, the study of (p_1, p_2) -Laplacian systems has yielded crucial insights into nonlinear interactions between equations and the impact of coupling on system behavior. These systems offer profound mathematical implications, aiding in identifying solution existence, multiplicity, stability properties, and pattern or singularity formation. Simultaneously, their relevance extends to various physical applications. In fluid dynamics, they serve as models for intricate flows (such as multiphase flows or flows in porous media), and resonance analysis within these systems illuminates resonance-induced instabilities, vortex shedding, and wave amplification, thereby providing valuable insights into fluid flow dynamics. Similarly, in materials science and solid mechanics, these systems capture phenomena, such as crack propagation and fracture mechanics.

On the other hand, convolution is crucial for understanding how systems process input. In parametric elliptic systems with convolution and convection terms at resonance, studying convolution is vital. It reveals the intricate interplay between parameters and system dynamics, providing profound insights into the phenomena of resonance. This operation is extensively used in signal processing, particularly in convolutional neural networks for image analysis (Goodfellow et al. [11]), and in physics and engineering for solving differential equations and understanding complex systems (Strang [19] and Evans [9]).

In this paper, given a bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) with Lipschitz boundary, we will study the following parametric elliptic system with convolution and convection terms

at resonance

$$\begin{cases} -\Delta_{w_1, p_1} u - \mu_1 \Delta_{w_1, q_1} u = \lambda_1 w(x) |u|^{\alpha_1-2} u |v|^{\alpha_2} \\ \quad + f(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v)) + h_1(x) & \text{in } \Omega, \\ -\Delta_{w_2, p_2} v - \mu_2 \Delta_{w_2, q_2} v = \lambda_1 w(x) |v|^{\alpha_2-2} v |u|^{\alpha_1} \\ \quad + g(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v)) + h_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where, for $i \in \{1, 2\}$, we have $-\Delta_{w_i, p_i} u = -\operatorname{div}(w_i(x) |\nabla u|^{p_i-2} \nabla u)$, $1 < q_i < p_i < N$, $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are functions of Carathéodory satisfying some conditions that will be specified later, $w \in C(\Omega) \cap L^\infty(\Omega)$ satisfying $w^+ = \max\{w(x), 0\} \neq 0$, μ_i are positive constants and $h_i \in (W_0^{1, p_i}(w_i, \Omega))^*$. Let be associated to system (1.1) the following eigenvalue problem:

$$\begin{cases} -\operatorname{div}(w_1(x) |\nabla u|^{p_1-2} \nabla u) = \lambda w(x) |u|^{\alpha_1-2} u |v|^{\alpha_2} & \text{in } \Omega, \\ -\operatorname{div}(w_2(x) |\nabla v|^{p_2-2} \nabla v) = \lambda w(x) |v|^{\alpha_2-2} v |u|^{\alpha_1} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where, for $i \in \{1, 2\}$, α_i verify

$$\alpha_1, \alpha_2 > 0 \quad \text{and} \quad \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = 1 \quad \text{for all } x \in \Omega.$$

Define the functionals $\phi, \varphi : X = W_0^{1, p_1}(w_1, \Omega) \times W_0^{1, p_2}(w_2, \Omega) \rightarrow \mathbb{R}$ as

$$\begin{cases} \phi(u, v) = \frac{\alpha_1}{p_1} \int_{\Omega} w_1(x) |\nabla u|^{p_1} dx + \frac{\alpha_2}{p_2} \int_{\Omega} w_2(x) |\nabla v|^{p_2} dx, \\ \varphi(u, v) = \int_{\Omega} w(x) |u|^{\alpha_1-1} |v|^{\alpha_2-1} u v dx, \end{cases}$$

and the manifold

$$\Sigma = \{(u, v) \in X : \varphi(u, v) = 1\}.$$

One can easily prove that $\phi(u, v)$ and $\varphi(u, v)$ are (p, q) -homogeneous, i.e.,

$$\phi(t^{1/p_1} u, t^{1/p_2} v) = t \phi(u, v) \quad \text{and} \quad \varphi(t^{1/p_1} u, t^{1/p_2} v) = t \varphi(u, v)$$

for all $t > 0$ and $(u, v) \in X$. We also have Σ as a symmetric manifold in X . Using the same arguments as in Zographopoulos [21], the eigenvalue problem (1.2) admits a sequence of eigenvalues where they can be variationally characterized as follows:

$$\lambda_k = \inf_{A \in \Sigma_k} \sup_{(u, v) \in A} \phi(u, v),$$

where

$$\Sigma_k = \{A \in \Sigma : \text{there exists a continuous, odd and surjective function } \psi : S^{k-1} \rightarrow A\},$$

with S^{k-1} is the unit sphere in $\mathbb{R}^k$. We also have

$$0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \dots, \quad \lambda_k \rightarrow +\infty \quad \text{as } k \rightarrow +\infty.$$

Now, let us define the following:

$$\lambda'_1 = \inf_{(u,v) \in \Sigma} \frac{\alpha_1}{p_1} \int_{\Omega} w_1(x) |\nabla u|^{p_1} dx + \frac{\alpha_2}{p_2} \int_{\Omega} w_2(x) |\nabla v|^{p_2} dx. \quad (1.3)$$

Hence, λ'_1 is the first eigenvalue of (1.2) as well, and taking the definition of λ_1 into account, one can easily derive that $\lambda'_1 = \lambda_1$. Moreover, λ_1 is a simple, positive, and isolated principal eigenvalue of (1.2). In view of the fact that $\phi(u, v)$ and $\varphi(u, v)$ are (p, q) -homogeneous, the eigenfunction space corresponding to the first eigenvalue λ_1 is

$$E_{\lambda_1} = \{(u, v) \in X : \phi(u, v) = \lambda_1 \varphi(u, v)\},$$

where $(t^{1/p_1} u, t^{1/p_2} v) \in E_{\lambda_1}$ for all $t \geq 0$ and $(u, v) \in E_{\lambda_1}$. In addition, there exists $t_0 > 0$ such that

$$\|t_0^{1/p_1} u_0\|_{W_0^{1,p_1}(w_1, \Omega)}^{p_1} + \|t_0^{1/p_2} v_0\|_{W_0^{1,p_2}(w_2, \Omega)}^{p_2} = 1,$$

where (u_0, v_0) is a normalized eigenvalue satisfying

$$\|u_0\|_{W_0^{1,p_1}(w_1, \Omega)} + \|v_0\|_{W_0^{1,p_2}(w_2, \Omega)} = 1.$$

It can be seen that E_{λ_1} is not a linear subspace of X of dimension one, unlike some results in the literature, see, e.g., Alves et al. [1], where this property holds true for a problem with a single equation. This distinction can be clarified by examining the structure of E_{λ_1} . Consider two elements $(t_1^{1/p_1} u, t_1^{1/p_2} v)$ and $(t_2^{1/p_1} u, t_2^{1/p_2} v)$ belonging to E_{λ_1} , where $t_1, t_2 > 0$. Their sum, given by $(t_1^{1/p_1} u + t_2^{1/p_1} u, t_1^{1/p_2} v + t_2^{1/p_2} v)$, generally fails to satisfy the defining scaling properties of E_{λ_1} due to the lack of a single parameter $t > 0$ satisfying both components simultaneously. Similarly, applying scalar multiplication $c \in \mathbb{R}$ to $(t^{1/p_1} u, t^{1/p_2} v)$ does not yield an element of the same form unless $c = \pm 1$. These observations underscore that E_{λ_1} does not exhibit closure under addition or scalar multiplication, as required for a linear subspace. Instead, its structure reflects a symmetric, non-linear manifold arising from the inherent (p, q) -homogeneity of the functionals ϕ and φ .

Recently, Ouannasser and El Hachimi [18] studied a similar problem as in (1.1) for the non-resonance case, but with variable exponents. More specifically, they obtained existence and uniqueness results for the following system:

$$\begin{cases} -\Delta_{p_1(x)} u_1 - \mu_1 \Delta_{q_1(x)} u_1 = f_1(x, u_1, u_2, \nabla u_1, \nabla u_2) + h_1(x) & \text{in } \Omega, \\ -\Delta_{p_2(x)} u_2 - \mu_2 \Delta_{q_2(x)} u_2 = f_2(x, u_1, u_2, \nabla u_1, \nabla u_2) + h_2(x) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

In addition, we also refer to another interesting paper that has dealt with non-resonance for quasilinear elliptic problems, see El Hachimi and Gossez [8]. Because of the lack of

information on the first eigenvalue in the case of variable exponents when dealing with resonance, we limit ourselves to the study of problem (1.1) for constant exponents. Even when the exponents are constant, there is limited knowledge regarding resonance and non-resonance issues in the context of the Lusternik–Schnirelmann spectrum for the eigenvalue problem (1.2) of the function

$$(s, t) \mapsto \frac{f(x, s, t, \xi, \eta)s + g(x, s, t, \xi, \eta)t}{w(x)|s|^{\alpha_1}|t|^{\alpha_2}}. \quad (1.5)$$

The same observation holds true when considering gradient-type systems, and it applies to functions f and g that do not have a dependency on ∇u and ∇v within the function

$$(s, t) \mapsto \frac{F(x, s, t)}{w(x)|s|^{\alpha_1}|t|^{\alpha_2}},$$

where $\frac{\partial F}{\partial s}(x, s, t) = f(x, s, t)$. The study of resonance has been of a great deal of importance. In our case, it follows that the function should be, in some sense, greater than the first eigenvalue but remain below the second eigenvalue. In other words, we have

$$\begin{aligned} \lambda_1 < \limsup_{|(\rho_1 \star u, \rho_2 \star v)|_\infty \rightarrow \infty} & \frac{f(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v))u}{w(x)|\rho_1 \star u|^{\alpha_1}|\rho_2 \star v|^{\alpha_2}} \\ & + \frac{g(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v))v}{w(x)|\rho_1 \star u|^{\alpha_1}|\rho_2 \star v|^{\alpha_2}} < \lambda_2, \end{aligned}$$

with $|(\rho_1 \star u, \rho_2 \star v)|_\infty = \max\{|\rho_1 \star u|, |\rho_2 \star v|\}$. There are many works dedicated to the study of quasilinear elliptic problems at resonance, we cite Alves et al. [1], Benouhiba and Belyacine [2], De Nápoli and Mariani [5], Haddaoui et al. [12], Ou [15, 16], Ou and Tang [17], and the references therein. Most of these studies established the existence of weak solutions under Landesman–Lazer-type conditions. For example, Ou [15] studied the following system at resonance:

$$\begin{cases} -\Delta_p u = \lambda_1 a(x)|u|^{p-2}u + \lambda_1 b(x)|u|^\alpha|v|^\beta v + g_1(x, u) - h_1(x) & \text{in } \Omega, \\ -\Delta_q v = \lambda_1 c(x)|v|^{q-2}v + \lambda_1 b(x)|u|^\alpha|v|^\beta v + g_2(x, v) - h_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.6)$$

where he established the existence of solutions for the system (1.6) under Landesman–Lazer-type conditions.

In this work, we extend the aforementioned works to a parametric (p_1, p_2) -Laplacian system, where the nonlinearities depend on the gradient of the solution, and we obtain the existence of the solution without using Landsman–Lazer-type conditions. In addition, we adopt a distinct parameterization involving functions ρ_1 and ρ_2 in $L^1(\mathbb{R}^N)$ incorporated through convolutions $\rho_1 \star u$ and $\rho_2 \star v$ with the solution (u, v) in $W_0^{1,p_1}(w_1, \Omega) \times W_0^{1,p_2}(w_2, \Omega)$. To facilitate our analysis, it is convenient to consider the weighted Sobolev space $W_0^{1,p_1}(w_1, \Omega)$ embedded in $W^{1,p_1}(w_1, \mathbb{R}^N)$ by associating each $u \in W_0^{1,p_1}(w_1, \Omega)$

with its extension $\tilde{u} \in W^{1,p_1}(w_1, \mathbb{R}^N)$, taking the value zero outside Ω , and henceforth replicating a similar approach with $v \in W_0^{1,p_2}(w_2, \Omega)$.

The lack of variational structure in problem (1.1), caused by convection terms, prevents the use of variational methods. Instead, we employ a topological method based on the surjectivity result of pseudomonotone operators. To the best of our knowledge, this is the first paper that deals with resonance for parametric elliptic systems where the nonlinearities are dependent on the gradient and the convolution of the solution, and without using Landesman–Lazer-type conditions. Such problems bring about many issues to overcome, such as the interaction of the variational spectrum with the function given in (1.5).

Similarly, we address the scenario of a gradient-type system and establish the existence of at least one weak solution, following a variational method similar to that used by El Hachimi and de Thélin [7]. More specifically, we deal with the following system at resonance:

$$\begin{cases} -\Delta_{w_1,p_1} u - \mu_1 \Delta_{w_1,q_1} u = \lambda_1 w(x) |u|^{\alpha_1-2} u |v|^{\alpha_2} \\ \quad + \frac{\partial F}{\partial u}(x, \rho_1 \star u, \rho_2 \star v) + h_1(x) & \text{in } \Omega, \\ -\Delta_{w_2,p_2} v - \mu_2 \Delta_{w_2,q_2} v = \lambda_1 w(x) |v|^{\alpha_2-2} v |u|^{\alpha_1} \\ \quad + \frac{\partial F}{\partial v}(x, \rho_1 \star u, \rho_2 \star v) + h_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.7)$$

Many studies have dealt with similar problems to the system (1.7), studied their resonance, and obtained the existence of solutions using different methods, such as Zhao and Tang [20]. In this section, we extend the works of Lv and Ou [14], El Hachimi and de Thélin [7], Ouannasser and El Hachimi [18], Ou [16], De Nápoli, and Mariani [5], Zhao and Tang [20], and deal with a more general class of systems where we obtain the existence of a solution at resonance.

The remainder of this paper is organized as follows. In Section 2, we present the necessary elements regarding the weighted Sobolev spaces in relation to our system (1.1). In addition, we provide a few convolution properties in the case of weighted Sobolev spaces. Moreover, we establish the existence of at least one weak solution using the surjectivity result of pseudomonotone operators. Finally, in Section 3, we delve into the case of a gradient-type system at resonance and prove the existence of a solution using a variational approach.

2. Functional framework and main results

For $i \in \{1, 2\}$, let w_i be a measurable and a.e. finite function in $\mathbb{R}^N$ such that

- (a₁) $w_i \in L_{loc}^1(\Omega)$ and $w_i^{-\frac{1}{p_i-1}} \in L_{loc}^1(\Omega)$,
- (a₂) $w_i^{-s_i} \in L^1(\Omega)$ for $s_i \in]\frac{N}{p_i}, +\infty[\cap [\frac{1}{p_i-1}, +\infty[$.

In what follows, let $X = W_0^{1,p_1}(w_1, \Omega) \times W_0^{1,p_2}(w_2, \Omega)$ be the product space endowed with the norm

$$\|(u, v)\|_X = \max \left\{ \|u\|_{W_0^{1,p_1}(w_1, \Omega)}, \|v\|_{W_0^{1,p_2}(w_2, \Omega)} \right\},$$

where

$$\begin{aligned} \|u\|_{W_0^{1,p_1}(w_1, \Omega)} &= \left(\int_{\Omega} w_1(x) |\nabla u|^{p_1} dx \right)^{\frac{1}{p_1}}, \\ \|v\|_{W_0^{1,p_2}(w_2, \Omega)} &= \left(\int_{\Omega} w_2(x) |\nabla v|^{p_2} dx \right)^{\frac{1}{p_2}} \end{aligned}$$

for every $(u, v) \in X$. It is important to note that condition (a_1) is mandatory in order for $W_0^{1,p_i}(w_i, \Omega)$ to be a Banach space, for $i \in \{1, 2\}$. Furthermore, we recall a few more results related to this space, such as continuous and compact embeddings.

Theorem 2.1 (Kufner and Opic [13]). *Assume that (a_1) is verified. Then, for $i \in \{1, 2\}$, the weighted Sobolev space $W_0^{1,p_i}(w_i, \Omega)$ is a Banach space.*

Theorem 2.2 (Drábek et al. [6]). *Let $\Omega \subset \mathbb{R}^N$ be an open set and assume that (a_1) is verified. Then, for $i \in \{1, 2\}$, we have the following continuous and compact embeddings.*

- (i) *For any r_i verifying $1 \leq r_i \leq p_{i,s_i}^* = \frac{Np_i s_i}{N(s_i+1)-p_i s_i}$ for $p_i s_i < N(s_i+1)$, and $r_i \geq 1$ is arbitrary for $p_i s_i \geq N(s_i+1)$, the embedding $W_0^{1,p_i}(w_i, \Omega) \hookrightarrow L^{r_i}(\Omega)$ is continuous.*
- (ii) *For any r_i verifying $1 \leq r_i < p_{i,s_i}^* = \frac{Np_i s_i}{N(s_i+1)-p_i s_i}$, the embedding $W_0^{1,p_i}(w_i, \Omega) \hookrightarrow L^{r_i}(\Omega)$ is compact.*
- (iii) *In particular, if $s_i > \frac{N}{p_i}$ then $p_{i,s_i}^* > p_i$. Consequently, the embedding $W_0^{1,p_i}(w_i, \Omega) \hookrightarrow L^{r_i}(\Omega)$ is compact.*

Remark 2.3. Using Theorem 2.2, and for every $r_i \in [1, p_{i,s_i}^*]$ with $i \in \{1, 2\}$, there exists a positive constant S_{r_i} such that

$$\|u\|_{L^{r_i}(\Omega)} \leq S_{r_i} \|u\|_{W_0^{1,p_i}(w_i, \Omega)} \quad (2.1)$$

for all $u \in W_0^{1,p_i}(w_i, \Omega)$.

To obtain a well-defined convolution, $\rho_1 \star u$ (resp., $\rho_2 \star v$), with $\rho_1 \in L^1(\mathbb{R}^N)$ (resp., $\rho_2 \in L^1(\mathbb{R}^N)$) and $u \in W_0^{1,p_1}(w_1, \Omega)$ (resp., $v \in W_0^{1,p_2}(w_2, \Omega)$), it should be noted that the space $W_0^{1,p_1}(w_1, \Omega)$ (resp., $W_0^{1,p_2}(w_2, \Omega)$) is embedded in $W^{1,p_1}(w_1, \mathbb{R}^N)$ (resp., $W^{1,p_2}(w_2, \mathbb{R}^N)$) by identifying every $u \in W_0^{1,p_1}(w_1, \Omega)$ (resp., $v \in W_0^{1,p_2}(w_2, \Omega)$) with its extension equal to zero outside Ω . Henceforth, the convolution $\rho_1 \star u$ can be defined as follows:

$$\rho_1 \star u = \int_{\mathbb{R}^N} \rho_1(x-y) u(y) w_1(y) dy,$$

see Brezis [3, Sections 4.4 and 9.1]. Furthermore, the weak partial derivatives of $\rho_1 \star u$ are given by

$$\frac{\partial}{\partial x_j}(\rho_1 \star u) = \rho_1 \star \frac{\partial u}{\partial x_j} \in L^{p_1}(w_1, \mathbb{R}^N) \quad (2.2)$$

for all $j \in \{1, \dots, N\}$. Now, using Fubini's and Tonelli's theorems alongside Hölder's inequality, we derive

$$\|\rho_1 \star u\|_{L^{r_1}(w_1, \mathbb{R}^N)} \leq \|\rho_1\|_{L^1(\mathbb{R}^N)} \|u\|_{L^{r_1}(w_1, \Omega)} \quad (2.3)$$

for $r_1 \in [1, p_{1,s_1}^*]$. In addition, we have

$$\|\rho_1 \star \frac{\partial u}{\partial x_j}\|_{L^{p_1}(w_1, \mathbb{R}^N)} \leq \|\rho_1\|_{L^1(\mathbb{R}^N)} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^{p_1}(w_1, \Omega)} \quad (2.4)$$

for all $j \in \{1, \dots, N\}$. From (2.2), (2.3), and (2.4), we derive that $u \in W_0^{1,p_1}(w_1, \Omega) \mapsto \rho_1 \star u \in W^{1,p_1}(w_1, \mathbb{R}^N)$ is continuous. Now, using Minkowski's inequality, the convexity of $t \mapsto t^{p_1}$ on $\mathbb{R}^+$ alongside (2.2) and (2.4), we find

$$\begin{aligned} \|\nabla(\rho_1 \star u)\|_{L^{p_1}(w_1, \mathbb{R}^N)}^{p_1} &= \int_{\Omega} w_1(x) |\nabla(\rho_1 \star u)|^{p_1} dx \\ &= \int_{\Omega} w_1(x) \left(\sum_{j=1}^N \left(\rho_1 \star \frac{\partial u}{\partial x_j} \right)^2 \right)^{\frac{p_1}{2}} dx \\ &\leq \int_{\mathbb{R}^N} w_1(x) \left(\sum_{j=1}^N \left| \rho_1 \star \frac{\partial u}{\partial x_j} \right| \right)^{p_1} dx \\ &\leq N^{p_1-1} \sum_{j=1}^N \left\| \rho_1 \star \frac{\partial u}{\partial x_j} \right\|_{L^{p_1}(w_1, \mathbb{R}^N)}^{p_1} \\ &\leq N^{p_1-1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{p_1} \sum_{j=1}^N \left\| \frac{\partial u}{\partial x_j} \right\|_{L^{p_1}(w_1, \Omega)}^{p_1} \\ &\leq N^{p_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{p_1} \|u\|_{W_0^{1,p_1}(w_1, \Omega)}^{p_1}. \end{aligned} \quad (2.5)$$

We adopt a similar approach when $\rho_2 \in L^1(\mathbb{R}^N)$ and $v \in W_0^{1,p_2}(w_2, \Omega)$ to obtain the same estimations. Next, we define

$$\begin{aligned} \bar{N}_f &: W_0^{1,p_1}(w_1, \mathbb{R}^N) \times W_0^{1,p_2}(w_2, \mathbb{R}^N) \subset L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N) \rightarrow L^{r'_1}(\mathbb{R}^N) \times L^{r'_2}(\mathbb{R}^N), \\ \bar{N}_g &: W_0^{1,p_1}(w_1, \mathbb{R}^N) \times W_0^{1,p_2}(w_2, \mathbb{R}^N) \subset L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N) \rightarrow L^{r'_1}(\mathbb{R}^N) \times L^{r'_2}(\mathbb{R}^N), \end{aligned}$$

the Nemytskij operators associated with f and g . In addition, let

$$\begin{aligned} j_1^* &: L^{r'_1}(\mathbb{R}^N) \times L^{r'_2}(\mathbb{R}^N) \rightarrow (W_0^{1,p_1}(w_1, \mathbb{R}^N))^* \times (W_0^{1,p_2}(w_2, \mathbb{R}^N))^*, \\ j_2^* &: L^{r'_1}(\mathbb{R}^N) \times L^{r'_2}(\mathbb{R}^N) \rightarrow (W_0^{1,p_1}(w_1, \mathbb{R}^N))^* \times (W_0^{1,p_2}(w_2, \mathbb{R}^N))^* \end{aligned}$$

be the adjoint operators for the embeddings

$$\begin{aligned} j_1 &: W_0^{1,p_1}(w_1, \mathbb{R}^N) \times W_0^{1,p_2}(w_2, \mathbb{R}^N) \rightarrow L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N), \\ j_2 &: W_0^{1,p_1}(w_1, \mathbb{R}^N) \times W_0^{1,p_2}(w_2, \mathbb{R}^N) \rightarrow L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N). \end{aligned}$$

Furthermore, we define

$$\begin{aligned} N_f &:= j_1^* \circ \bar{N}_f : W_0^{1,p_1}(w_1, \mathbb{R}^N) \times W_0^{1,p_2}(w_2, \mathbb{R}^N) \\ &\rightarrow (W_0^{1,p_1}(w_1, \mathbb{R}^N))^* \times (W_0^{1,p_2}(w_2, \mathbb{R}^N))^*, \\ N_g &:= j_2^* \circ \bar{N}_g : W_0^{1,p_1}(w_1, \mathbb{R}^N) \times W_0^{1,p_2}(w_2, \mathbb{R}^N) \\ &\rightarrow (W_0^{1,p_1}(w_1, \mathbb{R}^N))^* \times (W_0^{1,p_2}(w_2, \mathbb{R}^N))^*. \end{aligned}$$

Let the map $\mathcal{B} : W_0^{1,p_1}(w_1, \Omega) \times W_0^{1,p_2}(w_2, \Omega) \rightarrow W^{1,p_1}(w_1, \mathbb{R}^N) \times W^{1,p_2}(w_2, \mathbb{R}^N)$, which is given by the extension outside Ω with 0, to obtain $\mathcal{B}(u, v) = (\tilde{u}, \tilde{v})$ as mentioned in Section 1. We denote by

$$\mathcal{B}^* : (W^{1,p_1}(w_1, \mathbb{R}^N))^* \times (W^{1,p_2}(w_2, \mathbb{R}^N))^* \rightarrow (W_0^{1,p_1}(w_1, \Omega))^* \times (W_0^{1,p_2}(w_2, \Omega))^*$$

the adjoint mapping of $\mathcal{B}$, and we introduce $\mathcal{I} : X \rightarrow X^*$ as a nonlinear operator defined by

$$\begin{aligned} \mathcal{I}(u, v) &:= \tilde{\mathcal{I}}(u, v) - \lambda_1 A(u, v) - \mathcal{B}^*(N_f(\rho_1 \star \tilde{u}, \rho_2 \star \tilde{v})) \\ &\quad - \mathcal{B}^*(N_g(\rho_1 \star \tilde{u}, \rho_2 \star \tilde{v})) - \int_{\Omega} h_1 u dx - \int_{\Omega} h_2 v dx \end{aligned} \quad (2.6)$$

for all $(u, v) \in X$, where

$$\begin{aligned} \tilde{\mathcal{I}}(u, v) &= \int_{\Omega} \frac{w_1(x)}{p_1} |\nabla u|^{p_1} + \frac{\mu_1 w_1(x)}{q_1} |\nabla u|^{q_1} dx \\ &\quad + \int_{\Omega} \frac{w_2(x)}{p_2} |\nabla v|^{p_2} + \frac{\mu_2 w_2(x)}{q_2} |\nabla v|^{q_2} dx, \end{aligned} \quad (2.7)$$

and

$$A(u, v) = \int_{\Omega} w(x) |u|^{\alpha_1} |v|^{\alpha_2} dx.$$

Definition 2.4. Let X be a reflexive Banach space, X^* its dual space and denote by $\langle \cdot, \cdot \rangle$ the duality pairing. Consider an application $\mathcal{I} : X \rightarrow X^*$. Then, $\mathcal{I}$ is called

(a) to verify the (S^+) -property if

$$u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n \rightarrow \infty} \langle \mathcal{I}u_n, u_n - u \rangle \leq 0 \text{ imply } u_n \rightarrow u \text{ in } X,$$

(b) pseudomonotone if

$$\begin{aligned} u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n \rightarrow \infty} \langle \mathcal{I}u_n, u_n - u \rangle \leq 0 &\text{ imply } \mathcal{I}u_n \\ &\rightharpoonup \mathcal{I}u \text{ and } \langle \mathcal{I}u_n, u_n \rangle \rightarrow \langle \mathcal{I}u, u \rangle, \end{aligned}$$

(c) coercive if

$$\lim_{\|u\|_X \rightarrow \infty} \frac{\langle \mathcal{I}u, u \rangle}{\|u\|_X} = \infty.$$

In order to prove the existence result, we need the following assumptions.

(H₁) $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions such that, for $i \in \{1, 2\}$, there exist three positive continuous functions α_i^* , q_i^* , and δ_i , and a nonnegative function $e_i \in L^{\frac{\gamma_i}{\gamma_i-1}}(\Omega)$ with $1 < \gamma_i < p_{i,s_i}^*$, satisfying the following conditions:

$$0 < \alpha_i^* < p_{i,s_i}^* - 1, \quad 0 < \delta_i < \frac{q_i}{(q_i^*)'}, \quad \frac{\alpha_1^*}{\gamma_1} + \frac{\alpha_2^*}{\gamma_2} = \frac{\gamma_1 - 1}{\gamma_1},$$

and nonnegative constants a_i, b_i , such that for a.e. $x \in \Omega$ and all $s, t \in \mathbb{R}$, we have

$$|f(x, s, t, \xi, \eta)| \leq a_1 |s|^{\alpha_1^*} |t|^{\alpha_2^*} + b_1 \left(|\xi|^{\delta_1} + |\eta|^{\frac{\delta_1 q_2}{q_1}} \right) + e_1(x)$$

and

$$|g(x, s, t, \xi, \eta)| \leq a_2 |s|^{\alpha_1^*} |t|^{\alpha_2^*} + b_2 \left(|\xi|^{\frac{\delta_2 q_1}{q_2}} + |\eta|^{\delta_2} \right) + e_2(x).$$

(H₂) $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \exists C_\varepsilon \in L^1(\Omega), \forall (s, t) \in \mathbb{R}^2$ with $\max\{|s|, |t|\} > \delta_\varepsilon, \forall (\xi, \eta) \in \mathbb{R}^{2N}, \forall x \in \Omega$, we have

$$\begin{aligned} & f(x, s, t, \xi, \eta)s + g(x, s, t, \xi, \eta)t \\ & \leq (\lambda_2 + \varepsilon)w(x)|s|^{\alpha_1}|t|^{\alpha_2} + c(w_1(x)|\xi|^{q_1} + w_2(x)|\eta|^{q_2}) + C_\varepsilon(x), \end{aligned}$$

where the constant c satisfies

$$0 < c < \frac{\min\{\mu_1, \mu_2\}}{\max\{N^{q_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{q_1}, N^{q_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{q_2}\}}.$$

The main tool in this section is based on the surjectivity result for pseudomonotone operators from Carl et al. [4, Theorem 2.99], which we now state.

Theorem 2.5 (Carl et al. [4]). *Let X be a real reflexive Banach space, and assume that $\mathcal{I} : X \rightarrow X^*$ is a bounded, pseudomonotone, and coercive operator. Then, there exists a solution to the equation $\mathcal{I}(u) = v$, where $v \in X^*$.*

Lemma 2.6. *The operator $\tilde{\mathcal{I}} : X \rightarrow X^*$ defined in (2.7) is bounded, continuous, monotone (hence maximal monotone), and of type (S^+) .*

Proof. The proof bears resemblance to the proof given by Fan and Zhang [10, Theorem 3.1] for a single equation when $\mu_i = 0$ for $i \in \{1, 2\}$ and is omitted here. ■

Theorem 2.7. *Suppose that the hypotheses (H₁) and (H₂) are verified. Then, for all $(h_1, h_2) \in X^*$, the system (1.1) admits at least one weak solution $(u, v) \in X$.*

Proof. First, it is worth mentioning that both the Nemytskij operators N_f and N_g are well-defined based on hypothesis (H_1) . The arguments above, along with (2.5), imply that operator $\mathcal{I}$ is bounded and continuous. Going back to (2.6), the couple $(u, v) \in X$ is said to be a weak solution for the system (1.1) if and only if

$$\langle \mathcal{I}(u, v), (\varphi, \psi) \rangle = 0 \quad \text{for all } (\varphi, \psi) \in X.$$

Hence, proving the surjectivity of the operator $\mathcal{I} : X \rightarrow X^*$ allows us to obtain the existence of a weak solution. In this regard, in order to prove that $\mathcal{I}$ is surjective, we employ Theorem 2.5. Using hypothesis (H_1) alongside Hölder's inequality, we find

$$\begin{aligned} & \left| \int_{\Omega} f(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v)) w_1 dx \right| \\ & \leq \|e_1\|_{L^{\frac{\gamma_1}{\gamma_1-1}}(\Omega)} \|w_1\|_{L^{\gamma_1}(\Omega)} + a_1 \|\rho_1 \star u\|_{L^{\gamma_1}(\mathbb{R}^N)}^{\alpha_1^*} \|\rho_2 \star v\|_{L^{\gamma_2}(\mathbb{R}^N)}^{\alpha_2^*} \|w_1\|_{L^{\gamma_1}(\Omega)} \\ & \quad + b_1 \left(\|\nabla(\rho_1 \star u)\|_{L^{q_1}(\mathbb{R}^N)}^{\delta_1} + \|\nabla(\rho_2 \star v)\|_{L^{q_2}(\mathbb{R}^N)}^{\delta_1 \frac{q_2}{q_1}} \right) \|w_1\|_{L^{\frac{q_1}{q_1-\delta_1}}(\Omega)} \end{aligned}$$

for all $(u, v) \in X$, where (u, v) is identified with $\mathcal{B}(u, v) = (\tilde{u}, \tilde{v})$. Moreover, using relations (2.1) and (2.3) together, we obtain

$$\begin{aligned} & \left| \int_{\Omega} f(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v)) w_1 dx \right| \\ & \leq \|e_1\|_{L^{\frac{\gamma_1}{\gamma_1-1}}(\Omega)} \|w_1\|_{L^{\gamma_1}(\Omega)} \\ & \quad + a_1 \|\rho_1\|_{L^1(\mathbb{R}^N)}^{\alpha_1^*} S_{\gamma_1}^{\alpha_1^*} \|u\|_{W_0^{1,p_1}(w_1, \Omega)}^{\alpha_1^*} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{\alpha_2^*} S_{\gamma_2}^{\alpha_2^*} \|v\|_{W_0^{1,p_2}(w_2, \Omega)}^{\alpha_2^*} \|w_1\|_{L^{\gamma_1}(\Omega)} \\ & \quad + b_1 \left(\|\nabla(\rho_1 \star u)\|_{L^{q_1}(\mathbb{R}^N)}^{\delta_1} + \|\nabla(\rho_2 \star v)\|_{L^{q_2}(\mathbb{R}^N)}^{\delta_1 \frac{q_2}{q_1}} \right) \|w_1\|_{L^{\frac{q_1}{q_1-\delta_1}}(\Omega)}. \end{aligned}$$

Ultimately, by relation (2.5), it follows that

$$\begin{aligned} & \left| \int_{\Omega} f(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v)) w_1 dx \right| \\ & \leq \|e_1\|_{L^{\frac{\gamma_1}{\gamma_1-1}}(\Omega)} \|w_1\|_{L^{\gamma_1}(\Omega)} \\ & \quad + a_1 \|\rho_1\|_{L^1(\mathbb{R}^N)}^{\alpha_1^*} S_{\gamma_1}^{\alpha_1^*} \|u\|_{W_0^{1,p_1}(w_1, \Omega)}^{\alpha_1^*} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{\alpha_2^*} S_{\gamma_2}^{\alpha_2^*} \|v\|_{W_0^{1,p_2}(w_2, \Omega)}^{\alpha_2^*} \|w_1\|_{L^{\gamma_1}(\Omega)} \\ & \quad + b_1 \left(N^{\delta_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{\delta_1} \|u\|_{W_0^{1,p_1}(w_1, \Omega)}^{\delta_1} \right. \\ & \quad \left. + N^{\delta_1 \frac{q_2}{q_1}} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{\delta_1 \frac{q_2}{q_1}} \|v\|_{W_0^{1,p_2}(w_2, \Omega)}^{\delta_1 \frac{q_2}{q_1}} \right) \|w_1\|_{L^{\frac{q_1}{q_1-\delta_1}}(\Omega)}. \end{aligned} \tag{2.8}$$

Applying similar arguments for g , we derive

$$\begin{aligned}
& \left| \int_{\Omega} g(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v)) w_2 dx \right| \\
& \leq \|e_2\|_{L^{\frac{\gamma_2}{\gamma_2-1}}(\Omega)} \|w_2\|_{L^{\gamma_2}(\Omega)} \\
& + a_2 \|\rho_1\|_{L^1(\mathbb{R}^N)}^{\alpha_1^*} S_{\gamma_1}^{\alpha_1^*} \|u\|_{W_0^{1,p_1}(w_1,\Omega)}^{\alpha_1^*} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{\alpha_2^*} S_{\gamma_2}^{\alpha_2^*} \|v\|_{W_0^{1,p_2}(w_2,\Omega)}^{\alpha_2^*} \|w_2\|_{L^{\gamma_2}(\Omega)} \quad (2.9) \\
& + b_2 \left(N^{\delta_2 \frac{q_1}{q_2}} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{\delta_2 \frac{q_1}{q_2}} \|u\|_{W_0^{1,p_1}(w_1,\Omega)}^{\delta_2 \frac{q_1}{q_2}} \right. \\
& \quad \left. + N^{\delta_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{\delta_2} \|v\|_{W_0^{1,p_2}(w_2,\Omega)}^{\delta_2} \right) \|w_2\|_{L^{\frac{q_2}{q_2-\delta_2}}(\Omega)}
\end{aligned}$$

for all $(u, v) \in X$.

Now, let us prove that $\mathcal{I} : X \rightarrow X^*$ is pseudomonotone. To do this, let $(u_n, v_n)_{n \in \mathbb{N}} \subset W_0^{1,p_1}(w_1, \Omega) \times W_0^{1,p_2}(w_2, \Omega)$ be a sequence such that

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{in } X \quad (2.10)$$

and

$$\limsup_{n \rightarrow \infty} \langle \mathcal{I}(u_n, v_n), (u_n - u, v_n - v) \rangle \leq 0. \quad (2.11)$$

Because $(u_n, v_n)_{n \in \mathbb{N}}$ is bounded in X , then there exist two constants $C_1, C_2 > 0$ such that

$$\begin{cases} N \|\rho_1\|_{L^1(\mathbb{R}^N)} \|u_n\|_{W_0^{1,p_1}(w_1,\Omega)} \leq C_1, \\ N \|\rho_2\|_{L^1(\mathbb{R}^N)} \|v_n\|_{W_0^{1,p_2}(w_2,\Omega)} \leq C_2. \end{cases} \quad (2.12)$$

Using (2.8) and (2.12), we obtain

$$\begin{aligned}
& \left| \int_{\Omega} f(x, \rho_1 \star u_n, \rho_2 \star v_n, \nabla(\rho_1 \star u_n), \nabla(\rho_2 \star v_n))(u_n - u) dx \right| \\
& \leq \|e_1\|_{L^{\frac{\gamma_1}{\gamma_1-1}}(\Omega)} \|u_n - u\|_{L^{\gamma_1}(\Omega)} + a_1 S_{\gamma_1}^{\alpha_1^*} C_1^{\alpha_1^*} S_{\gamma_2}^{\alpha_2^*} C_2^{\alpha_2^*} \|u_n - u\|_{L^{\gamma_1}(\Omega)} \\
& + b_1 (C_1^{\delta_1} + C_2^{\delta_1 \frac{q_2}{q_1}}) \|u_n - u\|_{L^{\frac{q_1}{q_1-\delta_1}}(\Omega)}
\end{aligned}$$

for all $n \in \mathbb{N}$. Similarly for g , using (2.9) and (2.12), we find

$$\begin{aligned}
& \left| \int_{\Omega} g(x, \rho_1 \star u_n, \rho_2 \star v_n, \nabla(\rho_1 \star u_n), \nabla(\rho_2 \star v_n))(v_n - v) dx \right| \\
& \leq \|e_2\|_{L^{\frac{\gamma_2}{\gamma_2-1}}(\Omega)} \|v_n - v\|_{L^{\gamma_2}(\Omega)} + a_2 S_{\gamma_1}^{\alpha_1^*} C_1^{\alpha_1^*} S_{\gamma_2}^{\alpha_2^*} C_2^{\alpha_2^*} \|v_n - v\|_{L^{\gamma_1}(\Omega)} \\
& + b_1 (C_1^{\delta_2 \frac{q_1}{q_2}} + C_2^{\delta_2}) \|v_n - v\|_{L^{\frac{q_2}{q_2-\delta_2}}(\Omega)}
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking into account $(u_n, v_n) \rightarrow (u, v)$ in X as well as the compact embedding of $W_0^{1,p_i}(w_i, \Omega)$ in $L^{r_i}(\Omega)$, $L^{r_i}(\Omega)$ and $L^{\frac{q_i}{q_i-\delta_i}}(\Omega)$ for $i \in \{1, 2\}$, it follows that

$$\begin{cases} \lim_{n \rightarrow \infty} \int_{\Omega} f(x, \rho_1 \star u_n, \rho_2 \star v_n, \nabla(\rho_1 \star u_n), \nabla(\rho_2 \star v_n))(u_n - u) dx = 0, \\ \lim_{n \rightarrow \infty} \int_{\Omega} g(x, \rho_1 \star u_n, \rho_2 \star v_n, \nabla(\rho_1 \star u_n), \nabla(\rho_2 \star v_n))(v_n - u) dx = 0. \end{cases} \quad (2.13)$$

Using (2.13), we deduce that

$$\limsup_{n \rightarrow \infty} \langle \tilde{\mathcal{I}}(u_n, v_n), (u_n - u, v_n - v) \rangle = \limsup_{n \rightarrow \infty} \langle \mathcal{I}(u_n, v_n), (u_n - u, v_n - v) \rangle \leq 0. \quad (2.14)$$

Because $\tilde{\mathcal{I}}$ satisfies the (S^+) -property as seen in Lemma 2.6, and from (2.10) and (2.14), we obtain

$$(u_n, v_n) \rightarrow (u, v) \quad \text{in } X.$$

Ultimately, as $\mathcal{I}$ is continuous, we find $\mathcal{I}(u_n, v_n) \rightarrow \mathcal{I}(u, v)$ in X^* . Thus, $\mathcal{I}$ is pseudomonotone.

The final step is to verify that the operator $I : X \rightarrow X^*$ is coercive. Using hypothesis (H_2) and (2.5), we find

$$\begin{aligned} & \int_{\Omega} f(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v))u \\ & \quad + g(x, \rho_1 \star u, \rho_2 \star v, \nabla(\rho_1 \star u), \nabla(\rho_2 \star v))v dx \\ & \leq \int_{\Omega} (\lambda_2 + \varepsilon)w(x)|\rho_1 \star u|^{\alpha_1}|\rho_2 \star v|^{\alpha_2} dx \\ & \quad + \int_{\Omega} c(w_1(x)|\nabla(\rho_1 \star u)|^{q_1} + w_2(x)|\nabla(\rho_2 \star v)|^{q_2})dx + C_{\varepsilon}(x) \\ & \leq \int_{\Omega} \left(\frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \left(\frac{\alpha_1}{p_1} w_1(x)|\nabla(\rho_1 \star u)|^{p_1} + \frac{\alpha_2}{p_2} w_2(x)|\nabla(\rho_2 \star v)|^{p_2} \right) dx \\ & \quad + cN^{q_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{q_1} \int_{\Omega} w_1(x)|\nabla u|^{q_1} dx + cN^{q_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{q_2} \int_{\Omega} w_2(x)|\nabla v|^{q_2} dx \\ & \quad + \|C_{\varepsilon}(x)\|_{L^1(\Omega)}. \end{aligned} \quad (2.15)$$

On the other hand, we have

$$\begin{aligned} \langle \tilde{\mathcal{I}}(u, v), (u, v) \rangle & \geq \int_{\Omega} w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2} dx \\ & \quad + \min\{\mu_1, \mu_2\} \int_{\Omega} w_1(x)|\nabla u|^{q_1} + w_2(x)|\nabla v|^{q_2} dx. \end{aligned} \quad (2.16)$$

Now, using the embedding theorem, there exist $\theta_0, \theta_1 > 0$ such that

$$\begin{aligned} & \delta \left[\int_{\Omega} (w_1(x)|\nabla u|^{q_1} + w_2(x)|\nabla v|^{q_2}) dx \right] \\ & \geq \theta_0 \left[\int_{\Omega} (w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}) dx \right] - \theta_1, \end{aligned} \quad (2.17)$$

where

$$\delta = \min\{\mu_1, \mu_2\} - c \max \left\{ N^{q_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{q_1}, N^{q_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{q_2} \right\}.$$

Therefore, from (2.15) and (2.16), we deduce that

$$\begin{aligned} & \langle \mathcal{I}(u, v), (u, v) \rangle \\ & \geq \int_{\Omega} w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2} dx \\ & \quad + \delta \left(\int_{\Omega} w_1(x)|\nabla u|^{q_1} + w_2(x)|\nabla v|^{q_2} dx \right) \\ & \quad - \int_{\Omega} \left(\frac{\alpha_1}{p_1} w_1(x)|\nabla u|^{p_1} + \frac{\alpha_2}{p_2} w_2(x)|\nabla v|^{p_2} \right) dx \\ & \quad - \left(\frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \int_{\Omega} \left(\frac{\alpha_1}{p_1} w_1(x)|\nabla(\rho_1 \star u)|^{p_1} + \frac{\alpha_2}{p_2} w_2(x)|\nabla(\rho_2 \star v)|^{p_2} \right) dx \\ & \quad - \|C_{\varepsilon}(x)\|_{L^1(\Omega)} - \|(u, v)\|_X \|(h_1, h_2)\|_{X^*}. \end{aligned}$$

Now, using (2.17), it follows that

$$\begin{aligned} & \langle \mathcal{I}(u, v), (u, v) \rangle \\ & \geq (1 + \theta_0) \left[\int_{\Omega} w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2} dx \right] - \theta_1 \\ & \quad - \frac{\min\{\alpha_1, \alpha_2\}}{\max\{p_1, p_2\}} \left(\int_{\Omega} w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2} dx \right) - \|C_{\varepsilon}(x)\|_{L^1(\Omega)} \\ & \quad - \left(\frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \left[\frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{p_1} \int_{\Omega} w_1(x)|\nabla u|^{p_1} dx \right. \\ & \quad \quad \left. + \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{p_2} \int_{\Omega} w_2(x)|\nabla v|^{p_2} dx \right] \\ & \quad - \|(u, v)\|_X \|(h_1, h_2)\|_{X^*} \\ & \geq \left[(1 + \theta_0) - \frac{\min\{\alpha_1, \alpha_2\}}{\max\{p_1, p_2\}} \right. \\ & \quad \left. - \left(\frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \max \left\{ \frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{p_1}, \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{p_2} \right\} \right] \\ & \quad \cdot \left(\int_{\Omega} w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2} dx \right) - \theta_1 - \|C_{\varepsilon}(x)\|_{L^1(\Omega)} \\ & \quad - \|(u, v)\|_X \|(h_1, h_2)\|_{X^*} \end{aligned}$$

$$\begin{aligned}
&\geq \left[(1 + \theta_0) - \frac{\min\{\alpha_1, \alpha_2\}}{\max\{p_1, p_2\}} \right. \\
&\quad \left. - \left(\frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \max \left\{ \frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{p_1}, \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{p_2} \right\} \right] \\
&\quad \cdot \left(\|u\|_{W_0^{1,p_1}(w_1, \Omega)}^{p_1} + \|v\|_{W_0^{1,p_2}(w_2, \Omega)}^{p_2} \right) - \theta_1 - \|C_\varepsilon(x)\|_{L^1(\Omega)} - \frac{\alpha_0}{\tau} \|(u, v)\|_X^\tau \\
&\quad - \frac{1}{\alpha_0 \tau'} \|(h_1, h_2)\|_{X^*}^{\tau'},
\end{aligned}$$

where

$$0 < \alpha_0 < \tau < \min\{p_1, p_2\} \quad \text{with } \tau' = \frac{\tau}{\tau - 1}.$$

Thus, we obtain

$$\begin{aligned}
&\langle \mathcal{I}(u, v), (u, v) \rangle \\
&\geq \left[1 + \theta_0 - \frac{\min\{\alpha_1, \alpha_2\}}{\max\{p_1, p_2\}} - \frac{\alpha_0}{\tau} \right. \\
&\quad \left. - \left(\frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \max \left\{ \frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{p_1}, \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{p_2} \right\} \right] \|(u, v)\|_X^\tau \\
&\quad - \theta_1 - \|C_\varepsilon(x)\|_{L^1(\Omega)} - \frac{1}{\alpha_0 \tau'} \|(h_1, h_2)\|_{X^*}^{\tau'}.
\end{aligned}$$

Choosing appropriate values for ε and α_0 such that

$$\begin{aligned}
1 + \theta_0 &> \frac{\min\{\alpha_1, \alpha_2\}}{\max\{p_1, p_2\}} \\
&\quad + \left(\frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \max \left\{ \frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{p_1}, \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{p_2} \right\} + \frac{\alpha_0}{\tau}
\end{aligned}$$

leads to the coerciveness of $\mathcal{I}$.

Thus, all assumptions of Theorem 2.5 are verified. Therefore, there exists $(u, v) \in X$ such that $\mathcal{I}(u, v) = 0$. This completes the proof. $\blacksquare$

Example 2.8. Let the functions f and g be defined by

$$f(x, s, t, \xi, \eta) = |s|^{\alpha_1-2} s |t|^{\alpha_2} \left(a_1(x) + \frac{d_1(x)}{1 + |s|^{\alpha_1} |t|^{\alpha_2}} |\xi|^{q_1} \right),$$

and

$$g(x, s, t, \xi, \eta) = |s|^{\alpha_1} |t|^{\alpha_2-2} t \left(a_2(x) + \frac{d_2(x)}{1 + |s|^{\alpha_1} |t|^{\alpha_2}} |\eta|^{q_2} \right),$$

where α_1 and α_2 are defined in Section 1, and $a_1, a_2, d_1, d_2 \in L^\infty(\Omega)$ with the condition $\|a_1 + a_2\|_{L^\infty(\Omega)} < \lambda_2$ are satisfied. Then, f and g satisfy the hypotheses in Theorem 2.7.

3. Case of a gradient-type system

In this section, we focus our study on a gradient-type system. In other words, we consider the following system:

$$\begin{cases} -\Delta_{w_1, p_1} u - \mu_1 \Delta_{w_1, q_1} u = \lambda_1 w(x) |u|^{\alpha_1-2} u |v|^{\alpha_2} \\ \quad + \frac{\partial F}{\partial u}(x, \rho_1 \star u, \rho_2 \star v) + h_1(x) & \text{in } \Omega, \\ -\Delta_{w_2, p_2} v - \mu_2 \Delta_{w_2, q_2} v = \lambda_1 w(x) |v|^{\alpha_2-2} v |u|^{\alpha_1} \\ \quad + \frac{\partial F}{\partial v}(x, \rho_1 \star u, \rho_2 \star v) + h_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.1)$$

Similarly, in order to prove the existence of at least one weak solution, we need the following assumptions on F .

(F₁) $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t} : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions such that there exist positive continuous functions α_i^* , nonnegative functions $e_i \in L^1(\Omega)$ verifying $1 < \alpha_i^* < p_{i, s_i}^*$, and nonnegative constants a_i for $i \in \{1, 2\}$, such that for all $(s, t) \in \mathbb{R}^2$, we have

$$\left| \frac{\partial F}{\partial s}(x, s, t) \right| \leq a_1 w(x) |s|^{\alpha_1^*-1} |t|^{\alpha_2^*} + e_1(x)$$

and

$$\left| \frac{\partial F}{\partial t}(x, s, t) \right| \leq a_2 w(x) |s|^{\alpha_1^*} |t|^{\alpha_2^*-1} + e_2(x).$$

(F₂) There exist a nonnegative constant C_1 and a nonnegative function $C_2 \in L^1(\Omega)$ such that, for all $x \in \Omega$, we have

$$|F(x, s, t)| \leq C_1 w(x) |s|^{\alpha_1} |t|^{\alpha_2} + C_2(x).$$

(F₃) $F_\infty(x) \equiv \limsup_{|(s,t)| \rightarrow +\infty} \frac{F(x, s, t)}{w(x) |s|^{\alpha_1} |t|^{\alpha_2}} < \lambda_2$ uniformly for a.e. $x \in \Omega$.

Define

$$\mathcal{I}(u, v) = \tilde{\mathcal{I}}(u, v) - \lambda_1 A(u, v) - \int_{\Omega} F(x, \rho_1 \star u, \rho_2 \star v) dx - \langle h_1, u \rangle - \langle h_2, v \rangle, \quad (3.2)$$

where

$$\begin{aligned} \tilde{\mathcal{I}}(u, v) &= \int_{\Omega} \frac{w_1(x)}{p_1} |\nabla u|^{p_1} + \frac{\mu_1 w_1(x)}{q_1} |\nabla u|^{q_1} dx \\ &\quad + \int_{\Omega} \frac{w_2(x)}{p_2} |\nabla v|^{p_2} + \frac{\mu_2 w_2(x)}{q_2} |\nabla v|^{q_2} dx, \end{aligned} \quad (3.3)$$

and

$$A(u, v) = \int_{\Omega} w(x) |u|^{\alpha_1} |v|^{\alpha_2} dx.$$

Next, we state and prove the existence result for the system (3.1) at resonance.

Theorem 3.1. *Suppose that hypotheses (F₁)–(F₃) are verified. Then, the system (3.1) admits at least one weak solution.*

Proof. In order to obtain the desired result, we need to minimize the energy functional associated with the system (3.1), that is,

$$\mathcal{I}(u, v) = \tilde{\mathcal{I}}(u, v) - \lambda_1 A(u, v) - \int_{\Omega} F(x, \rho_1 \star u, \rho_2 \star v) dx - \langle h_1, u \rangle - \langle h_2, v \rangle.$$

To this end, we show that $\mathcal{I}$ is weakly lower semicontinuous and coercive. Let $(u_n, v_n)_{n \in \mathbb{N}}$ be a sequence such that

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{in } X.$$

Taking the compact embedding $W_0^{1,p_i}(w_i, \Omega) \hookrightarrow L^{r_i}(\Omega)$ into consideration, we obtain

$$(u_n, v_n) \rightarrow (u, v) \quad \text{in } L^{r_1}(\Omega) \times L^{r_2}(\Omega).$$

Since $r_i \in [1, p_{i,s_i}^*]$ for $i \in \{1, 2\}$. In addition, there exists $(\theta_1, \theta_2) \in L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ such that

$$\begin{cases} |u_n(x)| \leq \theta_1(x), \\ |v_n(x)| \leq \theta_2(x) \end{cases}$$

for a.e. $x \in \Omega$. Therefore, using hypothesis (F₂) as well as the properties of convolution and mollifiers, we obtain

$$\begin{aligned} F(x, \rho_1 \star u_n(x), \rho_2 \star v_n(x)) &\leq C_1 w(x) |\rho_1 \star u_n|^{\alpha_1} |\rho_2 \star v_n|^{\alpha_2} + C_2(x) \\ &\leq C_1 w(x) |\rho_1 \star \theta_1|^{\alpha_1} |\rho_2 \star \theta_2|^{\alpha_2} + C_2(x). \end{aligned}$$

Using the fact that $F(x, \rho_1 \star u_n(x), \rho_2 \star v_n(x)) \rightarrow F(x, \rho_1 \star u(x), \rho_2 \star v(x))$, for a.e. $x \in \Omega$ and applying Fatou's lemma, we derive

$$\limsup_{n \rightarrow +\infty} \int_{\Omega} F(x, \rho_1 \star u_n(x), \rho_2 \star v_n(x)) dx \leq \int_{\Omega} F(x, \rho_1 \star u(x), \rho_2 \star v(x)) dx.$$

Thus, we find

$$\begin{aligned} \mathcal{I}(u, v) &= \tilde{\mathcal{I}}(u, v) - \lambda_1 A(u, v) - \int_{\Omega} F(x, \rho_1 \star u, \rho_2 \star v) dx - \langle h_1, u \rangle - \langle h_2, v \rangle \\ &\leq \liminf_{n \rightarrow +\infty} \tilde{\mathcal{I}}(u_n, v_n) - \lambda_1 \lim_{n \rightarrow +\infty} A(u_n, v_n) \\ &\quad - \limsup_{n \rightarrow +\infty} \int_{\Omega} F(x, \rho_1 \star u_n, \rho_2 \star v_n) dx - \lim_{n \rightarrow +\infty} \langle h_1, u_n \rangle - \lim_{n \rightarrow +\infty} \langle h_2, v_n \rangle \\ &\leq \liminf_{n \rightarrow +\infty} \left\{ \tilde{\mathcal{I}}(u_n, v_n) - \lambda_1 A(u_n, v_n) - \int_{\Omega} F(x, \rho_1 \star u_n, \rho_2 \star v_n) dx \right. \\ &\quad \left. - \langle h_1, u_n \rangle - \langle h_2, v_n \rangle \right\} \\ &\leq \liminf_{n \rightarrow +\infty} \mathcal{I}(u_n, v_n). \end{aligned}$$

Thus, $\mathcal{I}$ is weakly lower semicontinuous.

The final step involves proving that $\mathcal{I}$ is coercive. Using hypotheses (F₃), and for all $\varepsilon > 0$, there exist $\delta_\varepsilon > 0$ and $C_\varepsilon \in L^1(\Omega)$ such that

$$F(x, \rho_1 \star u, \rho_2 \star v) \leq (F_\infty(x) + \varepsilon)w(x)|\rho_1 \star u|^{\alpha_1}|\rho_2 \star v|^{\alpha_2} + C_\varepsilon(x) \quad (3.4)$$

for every $(u, v) \in \mathbb{R}^2$ satisfying

$$|(\rho_1 \star u, \rho_2 \star v)|_\infty > \delta_\varepsilon$$

for a.e. $x \in \Omega$. On the other hand, applying the embedding theorem, there exist $\eta_0, \eta_1 > 0$ such that

$$\begin{aligned} \min\{\mu_1, \mu_2\} \int_\Omega \frac{w_1(x)}{q_1} |\nabla u|^{q_1} + \frac{w_2(x)}{q_2} |\nabla v|^{q_2} dx \\ \geq \eta_0 \int_\Omega \frac{w_1(x)}{p_1} |\nabla u|^{p_1} + \frac{w_2(x)}{p_2} |\nabla v|^{p_2} dx - \eta_1. \end{aligned} \quad (3.5)$$

Therefore, from (2.16) and (3.4), we derive

$$\begin{aligned} \mathcal{I}(u, v) &\geq \int_\Omega \frac{w_1(x)}{p_1} |\nabla u|^{p_1} + \frac{\mu_1 w_1(x)}{q_1} |\nabla u|^{q_1} dx \\ &\quad + \int_\Omega \frac{w_2(x)}{p_2} |\nabla v|^{p_2} + \frac{\mu_2 w_2(x)}{q_2} |\nabla v|^{q_2} dx \\ &\quad - \lambda_1 \int_\Omega w(x) |u|^{\alpha_1} |v|^{\alpha_2} dx \\ &\quad - (\lambda_2 + \varepsilon) \int_\Omega w(x) |\rho_1 \star u|^{\alpha_1} |\rho_2 \star v|^{\alpha_2} dx \\ &\quad - \int_\Omega C_\varepsilon(x) dx - \|(u, v)\|_X \|(h_1, h_2)\|_{X^*}. \end{aligned}$$

It follows then

$$\begin{aligned} \mathcal{I}(u, v) &\geq \int_\Omega \left(\frac{w_1(x)}{p_1} |\nabla u|^{p_1} + \frac{w_2(x)}{p_2} |\nabla v|^{p_2} \right) dx \\ &\quad + \min\{\mu_1, \mu_2\} \int_\Omega \left(\frac{w_1(x)}{q_1} |\nabla u|^{q_1} + \frac{w_2(x)}{q_2} |\nabla v|^{q_2} \right) dx - \|C_\varepsilon(x)\|_{L^1(\Omega)} \\ &\quad - \int_\Omega \left(\frac{\alpha_1 w_1(x)}{p_1} |\nabla u|^{p_1} + \frac{\alpha_2 w_2(x)}{p_2} |\nabla v|^{p_2} \right) dx - \|(u, v)\|_X \|(h_1, h_2)\|_{X^*} \\ &\quad - \left(\frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \int_\Omega \left(\frac{\alpha_1 w_1(x)}{p_1} |\nabla(\rho_1 \star u)|^{p_1} + \frac{\alpha_2 w_2(x)}{p_2} |\nabla(\rho_2 \star v)|^{p_2} \right) dx. \end{aligned}$$

Now, using (3.5), we find

$$\begin{aligned}
\mathcal{I}(u, v) &\geq \frac{(1 + \eta_0)}{\max\{p_1, p_2\}} \int_{\Omega} (w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}) dx - \eta_1 - \|C_{\varepsilon}(x)\|_{L^1(\Omega)} \\
&\quad - \frac{\min\{\alpha_1, \alpha_2\}}{\max\{p_1, p_2\}} \int_{\Omega} (w_1(x)|\nabla u|^{p_1} + w_2(x)|\nabla v|^{p_2}) dx - \|(u, v)\|_X \|(h_1, h_2)\|_{X^*} \\
&\quad - \left(\frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \left[\frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{p_1} \int_{\Omega} w_1(x)|\nabla u|^{p_1} dx \right. \\
&\quad \quad \left. + \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{p_2} \int_{\Omega} w_2(x)|\nabla v|^{p_2} dx \right] \\
&\geq \left[\frac{(1 + \eta_0)}{\max\{p_1, p_2\}} - \frac{\min\{\alpha_1, \alpha_2\}}{\max\{p_1, p_2\}} \right. \\
&\quad \left. - \left(\frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \max \left\{ \frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{p_1}, \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{p_2} \right\} \right] \\
&\quad \cdot \left(\|u\|_{W_0^{1,p_1}(w_1,\Omega)}^{p_1} + \|v\|_{W_0^{1,p_2}(w_2,\Omega)}^{p_2} \right) - \eta_1 - \|C_{\varepsilon}(x)\|_{L^1(\Omega)} - \frac{\beta_0}{\rho} \|(u, v)\|_X^{\rho} \\
&\quad - \frac{1}{\beta_0 \rho'} \|(h_1, h_2)\|_{X^*}^{\rho'},
\end{aligned}$$

where $0 < \beta_0 < \rho < \min\{p_1, p_2\}$, with $\rho' = \frac{\rho}{\rho-1}$. Thus, we obtain

$$\begin{aligned}
\mathcal{I}(u, v) &\geq \left[\frac{(1 + \eta_0) - \min\{\alpha_1, \alpha_2\}}{\max\{p_1, p_2\}} \right. \\
&\quad \left. - \left(\frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \max \left\{ \frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{p_1}, \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{p_2} \right\} - \frac{\beta_0}{\rho} \right] \|(u, v)\|_X^{\rho} \\
&\quad - \|C_{\varepsilon}(x)\|_{L^1(\Omega)} - \frac{1}{\beta_0 \rho'} \|(h_1, h_2)\|_{X^*}^{\rho'} - \eta_1.
\end{aligned}$$

Choosing appropriate values for ε and β_0 such that

$$\frac{(1 + \eta_0) - \min\{\alpha_1, \alpha_2\}}{\max\{p_1, p_2\}} > \left(\frac{\lambda_2 + \varepsilon}{\lambda_1} \right) \max \left\{ \frac{\alpha_1}{p_1} N^{p_1} \|\rho_1\|_{L^1(\mathbb{R}^N)}^{p_1}, \frac{\alpha_2}{p_2} N^{p_2} \|\rho_2\|_{L^1(\mathbb{R}^N)}^{p_2} \right\} + \frac{\beta_0}{\rho},$$

leads to the coerciveness of $\mathcal{I}$.

Hence, $\mathcal{I}$ is weakly lower semicontinuous and coercive, then $\mathcal{I}$ admits a minimum point (u, v) in X . Owing to hypothesis (F_1) , $\mathcal{I}$ is a C^1 -functional. Furthermore, $(u, v) \in X$ is a weak solution for the system (3.1) if and only if (u, v) is a critical point. This completes the proof. $\blacksquare$

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