Endpoint estimates
for higher order Marcinkiewicz multipliers

Odysseas Bakas, Valentina Ciccone, Ioannis Parissis and Marco Vitturi

Abstract. We consider Marcinkiewicz multipliers of any lacunary order defined by means of uniformly bounded variation on each lacunary Littlewood–Paley interval of some fixed order $\tau \geq 1$. We prove the optimal endpoint bounds for such multipliers as a corollary of a more general endpoint estimate for a class of multipliers introduced by Coifman, Rubio de Francia and Semmes, and further studied by Tao and Wright. Our methods also yield the best possible endpoint mapping property for higher order Hörmander–Mihlin multipliers, namely, multipliers which are singular on every point of a lacunary set of order τ. These results can be considered as endpoint versions of corresponding results of Sjögren and Sjölin. Finally, our methods generalize a weak square function characterization of the space $L \log^{1/2} L$ in terms of a square function introduced by Tao and Wright: we realize such a weak characterization as the dual of the Chang–Wilson–Wolff inequality, thus giving corresponding weak square function characterizations for the spaces $L \log^{\tau/2} L$ for general integer orders $\tau \geq 1$.

1. Introduction

Our topic is endpoint estimates for Marcinkiewicz-type multipliers on the real line. We recall that a Marcinkiewicz multiplier is a bounded function $m : \mathbb{R} \to \mathbb{C}$ which has bounded variation on each Littlewood–Paley interval

$$L_k := (-2^{k+1}, -2^k] \cup [2^k, 2^{k+1}),$$

uniformly in $k \in \mathbb{Z}$. It is well known that the operator $\mathrm{T}_m f := (m \widehat{f})^\vee$ is bounded on $L^p(\mathbb{R})$ for all $p \in (1, \infty)$. Endpoint estimates for Marcinkiewicz multipliers were proved by Tao and Wright in [17] where the authors prove that they locally map $L \log^{1/2} L$ into weak L^1.

A prototypical Marcinkiewicz multiplier is given by the signed sum

$$\sum_{k \in \mathbb{Z}} \varepsilon_k \mathbf{1}_{L_k}, \quad \varepsilon_k \in \{-1, +1\},$$

while an orthogonality argument provides the link between Marcinkiewicz multipliers and the classical Littlewood–Paley square function

$$\mathrm{LP}_1 f(x) := \Big(\sum_{k \in \mathbb{Z}} |\mathrm{P}_k f|^2 \Big)^{1/2} = \Big(\mathbf{E} \Big| \sum_{k \in \mathbb{Z}} \varepsilon_k \mathrm{P}_k f \Big|^2 \Big)^{1/2}, \quad \mathrm{P}_k f := (\mathbf{1}_{L_k} \widehat{f})^\vee,$$

the expectation being over choices of independent random signs.

In the present paper, we are interested in higher order versions of Marcinkiewicz multipliers. In order to motivate such a study, it is very natural to consider square functions that project to Littlewood–Paley intervals given by lacunary sets of order 2 or higher. For example, letting

$$L_{(k,m)} := \{\xi \in \mathbb{R} : |\xi| \in (2^k + 2^{m-1}, 2^k + 2^m] \cup [2^{k+1} - 2^m, 2^{k+1} - 2^{m-1})\}, \quad k > m,$$

denote the family of Littlewood–Paley intervals of second order, we naturally define

$$\mathrm{LP}_2 f := \Big(\sum_{\substack{(k,m) \in \mathbb{Z}^2 \\ k > m}} |\mathrm{P}_{(k,m)} f|^2 \Big)^{1/2}$$

$$= \Big(\mathbf{E} \Big| \sum_{\substack{(k,m) \in \mathbb{Z}^2 \\ k > m}} \varepsilon_{k,m} \mathrm{P}_{(k,m)} f \Big|^2 \Big)^{1/2}, \quad \mathrm{P}_{(k,m)} f := (\mathbf{1}_{L_{(k,m)}} \widehat{f})^\vee,$$

initially for Schwartz functions with compactly supported Fourier transform. This is a second order Littlewood–Paley square function, while the multiplier

$$\sum_{\substack{(k,m) \in \mathbb{Z}^2 \\ k > m}} \varepsilon_{(k,m)} \mathbf{1}_{L_{(k,m)}}, \quad \varepsilon_{(k,m)} \in \{-1, +1\},$$

can be considered as a prototypical Marcinkiewicz multiplier of order 2. A Littlewood–Paley partition $\{L : L \in \Lambda_\tau\}$ of lacunary order $\tau > 1$ is naturally produced by iterating Whitney decompositions inside each Littlewood–Paley interval of order $\tau - 1$. Accordingly, a Marcinkiewicz multiplier of order τ is a bounded function which has bounded variation uniformly on all Littlewood–Paley intervals of order τ. Likewise, we define the Littlewood–Paley square function of order τ as follows:

$$\mathrm{LP}_\tau f := \Big(\sum_{L \in \Lambda_\tau} |\mathrm{P}_L f|^2 \Big)^{1/2} = \Big(\mathbf{E} \Big| \sum_{L \in \Lambda_\tau} \varepsilon_L \mathrm{P}_L f \Big|^2 \Big)^{1/2}, \quad \mathrm{P}_L f := (\mathbf{1}_L \widehat{f})^\vee.$$

With precise definitions to follow, a punchline result of this paper is the following.

Theorem A. *If m is a Marcinkiewicz multiplier of order $\tau \in \mathbb{N}$, then T_m satisfies the estimate*

$$|\{x \in \mathbb{R} : |\mathrm{T}_m f(x)| > \alpha\}| \lesssim \int_{\mathbb{R}} \frac{|f|}{\alpha} \Big(\log\Big(e + \frac{|f|}{\alpha}\Big) \Big)^{\tau/2}, \quad \alpha > 0.$$

The same is true for the Littlewood–Paley square function LP_τ of order τ. In both cases, the endpoint estimates are best possible in the sense that the exponent $\tau/2$ in the right-hand side cannot be replaced by any smaller exponent.

We will deduce Theorem A as a consequence of the more general Theorem B below which applies to the wider class of $R_{2,\tau}$ multipliers.

1.1. Lacunary sets of higher order

In order to describe the classes of higher order multipliers we are interested in, it will be necessary to introduce some notation for lacunary sets of general order. The standard Littlewood–Paley partition of the real line is the collection of intervals $\Lambda_1 := \{\pm[2^k, 2^{k+1}):$ $k \in \mathbb{Z}\}$ and it is a Whitney decomposition of $\mathbb{R} \setminus \{0\}$. For a finite dyadic interval $I \subset \mathbb{R}$ the standard Whitney partition $\mathcal{W}(I)$ of I is the collection of the maximal dyadic subintervals $L \subset I$ such that $\text{dist}(L, \mathbb{R} \setminus I) = |L|$. Now, for any integer $\tau > 1$, we set

$$\Lambda_\tau := \bigcup_{I \in \Lambda_{\tau-1}} \mathcal{W}(I)$$

and call Λ_τ the standard Littlewood–Paley collection of intervals of order τ. We denote by lac_τ the collection of all endpoints of intervals in Λ_τ. Observe that, as in [3], the set lac_τ has the explicit representation

$$\text{lac}_\tau = \{\pm 2^{n_1} \pm 2^{n_2} + \cdots \pm 2^{n_\tau} : n_1 > n_2 > \cdots > n_\tau, \, n_j \in \mathbb{Z} \text{ for all } j\}.$$

For uniformity in the notation, we also set $\Lambda_0 = \{(-\infty, 0), (0, +\infty)\}$ and $\text{lac}_0 := \{0\}$. It will be useful throughout the paper to truncate the scales of lacunary intervals and numbers by defining

$$\Lambda_\tau^n := \{L \in \Lambda_\tau : |L| \geq n\}, \quad n \in 2^{\mathbb{Z}}.$$

Accordingly, lac_τ^n denotes endpoints of intervals in Λ_τ^n.

We need a smooth way to project to frequency intervals in Λ_τ. For this, we consider a smooth even function $0 \leq \eta \leq 1$ such that η is identically 1 on $[-1/2, 1/2]$ and vanishes off $[-5/8, 5/8]$. For a positive integer τ and $L \in \Lambda_\tau$, we define the (rescaled) L-th frequency component of some multiplier $m: \mathbb{R} \to \mathbb{C}$ as

$$m_L(\xi) := \eta(\xi) m(c_L + \xi|L|), \quad \xi \in \mathbb{R},$$

with c_L denoting the center of L.

1.2. Higher order multipliers and endpoint estimates

With this notation at hand we will say that $m: \mathbb{R} \to \mathbb{C}$ is a Hörmander–Mihlin multiplier of order τ if

$$\|m\|_{H_\tau} := \sup_{|\alpha| \leq M} \sup_{L \in \Lambda_\tau} \|\partial^\alpha m_L\|_{L^\infty} < +\infty,$$

for some sufficiently large positive integer M which we will not keep track of. Note that the higher order Hörmander–Mihlin condition is essentially the natural assertion

$$|\partial^\alpha m(\xi)| \lesssim \text{dist}(\xi, \text{lac}_{\tau-1})^{-\alpha}, \quad \xi \in \mathbb{R} \setminus \text{lac}_{\tau-1}.$$

Likewise, we will say that a bounded function $m\colon \mathbb{R} \to \mathbb{C}$ is a Marcinkiewicz multiplier of order $\tau \in \mathbb{N}$ if the components m_L have bounded variation uniformly in $L \in \Lambda_\tau$. Here we use the standard variation norms defined for $r \in [1, \infty]$ as follows:

$$\|F\|_{V_r} := \sup_N \sup_{x_0 < \cdots < x_N} \Big(\sum_{0 \le k \le N} |F(x_{k+1}) - F(x_k)|^r \Big)^{1/r}.$$

Note that usually Marcinkiewicz multipliers are defined by asking that the pieces $m\mathbf{1}_L$ have bounded 1-variation, uniformly in L. One can check that our definition, using the smooth cutoff η, is equivalent to the classical one. For one inequality of this equivalence, we just use that $\eta \equiv 1$ on $[-1/2, 1/2]$, while for the converse inequality it suffices to notice that $\|FG\|_{V_1} \lesssim \|F\|_{V_1}\|G\|_{V_1}$ together with the fact that the support of η is contained in three adjacent intervals of length 1. We will actually consider the wider class of $R_{2,\tau}$-multipliers defined below.

Definition 1.1. Let $\mathcal{R}$ to be the space of all functions of the form

$$m = \sum_I c_I \mathbf{1}_I$$

with I ranging over a family of disjoint arbitrary subintervals in $[1, 2)$ and the coefficients $\{c_I\}_I$ satisfying

$$\sum_I |c_I|^2 \le 1.$$

Then $\bar{\mathcal{R}}$ is the Banach space of functions $m := \sum_a \lambda_a m_a$ with $\sum_a |\lambda_a| < +\infty$, we equip $\bar{\mathcal{R}}$ with the norm

$$\|m\|_{\bar{\mathcal{R}}} := \inf \Big\{ \sum_a |\lambda_a| : m = \sum_a \lambda_a m_a, \quad m_a \in \mathcal{R} \Big\}.$$

For $\tau \in \mathbb{N}$, we say that the bounded function $m\colon \mathbb{R} \to \mathbb{C}$ is an $R_{2,\tau}$-multiplier if

$$\|m\|_{R_{2,\tau}} := \sup_{L \in \Lambda_\tau} \|m_L\|_{\bar{\mathcal{R}}} < +\infty.$$

Note that the class $R_{2,\tau}$ contains all Marcinkiewicz multipliers of order τ, as well as the Hörmander–Mihlin multipliers of order τ. This follows by the fact that Hörmander multipliers of order $\tau \ge 1$ are Marcinkiewicz multipliers of the same order and the latter belong to the class $\mathcal{V}_{1,\tau}$ consisting of functions which have uniformly bounded 1-variation on each lacunary interval of order τ; the inclusion relationship then follows, for example, by the fact that $\mathcal{V}_{1,\tau} \subset R_{2,\tau}$, proved in Lemma 2 of [6]. Our main result proves the sharp endpoint bound for multipliers in the class $R_{2,\tau}$.

Theorem B. *Let τ be a positive integer and $m \in R_{2,\tau}$. Then the operator $\mathrm{T}_m f := (m\hat{f})^\vee$ satisfies*

$$|\{x \in \mathbb{R} : |\mathrm{T}_m f(x)| > \alpha\}| \lesssim \int_{\mathbb{R}} \frac{|f|}{\alpha} \Big(\log\Big(e + \frac{|f|}{\alpha}\Big) \Big)^{\tau/2}, \quad \alpha > 0.$$

Furthermore, this estimate is best possible in the sense that the exponent $\tau/2$ in the right-hand side of the estimate cannot be replaced by any smaller exponent. The implicit constant depends only on τ and the $R_{2,\tau}$-norm of m.

For $\tau = 1$, the local version of the theorem above is contained in [17]. We note that Theorem B easily implies the following local estimate: For every interval I and $m \in R_{2,\tau}$ there holds

$$|\{x \in I : |\mathrm{T}_m f(x)| > \alpha\}| \lesssim \frac{1}{\alpha} \int_I |f|\Big(\log\Big(e + \frac{|f|}{\langle |f|\rangle_I}\Big)\Big)^{\tau/2}, \quad \alpha > 0, \ \mathrm{supp} f \subset I,$$

where $\langle |f|\rangle_I := |I|^{-1}\|f\|_{L^1(I)}$. The global estimate of Theorem B appears to be new even in the first order case $\tau = 1$, although a proof of a global result can be deduced for the first order case $\tau = 1$ from the methods in [17] without much additional work.

While Hörmander–Mihlin multipliers are $R_{2,\tau}$ multipliers, they are in general much better-behaved as the case $\tau = 1$ suggests: indeed, for $\tau = 1$ Hörmander–Mihlin multipliers map L^1 to $L^{1,\infty}$, in contrast to the sharpness of the $L\log^{1/2} L \to L^{1,\infty}$ estimate for general Marcinkiewicz or $R_{2,1}$ multipliers. In analogy to the Littlewood–Paley square function LP_τ of order τ, it is natural to define a smooth version as follows. For $C > 0$ $M \in \mathbb{N}$ and $L \in \Lambda_\tau$, we consider the class of bump functions

$$\Phi_{L,M} := \Big\{ \phi_L : \mathrm{supp}(\phi_L) \subseteq \frac{5}{4}L, \ \sup_{\alpha \le M} |L|^\alpha \|\partial^\alpha \phi_L\|_{L^\infty} \le 10^{10} \Big\}.$$

Now for some fixed large positive integer M (whose precise value is inconsequential) suppose that $\phi_L \in \Phi_{L,M}$ for all $L \in \Lambda_\tau$ and define, initially for f in the Schwartz class,

$$\mathrm{S}_\tau f := \Big(\sum_{L \in \Lambda_\tau} |\Delta_L f|^2 \Big)^{1/2}, \quad \Delta_L f(x) := \int_{\mathbb{R}} \phi_L(\xi) \widehat{f}(\xi) e^{2\pi i x\xi} \, \mathrm{d}\xi, \quad x \in \mathbb{R}.$$

The following theorem is the sharp endpoint estimate for higher order Hörmander–Mihlin multipliers and corresponding square functions.

Theorem C. *Let τ be a positive integer and let $m \in H_\tau$ be a Hörmander–Mihlin multiplier of order τ. Then*

$$|\{x \in \mathbb{R} : |\mathrm{T}_m f(x)| > \alpha\}| \lesssim \int_{\mathbb{R}} \frac{|f|}{\alpha}\Big(\log\Big(e + \frac{|f|}{\alpha}\Big)\Big)^{(\tau-1)/2}, \quad \alpha > 0.$$

The same holds for the smooth Littlewood–Paley square function S_τ of order τ, and these results are best possible. The implicit constant depends only on τ and the H_τ-norm of m, and also on M in the case of square functions.

The case $\tau = 1$ of this theorem is classical. The local version of the case of Hörmander–Mihlin multipliers of order $\tau = 2$ is implicit in [17] as it can be proved by combining Proposition 5.1 in [17] with Proposition 4.1 in [17]. All the higher order cases for the multipliers of the class H_τ appear to be new.

1.3. The Chang–Wilson–Wolff inequality and a square function for $L\log^{\tau/2} L$

Throughout this section we work on the probability space $([0, 1], \mathrm{d}x)$ unless otherwise stated. A central result in the approach in [17] was a weak characterization of the space

$L \log^{1/2} L$ in terms of an integrable square function, inspired by the analogous and better-known characterisation of the Hardy space H^1. More precisely, the authors in [17] prove that if $f \in L \log^{1/2} L$ and f has mean zero then for each $L \in \Lambda_1$ one can construct nonnegative functions F_L, such that

$$(1.1) \quad |\Delta_L f| \lesssim F_L * \varphi_{|L|^{-1}} \quad \text{for all } L \in \Lambda_1, \qquad \int_{\mathbb{R}} \Big(\sum_{L \in \Lambda_1} |F_L|^2 \Big)^{1/2} \lesssim \|f\|_{L \log^{1/2} L},$$

where Δ_L is as in Section 1.2 and

$$\varphi_\lambda(x) := \lambda^{-1} \varphi(x/\lambda) := \lambda^{-1} (1 + |x/\lambda|^2)^{-3/4}, \quad x \in \mathbb{R}.$$

Here and throughout the paper, we use local Orlicz norms and corresponding notation as described in Section 2.1.

There is a dyadic version: denoting by $\mathcal{D}_k$ the dyadic subintervals of $[0, 1]$ of length 2^{-k}, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we consider the conditional expectation and martingale differences

$$\mathbf{E}_k f := \sum_{I \in \mathcal{D}_k} \langle f \rangle_I \mathbf{1}_I, \quad \mathbf{D}_k f := \mathbf{E}_k f - \mathbf{E}_{k-1} f, \quad k \geq 1, \quad \mathbf{D}_0 f := \mathbf{E}_0 f, \quad f \in L^1.$$

For future reference, we record the definition of the dyadic martingale square function

$$\mathrm{S}_{\mathcal{M}} f := \Big(\sum_{k \geq 1} |\mathbf{D}_k f|^2 \Big)^{1/2}.$$

The dyadic analogue of (1.1) is that if $f \in L \log^{1/2} L$, then for each $k \in \mathbb{N}_0$ there exist functions f_k, such that

$$(1.2) \quad |\mathbf{D}_k f| \leq \mathbf{E}_k |f_k| \quad \text{for all } k \in \mathbb{N}_0, \qquad \int_{[0,1]} \Big(\sum_{k \geq 0} |f_k|^2 \Big)^{1/2} \lesssim \|f\|_{L \log^{1/2} L}.$$

In fact, the authors in [17] first prove (1.2) by constructing the functions f_k through a rather technical induction scheme, and then deduce (1.1) from (1.2) via a suitable averaging argument.

Several remarks are in order. Firstly, one notices that (1.2) combined with a simple duality argument based on the fact that $\exp(L^2) = (L \log^{1/2} L)^*$ implies the Chang–Wilson–Wolff inequality

$$(1.3) \qquad \|f - \mathbf{E}_0 f\|_{\exp(L^2)} \lesssim \|\mathrm{S}_{\mathcal{M}} f\|_{L^\infty}.$$

Estimate (1.3) was first proved in [5]; see also the monograph [20] for an in-depth discussion of exponential square integrability in relation to discrete and continuous square functions in analysis. Thus, the proof of (1.2) in [17] is of necessity somewhat hard as it reproves (1.3).

A second observation that goes back to [17], see also [14] for an analogous remark on the dual side, is that (1.1) implies the weaker estimate

$$(1.4) \qquad \Big(\sum_{L \in \Lambda_1^1} \|\Delta_L f\|_{L^1}^2 \Big)^{1/2} \lesssim \|f\|_{L \log^{1/2} L}.$$

Indeed, (1.4) follows by (1.1) and the Minkowski integral inequality. Alternatively, as observed in [14], the dual of (1.4) is a –again weaker– consequence of the Chang–Wilson–Wolff inequality (1.3).

Finally, a consequence of (1.4) is the Zygmund inequality

$$\Big(\sum_{\lambda \in \mathrm{lac}_1^1} |\widehat{f}(\lambda)|^2 \Big)^{1/2} \lesssim \|f\|_{L \log^{1/2} L}.$$

See, for example, Theorem 7.6 in Chapter XII of [22]. Indeed, if L_λ is an interval which has λ as an endpoint, we have $|\widehat{f}(\lambda)| \leq \|(\Delta_{L_\lambda} f)^\wedge\|_{L^\infty} \leq \|\Delta_{L_\lambda} f\|_{L^1}$ for a suitable choice of symbol in the definition of the Littlewood–Paley projection and Zygmund's inequality follows by (1.4).

All the estimates above have a higher order counterpart which plays an important role in our investigations in this paper. However, our point of view is somewhat different than in [17]. Firstly, we want to emphasize that the proof of our main theorem, Theorem B, hinges on a higher order version of the generalized Zygmund inequality (1.4) which loosely has the form

$$(1.5) \qquad \Big(\sum_{L \in \Lambda_\tau^1} \|\Delta_L f\|_{L^1}^2 \Big)^{1/2} \lesssim \|f\|_{L \log^{\tau/2} L}, \quad \tau \in \mathbb{N}.$$

Estimates of the form (1.5) will be referred to as *generalized Zygmund–Bonami inequalities* and will be stated precisely and proved in Section 4. The terminology comes from the fact that they imply the higher order version of Zygmund's inequality, due to Bonami [3], and which can be stated as follows:

$$(1.6) \qquad \Big(\sum_{\lambda \in \mathrm{lac}_\tau^1} |\widehat{f}(\lambda)|^2 \Big)^{1/2} \lesssim \|f\|_{L \log^{\tau/2} L}, \quad \tau \in \mathbb{N}.$$

A novelty in our approach is the realization that the weak square function characterization (1.2) of the space $L \log^{1/2} L$, in the dyadic case, is precisely the dual estimate of the Chang–Wilson–Wolff inequality (1.3). This relies on a duality argument involving quotient spaces which is inspired by the work of Bourgain, [4]. We can then use the Chang–Wilson–Wolff inequality for general order of integrability, see Section 3.1,

$$(1.7) \qquad \|f - \mathbf{E}_0 f\|_{\exp(L^{2/(\sigma+1)})} \lesssim \|S_{\mathcal{M}} f\|_{\exp(L^{2/\sigma})}, \quad \sigma \geq 0,$$

to conclude the weak square function characterization of the space $L \log^{(\sigma+1)/2} L$ given in the following theorem.

Theorem D. *If* $f \in L \log^{(\sigma+1)/2} L$ *for some* $\sigma \geq 0$, *then for each* $k \in \mathbb{N}_0$, *there exist functions* f_k *such that*

$$\mathbf{D}_k f = \mathbf{D}_k f_k \quad \text{for all } k \in \mathbb{N}_0, \qquad \Big\| \Big(\sum_{k \geq 0} |f_k|^2 \Big)^{1/2} \Big\|_{L \log^{\sigma/2} L} \lesssim \|f\|_{L \log^{(\sigma+1)/2} L}.$$

The implicit constant depends only on σ.

We will prove Theorem D in Section 3 as a consequence of (1.7). While this is a rather deep implication, as in the case $\sigma = 0$, it is not hard to see that the conclusion of Theorem D combined with the fact $\exp(L^{2/\sigma}) = (L \log^{\sigma/2} L)^*$ actually implies the Chang–Wilson–Wolff inequality (1.7) for the same value of σ. We note that while the conclusion of Theorem D and of the subsequent corollary below are already in [17] for the case $\sigma = 0$; our approach provides an alternative proof even for $L \log^{1/2} L$. This approach has the advantage of being able to deal with all spaces $L \log^{(\sigma+1)/2} L$ at once, hence leading to the more general conclusion of Theorem D.

As in the case $\sigma = 0$, Theorem D readily implies the continuous version below.

Corollary E. *Let $J \subset \mathbb{R}$ be a finite interval, $\sigma \geq 0$ and $f \in L \log^{(\sigma+1)/2} L(J)$. Then, for each $L \in \Lambda_1$ with $|L| \geq |J|^{-1}$, there exists a nonnegative function F_L such that for every $\gamma \geq 1$,*

$$|\Delta_L f| \lesssim F_L * \varphi_{(|L||J|)^{-1}},$$

$$\left\| \left(\sum_{L \in \Lambda_1^{|J|^{-1}}} |F_L|^2 \right)^{1/2} \right\|_{L \log^{\sigma/2} L(\gamma J, \, dx/|J|)} \lesssim \|f\|_{L \log^{(\sigma+1)/2} L(J, \, dx/|J|)},$$

with implicit constant depending only on γ and σ. If, in addition, $\int_J f = 0$, then the conclusion holds for all $L \in \Lambda_1$ with the summation extending over all $L \in \Lambda_1$. With or without this additional assumption, for $|L| \geq |J|^{-1}$, the functions F_L are supported in $5J$. The implicit constant depends only on γ and σ, as indicated.

1.4. Background and history

The fact that Marcinkiewicz multipliers are L^p-bounded is classical; see, e.g., Theorem 8.13 in [8]. The first endpoint result concerning multiplier operators of Marcinkiewicz-type is arguably a theorem due to Bourgain [4], which asserts that, in the periodic setting, the classical Littlewood–Paley square function LP_1 has operator norm $\|\text{LP}_1\|_{p \to p} \simeq (p-1)^{-3/2}$ as $p \to 1^+$. Tao and Wright proved in [17] the optimal local endpoint estimate $L \log^{1/2} L \to L^{1,\infty}$ for the class of $R_2 = R_{2,1}$ multipliers, which contains Marcinkiewicz multipliers. It was later observed in [1] that Bourgain's estimate follows by the endpoint bound of [17] combined with a randomization argument and Tao's converse extrapolation theorem from [16]. Recently, Lerner proved in [10] effective weighted bounds for the classical Littlewood–Paley square function LP_1; these weighted bounds imply the correct p-growth for the $L^p \to L^p$ norms of these operators as $p \to 1^+$. In addition, as observed in [2], the arguments of [10] can be used to establish weighted A_2 estimates for LP_τ that imply sharp $L^p \to L^p$ estimates for LP_τ as $p \to 1^+$ for any order τ. The class R_2 contains all multipliers m whose pieces m_L have bounded q-variation uniformly in $L \in \Lambda_1$, for all $1 \leq q < 2$; see [6], where the authors showed that all R_2 multipliers are bounded on L^p for $p \in (1, \infty)$.

As already discussed, the authors in [17] rely on the weak square function characterization of $L \log^{1/2} L$ as in (1.1) for their proof. Our argument here is a bit different, relying on the weaker generalized Zygmund–Bonami inequality instead; a hint of a different proof already appears in p. 540 of [17]. The Zygmund inequality first appeared in [21] in its dual form; see also Theorem 7.6 in Chapter XII of [22]. The higher lacunarity order (1.6) is

due to Bonami and it is contained in [3]. We note that our results provide an alternative proof for the case of finite order lacunary sets. On the other hand, a dual version of the generalized Zygmund–Bonami inequality in the first order case (that is, inequality (1.4)) appears in [14].

The L^p-boundedness of Marcinkiewicz multipliers of order one and higher in the periodic setting was established by Marcinkiewicz in [11]; see also Gaudry's paper [9]. Generalized versions of Hörmander–Mihlin and Marcinkiewicz multipliers, together with their square function counterparts of higher order, have been introduced in [15] in a very broad context. There the authors proved the equivalence of L^p-boundedness between different classes of such multipliers. Our setup is focused on the finite order lacunary case and provides the optimal endpoint bounds for such classes.

1.5. Structure

The general structure of the rest of this paper is as follows. Section 2 contains some basic facts and properties of Orlicz spaces, together with a small toolbox for dealing with lacunary sets; the reader is encouraged to skip this section on a first reading and only consult it when necessary. In Section 3 we will prove Theorem D and Corollary E. In Section 4 we will critically use Corollary E in order to conclude the generalized Zygmund–Bonami inequality of arbitrary order alluded to above. This inequality will be stated and proved in different versions which can be local or non-local, depending on the type of cancellation assumptions we impose. The reader can find the corresponding statements in Propositions 4.1 and 4.4, see also Corollary 4.6. In Section 5 we present the details of a Calderón–Zygmund decomposition for the Orlicz space $L \log^{\sigma/2} L$, adapted to the needs of this paper. The proof of Theorem B takes up the best part of Section 6, where the Calderón–Zygmund decomposition of Section 5 is combined with the generalized Zygmund–Bonami inequality of Section 4. The proofs of Theorem A and Theorem C are discussed in Section 6.2 as a variation of the proof of Theorem B.

2. Preliminaries and notation

In this section, we collect several background definitions and notations that will be used throughout the paper.

2.1. Some basic facts for certain classes of Orlicz spaces

We adopt standard nomenclature for Young functions and Orlicz spaces as, for example, in Chapter 10 of [20]. Given a Young function $\Phi \colon [0, \infty] \to [0, \infty]$, we will use the following notation for local L^{Φ} averages: for a finite interval $I \subset \mathbb{R}$,

$$\langle |f| \rangle_{\Phi, I} := \inf \Big\{ \lambda > 0 : \frac{1}{|I|} \int_I \Phi \Big(\frac{|f(x)|}{\lambda} \Big) \, \mathrm{d}x \leq 1 \Big\}.$$

For the usual local L^p averages, we just set

$$\langle |f| \rangle_{p, I} := |I|^{-1/p} \| f \|_{L^p(I)} \quad \text{for } 1 \leq p < \infty.$$

For $\sigma \geq 0$, we use the Young function

$$B_\sigma(t) := t \left(\log(e + t)\right)^\sigma$$

to define local $L \log^\sigma L$-spaces and we will also write,

$$\|f\|_{L \log^\sigma L(I, \mathrm{d}x/|I|)} := \langle|f|\rangle_{B_\sigma, I} \simeq \frac{1}{|I|} \int_I |f(x)| \left(\log\left(e + \frac{|f(x)|}{\langle|f|\rangle_{1,I}}\right)\right)^\sigma \mathrm{d}x.$$

The last approximate equality can be found in Theorem 10.8 of [20]. For future reference it is worth noting that the function B_σ is submultiplicative and thus doubling; see Section 5.2 of [7]. We will write instead $L^{B_\sigma}(\mathbb{R})$ to denote the (global) space of measurable functions f such that $\int_{\mathbb{R}} B_\sigma(|f|) < +\infty$.

Note that the dual Young function of B_σ can be taken to coincide with $E_{\sigma^{-1}}(t) := \exp(c_\sigma t^{1/\sigma}) - 1$ for $t \gtrsim 1$; here we insist on the equality only for sufficiently large values of t. With this function, we define the local $\exp(L^{1/\sigma})$ norms and the Hölder inequality $\langle|fg|\rangle_{1,I} \lesssim \langle|f|\rangle_{B_\sigma, I} \langle|f|\rangle_{E_{\sigma^{-1}}, I}$ holds. We reserve the notation $L \log^\sigma L$ and $\exp(L^{1/\sigma})$ for the case $I = [0, 1]$ and the space of functions supported in $[0, 1]$, for which

$$\|f\|_{L \log^\sigma L} := \langle|f|\rangle_{B_\sigma, [0,1]} < +\infty,$$
$$\|f\|_{\exp(L^{1/\sigma})} := \langle|f|\rangle_{E_{\sigma^{-1}}, [0,1]} \simeq \sup_{p \geq 2} p^{-\sigma} \|f\|_p < +\infty,$$

respectively (see Section 2.2.4 of [18] for the last approximate equality); there holds $\exp(L^{1/\sigma}) \cong (L \log^\sigma L)^*$. For $\sigma = 0$, we adopt the convention that $L \log^\sigma L = L^1$ and $\exp(L^{1/\sigma}) = L^\infty$. The Minkowski-type integral inequality

$$\left\|\{\|f_k\|_{L \log^\sigma L}\}\right\|_{\ell_k^2} \lesssim \left\|\|\{f_k\}\|_{\ell_k^2}\right\|_{L \log^\sigma L}$$

will be used with no particular mention. Its proof can be obtained by a simple duality argument.

2.2. Some tools for handling lacunary sets

We introduce some useful notions concerning lacunary sets of arbitrary order. Let $\tau \geq 1$ and $L \in \Lambda_\tau$. We will denote by $\widehat{L}$ the unique interval $\widehat{L} \in \Lambda_{\tau-1}$ such that $L \subset \widehat{L}$ and call $\widehat{L}$ the (lacunary) *parent* of L. Furthermore, we will denote by $\lambda(L)$ the unique element $\lambda \in \mathrm{lac}_{\tau-1}$ such that $\mathrm{dist}(L, \mathbb{R} \setminus \widehat{L}) = \mathrm{dist}(L, \lambda) = |L|$. We note that $\lambda(L)$ is one of the endpoints of $\widehat{L}$. These definitions also make sense in the case $\tau = 1$, keeping in mind the definitions of Λ_0 and lac_0.

If $L \in \Lambda_\tau$, then $L^* := L - \lambda(L) \in \Lambda_1$; in fact, L^* is one of the intervals $(-2|L|, -|L|)$ or $(|L|, 2|L|)$ depending on the original relative position of L with respect to $\lambda(L)$. The point of the definitions above is that if $L \in \Lambda_\tau$, then upon fixing a suitable choice of bump functions $\phi_L \in \Phi_{L,M}$, we can write the identity

$$\Delta_L f = e^{2\pi i \lambda(L) \cdot} \Delta_{L^*}(e^{-2\pi i \lambda(L) \cdot} \Delta_{\widehat{L}} f) = e^{2\pi i \lambda(L) \cdot} \Delta_{L^*}(e^{-2\pi i \lambda(L) \cdot} f).$$

This will be crucially used in several parts of the recursive arguments in the paper. We will also use the intuitive notation $\Delta_{|L|} := \Delta_{L^* \cup (-L^*)}$ for the smooth Littlewood–Paley

projection of first order at frequencies $|\xi| \simeq |L|$, which takes advantage of the fact that L^* essentially only depends on the length of L. The following notation will be useful to localize in a certain lacunary parent:

$$\Lambda_\tau^n(L') := \{L \in \Lambda_\tau^n : L \subset L'\}, \quad L' \subset \mathbb{R},$$

similarly we define $\Lambda_\tau(L')$. Note that if $L' \in \Lambda_{\tau-1}$ and $L \in \Lambda_\tau(L')$, then necessarily $\hat{L} = L'$.

The following simple lemma relies on the fact that lacunary sets are invariant under dyadic dilations with respect to the origin and will be used to allow rescaling of intervals of dyadic length to $[0, 1]$.

Lemma 2.1. *Let $\tau \in \mathbb{N}$ and $a \in 2^{\mathbb{Z}}$. Then $a^{-1}\mathrm{lac}_\tau^a := \{\lambda/a : \lambda \in \mathrm{lac}_\tau^a\} = \mathrm{lac}_\tau^1$.*

To showcase the typical application of this lemma, let $J \subset \mathbb{R}$ be an interval of dyadic length and $a = \{a_\lambda\}_{\lambda \in \mathrm{lac}_\tau}$ a finite collection of complex coefficients. By a standard change of variables,

$$(2.1) \quad p_a(y) := \sum_{\lambda \in \mathrm{lac}_\tau^{|J|^{-1}}} a_\lambda e^{i\lambda y}, \quad \langle|p_a|\rangle_{p,J}^p = \int_{[0,1]} \left| \sum_{\lambda \in |J|\mathrm{lac}_\tau^{|J|^{-1}}} a_{\lambda|J|^{-1}} e^{i\lambda y} \right|^p \mathrm{d}y,$$

and we crucially note that the sum on the right-hand side is for $\lambda \in |J|\mathrm{lac}_\tau^{|J|^{-1}} = \mathrm{lac}_\tau^1$ because of the lemma. Of course, the same change of variables will be valid for $\langle|p_a|\rangle_{\Phi,J}$ for any Young function Φ. We will use this rescaling argument in several places in the paper.

2.3. Other notation

For any function g and $\lambda > 0$, we write $g_\lambda(x) := \lambda^{-1}g(x/\lambda)$ for the L^1-rescaling. Two special kinds of bump functions will appear. Firstly, $\omega(x) := (1 + |x|^2)^{-N/2}$ is the smooth tailed indicator of $[-1/2, 1/2]$ with N any large positive integer. It will be enough to take $N = 10$ for the arguments in this paper, but more decay is available if needed. We will also write $\varphi(x) := (1 + |x|^2)^{-3/4}$, which is still an L^1-bump but has only moderate decay. In some cases we are restricted to using φ, most notably in the statement and proof of Corollary E.

3. A weak square function characterization of $L \log^{\sigma/2} L$

In this section, we provide the proof of Theorem D as a consequence of the Chang–Wilson–Wolff inequality of general order (1.7). The conclusion of Corollary E will then follow by a standard averaging argument using almost orthogonality between the continuous Littlewood–Paley projections and martingale differences.

3.1. Proof of Theorem D

We recall that we work on the probability space $([0, 1], \mathrm{d}x)$. It clearly suffices to prove the theorem for $k \geq 1$, as for $k = 0$, we can set $f_0 := \mathbf{E}_0 f = \mathbf{D}_0 f$. Our starting point is

the Chang–Wilson–Wolff inequality of general order of integrability, (1.7). This is pretty standard, but a quick proof can be produced by using the usual Chang–Wilson–Wolff inequality (1.3) in the form

$$p^{-\sigma/2} \| f - \mathbf{E}_0 f \|_p \lesssim p^{-\sigma/2} p^{1/2} \| S_{\mathcal{M}} f \|_p, \quad p \geq 2, \ \sigma \geq 0,$$

which readily implies

$$\| f - \mathbf{E}_0 f \|_{\exp(L^{2/(\sigma+1)})} \simeq \sup_{p \geq 2} \frac{\| f - \mathbf{E}_0 f \|_p}{p^{(\sigma+1)/2}} \lesssim \sup_{p \geq 2} \frac{\| S_{\mathcal{M}} f \|_p}{p^{\sigma/2}} \simeq \| S_{\mathcal{M}} f \|_{\exp(L^{2/\sigma})},$$

which is (1.7). Observe that (1.7) has the form

$$(3.1) \qquad \Big\| \sum_{k \geq 1} g_k \Big\|_{\exp(L^{2/(\sigma+1)})} \lesssim \Big\| \Big(\sum_{k \geq 1} |g_k|^2 \Big)^{1/2} \Big\|_{\exp(L^{2/\sigma})}, \quad g_k = \mathbf{D}_k f.$$

We will write (3.1) as a continuity property for the operator $T(\{g_k\}_k) := \sum_k g_k$ between suitable Banach spaces. To that end, let us consider the subspace of $L \log^{\sigma/2} L([0,1]; \ell^2)$ given by

$$Y := \{ \{\psi_k\}_k \in L \log^{\sigma/2} L([0,1]; \ell^2) : \mathbf{D}_k \psi_k = 0 \text{ for all } k \in \mathbb{N} \}.$$

We observe that Y is closed. To see this consider a sequence $(\psi^n)_n \subset Y$ with $\psi^n = \{\psi_k^n\}_k$ converging to some $\psi = \{\psi_k\}_k$ in $L \log L^{\sigma/2} L([0,1]; \ell^2)$. Clearly the limit ψ belongs to the space $L \log L^{\sigma/2} L([0,1]; \ell^2)$, the latter being a Banach space and, additionally, ψ_k^n converges to ψ_k in $L \log^{\sigma/2} L([0,1])$, and so also in $L^1([0,1])$, uniformly in k. Now it follows by Fatou's lemma that for each $k \in \mathbb{N}$, there holds

$$\| \liminf_{n \to \infty} |\mathbf{D}_k(\psi_k^n - \psi_k)| \|_{L^1([0,1])} \leq \liminf_{n \to \infty} \| \psi_k - \psi_k^n \|_{L^1([0,1])} = 0,$$

yielding $\mathbf{D}_k \psi_k^n = \mathbf{D}_k \psi_k = 0$ a.e., where we also used the uniform boundedness of $\mathbf{D}_k$ on $L^1([0,1])$.

Since $(L \log^{\sigma/2} L([0,1]; \ell^2))^* \cong \exp(L^{2/\sigma})([0,1]; \ell^2)$, the annihilator of Y is given equivalently by

$$Y^\perp = \Big\{ \{g_k\}_k \in \exp(L^{2/\sigma})([0,1]; \ell^2) : \int \Big(\sum_k g_k \psi_k \Big) = 0 \text{ for all } \{\psi_k\}_k \in Y \Big\}.$$

Since Y is a closed subspace of $L \log^{\sigma/2} L([0,1]; \ell^2)$ we have (see Theorem 4.9 in [13]) that $(L \log^{\sigma/2} L([0,1]; \ell^2)/Y)^*$ is isometrically isomorphic to $Y^\perp$. We equip $Y^\perp$ with the norm appearing on the right-hand side of (3.1). We will use the following fact.

Lemma 3.1. *If $\{g_k\}_k \in Y^\perp$, then $\mathbf{D}_k g_k = g_k$ for every $k \in \mathbb{N}$.*

Proof. Fix an index $k_0 \in \mathbb{N}$, let $\psi \in L \log^{\sigma/2} L$ be arbitrary and let $\{\psi_k\}_k$ be defined by

$$\psi_k := \begin{cases} 0 & \text{if } k \neq k_0, \\ \psi - \mathbf{D}_{k_0} \psi & \text{otherwise.} \end{cases}$$

Clearly, $\{\psi_k\}_k \in L \log^{\sigma/2} L([0,1]; \ell^2)$; moreover, $\mathbf{D}_{k_0}(\psi - \mathbf{D}_{k_0}\psi) = 0$, so that $\{\psi_k\}_k \in Y$. By the definition of $Y^\perp$, we have

$$0 = \int \sum_k g_k \psi_k = \int g_{k_0}(\psi - \mathbf{D}_{k_0}\psi) = \int (g_{k_0} - \mathbf{D}_{k_0} g_{k_0})\psi,$$

where we have used the fact that $\mathbf{D}_k$ is self-adjoint; but this is only possible for arbitrary ψ if $\mathbf{D}_{k_0} g_{k_0} = g_{k_0}$, as claimed. ∎

Now (3.1) can be written in the form

$$(3.2) \qquad \|\mathrm{T}(\{g_k\}_k)\|_{\exp(L^{2/(\sigma+1)})} \lesssim \|\{g_k\}_k\|_{Y^\perp}, \quad \mathrm{T}(\{g_k\}_k) := \sum_k g_k.$$

Let $X := L \log^{\sigma/2} L([0,1]; \ell^2)$ and denote by X_N and Y_N the functions in X and Y, respectively, which are constant on dyadic intervals of length smaller than 2^{-N}. In particular, such functions f have finite Haar expansion, which implies the a priori qualitative property that the spaces X_N and Y_N are finite dimensional. We note that $(X_N/Y_N)^*$ is isometrically isomorphic to $Y_N^\perp$. By the Riesz representation theorem, we then get that

$$\|\{\mathbf{D}_k f\}_k\|_{X_N/Y_N} \leq \sup_{\substack{\{g_k\}_k \in Y_N^\perp: \\ \|\{g_k\}_k\|_{Y_N^\perp} \leq 1}} \left|\int \sum_k g_k \mathbf{D}_k f\right| = \sup_{\substack{\{g_k\}_k \in Y_N^\perp: \\ \|\{g_k\}_k\|_{Y_N^\perp} \leq 1}} \left|\int \mathrm{T}(\{g_k\}_k) f\right|,$$

where we also used Lemma 3.1 in passing to the equality in the right-hand side above. Using (3.2) together with Hölder's inequality in Orlicz spaces, it follows that

$$\|\{\mathbf{D}_k f_N\}_k\|_{X/Y} = \inf_{\{\psi_k\}_k \in Y} \left\|\left(\sum_k |\mathbf{D}_k f_N + \psi_k|^2\right)^{1/2}\right\|_{L \log^{\sigma/2} L} \lesssim \|f_N\|_{L \log^{(\sigma+1)/2} L},$$

where f_N is the truncation of the Haar series of $f \in X$ at scale 2^{-N}. We stress that the approximate inequality above holds uniformly for all $N \in \mathbb{N}$. This inequality extends to all $f \in L \log^{(\sigma+1)/2} L$ by a standard approximation argument, using the fact that the truncated Haar series of functions $f \in L \log^{(\sigma+1)/2} L$ converge to f in $L \log^{(\sigma+1)/2} L$, see [12]. The extension of the operator $f \mapsto \{\mathbf{D}_k f\}_k$ is the obvious one given by the same expression.

In order to conclude the proof of the theorem, we notice that for every $\{\psi_k\}_k \in Y$, there holds $\mathbf{D}_k f = \mathbf{D}_k(\mathbf{D}_k f + \psi_k)$, and the last inequality guarantees the existence of a vector $\{\psi_k\}_k \in Y$ such that the functions $f_k := \mathbf{D}_k f + \psi_k$ satisfy the conclusion of the theorem.

3.2. Proof of Corollary E

The corollary follows from the dyadic case of Theorem D via an averaging argument which is essentially identical to the one in Section 9 of [17]; see also [19], where the argument of Section 9 in [17] is explained in detail. We can clearly assume that $\gamma \geq 3$ and, by affine invariance, we can take $\gamma J = [0,1]$, so that $|J| = \gamma^{-1}$, and supp $f \subset [1/3, 2/3]$.

For all $\theta \in [-1/3, 1/3]$, we define $f_\theta(x) := f(x - \theta)$. By Theorem D, for each $\theta \in [-1/3, 1/3]$ and $k \geq 0$, there exists a function $f_{\theta,k}$ such that $\mathbf{D}_k f_\theta = \mathbf{D}_k f_{\theta,k}$ and

$$\left\| \left(\sum_{k \geq 0} |f_{\theta,k}|^2 \right)^{1/2} \right\|_{L \log^{\sigma/2} L} \lesssim \|f_\theta\|_{L \log^{(\sigma+1)/2} L} = \|f\|_{L \log^{(\sigma+1)/2} L}.$$

Setting, for $L \in \Lambda_1^\gamma$,

$$F_L(x) := \sum_{k \in \mathbb{N}_0} 2^{-|\log_2 |L| - k|/2} \int_{[-1/3, 1/3]} |f_{\theta,k}(x + \theta)| \, d\theta$$

and arguing as in [17], we see that $|\Delta_L f(x)| \lesssim F_L * \varphi_{|L|^{-1}}$ and

$$\left(\sum_{L \in \Lambda_1^\gamma} |F_L(x)|^2 \right)^{1/2} \lesssim \int_{[-1/3, 1/3]} \left(\sum_{k \geq 0} |f_{\theta,k}(x + \theta)|^2 \right)^{1/2} d\theta.$$

By the Minkowski integral inequality for the space $L \log^{\sigma/2} L$, we have

$$\left\| \left(\sum_{L \in \Lambda_1^\gamma} |F_L|^2 \right)^{1/2} \right\|_{L \log^{\sigma/2} L} \lesssim \int_{[-1/3, 1/3]} \left\| \left(\sum_{k \geq 0} |f_{\theta,k}(\cdot + \theta)|^2 \right)^{1/2} \right\|_{L \log^{\sigma/2} L} d\theta,$$

and the proof follows for $L \in \Lambda_1^\gamma$. Under the additional cancellation assumption $\int_{[0,1]} f = 0$, we consider also $L \in \Lambda_1$ with $|L| < \gamma$, and for these we define $F_L := |\Delta_L f|$ and note that $|\Delta_L f| \lesssim \varphi_{|L|^{-1}} * F_L$. Using the cancellation condition, we have also

$$|\Delta_L f| \lesssim \varphi_{|L|^{-1}} * (|L| \|f\|_{L^1} \mathbf{1}_{[0,1]}),$$

which readily yields the estimate $\| \|\{F_L\}_{|L| < \gamma} \|_{\ell^2} \|_{L^1} \lesssim \|f\|_{L^1}$ and the proof is complete. Note that we used that since $\tau = 1$, there are at most two intervals $L \in \Lambda_1$ of any given length.

4. Generalized Zygmund–Bonami inequalities

In this section, we prove the versions of the generalized Zygmund–Bonami inequality presented in the introduction, where the Littlewood–Paley projections Δ_L for $L \in \Lambda_1$ are replaced by their τ-order counterparts Δ_L for $L \in \Lambda_\tau$, where τ is an arbitrary positive integer. As already discussed, the estimate corresponding to order $\tau = 1$ is (1.4) and it follows rather easily from the case $\sigma = 0$ of Corollary E. For $\tau > 1$, we first state the generalized Zygmund–Bonami inequalities in the case of $L \in \Lambda_\tau$ with $|L| \geq |J|^{-1}$, where J is an interval in which f is supported; this is the harder and deeper case. In the rest of the section we will also provide the statements and proofs for the easier case $|L| < |J|^{-1}$; the latter will rely on pointwise estimates for $\Delta_L f$ and recursive arguments, assuming suitable cancellation conditions for f in the same spirit as Corollary E.

4.1. The main term in the generalized Zygmund–Bonami inequalities

We encourage the reader to keep in mind the notation of Section 2 for the local Orlicz norms and the definitions concerning lacunary sets from Section 2.2 for the rest of this section. Our first result below gives a version of the generalized Zygmund–Bonami inequality, in which the intervals L are restricted to those for which $|L| \geq |J|^{-1}$, as anticipated above.

Proposition 4.1. *Let $J \subset \mathbb{R}$ be a finite interval and let f be a compactly supported function with* $\mathrm{supp}(f) \subseteq J$. *Let τ be a positive integer, $\sigma \geq 0$ and $\gamma > 1$. There holds*

$$\Big(\sum_{L \in \Lambda_\tau^{|J|^{-1}}} \langle |\Delta_L f| \rangle^2_{B_{\sigma/2}, \gamma J} \Big)^{1/2} \lesssim_{\sigma, \tau, \gamma} \langle |f| \rangle_{B_{(\sigma+\tau)/2}, J}$$

and

$$\sum_{L \in \Lambda_\tau^{|J|^{-1}}} \|\Delta_L f\|^2_{L^2(\mathbb{R} \setminus \gamma J)} \lesssim_{\tau, \gamma} |J| \langle |f| \rangle^2_{B_{(\tau-1)/2}, J}.$$

Proof. The proof is by way of induction on τ, with the base case $\tau = 1$ being an easy consequence of Corollary E, as we shall now illustrate. Indeed, let $C_1(\sigma, \tau, \gamma)$ and $C_2(\tau, \gamma)$ denote the best constants in the first and in the second estimates in the statement, respectively. Corollary E implies that for $L \in \Lambda_1^{|J|^{-1}}$, we have

$$\langle |\Delta_L f| \rangle_{B_{\sigma/2}, \gamma J} \lesssim \langle |\varphi_{(|L||J|)^{-1}} * F_L| \rangle_{B_{\sigma/2}, \gamma J} \lesssim \langle |F_L| \rangle_{B_{\sigma/2}, \gamma J},$$

using Young's convolution inequality and the L^1-normalization of each $\varphi_{(|L||J|)^{-1}}$. Now the proof of the first estimate in the conclusion for $\tau = 1$ can be concluded by yet another application of Minkowski's inequality, this time to yield that the left-hand side of the first estimate in the conclusion is bounded by a constant multiple of

$$\Big(\sum_{|L| \geq |J|^{-1}} \langle |F_L| \rangle^2_{B_{\sigma/2}, \gamma J} \Big)^{1/2} \lesssim \Big\langle \Big(\sum_{|L| \geq |J|^{-1}} |F_L|^2 \Big)^{1/2} \Big\rangle_{B_{\sigma/2}, \gamma J} \lesssim \langle |f| \rangle_{B_{(\sigma+1)/2}, J},$$

by the estimate for the square function of the $\{F_L\}_L$ in Corollary E. It follows that $C_1(\sigma, 1, \gamma) < +\infty$ for all nonnegative integers σ and $\gamma > 1$.

For the second estimate, we have, for $|L| \geq |J|^{-1}$ and $x \in \mathbb{R} \setminus \gamma J$,

$$(4.1) \quad |\Delta_L f(x)| \lesssim |L||J|(1 + |L||x - c_J|)^{-10} \langle |f| \rangle_{1, J} \lesssim \omega_{|L|^{-1}} * (\langle |f| \rangle_{1, J} \mathbf{1}_J)(x),$$

with $\omega_{|L|^{-1}}$ as given in Section 2. Using the first approximate inequality above, the square of the left-hand side of the second estimate in the conclusion of the proposition can be estimated by a constant multiple of

$$\sum_{L \in \Lambda_1^{|J|^{-1}}} \langle |f| \rangle^2_{1, J} (|L||J|)^2 |L|^{-20} \int_{\mathbb{R} \setminus \gamma J} |x - c_J|^{-20} \, \mathrm{d}x$$

$$\lesssim \langle |f| \rangle^2_{1, J} \sum_{L \in \Lambda_1 : |L||J| \geq 1} (|L||J|)^{-18} |J|,$$

which sums to the desired quantity since, for $\tau = 1$, there is exactly one interval $L \in \Lambda_1$ per dyadic scale. This shows that $C_2(1, \gamma) < +\infty$ for all $\gamma > 1$.

Consider now the case $\tau > 1$ and let $a > 1$ be such that $\gamma = a^\tau$. Recalling the discussion in Section 2.2, we write

$$(4.2) \quad |\Delta_L f| \leq |\Delta_{L^*}(\mathbf{1}_{a^{\tau-1}J} e^{-2\pi i \lambda(L) \cdot} \Delta_{\widehat{L}} f)| + |\Delta_{L^*}(\mathbf{1}_{\mathbb{R} \setminus a^{\tau-1}J} e^{-2\pi i \lambda(L) \cdot} \Delta_{\widehat{L}} f)|$$
$$=: |\Delta_{L^*} f_{1,L}| + |\Delta_{L^*} f_{2,L}|.$$

For clarity, we remind the reader that $\lambda(L)$ is either of the endpoints of $\widehat{L}$ (depending on the position of L), and therefore we can partition the intervals L into two families such that $f_{1,L}, f_{2,L}$ *actually depend only on* $\widehat{L}$, this will be relevant below. Fixing for a moment $L' \in \Lambda_{\tau-1}^{|J|^{-1}}$, we note that for any $L \in \Lambda_\tau^{|J|^{-1}}(L')$, there holds $\widehat{L} = L'$, and so

$$|f_{1,L}| = |\Delta_{\widehat{L}} f| \mathbf{1}_{a^{\tau-1}J} = |(\Delta_{L'} f) \mathbf{1}_{a^{\tau-1}J}|.$$

As the collection

$$\{L^* : L \in \Lambda_{\tau-1}^{|J|^{-1}}(L')\} \subset \Lambda_1^{|J|^{-1}},$$

we can use the conclusion of proposition for $\tau = 1$ to estimate, for fixed $L' \in \Lambda_{\tau-1}^{|J|^{-1}}$,

$$\Big(\sum_{L \in \Lambda_\tau^{|J|^{-1}}(L')} \langle |\Delta_{L^*} f_{1,L}| \rangle_{B_{\sigma/2}, a^\tau J}^2 \Big)^{1/2} \leq C_1(\sigma, 1, a) \langle |\Delta_{L'} f| \rangle_{B_{(\sigma+1)/2}, a^{\tau-1}J}.$$

Thus, we can recursively estimate

$$\Big(\sum_{L' \in \Lambda_{\tau-1}^{|J|^{-1}}} \sum_{L \in \Lambda_\tau^{|J|^{-1}}(L')} \langle |\Delta_{L^*} f_{1,L}| \rangle_{B_{\sigma/2}, a^\tau J}^2 \Big)^{1/2}$$
$$\leq C_1(\sigma, 1, a) \Big(\sum_{L' \in \Lambda_{\tau-1}^{|J|^{-1}}} \langle |\Delta_{L'} f| \rangle_{B_{(\sigma+1)/2}, a^{\tau-1}J}^2 \Big)^{1/2}$$
$$\leq C_1(\sigma, 1, a) C_1(\sigma+1, \tau-1, a^{\tau-1}) \langle |f| \rangle_{B_{(\tau+\sigma)/2}, J},$$

which takes care of the contribution of the $f_{1,L}$'s. Considering now the $f_{2,L}$'s, by Hölder's inequality for Orlicz spaces, we have that $\langle |\Delta_{L^*} f_{2,L}| \rangle_{B_{\sigma/2}, a^\tau J} \leq \langle |\Delta_{L^*} f_{2,L}| \rangle_{2, a^\tau J}$ and therefore, for any fixed $L' \in \Lambda_{\tau-1}^{|J|^{-1}}$,

$$\sum_{L \in \Lambda_\tau^{|J|^{-1}}(L')} \langle |\Delta_{L^*} f_{2,L}| \rangle_{B_{\sigma/2}, a^\tau J}^2 \lesssim \frac{1}{a^\tau |J|} \int_{\mathbb{R}} |f_{2,L}|^2 = \frac{1}{a^\tau |J|} \int_{\mathbb{R} \setminus a^{\tau-1}J} |\Delta_{L'} f|^2,$$

where we have used the $L^2 \to L^2$ boundedness of the smooth Littlewood–Paley square function together with the fact remarked above that $f_{2,L}$ depends essentially only on $\widehat{L} = L'$. It follows that we can bound recursively:

$$\Big(\sum_{\substack{L' \in \Lambda_{\tau-1}^{|J|^{-1}} \\ |L| \geq |J|^{-1}}} \sum_{L \in \Lambda_\tau(L')} \langle |\Delta_{L^*} f_{2,L}| \rangle_{B_{\sigma/2}, a^\tau J}^2 \Big)^{1/2} \lesssim_\sigma \Big(\frac{1}{a^\tau |J|} \sum_{L' \in \Lambda_{\tau-1}^{|J|^{-1}}} \int_{\mathbb{R} \setminus a^{\tau-1}J} |\Delta_{L'} f|^2 \Big)^{1/2}$$
$$\leq \Big(\frac{C_2(\tau-1, a^{\tau-1})}{a^\tau |J|} |J| \langle |f| \rangle_{B_{(\tau-2)/2}, J}^2 \Big)^{1/2} \lesssim_{\sigma, \tau} (C_2(\tau-1, a^{\tau-1})/a^\tau)^{1/2} \langle |f| \rangle_{B_{(\sigma+\tau)/2}, J}.$$

This proves that

$$C_1(\sigma, \tau, a^\tau) \leq C_1(\sigma, 1, a)\, C_1(\sigma + 1, \tau - 1, a^{\tau-1}) + c_{\sigma,\tau}\, C_2(\tau - 1, a^{\tau-1})^{1/2} a^{-\tau/2}$$

for some numerical constant $c_{\sigma,\tau}$ depending only on σ, τ.

We move to the proof of the inductive step for the L^2-estimate and we use again the splitting of (4.2). For the term corresponding to the $f_{1,L}$'s, we can estimate again recursively:

$$\sum_{L' \in \Lambda_{\tau-1}^{|J|-1}} \sum_{L \in \Lambda_\tau^{|J|-1}(L')} \|\Delta_{L^*} f_{1,L}\|_{L^2(\mathbb{R}\setminus a^\tau J)}^2 \leq C_2(1,a)|a^{\tau-1}J| \sum_{L' \in \Lambda_{\tau-1}^{|J|-1}} \langle |\Delta_{L'} f| \rangle_{1, a^{\tau-1} J}^2$$

$$\leq C_2(1,a)\, C_1(0, \tau - 1, a^{\tau-1}) a^{\tau-1} |J| \langle |f| \rangle_{B_{(\tau-1)/2,J}}^2.$$

Finally, for the contribution of the $f_{2,L}$'s, we use again the $L^2 \to L^2$ boundedness of the smooth Littlewood–Paley square function and the inductive hypothesis to estimate

$$\sum_{L' \in \Lambda_{\tau-1}^{|J|-1}} \sum_{L \in \Lambda_\tau^{|J|-1}(L')} \|\Delta_{L^*} f_{2,L}\|_{L^2(\mathbb{R}\setminus a^\tau J)}^2 \lesssim \sum_{L' \in \Lambda_{\tau-1}^{|J|-1}} \|\Delta_{L'} f\|_{L^2(\mathbb{R}\setminus a^{\tau-1} J)}^2$$

$$\leq C_2(\tau - 1, a^{\tau-1}) |J| \langle |f| \rangle_{B_{(\tau-1)/2,J}}^2.$$

We have thus shown that for some numerical constant $c'_{\sigma,\tau}$,

$$C_2(\tau, a^\tau) \leq C_2(1,a)\, C_1(0, \tau - 1, a^{\tau-1}) a^{\tau-1} + c'_{\sigma,\tau}\, C_2(\tau - 1, a^{\tau-1}).$$

This completes the proof of the inductive step and hence the proof of the proposition. ∎

Remark 4.2. The first estimate in Proposition 4.1 implies the Zygmund–Bonami inequality of general order. Indeed, assume for a moment that $J = [0, 1]$, and for $\lambda \in \mathrm{lac}_\tau^1$, let $L_\lambda \in \Lambda_\tau$ be an interval that has λ as an endpoint. We have

$$|\widehat{f}(\lambda)| = |\widehat{\Delta_{L_\lambda}(f)}(\lambda)| \leq \|\Delta_{L_\lambda}(f)\|_{L^1}$$

for a suitable choice of symbol in the definition of Δ_L, and so the first estimate of the proposition for $\sigma = 0$ implies

$$\left(\sum_{\lambda \in \mathrm{lac}_\tau^1} |\widehat{f}(\lambda)|^2 \right)^{1/2} \lesssim \|f\|_{L \log^{\tau/2} L},$$

which is the Zygmund–Bonami inequality of lacunary order τ. Dualizing and rescaling as in (2.1) tells us that for any finite interval $J \subset \mathbb{R}$ with $|J| \in 2^{\mathbb{Z}}$, we have

$$p(y) := \sum_{\lambda \in \mathrm{lac}_\tau^{|J|-1}} a_\lambda e^{-2\pi i \lambda y}, \quad \|\{a_\lambda\}_\lambda\|_{\ell^2} = 1 \implies \langle |p| \rangle_{E_{2/\tau,J}} \lesssim 1.$$

This formulation of the higher order Zygmund–Bonami inequality will be used in several points in the rest of the paper. As it follows from Proposition 4.1 above, this makes the proofs in the paper somewhat self contained.

Remark 4.3. One can easily verify that the L^2-estimate of Proposition 4.1 can be upgraded to the following form for "molecules". Let $\mathcal{J}$ be a family of pairwise disjoint intervals and $f = \sum_{J \in \mathcal{J}} b_J$, where $\mathrm{supp}(b_J) \subset J$ for each $J \in \mathcal{J}$. For every positive integer τ and $\gamma \geq 2$, there holds

$$\sum_{L \in \Lambda_\tau} \Big\| \sum_{J:|J| \geq |L|^{-1}} \Delta_L(b_J) \mathbf{1}_{\mathbb{R} \setminus \gamma J} \Big\|^2_{L^2(\mathbb{R})} \lesssim \sum_{J \in \mathcal{J}} |J| \langle |b_J| \rangle^2_{B_{(\tau-1)/2}, J}.$$

Indeed, an inductive proof is again available. The case $\tau = 1$ follows by the same pointwise estimate (4.1), which implies that

$$\sum_{J:|J| \geq |L|^{-1}} |\Delta_L(b_J)| \mathbf{1}_{\mathbb{R} \setminus \gamma J} \lesssim \omega_{|L|^{-1}} * \Big(\sum_{|J| \geq |L|^{-1}} \langle |b_J| \rangle_{1,J} \frac{\mathbf{1}_J}{(|L||J|)^5} \Big),$$

which sums using that there are at most two intervals $L \in \Lambda_1$ of any given length. The inductive step relies again on the identity (4.2), applied to each b_J in place of f. Then the contribution of the first term is estimated by an appeal to the case $\tau = 1$ followed by an application of the first estimate in Proposition 4.1. The contribution of the second term in (4.2) is estimated by the Littlewood–Paley inequalities and the inductive hypothesis. We omit the details.

We proceed to prove the easier range, corresponding to $|L| < |J|^{-1}$. As in Corollary E, we require cancellation conditions, which in the case at hand amount to vanishing Fourier coefficients of the function at lacunary frequencies corresponding to order $\tau - 1$. In this range, we can prove the stronger L^2 inequality that follows. For simplicity, we state the result below for f with support of dyadic length, but it is obvious that this is no real restriction.

Proposition 4.4. *Let $J \subset \mathbb{R}$ be a finite interval of dyadic length and f be a compactly supported function with $\mathrm{supp}(f) \subseteq J$. Let τ be a positive integer. We assume that $\hat{f}(\lambda) = 0$ for all $\lambda \in \mathrm{lac}_0^{|J|^{-1}} \cup \cdots \cup \mathrm{lac}_{\tau-1}^{|J|^{-1}}$. Then*

$$\sum_{\substack{L \in \Lambda_\tau: \\ |L| < |J|^{-1}}} \| \Delta_L f \|^2_{L^2(\mathbb{R})} \lesssim |J| \langle |f| \rangle^2_{B_{(\tau-1)/2}, J}.$$

Proof. We argue by induction as in the proof of Proposition 4.1. Let us denote by $C(\tau)$ the best constant in the inequality we intend to prove. Note that, for $\tau = 1$, the assumption reads $\int_J f = 0$ and the conclusion $C(1) < +\infty$ follows immediately by the pointwise estimate

$$(4.3) \qquad |\Delta_L(f)(x)| \lesssim_M |L|^2 |J|^2 (1 + |L||x - c_J|)^{-M} \langle |f| \rangle_{1,J}$$
$$\lesssim \omega_{|L|^{-1}} * (|L||J| \langle |f| \rangle_{1,J} \mathbf{1}_J)$$

for any large positive integer M, where c_J is the center of J and $|L||J| < 1$. In order to see the first estimate above, let us take $\phi_L \in \Phi_{L,M}$ to be the symbol of Δ_L which can be written in the form $\phi_L(x) = e^{i2\pi i c_L x} |L| \phi(|L|x)$, with c_L denoting the center of $L \in \Lambda_1$.

We compute, using the cancellation of f and the mean value theorem,

$$|\Delta_L(f)(x)| \le \int_J |\phi_L(x-y) - \phi_L(x-c_J)| \, |f(y)| \, \mathrm{d}y \lesssim \int_J |J| \sup_{z \in J} |\phi'_L(x-z)| \, |f(y)| \, \mathrm{d}y.$$

Using that $|c_L| \simeq \mathrm{dist}(L, 0) \simeq |L|$, we have, for $z \in J$,

$$\begin{aligned}
|\phi'_L(x-z)| &\lesssim c_L |L| \phi(|L|(x-z)) + |L|^2 \phi(|L|(x-z)) \\
&\lesssim |L|^2 (1 + |L||x-z|)^{-M} \simeq |L|^2 (1 + |L||x - c_J|)^{-M}.
\end{aligned}$$

The last approximate equality can be checked by considering the cases $x \in 3J$ and $x \notin 3J$ separately, remembering that $|L||J| < 1$. The combination of the last two displays yields the first estimate in (4.3). The second estimate in (4.3) follows by the first since

$$(1 + |L||x - c_J|)^{-M} \simeq (1 + |L||x - z|)^{-M}$$

for $z \in J \supseteq \mathrm{supp}(f)$.

For $\tau > 1$, we first do the same reduction as in the proof of Proposition 4.1. For $\tau > 1$, we can estimate

$$\begin{aligned}
(4.4) \qquad \sum_{\substack{L \in \Lambda_\tau \\ |L| < |J|^{-1}}} \|\Delta_L f\|_{L^2(\mathbb{R})}^2 &\le \sum_{\substack{L' \in \Lambda_{\tau-1} \\ |L'| < |J|^{-1}}} \sum_{L \in \Lambda_\tau(L')} \|\Delta_{L^*}(e^{-2\pi i \lambda(L) \cdot} \Delta_{L'} f)\|_{L^2(\mathbb{R})}^2 \\
&\quad + \sum_{L' \in \Lambda_{\tau-1}^{|J|^{-1}}} \sum_{\substack{L \in \Lambda_\tau(L') \\ |L| < |J|^{-1}}} \|\Delta_{L^*}(e^{-2\pi i \lambda(L) \cdot} f)\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

The first summand above is estimated by $\lesssim C(\tau - 1)|J| \langle |f| \rangle_{B_{(\tau-2)/2, J}}^2$ by using the L^2-bound for the smooth Littlewood–Paley square function and the inductive hypothesis.

The second summand above can be estimated by

$$\begin{aligned}
\sum_{\ell : 2^\ell < |J|^{-1}} \sum_{L' \in \Lambda_{\tau-1}^{|J|^{-1}}} \sum_{\substack{L \in \Lambda_\tau(L') \\ |L| = 2^\ell}} &\|\Delta_{2^\ell}(e^{-2\pi i \lambda(L) \cdot} f)\|_{L^2(\mathbb{R})}^2 \\
&= \sum_{\ell : 2^\ell < |J|^{-1}} \sum_{\lambda \in \mathrm{lac}_{\tau-1}^{|J|^{-1}}} \sum_{\substack{L \in \Lambda_\tau \\ \lambda(L) = \lambda \\ |L| = 2^\ell}} \|\Delta_{2^\ell}(e^{-2\pi i \lambda \cdot} f)\|_{L^2(\mathbb{R})}^2 \\
&\lesssim \sum_{\ell : 2^\ell < |J|^{-1}} \sum_{\lambda \in \mathrm{lac}_{\tau-1}^{|J|^{-1}}} \|\Delta_{2^\ell}(e^{-2\pi i \lambda \cdot} f)\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

where, in passing to the last line, we used that for each $\lambda \in \mathrm{lac}_{\tau-1}$ there are at most $O_\tau(1)$ intervals $L \in \Lambda_\tau$ of fixed length with $\lambda(L) = \lambda$. Fixing for a moment $2^\ell < |J|^{-1}$ and $x \in \mathbb{R}$, we write

$$\sum_{\lambda \in \mathrm{lac}_{\tau-1}^{|J|^{-1}}} |\Delta_{2^\ell}(e^{-2\pi i \lambda \cdot} f)(x)|^2 = \left| \Delta_{2^\ell} \left(\sum_{\lambda \in \mathrm{lac}_{\tau-1}^{|J|^{-1}}} a_\lambda e^{-2\pi i \lambda \cdot} f \right)(x) \right|^2 =: |\Delta_{2^\ell}(p_{x,\ell} f)(x)|^2,$$

where $\{a_\lambda\}_\lambda = \{a_\lambda(x, \ell)\}_\lambda$ is in the unit ball of ℓ_λ^2, and $p_{x,\ell}$, implicitly defined above, is as in (2.1). Using the cancellation assumptions on f, we see that $\int_J p_{x,\ell} f = 0$, so by appealing to (4.3), we get

$$|\Delta_{2^\ell}(f p_{x,k_\tau})| \lesssim \omega_{2^{-\ell}} * (2^\ell |J| \langle |f p_{x,\ell}| \rangle_{1,J} \mathbf{1}_J) \lesssim \omega_{2^{-\ell}} * (2^\ell |J| \langle |f| \rangle_{B_{(\tau-1)/2},J} \mathbf{1}_J),$$

where we used the Hölder inequality in Orlicz spaces together with the Zygmund–Bonami inequality of order $\tau - 1$ from Remark 4.2 to control $\langle |p_{x,\ell}| \rangle_{E_{2/(\tau-1)}} \lesssim 1$. Squaring the estimate in the last display, integrating, and then summing for $2^\ell < |J|^{-1}$ yields that the second summand in (4.4) is controlled by a constant multiple of $|J| \langle |f| \rangle_{B_{(\tau-1)/2},J}^2$. We have proved that $C(\tau) \lesssim (1 + C(\tau - 1))$ and this concludes the proof of the inductive step and of the proposition. ∎

Remark 4.5. As in Remark 4.3 there is an upgrade of the L^2-estimate of Proposition 4.4 from "atoms" to "molecules" $f = \sum_{J \in \mathcal{J}} b_J$, where $\mathcal{J}$ is a family of pairwise disjoint dyadic intervals and each b_J satisfies the cancellation assumptions of Proposition 4.4, namely,

$$\sum_{L \in \Lambda_\tau} \left\| \sum_{J : |J| < |L|^{-1}} \Delta_L(b_J) \right\|_{L^2(\mathbb{R})}^2 \lesssim \sum_{J \in \mathcal{J}} |J| \langle b_J \rangle_{B_{(\tau-1)/2},J}^2.$$

The base case $\tau = 1$ is essentially identical to the corresponding step in the proof of Proposition 4.4, relying on the pointwise estimate

$$\left| \Delta_L \left(\sum_{J : |L| < |J|^{-1}} b_J \right) \right| \lesssim \omega_{|L|^{-1}} * \left(|L| \sum_{J : |L| < |J|^{-1}} |J| \langle |b_J| \rangle_{1,J} \mathbf{1}_J \right).$$

This is a consequence of (4.3) using the cancellation assumption $\int b_J = 0$ for each $J \in \mathcal{J}$. For the inductive step with $\tau > 1$, denoting again by $C(\tau)$ the best constant in the desired L^2-estimate, we clearly have that

$$\sum_{L \in \Lambda_\tau} \left\| \sum_{J : |J| < |L|^{-1}} \Delta_L(b_J) \right\|_{L^2(\mathbb{R})}^2 \lesssim \sum_{L \in \Lambda_\tau} \left\| \sum_{J : |\hat{L}|^{-1} \leq |J| < |L|^{-1}} \Delta_{|L|}(e^{-2\pi i \lambda(L)\cdot} b_J) \right\|_{L^2(\mathbb{R})}^2$$

$$+ C(\tau - 1) \sum_{J \in \mathcal{J}} |J| \langle b_J \rangle_{B_{(\tau-2)/2},J}^2.$$

Using a linearization trick as in the proof of Proposition 4.4, we have

$$\sum_{L \in \Lambda_\tau} \left| \sum_{J : |\hat{L}|^{-1} \leq |J| < |L|^{-1}} \Delta_{|L|}(e^{-2\pi i \lambda(L)\cdot} b_J)(x) \right|^2$$

$$= \sum_{\ell \in \mathbb{Z}} \left| \sum_{J : |J| < 2^{-\ell}} \Delta_{2^\ell} \left(\sum_{\substack{L \in \Lambda_\tau \\ |L| = 2^\ell, |\hat{L}|^{-1} \leq |J|}} a_L e^{-2\pi i \lambda(L)\cdot} b_J \right)(x) \right|^2.$$

for some collection $\{a_L\}_{L \in \Lambda_\tau} = \{a_L(x, \ell)\}_{L \in \Lambda_\tau}$ in the unit ball of ℓ_L^2.

Fixing for the moment $\ell \in \mathbb{Z}$ and $x \in \mathbb{R}$, we have

$$
p_{x,\ell} := \sum_{\substack{L \in \Lambda_\tau \\ |L|=2^\ell, |\widehat{L}| \geq |J|^{-1}}} a_L \, e^{-2\pi i \lambda(L) \cdot} = \sum_{L' \in \Lambda_{\tau-1}^{|J|^{-1}}} \left(\sum_{\substack{L \in \Lambda_\tau(L') \\ |L| 2^\ell}} a_L \right) e^{-2\pi i \lambda(L) \cdot}
$$

$$
=: \sum_{\lambda \in \mathrm{lac}_{\tau-1}^{|J|^{-1}}} \beta_\lambda \, e^{-2\pi i \lambda \cdot},
$$

with $\|\{\beta_\lambda\}\|_{\ell_\lambda^2} = O(1)$. Here we used that there at most $O(1)$ intervals $L \in \Lambda_\tau$ with fixed length $|L| = 2^\ell$ inside $\widehat{L}$. Using the cancellation of $p_{x,\ell} b_J$, we can estimate pointwise

$$
|\Delta_{2^\ell}(p_{x,\ell} b_J)| \lesssim 2^\ell |J| \langle |p_{x,J} b_J| \rangle_{1,J} \omega_{2^\ell} * \mathbf{1}_J,
$$

and by Remark 4.2 and Hölder's inequality for Orlicz spaces, we have that

$$
\langle |p_{x,\ell} b_J| \rangle_{1,J} \lesssim \langle |b_J| \rangle_{B_{(\tau-1)/2}, J}.
$$

With this information the proof of the estimate can now be completed summing over $|J| 2^\ell < 1$ as in the proof of Proposition 4.4.

We conclude this section by recording the generalized Zygmund–Bonami inequality under cancellation conditions. This is just a combination of Propositions 4.1 and 4.4.

Corollary 4.6. Let σ be a nonnegative real number and τ a positive integer. Assume that $\mathrm{supp}(f) \subset J$ for some finite interval J and that $\widehat{f}(\lambda) = 0$ for all $\lambda \in \mathrm{lac}_0^{|J|^{-1}} \cup \cdots \cup \mathrm{lac}_{\tau-1}^{|J|^{-1}}$. Then

$$
\left(\sum_{L \in \Lambda_\tau} \langle |\Delta_L(f)| \rangle_{B_{\sigma/2}, \gamma J}^2 \right)^{1/2} \lesssim \langle |f| \rangle_{B_{(\sigma+\tau)/2}, J}.
$$

5. An $L^{B_{\sigma/2}}(\mathbb{R})$ Calderón–Zygmund decomposition

We describe in this section a Calderón–Zygmund decomposition adapted to the (global) Orlicz space $L^{B_{\sigma/2}}(\mathbb{R})$ for $\sigma \geq 0$. Such a Calderón–Zygmund decomposition, which is influenced by the one appearing in Appendix A of [7], is available to us because of the specific choice of the Young function $B_{\sigma/2}$ and it is adapted to the finite order lacunary setup.

Recall that for $\sigma \geq 0$, we write $f \in L^{B_{\sigma/2}}(\mathbb{R})$ if for some (or equivalently all) $\lambda > 0$, there holds

$$
\int_{\mathbb{R}} B_{\sigma/2}\left(\frac{|f(x)|}{\lambda} \right) dx < +\infty.
$$

There is an Orlicz maximal operator associated with B_σ,

$$
\mathrm{M}_{B_{\sigma/2}} f(x) := \sup_{Q \ni x} \langle |f| \rangle_{B_{\sigma/2}, Q}, \quad x \in \mathbb{R},
$$

with the supremum being over all intervals Q of $\mathbb{R}$ containing x. The dyadic version of $\mathrm{M}_{B_{\sigma/2}}$ is defined similarly, with the supremum over all dyadic intervals $Q \in \mathcal{D}$, where $\mathcal{D}$ is some dyadic grid. We will write $\mathrm{M}_{B_{\sigma/2}, \mathcal{D}}$ for the dyadic version. Below we denote by $Q \in \mathcal{D}$ a dyadic interval, $Q^{(1)}$ its dyadic parent and set $Q^{(k+1)}$ to be the dyadic parent of $Q^{(k)}$.

Remark 5.1 (Existence of stopping intervals). For the Calderón–Zygmund decomposition, we choose stopping intervals that are maximal under the condition $\langle |f| \rangle_{B_{\sigma/2}, I} > \lambda$. The existence of these stopping intervals relies on the following fact: if $f \in L^{B_{\sigma/2}}(\mathbb{R})$ for some $\sigma \geq 0$ and I is a dyadic interval in some grid $\mathcal{D}$, then $\langle |f| \rangle_{B_{\sigma/2}, I^{(k)}} \to 0$ as $k \to +\infty$. This can be easily proved using, for example, the fact that the Young function $B_{\sigma/2}$ is submultiplicative.

Proposition 5.2. *Let σ be a fixed nonnegative integer, $f \in L^{B_{\sigma/2}}(\mathbb{R})$, and $\alpha > 0$. There exist a collection $\mathcal{J}$ of pairwise disjoint dyadic intervals J and a decomposition of f,*

$$f = g + b_{\mathrm{canc},\sigma} + b_{\mathrm{lac},\sigma},$$

such that the following hold:

(i) *The function g satisfies $\|g\|_{L^\infty(\mathbb{R})} \lesssim \alpha$ and $\|g\|_{L^1(\mathbb{R})} \lesssim \|f\|_{L^1(\mathbb{R})}$.*

(ii) *The function $b_{\mathrm{canc},\sigma}$ is supported in $\bigcup_{J \in \mathcal{J}} J$ and, in particular,*

$$b_{\mathrm{canc},\sigma} = \sum_{J \in \mathcal{J}} b_J, \quad \mathrm{supp}(b_J) \subseteq J, \quad \widehat{b}_J(\lambda) = 0 \text{ for all } \lambda \in \mathrm{lac}_0 \cup \cdots \cup \mathrm{lac}_\sigma.$$

Furthermore, we have that $\langle |b_J| \rangle_{B_{\sigma/2}, J} \lesssim \alpha$ for all $J \in \mathcal{J}$ and

$$\sum_{J \in \mathcal{J}} |J| \leq \int_{\mathbb{R}} B_{\sigma/2}\left(\frac{|f|}{\alpha}\right).$$

(iii) *The function $b_{\mathrm{lac},\sigma}$ is also supported on $\bigcup_{J \in \mathcal{J}} J$ and satisfies*

$$\|b_{\mathrm{lac},\sigma}\|^2_{L^2(\mathbb{R})} \lesssim \sum_{J \in \mathcal{J}} |J| \langle |b_J| \rangle^2_{B_{\sigma/2}, J} \lesssim \alpha^2 \int_{\mathbb{R}} B_{\sigma/2}\left(\frac{|f|}{\alpha}\right).$$

Proof. We begin by recalling that $f \in L^{B_{\sigma/2}}(\mathbb{R})$ implies that $\int_{\mathbb{R}} B_{\sigma/2}(|f|/\alpha) < +\infty$ for all $\alpha > 0$. By Remark 5.1 and Theorem 5.5 in [7], we have that the dyadic Orlicz maximal operator $M_{B_{\sigma/2}, \mathcal{D}}$ satisfies

$$|E_\alpha| := |\{x \in \mathbb{R} : M_{B_{\sigma/2}, \mathcal{D}} f(x) > \alpha\}| \leq \int_{\mathbb{R}} B_{\sigma/2}\left(\frac{|f|}{\alpha}\right), \quad \alpha > 0.$$

Letting $\mathcal{J}$ denote the collection of maximal dyadic intervals contained in E_α, we have that for every $J \in \mathcal{J}$,

$$\alpha < \langle |f| \rangle_{B_{\sigma/2}, J} \leq 2\alpha, \quad \sum_{J \in \mathcal{J}} |J| \leq \int_{\mathbb{R}} B_{\tau/2}\left(\frac{|f|}{\alpha}\right).$$

The upper bound in the approximate inequality of the leftmost estimate above follows by the maximality of J and the convexity of the Young function of $B_{\sigma/2}$, which implies that

$$\langle |f| \rangle_{B_{\sigma/2}, J} \leq \rho \langle |f| \rangle_{B_{\sigma/2}, \rho J}, \quad \rho > 1,$$

see Proposition A.1 in [7] and equation (5.2) in [7]. One routinely checks that the function $g := f \mathbf{1}_{\mathbb{R} \setminus \bigcup_{J \in \mathcal{J}} J}$ satisfies (i).

For the "atoms", we set $f_J := f \mathbf{1}_J$ and define

$$b_{J,\mathrm{lac},\sigma}(y) := \sum_{\rho=0}^{\sigma} \Big(\sum_{\lambda \in \mathrm{lac}_\rho^{|J|^{-1}}} \hat{f}_J(\lambda) e^{2\pi i \lambda y} \Big) \frac{\mathbf{1}_J(y)}{|J|}, \quad b_J := f_J - b_{J,\mathrm{lac},\sigma},$$

and

$$b_{\mathrm{lac},\sigma} := \sum_{J \in \mathcal{J}} b_{J,\mathrm{lac},\sigma} \quad \text{and} \quad b_{\mathrm{canc},\sigma} := \sum_{J \in \mathcal{J}} b_J.$$

The cancellation conditions of (ii) for $b_{\mathrm{canc},\sigma}$ follow immediately by the definition above. Furthermore, by the Hölder inequality for Orlicz spaces and the Zygmund–Bonami inequality of order $\rho \in \{1, \ldots, \sigma\}$ as in Remark 4.2, one sees that

$$\langle |b_{J,\mathrm{lac},\sigma}| \rangle_{B_{\tau/2},J} \lesssim \langle |b_{J,\mathrm{lac},\sigma}| \rangle_{2,J} \lesssim \langle |f_J| \rangle_{B_{\sigma/2},J} \lesssim \alpha.$$

This and the triangle inequality also yield $\langle |b_J| \rangle_{B_{\sigma/2},J} \lesssim \langle |f_J| \rangle_{B_{\sigma/2},J} \lesssim \alpha$, thus completing the proof of the desired conclusions in (ii). Finally, for (iii) we estimate as above:

$$\|b_{\mathrm{lac},\sigma}\|_{L^2(\mathbb{R})}^2 \lesssim \sum_{J \in \mathcal{J}} |J| \langle |f_J| \rangle_{B_{\sigma/2},J}^2 \lesssim \alpha^2 \int_{\mathbb{R}} B_{\tau/2}\Big(\frac{|f|}{\alpha} \Big),$$

and the proof is complete. ∎

6. Proof of Theorem B and corollaries

In the first part of this section, we compile together the results of the previous sections to conclude the proof of Theorem B. In the second part, we show how to conclude our corollaries, namely, Theorems A and C.

6.1. Proof of Theorem B

Let us fix a positive integer τ and $m \in R_{2,\tau}$. Before entering the heart of the proof, we note that it suffices to prove the theorem for multipliers m having the form

$$m = \sum_{I \in \mathcal{J}} c_I \mathbf{1}_I,$$

where the family of intervals $\mathcal{J}$ has overlap at most N, for each $I \in \mathcal{J}$, there exists a unique $L = L_I \in \Lambda_\tau$ such that $I \subset L_I$ and for each fixed $L \in \Lambda_\tau$, there holds

$$\sum_{I : L_I = L} |c_I|^2 \leq N^{-1}.$$

See the analysis on p. 533 of [17] for the details of this approximation argument. For m of this form, we now can write

$$\mathrm{T}_m(f) = \sum_{I \in \mathcal{J}} c_I \mathrm{P}_I f = \sum_{L \in \Lambda_\tau} \sum_{I : L_I = L} c_I \mathrm{P}_I (\Delta_L f), \quad \mathrm{P}_I f := (\mathbf{1}_I \hat{f})^{\vee},$$

a fact that we will use repeatedly in what follows.

6.1.1. The upper bound in Theorem B. Let f be a function in $L^{B_\tau}(\mathbb{R})$ and let $\alpha > 0$ be fixed. We decompose f according to the Calderón–Zygmund decomposition in Proposition 5.2 with $\sigma = \tau$, yielding

$$f = g + b_{\mathrm{canc},\tau} + b_{\mathrm{lac},\tau}.$$

We directly estimate $g + b_{\mathrm{lac},\tau}$ in L^2 using (i) and (iii) of Proposition 5.2:

$$|\{x \in \mathbb{R} : |\mathrm{T}_m(g + b_{\mathrm{lac},\tau})(x)| > \alpha\}| \lesssim \frac{1}{\alpha^2} \|g + b_{\mathrm{lac},\tau}\|_{L^2(\mathbb{R})}^2 \lesssim \int_{\mathbb{R}} B_{\tau/2}\Big(\frac{|f|}{\alpha}\Big).$$

The main part of the proof deals with the bad part $b_{\mathrm{canc},\tau} = \sum_{J \in \mathcal{J}} b_J$, and it suffices to estimate

$$\Big|\{x \in \mathbb{R} \setminus \bigcup_{J \in \mathcal{J}} 6J : |\mathrm{T}_m(b_{\mathrm{canc},\tau})| > \alpha\}\Big|$$

as the measure $|\bigcup_{J \in \mathcal{J}} 6J|$ satisfies the desired estimate by (ii) of Proposition 5.2. We will adopt the splitting

$$\mathrm{T}_m\Big(\sum_J b_J\Big) = \sum_I c_I \mathrm{P}_I \Big(\sum_{J : |J| \geq |L_I|^{-1}} \Delta_{L_I}(b_J) \mathbf{1}_{\mathbb{R} \setminus 3J}\Big) + \sum_I c_I \mathrm{P}_I \Big(\sum_{J : |J| < |L_I|^{-1}} \Delta_{L_I}(b_J)\Big)$$

$$+ \sum_I c_I \mathrm{P}_I \Big(\sum_{J : |J| \geq |L_I|^{-1}} \Delta_{L_I}(b_J) \mathbf{1}_{3J}\Big) =: \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

The main term is III. Indeed, we can estimate the term I in $L^2(\mathbb{R})$ using Remark 4.3, while II is also estimated in $L^2(\mathbb{R})$ using Remark 4.5 this time. Note that each b_J has the required cancellation by (ii) of Proposition 5.2. Using also (ii) of Proposition 5.2 to control the averages $\langle|b_J|\rangle_{B_{(\tau-1)/2},J} \lesssim \langle|b_J|\rangle_{B_{\tau/2},J} \lesssim \alpha$, we have

$$|\{|\mathrm{I} + \mathrm{II}| > \alpha\}| \lesssim \frac{1}{\alpha^2} \sum_{J \in \mathcal{J}} |J| \langle|b_J|\rangle_{B_{(\tau-1)/2},J}^2 \lesssim \sum_{J \in \mathcal{J}} |J| \lesssim \int_{\mathbb{R}} B_\tau(|f|/\alpha)$$

as desired.

It remains to deal with III, and we make a further splitting. Let $k_I \in \mathbb{Z}$ be such that $2^{k_I} < |I| \leq 2^{k_I+1}$. Of course, we will always have that $|L_I| \geq 2_I^k$, since $I \subseteq L_I$. We write

$$\mathrm{III} = \sum_I c_I \mathrm{P}_I \Big(\sum_{J : 2^{-k_I} > |J| \geq |L_I|^{-1}} \Delta_{L_I}(b_J) \mathbf{1}_{3J}\Big) + \sum_I c_I \mathrm{P}_I \Big(\sum_{J : |J| \geq 2^{-k_I}} \Delta_{L_I}(b_J) \mathbf{1}_{3J}\Big)$$

$$=: \mathrm{III}_1 + \mathrm{III}_2.$$

We first handle the term III_1. Let Δ_I be the smooth frequency projections on the interval I as fixed in Section 2.2. Then, in particular, we can write $\mathrm{P}_I \Delta_I = \mathrm{P}_I$, and we have the familiar pointwise estimate

$$|\Delta_I(\Delta_{L_I}(b_J) \mathbf{1}_{3J})| \lesssim \omega_{|I|^{-1}} * (\langle|\Delta_{L_I}(b_J)|\rangle_{1,3J} \mathbf{1}_J),$$

as $|I|^{-1} \simeq 2^{-k_I} > |J|$. We thus get

$$|\{|\mathrm{III}_1| > \alpha\}| \lesssim \frac{1}{\alpha^2} \sum_{L \in \Lambda_\tau} N \sum_{I : L_I = L} |c_I|^2 \int_{\mathbb{R}} \left| \omega_{|I|^{-1}} * \left(\sum_{J : |J| \geq 2^{-k_I}} \langle |\Delta_L(b_J)| \rangle_{1,3J} \mathbf{1}_J \right) \right|^2$$

$$\lesssim \frac{1}{\alpha^2} \sum_{J \in \mathcal{J}} \sum_{L \in \Lambda_\tau^{|J|^{-1}}} |J| \langle |\Delta_L(b_J)| \rangle_{1,3J}^2 \lesssim \sum_{J \in \mathcal{J}} |J| \lesssim \int_{\mathbb{R}} B_{\tau/2}\left(\frac{|f|}{\alpha} \right),$$

where we used the ℓ^2-control on the coefficients $\{c_I\}_{L_I = L}$ in passing to the second line and the generalized Zygmund–Bonami inequality of Proposition 4.1 together with the properties of the Calderón–Zygmund decomposition in the penultimate approximate inequality.

The steps required for dealing with the term III_2 are essentially the same as those in [17]; however, as here we are dealing with a higher order setup, we include them for the sake of completeness. We will split the estimate for III_2 into two parts. In the first, we keep the part of the multiplier $\mathbf{1}_I = \mathbf{1}_{[\ell_I, r_I]}$ at scale $O(|J|^{-1})$ around its singularities which are at the endpoints. We make this precise now.

Let $0 \leq \psi_{I,J} \leq 1$ be a smooth bump which is 1 on the $(10|J|)^{-1}$-neighborhood of the endpoints $\{\ell_I, r_I\}$ of I and vanishes off the $(5|J|)^{-1}$-neighborhood of the endpoints, and satisfies $\|\partial^\alpha \psi_{I,J}\|_{L^\infty} \lesssim |J|^\alpha$ for all α up to some sufficiently large integer M. Letting $\Psi_{I,J}$ denote the operator with symbol $\psi_{I,J}$, we define

$$\mathcal{E}(\{b_J\}_{J \in \mathcal{J}}) := \sum_I c_I \mathrm{P}_I \left(\sum_{J : |J| \geq 2^{-k_I}} \Psi_{I,J}(\Delta_{L_I}(b_J) \mathbf{1}_{3J}) \right)$$

The following lemma shows that the operator $\mathcal{E}(\{b_J\}_{J \in \mathcal{J}})$ can be dealt with, again, by L^2-estimates.

Lemma 6.1. *We have the estimate*

$$\|\mathcal{E}(\{b_J\}_{J \in \mathcal{J}})\|_{L^2(\mathbb{R})}^2 \lesssim \sum_{J \in \mathcal{J}} |J| \langle |b_J| \rangle_{B_{\tau/2}, J}^2 \lesssim \alpha^2 \int_{\mathbb{R}} B_{\tau/2}\left(\frac{|f|}{\alpha} \right).$$

Proof. First note that by the overlap assumption on the intervals I, we have

$$\|\mathcal{E}(\{b_J\}_{J \in \mathcal{J}})\|_{L^2(\mathbb{R})}^2 \lesssim N \sum_{L \in \Lambda_\tau} \sum_{I : L_I = L} |c_I|^2 \int_{\mathbb{R}} \left(\sum_{J : |J| \geq 2^{-k_I}} |\Psi_{I,J}(\Delta_L(b_J) \mathbf{1}_{3J})| \right)^2.$$

The following pointwise estimate can be routinely verified:

$$|\Psi_{I,J}(\Delta_L(b_J) \mathbf{1}_{3J})(x)| \lesssim \mathrm{M}(\mathbf{1}_J)(x)^{10} \langle |\Delta_L(b_J)| \rangle_{1,3J}.$$

Using the Fefferman–Stein inequality and rearranging the sums, we can conclude that

$$\|\mathcal{E}(\{b_J\}_{J \in \mathcal{J}})\|_{L^2(\mathbb{R})}^2 \lesssim N \sum_k \sum_{L \in \Lambda_\tau^{2^k}} \sum_{\substack{I : L_I = L \\ \ell_I = \ell}} |c_I|^2 \sum_{|J| \geq 2^{-k}} \langle |\Delta_L(b_J)| \rangle_{1,3J}^2 |J|$$

$$\leq \sum_{J \in \mathcal{J}} |J| \sum_{L \in \Lambda_\tau^{|J|^{-1}}} \langle |\Delta_L(b_J)| \rangle_{1,3J}^2,$$

where we used the ℓ^2-control on the coefficients $\{c_I\}_{L_I=L}$ in passing to the second line. An appeal to the generalized Zygmund–Bonami inequality of order τ in Proposition 4.1 concludes the proof of the lemma. ∎

We are left with studying the contribution of the operator

$$\mathcal{L}(\{b_J\}_{J\in\mathcal{g}}) := \sum_I c_I P_I \sum_{|J|\geq 2^{-k_I}} (\mathrm{Id}-\Psi_{I,J})(\Delta_{L_I}(b_J)\mathbf{1}_{3J}).$$

For this, we consider the multiplier $\zeta_{I,J} := \mathbf{1}_I(1-\psi_{I,J})$ which is a smooth function with values in $[0,1]$, supported in I, is identically 1 on $|x-c_I| \lesssim |I|$ and drops to 0 with derivative $O(|J|)$ close to the endpoints of I. More generally, one easily checks that $\zeta_{I,J}$ satisfies

$$|\partial^\alpha \zeta_{I,J}| \lesssim |J|^\alpha \mathbf{1}_{I_{\mathrm{left}}(J)\cup I_{\mathrm{right}}(J)} \quad \text{for all } \alpha \geq 1,$$

where

$$I_{\mathrm{left}}(J) := [\ell_I + 10^{-1}|J|^{-1}, \ell_I + 5^{-1}|J|^{-1}] \subset I$$

and

$$I_{\mathrm{right}}(J) := [r_I - 5^{-1}|J|^{-1}, r_I - 10^{-1}|J|^{-1}] \subset I.$$

Remembering that we are dealing with the case $|I||J| \gtrsim 1$, we see that the function $\zeta_{I,J}$ has support of size $O(|I|)$ and α-derivatives of size $O(|J|^\alpha)$; thus the function $\zeta_{I,J}$ is not a good kernel. The important observation is however that the derivatives of $\check{\zeta}_{I,J}$ of order $\alpha \geq 1$ have support of size $|I_{\mathrm{left}}(J) \cup I_{\mathrm{right}}(J)| \simeq |J|^{-1}$.

Given an interval $J \subset \mathbb{R}$, we will also use an auxiliary function ρ_J defined as follows. We choose $0 \leq \rho \leq 1$ to be a smooth bump function which is identically 1 on $[-1,1]$ and vanishes off $[-3/2, 3/2]$, and define

$$\rho_J(x) := \rho(x/|J|) \quad \text{for } x \in \mathbb{R}.$$

Lemma 6.2. *Let I and J be intervals and let $\zeta_{I,J}$ and ρ_J be defined as above. If*

$$\check{\mu}_{I,J} := (1-\rho_J)\check{\zeta}_{I,J},$$

then for any nonnegative integers γ and β, there holds

$$|\partial_\xi^\beta \mu_{I,J}(\xi)| \lesssim |J|^\beta (1+|J|\,\mathrm{dist}(\xi, \mathbb{R}\setminus I))^{-\gamma}, \quad \xi \in \mathbb{R}.$$

Proof. We begin by noting that since $\zeta_{I,J}$ is a Schwartz function and $(1-\rho_J)$ is a smooth bounded function, we have we have that $(1-\rho_J)\check{\zeta}_{I,J}$ is a Schwartz function. Furthermore, by the comments preceding the statement of the lemma, we have that $\zeta_{I,J}$ satisfies

$$|\partial_\xi^\alpha \zeta_{I,J}| \lesssim |J|^\alpha \mathbf{1}_{I_{\mathrm{left}}(J)\cup I_{\mathrm{right}}(J)} \quad \text{for all } \alpha \geq 1.$$

Note that, by symmetry, it suffices to prove the estimate for $\xi \in \mathbb{R}$ such that $\mathrm{dist}(\xi, \mathbb{R}\setminus I) = |\xi - \ell_I|$, where we remember that $I = [\ell_I, r_I]$. For simplicity, we will write $I(J)$ for $I_{\mathrm{left}}(J)$. Thus, the conclusion of the lemma reduces to showing

$$|\partial_\xi^\beta \mu_{I,J}(\xi)| \lesssim_\beta |J|^\beta (1+|J||\xi-\ell_I|)^{-\gamma}.$$

We will henceforth drop the subindices I, J in order to simplify the notation. We record the following standard integration by parts identity: for nonnegative integers γ, ν, we have

$$(\partial_x - 2\pi i \ell_I)^\gamma [\check{\zeta}(x)] = (-1)^\nu \frac{(2\pi i)^{\gamma - \nu}}{x^\nu} \int_\mathbb{R} \partial_\xi^\nu [(\xi - \ell_I)^\gamma \zeta(\xi)] e^{2\pi i x \xi} \, d\xi.$$

In order to make sure that all terms in $\partial_\xi^\nu [(\xi - \ell_I)^\gamma \zeta(\xi)]$ contain at least one derivative, we take $\nu > \gamma$. Then we have

$$|\partial_\xi^\nu [(\xi - \ell_I)^\gamma \zeta(\xi)]| \lesssim \sum_{k=0}^\gamma |\partial_\xi^k (\xi - \ell_I)^\gamma \partial_\xi^{\nu-k} [\zeta(\xi)]|$$

$$\lesssim \sum_{k=0}^\gamma |\xi - \ell_I|^{\gamma-k} |J|^{\nu-k} \mathbf{1}_{I(J)}(\xi) \lesssim |J|^{\nu-\gamma} \mathbf{1}_{I(J)}(\xi),$$

provided that $\nu > \gamma$. Plugging this estimate into our integration by parts identity, we get

(6.1)
$$|(\partial_x - 2\pi i \ell_I)^\gamma [\check{\zeta}(x)]| \lesssim \frac{|J|^{\nu-\gamma-1}}{|x|^\nu}, \quad \nu > \gamma.$$

Using this estimate, for nonnegative integers β, γ, we have

$$\partial_\xi^\beta [\mu(\xi)] = \frac{(-2\pi i)^\beta}{(2\pi i (\xi - \ell_I))^\gamma} \int_\mathbb{R} (\partial_x - 2\pi i \ell_I)^\gamma [x^\beta (1 - \rho(x/|J|)) \check{\zeta}(x)] e^{-2\pi i x \xi} \, dx.$$

Using (6.1) with ν large, together with the fact that $\mathrm{supp}(1 - \rho_J) \subset \{|x| \gtrsim |J|\}$ and that $\mathrm{supp}(\partial_x[\rho_J]) \subset \{|x| \simeq |J|\}$, and combining with the previous identity yields

$$|\partial_\xi^\beta [\mu(\xi)]| \lesssim \frac{1}{|\xi - \ell_I|^\gamma} \int_\mathbb{R} \sum_{\substack{k_1 + k_2 + k_3 = \gamma \\ k_1 \leq \beta}} |x|^{\beta - k_1} |\partial_x^{k_2} (1 - \rho(x/|J|)) (\partial_x - 2\pi i \ell_I)^{k_3} [\check{\zeta}(x)]| \, dx$$

$$\leq \sum_{\substack{k_1 + k_3 = \gamma \\ k_1 \leq \beta}} \frac{|J|^{\nu - k_3 - 1}}{|\xi - \ell_I|^\gamma} \int_{|x| \gtrsim |J|} |x|^{\beta - k_1 - \nu} \, dx$$

$$+ \sum_{\substack{k_1 + k_2 + k_3 = \gamma \\ k_1 \leq \beta, k_2 \geq 1}} \frac{|J|^{\nu - k_3 - 1}}{|\xi - \ell_I|^\gamma} \int_{|x| \simeq |J|} |x|^{\beta - k_1 - \nu} |J|^{-k_2} \, dx \lesssim \frac{|J|^{\beta - \gamma}}{|\xi - \xi_J|^\gamma}.$$

Combining this estimate for general γ with the special case $\gamma = 0$ yields the conclusion of the lemma. ∎

We can now prove the desired estimate for the remaining term.

Lemma 6.3. *There holds*

$$\int_{\mathbb{R} \setminus \bigcup_{J \in \mathscr{J}} 6J} |\mathscr{L}(\{b_J\}_{J \in \mathscr{J}})| \lesssim \sum_{J \in \mathscr{J}} |J| \langle |b_J| \rangle_{B_{\tau/2}, J}.$$

Proof. For convenience, we set

$$L_{I,J} := P_I (\mathrm{Id} - \Psi_{I,J}), \quad F_{I,J} := \Delta_{L_I}(b_J)\mathbf{1}_{3J},$$

$$\mathcal{L}(\{b_J\}_{J\in\mathcal{J}}) = \sum_I \sum_{J:|J|\geq 2^{-k_I}} c_I L_{I,J}(F_{I,J}).$$

We immediately note that it will be enough to prove the desired estimate for a single b_J, and then sum the estimates. Furthermore, by translation and scale invariance, it will be enough to assume that $J = [-|J|/2, |J|/2]$; here we critically use that the operator $L_{I,J}$ depends only on the length and not on the position of J. The left-hand side in the conclusion of the lemma for a single such b_J can be estimated by

$$A_J := \int_{|x|\geq 3|J|} \Big| \sum_{I:\, 2^{k_I}\geq|J|^{-1}} c_I L_{I,J} F_{I,J} \Big| = \int_{|x|\geq 3|J|} \Big| \sum_{I:\, 2^{k_I}\geq|J|^{-1}} c_I (\check{\zeta}_{I,J} * F_{I,J})(x) \Big|\, dx$$

$$\lesssim |J|^{-1/2} \Big(\int_{|x|>3|J|} |x|^2 \Big| \sum_{I:\, 2^{k_I}\geq 1} c_I (\check{\zeta}_{I,J} * F_{I,J})(x) \Big|^2 dx \Big)^{1/2}.$$

Now let ρ be as before. It is then the case that for $|x| \geq 3$ and $|y| \leq 3/2$, we have

$$|x - y| \geq \frac{1}{2}|x| \geq \frac{3}{2}, \quad 1 - \rho\Big(\frac{x-y}{|J|}\Big) = 1 - \rho_J(x-y) = 1$$

for such pairs (x, y). As $F_{I,J}$ is supported in $[-3|J|/2, 3|J|/2]$, for all $|x| \geq 3|J|$, we have that

$$(\check{\zeta}_{I,J} * F_{I,J})(x) = \int_{[-3|J|/2,\, 3|J|/2]} F_{I,J}(y)\check{\zeta}_{I,J}(x-y)(1 - \rho_J(x-y)))\, dy.$$

Using this identity and setting $\check{\mu}_{I,J} := (1 - \rho_J)\check{\zeta}_{I,J}$, we get

$$A_J \lesssim \Big(\int_{\mathbb{R}} \Big| \sum_{I:\, 2^{k_I}\geq|J|^{-1}} c_I \partial_\xi[\hat{F}_{I,J}(\xi)\mu_{I,J}(\xi)] \Big|^2 d\xi \Big)^{1/2}.$$

Using the elementary estimates

$$\|\hat{F}_{I,J}\|_{L^\infty(\mathbb{R})} \leq \|F_{I,J}\|_{L^1(\mathbb{R})} \quad \text{and} \quad \|\partial_\xi \hat{F}_{I,J}\|_{L^\infty(\mathbb{R})} \leq |J|\|F_{I,J}\|_{L^1(\mathbb{R})},$$

together with the estimate of Lemma 6.2, for $\beta \in \{0, 1\}$, we get, for γ a large positive integer of our choice,

$$\partial_\xi^\beta[\mu_{I,J}(\xi)] \lesssim \frac{|J|^\beta}{(1 + |J|\,\mathrm{dist}(\xi, \mathbb{R}\setminus I))^\gamma} \lesssim |J|^\beta\, \mathrm{M}(\mathbf{1}_{I_{\mathrm{left}}(J)\cup I_{\mathrm{right}}(J)})^\gamma.$$

Hence, by using the Fefferman–Stein inequality, the Cauchy–Schwarz inequality, and the N-overlap assumption on the intervals I, we get

$$A_J \lesssim |J|^{-1/2} \Big(\sum_{2^{k_I}\geq|J|^{-1}} |c_I|^2 |J|^4 \langle|\Delta_{L_I}(b_J)|\rangle_{1,3J}^2 N|I_{\mathrm{left}}(J)\cup I_{\mathrm{right}}(J)| \Big)^{1/2}.$$

Here note that we use that $I_{\text{left}}(J) \cup I_{\text{right}}(J) \subsetneq I$, by construction, and hence

$$\sum_{2_I^k \geq |J|^{-1}} \mathbf{1}_{I_{\text{left}}(J) \cup I_{\text{right}}(J)} \leq \sum_I \mathbf{1}_I \leq N.$$

Further, using also the control on the ℓ^2-norm of the sequence $\{c_I\}_I$ yields

$$A_J \lesssim |J|^{-1/2} \Big(\sum_{L \in \Lambda_\tau^{|J|^{-1}}} |J|^3 \langle |\Delta_L(b_J)| \rangle_{1,3J}^2 \Big)^{1/2}$$

$$= |J| \Big(\sum_{L \in \Lambda_\tau^{|J|^{-1}}} \langle |\Delta_L(b_J)| \rangle_{1,3J}^2 \Big)^{1/2} \lesssim |J| \langle |b_J| \rangle_{B_{\tau/2},J}$$

by the generalized Zygmund–Bonami inequality of Proposition 4.1. This concludes the proof of the lemma. ∎

Using Lemmas 6.1 and 6.3, we complete the estimate for the term III and with that the proof of the endpoint bound of the theorem.

6.1.2. Optimality in Theorem B. We briefly comment on the optimality of the Young function $t \mapsto t(\log(e + t))^{\tau/2}$ in the upper bound of the theorem. Suppose that $r > 0$ is such that whenever T_m is an $R_{2,\tau}$ multiplier operator, then the bound of Theorem B holds with r in the place of τ. Since T_m is L^2-bounded, it follows by a Marcinkiewicz interpolation type of argument that the $L^p(\mathbb{R})$-bounds for the Littlewood–Paley square function LP_τ of order τ can be estimated by

$$\|\text{LP}_\tau\|_{p \to p} \lesssim \Big(\mathbf{E} \Big\| \sum_{L \in \Lambda_\tau} \varepsilon_L P_L \Big\|_{p \to p}^p \Big)^{1/p} \leq \sup_{\|m\|_{R_{2,\tau}}=1} \|T_m\|_{L^p \to L^p} \lesssim (p-1)^{-(r+1)}$$

as $p \to 1^+$, where the expectation in the display above is over independent choices of random signs $\{\varepsilon_L\}_L$. However, a modification of an example in [4], see Section 3 of [2], shows that the estimate in the display above does not hold for $r < \tau/2$. This argument also shows that our theorem implies that the L^p bounds for $R_{2,\tau}$ multipliers are $O(\max(p, p')^{1+\tau/2})$.

Alternatively, sharpness can also be obtained by adapting the corresponding argument in Section 3.2 of [17] to the higher order case. Let us briefly outline the second order case. For a smooth function ψ supported in $[-1/2, 1/2]$, with $\psi(0) = 1$, and $(k, l) \in \mathbb{Z}^2$, with $k > l$, we consider the multiplier $m_{(k,l)}$ given by

$$m_{(k,l)}(\xi) := m_0\Big(\frac{\xi - 2^k}{2^{l-1}} \Big), \quad \text{where } m_0(\xi) := \psi(\xi - 1)\mathbf{1}_{[1,\infty)}(\xi), \ \xi \in \mathbb{R}.$$

One then has

$$m_{(k,l)}^{\vee}(x) = \frac{e^{i2\pi 2^k x}}{i2\pi x} + O\Big(\frac{1}{2^l |x|} \Big), \quad |x| \gtrsim 2^{-l}.$$

For $N \in \mathbb{N}$, that will be eventually sent to infinity, we consider the second order ℓ^2-valued multiplier operator

$$T_N(g) := \{T_{m_{(k,l)}}(g)\}_{1 \leq l < k \leq N}.$$

Consider a smooth function f such that $\widehat{f}$ is supported in $[-4, 4]$ and $\widehat{f}(\xi) = 1$ for all $\xi \in [-2, 2]$. We then set

$$f_N(x) := 2^N f(2^N x), \quad x \in \mathbb{R}.$$

For $r > 0$, we have

$$\|T_N(f_N)(x)\|_{\ell^2} \gtrsim \frac{N}{|x|} \quad \text{if } |x| \geq 2^{-5N/8}$$

and

$$\int_{\mathbb{R}} \frac{|f_N(x)|}{\alpha} \left(\log\left(e + \frac{|f_N(x)|}{\alpha}\right)\right)^r \mathrm{d}x \lesssim \frac{1}{\alpha}\left(\log\left(e + \frac{2^N}{\alpha}\right)\right)^r \quad \text{for } \alpha > 0.$$

Hence, if we choose $\alpha = 2^{5N/8}$, then

$$\alpha^{-1}(\log(e + \alpha^{-1} 2^N))^r \simeq 2^{-5N/8} N^r$$

and

$$|\{x \in [-1/2, 1/2] : \|T_N(f_N)(x)\|_{\ell^2} > \alpha\}| \geq |\{2^{-5N/8} \leq x \leq 1/4 : N/|x| \gtrsim 2^{5N/8}\}|$$
$$\simeq N\alpha^{-1}.$$

To complete the proof, define $g_N := f_N \chi_{[1/2,1/2]}$ so that g_N is supported in $[-1/2, 1/2]$ and $\|g_N\|_{L \log^r L([-1/2,1/2])} \lesssim N^r$. Moreover, for all $1 \leq l \leq k \leq N$, one has

$$|T_{m_{k,l}}(f_N - g_N)(x)| \lesssim 2^{-2N} \quad \text{for all } x \in [2^{-5N/8}, 1/4],$$

and hence

$$\|\|T_N(g_N)\|_{\ell^2}\|_{L^{1,\infty}([-1/2,1/2])} \gtrsim N.$$

It follows from Khintchine's inequality that there exists a choice of signs $\varepsilon_{k,\ell}$, depending on g_N, such that

$$\left\|\sum_{1 \leq l < k \leq N} \varepsilon_{k,\ell} T_{m_{k,l}}(g_N)\right\|_{L^{1,\infty}([-1/2,1/2])} \gtrsim N,$$

from which it follows that $r \geq 1 = \tau/2$.

6.2. Proof of Theorems A and C

We begin by explaining the modifications needed in order to obtain a proof of the endpoint bounds in Theorems A and C.

6.2.1. Proof of Theorem A. Since Marcinkiewicz multipliers of order τ are contained in the class $R_{2,\tau}$, we only need to briefly discuss the conclusion of Theorem A for the Littlewood–Paley square function. Note that the proof of Theorem B relies on $L^2(\mathbb{R})$ estimates and L^1-type estimates. Then we can repeat the proof for the operator

$$|\mathrm{LP}_\tau f| \simeq \mathbf{E}\left|\sum_{L \in \Lambda_\tau} \varepsilon_L \mathrm{P}_L f\right| \simeq \left(\mathbf{E}\left|\sum_{L \in \Lambda_\tau} \varepsilon_L \mathrm{P}_L f\right|^2\right)^{1/2}$$

using the first approximate equality whenever L^1-estimates are needed, and the second one for the L^2-estimates. We omit the details. The optimality follows by the discussion in Section 6.1.2.

6.2.2. Proof of Theorem C. We proceed now to prove Theorem C concerning endpoint bounds for higher order Hörmander–Mihlin multipliers and smooth Littlewood–Paley square functions, which requires just small modifications compared to the proof of Theorem B. Consider a positive integer τ and $f \in L^{B_{\tau-1}}$; we apply the Calderón–Zygmund decomposition of Proposition 5.2 with $\sigma = \tau - 1$ at some fixed level $\alpha > 0$ to write $f = g + b_{\mathrm{canc},\tau-1} + b_{\mathrm{lac},\tau-1}$, and let $\mathcal{J}$ be the collection of stopping intervals. The good part $g + b_{\mathrm{lac},\tau-1}$ is estimated in L^2 by the L^2-bounds of the operator T_m, using that

$$\|g + b_{\mathrm{lac},\tau-1}\|_{L^2(\mathbb{R})}^2 \lesssim \alpha^2 \int_{\mathbb{R}} B_{(\tau-1)/2}\Big(\frac{|f|}{\alpha}\Big),$$

by the Calderón–Zygmund decomposition. As before, it remains to estimate the part of the operator acting on the cancellative atoms. We consider a partition of unity $\{\widetilde{\phi}_L\}_{L \in \Lambda_{\tau-1}}$ subordinated to the collection of Littlewood–Paley intervals $\Lambda_{\tau-1}$, with $\widetilde{\phi}_L \in \Phi_{L,M}$ for each L. We set

$$\widetilde{\Delta}_L(g) := (\widetilde{\phi}_L \widehat{g})^{\vee}, \quad \mathrm{Id} = \sum_{L \in \Lambda_\sigma} \widetilde{\Delta}_L.$$

With Δ_L the smooth Littlewood–Paley projections as fixed in Section 2.2, we have that $\Delta_L \widetilde{\Delta}_L = \widetilde{\Delta}_L$. We have thus the decomposition

$$\mathrm{T}_m = \sum_{L \in \Lambda_{\tau-1}} \mathrm{T}_m \widetilde{\Delta}_L =: \sum_{L \in \Lambda_{\tau-1}} \mathrm{T}_L = \sum_{L \in \Lambda_{\tau-1}} \mathrm{T}_L \Delta_L,$$

and let ζ_L denote the Fourier multiplier of the operator T_L. We then estimate

$$\mathrm{T}_m\Big(\sum_J b_J\Big) = \sum_{L \in \Lambda_{\tau-1}} \mathrm{T}_L\Big(\sum_{J:|J|\geq|L|^{-1}} \Delta_L(b_J)\mathbf{1}_{\mathbb{R}\setminus 3J}\Big) + \sum_{L \in \Lambda_{\tau-1}} \mathrm{T}_L\Big(\sum_{J:|J|<|L|^{-1}} \Delta_L(b_J)\Big)$$

$$+ \sum_{L \in \Lambda_{\tau-1}} \mathrm{T}_L\Big(\sum_{J:|J|\geq|L|^{-1}} \Delta_L(b_J)\mathbf{1}_{3J}\Big) =: \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

As in the proof of Theorem B, Remarks 4.3 and 4.5 take care of the terms I and II, respectively, by using L^2-bounds for each T_L and L^2-orthogonality for smooth Littlewood–Paley projections of order τ. Once again the main term is III.

We will split III into two parts, which are defined in the same way as the operators $\mathcal{E}$ and $\mathcal{L}$ from the proof of Theorem B, with the role of the interval I being replaced by an interval $L \in \Lambda_{\tau-1}$. For the first part, consider for each L nand J the function $\psi_{L,J}$ as defined before the proof of Lemma 6.1. Defining

$$\mathcal{E}(\{b_J\}_{J\in\mathcal{J}}) := \sum_{L\in\Lambda_{\tau-1}} \mathrm{T}_L\Big(\sum_{J:|J|\geq|L|^{-1}} \Psi_{L,J}(\Delta_L(b_J)\mathbf{1}_{3J})\Big),$$

and following the same steps as in the proof of Lemma 6.1, we get

$$|\{|\mathcal{E}(\{b_J\}_{J\in\mathcal{J}})| > \alpha\}| \lesssim \frac{1}{\alpha^2} \sum_{J\in\mathcal{J}} |J| \sum_{L\in\Lambda_{\tau-1}^{|J|^{-1}}} \langle|\Delta_L(b_J)|\rangle_{1,3J}^2 \lesssim \int_{\mathbb{R}} B_{(\tau-1)/2}\Big(\frac{|f|}{\lambda}\Big),$$

by the generalized Zygmund–Bonami inequality of Proposition 4.1 and the properties of the Calderón–Zygmund decomposition. It remains to deal with the operator

$$\mathscr{L}(\{b_J\}_{J \in \mathscr{J}}) := \sum_{L \in \Lambda_{\tau-1}} \sum_{|J| \geq |L|^{-1}} \mathrm{T}_L(\mathrm{Id} - \Psi_{L,J})(\Delta_L(b_J)\mathbf{1}_{3J}).$$

Letting $\zeta_{L,J}$ be the Fourier multiplier of the operator $\mathrm{T}_L(\mathrm{Id} - \Psi_{L,J})$ and ρ_J as in the statement of Lemma 6.2, we set $\tilde{\mu}_{L,J} := (1 - \rho_J)\check{\zeta}_{L,J}$. Lemma 6.2 for $I = L \in \Lambda_{\tau-1}$ yields the estimate

$$(6.2) \qquad |\partial_\xi^\beta \mu_{L,J}(\xi)| \lesssim |J|^\beta (1 + |J| \operatorname{dist}(\xi, \mathbb{R} \setminus L))^{-\gamma}, \quad \xi \in \mathbb{R}.$$

The proof for the operator $\mathscr{L}$ is then completed in the by now usual way. First, we have

$$|\{|\mathscr{L}(\{b_J\}_{J \in \mathscr{J}})| > \alpha\}| \leq \frac{1}{\alpha} \sum_{J \in \mathscr{J}} |J|^{-1/2} \Big\| \partial_\xi \Big(\sum_{L \in \Lambda_{\tau-1}^{|J|^{-1}}} \mu_{L,J} \widehat{F}_{L,J} \Big) \Big\|_{L^2(\mathbb{R})},$$

with $F_{L,J} := \widetilde{\Delta}_L(b_J)\mathbf{1}_{3J}$. Now (6.2) implies that

$$\Big| \partial_\xi \Big(\sum_{L \in \Lambda_\sigma^{|J|^{-1}}} \mu_{L,J} \widehat{F}_{L,J} \Big) \Big| \lesssim |J|^2 \mathrm{M}(\mathbf{1}_{L_J})^\gamma \langle |\widetilde{\Delta}_L(b_J)| \rangle_{1,3J},$$

where

$$L_J := [\xi_L + (10|J|)^{-1}, \xi_L + (5|J|)^{-1}] \subset L.$$

Thus, the estimates above together with the Fefferman–Stein inequality, the Cauchy-Schwarz inequality and the generalized Zygmund–Bonami inequality complete the estimate for the operator $\mathscr{L}$ and with that the upper bound of Theorem C for Hörmander–Mihlin multipliers of order τ. The proof for the smooth Littlewood–Paley square function of order τ follows the same randomization argument as the one used in the proof of Theorem A.

Finally, the optimality of the power $(\tau - 1)/2$ on the endpoint inequality can be checked by testing a local endpoint $L \log^r L \to L^{1,\infty}$ inequality for the smooth Littlewood–Paley square function of order τ on a smooth bump function supported in a small neighborhood of the origin. A routine calculation shows that necessarily $r \geq (\tau - 1)/2$. Note also that a local $L \log^r L \to L^{1,\infty}$ bound for Hörmander–Mihlin multipliers implies the corresponding endpoint square function estimate by a randomization argument as in [1].

References

[1] Bakas, O.: Endpoint mapping properties of the Littlewood–Paley square function. *Colloq. Math.* **157** (2019), no. 1, 1–15. Zbl 1421.42009 MR 3956881

[2] Bakas, O.: Sharp asymptotic estimates for a class of Littlewood–Paley operators. *Studia Math.* **260** (2021), no. 2, 195–206. Zbl 1473.42023 MR 4289669

[3] Bonami, A.: Étude des coefficients de Fourier des fonctions de $L^p(G)$. *Ann. Inst. Fourier (Grenoble)* **20** (1970), no. 2, 335–402 (1971). Zbl 0195.42501 MR 283496

[4] Bourgain, J.: On the behavior of the constant in the Littlewood–Paley inequality. In *Geometric aspects of functional analysis (1987–88)*, pp. 202–208. Lecture Notes in Math. 1376, Springer, Berlin, 1989. Zbl 0706.42008 MR 1008724

[5] Chang, S.-Y. A., Wilson, J. M. and Wolff, T. H.: Some weighted norm inequalities concerning the Schrödinger operators. *Comment. Math. Helv.* **60** (1985), no. 2, 217–246. Zbl 0575.42025 MR 800004

[6] Coifman, R., Rubio de Francia, J. L. and Semmes, S.: Multiplicateurs de Fourier de $L^p(\mathbb{R})$ et estimations quadratiques. *C. R. Acad. Sci. Paris Sér. I Math.* **306** (1988), no. 8, 351–354. Zbl 0643.42010 MR 934617

[7] Cruz-Uribe, D. V., Martell, J. M. and Pérez, C.: *Weights, extrapolation and the theory of Rubio de Francia*. Oper. Theory Adv. Appl. 215, Birkhäuser/Springer Basel AG, Basel, 2011. Zbl 1234.46003 MR 2797562

[8] Duoandikoetxea, J.: *Fourier analysis*. Grad. Stud. Math. 29, American Mathematical Society, Providence, RI, 2001. Zbl 0969.42001 MR 1800316

[9] Gaudry, G. I.: Littlewood–Paley theorems for sum and difference sets. *Math. Proc. Cambridge Philos. Soc.* **83** (1978), no. 1, 65–71. Zbl 0368.05010 MR 470576

[10] Lerner, A. K.: Quantitative weighted estimates for the Littlewood–Paley square function and Marcinkiewicz multipliers. *Math. Res. Lett.* **26** (2019), no. 2, 537–556. Zbl 1428.42029 MR 3999554

[11] Marcinkiewicz, J.: Sur les multiplicateurs des séries de Fourier. *Stud. Math.* **8** (1939), 78–91. Zbl 0020.35403

[12] Oswald, P.: On some convergence properties of Haar–Fourier series in the classes $\varphi(L)$. *Acta Math. Hungar.* **42** (1983), no. 3-4, 279–293. Zbl 0533.42019 MR 722843

[13] Rudin, W.: *Functional analysis*. Second edition. International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1991. Zbl 0867.46001 MR 1157815

[14] Seeger, A. and Trebels, W.: Low regularity classes and entropy numbers. *Arch. Math. (Basel)* **92** (2009), no. 2, 147–157. Zbl 1179.46032 MR 2481510

[15] Sjögren, P. and Sjölin, P.: Littlewood–Paley decompositions and Fourier multipliers with singularities on certain sets. *Ann. Inst. Fourier (Grenoble)* **31** (1981), no. 1, vii, 157–175. Zbl 0437.42011 MR 613033

[16] Tao, T.: A converse extrapolation theorem for translation-invariant operators. *J. Funct. Anal.* **180** (2001), no. 1, 1–10. Zbl 0987.47021 MR 1814420

[17] Tao, T. and Wright, J.: Endpoint multiplier theorems of Marcinkiewicz type. *Rev. Mat. Iberoamericana* **17** (2001), no. 3, 521–558. Zbl 1014.42010 MR 1900894

[18] Triebel, H.: Approximation numbers and entropy numbers of embeddings of fractional Besov–Sobolev spaces in Orlicz spaces. *Proc. London Math. Soc. (3)* **66** (1993), no. 3, 589–618. Zbl 0792.46024 MR 1207550

[19] Vitturi, M.: A Chang–Wilson–Wolff inequality using a lemma of Tao–Wright. *Almost Originality blog*, 2019, https://almostoriginality.wordpress.com/2019/11/21/the-chang-wilson-wolff-inequality-using-a-lemma-of-tao-wright/, visited on 30 November 2024.

[20] Wilson, M.: *Weighted Littlewood–Paley theory and exponential-square integrability*. Lecture Notes in Math. 1924, Springer, Berlin, 2008. Zbl 1138.42011 MR 2359017

[21] Zygmund, A.: On lacunary trigonometric series. *Trans. Amer. Math. Soc.* **34** (1932), no. 3, 435–446. Zbl 0005.06303 MR 1501647

[22] Zygmund, A.: *Trigonometric series. Vol. I, II*. Third edition. Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002. Zbl 1084.42003 MR 1963498

Odysseas Bakas
Department of Mathematics, University of Patras
Panepistimioupolis Rio Patras, 26504 Patras, Greece;
obakas@upatras.gr

Valentina Ciccone
Hausdorff Center for Mathematics, Universität Bonn
Endenicher Allee 60, 53115 Bonn, Germany;
ciccone@math.uni-bonn.de

Ioannis Parissis
Departamento de Matemáticas, Universidad del País Vasco
Aptdo. 644, 48080 Bilbao;
Ikerbasque, Basque Foundation for Science
Bilbao, Spain;
ioannis.parissis@ehu.es

Marco Vitturi
Department of Mathematics, Munster Technological University
Bishopstown, Cork, Ireland;
marco.vitturi@mtu.ie