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The Riemannian and symplectic geometry of the space of generalized Kähler structures

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Abstract. On a compact complex manifold (M, J) endowed with a holomorphic Poisson tensor π_J and a de Rham class $\alpha \in H^2(M, \mathbb{R})$, we study the space of generalized Kähler (GK) structures defined by a symplectic form $F \in \alpha$ and whose holomorphic Poisson tensor is π_J. We define a notion of generalized Kähler class of such structures, and use the moment map framework of Boulanger (2019) and Goto (2020) to extend the Calabi program to GK geometry. We obtain generalizations of the Futaki–Mabuchi extremal vector field (1995) and the Calabi–Lichnerowicz–Matsushima result (1982, 1958, 1957) for the Lie algebra of the group of automorphisms of (M, J, π_J). We define a closed 1-form on a GK class, which yields a generalization of the Mabuchi energy and thus a variational characterization of GK structures of constant scalar curvature. Next we introduce a formal Riemannian metric on a given GK class, generalizing the fundamental construction of Mabuchi–Semmes–Donaldson (1987, 1992, 1997) We show that this metric has nonpositive sectional curvature, and that the Mabuchi energy is convex along geodesics, leading to a conditional uniqueness result for constant scalar curvature GK structures. We finally examine the toric case, proving the uniqueness of extremal generalized Kähler structures and showing that their existence is obstructed by the uniform relative K-stability of the corresponding Delzant polytope. Using the resolution of the Yau–Tian–Donaldson conjecture in the toric case by Chen–Cheng (2021) and He (2019), we show in some settings that this condition suffices for existence and thus construct new examples.

1. Introduction

E. Calabi [11] initiated a far-reaching program of finding, on a given compact Kähler manifold (M, J), a canonical representative of the space $\mathcal{K}_\alpha$ of Kähler metrics that belong to a fixed de Rham class $\alpha \in H^2(M, \mathbb{R})$. He proposed as a candidate of such representative the notion of *extremal Kähler metric*, i.e., one whose scalar curvature Scal_ω defines a Hamiltonian vector field $\chi = -\omega^{-1}(d\,\mathrm{Scal}_\omega)$ satisfying $\mathcal{L}_\chi J = 0$. This problem unifies the existence problems for constant scalar curvature (csc) and

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Kähler–Einstein metrics and represents one of the most active areas of research in Kähler geometry during the last half-century. The central conjecture in the field, still open in full generality, is the *Yau–Tian–Donaldson* (YTD) conjecture. It states, broadly speaking, that the full obstruction for $\mathcal{K}_\alpha$ to admit an extremal Kähler metric can be expressed in terms of a complex-algebraic notion of stability of (M, J, α) [21, 62, 63, 68]. This correspondence, if established, will have further deep implications for the definition of well-behaved moduli spaces of Kähler manifolds [19, 25].

Only two years after Calabi's seminal paper appeared, an extension of Kähler geometry emerged from studies in $(2, 2)$ supersymmetric quantum field theory in physics [28]. These geometric structures were later rediscovered, and given the name of *generalized Kähler (GK) structures*, in the context of Hitchin's generalized geometry program [38, 40, 46]. In the ensuing decades it has become clear that GK geometry is a deeply structured extension of Kähler geometry with novel implications for complex, symplectic and Poisson geometry. In this paper we motivate a natural extension of the Calabi program to the setting of GK structures compatible with a given holomorphic Poisson tensor π_J and a de Rham class $\alpha \in H^2(M, \mathbb{R})$.

1.1. Generalized Kähler structures of symplectic type

To begin we first recall the classical *biHermitian* definition of generalized Kähler structure [28]. Here a GK structure consists of a quadruple (g, b, I, J) of a Riemannian metric g with compatible integrable complex structures I, J, and a 2-form b, satisfying

$$d_I^c \omega_I = db = -d_J^c \omega_J,$$

where $\omega_I = gI$, $\omega_J = gJ$ are the fundamental 2-forms of (g, I) and (g, J) and $d_I^c = \sqrt{-1}(\bar{\partial}_I - \partial_I)$. The case when $I = J$ and $b = 0$ gives rise to a Kähler metric (g, J). A key point, observed by Hitchin [47], is that for any GK structure the tensor

$$\pi := \frac{1}{2}[I, J]g^{-1}$$

is a bivector which defines a real Poisson structure whereas the complex bivectors

$$\pi_J := \pi - \sqrt{-1}J\pi, \quad \pi_I := \pi - \sqrt{-1}I\pi$$

are *holomorphic Poisson structures* defined respectively on (M, J) and (M, I). We will be interested in the special case of GK structures of *symplectic type*. In this setting we fix a complex manifold (M, J) and consider a symplectic form F on M which *tames* J, i.e., such that

$$-FJ = g + b,$$

where g is a Riemannian metric and b is a 2-form on M. Given this, we define a second g-compatible almost complex structure by

$$I = -F^{-1} J^* F.$$

If I is also integrable, then (g, b, I, J) is GK as defined above. Note that for a fixed complex structure J, the entire quadruple (g, b, I, J) is encoded by F, while the integrability of I places a further nonlinear, first-order differential condition on F. We denote by $\mathcal{GK}_{\pi,\alpha}$ the space of such symplectic-type generalized Kähler structures, compatible with the holomorphic Poisson manifold (M, J, π_J) and satisfying $[F] = \alpha$. In the case when $\pi_J = 0$, the space $\mathcal{GK}_{0,\alpha}$ is just the space $\mathcal{K}_\alpha$ of Kähler metrics in a fixed cohomology class.

A fundamental issue in extending the Calabi program to this setting is the nonlinear structure of $\mathcal{GK}_{\pi,\alpha}$. In the Kähler case, using the dd^c_J-lemma, the space $\mathcal{K}_\alpha$ is a convex-linear, Fréchet manifold modeled on $C^\infty(M, \mathbb{R})/\mathbb{R}$. Such a description is no longer globally possible in the GK case due to the integrability condition on I (cf. [9, 55] for results on *local* generalized Kähler potentials). Nonetheless, a natural notion of *generalized Kähler class* has now emerged [9, 31, 41], which here consists of GK structures defined by deforming an element $F_0 \in \mathcal{GK}_{\pi,\alpha}$ by a smooth path of functions $\phi_t \in C^\infty(M, \mathbb{R})/\mathbb{R}$ in the following nonlinear way [41]:

$$F_{\phi_t} := F_0 + \int_0^t dd^c_{I_s} \phi_s \, ds, \quad I_s := (\Phi_s)_* I_0 (\Phi_s)_*^{-1},$$

where Φ_t is the isotopy of diffeomorphisms corresponding to the time dependent vector field $-\pi(d\phi_t)$. It turns out that if F_{ϕ_t} tames J (an open condition), then (F_{ϕ_t}, J) gives rise to an element of $\mathcal{GK}_{\pi,\alpha}$ with I_t as defined above. In Proposition 2.17 we prove that any GK structure $F \in \mathcal{GK}_{\pi,\alpha}$ in the C^∞ path-connected component of F_0 can be obtained by the above deformation. In what follows, we denote this path-connected component by $\mathcal{GK}^0_{\pi,\alpha}(F_0)$, or simply $\mathcal{GK}^0_{\pi,\alpha}$ if the underlying base point is clear. Thus the space $\mathcal{GK}^0_{\pi,\alpha}$ will be referred to as a *generalized Kähler class* and we show in Lemma 2.16 that $\mathcal{GK}^0_{\pi,\alpha}$ is an integrable (formal) submanifold of a distribution of vector fields on a formal Fréchet manifold. Our point of view in this paper is that $\mathcal{GK}^0_{\pi,\alpha}$ is the right substitute of the Kähler class $\mathcal{K}_\alpha$ in the symplectic-type generalized Kähler setting.

1.2. Scalar curvature as moment map

The second problem of extending the Calabi problem to the space $\mathcal{GK}^0_{\pi,\alpha}$ stems from the fact that in the generalized Kähler setting, there is no connection preserving all the structure, and thus there is no obvious way to define a scalar curvature. Such a definition was first proposed by Boulanger [10], building on an unpublished work of Gauduchon [30]. It uses the approach developed by Fujiki [23] and Donaldson [20] in the Kähler case, who recast the Calabi program as a formal GIT problem of finding zeros of a momentum map. In their set up, the "manifold" is the space $\mathcal{AK}_{\omega_0}$ (a formal Fréchet manifold) of all almost complex structures J on M compatible with a fixed symplectic form ω_0, acted upon by the group $\mathrm{Ham}(M, \omega_0)$ of ω_0-Hamiltonian diffeomorphisms. It turns out that $\mathcal{AK}_{\omega_0}$ admits a formal Kähler structure $(\boldsymbol{\Omega}, \boldsymbol{J})$ such that $\mathrm{Ham}(M, \omega_0)$ acts in a Hamiltonian way with momentum map identified, at any integrable almost complex structure $J \in \mathcal{AK}_{\omega_0}$, with

$$\langle \boldsymbol{\mu}(J), f \rangle = -\int_M \mathrm{Scal}_{(\omega_0, J)} f \, dV_{\omega_0}, \quad f \in C^\infty(M, \mathbb{R}), \int_M f \, dV_{\omega_0} = 0,$$

where $\mathrm{Scal}_{(\omega_0,J)}$ is the scalar curvature of the corresponding Kähler structure (ω_0, J) and the Lie algebra of $\mathrm{Ham}(M, \omega_0)$ is identified with the vector space of zero mean smooth functions endowed with the Poisson bracket with respect to ω_0. Using Moser's lemma, $\mathcal{K}_{[\omega_0]}$ can be mapped to a subset of $\mathcal{AK}_{\omega_0}$. As observed in [20], the image of $\mathcal{K}_{[\omega_0]}$ belongs to the *complexified orbit* of $\mathrm{Ham}(M, \omega_0)$ and is transversal to the orbits of $\mathrm{Ham}(M, \omega_0)$ inside this complexified orbit. Thus, the Calabi problem becomes the familiar GIT problem of finding zeros J of $\boldsymbol{\mu}$ in a given complexified orbit of the group action.

In [10], the Donaldson–Fujiki setting is extended to the space $\mathcal{AGK}_{F_0}$ of *almost generalized Kähler structures of symplectic type* compatible with a fixed symplectic form F_0, i.e., the space of all almost complex structures J on M such that F_0 *tames* J. It is shown in [10] that $\mathcal{AGK}_{F_0}$ admits a formal Kähler structure such that $\mathcal{AK}_{F_0}$ is a formal Kähler submanifold; when $(M, F_0, \mathbb{T})$ is a compact toric manifold, Boulanger also proves that the equivariant part $\mathrm{Ham}^{\mathbb{T}}(M, F_0)$ acts in a Hamiltonian way on the invariant part $\mathcal{AGK}^{\mathbb{T}}_{F_0}$. Furthermore, in the 4-dimensional toric case, Boulanger identifies the corresponding momentum map $\boldsymbol{\mu}(J)$ for certain generalized Kähler structures $J \in \mathcal{AGK}^{\mathbb{T}}_{F_0}$ with the smooth function

$$\mathrm{Gscal}_{(F_0,J)} := \mathrm{Scal}_g - \frac{1}{12}|db|^2_g + 2\Delta_g \Psi - |d\Psi|^2_g, \quad \Psi = -\log \frac{dV_{F_0}}{dV_g}, \tag{1.1}$$

where Scal_g denotes the scalar curvature of g, $\Delta_g = -d^*d$ is the corresponding Laplace operator, and dV_{F_0} and dV_g are the volume forms associated to F_0 and g, respectively. Surprisingly, this function precisely corresponds to the density of the string effective action for $H = db$ (see [59]), and is thus a natural candidate for scalar curvature on physical grounds.

The crucial insight allowing for the extension of Donaldson–Fujiki approach from the set of *compatible* almost complex structures $\mathcal{AK}_{F_0}$ to the set of *tame* almost complex structures $\mathcal{AGK}_{F_0}$ comes from generalized geometry. It turns out that every $J \in \mathcal{AGK}_{F_0}$ gives rise to a *generalized almost complex structure* on $TM \oplus T^*M$

(see Proposition 2.4). This observation allows to identify the set of *linear* almost complex structures tamed by $(F_0)_x, x \in M$ with the noncompact symmetric space

$$S = U(n,n)/U(n) \times U(n).$$

Thus the Fréchet manifold $\mathcal{AGK}_{F_0}$ induces the formal Kähler structure from the one on S. This idea was used by Goto [34] who generalized the work of [10] and established without any restrictions on (M, F_0) that $\mathrm{Ham}(M, F_0)$ acts in a Hamiltonian way on $\mathcal{AGK}_{F_0}$. The associated moment map was taken to be the definition of scalar curvature, and Goto furthermore gave an expression for this curvature in terms of the underlying generalized complex structures $(\mathbb{J}, \mathbb{I})$ on $TM \oplus T^*M$, which are associated to the biHermitian data (g, b, I, J) via Gualtieri's map [38]. Despite this formula, there is no straightforward way to express Goto's scalar curvature in terms of the underlying biHermitian geometry, as it relies on local sections of the underlying generalized canonical bundles. Our first main result resolves the apparent ambiguities between the different approaches to scalar curvature and confirms that the moment map of [34] in the general integrable case is given by (1.1).

Theorem 1.1 (Theorem 3.2). *The Goto moment map $\boldsymbol{\mu}$ computed at a generalized Kähler structure J is given by the formula* (1.1).

The proof relies on the local nondegenerate approximation technique introduced in [5]. We thus refer to the function $\mathrm{Gscal}_{(F,J)}$ associated to a symplectic-type generalized Kähler structure (F, J) via (1.1) as the *generalized Kähler scalar curvature* of (F, J).

1.3. Variational formulations and obstructions

Based on the discussion of the previous subsection, Goto's moment map suggests natural generalizations of the extremal and csc metrics in Kähler geometry. In particular, a generalized Kähler structure for which the vector field $\chi := -F^{-1}(d\,\mathrm{Gscal}_{(F,J)})$ preserves J will be called *extremal*, and a generalized Kähler structure for which $\mathrm{Gscal}(F, J) = \mathrm{const}$ will be called a *cscGK* structure. As an initial fundamental step in understanding the existence and uniqueness of extremal GK structures, we show in Lemma 4.1 that the space $\mathcal{GK}^0_{\pi,\alpha}$ is mapped via Moser's lemma into a formal "complexified orbit" for the action of $\mathrm{Ham}(M, F_0)$ on $\mathcal{AGK}_{F_0}$, being transversal to the $\mathrm{Ham}(M, F_0)$-orbits inside it. This is precisely as in the description [20] in the Kähler setting. Thus we have the familiar GIT setup and we naturally arrive at the following.

Generalized Kähler Calabi program. *Express the obstructions to the existence of extremal generalized Kähler structures in $\mathcal{GK}^0_{\pi,\alpha}$ in the form of a complex algebraic notion of stability of (M, J, π_J, α). When they exist show that such structures are* *unique up to the action of the connected component of the identity* $\mathrm{Aut}_0(J,\pi_J)$ *of the automorphism group* $\mathrm{Aut}(J,\pi_J)$ *of* (M,J,π_J).

As a direct consequence of the GIT framework we provide an alternative characterization of extremal metrics as critical points of a Calabi functional **Ca** on $\mathcal{GK}_{\pi,\alpha}$, see Definition 4.3 and Proposition 4.4. Next, in Section 4.3, we introduce a proxy for the Mabuchi energy by defining a closed 1-form $\boldsymbol{\tau}$ on $\mathcal{GK}_{\pi,\alpha}$ which vanishes at cscGK structures. A primitive of this 1-form would provide an analogue of Mabuchi's K-energy, which is a central object in the YTD conjecture concerning the Kähler Calabi problem.

Next we prove a structure theorem for the complex Lie group $\mathrm{Aut}(J,\pi_J)$ of automorphisms of the complex Poisson manifold (M,J,π_J), similar to the well-known results by Matsushima [58] in the Kähler–Einstein case, Lichnerowicz [54] in the constant scalar curvature Kähler case, and Calabi [12] in the extremal Kähler case. Such results are obtained by Goto [36] for the Lie algebra $\mathfrak{g}$ of reduced automorphisms of one of the generalized complex structures associated to a symplectic-type cscGK structure. In the special case when the first Betti number of M is zero, and (F,J) is a small Poisson deformation of a Kähler structure, Goto proved that $\mathfrak{g}_0$ is the Lie algebra of holomorphic vector fields preserving the underlying Poisson tensor. Compared to this, our result is obtained as a direct corollary of our formal GIT framework, which has the advantages of treating immediately the case of extremal GK structures, and being directly formulated in terms of the complex Poisson manifold (M,J,π_J) and its *reduced* automorphism group $\mathrm{Aut}_{\mathrm{red}}(J,\pi_J)$ (see Definitions 4.9 and 4.12) without any extra technical assumptions, thus sharpening [36, Theorems 6.5 and 9.11].

Theorem 1.2 (Calabi–Lichnerowicz–Matsushima obstruction, Theorem 4.13). *Suppose* $F\in\mathcal{GK}_{\pi,\alpha}$ *is an extremal generalized Kähler structure with the holomorphic extremal vector field*

$$\chi=-F^{-1}\big(d\,\mathrm{Gscal}_{(F,J)}\big).$$

Denote by $\mathrm{Aut}_{\mathrm{red}}(J,\pi_J)^\chi\subset\mathrm{Aut}_{\mathrm{red}}(J,\pi_J)$ *the connected subgroup preserving* χ. *Then the group*

$$K:=\big(\mathrm{Aut}_{\mathrm{red}}(J,\pi_J)\cap\mathrm{Ham}(M,F)\big)_0$$

is a maximal compact subgroup of $\mathrm{Aut}_{\mathrm{red}}(J,\pi_J)$ *and* $\mathrm{Aut}_{\mathrm{red}}(J,\pi_J)^\chi=K^{\mathbb{C}}$. *In particular,* (F,J) *must be invariant under a maximal real torus in* $\mathrm{Aut}_{\mathrm{red}}(J,\pi_J)$ *containing the one-parameter subgroup* $\exp(t\chi)$. *If* F *is cscGK, then*

$$\chi=0\quad\textit{and}\quad\mathrm{Aut}_{\mathrm{red}}(J,\pi_J)=K^{\mathbb{C}}.$$

In Section 4.5 we provide an intrinsic description of the extremal vector field of a potential extremal GK structure in $\mathcal{GK}_{\pi,\alpha}$. It provides an efficient obstruction for the existence of cscGK metrics in a given GK class and is instrumental in our treatment of the toric case. In the Kähler case this description was obtained by Futaki–Mabuchi [24].

Theorem 1.3 (Extremal vector field, Theorem 4.17). *Given a torus $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(J, \pi_J)$ let $\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ be the space of $\mathbb{T}$-invariant generalized Kähler structures. Then, for any $F_0 \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$, $\mathbb{T} \subset \mathrm{Ham}(M, F_0)$ and we denote by $\Pi_{F_0}: C^\infty(M,\mathbb{R}) \mapsto C^\infty(M,\mathbb{R})$ the $L^2(M, dV_F)$ projection onto the space of F_0-Hamiltonians of $\mathbb{T}$. Moreover, the vector field*

$$\chi = -F_0^{-1}\big(d\Pi_{F_0}(\mathrm{Gscal}_{(F_0,J)})\big) \tag{1.2}$$

is independent of the choice of a $\mathbb{T}$-invariant symplectic form $F_0 \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$. If, furthermore, $\mathbb{T}$ is a maximal torus in $\mathrm{Aut}_{\mathrm{red}}(J, \pi_J)$, then $F \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ is an extremal structure if and only if

$$\mathrm{Gscal}_{(F,J)} - \Pi_F\big(\mathrm{Gscal}_{(F,J)}\big) = 0.$$

In particular, the underlying extremal vector field $-F^{-1}(d\,\mathrm{Gscal}_{(F,J)})$ is necessarily given by (1.2).

Using the Mabuchi 1-form $\boldsymbol{\tau}$ on $\mathcal{GK}^0_{\pi,\alpha}$, we define a generalized Kähler analogue of the classical obstruction for the existence of cscK metrics in Kähler geometry – the Futaki character of the Lie algebra $\mathfrak{h}_{\mathrm{red}}(J, \pi_J)$ of $\mathrm{Aut}_{\mathrm{red}}(J, \pi_J)$.

Theorem 1.4 (Futaki character, Theorem 4.18). *Let $F \in \mathcal{GK}_{\pi,\alpha}$ be a generalized Kähler structure on (M, J, π_J). Define a linear map $\mathcal{F}_{(F,J)}: \mathfrak{h}_{\mathrm{red}}(J, \pi_J) \to \mathbb{R}$ by*

$$\mathcal{F}_{(F,J)}(X) = \int_M \psi\, \mathrm{Gscal}_{(F,J)}\, dV_F, \quad X = F^{-1}(d\phi + I d\psi).$$

Then $\mathcal{F}_{(F,J)}$ is independent of $F \in \mathcal{GK}^0_{\pi,\alpha}$ and vanishes on the commutator

$$[\mathfrak{h}_{\mathrm{red}}(J, \pi_J), \mathfrak{h}_{\mathrm{red}}(J, \pi_J)].$$

In particular, $\mathcal{F}_{(F,J)}$ is a character of $\mathfrak{h}_{\mathrm{red}}(J, \pi_J)$ and is identically zero if $\mathcal{GK}^0_{\pi,\alpha}$ admits a cscGK metric.

Remark 1.5. A conceptually distinct notion of Futaki invariant has previously appeared in the context of generalized geometry, providing obstructions to the existence of solutions of the Hull–Strominger system [26].

1.4. Formal metric structure and uniqueness

In the Kähler setting, the GIT framework is naturally completed by a formal Riemannian metric, known as the Mabuchi–Semmes–Donaldson metric [20, 56, 60]. This metric formally gives $\mathcal{K}_\alpha$ the structure of a symmetric space of nonpositive curvature.

Furthermore, the Mabuchi K-energy, whose critical points are the constant scalar curvature Kähler metrics, is convex along geodesics of $\mathcal{K}_\alpha$, a key point leading to uniqueness of csc Kähler metrics [8, 15, 17]. We show that this formal picture extends to the symplectic-type GK setting.

To begin, the tangent space $\mathbf{T}_F(\mathcal{GK}_{\pi,\alpha})$ of $\mathcal{GK}_{\pi,\alpha}$ at a point F can be identified with

$$\mathbf{T}_F(\mathcal{GK}^0_{\pi,\alpha}) \simeq C^\infty(M,\mathbb{R})/\mathbb{R}.$$

We define a formal Riemannian metric at a point $F \in \mathcal{GK}_{\pi,\alpha}$ by

$$\langle\!\langle \phi_1, \phi_2 \rangle\!\rangle_F := \int_M \phi_1\phi_2 \frac{F^n}{n!}, \quad \phi_i \in C^\infty_0(M, dV_F),\ i = 1, 2,$$

where $C^\infty_0(M, dV_F)$ is the space of functions with zero average against the symplectic volume form dV_F. We first give a generalization of the formal symmetric space structure established by Mabuchi in the Kähler case.

Theorem 1.6 (Theorem 5.6). *The curvature tensor $\mathcal{R}$ of $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ at a point $F \in \mathcal{GK}_{\pi,\alpha}$ is given by*

$$\big(\mathcal{R}_{[\phi_1],[\phi_2]}[\phi_3]\big)_F = -\big\{\{\phi_1, \phi_2\}_F, \phi_3\big\}_F,$$

where $\{\cdot, \cdot\}_F$ denotes the Poisson bracket of functions with respect to the symplectic form F and $[\phi_i] \in C^\infty(M,\mathbb{R})/\mathbb{R}$, $i = 1, 2, 3$ are tangent vectors in $\mathbf{T}_F(\mathcal{GK}^0_{\pi,\alpha})$. In particular, the sectional curvature of $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is everywhere nonpositive.

The formal Riemannian metric naturally determines a notion of geodesic. These are smooth curves F_t in $\mathcal{GK}^0_{\pi,\alpha}$ for which the corresponding velocity

$$\phi_t \in C^\infty_0(M, dV_{F_t}) \cong T_{F_t}\big((\mathcal{GK}^0_{\pi,\alpha})\big)$$

satisfies

$$\dot\phi_t = \operatorname{tr}_{F_t}\, d\phi_t \wedge J\, d\phi_t.$$

In the Kähler case, letting $F_t = F_0 + dJ\, d\psi_t$ we have $\dot\psi_t = \phi_t$ and one recovers the familiar Mabuchi geodesic equation for ψ_t. We observe that a special class of geodesics are generated by the flow $\exp(-tJY)$ of any F_0-Hamiltonian Killing vector field Y of (F_0, J) (cf. Proposition 5.5). Analogous to the geodesic convexity of the Mabuchi energy in the Kähler setting, we show that the Mabuchi 1-form $\boldsymbol{\tau}$ increases along a smooth geodesic in $\mathcal{GK}_{\pi,\alpha}$, and is identically zero along a geodesic precisely when the latter is induced by a Hamiltonian Killing field as above. This leads to a formal proof of uniqueness.

Corollary 1.7 (Corollary 5.8). *Suppose $F_0, F_1 \in \mathcal{GK}_{\pi,\alpha}$ are cscGK structures connected by a smooth geodesic F_t. Then there exists $Y \in \mathfrak{h}_{\mathrm{red}}(J, \pi_J)$ such that $F_t = \Phi_t^* F_0$, where $\Phi_t = \exp(-tJY) \in \mathrm{Aut}_{\mathrm{red}}(J, \pi_J)$.*

Turning this into a genuine proof of uniqueness requires developing the theory of geodesics in $\mathcal{GK}_{\pi,\alpha}$. Such curves do not readily reduce to a Monge–Ampère equation, as exploited in the construction of (weak) geodesics in [15]. However, there is a generalization of the Semmes construction [60] expressing the geodesic equation as a prescribed volume form problem for a natural family of symplectic forms on an augmented spacetime track (cf. Remark 5.4).

1.5. The toric case

In the final portion of the paper we consider the case when $(M, J, \mathbb{T}^{\mathbb{C}})$ is a smooth projective toric variety and π_J a $\mathbb{T}^{\mathbb{C}}$-invariant Poisson tensor. In this case, the $\mathbb{T}$-invariant symplectic-type generalized Kähler structures have been studied in [7, 67] where they were described in terms of a smooth convex function defined on the interior of the Delzant polytope of $(M, F, \mathbb{T})$ and a bivector $\pi_J \in \wedge^2(\mathrm{Lie}(\mathbb{T}^{\mathbb{C}}))$. This is analogous to the Abreu–Guillemin description [1, 43] of toric Kähler structures. Building on this theory, we are able to obtain an almost complete picture for the generalized Kähler Calabi problem on a toric variety.

Theorem 1.8. *Suppose $(M, J, \pi_J, \mathbb{T}^{\mathbb{C}})$ is a toric projective holomorphic Poisson manifold and F_0 a $\mathbb{T}$-invariant π_J-compatible symplectic-type generalized Kähler structure in the de Rham class α. Then*

(a) (Corollary 6.1) *any extremal generalized Kähler structure in $\mathcal{GK}_{\pi,\alpha}$ is isometric to a $\mathbb{T}$-invariant such structure;*

(b) (Proposition 6.5 and Theorem 6.11) *the space of $\mathbb{T}$-invariant generalized Kähler structures compatible with π and $\alpha = [F_0]$ is path connected:*

$$(\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha})^0 = \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}.$$

Furthermore, any two extremal generalized Kähler structures in $\mathcal{GK}^0_{\pi,\alpha}$ are isometric by an element of $\mathrm{Aut}_0(J, \pi_J)$*;*

(c) (Theorem 6.13) *if $\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ admits an extremal generalized Kähler structure then the Delzant polytope of α is uniform relative K-stable;*

(d) (Theorems 6.16 and 6.19) *if the Delzant polytope of $(M, J, \alpha, \mathbb{T}^{\mathbb{C}})$ is uniform relative K-stable, and $\pi_J \in \wedge^2(\mathrm{Lie}(\mathbb{T}^C))$, then there exists an $\varepsilon > 0$, such that $\mathcal{GK}_{t\pi,\alpha}$ admits an extremal generalized Kähler metric for all $t \in \mathbb{R}$, $|t| < \varepsilon$.*

Statement (a) follows formally from the Matsushima-type obstruction of Theorem 4.13. The path-connectedness in (b) follows from the Abreu–Guillemin description of toric generalized Kähler structures obtained in [10, 67] and reviewed in Section 6 below. The uniqueness relies on establishing the smooth geodesic convexity of $\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$. In the toric Kähler case, this is due to Guan [42] who showed that the geodesics of $\mathcal{K}_\alpha^{\mathbb{T}}$ become linear segments in the Abreu–Guillemin description. In Proposition 6.6 we show this is still the case in the generalized Kähler case, but only for a special class of Poisson tensors π_J. To obtain the full claim, we argue in Corollary 6.8 that the existence problem for an extremal generalized Kähler structure can be reduced to this subclass of Poisson tensors. The statement in (c) makes a direct connection with the YTD conjecture which is now established for toric varieties due to the work of X. X. Chen and J. Cheng [16] in the constant scalar curvature case and its extension by W. He [44] to the extremal case. The condition of uniform relative K-stability of the polytope was introduced by Donaldson [21] and is now known [45] that it is equivalent to the uniform relative K-stability on toric test configurations of the corresponding polarized toric variety. Our proof of (c) adapts the original arguments [14] in the toric Kähler case to the generalized Kähler context, by replacing the hessian of the Kähler potential with a suitable positive-definite smooth symmetric-matrix valued function on the Delzant polytope, coming from the Abreu–Guillemin description. We also notice that (c) provides many constructible examples of toric varieties which do not admit extremal generalized Kähler structures. The final point (d) follows from the existence in α of a toric extremal Kähler metric [16, 44] and an adaptation of the LeBrun–Simanca openness result [50] where instead of varying the Kähler class we vary the Poisson tensor. This yields new examples of cscGK structures even on $\mathbf{CP}^2$, as the existence results obtained in [10, 34, 36] apply only to a special class of toric Poisson tensors.

2. Generalized Kähler structures of symplectic type

2.1. Conventions

For a Hermitian manifold (M, g, J) we consider the Riemannian metric g as a field of isomorphisms

$$g: TM \to T^*M, \quad X \to g(X, \cdot),\ X \in TM,$$

with inverse $g^{-1}: T^*M \to TM$, which leads to a definition of inner product on T^*M. To simplify some of the calculations, we define the action of J on T^*M as

$$J\xi := -J^*\xi = -\xi \circ J,$$

so that J commutes with $g: TM \to T^*M$ for any J-compatible metric. More generally, we will implicitly consider a field of bilinear forms $B \in C^\infty(T^*M \otimes T^*M)$ on TM (such as a Riemannian metric g or a symplectic form F on M) as a field of endomorphisms

$$B: TM \to T^*M, \quad X \to B(X, \cdot).$$

In case B is nondegenerate we denote by B^{-1} the inverse endomorphism. Similarly, a section of $\Pi \in C^\infty(TM \otimes TM)$ will be viewed as a field of endomorphisms

$$\Pi\colon T^*M \to TM, \quad \alpha \to \Pi(\alpha, \cdot).$$

We denote by $\langle \cdot, \cdot \rangle$ the natural pairing between TM and T^*M. If

$$B \in C^\infty(T^*M \otimes T^*M) \quad \text{and} \quad \Pi \in C^\infty(TM \otimes TM)$$

is skew, then we have

$$\langle \Pi, B \rangle = -\operatorname{tr}(\Pi \circ B).$$

For a 2-form ψ, we will let

$$\psi^{[k]} := \frac{1}{k!}\psi^k.$$

Our convention for the trace $\operatorname{tr}_F(\psi)$ of a 2-form ψ with respect to a symplectic form F is then

$$\operatorname{tr}_F(\psi) := \frac{\psi \wedge F^{[n-1]}}{F^{[n]}} = \frac{1}{2}\operatorname{tr}(F^{-1}\psi).$$

The volume form of a symplectic structure is

$$dV_F := F^{[n]},$$

and the Riemannian volume form of the Hermitian structure (g, J) is

$$dV_g = \omega_J^{[n]}, \quad \omega_J := gJ.$$

Furthermore, given a symplectic form F, we let $C_0^\infty(M, dV_F)$ denote the space of *normalized* smooth functions with zero average relative to dV_F:

$$C_0^\infty(M, dV_F) = \left\{ \phi \in C^\infty(M, \mathbb{R}) \mid \int_M \phi \, dV_F = 0 \right\}.$$

2.2. Symplectic-type generalized Kähler structures

In this subsection we recall the biHermitian formulation of generalized Kähler geometry, and the basic properties of the associated Poisson structures.

Definition 2.1. Given a smooth manifold M endowed with a closed 3-form H_0, we say that (g, b, I, J) on (M, H_0) is a *generalized Kähler structure (GK structure)* if I and J are integrable complex structures, b is a 2-form, g is Riemannian metric compatible with both I and J, and furthermore

$$d_I^c \omega_I = H_0 + db = -d_J^c \omega_J.$$

Associated to this structure we define the tensor

$$\pi := \frac{1}{2}[I, J]g^{-1} \in \wedge^2(TM).$$

By [6, 47], π is the real part of holomorphic $(2,0)$-bivectors with respect to I and J, and we define

$$\pi_J := \pi - \sqrt{-1}J\pi, \quad \pi_I := \pi - \sqrt{-1}I\pi. \tag{2.1}$$

Definition 2.2. Given a complex manifold (M, J), suppose there exists a symplectic form F such that

$$g := -(FJ)^{\mathrm{sym}}, \quad I := -F^{-1}J^*F$$

define respectively a Riemannian metric g and an integrable almost complex structure I. Then, setting

$$H_0 = 0, \quad b := -(FJ)^{\mathrm{skew}},$$

the data (g, b, I, J) satisfies Definition 2.1 and is referred to as *symplectic-type generalized Kähler structure* on (M, J). Since (F, J) algebraically determine the tuple (g, b, I, J), by abuse of notation we will often refer to (F, J) as a symplectic-type generalized Kähler structure.

Remark 2.3. Elementary linear algebra yields that for a symplectic-type GK structure one has

$$F(I + J) = -2g, \tag{2.2}$$

so that $\det(I + J) \neq 0$ on M. Notice that I coincides with J (and is then automatically integrable) precisely when F is of type $(1,1)$ with respect to J, i.e., defines a Kähler structure (g, J).

From the point of view of generalized geometry [38, 40], adopted in [34, 36], symplectic-type generalized Kähler structures correspond to pairs of commuting *generalized complex structures*. To make this connection more precise, we explicitly express the underlying (almost) generalized complex structures in terms of (F, J). Recall that $TM \oplus T^*M$ has a natural symmetric pairing $\langle \cdot, \cdot \rangle$ of signature $(2n, 2n)$ induced by the contraction between TM and T^*M. Then following [38, Section 6.4], we obtain the next proposition.

Proposition 2.4. *Given a symplectic structure (M, F) taming an almost complex structure J, we denote as above*

$$I = -F^{-1}J^*F, \quad g = -(FJ)^{\mathrm{sym}}, \quad b = -(FJ)^{\mathrm{skew}}.$$

Then (F, J) gives rise to a pair of commuting generalized almost complex structures $\mathbb{J}, \mathbb{I} \in O(TM \oplus T^*M, \langle \cdot, \cdot \rangle)$ orthogonal with respect to the standard pairing $\langle \cdot, \cdot \rangle$

on $TM \oplus T^*M$:

$$\mathbb{J} = \mathbb{J}_{P,Q} = \begin{pmatrix} P & QF^{-1} \\ -FQ & FPF^{-1} \end{pmatrix}, \quad \mathbb{I} = \mathbb{J}_F = \begin{pmatrix} 0 & -F^{-1} \\ F & 0 \end{pmatrix}, \tag{2.3}$$

where $P, Q \in C^\infty(M, \mathrm{End}(TM))$ are given by

$$P := -2(I + J)^{-1}, \quad Q := (J - I)(I + J)^{-1}.$$

Furthermore, $\mathbb{I}$ is integrable if and only if F is closed, and $(\mathbb{J}, \mathbb{I})$ are both integrable if and only if F is symplectic, and (J, I) are integrable. Conversely any generalized almost complex structure $\mathbb{J} \in O(TM \oplus T^*M, \langle\cdot,\cdot\rangle)$ commuting with $\mathbb{J}_F$ is necessarily given by (2.3) for some almost complex structures I and J.

We note that fundamental works of Goto and Gualtieri [32, 33, 41] show that on a compact *Kähler* manifold endowed with a holomorphic Poisson tensor π_J, there always exist symplectic-type GK structures with Poisson tensor π_J. Furthermore, there are natural deformation spaces, reviewed below, which naturally fix the holomorphic Poisson geometry. We thus define the space of GK structures on such a background.

Definition 2.5. A symplectic-type generalized Kähler structure F on (M, J) is called *π_J-compatible* if the holomorphic Poisson tensor associated to (g, b, I, J) via (2.1) equals π_J. As $\pi = \mathrm{Re}(\pi_J)$ determines π_J on (M, J), we let $\mathcal{GK}_\pi$ denote the space of π_J-compatible symplectic-type GK structures. The de Rham class $\alpha := [F] \in H^2(M, \mathbb{R})$ of a generalized Kähler structure in $\mathcal{GK}_\pi$ will be called a *compatible de Rham class*. We denote by $\mathcal{GK}_{\pi,\alpha}$ the space of π_J-compatible symplectic-type generalized Kähler structures within a fixed π_J-compatible de Rham class α.

2.3. Algebraic identities on a symplectic-type generalized Kähler manifold

In this subsection we collect some elementary identities for a symplectic-type GK manifold (M, g, I, J, F) which are useful in the sequel. Recall the basic expressions (see (2.2))

$$F = -2g(I + J)^{-1}, \quad F^{-1} = -\frac{1}{2}(I + J)g^{-1}. \tag{2.4}$$

We compute for any 1-forms α, β

$$\mathrm{tr}_F(\alpha \wedge \beta) = \frac{1}{2}\,\mathrm{tr}\big(F^{-1}(\alpha \wedge \beta)\big) = \frac{1}{2}\langle (I + J)\alpha, \beta\rangle_g, \tag{2.5}$$

where $\langle\cdot,\cdot\rangle_g$ denotes the inner product on T^*M induced by g and have used the fact that I, J are skew with respect to g. In particular, using (2.4), the g-orthogonality of I and J, and the identity

$$I(I + J) = (I + J)J, \tag{2.6}$$

we get

$$\mathrm{tr}_F(\alpha \wedge J\beta) = -\mathrm{tr}_F(I\alpha \wedge \beta), \tag{2.7}$$

or, equivalently,

$$\alpha \wedge J\beta \wedge F^{[n-1]} = -I\alpha \wedge \beta \wedge F^{[n-1]}. \tag{2.8}$$

Recall that the Poisson tensor π is given by

$$\pi = \frac{1}{2}[I, J]g^{-1} = \frac{1}{2}(I - J)(I + J)g^{-1}.$$

Using (2.5), the fact that I and J are skew, and (2.7), we compute for any 1-forms α, β

$$\begin{aligned}\langle \pi, \alpha \wedge \beta \rangle &= -\frac{1}{4}\,\mathrm{tr}\big(\pi \circ (\alpha \wedge \beta)\big) \\ &= -\frac{1}{4}\big(\langle \alpha, (I - J)(I + J)\beta\rangle_g - \langle (I - J)(I + J)\alpha, \beta\rangle_g\big) \\ &= -\frac{1}{2}\langle (I + J)(I - J)\alpha, \beta\rangle = \mathrm{tr}_F\big((J - I)\alpha \wedge \beta\big). \end{aligned} \tag{2.9}$$

We next establish the following identity, which holds for any 1-forms $\alpha_1, \alpha_2, \beta_1, \beta_2$:

$$\begin{aligned}\mathrm{tr}\big(F^{-1}(\beta_1 \wedge \beta_2)F^{-1}(\alpha_1 \wedge \alpha_2)\big) &= \frac{1}{2}\langle (I + J)\alpha_1 \wedge (I + J)\alpha_2, \beta_1 \wedge \beta_2\rangle_g \\ &= \frac{1}{2}\langle \alpha_1 \wedge \alpha_2, (I + J)\beta_1 \wedge (I + J)\beta_2\rangle_g. \end{aligned} \tag{2.10}$$

To this end, we use (2.4), (2.6) and the fact that I and J are skew with respect to g to compute

$$\begin{aligned}&4\,\mathrm{tr}\big(F^{-1}(\beta_1 \wedge \beta_2) \circ F^{-1}(\alpha_1 \wedge \alpha_2)\big) \\ &= \mathrm{tr}\big((\beta_1 \otimes (I + J)\beta_2^\sharp - \beta_2 \otimes (I + J)\beta_1^\sharp) \\ &\qquad \circ (\alpha_1 \otimes (I + J)\alpha_2^\sharp - \alpha_2 \otimes (I + J)\alpha_1^\sharp)\big) \\ &= \big(\langle (I + J)\alpha_2, \beta_1\rangle_g \langle \alpha_1, (I + J)\beta_2\rangle_g - \langle \beta_1, (I + J)\alpha_1\rangle_g \langle \alpha_2, (I + J)\beta_2\rangle_g \\ &\qquad - \langle (I + J)\beta_1, \alpha_1\rangle_g \langle \beta_2, (I + J)\alpha_2\rangle_g + \langle \beta_2, (I + J)\alpha_1\rangle_g \langle (I + J)\beta_1, \alpha_2\rangle_g\big) \\ &= 2\big(\langle (I + J)\alpha_1, \beta_1\rangle_g \langle (I + J)\alpha_2, \beta_2\rangle_g - \langle (I + J)\alpha_1, \beta_2\rangle_g \langle (I + J)\alpha_2, \beta_1\rangle_g\big). \end{aligned}$$

A useful ramification of (2.10) is the formula

$$\mathrm{tr}\big(F^{-1}(\beta \wedge I\beta) \circ F^{-1}(\alpha \wedge J\alpha)\big) = \frac{1}{2}\big(\langle (I + J)\alpha, \beta\rangle_g^2 + \langle (I + J)\alpha, I\beta\rangle_g^2\big). \tag{2.11}$$

Notice finally that (2.10) and (2.4) also yield

$$\mathrm{tr}\big(F^{-1}(I\beta_1 \wedge I\beta_2)F^{-1}(\alpha_1 \wedge \alpha_2)\big) = \mathrm{tr}\big(F^{-1}(\beta_1 \wedge \beta_2)F^{-1}(J\alpha_1 \wedge J\alpha_2)\big). \tag{2.12}$$

2.4. An integration by parts formula

We will frequently use in this paper the following basic integration by parts identity.

Lemma 2.6. *For (F, J) a symplectic-type generalized Kähler structure and for any $\phi, \psi \in C^\infty(M)$, one has*

$$\int_M \phi \operatorname{tr}_F(dI\, d\psi) F^{[n]} = \int_M \psi \operatorname{tr}_F(dJ\, d\phi) F^{[n]}. \tag{2.13}$$

Proof. We use (2.8) to compute

$$\begin{aligned}\int_M \phi \operatorname{tr}_F(dI\, d\psi) F^{[n]} &= \int_M \phi\, dI\, d\psi \wedge F^{[n-1]} = -\int_M d\phi \wedge I\, d\psi \wedge F^{[n-1]} \\ &= \int_M J\, d\phi \wedge d\psi \wedge F^{[n-1]} = \int_M \psi\, dJ\, d\phi \wedge F^{[n-1]} \\ &= \int_M \psi \operatorname{tr}_F(dJ\, d\phi) F^{[n]}.\end{aligned}$$

■

2.5. Consequences of the Generalized Kähler Hodge theory

Generalized Kähler geometry comes with a rich Hodge theory extending the classical Hodge theory in the Kähler setting. In this section we collect several specific corollaries of the Hodge theory and underlying identities on a generalized Kähler manifold of symplectic type (M, F, J). The proof of these results uses the technical formalism of generalized Kähler Hodge theory, which is independent of the rest of the paper. For this reason we postpone the proofs until the Appendix A, where we provide the necessary background.

Theorem 2.7. *Let (M, F, J) be a compact generalized Kähler manifold of symplectic type with $I = -F^{-1}J^*F$. Then*

(1) *any J-holomorphic p-form $\xi \in \wedge_J^{p,0}(M)$ is closed;*

(2) *any real differential 1-form ξ such that $(d\xi)_I^{2,0} = 0$ admits a unique decomposition*

$$\xi = (I + J)\eta + d\phi + I\, d\psi, \tag{2.14}$$

where η and $J\eta$ are closed, and $\phi, \psi \in C^\infty(M, \mathbb{R})$;

(3) *the map $\xi \mapsto \eta - \sqrt{-1}J\eta$, where η is determined by (2.14), induces an isomorphism*

$$H^1(M, \mathbb{R}) \simeq H^{1,0}_{\bar\partial_J}(M);$$

(4) *any exact 2-form α of type $(1, 1)$ with respect to I is $\alpha = dd_I^c\psi$ for some $\psi \in C^\infty(M, \mathbb{R})$.*

The same statements hold after changing the roles of I and J.

Remark 2.8. It follows from the above that any $a \in H^1(M, \mathbb{R})$ has a unique closed and d^c_J-closed representative $\xi \in \wedge^1(M)$. In particular, $b_1 = \dim H^1(M, \mathbb{R})$ must be even.

2.6. Hamiltonian symplectic-type generalized Kähler deformations

Let (M, ω_0, J) be a compact Kähler manifold. A key feature of Kähler geometry is the possibility to deform ω_0 by smooth functions as follows: given $\varphi \in C^\infty(M, \mathbb{R})/\mathbb{R}$ we can define a new (1,1)-form

$$\omega_\varphi := \omega_0 + dd^c_J \varphi,$$

and as long as ω_φ is positive definite (M, ω_φ, J) is again a Kähler manifold. This procedure is reversible, due to the dd^c_J-lemma: a Kähler metric ω on (M, J) is of the form $\omega = \omega_\varphi$ for some smooth function φ (uniquely defined up to an additive constant) if and only if $\omega \in \alpha = [\omega_0] \in H^2(M, \mathbb{R})$. In particular, the space $\mathcal{K}_\alpha$ of all Kähler metrics within a given de Rham class α has a structure of a Fréchet manifold modeled on the vector space $C^\infty(M, \mathbb{R})/\mathbb{R}$: for any $\phi \in C^\infty(M, \mathbb{R})/\mathbb{R}$, there is an infinitesimal deformation ω_t of ω in $\mathcal{K}_\alpha$, such that

$$\frac{\partial}{\partial t}\omega_t = dd^c_J \phi.$$

There is a similar construction in generalized Kähler geometry, which was first presented in the context of generalized Kähler structures with nondegenerate Poisson tensor on 4-dimensional manifolds in [6, Section 4.2], where it is attributed to Joyce (see also [48]). Gualtieri in [39, Section 7] defined, more generally, Hamiltonian deformations of symplectic-type generalized Kähler manifolds whereas [31] defined a version adapted to the general biHermitian case. We recall the construction of [39].

Theorem 2.9 ([39]). *Let (M, J, F_0) be a compact symplectic-type generalized Kähler manifold with the second complex structure I_0 and real Poisson tensor π. Let $\phi_t \in C^\infty(M, \mathbb{R})$, $t \in (-\varepsilon, \varepsilon)$ be a one-parameter family of smooth functions on M, $X_{\phi_t} := \pi(-d\phi_t)$ the time dependent π-Hamiltonian vector field, and Φ_t, $\Phi_0 = \mathrm{Id}$ the corresponding isotopy of diffeomorphisms. Define*

$$F_t := F_0 + \int_0^t (dd^c_{I_s}\phi_s)\, ds, \quad I_s = \Phi_s \cdot I_0 := (\Phi_s)_* I_0 (\Phi_s)_*^{-1}.$$

Then F_t defines a one-parameter family of symplectic-type generalized Kähler structures on (M, J) as long as $(F_t)^{1,1}_J > 0$. Furthermore, $I_t = -F_t^{-1} J^* F_t = \Phi_t \cdot I_0$.

Definition 2.10 (Hamiltonian deformations of symplectic-type generalized Kähler structures). The family of symplectic-type generalized Kähler structures F_t on the manifold (M, J, π_J) given by Theorem 2.9 for some $\phi_t \in C^\infty(M, \mathbb{R})$ will be referred to as a *Hamiltonian deformation* of F_0. Under this deformation

$$\frac{\partial}{\partial t} J = 0, \quad \frac{\partial}{\partial t} I = -\mathcal{L}_{X_\phi} I = -\pi \circ (dd^c_I \phi), \quad \frac{\partial}{\partial t} F = dd^c_I \phi, \quad \frac{\partial}{\partial t} \pi = 0. \quad (2.15)$$

It follows that the Hamiltonian deformations are π_J-compatible and preserve the given symplectic generalized Kähler class $\alpha = [F_0]$, i.e., $F_t \in \mathcal{GK}_{\pi,\alpha}$.

2.7. Formal manifold structure

According to [39], the space $\mathcal{GK}_\pi$ is parametrized by the closed 2-forms F which tame J and satisfy the algebraic (zero order) identity

$$FJ + J^* F - F \circ \pi \circ F = 0. \quad (2.16)$$

Lemma 2.11. *If (J, π) is a holomorphic Poisson structure and symplectic form F solves (2.16), then $I := -F^{-1} J^* F = J - \pi F$ is integrable.*

Proof. The proof of this claim is essentially contained in [39] (see also [41]). We sketch the proof here for reader's convenience, and refer the reader to [39] for the necessary definitions.

Condition (2.16) can be equivalently stated as the following identity for the operators in $O(TM \oplus T^*M, \langle \cdot, \cdot \rangle)$:

$$e^F \begin{pmatrix} J & \pi \\ 0 & -J^* \end{pmatrix} e^{-F} = \begin{pmatrix} I & \pi \\ 0 & -I^* \end{pmatrix}, \quad e^F = \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix}.$$

According to [39, Proposition 5] an upper triangular generalized almost complex structure of the form $\left(\begin{smallmatrix} J & \pi \\ 0 & -J^* \end{smallmatrix}\right)$ is integrable if and only if J is an integrable complex structure in the classical sense, and π is the real part of a holomorphic Poisson tensor. On the other hand, conjugation by a matrix e^F, where F is a closed form preserves the integrability of the underlying generalized complex structure. Thus given that $\left(\begin{smallmatrix} J & \pi \\ 0 & -J^* \end{smallmatrix}\right)$ is integrable, $\left(\begin{smallmatrix} I & \pi \\ 0 & -I^* \end{smallmatrix}\right)$ must be integrable itself, implying that the classical almost complex structure I is also integrable. ∎

Remark 2.12. One can derive Lemma 2.11 directly in the "classical" bihermitian language. We are grateful to the anonymous referee for pointing out to us the identity (2.17) below. Given a symplectic manifold (M, F), endowed with an F-tamed integrable complex structure J:

$$\big(\pi \circ (dd^c_J \phi) \circ + (\mathcal{L}_{\pi(d\phi)} J)\big)^* (d\psi) = 4F\big(N_I(X_\phi, X_\psi)\big), \quad (2.17)$$

where ϕ, ψ are any smooth functions, $X_\psi := -F^{-1}(d\psi)$, $X_\phi := -F^{-1}(d\phi)$ are the corresponding F-Hamiltonian vector fields, and

$$N_I(X,Y) := \frac{1}{4}\big([IX,IY] - I[IX,Y] - I[X,IY] - [X,Y]\big)$$

is the Nijenhuis tensor of the almost complex structure $I := -F^{-1}J^*F$. Notice that if we assume, furthermore, that π_J is a holomorphic-Poisson tensor on (M,J), it follows from the definition of the $\overline{\partial}_J$-operator (acting on TM by $\overline{\partial}_J X = -\frac{1}{2}J\mathcal{L}_X J$) that for any smooth function ϕ, we have

$$\mathcal{L}_{\pi(d\phi)}J = -\pi \circ (dd^c_J\phi).$$

Combined with (2.17), one thus gets an alternative derivation of Lemma 2.11, obtained entirely in terms of the underlying bi-Hermitian geometry. Formula (2.17) follows easily by taking the Lie derivative $\mathcal{L}_{IX_\phi}$ of the identity

$$F(IX_\psi, Y) + F(X_\psi, JY) = 0,$$

and using

$$\mathcal{L}_{IX_\phi}F = d\big(F(IX_\phi)\big) = -d\big(J^*F(X_\phi)\big) = -dd^c_J\phi, \quad IX_\phi = JX_\phi + \pi(d\phi),$$
$$F\big((\mathcal{L}_{X_\phi}I)X, Y\big) + F\big(X, (\mathcal{L}_{X_\phi}J)Y\big) = 0.$$

Lemma 2.11 motivates us to consider a bigger space of structures satisfying the algebraic constraints but not the integrability condition of F being closed, which is useful from an analytic perspective.

Definition 2.13. Given a complex manifold (M,J) with real Poisson tensor π, let

$$\mathcal{AGK}_\pi = \{F \in \wedge^2(M) \mid FJ + J^*F - F\circ\pi\circ F = 0\}.$$

Remark 2.14. By differentiating the defining relation along a path F_s, we see that

$$0 = \dot{F}J + J^*\dot{F} - (\dot{F}\circ\pi\circ F + F\circ\pi\circ\dot{F}) = 2(\dot{F}I)_I^{(2,0)+(0,2)},$$

where we have used that $I - J = -\pi\circ F$. In particular, the tangent space of $\mathcal{AGK}_\pi$ at F is

$$\mathbf{T}_F(\mathcal{AGK}_\pi) = \wedge^{1,1}_I(M), \quad I := -F^{-1}J^*F. \tag{2.18}$$

Definition 2.15. For any smooth (time independent) function $\phi \in C^\infty(M,\mathbb{R})$, we define a vector field $\mathbf{X}_\phi$ on $\mathcal{AGK}_\pi$ given by

$$\mathbf{X}_\phi(F) := (dd^c_I\phi)^{1,1}_I, \quad I := -F^{-1}J^*F, \tag{2.19}$$

and call it a *fundamental vector field* associated to ϕ. We denote by $\mathbf{D}$ the distribution generated by the fundamental vector fields on $\mathcal{AGK}_\pi$.

Given a $\phi \in C^\infty(M,\mathbb{R})$, by (2.15), for any $F \in \mathcal{GK}_\pi$ the Hamiltonian flow construction with the (time independent) function ϕ and starting at F produces a smooth path F_t representing $\mathbf{X}_\phi(F)$; it follows that F_t is an integral curve of $\mathbf{X}_\phi$. We have the following consequence of this fact.

Lemma 2.16. *The subset $\mathcal{GK}_\pi \subset \mathcal{AGK}_\pi$ is a (formal) integrable submanifold of* $\mathbf{D}$*. In particular, when restricted to $\mathcal{GK}_\pi$,* $\mathbf{D}$ *is involutive. More precisely, on $\mathcal{GK}_\pi$ we have*

$$[\mathbf{X}_\phi, \mathbf{X}_\psi] = -\mathbf{X}_{\{\phi,\psi\}_\pi},$$

where $\{\phi,\psi\}_\pi := \pi(d\phi, d\psi)$ is the π-Poisson bracket.

Proof. The first part follows from the facts that $\mathcal{GK}_\pi$ is $\mathbf{D}$-invariant, i.e., the integral curves of $\mathbf{X}_\phi$ preserve $\mathcal{GK}_\pi$, and at each point $F \in \mathcal{GK}_\pi$, by (2.15), $\mathbf{D}_F$ coincides with the sub-space of tangent vectors in $\mathbf{T}_F(\mathcal{AGK}_\pi)$ generated by smooth paths in $\mathcal{GK}_\pi$, i.e.,

$$\mathbf{T}_F(\mathcal{GK}_\pi) = \mathbf{D}_F \subset \mathbf{T}_F(\mathcal{AGK}_\pi).$$

We now compute the vector field bracket of $\mathbf{X}_\phi$ and $\mathbf{X}_\psi$. At each point F, we let F_t denote the Hamiltonian deformation with respect to ϕ and F_s the Hamiltonian deformation with respect to ψ. We also denote by Φ_t and Ψ_s the flows of the Poisson vector fields $X_\phi := -\pi(d\phi)$ and $X_\psi := -\pi(d\psi)$ on M. Noting that F_t and F_s are respectively integral curves of $\mathbf{X}_\phi$ and $\mathbf{X}_\psi$, we compute

$$\begin{aligned}
[\mathbf{X}_\phi, \mathbf{X}_\psi](F_0) &= \frac{d}{dt}\Big|_{t=0}\Big(\frac{d}{ds}_{|s=0}\big((F_t)_s\big)_{-t}\Big) \\
&= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}_{|s=0}\Big(\int_0^t (dd^c_{\Phi_r\cdot I_0}\phi)\,dr \\
&\qquad + \int_0^s (dd^c_{\Psi_p\Phi_t\cdot I_0}\psi)\,dp - \int_0^t (dd^c_{\Phi_{-q}\Psi_s\Phi_t\cdot I_0}\phi)\,dq\Big) \\
&= d\big(-(\mathcal{L}_{X_\phi} I_0^*)(d\psi) + (\mathcal{L}_{X_\psi} I_0^*)(d\phi)\big) \\
&= d\big(\iota_{X_\psi}(dd^c_{I_0}\phi) - \iota_{X_\phi}(dd^c_{I_0}\psi)\big) \\
&= -\big(\mathcal{L}_{X_\phi}(dd^c_{I_0}\psi) - \mathcal{L}_{X_\psi}(dd^c_{I_0}\phi)\big) \\
&= d\big((\mathcal{L}_{X_\phi} I_0^*)(d\psi) - (\mathcal{L}_{X_\psi} I_0^*)(d\phi)\big) - dd^c_{I_0}(\mathcal{L}_{X_\phi}\psi - \mathcal{L}_{X_\psi}\phi) \\
&= -2dd^c_{I_0}\{\phi,\psi\}_\pi + d\big((\mathcal{L}_{X_\phi} I_0^*)(d\psi) - (\mathcal{L}_{X_\psi} I_0^*)(d\phi)\big),
\end{aligned}$$

where we have used formulae (2.15) to pass from the third line to the fourth and the identity $\mathcal{L}_{X_\phi}\psi = \{\phi,\psi\}_\pi$ to obtain the last line. Finally, we arrive at

$$[\mathbf{X}_\phi, \mathbf{X}_\psi](F_0) = -dd^c_{I_0}\{\phi,\psi\}_\pi = -\mathbf{X}_{\{\phi,\psi\}_\pi}(F_0).$$

by comparing the third and the last lines in the above equalities. ∎

Note that at this point we have defined the natural class of Hamiltonian deformations in Definition 2.10, but on the other hand it is natural to consider the space of GK structures in $\mathcal{G}K_{\pi,\alpha}$ with fixed cohomological background data. We next show that these constructions are the same, in particular showing that any smooth path $F_t \in \mathcal{GK}_{\pi,\alpha}$ is given by the construction of Theorem 2.9.

Proposition 2.17. *Let $F_t \in \mathcal{GK}_{\pi,\alpha}$ be a smooth path of generalized Kähler structures of symplectic type. Denote the underlying second complex structure by I_t. Then*

$$\frac{dF_t}{dt} = dd^c_{I_t}\phi_t$$

for some $\phi_t \in C^\infty(M,\mathbb{R})$.

Proof. Let $\dot{F} := \frac{dF_t}{dt}$. Since $[F_t] \in H^2(M,\mathbb{R})$ is fixed, we have $\dot{F} = d\xi$. On the other hand, by (2.18) $\dot{F}$ must be of type $(1,1)$ with respect to I_t. Thus by part (4) of Theorem 2.7 applied to complex structure I_t, we conclude that $\dot{F} = dd^c_{I_t}\phi_t$ for some $\phi_t \in C^\infty(M,\mathbb{R})$, as claimed. ∎

Remark 2.18. On a compact manifold $dd^c_I\phi$ is nonzero unless $\phi \in C^\infty(M,\mathbb{R})$ is a constant. Thus, in view of Proposition 2.17, there is an identification of the tangent space to $\mathcal{GK}_{\pi,\alpha}$ at $F \in \mathcal{GK}_{\pi,\alpha}$:

$$\mathbf{T}_F(\mathcal{GK}_{\pi,\alpha}) \simeq C^\infty(M,\mathbb{R})/\mathbb{R}.$$

3. The generalized Kähler scalar curvature as a momentum map

Central to our understanding of the YTD conjecture in the Kähler setting is the GIT formulation due to Fujiki–Donaldson [20, 23]. The fundamental point in this framework is that the moment map for the space of Hamiltonian diffeomorphisms of a fixed Kähler form acting on complex structures is given by the scalar curvature. In attempting to extend this circle of ideas to generalized Kähler geometry one is faced with the subtle issue that there is not an obvious choice of scalar curvature due to the lack of a connection preserving all structure. Rather, the works of Boulanger–Goto [10,34] take the point of view of defining a natural Hamiltonian action on the space of GK structures and use its momentum map to *define* scalar curvature. In the work of Goto [35] this does lead to a general definition of scalar curvature for GK structures, although one expressed implicitly in terms of local defining spinors determining the underlying generalized complex structures. Furthermore, ideas from mathematical physics and generalized geometry (cf. [27,59,61]) suggest an explicit definition of scalar curvature for GK structures, and in special cases Boulanger and Goto have shown that this definition agrees with what arises from the moment map framework. In this section we will review the moment map construction, and furthermore close this circle of ideas by showing that the a priori different definitions of scalar curvature agree for all symplectic-type GK structures. The key input is the nondegenerate perturbation technique introduced in [5].

3.1. Scalar curvature of GK structures

We begin by explicitly stating our definition of scalar curvature of GK structures of symplectic type in terms of the biHermitian data. It was already noted by Boulanger in [10] that the moment map takes this explicit form in the toric setting, and this was inspirational for our work.

Definition 3.1. Let (g, b, I, J, F) be a symplectic-type generalized Kähler structure on M. The *generalized scalar curvature* is defined by

$$\mathrm{Gscal}_{(F,J)} := \mathrm{Scal}_g - \frac{1}{12}|db|_g^2 + 2\Delta_g \Psi - |d\Psi|_g^2, \quad \Psi = -\log \frac{dV_F}{dV_g}, \tag{3.1}$$

where Scal_g is the scalar curvature of g, $\Delta_g = -d^*d$ is the Laplacian, and $dV_F = F^{[n]}$, $dV_g = \omega_I^{[n]} = \omega_J^{[n]}$ are respectively the symplectic and Riemannian volume forms. Throughout the paper, we use the tensorial norm for forms, i.e., the norm of a p-form ψ is given by

$$|\psi|_g^2 := \sum_{i_1,\dots,i_p=1}^{2n} \psi(e_{i_1}, \dots, e_{i_p})^2,$$

where $\{e_i\}$ is any orthonormal frame of (T_pM, g).

In [34], Goto constructed a smooth function

$$\mathrm{Gscal}^{\mathrm{Goto}}_{(F,J)} \in C^\infty(M, \mathbb{R})$$

associated to a symplectic form F and a generalized almost complex structure $\mathbb{J} \in \mathrm{End}(TM \oplus T^*M)$ which via the Gualtieri map (2.3) is equivalent to an *almost* complex structure J tamed by F. Goto's motivation for defining this function originates from the formal momentum map picture, and we discuss it later. For now we record several important features of $\mathrm{Gscal}^{\mathrm{Goto}}_{(F,J)}$, which will be used throughout the paper.

(1) $\mathrm{Gscal}^{\mathrm{Goto}}_{(F,J)}$ at a point $x \in M$ depends algebraically on the second jets of F_x and J_x, is additive with respect to the Cartesian product, and vanishes on flat (linear) structures.

(2) A generalized *almost* complex structure $\mathbb{J}$ given by (2.3) determines a *canonical* complex line bundle $K_{\mathbb{J}}$ and there exists a representative $\rho_{(F,J)} \in \wedge^2(M)$ of $2\pi c_1(K_{\mathbb{J}}^{-1})$ such that

$$\mathrm{Gscal}^{\mathrm{Goto}}_{(F,J)} = 2\Big(\frac{\rho_{(F,J)} \wedge F^{[n-1]}}{F^{[n]}}\Big), \tag{3.2}$$

see [34, Definition 5.3 and Proposition 5.5].

(3) If (F, J) defines a genuine generalized Kähler structure, i.e., both complex structures J and $I = -F^{-1}J^*F$ are integrable and $I - J$ is invertible, then

$$\begin{aligned} \mathrm{Gscal}^{\mathrm{Goto}}_{(F,J)} &= -\operatorname{tr}_F\big(d(Fg^{-1}d\Phi)\big), \\ \Phi &= \frac{1}{2}\log\det(I-J) - \frac{1}{2}\log\det(I+J), \end{aligned} \tag{3.3}$$

see [34, Proposition 10.4].

Theorem 3.2. *If (F, J) is a symplectic-type generalized Kähler structure on M, then the generalized scalar curvature given by (3.1) coincides with Goto's scalar curvature*

$$\mathrm{Gscal}_{(F,J)} = \mathrm{Gscal}^{\mathrm{Goto}}_{(F,J)}.$$

Proof. Our goal is to give an explicit biHermitian expression for $\mathrm{Gscal}^{\mathrm{Goto}}_{(F,J)}$. We use the result of [5, Section 3], where we proved that possibly after taking the product with a flat factor $(\mathbb{C}, g_{\mathrm{flat}})$ any GK structure locally on an open dense set can be approximated in any $C^{k,\alpha}$ norm by a GK structure with invertible $I \pm J$. Let (F^l, J^l) be such a sequence of locally defined GK structures converging to a given (F_0, J_0) in the $C^{2,\alpha}$ norm. Since $\mathrm{Gscal}^{\mathrm{Goto}}$ depends continuously on the second jet of the underlying biHermitian data, we have

$$\mathrm{Gscal}^{\mathrm{Goto}}_{(F^l,J^l)} \mapsto \mathrm{Gscal}^{\mathrm{Goto}}_{(F_0,J_0)} \quad \text{as } l \to \infty.$$

Let (F, J) be any member of the sequence (F^l, J^l). Using (3.3) and the identities

$$\theta_I = gI\delta^g I, \quad \theta_J = gJ\delta^g J$$

for the Lee forms (where we let $\delta^g I := -\sum_{i=1}^{2n}(\nabla^g_{e_i} I)(e_i)$ and similarly for $\delta^g I$), we compute

$$\begin{aligned} \operatorname{tr}_F\big(d(Fg^{-1}d\Phi)\big) &= \sum_{i=1}^{2n}\big\langle (I+J)\nabla_{e_i}\big((I+J)^{-1}g^{-1}d\Phi\big), e_i\big\rangle_g \\ &= \Delta\Phi - \sum_{i=1}^{2n}\big\langle\big(\nabla_{e_i}(I+J)\big)(I+J)^{-1}g^{-1}d\Phi, e_i\big\rangle_g \end{aligned}$$

$$
\begin{aligned}
&= \Delta\Phi - \langle (I+J)^{-1} g^{-1} d\Phi, \delta^g I + \delta^g J\rangle \\
&= \Delta\Phi + \langle (I+J)^{-1} g^{-1} d\Phi, I\theta_I + J\theta_J\rangle \\
&= \Delta\Phi - \frac{1}{2}\langle d\log\det(I+J), d\Phi\rangle_g, \qquad (3.4)
\end{aligned}
$$

where in the last step we applied the identity $I\theta_I + J\theta_J = \frac{1}{2}(I+J)d\log\det(I+J)$, which holds on any symplectic-type generalized Kähler manifold; see [5, Proposition 4.3].

It remains to combine the identity (3.4) with a formula

$$
\begin{aligned}
\mathrm{Scal}_g - \frac{1}{12}|db|_g^2 = &-\frac{1}{2}\Delta_g\big(\log\det(I+J) + \log\det(I-J)\big) \\
&+ \frac{1}{4}\langle d\log\det(I+J), d\log\det(I-J)\rangle_g,
\end{aligned}
$$

(see [5, Lemma 4.5]) to conclude that

$$
\mathrm{Gscal}^{\mathrm{Goto}}_{(F,J)} = -\mathrm{tr}_F\big(d(Fg^{-1}d\Phi)\big) = \mathrm{Scal}_g - \frac{1}{12}|db|_g^2 + 2\Delta_g\Psi - |d\Psi|_g^2, \qquad (3.5)
$$

where $\Psi = \frac{1}{2}\log\det(I+J) = -\log\frac{dV_F}{dV_g}$. By (3.5) along the sequence (F^l, J^l), we have

$$
\mathrm{Gscal}^{\mathrm{Goto}}_{(F^l,J^l)} = \mathrm{Scal}_{g^l} - \frac{1}{12}|db^l|_{g^l}^2 + 2\Delta_{g^l}\Psi^l - |d\Psi^l|_{g^l}^2.
$$

Passing to the $C^{2,\alpha}$ limit $(F^l, J^l) \to (F_0, J_0)$, we conclude that

$$
\begin{gathered}
\mathrm{Gscal}^{\mathrm{Goto}}_{(F_0,J_0)} = \mathrm{Scal}_{g_0} - \frac{1}{12}|db_0|_{g_0}^2 + 2\Delta_{g_0}\Psi_0 - |d\Psi_0|_{g_0}^2 = \mathrm{Gscal}_{(F_0,J_0)}, \\
\Psi_0 = -\log\frac{dV_{F_0}}{dV_{g_0}}
\end{gathered}
$$

in general, without assuming that $I_0 - J_0$ is invertible. ■

In the view of the above proposition, we will denote Goto's scalar curvature simply by $\mathrm{Gscal}_{(F,J)}$ bearing in mind that it is given by (3.1) if (F, J) defines a generalized Kähler structure.

Remark 3.3. Goto [35] further introduced a generalized scalar curvature associated to an arbitrary generalized Kähler structure $(\mathbb{J}, \mathbb{I})$ on (M, H_0), $H_0 \in \wedge^3(M)$ and a volume form $d\mu_f = e^{-f}\, dV_g$. We will show in forthcoming work that similarly to Proposition 3.2 this quantity can be computed for (g, b, I, J) by an analogous formula

$$
\mathrm{Gscal}^f_{(\mathbb{J},\mathbb{I})} = \mathrm{Scal}_g - \frac{1}{12}|H|_g^2 + 2\Delta_g f - |df|_g^2,
$$

where $H = H_0 + db$.

3.2. Geometry of the space of almost generalized Kähler structures

On a compact smooth oriented manifold M, we consider the space $\mathcal{AC}$ of (oriented) almost complex structures J, endowed with the Fréchet topology of smooth sections of $T^*M \otimes TM$. Thus the tangent space of $\mathcal{AC}$ at a point J can be identified with the vector space of smooth sections $\dot{J}$ of $T^*M \otimes TM$ satisfying $J\dot{J} = -\dot{J}J$. Similarly, on a given compact symplectic manifold (M, F), we consider the subspace $\mathcal{AGK}_F \subset \mathcal{AC}$ of F-tamed almost complex structures, i.e.,

$$\mathcal{AGK}_F := \big\{J \in \mathcal{AC} \mid F_p(v, Jv) > 0,\ \forall 0 \neq v \in T_pM,\ \forall p \in M\big\}.$$

As the F-taming condition is open in the C^∞ topology, $\mathcal{AGK}_F$ is an open subspace of $\mathcal{AC}$, with the same tangent space at $J \in \mathcal{AGK}_F$. For each element $J \in \mathcal{AGK}_F$, we denote by $I := -F^{-1}J^*F^{-1} \in \mathcal{AGK}_F$ its F-conjugate. Writing $-FJ = g + b$ where g is the symmetric part and b is the skew-part of $-FJ$, the taming condition means that g is a Riemannian metric on M and b is a 2-form. It is easy to check that both J and its F-conjugate I are g-orthogonal, so we have a quadruple (g, b, I, J) which gives rise to a symplectic-type generalized Kähler structure if both I and J are integrable. Irrespective of the integrability of I and J, we will refer to (g, b, I, J) as an *almost generalized Kähler structure*. For any such structure, we still have

$$\det(I + J) \neq 0, \quad F = -2g(I + J)^{-1},$$
$$b = g(J - I)(I + J)^{-1} = -g(I + J)^{-1}(J - I).$$

Letting $P := (I + J)^{-1}$ and $Q := (J - I)(I + J)^{-1}$, we can still consider the endomorphisms $\mathbb{J} = \mathbb{J}_{P,Q}$ and $\mathbb{I} = \mathbb{J}_F$ of $TM \oplus T^*M$, introduced by (2.3); $(\mathbb{J}, \mathbb{I})$ give rise to a commuting pair of almost complex structures on $TM \oplus T^*M$ (called *generalized almost complex* structures), and a positive definite bilinear form $-\langle \mathbb{J}\mathbb{I}\cdot, \cdot\rangle$ (where $\langle \cdot, \cdot\rangle$ is the natural symmetric product on $TM \oplus T^*M$). Unlike $\mathbb{I} = \mathbb{J}_F$, the endomorphism $\mathbb{J}_{P,Q}$ will not be in general an *integrable* generalized almost complex structure. Thus dropping the integrability condition, we introduce the space

$$\mathcal{AGK}_{\mathbb{J}_F} := \big\{\mathbb{J} \in \mathrm{O}\big(TM \oplus T^*M, \langle \cdot, \cdot\rangle\big) \mid \mathbb{J}^2 = -\mathrm{Id},$$
$$\mathbb{J}_F\mathbb{J} = \mathbb{J}_F\mathbb{J},\ -\langle \mathbb{J}\mathbb{J}_F\cdot, \cdot\rangle > 0\big\}.$$

By Proposition 2.4, the correspondence $J \mapsto \mathbb{J} = \mathbb{J}_{P,Q}$ given by (2.3) provides an isomorphism of formal Fréchet manifolds

$$\gamma\colon \mathcal{AGK}_F \to \mathcal{AGK}_{\mathbb{J}_F}.$$

These Fréchet manifolds have the following tangent spaces at F (respectively $\mathbb{J}_F$)

$$\mathbf{T}_J(\mathcal{AGK}_F) = \big\{\dot{J} \in \mathrm{End}(TM) \mid J\dot{J} = -\dot{J}J\big\},$$
$$\mathbf{T}_{\mathbb{J}}(\mathcal{AGK}_{\mathbb{J}_F}) = \big\{\dot{\mathbb{J}} \in \mathfrak{o}\big(TM \oplus T^*M, \langle \cdot, \cdot\rangle\big) \mid \mathbb{J}\dot{\mathbb{J}} = -\dot{\mathbb{J}}\mathbb{J},\ [\mathbb{J}_F, \dot{\mathbb{J}}] = 0\big\}.$$

Both $\mathbf{T}_J(\mathcal{AGK}_F)$ and $\mathbf{T}_{\mathbb{J}}(\mathcal{AGK}_{\mathbb{J}_F})$ admit a formal almost complex structure, via the left multiplications by J and $\mathbb{J}$, respectively. It turns out that γ preserves these almost complex structures.

Lemma 3.4. *The isomorphism of Fréchet manifolds $\gamma\colon \mathcal{AGK}_F \mapsto \mathcal{AGK}_{\mathbb{J}_F}$ preserves the underlying almost complex structures. Namely, if $\dot{J}$ is a tangent vector at $J \in \mathcal{AGK}_F$, then*

$$\mathbb{J}\, d\gamma(\dot{J}) = d\gamma(J\dot{J}).$$

Proof. Along the map

$$\gamma\colon J \mapsto \begin{pmatrix} P & QF^{-1} \\ -FQ & FPF^{-1} \end{pmatrix},$$

where

$$P = -2(I+J)^{-1}, \quad Q = (J-I)(I+J)^{-1} = -(I+J)^{-1}(J-I),$$

we have

$$d\gamma(\dot{J}) = \begin{pmatrix} \dot{P} & \dot{Q}F^{-1} \\ -F\dot{Q} & F\dot{P}F^{-1} \end{pmatrix},$$

where

$$\dot{P} = 2(I+J)^{-1}(\dot{J}+\dot{I})(I+J)^{-1}, \quad \dot{Q} = 2(I+J)^{-1}(J\dot{J}-I\dot{I})(I+J)^{-1},$$

with

$$\dot{I} = -F^{-1}\dot{J}^*F, \quad I\dot{I} = -F^{-1}(J\dot{J})^*F.$$

Hence,

$$\begin{aligned} \mathbb{J}\, d\gamma(\dot{J}) &= \begin{pmatrix} P & QF^{-1} \\ -FQ & FPF^{-1} \end{pmatrix} \begin{pmatrix} \dot{P} & \dot{Q}F^{-1} \\ -F\dot{Q} & F\dot{P}F^{-1} \end{pmatrix} \\ &= \begin{pmatrix} P\dot{P} - Q\dot{Q} & (P\dot{Q}+Q\dot{P})F^{-1} \\ -F(P\dot{Q}+Q\dot{P}) & F(P\dot{P}-Q\dot{Q})F^{-1} \end{pmatrix}. \end{aligned}$$

We further compute

$$\begin{aligned} P\dot{P} - Q\dot{Q} &= -4(I+J)^{-2}(\dot{J}+\dot{I})(I+J)^{-1} \\ &\qquad - 2(J-I)(I+J)^{-2}(J\dot{J}-I\dot{I})(I+J)^{-1} \\ &= 2(I+J)^{-2}\big(-2(\dot{J}+\dot{I}) - (J-I)(J\dot{J}-I\dot{I})\big)(I+J)^{-1} \\ &= 2(I+J)^{-2}(-\dot{J}-\dot{I}+JI\dot{I}+IJ\dot{J})(I+J)^{-1} \\ &= 2(I+J)^{-1}(J\dot{J}+I\dot{I})(I+J)^{-1}, \end{aligned}$$

$$\begin{aligned} P\dot{Q} + Q\dot{P} &= -4(I+J)^{-2}(J\dot{J} - I\dot{I})(I+J)^{-1} \\ &\qquad + 2(J-I)(I+J)^{-2}(\dot{J}+\dot{I})(I+J)^{-1} \\ &= 2(I+J)^{-2}\big(-2(J\dot{J} - I\dot{I}) + (J-I)(\dot{J}+\dot{I})\big)(I+J)^{-1} \\ &= 2(I+J)^{-1}(-\dot{J}+\dot{I})(I+J)^{-1}. \end{aligned}$$

Comparing the resulting expressions with the entries of the matrix $d\gamma(J\dot{J})$ we conclude that

$$\mathbb{J}\, d\gamma(\dot{J}) = d\gamma(J\dot{J}),$$

as claimed. ■

Due to Lemma 3.4, we will abuse notations slightly and tacitly identify $\mathcal{AGK}_F$ and $\mathcal{AGK}_{\mathbb{J}_F}$ via the map γ. We will denote the underlying almost complex structure given by the left multiplication with J on $\mathbf{T}_J(\mathcal{AGK}_F)$ by $\mathbf{J}$.

Remark 3.5. The F-taming condition yields that any two elements $J_0, J \in \mathcal{AGK}_F$ are commensurable, i.e., $\det(J_0 + J) \neq 0$. Therefore, one can apply the Cayley transform with base-point J_0:

$$J \mapsto (J + J_0)^{-1}(J_0 - J),$$

to endow $\mathcal{AGK}_F$ with a structure of an open contractible subset $\mathbf{U}$ in a Fréchet complex vector space. It then follows from Lemma 3.4 and the identification in [29, Appendix B] of the induced complex structure on $\mathbf{U}$ with the complex multiplication by i that $\mathbf{J}$ is integrable.

Following Goto [34] and Gauduchon [30] we define a formal Kähler structure on $(\mathcal{AGK}_F, \mathbf{J})$. We use the identification $\gamma\colon \mathcal{AGK}_F \to \mathcal{AGK}_{\mathbb{J}_F}$ throughout.

Definition 3.6 (Formal Kähler structure on $\mathcal{AGK}_F$). Let $\dot{J}_1, \dot{J}_2 \in \mathbf{T}_J\mathcal{AGK}_F$ be two tangent vectors at $J \in \mathcal{AGK}_F$. Denote by $\dot{\mathbb{J}}_1 = d\gamma(\dot{J}_1)$, $\dot{\mathbb{J}}_2 = d\gamma(\dot{J}_2)$ the corresponding tangent vectors at $\mathbb{J} := \gamma(J) \in \mathcal{AGK}_{\mathbb{J}_F}$. One defines a 2-form

$$\mathbf{\Omega}(\dot{J}_1, \dot{J}_2) = \frac{1}{4}\int_M \operatorname{tr}(\mathbb{J}\dot{\mathbb{J}}_1\dot{\mathbb{J}}_2)F^{[n]}.$$

It is shown in [34] that $(\mathcal{AGK}_F, \mathbf{J}, \mathbf{\Omega})$ gives rise to a formal Kähler manifold with the underlying Riemannian metric

$$\boldsymbol{g}(\dot{J}_1, \dot{J}_2) = \frac{1}{4}\int_M \operatorname{tr}(\dot{\mathbb{J}}_1\dot{\mathbb{J}}_2)F^{[n]}.$$

One can trace through the identification $\gamma\colon \mathcal{AGK}_F \to \mathcal{AGK}_{\mathbb{J}_F}$ similarly to the proof of Lemma 3.4 and get a more explicit expression for $\mathbf{\Omega}$ and $\boldsymbol{g}$ which is useful in finding the linearization of scalar curvature. We leave the details to the reader.

Lemma 3.7. *The formal symplectic* $\boldsymbol{\Omega}$*-form and Riemannian metric* $\boldsymbol{g}$ *are given by*

$$\begin{aligned}\boldsymbol{\Omega}_J(\dot{J}_1, \dot{J}_2) = 2\int_M \operatorname{tr}\big((I+J)^{-2}\big[(J\dot{J}_1 + I\dot{I}_1)(I+J)^{-2}(\dot{J}_2 + \dot{I}_2) \\ + (\dot{J}_1 - \dot{I}_1)(I+J)^{-2}(J\dot{J}_2 - I\dot{I}_2)\big]\big) F^{[n]},\\ \boldsymbol{g}_J(\dot{J}_1, \dot{J}_2) = 2\int_M \operatorname{tr}\big((I+J)^{-2}\big[(\dot{J}_1 + \dot{I}_1)(I+J)^{-2}(\dot{J}_2 + \dot{I}_2) \\ - (J\dot{J}_1 - I\dot{I}_1)(I+J)^{-2}(J\dot{J}_2 - I\dot{I}_2)\big]\big) F^{[n]}.\end{aligned}$$

3.3. The momentum map

Let $\operatorname{Ham}(M, F)$ be the group of Hamiltonian diffeomorphisms of (M, F) whose Lie algebra $\mathfrak{ham}(M, F)$ is identified with the space $C_0^\infty(M, dV_F)$ of zero mean smooth functions on (M, F) by

$$C_0^\infty(M, dV_F) \ni f \mapsto -F^{-1}(df) \in \mathfrak{ham}(M, F),$$

thus giving rise to the usual Lie algebra isomorphism

$$\mathfrak{ham}(M, F) \simeq \big\{f \in C_0^\infty(M, dV_F), \{\cdot,\cdot\}_F\big\},$$

where $\{f, g\}_F = F^{-1}(df, dg)$ is the F^{-1}-Poisson pairing. The group $\operatorname{Ham}(M, F)$ naturally acts on $\mathcal{AGK}_F$ by the induced action on the underlying almost complex structures J:

$$\Phi \cdot J := (\Phi_*) J (\Phi_*)^{-1}, \quad \Phi \in \operatorname{Ham}(M, F),$$

leading to a Lie algebra representation

$$C_0^\infty(M, dV_F) \ni f \mapsto \mathbf{Y}_f(J) := \mathcal{L}_{F^{-1}(df)} J \in \mathbf{T}_J(\mathcal{AGK}_F).$$

From the point of view of generalized complex structures $(\mathbb{J}, \mathbb{J}_F)$ this action fixes $\mathbb{J}_F$ and pulls back $\mathbb{J} \in \operatorname{End}(TM \oplus T^*M)$. The crucial result of Goto [34] (see also [10] for the toric case) is that this action on the Kähler Fréchet manifold $(\mathcal{AGK}_F, \boldsymbol{\Omega}, \mathbf{J})$ is Hamiltonian. Proposition 3.2 above complements this with a concrete geometric interpretation of the momentum map in terms of the biHermitian geometry.

Theorem 3.8 ([34]). *The action of* $\operatorname{Ham}(M, F)$ *on* $(\mathcal{AGK}_F, \boldsymbol{\Omega}, \mathbf{J})$ *is Hamiltonian. Specifically, for each* $J \in \mathcal{AGK}_F$ *there exists a function* $\operatorname{Gscal}_{(F,J)}$ *such that the pairing*

$$\langle \boldsymbol{\mu}(J), f\rangle := -\int_M \operatorname{Gscal}_{(F,J)}\, f F^{[n]}, \quad f \in C_0^\infty(M, dV_F),$$

is a momentum map for the action, i.e., for any infinitesimal variation $\dot{J} = \frac{d}{dt}|_{t=0} J_t$, $J_t \in \mathcal{AGK}_F$ *and any smooth function* $f \in C_0^\infty(M, dV_F)$*, we have*

$$\boldsymbol{\Omega}\big(\mathcal{L}_{F^{-1}(df)} J, \dot{J}\big) = \int_M \Big(\frac{d}{dt}\operatorname{Gscal}_{(F,J_t)}\Big) f F^{[n]}.$$

We note that the setup in [34] is slightly more general than stated here, as it allows for a symplectic-type generalized complex structure $\mathbb{I}$ twisted by an additional B-*field transform*. However, in the special case when the B-field transform is trivial, the group acting on J (or equivalently on $\mathbb{J}$) reduces to $\operatorname{Ham}(M, F) \subset \operatorname{Diff}(M)$.

Remark 3.9 (Average value of Gscal). Using (3.2), we observe for any $J \in \mathcal{AGK}_F$ that the average value of Gscal

$$\begin{aligned}\bar{\mu} &:= \frac{\int_M \operatorname{Gscal}_{(F,J)} F^{[n]}}{\int_M F^{[n]}} \\ &= 2\frac{\int_M \rho_{(F,J)} \wedge F^{[n-1]}}{\int_M F^{[n]}} = 4\pi \frac{\langle c_1(K_{\mathbb{J}}^{-1}) \cdot [F]^{n-1}, [M]\rangle}{\langle [F]^n, [M]\rangle}\end{aligned}$$

is a topological constant depending only on $c_1(K_{\mathbb{J}}) \in H^2(M, \mathbb{Z})$, $[F] \in H^2(M, \mathbb{R})$.

Definition 3.10. We say that a symplectic-type generalized Kähler structure (F, J) is of *constant generalized scalar curvature*, and abbreviate *cscGK*, if $\operatorname{Gscal}_{(F,J)} = \bar{\mu}$ is a constant function. We say that (F, J) is an *extremal* generalized Kähler structure if the vector field $\chi := -F^{-1}(d \operatorname{Gscal}_{(F,J)})$ preserves J, i.e., $\mathcal{L}_\chi J = 0$. The vector χ will be referred to as the *extremal vector field*.

These definitions are motivated by the following immediate corollary of Theorem 3.8 (see also [36, Section 9]).

Corollary 3.11. *Let* (F, J) *be a symplectic-type generalized Kähler structure defined on a compact manifold* M*. Then*

- (F, J) *is cscGK if and only if* J *is a zero of the momentum map* $\boldsymbol{\mu}$ *of* $\mathcal{AGK}_F$*;*
- (F, J) *is extremal if and only if* J *is a critical point of the norm functional* $J \to \|\boldsymbol{\mu}(J)\|^2$ *on* $\mathcal{AGK}_F$*, where the norm is the* $L^2(M, F)$ *norm*

$$\|\boldsymbol{\mu}(J)\|^2 := \int_M \big(\operatorname{Gscal}_{(F,J)} - \bar{\mu}\big)^2 F^{[n]}.$$

Proof. It is immediate by Theorem 3.8 that (F, J) is cscGK if and only if J is a zero of the moment map $\boldsymbol{\mu}\colon \mathcal{AGK}_F \mapsto (C_0^\infty(M, \mathbb{R}))^*$. For the second part we compute the variation of

$$\|\boldsymbol{\mu}(J)\|^2 := \int_M \big(\operatorname{Gscal}_{(F,J)} - \bar{\mu}\big)^2 dV_F$$

along the corresponding Hamiltonian vector field $\chi := -F^{-1}(d\,\mathrm{Gscal}_{(F,J)})$. By Theorem 3.8, for any smooth path $J_t \in \mathcal{AGK}_F$ tangent to $\dot{J}$ at J, we have

$$\frac{d}{dt}\|\boldsymbol{\mu}(J_t)\|^2 = 2\boldsymbol{\Omega}_J(-\mathcal{L}_\chi J, \dot{J}).$$

As $\boldsymbol{\Omega}_J$ is nondegenerate, $J \in \mathcal{AGK}_F$ is a critical point of $\|\boldsymbol{\mu}(J)\|^2$ if and only if χ preserves J. ∎

4. The Calabi program for symplectic-type GK structures

Motivated by Calabi's program in Kähler geometry, which seeks extremal and constant scalar curvature Kähler metrics in $\mathcal{K}_\alpha$, the discussion in the previous section naturally leads one to ask a similar question in the symplectic-type generalized Kähler context. First in this section we show that the space $\mathcal{GK}^0_{\pi,\alpha}$ is the formal complexified orbit for $\mathrm{Ham}(M, F)$, further justifying it as a natural generalization of $\mathcal{K}_\alpha$. We then give variational characterizations of extremal and cscGK structures in terms of a Calabi functional and Mabuchi energy, respectively. Using this structure we establish the Calabi–Lichnerowicz–Matsushima-type obstruction, the existence of an extremal vector field, and the Futaki character discussed in the introduction. We end by computing the linearization of the scalar curvature, yielding a natural generalization of the Lichnerowicz operator in this setting.

4.1. The complexified orbits for the action of Ham(M, F)

We start by extending a key observation from [20] to the symplectic-type generalized Kähler setting.

Lemma 4.1. *Let (F_t, J_0) be a Hamiltonian flow deformation of a symplectic-type generalized Kähler structure (F_0, J_0), with respect to a time dependent smooth function ϕ_t. Let $\Phi_t \in \mathrm{Diff}(M)$, $\Phi_0 = \mathrm{Id}$ be the isotopy of diffeomorphisms corresponding to the time dependent vector field $Z_t := -F_t^{-1}(I_t\, d\phi_t)$. Let*

$$J_t := \Phi_t^{-1} \cdot J_0 := (\Phi_t)_*^{-1} J_0 (\Phi_t)_*, \quad \psi_t := \Phi_t^*(\phi_t), \quad Y_{\psi_t} := -F_0^{-1}(d\psi_t).$$

Then, for any t,

$$\Phi_t^*(F_t) = F_0, \quad \dot{J}_t = J_t(\mathcal{L}_{Y_{\psi_t}} J_t).$$

Proof. Using the fact that along the Hamiltonian flow deformation $\dot{F} = dd^c_I\phi$, $Z_t = -F_t^{-1}(I_t\, d\phi_t)$ defines a Moser isotopy, i.e., $\Phi_t^*(F_t) = F_0$. Furthermore, we compute

$$Z_t = -F_t^{-1}(d^c_{I_t}\phi_t) = \frac{1}{2}(I_t + J_0)I_t(d\phi_t) = \frac{1}{2}J_0(I_t + J_0)(d\phi_t) = -J_0 F_t^{-1}(d\phi_t).$$

Letting $J_t = (\Phi_t)_*^{-1} J_0 (\Phi_t)_*$, we get

$$\frac{d}{dt} J_t = \mathcal{L}_{(\Phi_t)^{-1} \cdot Z_t} J_t = \mathcal{L}_{J_t Y_{\psi_t}} J_t = J_t (\mathcal{L}_{Y_{\psi_t}} J),$$

as claimed. ■

Remark 4.2. By virtue of Lemma 3.4, it follows that $F_t \in \mathcal{GK}_{\pi,\alpha}$ is mapped via Φ_t into a curve inside the formal "complexified orbit" of $\mathrm{Ham}(M, F_0)$ in $(\mathcal{AGK}_{F_0}, \mathbf{J})$, which is transversal to the $\mathrm{Ham}(M, F_0)$ orbit of (F_0, J_0).

4.2. The generalized Kähler Calabi functional

We now consider an extension of the Calabi functional to our notion of a generalized Kähler class $\mathcal{GK}_{\pi,\alpha}$, which gives a further variational characterization of extremal metrics. This functional is related to the square norm of the momentum map studied in [36, Section 9] (see also Corollary 3.11), where the notion of extremal generalized Kähler structures was introduced.

Definition 4.3 (Generalized Kähler Calabi functional). We define the Calabi functional by

$$\mathbf{Ca}(F) = \int_M \mathrm{Gscal}^2_{(F,J)} F^{[n]}, \qquad F \in \mathcal{GK}_{\pi,\alpha}.$$

Proposition 4.4. *$F \in \mathcal{GK}_{\pi,\alpha}$ is a critical point of* **Ca** *if and only if F is extremal.*

Proof. This is similar to the proof of the second part of Corollary 3.11. Let F_t be a Hamiltonian flow starting at F and corresponding to ϕ_t. Using a Moser isotopy Φ_t as in Lemma 4.1, we can translate the above property with respect to a path $(F, J_t) \in \mathcal{AGK}_F$, and then compute

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \int_M \mathrm{Gscal}^2_{(F,J_t)} F^{[n]} &= \frac{d}{dt}\Big|_{t=0} \int_M \big(\mathrm{Gscal}_{(F,J_t)} - \bar{\mu}\big)^2 F^{[n]} \\ &= 2\mathbf{\Omega}(-\mathcal{L}_\chi J, \dot{J}) = -2\mathbf{\Omega}(\mathcal{L}_\chi J, J\mathcal{L}_Y J),\end{aligned}$$

where

$$\chi := -F^{-1}\big(d\,\mathrm{Gscal}_{(F,J_0)}\big) \quad \text{and} \quad Y = -F^{-1}(d\phi_0).$$

In the above equalities, we first used that the averaged generalized scalar curvature is constant in t, see Remark 3.9, while the third line follows from the computation in Corollary 3.11, and for the last line we have used Lemma 4.1. The result follows by specializing the above computation with $\phi_t := \mathrm{Gscal}_{(F_t,J_0)}$, we conclude that

$$\boldsymbol{g}(\mathcal{L}_\chi J, \mathcal{L}_\chi J) = 0$$

so that $\mathcal{L}_\chi J = 0$. ■

4.3. The generalized Kähler Mabuchi functional

In [56], Mabuchi further introduced his "K-energy" which is essential to the proof of uniqueness of cscK metrics. Formally it can be thought of as extending the moment map to the complexified orbit of the action of Hamiltonian diffeomorphisms of a Kähler form. A similar phenomenon exists here, although in general we only obtain a one-form corresponding to the differential of the Mabuchi energy. Specifically, we introduce a 1-form $\boldsymbol{\tau}$ on $\mathcal{GK}_{\pi,\alpha}$ by its value at fundamental vector fields $\mathbf{X}_\phi \in \mathbf{T}_F\mathcal{GK}_{\pi,\alpha}$, $\phi \in C_0^\infty(M, dV_F)$, see (2.19):

$$\boldsymbol{\tau}_F(\mathbf{X}_\phi) := -\int_M \phi\, \mathrm{Gscal}_{(F,J)}\, F^{[n]}, \quad F \in \mathcal{GK}_{\pi,\alpha}.$$

Proposition 4.5. *The* 1*-form* $\boldsymbol{\tau}$ *is closed.*

Proof. Using Lemma 2.16, we need to check that

$$\mathbf{X}_\psi\big(\boldsymbol{\tau}(\mathbf{X}_\phi)\big) - \mathbf{X}_\phi\big(\boldsymbol{\tau}(\mathbf{X}_\psi)\big) - \boldsymbol{\tau}\big(\mathbf{X}_{\{\phi,\psi\}_\pi}\big) = 0.$$

Let us examine the first term at a point $F \in \mathcal{GK}^0_{\pi,\alpha}$.

$$\mathbf{X}_\psi\big(\boldsymbol{\tau}(\mathbf{X}_\phi)\big) = -\frac{d}{dt}\int_M \phi\, \mathrm{Gscal}_{(F_t,J)} F_t^{[n]},$$

where F_t is a Hamiltonian generalized Kähler deformation of F generated by ψ. We use a Moser isotopy Φ_t corresponding to the vector field

$$Z_t = -F^{-1}(I_t\, d\psi) = -JF^{-1}(d\psi),$$

as in the proof of Lemma 4.1, and pull back by Φ_t the integrand of the above expression to yield

$$\mathbf{X}_\psi\big(\boldsymbol{\tau}(\mathbf{X}_\phi)\big) = -\frac{d}{dt}\int_M \phi_t\, \mathrm{Gscal}_{(F,J_t)} F^{[n]},$$

where J_t and ϕ_t are the pull-backs of $J_0 = J$ and $\phi_0 = \phi$ under Φ_t. We also put $\psi_t = \Phi_t^*(\psi)$. Using Theorem 3.8 and Lemma 4.1, we compute

$$\begin{aligned}
&-\frac{d}{dt}\int_M \phi_t\, \mathrm{Gscal}_{(F,J_t)} F^{[n]} \\
&\quad = \boldsymbol{\Omega}(\mathbf{Y}_{\phi_t}, \mathbf{J}\mathbf{Y}_{\psi_t}) - \int_M \mathrm{Gscal}_{(F,J_t)} d\phi_t(Z_t) F^{[n]} \\
&\quad = \boldsymbol{\Omega}(\mathbf{Y}_{\phi_t}, \mathbf{J}\mathbf{Y}_{\psi_t}) + \frac{1}{2}\int_M \mathrm{Gscal}_{(F,J_t)} \langle d\phi_t, J_t(I_t + J_t) d\psi_t\rangle_{g_t} F^{[n]} \\
&\quad = \boldsymbol{\Omega}(\mathbf{Y}_{\phi_t}, \mathbf{J}\mathbf{Y}_{\psi_t}) - \frac{1}{2}\int_M \mathrm{Gscal}_{(F,J_t)} \langle d\phi_t, d\psi_t\rangle_{g_t} F^{[n]} \\
&\qquad + \frac{1}{2}\int_M \mathrm{Gscal}_{(F,J_t)} \langle d\phi_t, J_t I_t d\psi_t\rangle_{g_t} F^{[n]}.
\end{aligned}$$

Noting that the two terms at the third line are symmetric in ϕ and ψ, we thus get from the above

$$\begin{aligned}\mathbf{X}_\psi\big(\boldsymbol{\tau}(\mathbf{X}_\phi)\big) - \mathbf{X}_\phi\big(\boldsymbol{\tau}(\mathbf{X}_\psi)\big) &= -\frac{1}{2}\int_M \mathrm{Gscal}_{(F_0,J_t)}\langle d\phi_t, [I_t, J_t]d\psi_t\rangle_{g_t} F_0^{[n]} \\ &= -\frac{1}{2}\int_M \mathrm{Gscal}_{(F_0,J_t)} d\phi_t\big(\pi_t(d\psi_t)\big) F_0^{[n]} \\ &= \int_M \mathrm{Gscal}_{(F_0,J_t)}\{\phi_t, \psi_t\}_\pi F_0^{[n]}.\end{aligned}$$

Pulling back via Φ_t^{-1}, we get the claim. ∎

As $\mathcal{GK}_{\pi,\alpha}$ may have a nontrivial topology, it is not immediately clear whether

$$\boldsymbol{\tau} = \mathbf{dM}$$

for some functional $\mathbf{M}\colon \mathcal{GK}_{\pi,\alpha} \to \mathbb{R}$.

Definition 4.6 (Generalized Kähler Mabuchi functional). We define the *Mabuchi functional* $\mathbf{M}_{F_0}$ to be the primitive of $\boldsymbol{\tau}$ (if it exists) satisfying $\mathbf{M}_{F_0}(F_0) = 0$.

4.4. The generalized Kähler Calabi–Lichnerowicz–Matsushima theorem

Here we prove a structural result on the group of Poisson automorphisms of an extremal GK structure, extending the classical results of Calabi–Lichnerowicz–Matsushima.

4.4.1. The reduced automorphism group. Let $\mathrm{Aut}_0(J, \pi_J)$ be the connected component of the (complex) automorphism group of J and π_J. We denote by

$$\mathfrak{h}(J, \pi_J) := \mathrm{Lie}\big(\mathrm{Aut}_0(J, \pi_J)\big)$$

its Lie algebra, which is a J-invariant subalgebra of the algebra of real holomorphic vector fields on (M, J). We first observe several basic structural results concerning $\mathfrak{h}(F, J)$, extending the theory in the Kähler case.

Let us denote by

$$\mathcal{H}^1_J := \big\{\xi \in \wedge^1(M) \mid d\xi = d^c_J\xi = 0\big\},$$

the space of d and d^c_J-closed 1-forms on (M, J): $\mathcal{H}^1_J$ is just the underlying real vector space of the space of holomorphic 1-forms on (M, J), which are automatically closed by Theorem 2.7.

Given $X \in \mathfrak{h}(J, \pi_J)$, we consider an infinitesimal variation $\mathcal{L}_X F$ of $F \in \mathcal{GK}_{\pi,\alpha}$. By Theorem 2.7, we conclude that 1-form $F(X)$ admits the decomposition

$$F(X) = (I + J)\eta_X + d\phi + I\, d\psi, \tag{4.1}$$

where $\eta_X \in \mathcal{H}^1_J$.

We note that for any $\eta \in \mathcal{H}^1_J$ and any $X \in \mathfrak{h}(J, \pi_J)$,

$$\mathcal{L}_X \eta = 0.$$

Indeed, as any element $\eta \in \mathcal{H}^1_J$ is uniquely determined by its cohomology class $[\eta] \in H^1(M, \mathbb{R})$, and since an infinitesimal symmetry of J preserves both the cohomology classes of closed forms and the space $\mathcal{H}^1_J$, the Lie derivative $\mathcal{L}_X \eta$ must vanish. This observation implies that $\eta(X)$ is a constant. We thus get a real linear map

$$\tau\colon \mathfrak{h}(J, \pi_J) \mapsto (\mathcal{H}^1_J)^*, \quad \tau(X)(\eta) = \eta(X).$$

Since for $X, Y \in \mathfrak{h}(J, \pi_J)$ we have $\eta(X) = \text{const}$ and $\eta(Y) = \text{const}$, and η is closed, it follows that $\eta([X, Y]) = 0$. Therefore, τ is a Lie algebra homomorphism from $\mathfrak{h}(J, \pi_J)$ to the abelian Lie algebra $(\mathcal{H}^1_J)^*$, and its kernel

$$\mathfrak{h}_{\text{red}}(J, \pi_J) := \text{Ker}(\tau) < \mathfrak{h}(J, \pi_J)$$

is an ideal. We have an alternative description of $\mathfrak{h}_{\text{red}}(J, \pi_J)$ as follows.

Lemma 4.7. *The Lie subalgebra $\mathfrak{h}_{\text{red}}(J, \pi_J) \subset \mathfrak{h}(J, \pi_J)$ consists of all elements $X \in \mathfrak{h}(J, \pi_J)$ which can be represented as*

$$X = F^{-1}(d\varphi) + JF^{-1}(d\psi), \quad \phi, \psi \in C^\infty(M, \mathbb{R}). \tag{4.2}$$

Proof. First we prove that X of the form (4.2) lies in $\text{Ker}\,\tau$. Indeed, for any 1-form $\eta \in \mathcal{H}^1_J$, we compute

$$\int_M \eta(X) F^{[n]} = \int_M \text{tr}_F\,(\eta \wedge d\varphi + J^*\eta \wedge d\psi) F^{[n]} = 0,$$

since the forms η, $J^*\eta$ and F are closed. Since $\eta(X)$ is constant on M, this implies that $\eta(X) = 0$, so that X belongs to the kernel of τ.

Now conversely assume that $X \in \mathfrak{h}(J, \pi_J)$ is any element with the corresponding decomposition (4.1). Using the computation in the first part of the proof, we know have

$$\eta_X(X) = \frac{1}{V} \int_M \text{tr}_F \big(\eta_X \wedge (I + J)\eta_X\big) F^{[n]} = \frac{1}{2V} \int_M |(I + J)\eta_X|^2_g F^{[n]} \geq 0,$$

where $V = \int_M F^{[n]}$ and g is the Riemannian metric determined by (F, J). The latter expression is nonzero unless $\eta_X = 0$, as claimed. ∎

Lemma 4.8. *If a vector field $X = F^{-1}(d\phi) + JF^{-1}(d\psi)$ preserves J, then it also automatically preserves π, i.e., $X \in \mathfrak{h}_{\text{red}}(J, \pi_J)$.*

Proof. Taking the Lie derivative $\mathcal{L}_X$ of the identity

$$FJ + J^* F - F \circ \pi \circ F = 0$$

and using the computations of Remark 2.14, we conclude

$$2\big((\mathcal{L}_X F) I\big)_I^{(2,0)+(0,2)} - F \circ \mathcal{L}_X \pi \circ F = 0.$$

Since $\mathcal{L}_X F = d(FJF^{-1}\, d\psi) = dI\, d\psi$ is of I-type (1,1), we conclude that $\mathcal{L}_X \pi = 0$, as claimed. ∎

Motivated by the classical Kähler setup, we make the following definition.

Definition 4.9. Lie algebra of *reduced automorphisms* of (J, π_J) is the kernel of $\tau\colon \mathfrak{h}(J, \pi_J) \to (\mathcal{H}_J^1)^*$, i.e.,

$$\mathfrak{h}_{\mathrm{red}}(J, \pi_J) := \big\{X \in \mathfrak{h}(J, \pi_J) \mid \eta(X) = 0 \text{ for any } \eta \in \mathcal{H}_J^1\big\}.$$

Equivalently, using Lemmas 4.7 and 4.8, $\mathfrak{h}_{\mathrm{red}}(J, \pi_J)$ can be described as

$$\mathfrak{h}_{\mathrm{red}}(J, \pi_J) := \big\{X = F^{-1}(d\varphi) + JF^{-1}(d\psi) \mid \mathcal{L}_X J = 0,\ \phi, \psi \in C^\infty(M, \mathbb{R})\big\}.$$

Remark 4.10. (1) The identification $\mathfrak{h}_{\mathrm{red}}(J, \pi_J) = \mathrm{Ker}(\tau)$ implies that the algebra of reduced automorphisms is independent of $F \in \mathcal{GK}_\pi$, since the space $\mathcal{H}_J^1$, the Lie algebra $\mathfrak{h}(J, \pi_J)$ and the homomorphism τ are all determined by the holomorphic data (J, π_J).

(2) Following the argument of LeBrun and Simanca [50], on a Kähler background (M, J), one can alternatively characterize the ideal $\mathfrak{h}_{\mathrm{red}}(J, \pi_J) < \mathfrak{h}(J, \pi_J)$ as the set of all holomorphic vector fields preserving π_J whose zero set is nonempty.

(3) We also observe that by the second equality in Definition 4.9, $\mathfrak{h}_{\mathrm{red}}(J, \pi_J)$ is invariant under J, so that it can be endowed with a complex Lie algebra structure.

Remark 4.11 (Comparison with the approach of Goto). To put our definition into context let us compare it to the approach in [36, Section 5.1]. Using the formalism of generalized geometry, Goto defines a complex Lie algebra

$$\mathfrak{g}_0 \subset \overline{L}_{\mathbb{J}} \subset \mathrm{End}(TM \oplus T^*M) \otimes \mathbb{C},$$

where $\overline{L}_{\mathbb{J}}$ is the $(-\sqrt{-1})$-eigenspace of $\mathbb{J} := \mathbb{J}_{P,Q}$ (see (2.3)). The Lie bracket on $\mathfrak{g}_0$ is given by the *Dorfman bracket* (see [27, Chapter 2]) on the sections of

$$\mathrm{End}(TM \oplus T^*M) \otimes \mathbb{C}.$$

The underlying *real* Lie algebra $\mathrm{Re}(\mathfrak{g}_0)$ is

$$\mathrm{Re}(\mathfrak{g}_0) = \big\{e \in TM \oplus T^*M \mid e = F^{-1}(d\phi) + \mathbb{J} F^{-1}(d\psi), \\ L_e^{\mathrm{Dor}}\mathbb{J} = 0, \phi, \psi \in C^\infty(M)\big\},$$

where L_e^{Dor} is the infinitesimal action of an element $e \in TM \oplus T^*M$ on $\mathbb{J}$ given by the Dorfman bracket. The Lie algebra structure on $\mathrm{Re}(\mathfrak{g}_0)$ is equivalently given by the F-Poisson bracket on the corresponding smooth complex-valued functions $\phi + i\psi$. One can show that $e := F^{-1}(d\phi) + \mathbb{J} F^{-1}(d\psi)$ is in $\mathrm{Re}(\mathfrak{g}_0)$ if and only if

$$X := F^{-1}(d\phi) + JF^{-1}(d\psi) \in \mathfrak{h}_{\mathrm{red}}(J, \pi_J).$$

This gives rise to a natural isomorphism of real Lie algebras

$$\mathfrak{h}_{\mathrm{red}}(J, \pi_J) \simeq \mathrm{Re}(\mathfrak{g}_0),$$

respecting the underlying complex structures and the F-Poisson bracket on $\{\phi + i\psi\}$.

As in the Kähler case (see, e.g., [29]), the Lie algebra homomorphism

$$\tau\colon \mathfrak{h}(J, \pi_J) \mapsto (\mathcal{H}_J^1)^*$$

can be integrated to a Lie group morphism

$$\hat{\tau}\colon \mathrm{Aut}_0(J, \pi_J) \to (\mathcal{H}_J^1)^*/\Gamma,$$

where $\Gamma = H_1(M, \mathbb{Z})$.

Definition 4.12 (Reduced automorphisms of (M, J, π_J)). The Lie group $\mathrm{Aut}_{\mathrm{red}}(J, \pi_J)$ of *reduced automorphisms* of (J, π_J) is the connected component of identity of the kernel of $\hat{\tau}$ inside $\mathrm{Aut}_0(J, \pi_J)$. Thus $\mathrm{Aut}_{\mathrm{red}}(J, \pi_J) \subset \mathrm{Aut}_0(J, \pi_J)$ is a closed subgroup with Lie algebra $\mathfrak{h}_{\mathrm{red}}(J, \pi_J)$.

4.4.2. The reduced automorphism group of an extremal generalized Kähler manifold. We note that the infinitesimal action of

$$X = F^{-1}(d\phi + I\, d\psi) \in \mathfrak{h}_{\mathrm{red}}(J, \pi_J)$$

on F is $\mathcal{L}_X F = dd_I^c\psi$, which vanishes if and only if $\psi = \mathrm{const}$. Thus we get a Lie subalgebra

$$\mathfrak{k}(F, J) = \big\{X \in \mathfrak{h}_{\mathrm{red}}(J, \pi_J) \mid \mathcal{L}_X F = 0\big\} \\ = \big\{X \in \mathfrak{h}(J, \pi_J) \mid X = F^{-1}(d\varphi)\big\} < \mathfrak{h}_{\mathrm{red}}(J, \pi_J).$$

We observe that $\mathfrak{k}(F, J)$ is the Lie algebra of the Lie group

$$K := \mathrm{Isom}(g) \cap \mathrm{Aut}(I) \cap \mathrm{Aut}_{\mathrm{red}}(J, \pi_J) = \mathrm{Ham}(M, F) \cap \mathrm{Aut}_{\mathrm{red}}(J, \pi_J).$$

Note that, since $\mathrm{Aut}(I)$ and $\mathrm{Aut}_{\mathrm{red}}(J, \pi_J)$ are closed Lie subgroups of $\mathrm{Diff}(M)$ and $\mathrm{Isom}(g)$ is compact, it follows that K is a *compact* Lie group.

Theorem 4.13. *Suppose $F \in \mathcal{GK}_{\pi,\alpha}$ is an extremal generalized Kähler structure with the holomorphic extremal vector field*

$$\chi = -F^{-1}\big(d\,\mathrm{Gscal}_{(F,J)}\big).$$

Denote by $\mathrm{Aut}_{\mathrm{red}}(J, \pi_J)^{\chi} \subset \mathrm{Aut}_{\mathrm{red}}(J, \pi_J)$ the connected subgroup preserving χ. Then the group

$$K = \mathrm{Aut}_{\mathrm{red}}(J, \pi_J) \cap \mathrm{Ham}(M, F)$$

is a maximal compact subgroup of $\mathrm{Aut}_{\mathrm{red}}(J, \pi_J)$ and $\mathrm{Aut}_{\mathrm{red}}(J, \pi_J)^{\chi} = K^{\mathbb{C}}$. In particular, (F, J) must be invariant under a maximal real torus in $\mathrm{Aut}_{\mathrm{red}}(J, \pi_J)$ containing the one-parameter subgroup $\exp(t\chi)$. If F is cscGK, then

$$\chi = 0 \quad \textit{and} \quad \mathrm{Aut}_{\mathrm{red}}(J, \pi_J) = K^{\mathbb{C}}.$$

Proof. We follow the recent treatment [51] which deduces the above properties by formal arguments from the moment map picture. Another relevant reference for this approach is [64]. We want to apply [51, Theorem 3.3]. In our setting, we consider the (holomorphic) action of $\mathrm{Ham}(M, F)$ on $(\mathcal{AGK}_F, \mathbf{J})$ endowed with the formal Kähler structure $\boldsymbol{\Omega}$, and with momentum map $\boldsymbol{\mu}\colon \mathcal{AGK}_F \to (C_0^{\infty}(M, dV_F))^*$:

$$\langle \boldsymbol{\mu}(J), \phi\rangle := -\int_M \mathrm{Gscal}_{(F,J)}\, \phi F^{[n]}, \quad J \in \mathcal{AGK}_F,\ \phi \in C_0^{\infty}(M, dV_F).$$

Recall that we identify the Lie algebra $\mathfrak{ham}(M, F)$ of Hamiltonian vector fields with the space of normalized smooth functions $C_0^{\infty}(M, dV_F)$ equipped with the F-Poisson bracket $\{\cdot, \cdot\}_F$, through the standard Lie algebra isomorphism

$$C_0^{\infty}(M, dV_F) \ni \phi \mapsto Y_\phi := -F^{-1}(d\phi) \in \mathfrak{ham}(M, F).$$

We further consider the ad-invariant inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $\mathfrak{ham}(M, F)$, defined by

$$\langle\!\langle \phi_1, \phi_2 \rangle\!\rangle := \int_M \phi_1\phi_2 F^{[n]}, \quad \phi_1, \phi_2 \in C_0^{\infty}(M, dV_F) \simeq \mathfrak{ham}(M, F).$$

The real Lie algebra $(C_0^{\infty}(M, dV_F), \{\cdot, \cdot\}_F) \simeq \mathfrak{ham}(M, J)$ can be complexified, and is then denoted by

$$\mathfrak{ham}(M, F)^{\mathbb{C}} = \big\{-F^{-1}(d\phi) - JF^{-1}(d\psi)\big\} \simeq \{\phi + i\psi\}, \quad \phi, \psi \in C_0^{\infty}(M, dV_F).$$

There is an "infinitesimal action" of $\mathfrak{ham}(M, F)^{\mathbb{C}}$ on $\mathcal{AGK}_F$, such that $\phi + i\psi$ is mapped to the vector field

$$J \to \mathcal{L}_{F^{-1}(d\phi)} J + \mathbf{J}\mathcal{L}_{F^{-1}(d\psi)} J \in \mathbf{T}_J(\mathcal{AGK}_F).$$

Suppose that $J \in \mathcal{AGK}_F$ is a generalized Kähler structure, i.e., J and I are both integrable. Then by Lemma 4.8 the stabilizer of J in $\mathfrak{ham}(M, F)^{\mathbb{C}}$ is $\mathfrak{h}_{\mathrm{red}}(J, \pi_J)$.

We now consider the functional $\|\boldsymbol{\mu}(J)\|^2$ on the formal Kähler Frèchet manifold $(\mathcal{AGK}_F, \mathbf{\Omega}, \mathbf{J})$ as in Corollary 3.11, which is the square-norm of the momentum map $\boldsymbol{\mu}$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Let $J \in \mathcal{AGK}_F$ be a generalized Kähler structure which is a critical point of $\|\boldsymbol{\mu}(J)\|^2$, i.e., an extremal generalized Kähler structure by Corollary 3.11. Then the extremal vector field

$$\chi := -F^{-1}\big(d\,\mathrm{Gscal}_{(F,J)}\big)$$

belongs to the center of $\mathfrak{k}(F, J)$. By applying the generalized Calabi–Lichnerowicz–Matsushima decomposition (see [51, Theorem 3.3]) to the convex functional $\|\boldsymbol{\mu}\|^2$ on

$$\big(\mathcal{AGK}_F, \mathbf{\Omega}, \mathbf{J}, \mathfrak{ham}(M, F), \langle\!\langle \cdot, \cdot \rangle\!\rangle\big),$$

there is a semidirect splitting $\mathfrak{h}_{\mathrm{red}}(J, \pi_J) = \mathfrak{h}_{\mathrm{red}}(J, \pi_J)^{\chi} + \mathfrak{s}(J, \pi_J)$ where the centralizer $\mathfrak{h}_{\mathrm{red}}(J, \pi_J)^{\chi}$ of χ is a reductive algebra satisfying

$$\mathfrak{h}_{\mathrm{red}}(J, \pi_J)^{\chi} = \mathfrak{k}(F, J) \otimes \mathbb{C},$$

whereas $\mathfrak{s}(J, \pi_J)$ is a solvable ideal. This shows that $\mathfrak{h}_{\mathrm{red}}(J, \pi_J)^{\chi}$ is a maximal reductive Lie subalgebra of $\mathfrak{h}_{\mathrm{red}}(J, \pi_J)$. The claims follow from this.

As $\mathfrak{k}(F, J)$ is the Lie algebra of K, it follows that $K < \mathrm{Aut}_{\mathrm{red}}(J, \pi_J)$ is a maximal compact subgroup, which is therefore connected as $\mathrm{Aut}_{\mathrm{red}}(J, \pi_J)$ is (see, e.g., [57, Theorem 11]). It then also follows from the above that $K^{\mathbb{C}} = \mathrm{Aut}_{\mathrm{red}}(J, \pi_J)^{\chi}$. ■

Remark 4.14. In the case $b_1(M) = 0$, we have $\mathcal{H}^1_J = \{0\}$, and then $\mathrm{Aut}_{\mathrm{red}}(J, \pi_J) = \mathrm{Aut}_0(J, \pi_J)$ is just the connected component of the identity of the group of Poisson automorphisms of (J, π_J). Furthermore, in this case $\mathfrak{k}(F, J)$ reduces to the Lie algebra of Killing fields of (F, J), so that K_0 is just the connected component of the isometry group of (F, J).

Remark 4.15. (1) Theorem 4.13 gives obstructions to the existence of extremal, non-cscGK generalized Kähler structures. We illustrate this on the following example. Let us consider the second Hirzebruch complex surface

$$(M, J) = \mathbf{F}_2 := \mathbf{P}\big(\mathcal{O} \oplus \mathcal{O}(-2)\big) \to \mathbf{CP}^1,$$

endowed with its Liouville Poisson structure π^L_J, that is, π^L_J is the inverse of the Liouville symplectic form on $\mathcal{O}(-2) = T^*\mathbf{CP}^1$. Let ω be a Kähler metric on (M, J), defining a de Rham class $\alpha = [\omega]$. By [33, Theorem 8.15] or [41, Corollary 7.3], one can deform ω in order to obtain on (M, J) a $t\pi^L_J$-compatible symplectic-type GK structure F_t, for $|t| < \varepsilon$ with $F_0 = \omega$. As $H^{0,2}(M, J) = 0$, the arguments in the proof of [41, Theorem 7.1] actually show that $F_t \in \alpha$. Notice that the automorphism group of the base $\mathbf{CP}^1$ preserves π_J^L and, in fact,

$$\mathrm{Aut}_0(J, \pi_J^L) = \mathrm{PGL}(2, \mathbb{C}).$$

On the other hand, the $\mathbb{C}^*$-action on the fibers of $\mathbf{F}_2$ preserves α and scales π_J^L, so by acting with an element of $\mathbb{C}^*$, the above results yield a symplectic-type generalized Kähler structure $F \in \mathcal{GK}_{\pi,\alpha}(M, J)$. As $\mathrm{PGL}(2, \mathbb{C})$ is a semisimple group (and thus has a trivial center), by Theorem 4.13 any extremal generalized structure in $\mathcal{GK}^0_{\pi,\alpha}(M, J)$ must be cscGK. Theorem 4.13 gives no further obstructions for the existence of a cscGK metric in $\mathcal{GK}^0_{\pi,\alpha}(M, J)$. It is thus interesting to know whether or not $\mathcal{GK}^0_{\pi,\alpha}(M, J)$ does admit a $\mathrm{PU}(2)$-invariant cscGK structure. At the same time, it is well known that $\mathcal{GK}_{0,\alpha}(M, J) = \mathcal{K}_\alpha(M, J)$ does not admit a Kähler metric of constant scalar curvature. In fact, it admits an extremal Kähler metric with nonconstant scalar curvature [11].

(2) Still considering the same example, an interesting phenomenon related to the connectedness of $\mathcal{GK}_{\pi,\alpha}$ and the size of π appears. Let us denote by I the second complex structure defined by (F, J). It is shown [48, Remark, p. 6] that

$$(M, I) \simeq \mathbf{CP}^1 \times \mathbf{CP}^1.$$

The corresponding holomorphic Poisson tensor π_I on (M, I) has zeros of order 2 along the diagonal $\Delta \subset \mathbf{CP}^1 \times \mathbf{CP}^1$. We find, therefore, that

$$\mathrm{Aut}_0(I, \pi_I) = \mathrm{PGL}(2, \mathbb{C}),$$

which is also the stabilizer of Δ inside $\mathrm{Aut}_0(M, I) = \mathrm{PGL}(2, \mathbb{C}) \times \mathrm{PGL}(2, \mathbb{C})$. Notice that, unlike (M, J, π_J^L), we cannot rescale π_I with an element of $\mathrm{Aut}_0(M, I)$. As

$$H^{2,0}(M, I) = H^{0,2}(M, I) = 0 \quad \text{and} \quad H^2(M, \mathbb{R}) = H^{1,1}(M, \mathbb{R}),$$

α is a $(1, 1)$-class on (M, I). As $F \in \alpha$ tames I, α satisfies the Nakai–Moishezon positivity condition [18] and thus is a Kähler class, i.e., there exists a Kähler structure $\omega' \in \alpha$ on (M, I). Using [33, Theorem 8.15] and [41, Corollary 7.3] on (M, I, ω'), one gets symplectic-type generalized Kähler structures $F'_t \in \mathcal{GK}_{t\pi,\alpha}(M, I)$, defined for $|t| < \varepsilon'$. Notice that the second complex structures J'_t are now biholomorphic to I: this follows for instance by the fact that $H^1(M, T_I^{1,0}M) = 0$, i.e., $\mathbf{CP}^1 \times \mathbf{CP}^1$ is rigid. If we were able to obtain such deformations up to $t = 1$, we will have for $F' := F'_1$ that $F, F' \in \mathcal{GK}_{\pi,\alpha}(M, I)$, but F' and F cannot be in the same connected component $\mathcal{GK}^0_{\pi,\alpha}(M, I)$ as J' and J are not biholomorphic.

4.5. The generalized Kähler extremal vector field

The momentum map interpretation of the generalized Kähler scalar curvature leads to an alternative intrinsic description of the extremal vector field.

Proposition 4.16. *Let $G \subset \mathrm{Ham}(M, F)$ be a compact subgroup with Lie algebra $\mathfrak{g}$. Let $\mathcal{AGK}_F^G$ be the space of G-invariant structures in $\mathcal{AGK}_F$. For an element $a \in \mathfrak{g}$ let $\phi := \phi_{a,F} \in C_0^\infty(M, dV_F)$ be the corresponding Hamiltonian. Then the integral*

$$\int_M \phi_{a,F}\ \mathrm{Gscal}_{(F,J)}\ dV_F$$

is independent of $J \in \mathcal{AGK}_F^G$. That is, the $L^2(M, dV_F)$-projection $\Pi_F(\mathrm{Gscal}_{(F,J)})$ of $\mathrm{Gscal}_{(F,J)}$ on the space of Hamiltonian potentials $\{\phi_{a,F} \mid a \in \mathfrak{g}\} \subset C_0^\infty(M, dV_F)$ is constant on $\mathcal{AGK}_F^G$. In particular, the vector field $\chi \in \mathfrak{g}$,

$$\chi = -F^{-1}\big(d\Pi_F(\mathrm{Gscal}_{(F,J)})\big)$$

is independent of $J \in \mathcal{AGK}_G^G$.

Proof. The claim follows from Theorem 3.8 along the lines of proof of Corollary 3.11. Let $J_0, J_1 \in \mathcal{AGK}_F^G$ be two invariant almost complex structure. Using the Cayley transform we can connect them via a path $J_t \in \mathcal{AGK}_F^G$:

$$J_t = (1 + tS)J_0(1 + tS)^{-1}, \quad S = (J_1 + J_0)^{-1}(J_0 - J_1).$$

Consider a Hamiltonian vector field $Y_\phi = -F^{-1}(d\phi) \in \mathfrak{g}$ on (M, F). Let $\mathbf{Y}_\phi(J) = -\mathcal{L}_{Y_\phi} J$ be the induced fundamental vector field on $\mathcal{AGK}_F$. As $\mathbf{Y}_\phi$ preserves any G-invariant element of $\mathcal{AGK}_F$, we have $\mathbf{Y}_\phi(J) = 0$ for $J \in \mathcal{AGK}_F^G$, and thus

$$\mathbf{\Omega}_J(\mathbf{Y}_\phi, \cdot) = 0.$$

We apply this to a path $J_t \in \mathcal{AGK}_F^G$: by the definition of momentum map, we have

$$\frac{d}{dt}\int_M \phi\, \mathrm{Gscal}_{(F,J_t)}\, dV_F = \mathbf{\Omega}_J(\mathbf{Y}_\phi, \dot{J}) = 0.$$

This shows that the $L^2(M, dV_F)$ projection $\mathrm{Gscal}_{(F,J_t)}$ onto the space of normalized Hamiltonian potentials of $\mathfrak{g}$ is constant. ■

Now we will change the point of view by fixing the *holomorphic* data (J, π_J) and varying $F \in \mathcal{GK}_{\pi,\alpha}$. Suppose (F, J) is an extremal generalized Kähler structure. Then by Theorem 4.13 the extremal vector field $\chi = -F^{-1}(d\, \mathrm{Gscal}_{(F,J)})$ is contained in the Lie algebra $\mathfrak{t}$ of a maximal torus $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(J, \pi_J) \cap \mathrm{Ham}(M, F)$. It turns out that by the means of Proposition 4.16, vector field χ can be defined intrinsically from any $\mathbb{T}$-invariant generalized Kähler structure F_0, using the $L^2(M, dV_F)$ projection

$$\Pi_{F_0}: C^\infty(M,\mathbb{R}) \mapsto C^\infty(M,\mathbb{R})$$

of $\mathrm{Gscal}_{(F_0,J)}$ onto the space of F_0-Hamiltonians of $\mathbb{T}$. In particular, χ being nonzero provides an obstruction for the existence of $\mathbb{T}$-invariant cscGK structures in $\mathcal{GK}_{\pi,\alpha}$.

Theorem 4.17 (Extremal vector field). *Given a torus* $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(J,\pi_J)$ *let* $\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ *be the space of* $\mathbb{T}$*-invariant generalized Kähler structures. Then, for any* $F_0 \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$, *necessarily* $\mathbb{T} \subset \mathrm{Ham}(M, F_0)$. *Moreover, the vector field*

$$\chi = -F_0^{-1}\big(d\Pi_{F_0}\big(\mathrm{Gscal}_{(F_0,J)}\big)\big) \tag{4.3}$$

is independent of the choice of a $\mathbb{T}$*-invariant symplectic form* $F_0 \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$. *If, furthermore,* $\mathbb{T}$ *is a maximal torus in* $\mathrm{Aut}_{\mathrm{red}}(J,\pi_J)$, *then* $F \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ *is an extremal structure if and only if*

$$\mathrm{Gscal}_{(F,J)} - \Pi_F\big(\mathrm{Gscal}_{(F,J)}\big) = 0.$$

In particular, the underlying extremal vector field $-F^{-1}(d\,\mathrm{Gscal}_{(F,J)})$ *is necessarily given by* (4.3).

Proof. First we prove that $\mathbb{T} \subset \mathrm{Aut}_{\mathrm{red}}(J,\pi_J)$ acts in a Hamiltonian fashion. Indeed, any vector $X \in \mathfrak{h}_{\mathrm{red}}(J,\pi_J)$ in its Lie algebra has a decomposition

$$X = F_0^{-1}(d\phi) + F_0^{-1}(I\, d\psi),$$

see Definition 4.9. Since F_0 is $\mathbb{T}$-invariant, we have $\mathcal{L}_X F_0 = 0$, which implies that $dd^c_I\psi = 0$, so $\psi = \mathrm{const}$ and X is a Hamiltonian vector field.

For the claim about the constancy of χ on $\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$, take another $F_1 \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$. Then applying an equivariant Moser isotopy we find a $\mathbb{T}$-equivariant diffeomorphism $\Phi\colon M \to M$ such that $\Phi^*(F_1) = F_0$. Then we have $J, \Phi^*(J) \in \mathcal{AGK}^{\mathbb{T}}_{F_0}$, and we can apply Proposition 4.16 to conclude that

$$\chi = -F_0^{-1}\big(d\Pi_{F_0}\big(\mathrm{Gscal}_{(F_0,J)}\big)\big) = -F_0^{-1}\big(d\Pi_{F_0}\big(\mathrm{Gscal}_{(F_0,\Phi^*(J))}\big)\big)$$

Since Φ is $\mathbb{T}$-equivariant, we observe that $(\Phi^*)\chi = \chi$ so that applying $(\Phi^*)^{-1}$ to the last term we deduce

$$-F_0^{-1}\big(d\Pi_{F_0}\big(\mathrm{Gscal}_{(F_0,J)}\big)\big) = -F_1^{-1}\big(d\Pi_{F_1}\big(\mathrm{Gscal}_{(F_1,J)}\big)\big),$$

as claimed.

Finally, assume that $\mathbb{T}$ is a maximal torus in $\mathrm{Aut}_{\mathrm{red}}(J,\pi_J)$ and $F \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ is an extremal generalized Kähler structure with the extremal vector field

$$\chi_{\mathrm{ext}} = -F\big(d\,\mathrm{Gscal}_{(F,J)}\big).$$

Since (F, J) and hence χ_{ext} is invariant under $\mathbb{T}$, by the maximality of $\mathbb{T}$ and Theorem 4.13 we have $\chi_{\mathrm{ext}} \in \mathfrak{t}$, so that

$$\Pi_F \operatorname{Gscal}_{(F,J)} = \operatorname{Gscal}_{(F,J)} \quad \text{and} \quad \chi_{\mathrm{ext}} = \chi.$$

Conversely, if $\Pi_F \operatorname{Gscal}_{(F,J)} = \operatorname{Gscal}_{(F,J)}$, then

$$\chi = -F^{-1}\big(d \operatorname{Gscal}_{(F,J)}\big) \in \mathfrak{t} \subset \mathfrak{h}_{\mathrm{red}}(J, \pi_J)$$

is holomorphic, and (F, J) is extremal. ∎

4.6. Futaki character

In this section we introduce a generalized Kähler analogue of the Futaki character on $\mathfrak{h}_{\mathrm{red}}(J, \pi_J)$. Recall that given a generalized Kähler structure $F \in \mathcal{GK}_{\pi,\alpha}$, we have a decomposition (see 4.9) of a holomorphic vector field $X \in \mathfrak{h}_{\mathrm{red}}(J, \pi_J)$:

$$X = F^{-1}\big(d\phi_{(X,F)}\big) + F^{-1}\big(I\, d\psi_{(X,F)}\big), \quad \phi_{(X,F)}, \psi_{(X,F)} \in C_0^{\infty}(M, dV_F).$$

Theorem 4.18 (Futaki character). *Let $F \in \mathcal{GK}_{\pi,\alpha}$ be a generalized Kähler structure on a (M, J, π_J). Define a homomorphism $\mathcal{F}_{(F,J)}\colon \mathfrak{h}_{\mathrm{red}}(J, \pi_J) \to \mathbb{R}$ by*

$$\mathcal{F}_{(F,J)}(X) = \int_M \psi_{(X,F)} \operatorname{Gscal}_{(F,J)}\, dV_F = -\boldsymbol{\tau}\big(\mathbf{X}_{\psi_{(X,F)}}\big).$$

Then $\mathcal{F}_{(F,J)}$ is independent of $F \in \mathcal{GK}^0_{\pi,\alpha}$ and vanishes on the commutator

$$[\mathfrak{h}_{\mathrm{red}}(J, \pi_J), \mathfrak{h}_{\mathrm{red}}(J, \pi_J)].$$

In particular, $\mathcal{F}_{(F,J)}$ is a character of $\mathfrak{h}_{\mathrm{red}}(J, \pi_J)$ and is identically zero if $\mathcal{GK}^0_{\pi,\alpha}$ admits a cscGK metric.

Proof. The infinitesimal action of the vector field $X \in \mathfrak{h}_{\mathrm{red}}(J, \pi_J)$ on $F \in \mathcal{GK}_{\pi,\alpha}$ is given by $\mathcal{L}_X F = dd^c_I \psi_{(X,F)}$ thus it induces a vector field $\mathbf{X}$ on $\mathcal{GK}_{\pi,\alpha}$, which at a point $F \in \mathcal{GK}_{\pi,\alpha}$ is given by

$$\mathbf{X}(F) = \mathbf{X}_{\psi_{(X,F)}}.$$

Now, the 1-form $\boldsymbol{\tau}$ on $\mathcal{GK}^0_{\pi,\alpha}$ is invariant under the vector field $\mathbf{X}$, since it is induced by an action of $\operatorname{Diff}(M)$ on the entire structure (F, J). Thus by Cartan's formula we have

$$0 = d\big(\boldsymbol{\tau}(\mathbf{X})\big) + d\boldsymbol{\tau}(\mathbf{X}, \cdot).$$

Since $\boldsymbol{\tau}$ is closed, it follows that $\boldsymbol{\tau}(\mathbf{X})$ is constant on $\mathcal{GK}^0_{\pi,\alpha}$ which is equivalent to the statement of the constancy of $\mathcal{F}_{(F,J)}$ on $\mathcal{GK}^0_{\pi,\alpha}$.

Now, if $Y \in \mathfrak{h}_{\rm red}(J, \pi_J)$ is another holomorphic vector field inducing a vector field $\mathbf{Y}$ on $\mathcal{GK}_{\pi,\alpha}$, then

$$\mathcal{F}_{(F,J)}\big([X,Y]\big) = -\boldsymbol{\tau}\big([\mathbf{X},\mathbf{Y}]\big) = d\boldsymbol{\tau}(\mathbf{X},\mathbf{Y}) + \mathbf{Y}\cdot\boldsymbol{\tau}(\mathbf{X}) - \mathbf{X}\cdot\boldsymbol{\tau}(\mathbf{Y}) = 0,$$

as claimed, so that $\mathcal{F}_{(F,J)}$ is a character of the Lie algebra $\mathfrak{h}_{\rm red}(J, \pi_J)$. ∎

4.7. Linearization of the normalized generalized scalar curvature

We present here the linearization of $\mathrm{Gscal}_{(F_t,J)}$ when F_t varies within a given generalized Kähler class $\mathcal{GK}_{\pi,\alpha}$, similar to [36, Section 6]. Motivated by Theorem 4.17 we will, more generally, fix a compact torus $\mathbb{T} \subset \mathrm{Aut}_{\rm red}(M, J, \pi_J)$ and consider the space $\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ of $\mathbb{T}$-invariant elements of $\mathcal{GK}_{\pi,\alpha}$, and define a notion of $\mathbb{T}$-*normalized* scalar curvature as follows.

Definition 4.19. For any $F \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$, denote by $\mathfrak{t}_F$ the vector space of smooth functions f such that $-F^{-1}(df) \in \mathfrak{t} := \mathrm{Lie}(\mathbb{T})$ and by Π_F the $L^2(M, dV_F)$-orthogonal projection of $C^\infty(M,\mathbb{R})$ to $\mathfrak{t}_F$. Then the $\mathbb{T}$-*normalized generalized scalar curvature* is given by

$$\mathrm{Gscal}^{\mathbb{T}}_{(F,J)} := \mathrm{Gscal}_{(F,J)} - \Pi_F\big(\mathrm{Gscal}_{(F,J)}\big).$$

Lemma 4.20. *Suppose that F_t is a smooth path of generalized Kähler structures in $\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ corresponding to a Hamiltonian deformation with a $\mathbb{T}$-invariant function $\phi \in (C^\infty(M,\mathbb{R}))^{\mathbb{T}}$. Then, for any smooth function $\psi \in (C^\infty(M,\mathbb{R}))^{\mathbb{T}}$, we have*

$$\int_M \psi\left(\frac{d}{dt}\Big|_{t=0}\mathrm{Gscal}^{\mathbb{T}}_{(F_t,J)}\right)F_0^{[n]} = -\mathbf{g}_J\big(\mathcal{L}_{F_0^{-1}(d\phi)}J, \mathcal{L}_{F_0^{-1}(d\psi)}J\big)$$
$$+ \int_M \psi\big(\mathrm{tr}_{F_0}\big(dd^c_J\phi - dd^c_I\phi + d\,\mathrm{Gscal}^{\mathbb{T}}_{(F_0,J)}\wedge J\,d\phi\big)\big)F_0^{[n]},$$

where $\mathbf{g}_J$ is defined in Lemma 3.7.

Proof. Notice that, by the assumption for $\mathbb{T}$ with respect to F_0, for any $F \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ we have a Lie algebra isomorphism

$$\big(\mathfrak{t}_F/\mathbb{R}, \{\cdot,\cdot\}_F\big) \simeq \big(\mathfrak{t}, [\cdot,\cdot]\big),$$

i.e., each vector field $Y \in \mathfrak{t}$ is F-Hamiltonian. Denote by $\mu_{F_0}\colon M \to \mathfrak{t}^*$ the momentum map of $\mathbb{T}$ with respect to F_0, with momentum image a polytope P. Using a $\mathbb{T}$-equivariant Moser isotopy with respect to $F \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ (notice that $(1-t)F_0 + tF$ defines a $\mathbb{T}$-invariant isotopy of symplectic forms as they tame J for all t), it follows that there is a uniquely determined momentum map $\mu_F\colon M \to$ P. Furthermore, the Π_{F_0}-projection of $\mathrm{Gscal}_{(F_0,J)}$ is of the form $\mu^*_{F_0}\ell$, where ℓ is an affine-linear function on $\mathfrak{t}^*$. By the $\mathbb{T}$-equivariant Moser lemma and Theorem 4.17, we have

$$\Pi_F\left(\mathrm{Gscal}_{(F,J)}\right) = \mu_F^*(\ell)$$

for the same affine-linear function ℓ. We now use a slight modification of the computation in the proof of Proposition 4.5: letting $\psi_t := \Phi_t^*(\psi)$ and $J_t := (\Phi_t)_* J (\Phi_t)_*^{-1}$, where Φ_t is the isotopy defined in Lemma 4.1, we have

$$\begin{aligned}
&\int_M \left(\frac{d}{dt}\Big|_{t=0} \mathrm{Gscal}^{\mathbb{T}}_{(F_t,J)}\right)\psi F_0^{[n]} \\
&\quad = \frac{d}{dt}\Big|_{t=0}\int_M \mathrm{Gscal}^{\mathbb{T}}_{(F_t,J)}\,\psi F_t^{[n]} - \int_M \mathrm{Gscal}^{\mathbb{T}}_{(F_0,J)}\,\psi\, d d_I^c\phi \wedge F_0^{[n-1]} \\
&\quad = \frac{d}{dt}\Big|_{t=0}\int_M \mathrm{Gscal}^{\mathbb{T}}_{(F_0,J_t)}\,\psi_t F_0^{[n]} - \int_M \mathrm{Gscal}^{\mathbb{T}}_{(F_0,J)}\,\psi\, d d_I^c\phi \wedge F_0^{[n-1]} \\
&\quad = \frac{d}{dt}\Big|_{t=0}\int_M \left(\mathrm{Gscal}_{(F_0,J_t)} - \mu^*_{F_0}(\ell)\right)\psi_t F_0^{[n]} - \int_M \mathrm{Gscal}^{\mathbb{T}}_{(F_0,J)}\,\psi\, d d_I^c\phi \wedge F_0^{[n-1]} \\
&\quad = -\mathbf{g}_J\left(\mathcal{L}_{F_0^{-1}(d\phi)}J, \mathcal{L}_{F_0^{-1}(d\psi)}J\right) + \frac{1}{2}\int_M \mathrm{Gscal}^{\mathbb{T}}_{(F_0,J)}\langle d\phi, J(I+J)\,d\psi\rangle_{g_0} F_0^{[n]} \\
&\qquad - \int_M \mathrm{Gscal}^{\mathbb{T}}_{(F_0,J)}\,\psi\, d d_I^c\phi \wedge F_0^{[n-1]}.
\end{aligned}$$

Notice that (cf. (2.5), (2.6))

$$\frac{1}{2}\langle d\phi, J(I+J)\,d\psi\rangle_{g_0} = \frac{1}{2}\langle d\psi, (I+J)J\,d\phi\rangle_{g_0} = -\operatorname{tr}_{F_0}(d\psi \wedge J\,d\phi).$$

Substituting in the previous identity and integrating by parts yields the claim. ∎

Corollary 4.21. *Suppose (F_0, J) is an extremal symplectic-type generalized Kähler structure and let $\mathbb{T}$ be a torus in* $\mathrm{Ham}(M, F_0) \cap \mathrm{Aut}(M, J)$ *containing the exponential of the extremal vector field $\chi = -F_0^{-1}(d\,\mathrm{Gscal}_{(F,J)})$. Then the linearization of $\mathrm{Gscal}^{\mathbb{T}}_{(F,J)}$ at F_0 in the space $\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ is a fourth-order linear operator on $(C^\infty(M,\mathbb{R}))^{\mathbb{T}}/\mathbb{R}$, whose kernel is the space of F_0-Hamiltonian holomorphic vector fields on (M, J) commuting with χ.*

Proof. By the definition of $\mathbb{T}$-normalized scalar curvature, if (F_0, J) is extremal then $\mathrm{Gscal}^{\mathbb{T}}_{(F_0,\mathbb{T})} = 0$. The first part of the claim then follows from Lemmas 4.20 and 3.7, noticing that (after integrating by parts) the right-hand side of the expression in Lemma 4.20 vanishes for $\psi = \mathrm{const}$. Furthermore, to identify the kernel of the linearization, denoted here T_0, we note using (2.13) that

$$\int_M T_0(\psi)\psi F_0^n = -\mathbf{g}_J\left(\mathcal{L}_{F_0^{-1}(d\psi)}J, \mathcal{L}_{F_0^{-1}(d\psi)}J\right),$$

and the claim follows. ∎

Remark 4.22. In order to obtain a formula for the linearization T_0 of $\mathrm{Gscal}^{\mathbb{T}}_{(F_0,J)}$, one needs to further integrate by parts the expression in the right-hand side of the integral formula above, using the rather involved definition of $\mathbf{g}_J$ in Lemma 3.7. We did not attempt to achieve this in the current paper.

5. Formal Riemannian structure on $\mathcal{GK}_{\pi,\alpha}$ and uniqueness

5.1. The Riemannian metric on $\mathcal{GK}_{\pi,\alpha}$ and geodesics

We now introduce a formal Riemannian metric on the space $\mathcal{GK}_{\pi,\alpha}$, generalizing the Mabuchi–Semmes–Donaldson Riemannian structure [20, 56, 60] on $\mathcal{K}_\alpha$. Recall that at any $F \in \mathcal{GK}_{\pi,\alpha}$, we showed that the tangent space $\mathbf{T}_F(\mathcal{GK}_{\pi,\alpha})$ is identified with $C^\infty(M,\mathbb{R})/\mathbb{R}$, see Remark 2.18. For any given $F \in \mathcal{GK}_{\pi,\alpha}$, we will further identify $C^\infty(M,\mathbb{R})/\mathbb{R}$ with the space of *F-normalized* smooth functions,

$$\mathbf{T}_F(\mathcal{GK}_{\pi,\alpha}) \simeq C^\infty_0(M, dV_F) := \left\{\phi \in C^\infty(M,\mathbb{R}) \mid \int_M \phi F^{[n]} = 0\right\}.$$

We can then define a (formal) Riemannian metric $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathcal{GK}_{\pi,\alpha}$, by letting

$$\langle\langle\phi_1,\phi_2\rangle\rangle_F := \int_M \phi_1\phi_2 F^{[n]}, \quad \phi_1,\phi_2 \in C^\infty_0(M, dV_F).$$

We note by Stokes' theorem that at any fixed point F, the inner product $\langle\langle\cdot,\cdot\rangle\rangle_F$ is ad-invariant with respect the Poisson bracket of F, i.e., satisfies

$$\langle\langle\{\phi_1,\phi_3\}_F,\phi_2\rangle\rangle_F + \langle\langle\phi_1,\{\phi_2,\phi_3\}_F\rangle\rangle_F = 0, \quad \forall \phi_1,\phi_2,\phi_3 \in C^\infty_0(M, dV_F). \quad (5.1)$$

Lemma 5.1. *Let $F \to \boldsymbol{\xi}_F \in \mathbf{T}_F(\mathcal{GK}_{\pi,\alpha}) = C^\infty_0(M, dV_F)$ be a formal vector field on $\mathcal{GK}_{\pi,\alpha}$. Then the Riemannian structure $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathcal{GK}$ admits a unique Levi–Civita connection $\mathcal{D}$, defined by*

$$(\mathcal{D}_\phi \boldsymbol{\xi})_F = \dot{\boldsymbol{\xi}}_F(\phi) - \mathrm{tr}_F(d\phi \wedge J\, d\boldsymbol{\xi}_F), \quad \phi \in C^\infty_0(M, dV_F) = \mathbf{T}_F(\mathcal{GK}_{\pi,\alpha}), \quad (5.2)$$

where $\dot{\boldsymbol{\xi}}_F(\phi) := \frac{d}{dt}|_{t=0}\boldsymbol{\xi}_{F_t}$, with F_t being the Hamiltonian deformation of F defined by ϕ.

Proof. We will first establish (5.2) for a fundamental vector field $\boldsymbol{\xi} = \mathbf{X}_\psi$ (cf. Definition 2.15). In what follows, we assume (without loss) that $\int_M F^{[n]} = 1$ for any $F \in \mathcal{GK}_{\pi,\alpha}$. We will identify $\mathbf{X}_\psi$ at $F \in \mathcal{GK}_{\pi,\alpha}$ with the normalized function given by

$$\boldsymbol{\xi}_F = \psi - \int_M \psi F^{[n]}.$$

Letting $\mathcal{D}$ denote the formal connection which preserves $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and is torsion-free and we will compute it using the Koszul's formula for fundamental vector fields $\boldsymbol{\xi}_i = \mathbf{X}_{\phi_i}$, $i = 1, 2, 3$. Applying Lemma 2.16, we have

$$2\langle\!\langle \mathcal{D}_{\boldsymbol{\xi}_1} \boldsymbol{\xi}_2, \boldsymbol{\xi}_3 \rangle\!\rangle = \mathbf{X}_{\phi_1} \cdot \langle\!\langle \mathbf{X}_{\phi_2}, \mathbf{X}_{\phi_3} \rangle\!\rangle + \mathbf{X}_{\phi_2} \cdot \langle\!\langle \mathbf{X}_{\phi_1}, \mathbf{X}_{\phi_3} \rangle - \mathbf{X}_{\phi_3} \cdot \langle\!\langle \mathbf{X}_{\phi_1}, \mathbf{X}_{\phi_2} \rangle\!\rangle \\ - \langle\!\langle \mathbf{X}_{\{\phi_1,\phi_2\}_\pi}, \mathbf{X}_{\phi_3} \rangle\!\rangle + \langle\!\langle \mathbf{X}_{\{\phi_2,\phi_3\}_\pi}, \mathbf{X}_{\phi_1} \rangle\!\rangle + \langle\!\langle \mathbf{X}_{\{\phi_1,\phi_3\}_\pi}, \mathbf{X}_{\phi_2} \rangle\!\rangle. \quad (5.3)$$

We compute at F (assuming without loss that $\int_M \phi_i F^{[n]} = 0$):

$$\mathbf{X}_{\phi_1} \cdot \langle\!\langle \mathbf{X}_{\phi_2}, \mathbf{X}_{\phi_3} \rangle\!\rangle = \int_M \phi_2\phi_3 (dd^c_I \phi_1) \wedge F^{[n-1]},$$
$$\mathbf{X}_{\phi_2} \cdot \langle\!\langle \mathbf{X}_{\phi_1}, \mathbf{X}_{\phi_3} \rangle\!\rangle = \int_M \phi_1\phi_3 (dd^c_I \phi_2) \wedge F^{[n-1]},$$
$$\mathbf{X}_{\phi_3} \cdot \langle\!\langle \mathbf{X}_{\phi_1}, \mathbf{X}_{\phi_2} \rangle\!\rangle = \int_M \phi_1\phi_2 (dd^c_I \phi_3) \wedge F^{[n-1]} = \int_M \phi_3 \big(dd^c_J(\phi_1\phi_2)\big) \wedge F^{[n-1]},$$

where for the last line we have used (2.13). It follows that

$$\begin{aligned} &\mathbf{X}_{\phi_1} \cdot \langle\!\langle \mathbf{X}_{\phi_2}, \mathbf{X}_{\phi_3} \rangle\!\rangle + \mathbf{X}_{\phi_2} \cdot \langle\!\langle \mathbf{X}_{\phi_1}, \mathbf{X}_{\phi_3} \rangle\!\rangle - \mathbf{X}_{\phi_3} \cdot \langle\!\langle \mathbf{X}_{\phi_1}, \mathbf{X}_{\phi_2} \rangle\!\rangle \\ &= \int_M \phi_3 \big(\phi_2 dd^c_I \phi_1 + \phi_1 dd^c_I \phi_2 - dd^c_I(\phi_1\phi_2) + d(I - J)d(\phi_1\phi_2)\big) F^{[n]} \\ &= \int_M \phi_3 \big(-d\phi_2 \wedge I\, d\phi_1 - d\phi_1 \wedge I\, d\phi_2 + d(I - J)d(\phi_1\phi_2)\big) \wedge F^{[n-1]} \\ &= \int_M \phi_3 \big((I + J)d\phi_1 \wedge d\phi_2 + d(I - J)d(\phi_1\phi_2)\big) \wedge F^{[n-1]} \\ &= \int_M \phi_3 \big((I + J)d\phi_1 \wedge d\phi_2 + \phi_2 d(I - J)(d\phi_1) + \phi_1 d(I - J)d\phi_2\big) \wedge F^{[n-1]}, \end{aligned}$$

where for passing from the fourth line to the fifth and from the fifth to the sixth we have used (2.8).

Similarly, using (2.8) and (2.9) we compute

$$\langle\!\langle \mathbf{X}_{\{\phi_1,\phi_2\}_\pi}, \mathbf{X}_{\phi_3} \rangle\!\rangle = \int_M \phi_3 \big((J - I)d\phi_1 \wedge d\phi_2\big) \wedge F^{[n-1]},$$
$$\begin{aligned} \langle\!\langle \mathbf{X}_{\{\phi_2,\phi_3\}_\pi}, \mathbf{X}_{\phi_1} \rangle\!\rangle &= \int_M \phi_1 \big((J - I)d\phi_2 \wedge d\phi_3\big) \wedge F^{[n-1]} \\ &= -\int_M \phi_3 \big(d\phi_1 \wedge (I - J)d\phi_2 + \phi_1 d(I - J)(d\phi_2)\big) \wedge F^{[n-1]}, \end{aligned}$$
$$\langle\!\langle \mathbf{X}_{\{\phi_1,\phi_3\}_\pi}, \mathbf{X}_{\phi_2} \rangle\!\rangle = -\int_M \phi_3 \big(d\phi_2 \wedge (I - J)d\phi_1 + \phi_2 d(I - J)d\phi_1\big) \wedge F^{[n-1]}.$$

Using (2.8), this yields

$$\langle\!\langle \mathbf{X}_{\{\phi_1,\phi_2\}_\pi}, \mathbf{X}_{\phi_3}\rangle\!\rangle - \langle\!\langle \mathbf{X}_{\{\phi_2,\phi_3\}_\pi}, \mathbf{X}_{\phi_1}\rangle\!\rangle - \langle\!\langle \mathbf{X}_{\{\phi_1,\phi_3\}_\pi}, \mathbf{X}_{\phi_2}\rangle\!\rangle$$
$$= \int_M \phi_3\big((J-I)(d\phi_1)\wedge d\phi_2 + \phi_1 d(I-J)d\phi_2 + \phi_2 d(I - J\, d\phi_1)\big)\wedge F^{[n-1]}.$$

Substituting the above expressions back into (5.3) and using (2.8) and that

$$\int_M \phi_3 F^{[n]} = 0,$$

we get

$$\begin{aligned}(\mathcal{D}_\phi \xi)_F &= -\operatorname{tr}_F(d\phi\wedge J\, d\psi) - \int_M \psi(dd^c_I\phi)\wedge F^{[n-1]} \\ &= -(d\phi\wedge J\, d\psi)\wedge F^{[n-1]}/F^{[n]} - \int_M \psi(dd^c_I\phi)\wedge F^{[n-1]}, \qquad (5.4)\end{aligned}$$

which is equivalent to (5.2) for the fundamental vector fields. Notice that the right-hand side of (5.4) integrates to zero against $F^{[n]}$ by (2.13).

We now show that (5.2) holds for a general vector field $F \to \xi_F$. Using metric compatibility and (5.4), we have

$$\begin{aligned}\langle\!\langle \mathcal{D}_\phi\xi, \mathbf{X}_\psi\rangle\!\rangle_F &= \big(\mathbf{X}_\phi\cdot\langle\!\langle \xi, \mathbf{X}_\psi\rangle\!\rangle\big)_F - \langle\!\langle \xi_F, (\mathcal{D}_\phi \mathbf{X}_\psi)_F\rangle\!\rangle_F \\ &= \frac{d}{dt}\Big|_{t=0}\int_M \xi_{F_t}\psi F_t^{[n]} + \int_M \xi_F(d\phi\wedge J\, d\psi)\wedge F^{[n-1]} \\ &= \int_M \dot{\xi}_F(\phi)\psi F^{[n]} + \int_M \xi_F\psi(dd^c_I\phi)\wedge F^{[n-1]} \\ &\quad + \int_M \xi_F(d\phi\wedge J\, d\psi)\wedge F^{[n-1]} \\ &= \int_M \big(\dot{\xi}_F(\phi) - \operatorname{tr}_F\big(d\phi\wedge J\, d\xi_F\big)\big)\psi F^{[n]},\end{aligned}$$

where for final line we applied (2.8) and integration by parts. ∎

From Lemma 5.1, we derive the corresponding geodesic equation which extends the expression found by Mabuchi [56] in the Kähler case.

Definition 5.2. Let F_t be a smooth path in $\mathcal{GK}_{\pi,\alpha}$, corresponding to a Hamiltonian deformation with respect to a path of smooth functions $\phi_t \in C^\infty_0(M, dV_{F_t})$. We say that F_t is a geodesic if ϕ_t satisfies

$$\begin{aligned}0 = \mathcal{D}_{\phi_t}\phi_t &= \dot{\phi}_t - \operatorname{tr}_{F_t}(d\phi_t\wedge J\, d\phi_t) \\ &= \dot{\phi}_t - \frac{1}{2}\big(g_t(d\phi_t, d\phi_t) + g_t(I_t\, d\phi_t, J\, d\phi_t)\big). \qquad (5.5)\end{aligned}$$

Notice that in the expression (5.5), ϕ_t is identified with the velocity of F_t, i.e., $\dot{F}_t = dI_t\, d\phi_t$. In the Kähler case, letting $F_t = \omega_t = \omega_0 + dJ\, d\phi_t$, we thus have $\phi_t = \dot{\psi}_t$, so (5.5) reduces to the familiar second order equation in terms of ψ_t.

Remark 5.3. One can more generally consider smooth solutions ϕ_t of (5.5), without assuming a priori that $\phi_t \in C_0^\infty(M, dV_{F_t})$. It then follows using (2.13) that

$$\frac{d}{dt}\int_M \phi_t F_t^{[n]} = \int_M (d\phi_t \wedge J\, d\phi_t) \wedge F^{[n-1]} + \int_M \phi_t\, dI_t\, d\phi_t \wedge F^{[n-1]} = 0.$$

The above shows that $\phi_t \in C_0^\infty(M, dV_{F_t})$ as soon as $\phi_0 \in C_0^\infty(M, dV_{F_0})$.

Remark 5.4. Let F_t be a smooth path in $\mathcal{GK}_{\pi,\alpha}$, corresponding to a Hamiltonian deformation with respect to a path of smooth functions $\phi_t \in C_0^\infty(M, dV_{F_t})$, $t \in [0, 1]$. Let $\widetilde{M} = M \times [0, 1] \times [0, 1]$, and denote the generic point in $\widetilde{M}$ as (p, t, s). Define

$$\widetilde{F}_{(p,t,s)} = F_{(p,t)} - d\phi_t \wedge ds + J\, d\phi_t \wedge dt + \dot{\phi}_t\, dt \wedge ds.$$

Then the geodesic equation is equivalent to $\widetilde{F}^{n+1} = 0$.

Proposition 5.5. *Let (F_0, J) be a symplectic-type generalized Kähler structure and Y a vector field preserving (F_0, J) which is also Hamiltonian with respect to F_0, i.e., there exists a smooth function ϕ_0 such that*

$$Y = -F_0^{-1}(d\phi_0) = \frac{1}{2}\big(I_0 \operatorname{grad}_{g_0} \phi_0 + J \operatorname{grad}_{g_0} \phi_0\big), \qquad \int_M \phi_0 F_0^n = 0.$$

Then the flow $\Phi_t = \exp(-tJY)$ of $-JY$ defines a geodesic $F_t := \Phi_t^*(F_0)$ in $\mathcal{GK}_{\pi,\alpha}$.

Proof. Indeed, we have that

$$\begin{aligned}\dot{F}_t &= \Phi_t^*(\mathcal{L}_{-JY} F_0) = \Phi_t^*\big(d\big(F_0 J F_0^{-1}(d\phi_0)\big)\big)\\ &= \Phi_t^*\big(-dI_0^*(d\phi_0)\big) = \Phi_t^*(dd^c_{I_0}\phi_0) = dd^c_{I_t}\phi_t,\end{aligned}$$

where $\phi_t := \Phi_t^*(\phi_0)$ satisfies $\int_M \phi_t F_t^n = 0$. Furthermore, as $\Phi_t \cdot Y = Y$, we have $Y = -F_t^{-1}(d\phi_t)$, so we compute

$$\begin{aligned}\frac{d}{dt}\phi_t &= \mathcal{L}_{-JY}\phi_t = \langle d\phi_t, -JY\rangle = \langle -J\, d\phi_t, F_t^{-1}(d\phi_t)\rangle\\ &= \frac{1}{2}\big(g_t(d\phi_t, d\phi_t) + g_t(J\, d\phi_t, I_t\, d\phi_t)\big),\end{aligned}$$

which is precisely the geodesic equation. ∎

5.2. Curvature

We next compute the curvature of the formal Riemannian connection $\mathcal{D}$. To this end, it is enough to consider fundamental vector fields $\boldsymbol{\xi}_i = \mathbf{X}_{\phi_i}$, $i = 1, 2, 3, 4$, see Definition 2.15, as they generate $\mathbf{T}_F(\mathcal{GK}_{\pi,\alpha})$ at any given point $F \in \mathcal{GK}_{\pi,\alpha}$. The following is an extension of a result by Mabuchi [56] to the symplectic-type GK case.

Theorem 5.6. *At any given point $F \in \mathcal{GK}_{\pi,\alpha}$, the curvature tensor $\mathcal{R}$ of $\mathcal{D}$ is given by*

$$(\mathcal{R}_{\mathbf{X}_{\phi_1},\mathbf{X}_{\phi_2}}\mathbf{X}_{\phi_3})_F = -\big\{\{\phi_1, \phi_2\}_F, \phi_3\big\}_F,$$

where $\{\cdot\,,\cdot\}_F$ denotes the Poisson bracket of functions with respect to the symplectic form F. In particular,

$$\langle\!\langle \mathcal{R}_{\mathbf{X}_{\phi_1},\mathbf{X}_{\phi_2}}\mathbf{X}_{\phi_1}, \mathbf{X}_{\phi_2}\rangle\!\rangle_F = -\langle\!\langle \{\phi_1, \phi_2\}_F, \{\phi_1, \phi_2\}_F\rangle\!\rangle_F,$$

showing that $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ has nonpositive sectional curvature at any point.

Proof. The second formula follows from the first by the ad-invariance of $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ (cf. equation (5.1)). Conversely, the first formula follows from the second as $\mathcal{R}$ is associated to a torsion-free Riemannian connection, and thus the sectional curvature determines the Riemannian curvature tensor. It is thus enough to establish the second formula.

At a given point F, by Lemma 2.16 the curvature is given by

$$(\mathcal{R}_{\mathbf{X}_{\phi_1},\mathbf{X}_{\phi_2}}\mathbf{X}_{\phi_3})_F = \big(-\mathcal{D}_{\phi_1}(\mathcal{D}_{\phi_2}\mathbf{X}_{\phi_3}) + \mathcal{D}_{\phi_2}(\mathcal{D}_{\phi_1}\mathbf{X}_{\phi_3}) - \mathcal{D}_{\{\phi_1,\phi_2\}_\pi}\mathbf{X}_{\phi_3}\big)_F. \quad (5.6)$$

With a small abuse of notation, we can ignore the normalizing additive constants of the smooth functions as they do not contribute to the desired formula for $\mathcal{R}_{\mathbf{X}_{\phi_1},\mathbf{X}_{\phi_2}}\mathbf{X}_{\phi_3}$. We will also drop the dependence on the basepoint F from the notation. Therefore, from (5.4), we can express the connection as

$$\mathcal{D}_\phi\mathbf{X}_\psi = -\operatorname{tr}_F(d\phi \wedge J\,d\psi) = -\frac{1}{2}\big(g(d\phi, d\psi) + g(I\,d\phi, J\,d\psi)\big),$$

where for the second equality we used the computation in (2.7). Letting $\dot{F} = dd^c_I\phi_1$ be the derivative of F in direction of $\mathbf{X}_{\phi_1}$, it follows from Lemma 5.1 and (2.8) that (up to an additive constant)

$$\begin{aligned}\mathcal{D}_{\phi_1}(\mathcal{D}_{\phi_2}\mathbf{X}_{\phi_3}) &= \frac{1}{2}\operatorname{tr}\big(F^{-1}\dot{F}F^{-1}(d\phi_2 \wedge J\,d\phi_3)\big)\\ &\quad + \operatorname{tr}_F\big(d\phi_1 \wedge J\,d\big(\operatorname{tr}_F(d\phi_2 \wedge J\,d\phi_3)\big)\big)\\ &= \frac{1}{2}\operatorname{tr}\big(F^{-1}(dd^c_I\phi_1)F^{-1}(d\phi_2 \wedge J\,d\phi_3)\big)\\ &\quad + \operatorname{tr}_F\big(d\big(\operatorname{tr}_F(d\phi_2 \wedge J\,d\phi_3) \wedge I\,d\phi_1\big)\big). \end{aligned} \quad (5.7)$$

We will use the following algebraic identity which can be deduced easily from Schur's lemma and holds for any 2-forms Φ, Ψ:

$$\Phi \wedge \Psi \wedge F^{[n-2]} = -\frac{1}{2}\operatorname{tr}(F^{-1}\Phi F^{-1}\Psi) F^{[n]} + (\operatorname{tr}_F \Phi)(\operatorname{tr}_F \Psi) F^{[n]}. \tag{5.8}$$

By (5.8), we get from (5.7) and integrating by parts

$$\begin{aligned}
\langle\!\langle \mathcal{D}_{\phi_1}(\mathcal{D}_{\phi_2}\mathbf{X}_{\phi_3}), \mathbf{X}_{\phi_4}\rangle\!\rangle = &- \int_M \phi_4\big(dd^c_I\phi_1 \wedge d\phi_2 \wedge J\,d\phi_3\big) \wedge F^{[n-2]}\\
&+ \int_M \phi_4\big(\operatorname{tr}_F(d\phi_2 \wedge J\,d\phi_3)\big) dd^c_I\phi_1 \wedge F^{[n-1]}\\
&+ \int_M \phi_4\big(d(\operatorname{tr}_F(d\phi_2 \wedge J\,d\phi_3)) \wedge I\,d\phi_1\big) \wedge F^{[n-1]}\\
= &\int_M \big(d\phi_4 \wedge I\,d\phi_1 \wedge d\phi_2 \wedge J\,d\phi_3\big) \wedge F^{[n-2]}\\
&+ \int_M \phi_4\big(I\,d\phi_1 \wedge d\phi_2 \wedge dd^c_J\phi_3\big) \wedge F^{[n-2]}\\
&- \int_M \big(\operatorname{tr}_F\big(d\phi_2 \wedge J\,d\phi_3\big) \operatorname{tr}_F(d\phi_4 \wedge I\,d\phi_1)\big) F^{[n]}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\langle\!\langle -\mathcal{D}_{\phi_1}(\mathcal{D}_{\phi_2}\mathbf{X}_{\phi_1}) + \mathcal{D}_{\phi_2}(\mathcal{D}_{\phi_1}\mathbf{X}_{\phi_1}), \mathbf{X}_{\phi_2}\rangle\!\rangle\\
&\quad = \int_M \big(d\phi_2 \wedge I\,d\phi_2 \wedge d\phi_1 \wedge J\,d\phi_1\big) \wedge F^{[n-2]}\\
&\qquad - \int_M \big(\operatorname{tr}_F(d\phi_1 \wedge J\,d\phi_1)\big)\big(\operatorname{tr}_F(d\phi_2 \wedge I\,d\phi_2)\big) F^{[n]}\\
&\qquad - \int_M \phi_2\big(d\phi_1 \wedge I\,d\phi_2 + I\,d\phi_1 \wedge d\phi_2\big) \wedge dd^c_J\phi_1 \wedge F^{[n-2]}\\
&\qquad + \int_M \big(\operatorname{tr}_F(d\phi_2 \wedge J\,d\phi_1)\big)\big(\operatorname{tr}_F(d\phi_2 \wedge I\,d\phi_1)\big) F^{[n]}.
\end{aligned}$$

Noting that from (2.9) we have

$$\{\phi_1, \phi_2\}_\pi = -\operatorname{tr}_F(d\phi_1 \wedge I\,d\phi_2) + \operatorname{tr}_F(d\phi_1 \wedge J\,d\phi_2),$$

we further compute, after integrating by parts and using (2.7) for the last equality,

$$\begin{aligned}
\langle\!\langle \mathcal{D}_{\{\phi_1,\phi_2\}_\pi}\mathbf{X}_{\phi_1}, \mathbf{X}_{\phi_2}\rangle\!\rangle &= -\int_M \phi_2\big(d\{\phi_1,\phi_2\}_\pi \wedge J\,d\phi_1\big) \wedge F^{[n-1]}\\
&= \int_M \{\phi_1,\phi_2\}_\pi d\phi_2 \wedge J\,d\phi_1 \wedge F^{[n-1]} + \int_M \phi_2\{\phi_1,\phi_2\}_\pi dd^c_J\phi_1 \wedge F^{[n-1]}
\end{aligned}$$

$$
\begin{aligned}
&= -\int_M \big(\mathrm{tr}_F(d\phi_1 \wedge I\, d\phi_2) - \mathrm{tr}_F(d\phi_1 \wedge J\, d\phi_2)\big)\,\mathrm{tr}_F(d\phi_2 \wedge J\, d\phi_1) F^{[n]} \\
&\quad - \int_M \phi_2\big(\mathrm{tr}_F(d\phi_1 \wedge I\, d\phi_2) - \mathrm{tr}_F(d\phi_1 \wedge J\, d\phi_2)\big)\,\mathrm{tr}_F(dd^c_J\,\phi_1) F^{[n]} \\
&= -\int_M \big(\mathrm{tr}_F(d\phi_2 \wedge J\, d\phi_1) - \mathrm{tr}_F(d\phi_2 \wedge I\, d\phi_1)\big)\,\mathrm{tr}_F(d\phi_2 \wedge J\, d\phi_1) F^{[n]} \\
&\quad - \int_M \phi_2\big(\mathrm{tr}_F(d\phi_1 \wedge I\, d\phi_2) + \mathrm{tr}_F(I\, d\phi_1 \wedge d\phi_2)\big)\,\mathrm{tr}_F(dd^c_J\phi_1) F^{[n]}.
\end{aligned}
$$

Substituting the latter two expressions above back in (5.6), and regrouping the terms, we get

$$
\begin{aligned}
\langle\!\langle \mathcal{R}_{\mathbf{X}_{\phi_1},\mathbf{X}_{\phi_2}}\mathbf{X}_{\phi_1}, \mathbf{X}_{\phi_2}\rangle\!\rangle = &\int_M \big(\mathrm{tr}_F(d\phi_2 \wedge J\, d\phi_1)\big)^2 F^{[n]} \\
&+ \int_M \big(d\phi_2 \wedge I\, d\phi_2 \wedge d\phi_1 \wedge J\, d\phi_1\big) \wedge F^{[n-2]} \\
&- \int_M \big(\mathrm{tr}_F(d\phi_1 \wedge J\, d\phi_1)\big)\big(\mathrm{tr}_F(d\phi_2 \wedge I\, d\phi_2)\big) F^{[n]} \\
&- \int_M \phi_2\big(d\phi_1 \wedge I\, d\phi_2 \wedge dd^c_J\,\phi_1\big) \wedge F^{[n-2]} \\
&+ \int_M \phi_2\big(\mathrm{tr}_F(d\phi_1 \wedge I\, d\phi_2)\,\mathrm{tr}_F(dd^c_J\,\phi_1)\big) F^{[n]} \\
&- \int_M \phi_2\big(I\, d\phi_1 \wedge d\phi_2 \wedge dd^c_J\,\phi_1\big) \wedge F^{[n-2]} \\
&+ \int_M \phi_2\big(\mathrm{tr}_F(I\, d\phi_1 \wedge d\phi_2)\big)\,\mathrm{tr}_F(dd^c_J\phi_1) F^{[n]}.
\end{aligned}
$$

We now apply (5.8) to each of the sums on the last three lines. The algebraic identity (2.11) shows that the last two lines cancel out, whereas (2.12) allows us to simplify the expressions at the first and second lines to

$$
\begin{aligned}
\langle\!\langle \mathcal{R}_{\mathbf{X}_{\phi_1},\mathbf{X}_{\phi_2}}\mathbf{X}_{\phi_1}, \mathbf{X}_{\phi_2}\rangle\!\rangle &= \int_M \big(\mathrm{tr}_F(d\phi_2 \wedge J\, d\phi_1)\big)^2 F^{[n]} \\
&\quad + \int_M \big(d\phi_2 \wedge I\, d\phi_2 \wedge d\phi_1 \wedge J\, d\phi_1\big) \wedge F^{[n-2]} \\
&\quad - \int_M \big(\mathrm{tr}_F(d\phi_1 \wedge J\, d\phi_1)\big)\big(\mathrm{tr}_F(d\phi_2 \wedge I\, d\phi_2)\big) F^{[n]} \\
&= \int_M \big(\mathrm{tr}_F(d\phi_2 \wedge J\, d\phi_1)\big)^2 F^{[n]} \\
&\quad - \frac{1}{2}\int_M \mathrm{tr}\big(F^{-1}(d\phi_2 \wedge I\, d\phi_2) F^{-1}(d\phi_1 \wedge J\, d\phi_1)\big) F^{[n]}
\end{aligned}
$$

$$= \int_M \left(\mathrm{tr}_F(d\phi_2 \wedge J\, d\phi_1)\right)^2 F^{[n]} - \int_M \left(\mathrm{tr}_F(d\phi_1 \wedge d\phi_2)\right)^2 F^{[n]}$$
$$- \int_M \left(\mathrm{tr}_F(d\phi_1 \wedge I\, d\phi_2)\right)^2 F^{[n]}$$
$$= - \int_M \left(\mathrm{tr}_F(d\phi_1 \wedge d\phi_2)\right)^2 F^{[n]},$$

where we have used (2.7) to obtain the last equality. The proposition is proved. ■

5.3. Formal uniqueness

The formal moment map picture yields that if it exists, $\mathbf{M}_{F_0}$ is convex on geodesics. More generally, the pairing of the 1-form $\boldsymbol{\tau}$ with the velocity of a geodesic is monotone.

Proposition 5.7. *Let F_t be a geodesic in $(\mathcal{GK}_{\pi,\alpha}, \langle\!\langle \cdot, \cdot \rangle\!\rangle)$, corresponding to a time dependent smooth function ϕ_t satisfying (5.5). Then*

$$\frac{d}{dt}\boldsymbol{\tau}(\mathbf{X}_{\phi_t}) \geq 0,$$

with equality for all t if and only if ϕ_t is a geodesic induced by a Hamiltonian Killing field as in Proposition 5.5.

Proof. We suppose (without loss, see Remark 5.3) that ϕ_t is dV_{F_t}-normalized, i.e.,

$$\int_M \phi_t F_t^{[n]} = 0.$$

We then have to compute

$$-\frac{d}{dt}\int_M \mathrm{Gscal}_{(F_t,J)}\phi_t F_t^{[n]}.$$

It is enough to establish the positivity of the derivative at $t = 0$. The computation at time t_0 will follow from the latter by considering the reparametrized geodesic ϕ_{t+t_0}. To this end, we apply the Moser isotopy Φ_t defined in Lemma 4.1. In particular, we first claim that $\psi_t := \Phi_t^*(\phi_t) = \phi_0$. Using the notation of Lemma 4.1, we compute

$$\begin{aligned}
\frac{d}{dt}\psi_t &= \Phi_t^*\left(\mathcal{L}_{Z_t}\phi_t + \dot{\phi}_t\right) \\
&= \Phi_t^*\left(\langle -F_t^{-1}(I_t\, d\phi_t), d\phi_t\rangle + \dot{\phi}_t\right) \\
&= \Phi_t^*\left(\mathrm{tr}_{F_t}(I_t\, d\phi_t \wedge d\phi_t) + \dot{\phi}_t\right) \\
&= \Phi_t^*\left(-\mathrm{tr}_{F_t}(d\phi_t \wedge J\, d\phi_t) + \dot{\phi}_t\right) = 0.
\end{aligned}$$

Using this fact we have

$$\begin{aligned}
-\frac{d}{dt}\Big|_{t=0}\int_M \mathrm{Gscal}_{(F_t,J)}\phi_t F_t^{[n]} &= -\frac{d}{dt}\Big|_{t=0}\int_M \Phi_t^*\big(\mathrm{Gscal}_{(F_t,J)}\phi_t F_t^{[n]}\big)\\
&= -\frac{d}{dt}\Big|_{t=0}\int_M \mathrm{Gscal}_{(F_0,J_t)}\phi_0 F_0^{[n]}\\
&= -\mathbf{\Omega}_J\big((-\mathcal{L}_{Y_{\phi_0}}J),\dot{J}\big)\\
&= \mathbf{\Omega}_J\big((-\mathcal{L}_{Y_{\phi_0}}J), J(-\mathcal{L}_{Y_{\phi_0}}J)\big) = \|\mathcal{L}_{Y_{\phi_0}}J\|^2_{\mathbf{g}_J}.
\end{aligned}$$

In the above equalities, $Y_{\phi_0} := -F_0^{-1}(d\phi_0)$ and for passing to the third line we have used that $\mathrm{Gscal}_{(F_0,J_t)}$ is a moment map for the action of $\mathrm{Ham}(M, F_0)$ on $(\mathcal{AGK}_F, \mathbf{\Omega})$, for passing to the fourth line we have used Lemma 4.1 to identify $\dot{J}$, and for passing to the last line we have used Lemma 3.4.

This yields the claimed inequality, and for equality to hold for all t, one must have that the vector field $Y_t := -F_t^{-1}(d\phi_t)$ is J-holomorphic. As it preserves F_t, Y_t thus preserves the whole biHermitian structure (g_t, b_t, I_t, J, F_t), i.e., it is a Killing field with potential ϕ_t. To obtain the claim, it is enough to show that $Y_t = Y$ is time independent, as the F_t-normalized potential of the Killing field Y with respect to F_t is unique by the maximum principle, and is given by the geodesic determined by Y via Proposition 5.5. Below we check that Y_t is time independent, by using the geodesic equation (5.5), expressed as $\dot{\phi} = \langle Y, I\,d\phi\rangle$:

$$\begin{aligned}
\dot{Y} &= F^{-1}\dot{F}F^{-1}(d\phi) - F^{-1}(d\dot{\phi}) = F^{-1}(-\iota_Y dd_I^c\phi - d\iota_Y I\phi)\\
&= F^{-1}(-\mathcal{L}_Y(I\,d\phi)) = F^{-1}(-I\mathcal{L}_Y\,d\phi) = 0,
\end{aligned}$$

where in the last line we have used that Y is Killing (and thus $\mathcal{L}_Y I = 0$) and

$$\mathcal{L}_Y\phi = -\langle F^{-1}(d\phi), d\phi\rangle = 0. \qquad \blacksquare$$

Corollary 5.8 (Conditional uniqueness). *Suppose $F_0, F_1 \in \mathcal{GK}_{\pi,\alpha}$ are cscGK structures connected by a smooth geodesic F_t. Then there exists $Y \in \mathfrak{h}_{\mathrm{red}}(J, \pi_J)$ such that $F_t = \Phi_t^* F_0$, where*

$$\Phi_t = \exp(-tJY) \in \mathrm{Aut}_{\mathrm{red}}(J, \pi_J).$$

Proof. Let $\phi_t \in C_0^\infty(M, \mathbb{R}), 0 \le t \le 1$ be a path inducing a geodesic between F_0 and F_1. Since F_0 and F_1 are cscGK, we have $\boldsymbol{\tau}\big|_{F_0} = \boldsymbol{\tau}\big|_{F_1} = 0$. By Proposition 5.7, function $t \mapsto \boldsymbol{\tau}(\mathbf{X}_{\phi_t})$ is nondecreasing, and being 0 at $t = 0$ and $t = 1$, it must vanish identically. Therefore, by the second part of Proposition 5.7, there exists

$$Y \in \mathfrak{h}_{\mathrm{red}}(J, \pi_J) \cap \mathfrak{ham}(M, F_0)$$

such that the flow of $-JY$ induces the path F_t. $\blacksquare$

6. The toric case

In this section we consider the case when $(M, J, \mathbb{T}^{\mathbb{C}})$ is a (projective) smooth toric variety under the effective action of a complex n-dimensional complex torus

$$\mathbb{T}^{\mathbb{C}} \simeq (\mathbb{C}^*)^n.$$

We denote by $\mathbb{T}$ the corresponding compact (real) n-dimensional torus and by $\mathfrak{t}$ its Lie algebra. We will assume that π_J is a $\mathbb{T}^{\mathbb{C}}$-invariant holomorphic Poisson tensor on (M, J), i.e., given by an element of $\wedge^2(\mathfrak{t} \otimes \mathbb{C})$ (see [49, Proposition 2.14]).

6.1. Reduction to invariant structures

As a corollary of Theorem 4.13 we have the following.

Corollary 6.1. *Suppose $(M, J, \mathbb{T}^{\mathbb{C}})$ is a smooth projective toric variety and π_J a $\mathbb{T}^{\mathbb{C}}$-invariant holomorphic Poisson tensor. Let F_0 be a $\mathbb{T}$-invariant symplectic-type generalized Kähler structure in $\mathcal{GK}_\pi$. Then, up to acting with an element of* $\mathrm{Aut}_0(J, \pi_J)$*, any extremal generalized Kähler structure $F \in \mathcal{GK}_\pi$ is $\mathbb{T}$-invariant.*

Proof. By the toric assumption on (M, J), the generators of the $(\mathbb{C}^*)^n$-action are holomorphic vector fields with zeros which are linearly independent at each point over the dense open orbit of $(\mathbb{C}^*)^n$ in M. This shows that $H^{1,0}_{\bar\partial}(M, J) = \{0\}$ and therefore $b_1(M) = 0$. By Remark 4.14,

$$\mathrm{Aut}_0(J, \pi_J) = \mathrm{Aut}_{\mathrm{red}}(J, \pi_J).$$

Note that $\mathbb{T}$ is a maximal compact torus in $\mathrm{Aut}_0(J, \pi_J)$. By Theorem 4.13, (F, J) is invariant under a maximal torus $\mathbb{T}'$ in $\mathrm{Aut}_0(J, \pi_J)$. As any two such tori are conjugated inside $\mathrm{Aut}_0(J, \pi_J)$, after acting with an element of $\mathrm{Aut}_0(J, \pi_J)$, we can assume that (F, J) is $\mathbb{T}$-invariant. ∎

6.2. Abreu–Guillemin-type description of toric structures

Because of Corollary 6.1, we focus from now on $\mathbb{T}$-invariant symplectic-type generalized Kähler structures. The corresponding spaces will be denoted by upper script $\mathbb{T}$: $\mathcal{GK}^{\mathbb{T}}_\pi$, $(\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha})^0$, $\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$, etc. As $b_1(M) = 0$, for any $F \in \mathcal{GK}^{\mathbb{T}}_\pi$ the action of $\mathbb{T}$ is F-Hamiltonian, i.e., (F, J) is toric in the sense of [7,67]. Toric symplectic-type generalized Kähler structures were extensively studied in [10,66,67], where an Abreu–Guillemin-type description [1,43] is obtained. We follow the notation of our previous work [7] where we have recast the classification of [10,66,67] in a formalism compatible with the one used in the current article. The following discussion is a brief summary of [7, Section 4]. We refer the reader to this work for further details and a literature review.

We thus assume that (M, g, J, I, F) is a compact symplectic-type generalized Kähler structure which admits an effective isometric and F-Hamiltonian action of a compact torus $\mathbb{T}$ with $2\dim_{\mathbb{R}}\mathbb{T} = \dim_{\mathbb{R}} M = 2n$. We denote by (P, L) the (labeled) Delzant polytope of (M, F) in $\mathfrak{t}^* \simeq \mathbb{R}^n$, and by $\mu = (\mu_1, \dots, \mu_n)\colon M \to \mathrm{P}$ the corresponding momenta. Here $\mathfrak{t} = \mathrm{Lie}(\mathbb{T})$ is the Lie algebra of $\mathbb{T}$ and we use a lattice basis (and its dual basis) to identify respectively $\mathfrak{t}$ and its dual vector space $\mathfrak{t}^*$ with $\mathbb{R}^n$; in particular, $\mathbb{T} = \mathbb{R}^n/2\pi\mathbb{Z}^n$. In this set up, it is observed by Guillemin [43] that there are canonical *angular coordinates* $d\theta = \{d\theta_1, \dots, d\theta_n\}$ defined on the dense open subset $\mathring{M} := \mu^{-1}(\mathring{\mathrm{P}})$ (where $\mathring{\mathrm{P}}$ is the interior of P). Furthermore, with respect to the coordinate system (μ, θ), the $\mathbb{T}$-invariant 2-form on $\mathring{M}$ given by

$$\omega := \langle d\mu, d\theta\rangle = \sum_{i=1}^{n} d\mu_i \wedge d\theta_i$$

is smoothly extendable to M and defines a symplectic structure (still denoted by ω). In general, F and ω are different symplectic forms, even though they are $\mathbb{T}$-equivariantly symplectomorphic, belong to the same de Rham class $[F] = [\omega]$, and share the same momentum coordinates and Delzant polytope (P, L). The main conclusion in [10, 67] is that, up to a $\mathbb{T}$-equivariant isometry, (F, J) is obtained as follows:

(i) There exists a unique, up to the addition of an affine-linear function on $\mathfrak{t}^*$, convex smooth function $u(x)$ defined on the interior $\mathring{P}$ such that

$$g_u := \sum_{i,j=1}^{n} u_{,ij}(\mu)\, d\mu_i\, d\mu_j + u^{,ij}(\mu)\, d\theta_i\, d\theta_j$$

is a ω-compatible Kähler structure (defined on $\mathring{M}$ and extendable to M) whose complex structure is

$$J_u := g_u^{-1}\omega = \sum_{i,j=1}^{n}\Big(u_{,ij}\, d\mu_i \otimes \frac{\partial}{\partial\theta_j} - u^{,ij}\, d\theta_i \otimes \frac{\partial}{\partial\mu_j}\Big).$$

(ii) There exist unique elements $A, B \in \wedge^2\mathfrak{t}$ such that (F, J) is obtained from (ω, J_u) by deformations of type A and B, i.e.,

$$J = \sum_{i,j=1}^{n}\Big((u_{,ij} + A_{ij})\, d\mu_i \otimes \frac{\partial}{\partial\theta_j} - (u_{,ij} + A_{ij})^{-1}_{ij}\, d\theta_i \otimes \frac{\partial}{\partial\mu_j}\Big),$$
$$F = \sum_{i=1}^{n} d\mu_i \wedge d\theta_i + \sum_{i,j=1}^{n} B_{ij}\, d\mu_i \wedge d\mu_j,$$

where in the above formulae $A = (A_{ij})$, $B = (B_{ij})$ using the chosen basis of $\mathfrak{t}$.

(iii) The corresponding Poisson tensor π_J is

$$\pi_J = \sum_{i,j=1}^{n} \big(A_{ij} + \sqrt{-1}B_{ij}\big)\Big(\frac{\partial}{\partial\theta_i} - \sqrt{-1}J\frac{\partial}{\partial\theta_i}\Big) \wedge \Big(\frac{\partial}{\partial\theta_j} - \sqrt{-1}J\frac{\partial}{\partial\theta_j}\Big). \quad (6.1)$$

Definition 6.2 (Abreu–Guillemin data). For a given toric symplectic-type GK structure $(M, F, J, \mathbb{T})$ with Delzant polytope (P, L), we will refer to (u, A, B) defined above as the corresponding *Abreu–Guillemin data*. We note that $A, B \in \wedge^2\mathfrak{t}$ determine and are determined by the holomorphic Poisson tensor π_J of (F, J) via (6.1). The convex function $u(x)$ is defined only up to additive affine-linear function, and the Kähler toric structure (ω, J_u, g_u) is referred to as *the toric Kähler reduction* of (F, J).

We now describe more precisely the space $\mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$ of smooth convex functions $u(x)$ on $\mathring{\mathrm{P}}$ appearing in Definition 6.2.

Lemma 6.3. *A smooth strictly convex function $u(x)$ on $\mathring{\mathrm{P}}$ belongs to $\mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$ if and only if the following conditions hold:*

(i) *$u(x)$ is smooth and strictly convex on $\mathring{\mathrm{P}}$ such that* $(\mathrm{Hess}(u) + \sqrt{-1}B)$ *is positive definite Hermitian form at any point of $\mathring{\mathrm{P}}$;*

(ii) *$u(x) = \frac{1}{2}\sum_{L\in\mathrm{L}} L(x)\log L(x) + v(x)$ with $v(x)$ smooth over* P*;*

(iii) *on the interior $\mathring{\mathrm{F}}$ of any face* $\mathrm{F} \subset \mathrm{P}$*,* $(\mathrm{Hess}(u) + \sqrt{-1}B)$ *is a smooth and positive definite Hermitian form on $\mathfrak{t}^*_{\mathrm{F}} \otimes \mathbb{C}$.*

The above conditions (i), (ii), (iii) *are equivalent to the conditions* (i), (ii), (iii)$'$*, where*

(iii)$'$ $(\mathrm{Hess}(u) + \sqrt{-1}B)^{-1}\mathbf{G}_0$ *extends smoothly on* P*, where*

$$\mathbf{G}_0 = \mathrm{Hess}\Big(\frac{1}{2}\sum_{L\in\mathrm{L}} L(x)\log L(x)\Big).$$

Proof. The conditions (i), (ii), (iii) extend the one obtained by Donaldson [22] in the Kähler case, and reflect the fact that (F, J) induces a symplectic structure taming J on the pre-image of each face F (which is a smooth toric sub-manifold of (M, J)). The proof is similar and left to Reader. One can alternatively use a Taylor expansion around a point of $\mathring{F}$ as in [3, Appendix A2] to show that (i), (ii), (iii) are equivalent to (i), (ii), (iii)$'$. The latter are necessary and sufficient for the extension of (F, J) to M by [67] (which in turn uses [4, Lemma 3 and Remark 4 (ii)]). ■

Lemma 6.4. $\mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L}) = \mathcal{S}_{0,B}(\mathrm{P}, \mathrm{L})$ *is a linearly convex subset of* $\mathcal{S}_{0,0}(\mathrm{P}, \mathrm{L})$*, which is a Fréchet space modeled on* $C^\infty(\mathrm{P})$.

Proof. The first claim follows from the description (i), (ii), (iii) of $\mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$ in Lemma 6.3 and the general relation of Hermitian forms

$$\big(\mathrm{Hess}(u) + \sqrt{-1}B\big)_{\mathfrak{t}^*_{\mathrm{F}}} \leq \mathrm{Hess}(u)_{\mathfrak{t}^*_{\mathrm{F}}}.$$

The convexity property also follows from (i), (ii), (iii). To observe the relevant Fréchet manifold structure of $\mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$, it is enough to show that for any $u \in \mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$ and $v \in C^\infty(\mathrm{P})$, $u + tv \in \mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$ for $|t| < \varepsilon(u, v)$. This follows easily from the description (i), (ii), (iii)′ of $\mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$ given in Lemma 6.3. ■

6.3. Invariant classes and geodesics

The description of $\mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$ gives the following connectedness result in the toric case.

Proposition 6.5. *Let (M, J_0, F_0) be a toric symplectic-type generalized Kähler structure corresponding to Abreu–Guillemin data (u_0, A, B), $u_0 \in \mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$. Denote by $\alpha = [F_0]$ the corresponding de Rham class in $H^2(M, \mathbb{R})$. Then $\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ is path-connected.*

Proof. If $F \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$, we can use the $\mathbb{T}$-equivariant Moser's lemma to send the $\mathbb{T}$-invariant generalized Kähler structure F (via an isotopy of $\mathbb{T}$-equivariant diffeomorphisms connecting the identity) to a $\mathbb{T}$-invariant GK structure (F_0, J), compatible with the symplectic form F_0. Acting by a further $\mathbb{T}$-equivariant F-symplectomorphism in $\mathrm{Symp}^{\mathbb{T}}_0(M, F_0)$ if necessary, we can also assume that (F_0, J_0) and (F_0, J) correspond to Abreu–Guillemin data (u_0, A, B), (u, A, B), $u, u_0 \in \mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$ and are respectively obtained by deformations of type A and B of two ω-compatible Kähler structures J_{u_0} and J_u with the same momentum-angular coordinates. As the space $\mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$ is linearly convex, letting

$$u_t := (1-t)u_0 + tu \in \mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$$

defines a smooth path J_t between J_0 and J inside the space of $\mathbb{T}$-invariant F_0-compatible generalized Kähler structures. Now, by [7, Lemma 3.6], we can find a $\mathbb{T}$-equivariant isotopy of diffeomorphisms, sending (F_0, J_t) to (F_t, J_0), preserving

$$[F_t] = [F_0] = \alpha.$$ ■

One of the key applications of the Abreu–Guillemin formalism for the theory of toric Kähler manifolds is that it transforms the rather complicated geodesic equation in the space $\mathcal{K}^{\mathbb{T}}_\alpha$ into the linear equation $\ddot{u} = 0$ for the corresponding symplectic potentials. This implies, in particular, that the space of $\mathbb{T}$-invariant Kähler metrics $\mathcal{K}^{\mathbb{T}}_\alpha$ is geodesically connected, which in turn yields the uniqueness (modulo $\mathbb{T}^{\mathbb{C}}$) of the constant scalar curvature metrics in that space (see [42]). We show below that these phenomena persist in the generalized Kähler setting. For simplicity, we check these facts under the assumption $A = 0$, which we can suppose without loss of generality if we study extremal generalized Kähler structures (as we show in the next subsection).

Proposition 6.6. *Let (F_0, J) be a toric symplectic-type generalized Kähler manifold with holomorphic Poisson tensor π_J given by (6.1) for $A = 0$ and $B \in \wedge^2\mathfrak{t}$. Denote by (ω_0, J) the corresponding $\mathbb{T}$-invariant Kähler reduction and by $\alpha := [F_0] = [\omega_0]$ the corresponding Kähler class. For any path $F_t \in \mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$, let ω_t be the unique Kähler metric in α which solves $\omega_t^n = F_t^n$ and denote by ϕ_t the unique $\mathbb{T}$-invariant smooth function such that*

$$\omega_t = \omega_0 + dd^c_J\phi_t, \qquad \int_M \dot{\phi}_t\omega_t^n = 0, \quad \phi_0 = 0.$$

Then F_t is obtained from the Hamiltonian deformation of F_0 with respect to $\dot{\phi}_t$ and the following conditions are equivalent:

- *ϕ_t is a geodesic in the space of $\mathbb{T}$-invariant ω_0-relative Kähler potentials;*
- *$\dot{\phi}_t$ is a geodesic on $\mathcal{GK}^0_{\pi,\alpha}$ in the sense of Definition 5.2;*
- *if $(u_t, 0, B)$ denotes the corresponding to Abreu–Guillemin data of $(F_t, J, \mathbb{T})$, then $\ddot{u}_t = 0$.*

Proof. By [43] (see also [7]), we can express, on $(\mathring{M}, J)$, $\omega_t = dd^c_J\varphi_t$ for some $\mathbb{T}$-invariant smooth function φ_t. More precisely, if (y, θ) are the (exponential) J-pluriclosed coordinates defined on the open dense orbit $\mathring{M} = \mathbb{T}^{\mathbb{C}} \cdot p_0$ of the complexified action $\mathbb{T}^{\mathbb{C}}$ (determined by fixing a base point $p_0 \in \mathring{M}$), there is a unique, up to the addition of an affine-linear function, function $\varphi_t(y)$ with the above property. One can further require that $\phi_t(y) := \varphi_t(y) - \varphi_0(y)$ extends smoothly to M, which leaves only an additive constant in the definition $\varphi_t(y)$ (and ϕ_t), once we have chosen φ_0. This constant is further determined via the normalization used in the lemma.

We now consider the Legendre transform, u_t, of φ_t: letting $\mu_t := \nabla\varphi_t$, (μ_t, θ) are momentum-angular coordinates of ω_t and the functions $u_t(x)$ defined by

$$\varphi_t(y) + u_t(\mu_t) = \langle y, \mu_t\rangle$$

are elements of $\mathcal{S}_{0,B}(\mathrm{P}, \mathrm{L})$, such that $(u_t, 0, B)$ are Abreu–Guillemin data of (F_t, J) pulled back by the diffeomorphism $\mu_t \to \mu_0$, $\theta \to \theta$, see [7] for details. Using the basic property of the Legendre transform $\mathbf{H} = (\mathrm{Hess}(u))^{-1} = \mathrm{Hess}(\varphi)$, we have

$$\omega_t = \sum_{i,j=1}^m \mathbf{H}_{ij}\, dy_i \wedge d\theta_j, \quad F_t = \sum_{i,j=1}^m \mathbf{H}_{ij}\, dy_i \wedge d\theta_j + (\mathbf{H}B\mathbf{H})_{ij}\, dy_i \wedge dy_j,$$

where $\mathbf{H}_t(y) = \mathrm{Hess}(\varphi_t)(y) = \mathrm{Hess}(u_t)^{-1}(\mu_t)$. Let $\dot\phi = \dot\varphi$ be a first order variation of ϕ_t, and $\dot\omega$, $\dot F$ and $\dot I$ denote the corresponding first order variations of ω_t, F_t and I_t. We have

$$\dot\omega = \sum_{i,j=1}^m \dot{\mathbf{H}}_{ij}\, dy_i \wedge d\theta_j, \quad \dot F = \sum_{i,j=1}^m \dot{\mathbf{H}}_{ij}\, dy_i \wedge \theta_j + (\dot{\mathbf{H}}B\mathbf{H} + \mathbf{H}B\dot{\mathbf{H}})_{ij}\, dy_i \wedge dy_j.$$

Using that (y,θ) and $(y, \overline{\theta} = \theta - 2B\mu_t)$ with $\mu_t = \nabla\varphi$ (see [7]) are respective pluriharmonic coordinates for J and I_t, we compute

$$\begin{aligned}
dd^c_J\dot\phi &= \sum_{i,j=1}^m \dot\phi_{,ij}\, dy_i \wedge d\theta_j = \sum_{i,j=1}^m \dot{\mathbf{H}}_{ij}\, dy_i \wedge d\theta_j, \\
dd^c_I\dot\phi &= \sum_{i=1}^m dI(\dot\phi_{,i}\, dy_i) = d\Big(\sum_{i=1}^m \dot\phi_{,i}\, d\theta_i - 2\sum_{i,j,k=1}^m \mathbf{H}_{jk}B_{ki}\dot\phi_{,i}\, dy_j\Big) \\
&= \sum_{i,j=1}^m \big(\dot{\mathbf{H}}_{ij}\, dy_i \wedge d\theta_j + 2(\mathbf{H}B\dot{\mathbf{H}})_{ij}\, dy_i \wedge dy_j\big) \\
&= \sum_{i,j=1}^m \big(\dot{\mathbf{H}}_{ij}\, dy_i \wedge d\theta_j + (\mathbf{H}B\dot{\mathbf{H}} + \dot{\mathbf{H}}B\mathbf{H})_{ij}\, dy_i \wedge dy_j\big),
\end{aligned}$$

which gives

$$\dot\omega = dd^c_J\dot\phi, \quad \dot F = dd^c_I\dot\phi.$$

The variation of $\dot I = -\frac{1}{2}\pi(dd^c_I\dot\phi)$ then follows from the general property of variations in $\mathcal{AGK}_{\pi,\alpha}$, see Section 2.7, and the dd^c_I-lemma. This shows that F_t is a Hamiltonian deformation of F_0 with function $\dot\phi_t$.

The equivalence of the first and third statements is established in [42]. The equivalence of the first and second statements follows from (5.5) and the general relation

$$df \wedge J\,df \wedge F^{n-1} = df \wedge J\,df \wedge \omega^{n-1}, \quad f = f(y),$$

which can be checked from the above expressions for ω and F. ∎

6.4. The generalized Kähler scalar curvature and extremal structures

In the above setting, a computation from [65] shows that Goto's scalar curvature of $(F, J, \mathbb{T})$ is given by an Abreu-type formula (compare with [1] in the Kähler case):

$$\mathrm{Gscal}_{(F,J)} = -\sum_{i,j=1}^n \frac{\partial^2}{\partial\mu_i\partial\mu_j}\big(\mathrm{Hess}(u) + \sqrt{-1}B\big)^{-1}_{ij}. \tag{6.2}$$

This can be also deduced from [10] (where the momentum map for the action of $\mathrm{Ham}^{\mathbb{T}}(M,F)$ on $\mathcal{AGK}^{\mathbb{T}}_F$ is identified with the right-hand side of (6.2)) and Theorem 3.8 above.

Using Theorem 4.13, for any $\mathbb{T}$-invariant *extremal* generalized Kähler structure (F,J) with Abreu–Guillemin data (u,A,B), $u \in \mathcal{S}_{A,B}(\mathrm{P},\mathrm{L})$, the corresponding generalized Kähler scalar curvature given by (6.2) is a pull-back by μ of an affine-linear function $\ell(x)$ on P. It follows that the extremal equation is

$$-\sum_{i,j=1}^{n} \frac{\partial^2}{\partial\mu_i\partial\mu_j}\big(\mathrm{Hess}(u)+\sqrt{-1}B\big)^{-1}_{ij} = \ell(\mu), \quad u \in \mathcal{S}_{A,B}(\mathrm{P},\mathrm{L}).$$

An important feature is that the above expression is independent of A. Indeed, the left-hand side is manifestly independent of A and we have $\mathcal{S}_{A,B}(\mathrm{P},\mathrm{L}) = \mathcal{S}_{0,B}(\mathrm{P},\mathrm{L})$. Furthermore, we have the following result.

Lemma 6.7. *If* (u,A,B), $u \in \mathcal{S}_{A,B}(\mathrm{P},\mathrm{L})$ *defines an extremal generalized Kähler structure with corresponding affine-linear function* ℓ, *then* $\ell = \ell_{\mathrm{ext}}$ *is the extremal affine-linear function of the labeled* (P,L), *introduced in* [21]. *In particular,* ℓ_{ext} *is independent of* A,B.

Proof. Let

$$\mathbf{X} := \mathrm{Re}\big(\mathrm{Hess}(u)+\sqrt{-1}B\big)^{-1}. \tag{6.3}$$

As $\mathrm{Im}(\mathrm{Hess}(u)+\sqrt{-1}B)^{-1}$ is skew-symmetric, (6.2) becomes

$$\mathrm{Gscal}_{(F,J)} = -\sum_{i,j=1}^{n} \mathbf{X}_{ij,ij} = \ell. \tag{6.4}$$

Notice that $\mathbf{X} = (\mathbf{G}+B\mathbf{H}B)^{-1}$ where $\mathbf{G} = \mathrm{Hess}(u)$ and $\mathbf{H} = \mathbf{G}^{-1}$. It then follows from the boundary conditions (i), (ii), (iii)$'$ in Lemma 6.3 that $\mathbf{X}^{-1} - \mathbf{G}_0$ is smooth and $\mathbf{X}\mathbf{G}_0$ is smooth and nondegenerate on P. These in turn yield that $\mathbf{X}$ is smooth on P and satisfies first order boundary conditions at each facet of P (see [4, Lemma 2, Remark 4 and Proposition 1]). One can then obtain from (6.3), by integrating by parts twice as in [21] (see also [2, Lemma 3.2]) that for any smooth function f on P, we have

$$\begin{aligned} 0 &= \int_{\mathrm{P}} f\Big(\sum_{i,j=1}^{n} \mathbf{X}_{ij,ij} + \ell\Big)\,dx \\ &= \int_{\mathrm{P}} f\ell\,dx - 2\int_{\partial\mathrm{P}} f\,d\sigma_{\mathrm{L}} + \int_{\mathrm{P}} \mathrm{tr}\big(\mathbf{X}\circ\mathrm{Hess}(f)\big)\,dx, \end{aligned} \tag{6.5}$$

where dx is the Lebesgue measure on $\mathfrak{t}^*$ associated to the chosen basis of $\mathfrak{t}$, and $d\sigma_{\mathrm{L}}$ is the induced measure by dx and the inward normal dL_j on each facet $\mathrm{P}_j \subset \partial\mathrm{P}$. Specializing the above formula for f affine-linear, we obtain that

$$-\int_{\mathrm{P}} \ell(x) f(x)\, dx + 2\int_{\partial \mathrm{P}} f(x)\, d\sigma_{\mathrm{L}} = 0 \quad \forall f \text{ affine-linear.}$$

This determines a unique affine-linear function ℓ, denoted by ℓ_{ext}. ■

Corollary 6.8. $(u, A, B) \in \mathcal{S}_{A,B}(\mathrm{P},\mathrm{L})$ *defines an extremal* $\mathbb{T}$*-invariant generalized Kähler structure if and only if* $(u, 0, B) \in \mathcal{S}_{0,B}(\mathrm{P},\mathrm{L})$ *defines an extremal* $\mathbb{T}$*-invariant generalized Kähler structure.*

Following [21], for any affine-linear function ℓ, we introduce a linear functional defined on the space of continuous functions on P by

$$\mathbf{F}_{\ell}(f) := -\int_{\mathrm{P}} \ell(x) f(x)\, dx + 2\int_{\partial \mathrm{P}} f(x)\, d\sigma_{\mathrm{L}}. \tag{6.6}$$

Definition 6.9. The *extremal affine-linear function* ℓ_{ext} of (P,L) is the unique affine-linear function ℓ such that $\mathbf{F}_{\ell}(f) = 0$ for any affine-linear function f. The corresponding functional $\mathbf{F}_{\ell_{\mathrm{ext}}}$ is called the *relative Donaldson–Futaki invariant* of (P,L).

With the above remarks in mind, we now consider the following PDE problem which is a modified version of Abreu's equation in [1]:

$$-\sum_{i,j=1}^{n} \frac{\partial^2}{\partial \mu_i \partial \mu_j}\big(\mathrm{Hess}(u) + \sqrt{-1}B\big)^{-1}_{ij} = \ell_{\mathrm{ext}}(\mu), \quad u \in \mathcal{S}_{0,B}(\mathrm{P},\mathrm{L}), \tag{6.7}$$

where ℓ_{ext} is the extremal affine-linear function defined in Definition 6.9.

6.5. The relative Mabuchi energy and the uniqueness of extremal generalized Kähler structures

The identification of $\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha}$ with the convex space $\mathcal{S}_{A,B}(\mathrm{P},\mathrm{L})$ can be used to integrate the formal closed 1-form $\boldsymbol{\tau}$ and obtain a well-defined Mabuchi functional. This has been indeed obtained in [7], where the following "toric" Mabuchi functional was introduced:

$$\begin{aligned}\mathbf{M}(u) := \mathbf{F}_a(u) &- \int_{\mathrm{P}} \log\det\big(\mathrm{Hess}(u) + \sqrt{-1}B\big)\, dx \\ &+ \int_{\mathrm{P}} \log\det\big(\mathrm{Hess}(u_0)\big)\, dx, \quad u \in \mathcal{S}_{A,B}(\mathrm{P},\mathrm{L}).\end{aligned}$$

In the above formula, $u_0 \in \mathcal{S}_{0,0}(\mathrm{P}, \mathrm{L})$ is the convex function associated to some background toric Kähler metric compatible with F, and a is the topological constant

$$a := 2\frac{\mathrm{Vol}(\partial\mathrm{P}, d\sigma_{\mathrm{L}})}{\mathrm{Vol}(\mathrm{P}, dx)},$$

which computes the average value of $\mathrm{Gscal}_{(F,J)}$ (see Remark 3.9), and $\mathbf{F}_a$ denotes the linear functional (6.6) defined with respect to the affine-linear function $\ell \equiv a$. Notice that the difference of the last two terms is well defined for $u \in \mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$, by virtue of the conditions (i), (ii), (iii)′ in Lemma 6.3. We can more generally introduce the *relative* Mabuchi energy

$$\mathbf{M}^{\ell_{\mathrm{ext}}}(u) := \mathbf{F}_{\ell_{\mathrm{ext}}}(u) - \int_{\mathrm{P}} \log\det\big(\mathrm{Hess}(u) + \sqrt{-1}B\big)\,dx + \int_{\mathrm{P}} \log\det\big(\mathrm{Hess}(u_0)\big)\,dx,$$

where, instead of the constant function a, we use the extremal affine-linear function ℓ_{ext} in the linear term.

Lemma 6.10. *Given a one-parameter family of symplectic potentials $u_s = u + s\dot{u}$ and fixed A, B, we have*

$$\big(\delta_u \mathbf{M}^{\ell_{\mathrm{ext}}}\big)(\dot{u}) = \int_{\mathrm{P}} \dot{u}\bigg(-\sum_{i,j=1}^{n} \frac{\partial^2}{\partial\mu_i\partial\mu_j}\big(\mathrm{Hess}(u) + \sqrt{-1}B\big)^{-1}_{ij} - \ell_{\mathrm{ext}}\bigg)\,dx.$$

In particular, the critical points of $\mathbf{M}^{\ell_{\mathrm{ext}}}$ correspond to extremal toric generalized Kähler structures with Poisson tensor corresponding to A, B. Furthermore,

$$\big(\delta_u^2 \mathbf{M}^{\ell_{\mathrm{ext}}}\big)(\dot{u}, \dot{u}) = \int_{\mathrm{P}} \mathrm{tr}\Big[\Big(\big(\mathrm{Hess}(u) + \sqrt{-1}B\big)^{-1}\,\mathrm{Hess}(\dot{u})\Big)^2\Big]\,dx \geq 0.$$

Proof. A direct computation shows

$$\begin{aligned}\frac{d}{ds}\big(\mathbf{M}^{\ell_{\mathrm{ext}}}(u_s)\big) &= \mathbf{F}_{\ell_{\mathrm{ext}}}(\dot{u}) - \int_{\mathrm{P}} \mathrm{tr}\Big(\big(\mathrm{Hess}(u) + \sqrt{-1}B\big)^{-1} \circ \mathrm{Hess}(\dot{u})\Big)\,dx \\ &= \mathbf{F}_{\ell_{\mathrm{ext}}}(\dot{u}) - \int_{\mathrm{P}} \mathrm{tr}\big(\mathbf{X} \circ \mathrm{Hess}(\dot{u})\big)\,dx,\end{aligned}$$

where $\mathbf{X} = \mathrm{Re}(\mathrm{Hess}(u) + \sqrt{-1}B)^{-1}$. The first identity follows from the above and using the second equality in (6.5), whereas the second identity follows by taking a second derivative in s. ∎

Theorem 6.11. *On a toric holomorphic Poisson manifold $(M, J, \pi_J, \mathbb{T})$ endowed with a generalized Kähler structure $F_0 \in \mathcal{GK}_{\pi}^{\mathbb{T}}$, the extremal generalized Kähler structures in $\mathcal{GK}_{\pi,\alpha}^{0}$ are unique modulo $\mathrm{Aut}_0(J, \pi_J)$. Furthermore, the symplectic potential in $\mathcal{S}_{A,B}(\mathrm{P}, \mathrm{L})$ of a $\mathbb{T}$-invariant extremal generalized Kähler structure in $\mathcal{GK}_{\pi,\alpha}^{\mathbb{T}}$ minimizes $\mathbf{M}^{\ell_{\mathrm{ext}}}$.*

Proof. By Corollary 6.1, it is enough to show uniqueness in $(\mathcal{GK}^{\mathbb{T}}_{\pi,\alpha})^0$, modulo the action of $\mathbb{T}^{\mathbb{C}}$. In this case, the problem is reduced to the uniqueness of (6.2) modulo affine-linear functions (see [7]), and the latter follows from the convexity of $\mathbf{M}^{\ell_{\rm ext}}$ established in Lemma 6.10 and the fact that $\mathcal{S}_{A,B}(\mathrm{P},\mathrm{L})$ is linearly convex (see Lemma 6.4). This also shows that the symplectic potential of an extremal generalized Kähler structures is a global minimizer of $\mathbf{M}^{\ell_{\rm ext}}$. ■

6.6. The obstruction theory

Definition 6.12 ([14,21]). We say that (P, L) is *uniform relative K-stable* if, for a chosen point $x_0 \in \mathring{\mathrm{P}}$, there exist uniform positive constants $\lambda = \lambda(\mathrm{P},\mathrm{L},x_0)$, $\delta = \delta(\mathrm{P},\mathrm{L},x_0)$ such that

$$\mathbf{F}_{\ell_{\rm ext}}(f) \geq \lambda \int_{\partial \mathrm{P}} f \, d\sigma_{\mathrm{L}} - \delta, \tag{6.8}$$

for any continuous convex function $f(x)$ on P, *normalized* (by adding an affine-linear function ℓ which does not affect the right-hand side) so that $f(x) \geq f(x_0) = 0$.

By the works [14, 21], if $(M, F, \mathbb{T})$ admits a compatible extremal Kähler metric, then (P, L) is necessarily uniform relative K-stable. The arguments readily extend to the generalized Kähler case.

Theorem 6.13. *Let* $(M, F, J, \mathbb{T})$ *an extremal toric symplectic-type generalized Kähler structure. Then the corresponding Delzant polytope* (P, L) *is uniform relative K-stable.*

Proof. Let (u, A, B) be Abreu–Guillemin data corresponding to $(F, J, \mathbb{T})$ and $\mathbf{X}$ the corresponding symmetric matrix valued function defined in (6.3). As we have already observed in the proof of Lemma 6.7, $\mathbf{X}$ is smooth on P, positive definite on $\mathring{\mathrm{P}}$, and satisfies the same first order boundary conditions at $\partial\mathrm{P}$ as $(\mathrm{Hess}(u_0))^{-1}$ for any u_0 corresponding to a compatible toric Kähler structure. Furthermore, the extremality of (F, J) is equivalent to (6.4). Notice that, using (6.4) and $\mathbf{X} > 0$ on $\mathring{\mathrm{P}}$, (6.5) already shows that for any smooth convex function f, $\mathbf{F}_{\ell_{\rm ext}}(f) \geq 0$ with equality if and only if f is affine-linear. The improvement to uniform relative K-stability of (P, L) is obtained in [14, Theorem 4.4 and Proposition 4.6]. The arguments of the proof of [14, Theorem 4.4] can be carried out by replacing the inverse Hessian $(v^{ij}) = (\mathrm{Hess}(u_{\rm ext}))^{-1}$ with the matrix valued function $\mathbf{X}$ defined in (6.3). ■

Remark 6.14. As we saw in the course of the proof of Lemma 6.7, $\mathbf{X}$ is positive-definite on $\mathring{\mathrm{P}}$ and satisfies the same boundary conditions on $\partial\mathrm{P}$ as $\mathbf{H} = \mathrm{Hess}(u)^{-1}$. By [4], $\mathbf{X}$ then defines an ω-compatible, $\mathbb{T}$-invariant Riemannian metric $\overline{g}$ on $\mathring{M}$

which is smoothly extendable to M, by the formula

$$\overline{g} = \sum_{i,j=1}^{n} \big(\mathbf{X}_{ij}^{-1}\, d\mu_i\, d\mu_j + \mathbf{X}_{ij}\, d\theta_i\, d\theta_j\big).$$

As $\mathbf{X}$ satisfies (6.4), this is an instance of *extremal almost-Kähler* structure in the sense of [53]. Legendre observed in [52] that the proof of the uniform relative K-stability of (P, L) in [14] extends to the case of extremal toric almost-Kähler structures as above. The proof of Theorem 6.13 can be alternatively deduced from her result.

Conversely, it is now established as a consequence of deep recent work by Chen–Cheng [16] with a supplement by [44], and previous work by Donaldson [21] and Zhou–Zhu [69], that any smooth toric variety with uniform relative K-stable Delzant polytope (P, L) in the sense of (6.8) admits a compatible toric extremal Kähler metric (see [2] for a survey of the proof).

Theorem 6.15 ([16,44]). *Let $(M, J, \mathbb{T}^{\mathbb{C}})$ be a smooth toric variety and α be a Kähler de Rham class on (M, J), with a corresponding uniform relative K-stable labeled Delzant polytope* (P, L). *Then $\mathcal{K}_{\alpha}^{\mathbb{T}}$ admits an extremal Kähler metric.*

Combining Theorems 6.13 and 6.15, we get the following.

Corollary 6.16. *Suppose $(M, F, J, \mathbb{T})$ is an extremal toric symplectic-type generalized Kähler structure. Then, $\alpha := [F]$ is a Kähler class which admits a $\mathbb{T}$-invariant extremal Kähler metric.*

Proof. This follows from Theorems 6.13 and 6.15, noticing that $\alpha = [F]$ is a Kähler class with Delzant polytope (P, L): we already saw that the Kähler reduction (ω, J_u) satisfies $\omega \in \alpha$ and J_u biholomorphic to J, see [7, Lemma 3.6]. ∎

Remark 6.17. By [21, 45], the uniform relative stability of (P, L) is equivalent to a notion of uniform relative K-stability of the underlying Kähler manifold (M, J, α). The above result strongly suggests that in general, there is a similar stability notion associated to an extremal generalized Kähler manifold of symplectic type.

6.7. An existence result à la LeBrun–Simanca

Theorem 6.15 motivates us to ask the following question.

Question 6.18. If $B \neq 0$, are there further obstructions, beyond the uniform relative K-stability of (P, L), to the existence of a solution of (6.7)?

Notice that, by Proposition 4.16 and a standard use of Moser's lemma as in the proof of Theorem 4.17, the restriction of the Futaki character $\mathcal{F}_{(F,J)}$ defined in Theorem 4.18 to the Lie algebra of complex torus $(\mathbb{C}^*)^n \subset \mathrm{Aut}_{\mathrm{red}}(J, \pi_J)$ is independent of the Poisson structure π_J, i.e., of B. It is interesting to extend this observation to the whole $\mathfrak{h}_{\mathrm{red}}(J, \pi_J)$.

We will show below that the answer to Question 6.18 is negative for sufficiently "small" B. This follows from a straightforward adaptation of the LeBrun–Simanca openness result [50] (compare with [36, Theorem 8.2] which requires trivial automorphism group).

Theorem 6.19. *Suppose $(M, J, \mathbb{T}^{\mathbb{C}})$ is a smooth toric variety which admits an extremal $\mathbb{T}$-invariant Kähler metric ω_0 in the de Rham class α. Let π_J be a $\mathbb{T}^{\mathbb{C}}$-invariant Poisson structure. Then, there exists $\varepsilon = \varepsilon(\omega_0, \pi_J) > 0$, such that for any $t \in \mathbb{R}$, $|t| < \varepsilon$, there exists an extremal generalized Kähler structure $F_t \in \mathcal{GK}^{\mathbb{T}}_{t\pi,\alpha}$.*

Proof. Let (P, L) be the Delzant polytope of $(M, \omega_0, \mathbb{T})$. By the $\mathbb{T}^{\mathbb{C}}$-invariance of π_J, it is of the form (6.1) for $A, B \in \wedge^2\mathfrak{t}$. The extremal Kähler metric (J, ω_0) then corresponds to a solution $u_0 \in \mathcal{S}_{0,0}(\mathrm{P}, \mathrm{L})$ at $t = 0$ of the family of PDE's

$$-\sum_{i,j=1}^{n} \frac{\partial^2}{\partial\mu_i \partial\mu_j}\big(\mathrm{Hess}(u_t) + \sqrt{-1}tB\big)^{-1}_{ij} = \ell_{\mathrm{ext}}(\mu),$$
$$u_t \in \mathcal{S}_{tA,tB}(\mathrm{P}, \mathrm{L}) = \mathcal{S}_{0,tB}(\mathrm{P}, \mathrm{L}). \tag{6.9}$$

We first notice that (6.9) is well defined for $|t| < \varepsilon_0$. Indeed, there exists $\varepsilon_0 > 0$ such that the closed 2-form

$$F_t = \omega_0 + t\sum_{i,j=1}^{n} B_{ij}\, d\mu_i \wedge d\mu_j$$

tames J for any $|t| < \varepsilon_0$; the A-transform of (F_t, J) with $A \in \wedge^2\mathfrak{t}$ then defines a generalized Kähler structure with data (u_0, tA, tB) for any $|t| < \varepsilon_0$. Thus, the spaces $\mathcal{S}_{tA,tB}(\mathrm{P}, \mathrm{L}) = \mathcal{S}_{0,tB}(\mathrm{P}, \mathrm{L})$ are nonempty.

Up to an equivariant diffeomorphism (obtained by using Moser's lemma), any element of $\mathcal{GK}^{\mathbb{T}}_{t\pi,\alpha}$ corresponds to a F-compatible toric generalized Kähler structure with data (u_t, tA, tB). It is thus enough to show that there exists $0 < \varepsilon < \varepsilon_0$ such that (6.9) has a solution $u_t \in \mathcal{S}_{tA,tB}(\mathrm{P}, \mathrm{L})$ for any $|t| < \varepsilon$. By Corollary 6.8, it is sufficient to show the latter holds for $A = 0$, or equivalently, we can assume without loss of generality that the Poisson tensor π_J is of the form (6.1) with $A = 0$.

In order to set up the problem in a form suitable for the application of the implicit function theorem, we will first apply the Legendre transform to (6.9) (assuming $A = 0$), as we detailed in the proof of Proposition 6.6: letting

$$y_t = \nabla u_t \quad \text{and} \quad \varphi_t(y_t) + u_t(\mu) = (y_t, \mu),$$

this introduces $\mathbb{T}$-invariant smooth functions $\phi_t(y_0) = \varphi_{u_t}(y_0) - \varphi_{u_0}(y_0)$ on M, such that

$$\omega_{\phi_t} := \omega_0 + dd^c_J\phi_t > 0,$$
$$F_{(\phi_t, tB)} := \omega_{\phi_t} + t\sum_{i,j=1}^n B_{ij}\, d\mu_i^{\phi_t} \wedge d\mu_j^{\phi_t}, \quad \mu^{\phi_t} := \mu + d^c_J\phi_t,$$

is a generalized Kähler structure in $\mathcal{GK}^{\mathbb{T}}_{t\pi,\alpha}(M, J)$ with $J = J_{u_0}$, $\mathbb{T}$-equivariantly isometric to the one corresponding to the data $(u_t, 0, tB)$. The process is invertible, by applying Legendre transform with respect to $\phi_t = \varphi_t + \phi_0$. Thus, finding solution of (6.9) is equivalent to finding $\phi_t \in C^\infty(M, \mathbb{R})^{\mathbb{T}}$ such that

$$\mathrm{Gscal}_{(F_{(\phi_t, tB)}, J)} = \ell_{\mathrm{ext}}(\mu^{\phi_t}). \tag{6.10}$$

We define a map

$$W : C^\infty(M, \mathbb{R})^{\mathbb{T}} \times \wedge^2\mathfrak{t} \to C^\infty(M, \mathbb{R})^{\mathbb{T}}, \quad W(\phi, B) = \mathrm{Gscal}_{(F_{(\phi,B)}, J)} - \ell_{\mathrm{ext}}(\mu^\phi).$$

Note that W is a nonlinear differential operator defined on an open subset of $0 \in C^\infty(M, \mathbb{R})^{\mathbb{T}}$. It can be extended as a C^1-map on suitable Sobolev spaces by standard theory. We are now in a position to apply the implicit function theorem to solve $\mathrm{Gscal}_{(F_{(\phi,tB)}, J)} = 0$, knowing that $\phi_0 = 0$ is a solution at $t = 0$.

The differential of W at $(0, 0)$, computed in the direction of $(\dot\phi, 0)$ is the linearization at $\phi = 0$ of the *normalized scalar curvature* $\mathrm{Scal}^{\mathbb{T}}_{\omega_\phi} := \mathrm{Scal}_{\omega_\phi} - \ell_{\mathrm{ext}}(\mu^\phi)$ with respect to $\mathbb{T}$ of the Kähler metric $\omega_\phi = \omega_0 + dd^c_J\phi$. As ω_0 is an extremal Kähler metric, $DW_{(0,0)}(\dot\phi, 0)$ is given by (see, for instance, [29, Lemma 5.2.9] applied to $G = \mathbb{T}$ or Corollary 4.21):

$$DW_{(0,0)}(\dot\phi, 0) = \mathbb{L}_{\omega_0}(\dot\phi),$$

where $\mathbb{L}_{\omega_0}$ is the Lichnerowicz Laplacian of ω_0. The kernel of $\mathbb{L}$ restricted to the space of $\mathbb{T}$-invariant smooth functions $C^\infty(M, \mathbb{R})^{\mathbb{T}}$ therefore consists of the $\mathbb{T}$-invariant ω_0-Killing potentials. Using the maximality of $\mathbb{T}$, these are precisely the pull-backs of affine-linear functions on P by the moment map μ. To remedy the nontriviality of $\mathrm{Ker}(\mathbb{L}_{\omega_0})$, LeBrun–Simanca [50] proposed to consider the modified operator

$$(\mathrm{Id} - \Pi_0)\big(\mathrm{Gscal}_{(F_{(\phi,B)}, J)} - \ell_{\mathrm{ext}}(\mu^\phi)\big),$$

where Π_0 denotes the $L^2(M, dV_{\omega_0})$-orthogonal projection to the vector space of pull-backs by μ of affine-linear functions. This operator now acts on a neighborhood of $0 \in (\mathrm{Id} - \Pi_0)(C^\infty(M, \mathbb{R})^{\mathbb{T}})$ and takes values in $(\mathrm{Id} - \Pi_0)(C^\infty(M, \mathbb{R})^{\mathbb{T}})$. The corresponding linearization is $(\mathrm{Id} - \Pi_0)\mathbb{L}_{\omega_0} = \mathbb{L}_{\omega_0}$ (as $\mathbb{L}_{\omega_0}$ is self-adjoint on $L^2(M, dV_{\omega_0})$) and it has a trivial kernel on $(\mathrm{Id} - \Pi_0)(C^\infty(M, \mathbb{R})^{\mathbb{T}})$. By the implicit function theorem, one can then find a family $\phi_t, \phi_0 = 0$ (in a suitable Sobolev space embedded in $C^4(M)$) such that

$$(\mathrm{Id} - \Pi_0)\big(\mathrm{Gscal}_{(F_{(\phi_t, tB)}, J)} - \ell_{\mathrm{ext}}(\mu^{\phi_t})\big) = 0.$$

Notice that, by the definition of ℓ_{ext} (see (6.4) and (6.5) with f affine-linear),

$$\Pi_t\big(\mathrm{Gscal}_{(F_{(\phi_t, tB)}, J)} - \ell_{\mathrm{ext}}(\mu^{\phi_t})\big) = 0,$$

where Π_t stands for the $L^2(M, dV_{F_{(\phi_t, tB)}})$ orthogonal projection to the space of pull-backs by μ^{ϕ_t} of affine-linear functions on P. As

$$\mathrm{Ker}(\mathrm{Id} - \Pi_0) \cap \mathrm{Ker}(\Pi_t) = 0$$

for t close to 0, we conclude that (6.10) holds. The bootstrapping argument for ϕ_t uses the ellipticity of the linearization of $\mathrm{Gscal}_{(F_\phi tB)}$, and the fact that the vector field $\omega_\phi^{-1}(d\ell_{\mathrm{ext}}(\mu^\phi))$ must be J-holomorphic (and thus smooth) as soon as $\phi \in C^4(M)$, concluding the proof. ∎

Remark 6.20. Theorem 6.19 yields new examples of cscGK structures even on $\mathbf{CP}^2$ endowed with a general toric Poisson tensor π_J. The existence results for such metrics obtained in [10, 34, 36] apply only for special toric Poisson tensors which correspond to $B = 0$ in our notation. In this case, (6.2) reduces to the usual Abreu equation and is trivially solved by the symplectic potential of the Fubini–Study metric.

In general, when we fix the generalized Kähler class $\mathcal{GK}^0_{\pi,\alpha}$ (and whence the cohomology class of the symplectic form and the Poisson tensor π_J), the existence problem for extremal generalized Kähler structures is not scale invariant. Our result above solves Question 6.18 only partially, as we have no control on how small the scale $t\pi_J$ is in Theorem 6.19.

A. Generalized Kähler Hodge theory

In this appendix we briefly review the Hodge theory on a generalized Kähler manifold focusing on the symplectic-type case and prove Theorem 2.7 as a corollary of the Generalized Kähler $\partial\bar{\partial}$-lemma. We refer the reader to [37] and [13] for a more detailed exposition. Because of Definition 2.2, we shall consider a generalized Kähler structure $(\mathbb{I}, \mathbb{J})$ on (M, H_0) with $H_0 = 0$.

We start with some linear-algebraic preliminaries. Elements of the *generalized tangent bundle* $TM \oplus T^*M$ naturally act on the space of differential forms $\wedge^\bullet(M)$ via the contraction and exterior product. This action extends to an action of the Clifford algebra $\mathrm{Cl}(TM \oplus T^*M, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ is the usual pairing between TM and T^*M.

This turns $\wedge^\bullet(M)$ into a Clifford module. The action on $TM \oplus T^*M$ of the commuting operators $\mathbb{J}$ and $\mathbb{I}$ from (2.3) extends to an action on $\wedge^\bullet(M)$, via the representation of the $\mathfrak{spin}(n,n)$ Lie algebra. We thus get a decomposition of $\wedge^\bullet(M) \otimes \mathbb{C}$ into the eigenspaces of $\mathbb{J}$ and $\mathbb{I}$:

$$\wedge^\bullet(M) \otimes \mathbb{C} = \bigoplus_{-n \le p,q \le n} U^{p,q},$$

where $U^{p,q}$ is the $(\sqrt{-1}p, \sqrt{-1}q)$ eigenspace of $(\mathbb{J}, \mathbb{I})$. One can show that $U^{p,q} = 0$ unless $|p-q|, |p+q| \le n$, and $\overline{U}^{p,q} = U^{-p,-q}$. For a form $\xi \in \wedge^\bullet(M) \otimes \mathbb{C}$ we will denote by $\xi^{p,q}$ its component in $U^{p,q}$. More concretely, we can describe the spaces $U^{p,q}$ as follows. Let

$$(TM \oplus T^*M) \otimes \mathbb{C} = L_+ \oplus L_- \oplus \overline{L}_+ \oplus \overline{L}_-$$

be the decomposition of the complexified generalized tangent bundle into the eigenspaces of $(\mathbb{J}, \mathbb{I})$:

$$\mathbb{J}\big|_{L_+ \oplus \overline{L}_-} = \sqrt{-1}\,\mathrm{Id}, \quad \mathbb{I}\big|_{L_+ \oplus L_-} = \sqrt{-1}\,\mathrm{Id}\,.$$

Specifically, using (2.3), we have

$$\begin{aligned} L_+ &= \{v - \sqrt{-1}F(v,\cdot) \mid v \in T_I^{1,0}M\}, \\ L_- &= \{v - \sqrt{-1}F(v,\cdot) \mid v \in T_J^{0,1}M\}. \end{aligned} \tag{A.1}$$

Then one can compute (see [13])

$$U^{0,n} = \mathrm{span}\big(e^{\sqrt{-1}F}\big), \quad U^{a-b,n-a-b} = \wedge^a \overline{L}_- \cdot \big(\wedge^b \overline{L}_+ \cdot U^{0,n}\big). \tag{A.2}$$

Since $FI = -J^*F$, and $v \cdot e^{\sqrt{-1}F} = \sqrt{-1}F(v,\cdot) \wedge e^{\sqrt{-1}F}$, it follows from (A.2) that

$$\begin{aligned} U^{1,n-1} &= \{\xi \wedge e^{\sqrt{-1}F} \mid \xi \in \wedge_I^{0,1}(M)\}, \\ U^{-1,n-1} &= \{\xi \wedge e^{\sqrt{-1}F} \mid \xi \in \wedge_J^{1,0}(M)\}. \end{aligned} \tag{A.3}$$

Now let us review the underlying differential complex of $(U^{\bullet,\bullet}, d)$. The integrability assumptions of the generalized Kähler structure $(\mathbb{J}, \mathbb{I})$ imply that $d\colon U^{\bullet,\bullet} \to U^{\bullet,\bullet}$ has four components of bidegrees $(\pm 1, \pm 1)$. We denote them respectively

$$\begin{aligned} &\delta_+ \colon U^{p,q} \mapsto U^{p+1,q-1}, \quad \delta_- \colon U^{p,q} \mapsto U^{p-1,q-1}, \\ &\overline{\delta}_+ \colon U^{p,q} \mapsto U^{p-1,q+1}, \quad \overline{\delta}_- \colon U^{p,q} \mapsto U^{p+1,q+1}, \end{aligned}$$

so that $d = \delta_+ + \delta_- + \overline{\delta}_+ + \overline{\delta}_-$.

Furthermore, there is a natural real pairing $\wedge^\bullet(M) \times \wedge^\bullet(M) \to \wedge^{2n}(M)$:

$$(\xi, \eta) := [\xi \wedge \sigma(\eta)]_{\mathrm{top}},$$

where $[\cdot]_{\mathrm{top}}$ is the top degree component, and $\sigma\colon \wedge^\bullet(M) \mapsto \wedge^\bullet(M)$ is the Clifford involution $\sigma(dx^1 \wedge \cdots \wedge dx^k) := dx^k \wedge \cdots \wedge dx^1$. Following [13,37], we define the linear operator $\star = \mathbb{I}\mathbb{J}$ on $U^{\bullet,\bullet}$ as follows:

$$\star\colon U^{\bullet,\bullet} \to U^{\bullet,\bullet}, \quad \star\big|_{U^{p,q}} = \left(\sqrt{-1}\right)^{p+q} \mathrm{Id}\,.$$

Since for $\alpha \in \wedge^\bullet(M) \otimes \mathbb{C}$ we have $\overline{\star\alpha} = \star\bar{\alpha}$, it follows that $\star$ is a real operator.

Remark A.1. In the standard Kähler setting the operator $\star$ is related to the classical Hodge star $*_g$ via the identity

$$(\xi, \star\eta) = \xi \wedge *_g \eta.$$

Our convention for $\star$ is consistent with [13, Section 2.1.1].

Note that in general, given $\xi \in \wedge^k(M)$, $\star\xi$ is not concentrated in $\wedge^{2n-k}(M)$, but rather has components of various degrees.

The key algebraic fact about generalized geometry is that $\star$ induces a positive definite Hermitian pairing on the space of complex-valued differential forms

$$G(\xi, \eta) := \int_M (\xi, \star\bar{\eta}), \quad \xi, \eta \in \wedge^\bullet(M) \otimes \mathbb{C}.$$

Example A.2. For $e^{\sqrt{-1}F} \in U^{0,n}$, we have

$$\star\overline{e^{\sqrt{-1}F}} = \left(\sqrt{-1}\right)^{-n} e^{-\sqrt{-1}F} \quad \text{and} \quad \sigma\left(e^{-\sqrt{-1}F}\right) = e^{\sqrt{-1}F},$$

so that

$$G\left(e^{\sqrt{-1}F}, e^{\sqrt{-1}F}\right) = \left(\sqrt{-1}\right)^{-n} \int_M \left(e^{\sqrt{-1}F}, e^{-\sqrt{-1}F}\right) = 2^n \int_M F^{[n]}.$$

Using this pairing and integration by parts we define the adjoint linear first order differential operator $d^*\colon \wedge^\bullet(M) \mapsto \wedge^\bullet(M)$ as follows:

$$G(d\xi, \eta) = G(\xi, d^*\eta).$$

Just like $\star$, operator d^* usually differs from its Riemannian analogue. The following result is the crucial fact of Hodge theory on M.

Theorem A.3 (Hodge identities, see [13, 37]). *On a compact generalized Kähler manifold there is a decomposition $d^* = \delta_+^* + \delta_-^* + \bar{\delta}_+^* + \bar{\delta}_-^*$, where $\delta_\pm^*$ and $\bar{\delta}_\pm^*$ are the adjoint operators of $\delta_\pm$ and $\bar{\delta}_\pm$. These operators satisfy the following identities:*

$$\delta_+^* = \bar{\delta}_+, \quad \delta_-^* = -\bar{\delta}_-, \quad 4\Delta = \Delta_{\delta_+} = \Delta_{\delta_-} = \Delta_{\bar{\delta}_+} = \Delta_{\bar{\delta}_-},$$

where $\Delta_d = dd^* + d^*d$, $\Delta_{\delta_+} = \delta_+\delta_+^* + \delta_+^*\delta_+$, and the remaining Laplacians are defined analogously.

This theorem has several important consequences. First, the cohomology of $U^{\bullet,\bullet}$ with respect to any of the differentials d, δ_+, δ_- are naturally isomorphic to each other and to the space of *harmonic forms*

$$\mathcal{H}^{p,q} = \{\xi \in U^{p,q} \mid \Delta\xi = 0\}. \tag{A.4}$$

Importantly, a form $\xi \in U^{p,q}$ of pure type is harmonic if and only if it is closed. Furthermore, every form $\xi \in U^{p,q}$ has a decomposition

$$\xi = \xi_h + \Delta\eta,$$

where $\xi_h \in \mathcal{H}^{p,q}$ is the harmonic part of ξ and $\eta \in U^{p,q}$. Another key consequence of the Hodge identities is the following $\delta_+\delta_-$-lemma for $(U^{p,q}, d)$.

Lemma A.4 ($\delta_+\delta_-$ and $\delta_+\overline{\delta}_-$-lemma [13, 37]). *On a compact generalized Kähler manifold, we have*

$$\begin{aligned}\mathrm{Im}(\delta_+) \cap \mathrm{Ker}(\delta_-) &= \mathrm{Ker}(\delta_+) \cap \mathrm{Im}(\delta_-) = \mathrm{Im}(\delta_+\delta_-),\\ \mathrm{Im}(\delta_+) \cap \mathrm{Ker}(\overline{\delta}_-) &= \mathrm{Ker}(\delta_+) \cap \mathrm{Im}(\overline{\delta}_-) = \mathrm{Im}(\delta_+\overline{\delta}_-).\end{aligned}$$

In the symplectic-type case we will use a more explicit description of $\mathcal{H}^{\pm 1,n-1}$. Using the identification (A.3) we conclude that there are natural isomorphisms given by wedging with $e^{-\sqrt{-1}F}$:

$$\begin{aligned}\mathcal{H}^{1,n-1} &\simeq \{\xi \in \wedge^{0,1}_I(M) \mid d\xi = 0\},\\ \mathcal{H}^{-1,n-1} &\simeq \{\xi \in \wedge^{1,0}_J(M) \mid d\xi = 0\},\\ \mathcal{H}^{1,n-1} \oplus \mathcal{H}^{-1,n-1} &\simeq H^1(M, \mathbb{C}).\end{aligned}$$

Lemma A.5. *Let (F, J) be a symplectic-type generalized Kähler structure on a compact manifold M. Given a 1-form $\xi \in \wedge^1(M) \otimes \mathbb{C}$, we decompose it as*

$$\xi = \xi_+ + \xi_-,$$

where

$$\xi_+ = I(I+J)^{-1}\xi + \sqrt{-1}(I+J)^{-1}\xi, \quad \xi_- = J(I+J)^{-1}\xi - \sqrt{-1}(I+J)^{-1}\xi,$$

so that

$$\xi \wedge e^{\sqrt{-1}F} = \xi_+ \wedge e^{\sqrt{-1}F} + \xi_- \wedge e^{\sqrt{-1}F}$$

is the decomposition of $\xi \wedge e^{\sqrt{-1}F} \in U^{1,n-1} \oplus U^{-1,n-1}$ by type. Then the maps

$$\xi \mapsto \left(\xi \wedge e^{\sqrt{-1}F}\right)_h^{1,n-1}, \quad \xi \mapsto \left(\xi \wedge e^{\sqrt{-1}F}\right)_h^{-1,n-1}$$

induce isomorphisms

$$i_+ \colon H^1(M,\mathbb{R}) \mapsto \mathcal{H}^{1,n-1}, \quad i_- \colon H^1(M,\mathbb{R}) \mapsto \mathcal{H}^{-1,n-1}.$$

Furthermore, any $[\xi] \in H^1(M,\mathbb{R})$ *can be represented by a closed,* d^c_J*-closed form* ξ_-.

Proof. First, we observe that the map $\xi \mapsto (\xi \wedge e^{\sqrt{-1}F})^{1,n-1}_h$ yields a well-defined map on $H^1(M,\mathbb{R})$. Indeed, for an exact 1-form df, we have

$$\big(df \wedge e^{\sqrt{-1}F}\big)^{1,n-1}_h = \big(\delta_+\big(f \wedge e^{\sqrt{-1}F}\big)\big)_h = 0.$$

Next we observe (using (A.3)) that the map

$$\wedge^1(M) \otimes \mathbb{C} \mapsto U^{1,n-1} \oplus U^{-1,n-1}, \quad \xi \mapsto \xi \wedge e^{\sqrt{-1}F}$$

establishes an isomorphism between

$$H^1(M,\mathbb{C}) \quad \text{and} \quad H^{1,n-1}(U^{\bullet,\bullet},d) \oplus H^{1,n-1}(U^{\bullet,\bullet},d) \simeq \mathcal{H}^{1,n-1} \oplus \mathcal{H}^{-1,n-1},$$

thus

$$2\dim_{\mathbb{R}} H^1(M,\mathbb{R}) = \dim_{\mathbb{R}} \mathcal{H}^{1,n-1} + \dim_{\mathbb{R}} \mathcal{H}^{-1,n-1}.$$

Therefore, it suffices to prove that the maps $i_\pm$ are injective. Indeed, consider a closed real form $\xi \in \wedge^1(M)$. Assume that $(\xi \wedge e^{\sqrt{-1}F})^{1,n-1}_h = 0$, so that the δ_+-closed form $(\xi \wedge e^{\sqrt{-1}F})^{1,n-1}$ is δ_+-exact:

$$\big(\xi \wedge e^{\sqrt{-1}F}\big)^{1,n-1} = \delta_+\big((\phi + \sqrt{-1}\psi)e^{\sqrt{-1}F}\big)$$

for some complex-valued function $\phi + \sqrt{-1}\psi \in C^\infty(M,\mathbb{C})$. Using the definition of ξ_+ and the explicit expression for δ_+, we can rewrite it as

$$\xi = d\psi + J\,d\phi.$$

Since ξ is closed, we conclude that $dd^c_J\phi = 0$. On a compact manifold M the latter implies that $\phi = \text{const}$ by the maximum principle. Thus $\xi = d\psi$ represents the trivial cohomology class, so the map i_+ is injective. Similarly, i_- is also injective, and by the dimension count both must by isomorphisms.

To prove that last claim, we observe that the map

$$\mathcal{H}^{-1,n-1} \to H^1(M,\mathbb{R}), \quad \xi_- \wedge e^{\sqrt{-1}F} \mapsto [\mathrm{Re}(\xi_-)]$$

between two vector spaces of equal dimensions is also injective for similar reasons as for i_-. Indeed, if $\mathrm{Re}(\xi_-) = d\phi$ is exact, then ϕ is dd^c_J-closed, which is only possible when ϕ is a constant. Therefore, the above map is an isomorphism, and every element in $H^1(M,\mathbb{R})$ has a closed, d^c_J-closed representative. ∎

Now we are ready to prove Theorem 2.7. Let us recall its statement.

Theorem A.6 (Theorem 2.7). *Let (M, F, J) be a compact generalized Kähler manifold of symplectic type with $I = -F^{-1}J^*F$. Then*

(1) *any J-holomorphic p-form $\xi \in \wedge_J^{p,0}(M)$ is closed;*

(2) *any real differential 1-form ξ such that $(d\xi)_I^{2,0} = 0$ admits a unique decomposition*

$$\xi = (I + J)\eta + d\phi + I\,d\psi, \tag{A.5}$$

where η and $J\eta$ are closed, and $\phi, \psi \in C^\infty(M, \mathbb{R})$.

(3) *the map $\xi \mapsto \eta - \sqrt{-1}J\eta$, where η is determined by (A.5), induces an isomorphism*

$$H^1(M, \mathbb{R}) \simeq H^{1,0}_{\overline{\partial}_J}(M);$$

(4) *any exact 2-form α of type $(1, 1)$ with respect to I is $\alpha = dd_I^c\psi$ for some $\psi \in C^\infty(M, \mathbb{R})$.*

The same statements hold after changing the roles of I and J.

Proof. First, we observe that from the identities (A.1) and (A.2) we get an isomorphism

$$\bigoplus_p \wedge_J^{p,0}(M) \simeq \bigoplus_p U^{-p,n-p}, \quad \xi \mapsto \xi \wedge e^{\sqrt{-1}F}. \tag{A.6}$$

Let $\xi \in \wedge_J^{p,0}(M)$ be a p-holomorphic form. Then $d\xi = \sqrt{-1}\partial_J\xi \in \wedge_J^{p+1,0}(M)$, and we claim that this form vanishes. Indeed, from isomorphism (A.6) we conclude that

$$\xi \wedge e^{\sqrt{-1}F}, d\xi \wedge e^{\sqrt{-1}F} \in \bigoplus_p U^{-p,n-p},$$

which implies that $\xi \wedge e^{\sqrt{-1}F}$ is δ_+-closed. Consider the form $\delta_-(\xi \wedge e^{\sqrt{-1}F})$. This form is δ_--exact and δ_+-closed, therefore by the $\delta_+\delta_-$-lemma it has to be in the image of

$$\delta_+\delta_-\colon \bigoplus_p U^{-p,n-p-2} \to \bigoplus_p U^{-p,n-p}.$$

As the former space is trivial, $\delta_-(\xi \wedge e^{\sqrt{-1}F})$ is zero. Similarly, applying the $\delta_+\overline{\delta}_-$-lemma we conclude that $\overline{\delta}_-(\xi \wedge e^{\sqrt{-1}F}) = 0$. Since for dimensional reasons we also have $\overline{\delta}_+(\xi \wedge e^{\sqrt{-1}F}) = 0$, it follows that $\xi \wedge e^{\sqrt{-1}F}$ is closed which is equivalent to ξ being closed. This proves (1).

To prove (2), we now assume that a real 1-form ξ satisfies $(d\xi)_I^{2,0} = 0$. We claim that the $U^{-2,n-2}$-component of $d(\xi \wedge e^{\sqrt{-1}F})$ vanishes. Indeed, for any two elements

$$v - \sqrt{-1}F(v, \cdot), w - \sqrt{-1}F(w, \cdot) \in L_+, \quad v, w \in T_I^{1,0}M,$$

we have the vanishing of the Clifford action

$$\begin{aligned}\big(v-\sqrt{-1}F(v,\cdot)\big)\cdot\big(w-\sqrt{-1}F(w,\cdot)\big)\cdot\big(d\xi\wedge e^{\sqrt{-1}F}\big)&=2(d\xi)(w,v)\wedge e^{\sqrt{-1}F}\\&=0,\end{aligned}$$

since $d\xi$ is of type $(1,1)$ with respect to I. Therefore, $(d(\xi\wedge e^{\sqrt{-1}F}))^{-2,n-2}=0$, so that the element $(\xi\wedge e^{\sqrt{-1}F})^{-1,n-1}$ is δ_--closed, and its Hodge decomposition (A.4) takes form

$$\xi_-\wedge e^{\sqrt{-1}F}=\big(J\eta-\sqrt{-1}\eta\big)\wedge e^{\sqrt{-1}F}+\delta_-\big((\phi-\sqrt{-1}\psi)e^{\sqrt{-1}F}\big)$$

for some real closed and d^c_J-closed 1-form η, and $\phi,\psi\in C^\infty(M,\mathbb{R})$. Collecting the imaginary degree 1 parts of the above identity and using $J(I+J)^{-1}=(I+J)^{-1}I$, we find

$$\xi=(I+J)\eta+d\phi+I\,d\psi.$$

This proves (2).

Claim (3) follows from Lemma A.5 and the fact that every holomorphic 1-form is closed.

Finally, for (4), we let $\alpha=d\xi$ to be a 2-form of I-type (1,1). Then by (2) we can decompose ξ as

$$\xi=(I+J)\eta+d\phi+I\,d\psi.$$

On the other hand, by (3) we can find a closed ξ'-form representing $[\xi']\in H^1(M,\mathbb{R})$ such that its decomposition has the same η-term:

$$\xi'=(I+J)\eta+d\phi'+Id\psi'.$$

Therefore, taking the exterior derivative of the difference $\xi-\xi'$, we find

$$\alpha=d\xi=d(\xi-\xi')=dI\,d(\psi-\psi'),$$

as claimed. ∎

Remark A.7. The Hodge Laplacian $\Delta_d=d^*d+dd^*$ can be restricted to the space of sections of the line bundle $U^{0,n}$. Since $U^{0,n}$ is trivialized by a nowhere vanishing section $e^{\sqrt{-1}F}$, we can identify $C^\infty(M,U^{0,n})$ with the space of complex-valued functions $C^\infty(M,\mathbb{C})$. Hence, we can interpret Δ_d as a *Generalized Kähler Laplacian* $\Delta^{GK}\colon C^\infty(M,\mathbb{C})\to C^\infty(M,\mathbb{C})$. Explicitly, this operator is given by the formula

$$\Delta^{GK}f=\mathrm{tr}_F\big(2d(I+J)^{-1}df+\sqrt{-1}d(I-J)(I+J)^{-1}df\big).$$

It follows that $\Delta^{GK}f$ is self-adjoint with respect to the Hermitian $L^2(M,dV_F)$-pairing on $C^\infty(M,\mathbb{C})$.

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