

# Which values of the volume growth and escape time exponent are possible for a graph?

Martin T. Barlow

## Abstract

Let  $\Gamma = (G, E)$  be an infinite weighted graph which is Ahlfors  $\alpha$ -regular, so that there exists a constant  $c$  such that  $c^{-1}r^\alpha \leq V(x, r) \leq cr^\alpha$ , where  $V(x, r)$  is the volume of the ball centre  $x$  and radius  $r$ . Define the escape time  $T(x, r)$  to be the mean exit time of a simple random walk on  $\Gamma$  starting at  $x$  from the ball centre  $x$  and radius  $r$ . We say  $\Gamma$  has escape time exponent  $\beta > 0$  if there exists a constant  $c$  such that  $c^{-1}r^\beta \leq T(x, r) \leq cr^\beta$  for  $r \geq 1$ . Well known estimates for random walks on graphs imply that  $\alpha \geq 1$  and  $2 \leq \beta \leq 1 + \alpha$ . We show that these are the only constraints, by constructing for each  $\alpha_0, \beta_0$  satisfying the inequalities above a graph  $\tilde{\Gamma}$  which is Ahlfors  $\alpha_0$ -regular and has escape time exponent  $\beta_0$ . In addition we can make  $\tilde{\Gamma}$  sufficiently uniform so that it satisfies an elliptic Harnack inequality.

## 0. Introduction

Let  $\Gamma = (G, E)$  be an infinite connected locally finite graph. We call  $a = (a_{xy}), x, y \in G$  a *conductance matrix* if  $a_{xy} \geq 0$  and  $a_{xy} = a_{yx}$  for all  $x, y \in G$  and in addition  $a$  is linked to the graph structure by the requirement that there exists  $C_1 > 0$  such that

$$\begin{aligned} a_{xy} &= 0 && \text{if } \{x, y\} \text{ is not an edge in } \Gamma, \\ a_{xy} &\geq C_1 > 0 && \text{if } \{x, y\} \in E. \end{aligned} \tag{0.1}$$

We call the pair  $(\Gamma, a)$  a *weighted graph*. We call the *natural weight* on  $\Gamma$  the weights given by taking the conductance matrix  $a$  to be the adjacency

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matrix of  $\Gamma$ ; that is

$$a_{xy} = \begin{cases} 1 & \text{if } \{x, y\} \in E, \\ 0 & \text{if } \{x, y\} \notin E. \end{cases}$$

Whenever we discuss below a graph without any weights specified, we will assume we are using the natural weights. We set  $\mu_x = \sum_y a_{xy}$ , and extend  $\mu$  to a measure on  $G$ . Let  $d(x, y)$  be the usual graph distance on  $G$ , and let for  $x \in G$ ,  $r \in (0, \infty)$ ,

$$B(x, r) = \{y : d(x, y) < r\}, \quad V(x, r) = \mu(B(x, r)).$$

We say that  $\Gamma$  is *Ahlfors  $\alpha$ -regular* (here  $\alpha \in (0, \infty)$ ) if there exists a constant  $c \geq 1$  such that the volume growth function  $V$  satisfies

$$c^{-1}r^\alpha \leq V(x, r) \leq cr^\alpha, \quad r \in [1, \infty), x \in G. \quad (V_\alpha)$$

Note that, with (0.1),  $(V_\alpha)$  implies that the vertex degree is uniformly bounded.

A random walk  $X = (X_n, n \geq 0, \mathbb{P}^x, x \in G)$  on  $(\Gamma, a)$  is a  $\mu$ -symmetric  $G$ -valued Markov chain with transition probabilities given by

$$p_{xy} = \mathbb{P}^x(X_{n+1} = y | X_n = x) = \frac{a_{xy}}{\mu_x}, \quad x, y \in G, n \geq 0.$$

The heat kernel on  $(\Gamma, a)$  is the density of  $X_n$  with respect to the measure  $\mu$ :

$$p_n(x, y) = \mathbb{P}^x(X_n = y) / \mu_y,$$

and is easily seen to be symmetric:  $p_n(x, y) = p_n(y, x)$ . For  $A \subset G$  write

$$T_A = \min\{n \geq 0 : X_n \in A\}, \quad T_x = T_{\{x\}},$$

and set

$$\tau_{x,r} = T_{B(x,r)^c} = \min\{n \geq 0 : d(x, X_n) \geq r\}.$$

We say that  $\Gamma$  satisfies  $(E_\beta)$  if for some constant  $c \geq 1$ ,

$$c^{-1}r^\beta \leq \mathbb{E}^x \tau_{x,r} \leq cr^\beta, \quad r \in [1, \infty), x \in G. \quad (E_\beta)$$

There has recently been much activity in the general area of geometry and heat kernels. While many of the questions in this field arose in the context of manifolds, they can also be posed for graphs, where the initial technical difficulties are fewer, but the same basic principles apply. The overall object is to relate geometric properties of these spaces (such as  $(V_\alpha)$  or the weaker volume doubling property), and analytic ones, such as the space satisfying various kinds of Sobolev, Poincaré or Harnack inequalities, with the global properties of the random walk  $X$  and its transition density  $p_n(x, y)$ .

In particular, it has been discovered that spaces satisfying  $(E_\beta)$  with  $\beta > 2$  provide natural families of examples of spaces which satisfy the elliptic Harnack inequality (see below), but fail to satisfy the stronger parabolic Harnack inequality. See [BB1], and [HSC] for a recent discussion.

The weighted graph  $(\Gamma, a)$  satisfies the *volume doubling* condition (VD) if there exists  $c > 1$  such that

$$V(x, 2R) \leq cV(x, R) \text{ for all } x \in G, R \geq 1. \tag{VD}$$

Volume doubling, together with a Poincaré inequality, is a necessary and sufficient condition for the parabolic Harnack inequality to hold – see [G], [SC] (for manifolds) and [D1] for graphs. The condition  $(V_\alpha)$  immediately implies (VD), but gives much more regularity in the spatial structure of  $\Gamma$ .

Probabilistic conditions like  $(E_\beta)$  have only been introduced more recently. In [GT1], [GT2] it is shown that, combined with  $(V_\alpha)$  or (VD) and an elliptic Harnack inequality,  $(E_\beta)$  yields very good upper and lower bounds on  $p_n(x, y)$ .

In this paper we answer the following question:

If  $(\Gamma, a)$  satisfies  $(V_\alpha)$  and  $(E_\beta)$  what values of  $(\alpha, \beta)$  are possible?

The theorem below is well known to experts, and follows easily from known estimates on random walks due to Varopoulos, Carne, Kesten, Kusuoka and Telcs; for completeness we give a quick proof in Section 1.

**Theorem 1** *If  $(\Gamma, a)$  is an infinite connected weighted graph satisfying (0.1),  $(V_\alpha)$  and  $(E_\beta)$  then  $\alpha \geq 1$ , and*

$$2 \leq \beta \leq 1 + \alpha. \tag{0.2}$$

We now recall the definition of the elliptic Harnack inequality. (See [D1] for the parabolic Harnack inequality, which has a more complicated definition, and is not used in this paper.)

**Definitions.** 1. Let  $A \subset G$ . We write  $\partial A = \{y \in A^c : d(x, y) = 1 \text{ for some } x \in A\}$  for the exterior boundary of  $A$ , and set  $\bar{A} = A \cup \partial A$ .

2. A function  $h : \bar{A} \rightarrow \mathbb{R}$  is *harmonic on  $A \subset G$*  if

$$\Delta h(x) = \frac{1}{\mu_x} \sum_y a_{xy}(h(y) - h(x)) = 0, \quad x \in A.$$

This is equivalent to the assertion that  $(h(X_{n \wedge T_{A^c}}), n \geq 0)$  is a martingale.

3.  $(\Gamma, a)$  satisfies an *elliptic Harnack inequality* (EHI) if there exists  $c_1 > 0$  such that, for any  $x \in G$ ,  $R \geq 1$ , and non-negative  $h : G \rightarrow \mathbb{R}$  harmonic in  $B(x, 2R)$ ,

$$\sup_{B(x,R)} h \leq c_1 \inf_{B(x,R)} h. \quad (0.3)$$

We have taken balls  $B(x, R) \subset B(x, 2R)$  just for simplicity: if  $K > 1$  and (0.3) holds whenever  $h \geq 0$  is harmonic in  $B(x, KR)$ , then an easy chaining argument gives (EHI) (for a different constant  $c_1$ ).

The main result of this paper is

**Theorem 2** *Let  $\alpha \geq 1$  and  $2 \leq \beta \leq 1 + \alpha$ . There there exists an infinite connected locally finite graph  $\tilde{\Gamma}$  which satisfies  $(V_\alpha)$ ,  $(E_\beta)$  and (EHI).*

### Examples.

1. The Euclidean space  $\mathbb{Z}^d$ ,  $d \geq 1$ , (with its natural graph structure and conductances) satisfies  $(V_d)$  and  $(E_2)$ , as well as (EHI). If  $d \geq 2$  then the graph  $\Gamma$  consisting of two copies of  $\mathbb{Z}^d$  with their origins identified satisfies  $(V_d)$  and  $(E_2)$ , but fails to satisfy (EHI).

2. The binary tree satisfies  $(E_1)$ , but since  $V(x, r) \approx 2^r$  it fails to satisfy  $(V_\alpha)$  for any  $\alpha$ . ((EHI) also fails.)

3. Examples of graphs with  $\beta > 2$  are provided by ‘pre-fractal’ graphs (see for example [J], [BB2], [GT1]). The condition  $(E_\beta)$  implies that the mean square displacement  $\mathbb{E}^x d(x, X_n)^2$  grows as  $n^{2/\beta}$ : if  $\beta > 2$  then this growth is sublinear and is referred to by physicists as ‘anomalous diffusion’. We call  $\beta$  the ‘anomalous diffusion exponent’, or the ‘escape time exponent’. (In the physics literature, or that on diffusions on fractals, one would write  $\alpha = d_f$ , the ‘fractal dimension’ and  $\beta = d_w$ , the ‘walk dimension’.)

4. Let  $\Gamma_i$ ,  $i = 1, 2$  satisfy  $(V_{\alpha_i})$ ,  $(E_{\beta_i})$ . The product graph  $\hat{\Gamma} = \Gamma_1 \times \Gamma_2$  can be defined by taking the edges of  $\hat{\Gamma}$  to be of the form  $\{(x, x_2), (x, y_2)\}$ , where  $\{x_2, y_2\} \in E_2$ , and  $\{(x_1, x_2), (y_1, x_2)\}$ , where  $\{x_1, y_1\} \in E_1$ . Then it is easy to see that  $\hat{\Gamma}$  satisfies  $(V_{\alpha_1 + \alpha_2})$  and  $(E_{\beta_1 \wedge \beta_2})$ .

The case  $\beta = 2$ ,  $\alpha \in [1, \infty)$ ,  $\alpha$  not an integer, has been treated (in the metric space context) in several recent papers. Following a question in [HS], Bourdon and Pajot [BoP] proved that the boundary of certain hyperbolic buildings satisfy  $(V_\alpha)$ , as well as an analytic condition (a weak (1,1) Poincaré inequality) which is strong enough to imply both  $(E_2)$  and (EHI). Here the possible values of  $\alpha$  form a countable dense subset in  $[1, \infty)$ . More recently, Laakso [L] has given another construction of metric spaces satisfying  $(V_\alpha)$  and a weak (1,1) Poincaré inequality, which permits any  $\alpha > 1$ . This is

done by taking the product of  $[0, 1]$  with a Cantor set and then identifying a dense set of points. The proof of Theorem 2 uses a similar construction, adapted to the graph case.

The examples in [BoP] and [L] disprove a conjecture made in [B1, Sec. 3].

We now outline the main steps in the proof of Theorem 2.

**Definition.** We say a weighted graph  $(\Gamma, a)$  is *very strongly recurrent* (VSR) if there exists  $p_0 > 0$  such that, for all  $R \geq 1$ ,  $x, y \in G$  with  $d(x, y) < R$ ,

$$\mathbb{P}^x(T_y < \tau_{x,2R}) \geq p_0. \tag{VSR}$$

**Remarks.** 1. The literature contains (at least) two distinct definitions of strong recurrence for graphs, in [T2] and [D2]. (VSR) is equivalent to the definition in [D2], and stronger than that in [T2].

2. It is easily seen that (VSR) implies recurrence; in Lemma 1.6 below we prove it implies (EHI).

3.  $\mathbb{Z}^1$  is very strongly recurrent, while  $\mathbb{Z}^2$  is recurrent, but not very strongly recurrent.

The following Proposition, which is proved in Section 1, implies that many of the graphs studied in the fractals literature (such as the pre-Sierpinski gasket) satisfy (VSR).

**Proposition 3** *Let  $(\Gamma, a)$  satisfy  $(V_\alpha)$ , and  $(E_\beta)$ .*

- (a) *If  $\beta \geq \alpha$  then  $\Gamma$  is recurrent.*
- (b) *If  $\beta > \alpha$  and  $\Gamma$  satisfies (EHI) then  $\Gamma$  satisfies (VSR).*
- (c) *If  $\beta = \alpha$  then  $\Gamma$  is recurrent but does not satisfy (VSR).*
- (d) *If  $\beta < \alpha$  and  $\Gamma$  satisfies (EHI) then  $\Gamma$  is transient.*

**Remark.** I do not know if the conditions  $(V_\alpha)$  and  $(E_\beta)$ , with  $\alpha > \beta$ , are enough to imply that  $\Gamma$  is transient.

**Proposition 4** *Let  $\alpha \geq 1$ . Then there exists a connected locally finite infinite graph  $\Gamma_\alpha$  satisfying  $(V_\alpha)$ ,  $(E_{1+\alpha})$ , (EHI), and (VSR).*

This is proved in Section 4, by adapting work of the author and Hambly (in [BH]) on mixtures of different types of Sierpinski gaskets to the case of graphs which are trees.

**Proposition 5** *Let  $(\Gamma, a)$  satisfy  $(V_\alpha)$ ,  $(E_\beta)$ , (VSR), and so (EHI). Let  $\lambda > 0$ . Then there exists a weighted graph  $(\tilde{\Gamma}, \tilde{a})$  satisfying  $(V_{\alpha+\lambda})$ ,  $(E_\beta)$  and (EHI).*

This is proved in Sections 2 and 3. In section 2 we construct  $\tilde{\Gamma}$  by taking the product of  $G$  with an ultrametric space  $U$ , and fitting in edges in such a way that  $\tilde{\Gamma}$  consists of a countable number of copies of  $\Gamma$ , connected at link points. Section 2 deals with the geometry of  $\tilde{\Gamma}$ , and proves that it satisfies  $(V_{\alpha+\lambda})$ . In section 3 we study random walks on  $\tilde{\Gamma}$ . It is easy to prove that  $(E_\beta)$  holds, but the elliptic Harnack inequality takes a little more work.

In Section 5 we conclude the paper with some additional examples, motivated by those in [D2], concerning the property (EHI). In particular we have:

**Theorem 6** *The elliptic Harnack inequality is not stable under products. That is, there exists a graph  $\Gamma$  which satisfies (EHI) such that the product graph  $\Gamma \times \Gamma$  does not satisfy (EHI).*

**Proof of Theorem 2.** Let  $\alpha \geq 1$  and  $\beta$  satisfy (0.2). By Proposition 4 there exists a graph  $\Gamma$  satisfying  $(V_{\beta-1})$ ,  $(E_\beta)$ , (EHI), and (VSR). By Proposition 5, taking  $\lambda = \alpha + 1 - \beta$ , there exists a graph  $\tilde{\Gamma}$  satisfying  $(V_\alpha)$ ,  $(E_\beta)$ , and (EHI). ■

Throughout this paper we use  $c, c', c''$  for positive constants which may change from line to line, and  $c_i$  for positive constants that are fixed for each section.  $C_i$  denote positive constants which are fixed for the whole paper.

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## 1. Weighted graphs and random walks

Throughout this section we take  $(G, a)$  to be an infinite connected locally finite weighted graph.

**Lemma 1.1** (See [Ku], [Ke]) *If  $\Gamma$  satisfies  $(V_\alpha)$  then there exists a constant  $c$  such that*

$$\mathbb{E}^x \tau_{x,2R} \geq c \frac{R^2}{(\log R)^{1/2}}, \quad x \in G, R \geq 1. \quad (1.1)$$

*If  $\Gamma$  also satisfies  $(E_\beta)$  then  $\beta \geq 2$ .*

**Remark.** See [BaP] for an example which shows that it is not possible to remove the  $\log R$  term in (1.1).

**Proof.** First, note that the final assertion is immediate from (1.1). By [C], [V], the transition probabilities  $p_n(x, y)$  for  $X$  satisfy

$$p_n(x, y) \leq ce^{-d(x,y)^2/2n}, \quad x, y \in G, \quad n \geq 1.$$

Let  $R \geq 1$ ,  $1 \leq n \leq R^2$  and set  $\lambda = R/n^{1/2} \geq 1$ . Set  $A_k = \{y : 2^k R < d(x, y) \leq 2^{k+1} R\}$ ; by  $(V_\alpha)$  we have  $\mu(A_k) \leq c2^{\alpha k} R^\alpha$ .

We have (see [B1, Lemma 3.7] for a similar calculation with more details)

$$\begin{aligned} \mathbb{P}^x(d(x, X_n) \geq R) &= \sum_{k=0}^{\infty} \sum_{y \in A_k} p_n(x, y) \mu_y \leq c \sum_{k=0}^{\infty} 2^{\alpha k} R^{\alpha} e^{-4^k R^2/2n} \\ &= cR^{\alpha} \sum_{k=0}^{\infty} 2^{\alpha k} e^{-4^k \lambda^2/2} \leq cR^{\alpha} e^{-\lambda^2/2}. \end{aligned} \quad (1.2)$$

For  $x \in G$ ,

$$\begin{aligned} \mathbb{P}^x(\tau_{x,2R} \leq n) &= \\ &= \mathbb{P}^x(\tau_{x,2R} \leq n, X_n \notin B(x, R)) + \mathbb{P}^x(\tau_{x,2R} \leq n, X_n \in B(x, R)). \end{aligned} \quad (1.3)$$

The second term in (1.3) equals, writing  $\tau = \tau_{x,2R}$ ,

$$\begin{aligned} \mathbb{E}^x \mathbf{1}_{(\tau \leq n)} \mathbb{P}^{X_{\tau}}(X_{n-\tau} \in B(x, R)) &\leq \mathbb{E}^x \mathbf{1}_{(\tau \leq n)} \mathbb{P}^{X_{\tau}}(d(X_0, X_{n-\tau}) \geq R) \\ &\leq \sup_{y \in G} \sup_{m \leq n} \mathbb{P}^y(d(y, X_m) \geq R) \\ &\leq cR^{\alpha} e^{-\lambda^2/2}, \end{aligned}$$

by (1.2). Since the first term in (1.3) also satisfies this bound, we deduce

$$\mathbb{P}^x(\tau_{x,2R} \leq R^2/\lambda) \leq c'R^{\alpha} e^{-\lambda^2/2} = c'e^{-\lambda^2/2 + \alpha \log R}.$$

So if we set  $\lambda = \theta(\log R)^{1/2}$  where  $\theta$  is a constant with  $\theta^2/2 > \alpha$  then

$$\mathbb{P}^x(\tau_{x,2R} \leq \theta^{-1} R^2 (\log R)^{-1/2}) \leq c'e^{-(\theta^2/2 - \alpha) \log R},$$

which implies (1.1). ■

The proof that  $\beta \leq 1 + \alpha$  uses the connection between random walks and electrical networks – see [DS]. If  $(\Gamma', a)$  is any finite weighted graph then (see [Tet])

$$\mathbb{E}^x T_y + \mathbb{E}^y T_x = \mu(G') R_e(x, y). \quad (1.4)$$

Here  $R_e(x, y)$  is the effective resistance between  $x$  and  $y$  in the network where the edge  $\{x', y'\}$  has conductivity  $a_{x'y'}$ . If we collapse the vertices in  $B(x, R)^c$  to a single vertex  $y$ , and discard the second term in (1.4) then we obtain an inequality proved in [T1]:

$$\mathbb{E}^x \tau_{x,R} \leq V(x, R) R_e(x, \partial B(x, R)). \quad (1.5)$$

**Lemma 1.2** *If  $\Gamma$  satisfies  $(V_{\alpha})$  and  $(E_{\beta})$  then  $\beta \leq 1 + \alpha$ .*

**Proof.** We can take  $R \in \mathbb{N}$ . As  $x$  and  $\partial B(x, R)$  are connected by a chain of exactly  $R$  wires, each of conductance at least  $C_1$ , we have  $R_e(x, \partial B(x, R)) \leq C_1^{-1} R$ . So using  $(V_{\alpha})$  and  $(E_{\beta})$ , (1.5) implies that  $cR^{\beta} \leq cR^{\alpha+1}$ , giving  $\beta \leq 1 + \alpha$ . ■

**Proof of Theorem 1.** Since  $G$  is infinite and  $\Gamma$  is connected, there exists a path to infinity from any point  $x$ . Hence  $V(x, R) \geq C_1 R$ , which implies that  $\alpha \geq 1$ . The remaining assertions are immediate from Lemmas 1.1 and 1.2.  $\blacksquare$

From [GT1], [GT2] we have the following estimates.

**Lemma 1.3** *Let  $\Gamma$  satisfy (VD),  $(E_\beta)$  and (EHI), and write  $\gamma = 1/(\beta - 1)$ . Suppose in addition that there exists a constant  $c$  such that  $\mu_x \leq c$  for all  $x \in G$ .*

(a) *There exist constants  $c_1 - c_4$  such that for  $n \geq 1$ ,  $x, y \in G$ ,*

$$\begin{aligned} p_n(x, y) &\leq c_1 V(x, n^{1/\beta})^{-1} \exp(-c_2(d(x, y)^\beta/n)^\gamma), \\ p_n(x, y) + p_{n+1}(x, y) &\geq c_3 V(x, n^{1/\beta})^{-1} \exp(-c_4(d(x, y)^\beta/n)^\gamma). \end{aligned}$$

(b) *For any  $x \in G$ ,  $n \geq 1$ ,  $R \geq 1$ ,*

$$\mathbb{P}^x(\tau_{x,R} \leq n) \leq c_5 \exp(-c_6(R^\beta/n)^\gamma).$$

(c) *There exists  $c_7$  such that*

$$\mathbb{P}^y(\tau_{x,R} > c_7 R^\beta) \leq \frac{1}{2}, \quad R \geq 1, y \in B(x, R).$$

Let  $x_0 \in G$ ,  $R \geq 1$ , and let  $B = B(x_0, 2R)$ ,  $B' = B(x_0, R)$ . Write  $\tau = \tau_{x_0, 2R}$ . Set

$$\bar{p}_n(x, y) = \mathbb{P}^x(X_n = y, \tau > n) \mu_y^{-1}$$

for the density of the process  $X$  killed on exiting  $B$ , and let

$$\hat{p}_n(x, y) = p_n(x, y) - \bar{p}_n(x, y) = \mathbb{P}^x(X_n = y, \tau \leq n) \mu_y^{-1}.$$

**Lemma 1.4** *Let  $\Gamma$  satisfy  $(V_\alpha)$ ,  $(E_\beta)$  and (EHI). For  $x, y \in B'$ ,*

$$\bar{p}_n(x, y) + \bar{p}_{n+1}(x, y) \geq c_8 n^{-\alpha/\beta} \exp(-c_9(d(x, y)^\beta/n)^\gamma), \quad n \leq R^\beta. \quad (1.6)$$

**Proof.** We begin by using the ‘there and back’ argument of [BB2] to bound  $\hat{p}$ . We have, for  $x, y \in B'$ ,

$$\begin{aligned} \mathbb{P}^x(X_n = y, \tau \leq n) &= \mathbb{P}^x(X_n = y, \tau \leq n/2) + \mathbb{P}^x(X_n = y, n/2 < \tau < n) \\ &\leq \mathbb{P}^x(X_n = y, \tau \leq n/2) \\ &\quad + \mathbb{P}^x(X_n = y, X_m \in B^c \text{ for some } n/2 < m < n). \end{aligned} \quad (1.7)$$

By time reversibility the second term equals

$$\begin{aligned} \mu_y \mu_x^{-1} \mathbb{P}^y(X_n = x, X_m \in B^c \text{ for some } 0 < m < n/2) &\leq \\ &\leq c \mathbb{P}^y(X_n = x, \tau \leq n/2). \end{aligned} \quad (1.8)$$



To bound the first term in (1.7) we have

$$\begin{aligned} \mathbb{P}^x(X_n = y, \tau \leq n/2) &\leq \mathbb{E}^x 1_{(\tau \leq n/2)} \mathbb{P}^{X_\tau}(X_{n-\tau} = y) \\ &\leq \mathbb{P}^x(\tau \leq n/2) \sup_{z \in G} \sup_{m \leq n/2} \mathbb{P}^z(X_{n-m} = y) \\ &\leq cn^{-\alpha/\beta} \exp(-c'(R^\beta/n)^\gamma); \end{aligned}$$

here we used Lemma 1.3(a) and (b) in the last line. The same bound controls the right hand side of (1.8). Therefore, for  $x, y \in B'$ ,

$$\bar{p}_n(x, y) + \bar{p}_{n+1}(x, y) \geq cn^{-\alpha/\beta} \left( \exp(-c'(d(x, y)^\beta/n)^\gamma) - c'' \exp(-c'''(R^\beta/n)^\gamma) \right).$$

It follows that there exist constants  $c_{10} - c_{12}$ , depending only on the constants above, such that

$$\begin{aligned} \bar{p}_n(x, y) + \bar{p}_{n+1}(x, y) &\geq c_{10} n^{-\alpha/\beta} \exp(-c_{11}(d(x, y)^\beta/n)^\gamma), \\ n &\leq c_{12} R^\beta, \quad d(x, y) \leq c_{12} R. \end{aligned}$$

This gives the lower bound (1.6) when  $x, y$  are sufficiently close together; to extend this to the case  $x, y \in B'$  we can use a standard chaining argument—see for example [B1, section 3].  $\blacksquare$

We write  $\bar{g}_B(x, y) = \sum_n \bar{p}_n(x, y)$  for the Green's function for  $X$  killed on exiting  $B = B(x_0, 2R)$ .

**Proposition 1.5** *Let  $\Gamma$  satisfy  $(V_\alpha)$ ,  $(E_\beta)$  (EHI).*

(a) *If  $\beta > \alpha$  then*

$$c_{13} R^{\beta-\alpha} \leq \bar{g}_B(x, y) \leq c_{14} R^{\beta-\alpha}, \quad x, y \in B'.$$

(b) *If  $\beta = \alpha$  then*

$$c_{13} \log R \leq \bar{g}_B(x, x) \leq c_{14} \log R, \quad x, y \in B'.$$

**Proof.** The lower bounds in (a) and (b) are immediate on summing the bounds in Lemma 1.4.

For the upper bounds, it is enough to take  $y = x$ , since  $\bar{g}_B(x, y) = \bar{g}_B(y, x) \leq \bar{g}_B(x, x)$ . Let  $m = c_7(2R)^\beta$ . Then by Lemma 1.3(c),

$$\begin{aligned} \mu_x \bar{g}_B(x, x) &= \mathbb{E}^x \sum_{n=0}^{\infty} 1_{(X_n=y, \tau > n)} \\ &= \mathbb{E}^x \sum_{n=0}^m 1_{(X_n=y, \tau > n)} + \mathbb{E}^x 1_{(\tau > m)} \mathbb{E}^{X_m} \sum_{n=0}^{\infty} 1_{(X_n=y, \tau > n)} \\ &\leq \mathbb{E}^x \sum_{n=0}^m 1_{(X_n=y)} + \frac{1}{2} \mu_x \bar{g}_B(x, x). \end{aligned}$$

So

$$\bar{g}_B(x, x) \leq c \sum_{n=0}^m p_n(x, x) \leq c' \sum_{n=0}^{c'R^\beta} n^{-\alpha/\beta},$$

which gives the upper bounds in both (a) and (b).  $\blacksquare$

**Proof of Proposition 3.** (a) The case when  $\beta > \alpha$  is easy; we have (let  $A = B(x, R)$ )

$$cR^\beta \leq \mathbb{E}^x \tau_{x,R} = \sum_{y \in B(x,R)} g_A(x, y) \mu_y \leq c'R^\alpha g_A(x, x).$$

So  $g_A(x, x) \geq cR^{\beta-\alpha}$ , and thus  $g(x, x) = \infty$ .

If  $\alpha = \beta$  then we consider the resistance from  $x_0 \in G$  to infinity. Let  $S_n = \{y : d(x_0, y) = 2^n\}$ , and write  $R_e(S_n, S_{n+1})$  for the effective resistance between  $S_n$  and  $S_{n+1}$ . Consider the finite graph  $\Gamma'$  where all vertices in  $B(x_0, 1+2^n)$  are collapsed to a single vertex  $a$  and all vertices in  $B(x_0, 2^{n+1} - 1)^c$  are collapsed to a vertex  $b$ . Then, using  $R'_e$  to denote effective resistance in  $\Gamma'$ ,  $R_e(S_n, S_{n+1}) = R'_e(a, b)$ . By (1.5)

$$E^a T_b \leq \mu(G') R'_e(a, b).$$

We have  $\mu(G') \leq c(2^n)^\alpha$ , while  $E^a T_b \geq c'(2^{n-1})^\beta$ . So  $R_e(S_n, S_{n+1}) \geq c'' > 0$ , with  $c''$  independent of  $n$ . Since  $R_e(x_0, S_n) \geq \sum_{k=0}^{n-1} R_e(S_k, S_{k+1}) \geq c''n$ , we deduce from [DS] that  $\Gamma$  is recurrent.

(b) Let  $R \geq 1$  and  $x, y \in G$  with  $d(x, y) < R$ . As above write  $B = B(x, 2R)$ ,  $B' = B(x, R)$ . For  $x, y \in B'$ , by Proposition 1.5,

$$\mathbb{P}^y(T_x < \tau_{x,2R}) = \frac{\bar{g}_B(y, x)}{\bar{g}_B(x, x)} \geq c,$$

which proves that  $\Gamma$  satisfies (VSR).

(c)  $\Gamma$  is recurrent by (a). Suppose  $\Gamma$  does satisfy (VSR) and  $(V_\alpha)$ . Let  $B = B(x, 2R)$ ,  $B' = B(x, R)$ . If  $y \in B'$  then by (VSR)  $\mathbb{P}^x(T_y < \tau_{x,2R}) = g_B(x, y)/g_B(y, y) \geq p_1$ . So

$$g_B(x, y) \geq p_1 g_B(y, y) \geq c \log R.$$

Hence

$$\mathbb{E}^x \tau_{x,2R} \geq \sum_{y \in B'} \mu_y g_B(x, y) \geq cV(x, R) \log R \geq cR^\alpha \log R.$$

Thus  $\Gamma$  does not satisfy  $(E_\alpha)$ .

(d) From Lemma 1.3(a) we have that  $g(x, x) = \sum_{n=0}^{\infty} p_n(x, x) < \infty$ .  $\blacksquare$

We conclude this section with the following easy Lemma.

**Lemma 1.6** *Suppose (VSR) holds for  $\Gamma$ . Then  $\Gamma$  satisfies (EHI).*

**Proof.** Let  $h \geq 0$  be harmonic on  $B(x, 2R)$ , and let  $x_0$  and  $x_1$  be the points in  $B = B(x, R)$  where  $h$  attains its minimum and maximum respectively. Then using (VSR) repeatedly we obtain  $\mathbb{P}^{x_0}(T_{x_1} < \tau_{x, 2R}) \geq c_{15}$ . So

$$h(x_0) \geq \mathbb{E}^{x_0} 1_{(T_{x_1} < \tau_{x, 2R})} h(x_1) \geq c_{15} h(x_1). \quad \blacksquare$$

## 2. Construction of Product Graph

Let  $\Gamma = (G, E)$  be an infinite locally finite connected graph. Further hypotheses will be added later in this section, but this is all that is needed at this point. We write  $\#(A)$  for the number of elements in the set  $A$ .

**Proposition 2.1** *Let  $b \geq 3$ . There exists a partition  $G = \cup_{n=0}^{\infty} A_n$  of  $G$  such that*

- (a)  $\{B(x, b^n)\}$ ,  $x \in \cup_{k=n}^{\infty} A_k$  are disjoint.
- (b) For each  $n$ ,  $G \subset \cup_{k=n}^{\infty} \cup_{x \in A_k} B(x, 3b^n)$ .
- (c) For each  $n$ ,  $G = \cup_{x \in A_n} B(x, 9b^n)$ .

**Proof.** Let  $D_0 = G$ . We construct a decreasing chain of sets  $D_k$ ,  $k \geq 0$ , by choosing, for each  $k$ ,  $D_{k+1}$  to be a maximal subset of  $D_k$  such that  $B(x, b^{k+1})$ ,  $x \in D_{k+1}$  are disjoint. We also need to ensure that  $\cap_{k=0}^{\infty} D_k = \emptyset$ . To do this, write  $G = \{z_1, z_2, \dots\}$ . If then  $D_n = \{z_{i_1}, z_{i_2}, \dots\}$ , with  $i_1 < i_2 < \dots$ , let  $z'$  be a closest point in  $D_n - \{z_{i_1}\}$  to  $z_{i_1}$ , and include  $z'$  in  $D_{n+1}$ . We will see below this implies that  $z_{i_1} \notin D_{n+1}$ . Let  $b_0 = 1/3$ , and  $b_n = \sum_{i=1}^n b^i$  for  $n \geq 1$ . Then as  $b \geq 3$ ,

$$2b_n + 1 = \frac{2b^{n+1} - b - 1}{b - 1} < \frac{2b}{b - 1} b^n \leq 3b^n \leq b^{n+1}, \quad n \geq 0.$$

The construction of the sets  $D_k$  gives:

$$B(x, b^k), \quad x \in D_k \text{ are disjoint, } \quad k \geq 0, \quad (2.1)$$

$$D_k \subset \cup_{x \in D_{k+1}} B(x, 2b^{k+1}), \quad k \geq 0. \quad (2.2)$$

(If (2.2) fails then  $D_{k+1}$  would not be maximal.) Note that iterating (2.2) we obtain:

$$\text{for any } x_0 \in G \text{ there exists } x_n \in D_n \text{ with } d(x_0, x_n) < 2b_n < 3b^n, \quad n \geq 0. \quad (2.3)$$

Let  $x \in D_n$ ; we bound the distance from  $x$  to its closest neighbour  $x'$  in  $D_n$ . Let  $m$  be the integer with  $2b_n \leq m < 2b_n + 1$ , and choose  $y \in G$  with  $d(x, y) = m$ . By (2.3) there exists  $x'' \in D_n$  with  $d(y, x'') < 2b_n$  – note this implies that  $x \neq x''$ . Thus for any  $x \in D_n$  there exists  $x' \in D_n$ , with  $d(x, x') < m + 2b_n$ . Choosing  $y'$  on a shortest path between  $x$  and  $x'$  with  $d(x, y') = m$  we have  $y' \in B(x, b^{n+1}) \cap B(x', b^{n+1})$ , so that  $x$  and  $x'$  cannot both be in  $D_{n+1}$ .

This calculation shows that  $z_{i_1} \notin D_{n+1}$ , and thus that  $\cap D_k = \emptyset$ . Set  $A_k = D_k - D_{k+1}$ : we have  $\cup_n A_n = G$ . (a) is now immediate from (2.1), and (b) from (2.3).

To prove (c), let  $x_0 \in G$ . By (2.3) there exists  $x_n \in D_n$  with  $d(x_0, x_n) < 2b_n$ , and if  $x'_n$  is a closest neighbour to  $x_n$  in  $D_n$  then  $d(x_n, x'_n) < 4b_n + 1$ , so that  $d(x_0, x'_n) < 6b_n + 1 < 9b^n$ . Since at least one of  $x_n$  and  $x'_n$  is in  $A_n$ , this proves (c). ■

**Lemma 2.2** *Let  $\Gamma$ ,  $(A_n)$  be as above, and let  $x \in G$ ,  $n \geq 1$ . Suppose that  $b \geq 9$ .*

- (a)  $B(x, b^n)$  contains at least one point in  $A_k$  for each  $k \leq n - 1$ .
- (b)  $B(x, 3b^n)$  contains at least one point in  $\cup_{k=n}^{\infty} A_k$ .
- (c)  $B(x, b^n)$  contains at most one point in  $\cup_{k=n}^{\infty} A_k$ .

**Proof.** (b) and (c) are immediate from Proposition 2.1(b) and (a). Let  $x \in G$  and  $k \leq n - 1$ . By Proposition 2.1(c) there exists  $y \in A_k$  such that  $d(x, y) < 9b^k \leq b^n$ . ■

Let  $M \geq 2$  be an integer, and  $U$  be a discrete ultrametric space with ‘family size’  $M$ . We set

$$U = \{u = (u_1, u_2, \dots), u_i \in \{0, 1, \dots, M - 1\}, \sum u_i < \infty\}.$$

We can regard the sequence  $(u_i)$  as the address of  $u \in U$ :  $u_1$  denotes the district,  $u_2$  the town,  $u_3$  the county etc. —all points  $u \in U$  have an address with components which are 0 from some point on. Set  $\delta(u, u) = 0$ , and for  $u \neq v$ ,

$$\delta(u, v) = \max\{i : u_i \neq v_i\};$$

$\delta$  is a metric on  $U$ , and the ultrametric property  $\delta(u, v) \leq \delta(u, w) \vee \delta(w, v)$  for  $u, v, w \in U$  is easy to verify. Write

$$C_n(u) = \{v \in U : \delta(u, v) \leq n\}$$

for the closed ball radius  $n$  in  $U$ , and note that  $\#(C_n(u)) = M^n$ .

Now let  $\Gamma = (G, E)$  be as above, and choose  $b \geq 9$ . Let  $(A_n)$  be a partition of  $G$  satisfying Proposition 2.1: we call the points in  $A_n$  *links of order  $n$* . (We could manage with smaller  $b$ , but there is no advantage in doing so, while a large  $b$  will allow us to use Lemma 2.2 to reduce the number of possible types of ‘high level’ link points in a ball.) We now construct a new graph  $\tilde{\Gamma}$ . Set  $\tilde{G} = G \times U$ . We define the edge set  $\tilde{E}$  for the graph  $\tilde{\Gamma} = (\tilde{G}, \tilde{E})$  as follows:

$$\begin{aligned} \{(x, u), (y, u)\} &\in \tilde{E} \text{ if } \{x, y\} \in E, \\ \{(x, u), (x, v)\} &\in \tilde{E} \text{ if } u_i = v_i, i \neq n, u_n \neq v_n, x \in A_n, n \geq 1. \end{aligned}$$

These are the only edges of  $\tilde{\Gamma}$ . Thus we construct links from  $(x, u)$  to points  $(x, v)$  at  $M - 1$  other levels of  $\tilde{G}$  if  $x \in G - A_0$ . If  $x$  has degree  $r$ , then  $(x, u) \in \tilde{G}$  has degree  $M - 1 + r$  if  $x \notin A_0$ , and degree  $r$  if  $x \in A_0$ . The graph  $\tilde{\Gamma}$  thus consists of a countable number of copies of  $\Gamma$ , glued together at the link points.

If  $\Gamma = (G, E, a)$  is a weighted graph then we define the weights  $\tilde{a}_{(x,u),(y,v)}$  by

$$\tilde{a}_{(x,u),(y,v)} = \begin{cases} a_{xy}, & \text{if } u = v, \\ 1, & \text{if } x = y \text{ and } \{(x, u), (y, v)\} \text{ is an edge in } \tilde{E}, \\ 0, & \text{otherwise.} \end{cases}$$

**Notation.** We denote points in  $\tilde{G}$  by  $\tilde{x} = (x, u)$ . We write  $\tilde{\mu}$  for the measure associated with  $\tilde{a}$ ,  $\tilde{B}(x, u, r)$  for the ball in  $\tilde{\Gamma}$  centre  $(x, u)$  and radius  $r$ , and  $\tilde{V}(x, u, r) = \tilde{\mu}(\tilde{B}(x, u, r))$ . For  $Q \subset G$  and  $\tilde{x} = (x, u) \in Q \times U$  let  $W_Q(\tilde{x})$  be the unique subset of  $U$  such that the connected component of  $Q \times U$  containing  $\tilde{x}$  is  $Q \times W_Q(\tilde{x})$ . Write

$$W_n(x, u) = W_{B(x, b^n)}(x, u).$$

Thus  $W_n(x, u)$  is the set of  $v \in U$  such that  $(x, v)$  is connected to  $(x, u)$  by a path  $(x_i, u^{(i)})$  with  $x_i \in B(x, b^n)$  for all  $i$ .

By Lemma 2.2 we obtain

**Lemma 2.3** *Exactly one of the following holds:*

$$W_n(x, u) = C_{n-1}(u), \tag{2.4}$$

$$W_n(x, u) = C_n(u), \tag{2.5}$$

$$W_n(x, u) = C_{n-1}(u) \cup C_{n-1}(u'), \text{ for some } u' \text{ with } \delta(u, u') > n. \tag{2.6}$$

**Lemma 2.4** For  $n \geq 1$ ,  $(x, u) \in \tilde{G}$ ,

$$\tilde{B}(x, u, b^n) \subset B(x, b^n) \times W_n(x, u) \subset \tilde{B}(x, u, 6b^n).$$

**Proof.** The first inclusion is clear from the definition of  $W_n(x, u)$ . To prove the second, let  $(y, v) \in B(x, b^n) \times W_n(x, u)$ . We begin by constructing a path from  $x$  to  $y$  which contains at least one point in  $A_k \cap B(x, b^n)$  for each  $k \geq 1$  for which this set is non-empty.

If  $1 \leq k \leq n-1$  then by Proposition 2.1(c) there exists  $x_k \in A_k$  with  $d(x, x_k) < 9b^k \leq b^n$ . So if  $\pi$  is the path which successively visits  $x, x_1, x_2, \dots, x_{n-1}, x$  the length of  $\pi$  is at most

$$18 \sum_{k=1}^{n-1} b^k \leq (9/4)b^n.$$

If (2.4) holds the path contains all the necessary link points; if not then there is an extra high level link point  $x' \in B(x, b^n) \cap A_m$  for some  $m \geq n$ , and another section of length at most  $2b^n$  is needed. Finally, we add a section from  $x$  to  $y$  of length again at most  $b^n$ .

To construct a path from  $(x, u)$  to  $(y, v)$  we simply add (if necessary) an extra ‘vertical’ edge of the form  $\{(x_k, u^{(k)}), (x_k, v^{(k)})\}$  at each of the link points, to switch the  $k$ th component of  $u$  to that of  $v$ . The overall length of the path is therefore at most  $(21/4)b^n + n \leq 6b^n$ . ■

**Proposition 2.5** (a) Let  $\alpha \geq 1$  and  $\lambda = \log M / \log b$ . Then  $\Gamma$  satisfies  $(V_\alpha)$  if and only if  $\tilde{\Gamma}$  satisfies  $(V_{\alpha+\lambda})$ .

(b)  $\tilde{\Gamma}$  satisfies  $(VD)$  if and only if  $\Gamma$  satisfies  $(VD)$ .

**Proof.** Since  $\Gamma$  is connected,  $\mu_x \geq C_1$  for all  $x \in G$ . Thus  $V(x, R) \geq C_1 \#B(x, R)$ . For  $(x, u) \in \tilde{G}$  we have  $\mu_x \leq \tilde{\mu}_{(x,u)} \leq \mu_x + (M-1)$ , and so

$$\begin{aligned} M^n V(x, b^n) &\leq \tilde{\mu}(B(x, b^n) \times W_n(x, u)) \\ &\leq M^n V(x, b^n) + (M-1)M^n \#(B(x, b^n)) \\ &\leq cM^n V(x, b^n). \end{aligned}$$

Using Lemma 2.4 we therefore have, since  $M = b^\lambda$ ,

$$\tilde{V}(x, u, b^n) \leq c(b^n)^\lambda V(x, b^n), \quad \tilde{V}(x, u, 6b^n) \geq c'(b^n)^\lambda V(x, b^n),$$

and (a) and (b) now follow easily. ■

### 3. Random walk on $\tilde{\Gamma}$

It will be technically easier to work with continuous time random walks in this section. Let  $(\Gamma, a) = (G, E, a)$  be a weighted graph. The continuous time random walk (CTRW) on  $(\Gamma, a)$  is the Markov process  $X = (X_t, t \in [0, \infty), \mathbb{P}^x, x \in G)$  with generator

$$\mathcal{L}f(x) = \sum_{y \in G} a_{xy}(f(y) - f(x)).$$

The process  $X$  waits at a vertex  $x$  for an exponential time with mean  $\mu_x^{-1}$ , and then jumps to one of the neighbours of  $x$ , moving to  $y$  with probability  $p_{xy} = a_{xy}/\mu_x$ . So, if we write  $S_i$  for the jump times, the process  $Z_n = X_{S_n}$  is exactly the discrete time random walk on  $\Gamma$  defined in the introduction.

Assume that

$$C_2 \leq \mu_x \leq C_3, \quad x \in G; \quad (3.1)$$

of course this condition follows from (0.1) and  $(V_\alpha)$ . Then we have  $\mathbb{E}(S_{n+1} - S_n | X_s, s \leq S_n) \in [C_3^{-1}, C_2^{-1}]$ , and it follows that  $(E_\beta)$  holds for the process  $X$  if and only if it holds for  $Z$ .

Now fix  $(\Gamma, a) = (G, E, a)$  satisfying (3.1), and let  $(\tilde{\Gamma}, \tilde{a})$  be the weighted graph constructed in the previous section. Let  $X$  be the CTRW on  $(\Gamma, a)$ . Then the CTRW  $\tilde{X}$  on  $\tilde{\Gamma}$  can be constructed from  $X$  and a process  $Y$  which is defined as follows.

Let  $\eta_k, k \geq 1$  be independent Poisson point processes with rate  $M$  on  $\mathbb{R}_+$ , and  $\theta_{k,n}, k \geq 1, n \geq 1$  be i.i.d.r.v. uniform on  $\{0, 1, \dots, M-1\}$ . Let  $N_t^k, t \geq 0$  be the counting processes associated with  $\eta_k$ . Write  $\sigma_{k,n}, n \geq 1$  for the points in  $\eta_k$ :  $\sigma_{k,n+1} - \sigma_{k,n}$  is an exponential r.v. with mean  $1/M$ . Given  $(X_t, t \geq 0)$  we construct  $Y_t = (Y_t^1, Y_t^2, \dots) \in U$ , with  $Y_0 = u = (u_1, \dots)$ , by taking  $Y_t^k$  to be constant on each interval  $[\sigma_{k,n}, \sigma_{k,n+1})$ , and setting

$$Y_{\sigma_{k,n+1}}^k = \begin{cases} Y_{\sigma_{k,n}}^k & \text{if } X_{\sigma_{k,n+1}} \notin A_k, \\ \theta_{k,n+1} & \text{if } X_{\sigma_{k,n+1}} \in A_k. \end{cases}$$

Note that the times of the point process  $\eta_k$  are not the jump times of  $Y^k$ :  $Y^k$  only makes a jump at one of these times if, first  $X$  is in  $A_k$ , and second, the r.v.  $\theta_{k,\cdot}$  gives a new state. The process  $\tilde{X}_t = (X_t, Y_t)$  is then a CTRW on  $\tilde{\Gamma}$ .

Define the filtration

$$\mathcal{G}_t = \sigma(X_s, N_s^k, s \leq t, k \geq 1), \quad t \geq 0.$$

Set also

$$\Lambda_k = \{\sigma_{k,n} : X_{\sigma_{k,n}} \in A_k, n \geq 1\} \subset [0, \infty).$$

Note that  $\Lambda_k \cap [0, t]$  is a  $\mathcal{G}_t$ -measurable random set. Let

$$\begin{aligned}\tau_{x,r} &= \inf\{t \geq 0 : X_t \notin B(x, r)\}, \\ \tilde{\tau}_{x,u,r} &= \inf\{t \geq 0 : \tilde{X}_t \notin \tilde{B}(x, u, r)\}, \\ \tau_n(x, u) &= \inf\{t \geq 0 : \tilde{X}_t \notin B(x, r) \times W_n(x, u)\}.\end{aligned}$$

**Lemma 3.1**  $(\tilde{\Gamma}, \tilde{a})$  satisfies  $(E_\beta)$  if and only if  $(\Gamma, a)$  satisfies  $(E_\beta)$ .

**Proof.** Note that  $\tilde{X}$  can only exit from  $B(x, b^n) \times W_n(x, u)$  when  $X$  exits from  $B(x, b^n)$ , so that  $\tau_n(x, u) = \tau_{x, b^n}$ . Hence using Lemma 2.4,

$$\mathbb{E}^{(x,u)} \tilde{\tau}_{x,u,6b^n} \geq \mathbb{E}^{(x,u)} \tau_n(x, u) = \mathbb{E}^x \tau_{x, b^n} \geq \mathbb{E}^{(x,u)} \tilde{\tau}_{x,u, b^n},$$

and  $(E_\beta)$  for  $\tilde{\Gamma}$  follows immediately.  $\blacksquare$

We now turn to the proof of the elliptic Harnack inequality on  $(\tilde{\Gamma}, \tilde{a})$ . For the remainder of this section we will assume that (VSR) holds for  $(\Gamma, a)$ .

**Lemma 3.2** *There exists  $c_1 > 0$  such that for  $k \geq 1$ ,  $x \in G$ ,*

$$\mathbb{P}^x(\Lambda_k \cap [0, \tau_{x, 18b^k}] \neq \emptyset) \geq c_1.$$

**Proof.** By Proposition 2.1(c) there exists  $x' \in A_k \cap B(\xi_r, 9b^k)$ . So, using (VSR), and writing  $\tau = \tau_{x, 18b^k}$ ,  $\mathbb{P}^x(T_{x'} < \tau) \geq p_0 > 0$ . If  $X_t = x'$  then the time to the next jump of  $X$  is exponential with rate  $\mu_x$ , while points in  $\eta_k$  occur at rate  $M$ . So the probability that a point in  $\eta_k$  will occur before  $X$  leaves  $x'$  is  $M/(M + \mu_x) \geq M/(M + C_3)$ . Combining these estimates completes the proof, with  $c_1 = p_0 M/(M + C_3)$ .  $\blacksquare$

Fix  $(x_0, u^0) \in \tilde{G}$ , let  $R \geq 1$ , let  $n$  be such that  $b^{n-1} \leq R < b^n$ , write

$$B = B(x_0, R), \quad B' = B(x_0, \frac{3}{2}R), \quad B^* = B(x_0, 2R),$$

and let  $\tau', \tau^*$  be the first exit times of  $X$  from  $B'$  and  $B^*$ . We begin by considering harmonic functions  $h$  on  $B^* \times W_{B^*}(u^0)$  of the form

$$h(x, u) = h^{z_0, v}(x, u) = \mathbb{P}^{(x,u)}(X_{\tau^*} = z_0, Y_{\tau^*} = v), \quad (3.2)$$

where  $z_0 \in \partial B^*$  and  $v \in W_{B^*}(u^0)$ . Let

$$H_k = \{\text{there exists } n \geq 0 \text{ with } \sigma_{k,n} < \tau' \text{ and } X_{\sigma_{k,n}} \in A_k\} = \{\Lambda_k \cap [0, \tau'] \neq \emptyset\}.$$

**Lemma 3.3** *For  $x \in B$*

$$\mathbb{P}^{(x,u)}(H_k^c) \leq c_2 e^{-c_3 b^{n-k}}, \quad 1 \leq k \leq n.$$



**Proof.** Define a sequence of stopping times, and points in  $\tilde{G}$  as follows.

$$T_0 = 0, \quad \xi_0 = x, \quad T_{r+1} = \tau_{\xi_r, 18b^k}, \quad \xi_{r+1} = X_{T_{r+1}}, \quad r \geq 0.$$

Let  $m \in \mathbb{Z}_+$  satisfy  $m \in [R/(36b^k) - 1, R/(36b^k)]$ ; then  $d(x, \xi_m) \leq 18b^k m < \frac{1}{2}R$ , so that  $T_m < \tau'$  for  $1 \leq r \leq m$ .

By Lemma 3.2 we obtain

$$\mathbb{P}^x(\Lambda_k \cap [T_r, T_{r+1}) \neq \emptyset | \mathcal{G}_{T_r}) \geq c_3. \quad (3.3)$$

Therefore

$$\mathbb{P}^x(H_k^c) \leq (1 - c_3)^m \leq \exp(-cm) \leq c' \exp(-c''b^{n-k}). \quad \blacksquare$$

Set

$$\bar{h}(x) = \mathbb{P}^x(X_{\tau^*} = z_0).$$

As  $\bar{h}$  is harmonic on  $\Gamma$ , it satisfies an elliptic Harnack inequality by Lemma 1.6. Write  $\bar{h}_{\max}$  for the maximum value of  $\bar{h}$  on  $B'$ . The next result bounds  $h$  above by  $\bar{h}_{\max}$ . Naturally we expect  $h$  to be maximised if  $v = u^0$ , but we do not need this.

**Proposition 3.4** *Let  $h$ ,  $B$ ,  $B'$ ,  $B^*$  be as above. Then for  $(x, u) \in B \times W_B(u^0)$ ,*

$$h(x, u) \leq c_4 M^{-n} \bar{h}_{\max}.$$

**Proof.** Define an integer valued random variable  $J$  by taking  $\{J = 0\} = H_1^c$ ,  $\{J = k\} = H_1 \cap \dots \cap H_k \cap H_{k+1}^c$  for  $1 \leq k \leq n-1$ , and  $\{J = n\} = H_1 \cap \dots \cap H_n$ . Note that  $J$  is  $\mathcal{G}_{\tau'}$  measurable.

Let  $k \geq 1$ , and define

$$\kappa = \begin{cases} \max\{n : \sigma_{k,n} \in \Lambda_k \cap [0, \tau^*)\} & \text{on } H_k, \\ 0 & \text{on } H_k^c. \end{cases}$$

Then  $Y_{\tau^*}^k = \theta_{k,\kappa}$  on  $H_k$ , so that (on  $H_k$ )  $Y_{\tau^*}^k$  is equal to a random variable which is independent of  $\mathcal{G}_\infty$ , and is uniformly distributed on  $\{0, \dots, M-1\}$ . So, if  $J = k$ , then the first  $k$  components of  $Y_{\tau^*}$  will have been randomised and so we have, for any  $v \in U$ ,

$$\mathbb{P}^{(x,u)}(X_{\tau^*} = z_0, Y_{\tau^*} = v | J = k, X_{\tau'} = y) \leq M^{-k} \bar{h}(y) \leq M^{-k} \bar{h}_{\max}. \quad (3.4)$$

Now choose  $j_0$  to be the smallest strictly positive integer so that  $Me^{-c_3bj_0} \leq 1/2$ . Then

$$\begin{aligned}
\sum_{k=0}^{n-j_0} M^{-k} \mathbb{P}^{(x,u)}(J=k) &\leq \sum_{k=0}^{n-j_0} M^{-k} \mathbb{P}^{(x,u)}(H_{k+1}^c) \leq \sum_{k=0}^{n-j_0} M^{-k} c_2 \exp(-c_3b^{n-k-1}) \\
&= c_2 M^{-n+j_0} \sum_{i=0}^{n-j_0} M^i \exp(-c_3b^{j_0-1}b^i) \leq c_2 M^{-n+j_0} \sum_{i=0}^{\infty} M^i \exp(-c_3b^{j_0-1}ib) \\
&= c_2 M^{-n+j_0} (1 - Me^{-c_3bj_0})^{-1} \leq 2c_2 M^{-n+j_0}. \tag{3.5}
\end{aligned}$$

So, using (3.4) and (3.5) we have

$$\begin{aligned}
h(x, u) &= \sum_{k=0}^n \sum_{y \in \partial B'} \mathbb{P}^{(x,u)}(X_{\tau^*} = z_0, Y_{\tau^*} = v | J=k, X_{\tau'} = y) \mathbb{P}^{(x,u)}(J=k, X_{\tau'} = y) \\
&\leq \sum_{k=0}^n M^{-k} \bar{h}_{\max} \sum_{y \in \partial B'} \mathbb{P}^{(x,u)}(J=k, X_{\tau'} = y) = \bar{h}_{\max} \sum_{k=0}^n M^{-k} \mathbb{P}^{(x,u)}(J=k) \\
&\leq \bar{h}_{\max} \left( 2c_2 M^{-n+j_0} + \sum_{k=n-j_0+1}^n M^{-k} \right) \leq c_4 \bar{h}_{\max} M^{-n}. \quad \blacksquare
\end{aligned}$$

**Proposition 3.5** *Let  $v \in W_{B'}$ , and  $h = h^{z_0, v}$  where  $z_0 \in \partial B^*$ . Then*

$$h(x, u) \geq c_5 M^{-n} \inf_{y \in B'} \bar{h}(y), \quad (x, u) \in B \times W_B(u^0).$$

**Proof.** Since  $b^{n-1} \leq R \leq b^n$ , by Lemma 2.2 the ball  $B$  contains link points in  $A_1, \dots, A_{n-2}$ . In addition  $B'$  may contain some additional ‘higher level’ link points. We will deal with the worst case, when both  $A_{n-1} \cap B' \neq \emptyset$  and  $D_n \cap B' \neq \emptyset$ : if either of these sets is empty, then the relevant part of the construction below can be omitted. Note that  $D_n \cap B'$  can contain at most one link point, in  $A_m$  say. Let  $n_0 \geq 1$  be the smallest integer such that  $c_2 e^{-c_3 b^{n_0}} / (1 - e^{-c_3 b^{n_0}}) < \frac{1}{2}$ . We begin by assuming that  $n \geq n_0$ .

We can use the symmetry of  $U$  to take  $u^0 = (0, \dots)$ . Then since  $v \in W_{B'}$ , we have  $v_i = 0$  for  $i \geq n, i \neq m$ . We now estimate from below the probability that  $X$  and  $\eta_k$  satisfy the following:

- (1) Each of the events  $H_k$ ,  $1 \leq k \leq n - j_0$  occur before time  $\tau'$ .
- (2)  $X$  then hits a link point in each of  $A_k \cap B$ , for  $n - j_0 + 1 \leq k \leq n - 2$  without leaving  $B^*$ .
- (3)  $X$  then hits the link points in  $A_{n-1} \cap B'$  and  $A_m \cap B'$  before  $\tau^*$ .

We will write  $F_i$ ,  $1 \leq i \leq 3$  for the events described above, and define stopping times  $T_i$  for the time this event is completed. More precisely, we set

$$\begin{aligned} T_1 &= \inf\{t \geq 0 : \Lambda_k \cap [0, t] \neq \emptyset, \text{ for each } k \text{ with } 1 \leq k \leq n - n_0\}, \\ T_2 &= \inf\{t \geq T_1 : \Lambda_k \cap [T_1, t] \neq \emptyset, \text{ for each } k \text{ with } n - n_0 < k \leq n - 2\}, \\ T_3 &= \inf\{t \geq T_2 : \Lambda_k \cap [T_2, t] \neq \emptyset, \text{ for each } k \text{ with } k = n - 1, m\}. \end{aligned}$$

We have, writing  $\mathbb{P} = \mathbb{P}^{(x,u)}$ ,

$$\begin{aligned} \mathbb{P}(F_1^c) &\leq \sum_{k=0}^{n-n_0} P(H_k^c) \leq \sum_{k=0}^{n-n_0} c_2 e^{-c_3 b^{n-k}} \\ &= c_2 \sum_{i=0}^{n-n_0} e^{-c_3 b^{n_0+i}} \leq c_2 \sum_{i=0}^{\infty} e^{-c_3 b^{n_0(i+1)}} \leq \frac{1}{2}, \end{aligned}$$

by the choice of  $n_0$ . So  $\mathbb{P}(F_1) \geq \frac{1}{2}$ . We have  $T_1 \leq \tau'$  on  $F_1$ . So, using (VSR) repeatedly to ‘move’  $X$  around in  $B$  and  $B'$  without leaving  $B^*$ , we deduce that

$$\mathbb{P}(F_2 \cap F_3 | \mathcal{G}_{T_1}) \geq c_6^{n_0} \quad \text{on } F_1.$$

Thus if  $F = \cap_{i=1}^3 F_i$  we have  $\mathbb{P}(F) \geq c_7 > 0$ , and  $X_{T_3} \in B'$  on  $F$ .

Since on the event  $F$  all the components of  $Y$  which can change while  $X$  remains in  $B^*$  have had an opportunity to do so, we have

$$\begin{aligned} h(x, u) &\geq \mathbb{E}^x 1_F \mathbb{E}^x(X_{\tau^*} = z_0, Y_{\tau^*} = v | \mathcal{G}_{T_3}) \\ &= \mathbb{E}^x 1_F \mathbb{E}^x(X_{\tau^*} = z_0 | \mathcal{G}_{T_3}) M^{-n} \\ &= M^{-n} \mathbb{E}^x 1_F \bar{h}(X_{T_3}) \geq c_7 M^{-n} \inf_{y \in B'} \bar{h}(y). \end{aligned}$$

If  $n < n_0$  then we can omit step (1) above, and in step (2) require that  $X$  hits each of  $A_k \cap B$ ,  $1 \leq k \leq n - 2$ , before  $\tau'$ . This event has probability bounded below by  $c_6^n \leq c_6^{n_0-2}$ . A similar argument to that above then completes the proof.  $\blacksquare$

**Lemma 3.6** *Let  $(x_0, u^0) \in \tilde{G}$ ,  $R \geq 1$ , and let  $h \geq 0$  be harmonic on  $Q_1 = B(x_0, 2R) \times W_{B(x_0, 2R)}(u^0)$ . Suppose that*

$$W_{B(x_0, R)}(u^0) = W_{B(x_0, 2R)}(u^0). \quad (3.6)$$

*Then there exists  $c_9$  such that, writing  $Q_0 = B(x_0, R) \times W_{B(x_0, R)}(u^0)$ ,*

$$\sup_{Q_0} h \leq c_9 \inf_{Q_0} h. \quad (3.7)$$

**Proof.** If  $h = h^{z_0, v}$ , with  $v \in W_{B(x_0, R)}(u^0)$  then (3.7) is immediate from the estimates in Propositions 3.4 and 3.5, and the elliptic Harnack inequality for  $(\Gamma, a)$ . Now if  $h \geq 0$  is harmonic in  $B(x_0, 2R) \times W_{B(x_0, 2R)}(u^0)$ , then  $h$  can be written

$$h(x, u) = \sum_{z_0 \in \partial B(x_0, 2R)} \sum_{v \in W_{B(x_0, 2R)}(u^0)} h(z_0, v) h^{z_0, v}(x, u),$$

so that (3.7) follows. ■

**Lemma 3.7** *Let  $(x_0, u^0) \in \tilde{G}$ ,  $n \geq 1$ , and let  $h \geq 0$  be harmonic on  $Q_1 = B(x_0, b^{n+1}) \times W_{n+1}(x_0, u^0)$ . Then writing  $Q_0 = B(x_0, b^n) \times W_n(u^0)$ ,*

$$\sup_{Q_0} h \leq c_9 \inf_{Q_0} h.$$

**Proof.** Set  $B_j = B(x_0, 2^j b^n)$  for  $j = 0, 1, 2, 3$ . By Lemma 2.2  $B_0$  contains link points in  $A_1, \dots, A_{n-1}$ , while  $B_3$  contains at most two additional kinds of link points – in  $A_n$  and, possibly, in  $A_m$  for some  $m \geq n + 1$ . So we must have  $W_{B_j} = W_{B_{j+1}}$  for at least one  $j \in \{0, 1, 2\}$ . So we can apply Lemma 3.6 for this  $B_j$  and  $B_{j+1}$ , and obtain, writing  $Q' = B_j \times W_{B_j}$ ,  $\sup_{Q_0} h \leq \sup_{Q'} h \leq c_9 \inf_{Q'} h \leq c_9 \inf_{Q_0} h$ . ■

We now obtain the elliptic Harnack inequality for  $(\tilde{\Gamma}, \tilde{a})$ .

**Theorem 3.8** *Suppose that  $(\Gamma, a)$  satisfies (VSR), and  $(\tilde{\Gamma}, \tilde{a})$  is constructed by the procedure of Section 2. Then  $(\tilde{\Gamma}, \tilde{a})$  satisfies (EHI).*

**Proof.** Let  $\tilde{x} = (x, u) \in \tilde{\Gamma}$ ,  $R \geq 1$  and  $h \geq 0$  be harmonic in  $\tilde{B}(\tilde{x}, 2R)$ . Choose  $n$  so that  $6b^{n+1} \leq 2R < 6b^{n+2}$ . Then by Lemma 2.4,  $h$  is harmonic in  $B(x, u, b^{n+1}) \times W_{n+1}(x, u)$ . Since, also by Lemma 2.4,  $\tilde{B}(x, u, R/(3b^2)) \subset B(x, u, b^n) \times W_n(x, u)$ , we obtain from Lemma 3.8

$$\sup_{\tilde{B}(x, u, R/(3b^2))} h \leq c_9 \inf_{\tilde{B}(x, u, R/(3b^2))} h.$$

This is the elliptic Harnack inequality, but with a tighter condition on the ratio of the sizes of the two balls. A routine chaining argument now gives the (EHI) in its standard form. ■

**Proof of Proposition 5.** This is immediate from Proposition 2.5, Lemma 3.1, and Theorem 3.8.

### 4. Construction of trees satisfying $(V_\alpha)$ and $(E_{1+\alpha})$

In this section we prove Proposition 4, by constructing a family of graphs which we will call Vicsek trees. These are most easily defined via their embedding in  $\mathbb{R}^N$ . Let  $N \geq 2$ , and let  $\mathcal{C}_N$  be the collection of unit cubes in  $\mathbb{R}^N$  with corners in  $\mathbb{Z}^N$  and edges parallel to the axes. We write  $\lambda\mathcal{C}_N = \{\lambda Q : Q \in \mathcal{C}_N\}$ . We call any connected set  $A \subset \mathbb{R}^N$  which is a union (finite or infinite) of cubes in  $\mathcal{C}_N$  a *cubical set*. As we will be working with cubes, we obtain some slight simplification if we use the  $L_\infty$  metric on  $\mathbb{R}^N$ , in which balls are cubes. For  $x = (x_1, \dots, x_N) \in \mathbb{R}^N$  we set  $|x|_\infty = \max\{x_1, \dots, x_N\}$ . We also write  $|A|$  for the Lebesgue measure of  $A \subset \mathbb{R}^N$ .

**Definition 4.1** Given a cubical set  $A$  we define a graph  $\Gamma = \Gamma(A)$ , which we call the *graph generated by  $A$* , as follows. The vertex set of  $\Gamma$  is the set of corners and centers of the cubes in  $\mathcal{C}_N$ . The edges of  $\Gamma$  connect the center of any cube  $Q \in \mathcal{C}_N$  with  $Q \subset A$  to each corner of that cube.

Note that since  $A$  is connected,  $\Gamma$  is also connected, and that each cube in  $A$  contains  $2^N$  edges in  $\Gamma(A)$ . We also remark that each edge of  $\Gamma$  has length  $\frac{1}{2}$  (in the  $L_\infty$  metric).

Given any suitably regular fractal  $F$  one can construct an infinite ‘pre-fractal’ graph  $\Gamma_F$  such that the large scale structure of  $\Gamma_F$  mimics the small scale structure of  $F$ . We begin by constructing a family of regular fractal subsets of  $[0, 1]^N$ .

Let  $L \geq 1$ ,  $N \geq 2$ ,  $F_0 = [0, 1]^N$ , and  $x_0 = (\frac{1}{2}, \dots, \frac{1}{2})$  be the center of  $F_0$ . Let  $J$  be the union of the  $2^N$  line segments connecting  $x_0$  with the corners of  $F_0$ , and let  $F_1$  be the union of the  $2^N L + 1$  cubes in  $(2L + 1)^{-1}\mathcal{C}_d$  with centers in  $J$ . (See Figure 1).

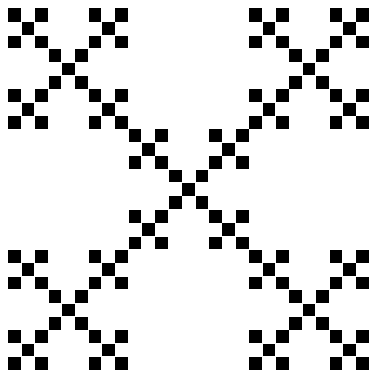


Figure 1: The set  $F_3$  in the case  $N = 2$ ,  $L = 1$ .

Label these cubes  $Q_i$ ,  $1 \leq i \leq 2^N L + 1$ , and for each  $i$  let  $\psi_i^{(N,L)}$  be the orientation preserving linear map which maps  $F_0$  onto  $Q_i$ . For compact sets  $K \subset \mathbb{R}^N$ , set

$$\Psi^{(N,L)}(K) = \bigcup_{i=1}^{2^N L + 1} \psi_i^{(N,L)}(K).$$

Thus we have  $F_1 = \Psi^{(N,L)}(F_0)$ . Now let  $F_{n+1} = \Psi^{(N,L)}(F_n)$ , and note that  $(F_n)$  is a decreasing sequence of compact sets. The intersection  $F$  is a fractal tree which contains  $J$  and has Hausdorff dimension

$$\alpha_{N,L} = \frac{\log(2^N L + 1)}{\log(2L + 1)}. \quad (4.1)$$

We next construct a graph  $\Gamma^{(N,L)}$  which has at the large scale the same structure as  $F$  at the small scale. (See [BCG] for some more details and pictures). Let

$$H_n = (2L + 1)^n F_n.$$

Thus  $H_n$  is a cubical set and is contained in  $[0, (2L + 1)^n]^N$ . It is easy to check that  $(H_n)$  is an increasing sequence of sets, and that if  $m > n$  then  $H_m \cap [0, (2L + 1)^n]^N = H_n$ . Set

$$H = \bigcup_{n=0}^{\infty} H_n.$$

Then  $H$  is a cubical set, and we define  $\Gamma^{(N,L)}$  to be the graph induced by  $H$ . (It is easy to see that  $\Gamma^{(N,L)}$  is a tree.)

**Lemma 4.2**  $\Gamma^{(N,L)}$  is an infinite connected graph which satisfies  $(V_{\alpha_{N,L}})$ ,  $(E_{1+\alpha_{N,L}})$ ,  $(EHI)$  and  $(VSR)$ .

This Lemma gives Proposition 4 for a countable dense set of  $\alpha$  in  $[1, \infty)$ . We will not prove it at this point, since it will follow from the more general construction below which is needed to prove the full version of Proposition 4.

We now consider fractals and graphs obtained by mixtures of the iterations  $\Psi^{(N,L)}$ . This was done for Sierpinski gaskets in the fractal case in [BH]. The construction here is quite similar, except for one point: here we work ‘outwards’, starting with the small scale structure, while [BH] worked ‘inwards’. We fix  $N$  in what follows, and let  $1 \leq L_1 < L_2$ . (We could allow more than two values of  $L$ , as in [BH], but this complicates the notation without giving us anything more.)

Let  $\Xi = \{1, 2\}^{\mathbb{N}}$  and let  $\xi = (\xi_1, \dots) \in \Xi$ : we call  $\xi$  an *environment sequence*. Let  $F_0^\xi = [0, 1]^N$ , and define

$$\begin{aligned} a_n &= \prod_{i=1}^n (2L_{\xi_i} + 1), \\ b_n &= \prod_{i=1}^n (2^N L_{\xi_i} + 1), \\ F_n^\xi &= \Psi^{(N, L_{\xi_n})}(F_{n-1}^\xi), \\ H_n^\xi &= a_n F_n^\xi, \\ H^\xi &= \bigcup_{i=1}^{\infty} H_n^\xi. \end{aligned}$$

If  $\xi$  is constant then we obtain one of the sets  $H^{(N, L_i)}$ .

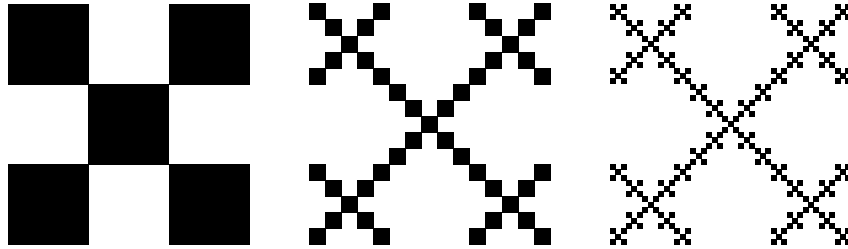


Figure 2: The sets  $F_1^\xi, F_2^\xi, F_3^\xi$  when  $N = 2, L_1 = 1, L_2 = 2, \xi = (1, 2, 1, \dots)$ .

It is straightforward to check the following properties of  $H_n^\xi$  and  $H^\xi$ .

**Lemma 4.3**

- (a)  $H_n^\xi \subset H_m^\xi$  if  $n \leq m$ .
- (b)  $H_n^\xi \subset [0, a_n]^N$ .
- (c) If  $m > n$  then  $H_m^\xi \cap [0, a_n]^N = H_n^\xi$ .
- (d)  $|H_n^\xi| = b_n$ .
- (e)  $H_n^\xi$  and  $H^\xi$  are cubical sets.

We call an *n-block* of  $H^\xi$  any subset of  $H^\xi$  isomorphic to  $H_n^\xi$ . The form of the *n-blocks* is determined by the elements  $\xi_1, \dots, \xi_n$  in the environment sequence:  $\xi_1$  determines the smallest scale structure, then  $\xi_2$  determines how these 1-blocks are pieced together to form the 2-blocks, and so on.

We define the graph  $\Gamma^\xi = (G^\xi, E^\xi)$  to be the graph induced by the cubical set  $H^\xi$ . It is clear that  $\Gamma^\xi$  is a tree. We work with the natural weights on  $\Gamma^\xi$ , write  $d$  for the usual graph distance, and  $B(x, r)$ ,  $V(x, r)$  for balls and the volume function.

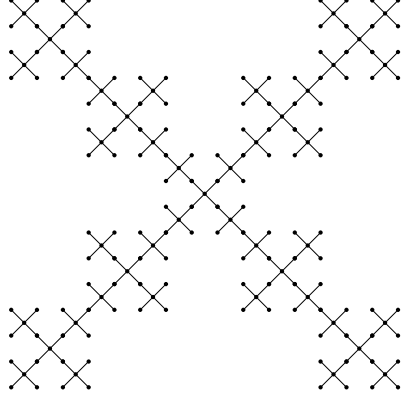


Figure 3: Part of the graph  $\Gamma^\xi$ , with  $N = 2$ ,  $L_1 = 1$ ,  $L_2 = 2$ ,  $\xi = (2, 1, 1, \dots)$ .

For the remainder of this section we will allow the constants  $c_i$  to depend on  $N$  and  $L_2$ , but not on the environment sequence  $\xi$ . (Since  $2L_1 + 1 \geq 3$  we do not need to include explicit dependence on  $L_1$ .) Note that the sequences  $a_n$ ,  $b_n$  satisfy

$$3a_n \leq a_{n+1} \leq (2L_2 + 1)a_n, \quad 5b_n \leq (2^N + 1)b_n \leq b_{n+1} \leq (2^N L_2 + 1)b_n. \quad (4.2)$$

For  $x \in G^\xi$  let  $D_k(x)$  be a  $k$ -block containing  $x$ . For some  $x$  there will be more than one of these – if so then we choose the one closest to the origin. We abuse notation and will also write  $D_k(x)$  for the subgraph of  $\Gamma^\xi$  induced by the cubical set  $D_k(x)$  – note that this subgraph contains  $2^N b_k$  edges. We have  $|D_k(x)| = b_k$ , and that the diameter of  $D_k(x)$  is  $a_k$  (in the  $L_\infty$  metric), and  $2a_k$  in the graph metric  $d$ .

We now consider the volume growth of  $\Gamma^\xi$ .

**Lemma 4.4** (a) *Let  $n \geq 1$  and  $a_n \leq R \leq a_{n+1}$ . Then for  $x \in G^\xi$ ,*

$$c_1 b_n \leq V(x, R) \leq c_2 b_n.$$

(b)  $\Gamma^\xi$  satisfies the volume doubling condition (VD).

**Proof.** (a) If  $y \in D_{n-1}(x)$  then  $d(x, y) \leq 2a_{n-1} < a_n \leq R$ . So  $D_{n-1}(x) \subset B(x, R)$ , and

$$V(x, R) \geq 2^N |D_{n-1}(x)| = 2^N b_{n-1} \geq c_1 b_n.$$



The upper bound is proved in a similar way.  $B(x, R)$  is contained in an  $L_\infty$  ball in  $\mathbb{R}^N$  of radius  $\frac{1}{2}a_{n+1}$ , and so cannot intersect more than  $2^N$   $(n + 1)$ -blocks. Hence

$$V(x, R) \leq 2^{2N} |D_{n+1}(x)| \leq c_2 b_n.$$

(b) is immediate from (a) and (4.2). ■

The exit times of balls are obtained in a similar way, but require a little more work.

**Lemma 4.5** *Let  $x \in G^\xi$ , let  $n \geq 1$ , and let  $x_0$  be the centre of  $D_n(x)$ . Let  $A$  be the set of  $2^N$  corners of  $D_n(x)$ . Then (for the discrete time simple random walk on  $\Gamma^\xi$ )*

$$\mathbb{E}^{x_0} T_A = \frac{1}{2} a_n b_n.$$

**Proof.** Let  $y_0 \in A$ . The graph  $D_n(x_0)$  consists of  $2^N$  identical subgraphs, each containing one corner of  $D_n(x_0)$  and connected at the centre  $x_0$ . By symmetry each of these subgraphs has  $b_n$  edges. Write  $\Gamma'$  for the subgraph containing  $y_0$ . Let  $X'$  be the simple random walk on  $\Gamma'$ . As  $\Gamma'$  is a tree, the effective resistance between any two points is just the graph distance between them, so  $R_e(x_0, y_0) = d(x_0, y_0) = a_n$ . Thus by (1.4) we have (for  $X'$ )

$$\mathbb{E}^{x_0} T_{y_0} + \mathbb{E}^{y_0} T_{x_0} = a_n b_n,$$

and so by symmetry  $\mathbb{E}^{x_0} T_{y_0} = \frac{1}{2} a_n b_n$ . Finally, again using the symmetry of  $D_n(x)$ , we have (for  $X$ )  $\mathbb{E}^{x_0} T_A = \frac{1}{2} a_n b_n$ . ■

**Proposition 4.6** *Let  $n \geq 1$  and  $a_n \leq R \leq a_{n+1}$ . Then for  $x \in G^\xi$*

$$c_1 a_n b_n \leq \mathbb{E}^x \tau_{x,R} \leq c_2 a_n b_n.$$

**Proof.** Since  $V(x, R) \leq c b_n$ , and the effective resistance from  $x$  to  $B(x, R)^c$  is at most  $a_{n+1}$ , the upper bound is clear from (1.5).

For the lower bound note first that  $D_{n-1}(x)$ , and every  $(n - 1)$ -block touching  $D_{n-1}(x)$ , are in  $B(x, R)$ . If we write  $C$  for the set of centers of these  $(n - 1)$ -blocks, any path from  $x$  to  $B(x, R)^c$  must pass through a point in  $C$ . So, using Lemma 4.5,  $\mathbb{E}^x \tau_{x,R} \geq \frac{1}{2} a_n b_n$ . ■

**Proposition 4.7**  $\Gamma^\xi$  *satisfies (EHI).*

**Proof.** We begin by proving the elliptic Harnack inequality for  $k$ -blocks, rather than balls. Call two  $k$ -blocks *adjacent* if they meet at a point, and write  $N_k(x)$  for the union of  $D_k(x)$  and the  $k$ -blocks adjacent to  $D_k(x)$ . (There will be between 1 and  $2^N$  of these.)

Suppose  $x \in G^\xi$ ,  $k \geq 1$  and  $h > 0$  is harmonic on  $N_k(x)$ . We prove that there exists  $c_3$ , depending only on  $d$  and  $L_2$  such that

$$\sup_{D_k(x)} h \leq c_3 \inf_{D_k(x)} h. \quad (4.3)$$

Write  $B = D_k(x)$ , and  $\Lambda = \partial(B^c)$ ; this is the sets of corner points in  $B$  which are also in a  $k$ -block adjacent to  $B$ . By the maximum principle  $h$  attains its maximum and minimum on  $B$  at points in  $\Lambda$  – at  $x_1$  and  $x_0$  say. Write  $B'$  for the  $k$ -block adjacent to  $B$  which contains  $x_0$ , and  $\Lambda'$  for the set of all the corners of  $B$  and  $B'$  except  $x_0$ . Consider the simple random walk  $X$  started at  $x_0$  and run until it first hits a point in  $\Lambda'$ . As  $x_1 \in \Lambda'$ , by symmetry

$$\mathbb{P}^{x_0}(X_{T_{\Lambda'}} = x_1) = \frac{1}{\#(\Lambda')} = \frac{1}{2^{N+1} - 2}.$$

So, since  $h(X)$  is a martingale, we obtain  $h(x_0) = \sum_{y \in \Lambda'} h(y) \mathbb{P}^{x_0}(X_{T_{\Lambda'}} = y) h(y) \geq ch(x_1)$ , proving (4.3).

A chaining argument now gives the elliptic Harnack inequality in its standard version, for balls in  $\Gamma^\xi$ . ■

**Remark 4.8** By looking at Green's functions, as in the proof of Proposition 3(c), we could also prove that  $\Gamma^\xi$  satisfies (VSR) for any  $\xi \in \Xi$ . (We cannot use Proposition 3(c) directly here, since  $\Gamma^\xi$  need not satisfy  $(V_\alpha)$ .)

We now investigate the conditions on  $\xi$  under which  $\Gamma^\xi$  satisfies  $(V_\alpha)$ . Fix  $\xi \in \Xi$ , and set

$$h_j(n) = n^{-1} \sum_{r=1}^n 1_{(\xi_r=j)}, \quad j = 1, 2.$$

Note that  $h_1(n) + h_2(n) = n$ . Write

$$l_j = 2L_j + 1, \quad m_j = 2^N L_j + 1, \quad j = 1, 2.$$

Then we have

$$a_n = l_1^{h_1(n)} l_2^{h_2(n)}, \quad \text{and } b_n = m_1^{h_1(n)} m_2^{h_2(n)}. \quad (4.4)$$

Elementary calculations show that the function  $f(x) = \log(2^N x + 1) / \log(2x + 1)$  is decreasing on  $[1, \infty)$ , so that we have

$$\frac{\log m_2}{\log l_2} < \frac{\log m_1}{\log l_1}. \quad (4.5)$$

**Proposition 4.9** (a) A necessary and sufficient condition that  $\Gamma^\xi$  satisfies  $V(\alpha)$  for some  $\alpha > 1$  is that there exists  $c_1 < \infty$  and  $p_1 \in [0, 1]$  such that

$$|h_1(n) - np_1| \leq c_1, \quad \text{for all } n \geq 1. \quad (4.6)$$

(b) If (4.6) holds, then, writing  $p_2 = 1 - p_1$ ,  $\Gamma^\xi$  satisfies  $(V_\alpha)$  and  $(E_{1+\alpha})$  with

$$\alpha = \frac{p_1 \log m_1 + p_2 \log m_2}{p_1 \log l_1 + p_2 \log l_2}. \quad (4.7)$$

**Proof.** Suppose first  $\Gamma^\xi$  satisfies  $V(\alpha)$ . Then there exist positive constants  $c_2, c_3$  such that for all  $x \in G^\xi$ ,  $n \geq 1$ ,

$$c_2 b_n \leq V(x, a_n) \leq c_3 b_n, \quad c_2 a_n^\alpha \leq V(x, a_n) \leq c_3 a_n^\alpha.$$

Hence  $(c_2/c_3)a_n^\alpha \leq b_n \leq (c_3/c_2)a_n^\alpha$  and so there exists  $c_4 < \infty$  such that  $|\log(b_n) - \alpha \log(a_n)| \leq c_4$  for all  $n \geq 1$ . So, using (4.4), the function (in  $n$ )

$$|h_1(n)(\log m_1 - \alpha \log l_1 - \log m_2 + \alpha \log l_2) + n(\log m_2 - \alpha \log l_2)| \quad (4.8)$$

is bounded by  $c_4$ . If  $\log m_1 - \alpha \log l_1 - \log m_2 + \alpha \log l_2 = 0$  then we would have  $\log m_2 - \alpha \log l_2 = 0$ , which would imply that

$$\frac{\log m_2}{\log l_2} = \frac{\log m_1}{\log l_1},$$

contradicting (4.5). So writing

$$p_1 = -\frac{\log m_2 - \alpha \log l_2}{\log m_1 - \alpha \log l_1 - \log m_2 + \alpha \log l_2}$$

we deduce from the boundedness of (4.8) that (4.6) holds.

Now suppose that (4.6) holds. Then by (4.4) we obtain

$$l_1^{np_1} l_2^{np_2} (l_1 l_2)^{-c_1} \leq a_n \leq l_1^{np_1} l_2^{np_2} (l_1 l_2)^{c_1} \\ m_1^{np_1} m_2^{np_2} (m_1 m_2)^{-c_1} \leq b_n \leq m_1^{np_1} m_2^{np_2} (m_1 m_2)^{c_1}.$$

This implies that

$$c_5 a_n^\alpha \leq b_n \leq c_6 a_n^\alpha, \quad n \geq 1,$$

with  $\alpha$  given by (4.7). The conditions  $(V_\alpha)$  and  $(E_{1+\alpha})$  now follow using Lemma 4.4 and Proposition 4.6.  $\blacksquare$

**Remark 4.10** See [BH, Theorem 6.2] for a similar result in the fractal context. The condition (4.6) on  $\xi$  is extremely strong, and shows (for example) that if the components  $\xi_i$  of  $\xi$  are chosen to be independent (non-trivial) random variables, then  $\Gamma^\xi$  fails to satisfy  $(V_\alpha)$  for any  $\alpha$ . Volume doubling, on the other hand, holds for all  $\xi \in \Xi$  by Lemma 4.4(b).

**Proof of Proposition 4.** If  $\alpha = 1$  we take  $\Gamma_\alpha$  to be  $\mathbb{Z}$ . So assume  $\alpha > 1$ . Let  $L_1 = 1$  and choose  $d \geq 2$  large enough so that  $\log m_1 / \log l_1 = \log(2^N + 1) / \log(2 + 1) > \alpha$ . Then choose  $L_2$  large enough so that  $\log m_2 / \log l_2 = \log(2^N L_2 + 1) / \log(2L_2 + 1) < \alpha$ . Therefore there exists  $p_1 \in (0, 1)$  such that

$$\alpha = \frac{p_1 \log m_1 + (1 - p_1) \log m_2}{p_1 \log l_1 + (1 - p_1) \log l_2}.$$

We can choose  $\xi \in \Xi$  so that  $|h_1(n) - np_1| \leq 1$ , so that the condition (4.6) of Proposition 4.9 is satisfied. Thus  $\Gamma^\xi$  satisfies  $(V_\alpha)$  and  $(E_{1+\alpha})$ . ■

## 5. Some additional examples

Let  $\Gamma_i = (G_i, E_i)$ ,  $i = 1, 2$  be two infinite connected graphs. Let  $z_i \in G_i$ . A *join* of  $\Gamma_1$  and  $\Gamma_2$  (at  $z_1, z_2$ ) is the graph  $\Gamma$  obtained by identifying  $z_1$  and  $z_2$ . Thus, formally,  $\Gamma$  has vertex set  $G_1 \cup G_2 - \{z_2\}$  and edge set

$$\begin{aligned} & \{ \{x_i, y_i\} : \{x_i, y_i\} \in E_i, x_i, y_i \neq z_i, i = 1, 2 \} \cup \{ \{z_1, x_1\} : \{z_1, x_1\} \in E_1 \} \\ & \cup \{ \{z_1, x_2\} : \{z_2, x_2\} \in E_2 \}. \end{aligned}$$

If  $(\Gamma_i, a^{(i)})$  are weighted graphs, then we define the weights  $a_{xy}$  for  $\Gamma$  by taking  $a_{xy} = a_{xy}^{(i)}$  if  $x, y \in G_i$ .

From [D2] we have:

**Lemma 5.1** *Let  $\Gamma_i$  be infinite connected weighted graphs, satisfying  $(V_{\alpha_i})$  and  $(E_{\beta_i})$ , with  $\alpha_i < \beta_i$ ,  $i = 1, 2$ . Let  $\Gamma$  be a join of  $\Gamma_1$  and  $\Gamma_2$ .*

- (a) *If  $\Gamma_i$  both satisfy (VSR), and  $\beta_1 - \alpha_1 = \beta_2 - \alpha_2$ , then  $\Gamma$  satisfies (EHI).*  
 (b) *If  $\alpha_1 \neq \alpha_2$  then  $\Gamma$  does not satisfy (VD).*

**Lemma 5.2** *There exists a transient graph which satisfies (EHI) but not (VD).*

**Proof.** Examples of recurrent graphs satisfying (EHI) but not (VD) were given in [D2]. Let  $1 \leq \alpha_1 < \alpha_2 < 2$ ,  $\beta_1 - \alpha_1 = \beta_2 - \alpha_2$ , and let  $\Gamma_i$  be graphs satisfying (EHI),  $(E_{\beta_i})$  and  $(V_{\alpha_i})$ . Such graphs exist by Theorem 2, and satisfy (VSR) by Proposition 3(b). Let  $\Gamma$  be a join of  $\Gamma_1$  and  $\Gamma_2$ ; by Lemma 5.1,  $\Gamma$  satisfies (EHI) but fails (VD).

To construct a transient graph choose  $b = 9$ , and take  $M \geq 2$  sufficiently large so that  $\lambda = \log M / \log b > \beta_1 - \alpha_1$ . Let  $\tilde{\Gamma}$  be a graph constructed from  $\Gamma$  by the procedure of Section 2. Then  $\tilde{\Gamma}$  fails (VD), but satisfies (EHI).

Write  $\tilde{\Gamma}_i$  for the subgraphs of  $\tilde{\Gamma}$  with vertex sets  $G_i \times U$ . Then each of these subgraphs satisfies  $(V_{\alpha_i + \lambda})$ ,  $(E_{\beta_i})$  and (EHI), and so is transient by Proposition 3(d). Since  $\tilde{\Gamma}$  contains a transient subgraph, it is transient. ■

**Proof of Theorem 6.** Let  $\Gamma = (G, E)$  be the graph constructed in Lemma 5.2. Write  $x_0$  for the common point of the components  $G_1$  and  $G_2$ . Note that  $\beta_1 < \beta_2$ . Now consider the product graph  $\Gamma^{(2)}$  with vertex set  $G \times G$  and edge set

$$E^{(2)} = \{ \{(x, y_1), (x, y_2)\} : \{y_1, y_2\} \in E \} \cup \{ \{(x_1, y), (x_2, y)\} : \{x_1, x_2\} \in E \}.$$

If  $X^{(1)}$  and  $X^{(2)}$  are independent copies of the continuous time random walk on  $\Gamma$  then  $Z_t = (X_t^{(1)}, X_t^{(2)})$ ,  $t \geq 0$  is a continuous time random walk on  $\Gamma^{(2)}$ .

We now show that (EHI) fails for  $\Gamma^{(2)}$ . We use  $B$  to denote balls in  $\Gamma$ . Let  $R \gg 1$ , and choose  $z_i \in G_i$  with  $d(z_i, x_0) > 4R$ . Write  $Q = B(z_1, 2R) \times B(z_2, 2R)$ , and  $\tau_Q$  for the exit time of  $Z$  from  $Q$ . Let  $\delta = (\beta_2 - \beta_1)/4$ , and for simplicity suppose that  $\beta_2 \leq 3$ . Write

$$\begin{aligned} \tau_1 &= \inf\{t \geq 0 : X_t^{(1)} \notin B(z_1, R)\}, \\ \tau_2(y) &= \inf\{t \geq 0 : X_t^{(2)} \notin B(y, R^{1-\delta/\beta_2})\}. \end{aligned}$$

The estimates in Lemma 1.3(b) and (c) also hold for the continuous time random walks  $X^{(i)}$ , and can be applied to the exit times  $\tau_i$  above as long as the relevant balls do not contain the join point  $x_0$ . So, if  $t = R^{\beta_1 + \delta}$  we deduce that

$$\mathbb{P}^{x_1}(\tau_1 > t) \leq c \exp(-c'R^\delta), \tag{5.1}$$

and if  $y \in B(z_2, R)$  then

$$\mathbb{P}^y(\tau_2(y) \leq t) \leq c \exp\left(-c' \left(\frac{R^{\beta_2(1-\delta/\beta_2)}}{t}\right)^{1/(\beta_2-1)}\right) \leq c \exp(-c'R^\delta). \tag{5.2}$$

Let  $D = \partial B(z_1, R) \times B(z_2, R^{1-\delta/\beta_2}) \subset \partial Q$  and let  $h$  be the harmonic function in  $Q$  with boundary values 1 on  $D$  and 0 on  $\partial Q - D$ , so that  $h((x_1, x_2)) = \mathbb{P}^{(x_1, x_2)}(\tau_Q \in D)$ . Note that  $h$  is harmonic on the ball in  $\Gamma^{(2)}$  with centre  $(z_1, z_2)$  and radius  $2R$ . The estimate (5.1) implies that

$$h(z_1, z_2) > 1 - c \exp(-c'R^\delta),$$

while by (5.2) if  $R^{1-\delta/\beta_2} < d(y, z_2) < R$  then

$$h(z_1, y) < c \exp(-c'R^\delta).$$

So we deduce that

$$\sup_{B(z, R)} h > c e^{c'R^\delta} \inf_{B(z, R)} h,$$

and since  $R$  can be as large as we like this shows that (EHI) fails on  $\Gamma^{(2)}$ . ■

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Martin T. Barlow  
Department of Mathematics  
University of British Columbia  
Vancouver, V6T 1Z2. Canada  
barlow@math.ubc.ca