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On the low regularity phase space of the Benjamin–Ono equation

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Abstract. In this paper, we prove that the Benjamin–Ono equation is globally in time C^0-well-posed in the Hilbert space $H^{-1/2,\sqrt{\log}}(\mathbb{T},\mathbb{R})$ of periodic distributions in $H^{-1/2}(\mathbb{T},\mathbb{R})$ with $\sqrt{\log}$-weights. The space $H^{-1/2,\sqrt{\log}}(\mathbb{T},\mathbb{R})$ can thus be considered as a maximal low regularity phase space for the Benjamin–Ono equation corresponding to the scale $H^s(\mathbb{T},\mathbb{R})$, $s > -1/2$.

Dedicated to the memory of our friend and collaborator Thomas Kappeler

1. Introduction

In this paper, we study the Benjamin–Ono equation on the torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$,

$$\partial_t u = \partial_x\big(|\partial_x| u - u^2\big), \tag{1}$$

where $u \equiv u(x,t)$, $x \in \mathbb{T}$, $t \in \mathbb{R}$ is real valued and $|\partial_x| : H_c^\beta \to H_c^{\beta-1}$, $\beta \in \mathbb{R}$, is the Fourier multiplier

$$|\partial_x| : \sum_{n\in\mathbb{Z}} \hat{v}(n) e^{\mathrm{inx}} \mapsto \sum_{n\in\mathbb{Z}} |n|\,\hat{v}(n) e^{\mathrm{inx}}, \tag{2}$$

where $\hat{v}(n)$, $n \in \mathbb{Z}$, are the Fourier coefficients of $v \in H_c^\beta$ and $H_c^\beta \equiv H^\beta(\mathbb{T},\mathbb{C})$ is the Sobolev space of complex valued distributions on the torus $\mathbb{T}$. Equation (1) was introduced in 1967 by Benjamin [2] and Davis and Acrivos [4] as a model for a special regime of internal gravity waves at the interface of two fluids. It is well known that (1) admits a Lax pair representation (cf. [16]) that leads to an infinite sequence of conserved quantities (cf. [3, 16]) and that it can be written in Hamiltonian form with Hamiltonian

$$\mathcal{H}(u) := \frac{1}{2\pi}\int_0^{2\pi} \Big(\frac{1}{2}\big(|\partial_x|^{1/2} u\big)^2 - \frac{1}{3}u^3\Big)dx \tag{3}$$

by the use of the Gardner bracket

$$\{F,G\}(u) := \frac{1}{2\pi}\int_0^{2\pi} \big(\partial_x \nabla_u F\big)\nabla_u G\, dx, \tag{4}$$

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where $\nabla_u F$ and $\nabla_u G$ are the L^2-gradients of $F, G \in C^1(H_c^\beta, \mathbb{R})$ at $u \in H_c^\beta$. By the Sobolev embedding $H_c^{1/2} \hookrightarrow L^3(\mathbb{T}, \mathbb{C})$, the Hamiltonian (3) is well defined and analytic on $H_c^{1/2}$, the *energy space* of (1). The problem of the existence and the uniqueness of the solutions of the Benjamin–Ono equation is well studied; see [6, 10, 13, 18] and references therein. We refer to [13, 18] for an excellent survey and a derivation of (1).

By using the Hamiltonian formalism for (1), it was recently proved in [6, 10] that for any $s > -1/2$, the Benjamin–Ono equation has a *homeomorphic* Birkhoff map

$$\Phi : H^s_{r,0} \to \mathfrak{h}^{\frac{1}{2}+s}_{r,0}, \quad u \mapsto \big((\overline{\Phi_{-n}(u)})_{n\le -1}, (\Phi_n(u))_{n\ge 1}, \Phi_0(u) = 0\big), \tag{5}$$

where, for $\beta \in \mathbb{R}$,

$$H^\beta_{r,0} := \big\{u \in H^\beta_c \mid \hat{u}(0) = 0, \bar{u} = u\big\} \tag{6}$$

and

$$\mathfrak{h}^\beta_{r,0} := \big\{z \in \mathfrak{h}^\beta_c \mid z_0 = 0, z_{-n} = \bar{z}_n \ \forall n \ge 1\big\} \tag{7}$$

is a real subspace in the Hilbert space of complex-valued sequences

$$\mathfrak{h}^\beta_c := \left\{(z_n)_{n\in\mathbb{Z}} \,\middle|\, \sum_{n\in\mathbb{Z}} \langle n\rangle^{2\beta} |z_n|^2 < \infty\right\}, \quad \langle n\rangle := \max(1, |n|), \tag{8}$$

equipped with the norm $\|z\|_{\mathfrak{h}^\beta_c} := \big(\sum_{n\in\mathbb{Z}} \langle n\rangle^{2\beta} |z_n|^2\big)^{1/2}$. For $\beta = 0$, we set

$$L^2_{r,0} \equiv H^0_{r,0}, \quad L^2_c \equiv H^0_c, \quad \ell^2_{r,0} \equiv \mathfrak{h}^0_{r,0}, \quad \ell^2_c \equiv \mathfrak{h}^0_c.$$

By [8,9], the Birkhoff map (5) is a *bianalytic* diffeomorphism. It transforms the trajectories of the Benjamin–Ono equation (1) into straight lines that have constant frequencies on any given isospectral set (infinite torus) of potentials of the corresponding Lax operator (see (12) below). In this sense, the Birkhoff map can be considered as a non-linear Fourier transform that significantly simplifies the solutions of (1). This fact allows us to prove that for any $-1/2 < s < 0$, (1) is globally C^0–well-posed on $H^s_{r,0}$ (see [10]), improving in this way the previously known well-posedness results (see [14, 15]). Additional applications of the Birkhoff map include the proof of the almost periodicity of the solutions of the Benjamin–Ono equation and the orbital stability of the Benjamin–Ono traveling waves (see [10, Theorems 3 and 4] and [9]).

In order to formulate our results, we define for any $\beta \in \mathbb{R}$ the Hilbert space of periodic distributions in H^β_c,

$$H_c^{\beta,\sqrt{\log}} \equiv H_c^{\beta,\sqrt{\log}}(\mathbb{T}, \mathbb{C}) := \left\{u \in H^\beta_c \,\middle|\, \sum_{n\in\mathbb{Z}} \langle n\rangle^{2\beta} \log(\langle n\rangle + 1)\, |\hat{u}(n)|^2 < \infty\right\}, \tag{9}$$

as well as the spaces

$$H^{\beta,\sqrt{\log}}_{r,0} := H^\beta_{r,0} \cap H^{\beta,\sqrt{\log}}_c,$$
$$\mathfrak{h}^{\beta,\sqrt{\log}}_{r,0} := \mathfrak{h}^\beta_{r,0} \cap \mathfrak{h}^{\beta,\sqrt{\log}}_c$$

and

$$\mathfrak{h}_c^{\beta,\sqrt{\log}} := \left\{ (z_n)_{n\in\mathbb{Z}} \in \mathfrak{h}_c^{\beta} \;\middle|\; \sum_{n\in\mathbb{Z}} \langle n\rangle^{2\beta} \log(\langle n\rangle + 1)\,|z_n|^2 < \infty \right\}.$$

Note that for any $s > -1/2$ we have the compact embedding

$$H^s_{r,0} \subsetneqq H^{-1/2,\sqrt{\log}}_{r,0}.$$

Our first result concerns the extension of the Birkhoff map (5) from $H^s_{r,0}$ with $s > -1/2$ to the space $H^{-1/2,\sqrt{\log}}_{r,0}$.

Theorem 1.1. *The Birkhoff map* (5) *extends to a homeomorphic map* $\Phi : H^{-1/2,\sqrt{\log}}_{r,0} \to \mathfrak{h}^{0,\sqrt{\log}}_{r,0}$.

As a consequence from this theorem, we obtain the following corollary.

Corollary 1.1. *The Benjamin–Ono equation* (1) *is globally in time* C^0*–well-posed in the phase space* $H^{-1/2,\sqrt{\log}}_{r,0}$*. More specifically, for any* $t \in \mathbb{R}$ *and* $s > -1/2$*, the flow map* $S^t : H^s_{r,0} \to H^s_{r,0}$ *defined in* [10, Theorem 1] *extends to a continuous flow map* $S^t : H^{-1/2,\sqrt{\log}}_{r,0} \to H^{-1/2,\sqrt{\log}}_{r,0}$*. Furthermore, for any* $T > 0$*, the associated solution map* $S : H^{-1/2,\sqrt{\log}}_{r,0} \to C([-T,T], H^{-1/2,\sqrt{\log}}_{r,0})$*,* $u_0 \mapsto \{t \mapsto S^t u_0, t \in [-T,T]\}$*, is continuous and the corresponding trajectories are almost periodic as functions from* $\mathbb{R}$ *to* $H^{-1/2,\sqrt{\log}}_{r,0}$.

Remark 1.1. Recently, Killip, Laurens, and Vişan [12] found a different proof of the well-posedness on H^s_r for every $s > -1/2$, which can be generalized to the Benjamin–Ono equation on the real line. It would be interesting to know whether the methods of [12] lead to a similar well-posedness result on $H^{-1/2,\sqrt{\log}}_r(\mathbb{R})$.

Remark 1.2. The first author recently derived in [5] an explicit formula for the solution of the Benjamin–Ono equation on the torus. It can be easily checked that this formula holds for every initial datum in $H^{-1/2,\sqrt{\log}}_{r,0}(\mathbb{T})$. It does not seem straightforward to get Corollary 1.1 from this formula only.

Next, we come to some important limitations of the above extension, which are specific to the bottom regularity $H^{-1/2,\sqrt{\log}}_{r,0}$. First, we start with the lack of weak continuity.

Proposition 1.1. *The map* Φ *is not weakly continuous from* $H^{-1/2,\sqrt{\log}}_{r,0}$ *to* $\mathfrak{h}^{0,\sqrt{\log}}_{r,0}$*, and the flow map of the Benjamin–Ono equation is not weakly continuous from* $H^{-1/2,\sqrt{\log}}_{r,0}$ *to* $H^{-1/2,\sqrt{\log}}_{r,0}$*. In fact, there exists a sequence of smooth initial data converging weakly to* 0 *in* $H^{-1/2,\sqrt{\log}}_{r,0}$*, and such that the sequence of corresponding solutions does not converge to* 0 *in* $\mathcal{D}'(\mathbb{T})$ *on any time interval* $[0,T]$ *with* $T > 0$.

The proof of Proposition 1.1 consists in revisiting the counterexample of [10] and in observing that the $H^{-1/2,\sqrt{\log}}_{r,0}$ regularity is critical in this construction.

The second limitation concerns the smoothness of the Birkhoff map and is in sharp contrast with the results of [8, 9].

Proposition 1.2. *The (bi-analytic) Birkhoff map* (5) *cannot be extended to an analytic map*

$$\Phi : H^{-1/2,\sqrt{\log}}_{r,0} \to \mathfrak{h}^{0,\sqrt{\log}}_{r,0}. \tag{10}$$

In fact, we prove that (5) cannot be extended to a C^2-map

$$\Phi : H^{-1/2,\sqrt{\log}}_{r,0} \to \mathfrak{h}^{0,\sqrt{\log}}_{r,0}$$

in a neighborhood of the origin.

Notation. In addition to the spaces introduced above, we will also use the Hardy space

$$H^\beta_+ := \{ f \in H^\beta_c \mid \hat{f}(n) = 0 \,\forall\, n < 0 \}, \quad \beta \in \mathbb{R} \tag{11}$$

as well as the spaces of complex-valued sequences

$$\mathfrak{h}^\beta_+ := \left\{ (z_n)_{n\geq 1} \,\middle|\, \sum_{n\geq 1} \langle n \rangle^{2\beta} |z_n|^2 < \infty \right\},$$

$$\mathfrak{h}^{\beta,\sqrt{\log}}_+ := \left\{ (z_n)_{n\geq 1} \in \mathfrak{h}^\beta_+ \,\middle|\, \sum_{n\geq 1} \langle n \rangle^{2\beta} \log(\langle n \rangle + 1)\, |z_n|^2 < \infty \right\}$$

and

$$\mathfrak{h}^\beta_{\geq 0} := \left\{ (z_n)_{n\geq 0} \,\middle|\, \sum_{n\geq 0} \langle n \rangle^{2\beta} |z_n|^2 < \infty \right\}.$$

We will denote the norm in H^β_c by $\|\cdot\|_\beta$ and set $\|\cdot\| := \|\cdot\|_0$. Similarly, the norm in $H^{\beta,\sqrt{\log}}_c$ will be denoted by $\|\cdot\|_{\beta,\sqrt{\log}}$, and the norm in $\mathfrak{h}^{\beta,\sqrt{\log}}_c$ (resp., $\mathfrak{h}^\beta_{\geq 0}$) will be denoted by $\|\cdot\|_{\mathfrak{h}^{\beta,\sqrt{\log}}_c}$ (resp. $\|\cdot\|_{\mathfrak{h}^\beta_{\geq 0}}$). For $\beta = 0$, we set $L^2_+ \equiv H^0_+$, $\ell^2_+ := \mathfrak{h}^0_+$, and $\ell^2_{\geq 0} := \mathfrak{h}^0_{\geq 0}$. We will also need the Banach space ℓ^1_+ of complex-valued absolutely summable sequences $(z_n)_{n\geq 1}$ and the quadratic forms

$$\langle f | g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}\, dx, \quad \langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(x) g(x)\, dx, \quad f, g \in L^2_c.$$

Now, take $u \in H^s_c$ with $s > -1/2$ and consider the pseudo-differential expression

$$L_u := D - T_u \tag{12}$$

where $D := -i\partial_x$ and $T_u : H^{1+s}_+ \to H^s_+$ is the *Toeplitz operator*

$$T_u f := \Pi(uf), \quad f \in H^{1+s}_+,$$

where $\Pi \equiv \Pi^+ : H^s_c \to H^s_+$ is the *Szegő projector*

$$\Pi : H^s_c \to H^s_+, \quad \sum_{n\in\mathbb{Z}} \hat{v}(n)e^{\mathrm{inx}} \mapsto \sum_{n\geq 0} \hat{v}(n)e^{\mathrm{inx}},$$

onto the Hardy space H^s_+, introduced in (11). Note that, when restricted to H^{1+s}_+, D coincides with the Fourier multiplier (2). An important role in the integrability of the Benjamun–Ono equation is played by the *shift operator* $S : H^\beta_+ \to H^\beta_+$, $f(x) \mapsto e^{ix} f(x)$, $\beta \in \mathbb{R}$ (cf. [6]). It is not hard to see (cf., e.g., [7, Lemma 1 (ii)]) that for any given $u \in H^s_c$ with $s > -1/2$ the pseudo-differential expressions T_u and L_u define bounded linear maps

$$T_u : H^{1+s}_+ \to H^s_+ \quad \text{and} \quad L_u : H^{1+s}_+ \to H^s_+. \tag{13}$$

By Corollary 5.2 below, this does *not* extend to log-spaces with $s = -1/2$.

2. The Lax operator in log-spaces

In this section, we establish the basic properties of the Lax operator (12) with potential $u \in H_c^{-1/2,\sqrt{\log}}$. In view of Corollary A.1, for $u \in H_c^{-1/2,\sqrt{\log}}$, the pseudo-differential expression $L_u \equiv D - T_u$ defines a continuous map

$$L_u : H^{1/2}_+ \to H^{-1/2}_+.$$

In what follows, we will think of L_u as an unbounded operator on $H^{-1/2}_+$ with domain $\mathrm{Dom}(L_u) = H^{1/2}_+$. Let us fix $u_0 \in L^2_c$ and choose $u \in B_{-1/2,\sqrt{\log}}(u_0)$,

$$B_{-1/2,\sqrt{\log}}(u_0) := \left\{u \in H_c^{-1/2,\sqrt{\log}} \,\middle|\, \|u - u_0\|_{-1/2,\sqrt{\log}} < 1/(4K_0)\right\}, \tag{14}$$

where $K_0 > 0$ is the constant appearing in Corollary A.1. As in [10], consider the sesquilinear form

$$\mathcal{Q}_{u_0,\lambda}(f, g) := \langle -i\partial_x f | g\rangle - \big\langle \Pi(u_0 f)\big| g\big\rangle - \lambda\langle f | g\rangle, \tag{15}$$

where

$$\lambda \in \Lambda_{u_0} := \big\{\lambda \in \mathbb{C} \,\big|\, \mathrm{Re}(\lambda) < -\big(1 + \eta(\|u_0\|)\big)\big\},$$
$$\eta(\|u_0\|) := C_0\|u_0\|\big(1 + \|u_0\|\big),$$

and $C_0 > 0$ is a positive constant defined below. One easily sees that [10, Lemma 3.2] continues to hold for complex-valued $u \in H^s_c$, $-1/2 < s \leq 0$. Then, we set $s = 0$ in [10, Lemma 3.2], choose $C_0 := 2C^2_{2,0}$, and argue as in the proof of [10, Lemma 3.3] to obtain the following lemma.

Lemma 2.1. *There exist a constant $K > 0$ such that for any $u_0 \in L^2_c$ and for any $f, g \in H^{1/2}_+$, one has*

$$\frac{1}{2}\,\|f\|^2_{1/2} \leq \big|\big\langle \mathcal{Q}_{u_0,\lambda} f \,\big|\, f\big\rangle\big|, \quad \big|\big\langle \mathcal{Q}_{u_0,\lambda} f \,\big|\, g\big\rangle\big| \leq \big(1 + |\lambda| + K\|u_0\|\big)\|f\|_{1/2}\|g\|_{1/2}$$

uniformly in $\lambda \in \Lambda_{u_0}$. *If* $u_0 \in L^2_{r,0}$ *and* $\lambda \in \Lambda_{u_0} \cap \mathbb{R}$, *then* $\langle \mathcal{Q}_{u_0,\lambda} f | f \rangle \ge 0$ *for any* $f \in H^{1/2}_+$.

Now, we apply the Lax–Milgram lemma to obtain from Lemma 2.1 that for any $\lambda \in \Lambda_{u_0}$ the continuous map

$$L_{u_0} - \lambda : H^{1/2}_+ \to H^{-1/2}_+$$

is a linear isomorphism such that

$$\big\|(L_{u_0} - \lambda)^{-1}\big\|_{H^{-1/2}_+ \to H^{1/2}_+} \le 2. \tag{16}$$

This implies that, for any $u \in B_{-1/2,\sqrt{\log}}(u_0)$ and $\lambda \in \Lambda_{u_0}$, we have

$$L_u - \lambda = L_{u_0} - \lambda - T_{\tilde u} = \big(I - T_{\tilde u}(L_{u_0} - \lambda)^{-1}\big)(L_{u_0} - \lambda), \tag{17}$$

where $\tilde u := u - u_0$ and I is the identity. It follows from Corollary A.1 and the fact that $u \in B_{-1/2,\sqrt{\log}}(u_0)$ that

$$\|T_{\tilde u}\|_{H^{1/2}_+ \to H^{-1/2}_+} \le K_0 \|\tilde u\|_{-1/2,\sqrt{\log}} < 1/4. \tag{18}$$

By combining this with (16), we see that $\|T_{\tilde u}(L_{u_0} - \lambda)^{-1}\|_{H^{-1/2}_+ \to H^{-1/2}_+} < 1/2$, and hence, in view of (17), the map $L_u - \lambda : H^{1/2}_+ \to H^{-1/2}_+$ is a linear isomorphism such that

$$(L_u - \lambda)^{-1} = (L_{u_0} - \lambda)^{-1} \sum_{k \ge 0} \big[T_{\tilde u}(L_{u_0} - \lambda)^{-1}\big]^k, \tag{19}$$

where the Neumann series converges in $\mathcal{L}(H^{-1/2}_+, H^{-1/2}_+)$ uniformly in $u \in B_{-1/2,\sqrt{\log}}(u_0)$ and $\lambda \in \Lambda_{u_0}$. In particular, the map

$$(L_u - \lambda)^{-1} : H^{-1/2}_+ \to H^{1/2}_+$$

is bounded for u and λ chosen as above. As a consequence, we obtain the following theorem.

Theorem 2.1. *For any given* $u \in H^{-1/2,\sqrt{\log}}_c$ *the pseudo-differential expression* (12) *defines a closed operator* L_u *on* $H^{-1/2}_+$ *with domain* $\mathrm{Dom}(L_u) = H^{1/2}_+$. *This operator has a compact resolvent, and hence, a discrete spectrum. Moreover, the following statements hold.*

(i) *Take* $u_0 \in L^2_c$ *and assume that* $u \in B_{-1/2,\sqrt{\log}}(u_0)$. *Then, the half-plane* Λ_{u_0} *belongs to the resolvent set of* L_u.

(ii) *The map*

$$(u, \lambda) \mapsto (L_u - \lambda)^{-1}, \quad B_{-1/2,\sqrt{\log}}(u_0) \times \Lambda_{u_0} \to \mathcal{L}\big(H^{-1/2}_+, H^{1/2}_+\big) \tag{20}$$

is well defined and analytic.

Proof of Theorem 2.1. We already proved that for a given $u_0 \in L^2_c$ and for any $u \in B_{-1/2,\sqrt{\log}}(u_0)$ and $\lambda \in \Lambda_{u_0}$ the map (20) is well defined. The analyticity of (20) follows from (16) and the uniform convergence of the Neumann series in (19) in $\mathcal{L}(H^{-1/2}_+, H^{-1/2}_+)$. Since the embedding $H^{1/2}_+ \subseteq H^{-1/2}_+$ is compact, the map $(L_u - \lambda)^{-1} : H^{-1/2}_+ \to H^{1/2}_+ \subseteq H^{-1/2}_+$ is compact for $\lambda \in \Lambda_{u_0}$. This proves that for $u \in B_{-1/2,\sqrt{\log}}(u_0)$ the unbounded operator L_u on $H^{-1/2}$ with domain $\mathrm{Dom}(L_u) = H^{1/2}_+$ is closed and has a compact resolvent. Since the radius of the ball $B_{-1/2,\sqrt{\log}}(u_0)$ is independent of the choice of $u_0 \in L^2_c$, the above holds for any $u \in H_c^{-1/2,\sqrt{\log}}$. ∎

Let us now assume that the potential u is *real-valued*, $u \in H_{r,0}^{-1/2,\sqrt{\log}}$, and set

$$B^{r,0}_{-1/2,\sqrt{\log}}(u_0) := B_{-1/2,\sqrt{\log}}(u_0) \cap H_{r,0}^{-1/2,\sqrt{\log}}.$$

We can then choose $u_0 \in L^2_{r,0}$ such that $u \in B^{r,0}_{-1/2,\sqrt{\log}}(u_0)$ and define the unbounded operator

$$L_u^{\mathrm{sym}} := L_u|_{\mathrm{Dom}(L_u^{\mathrm{sym}})} \tag{21}$$

on L^2_+ with domain

$$\mathrm{Dom}(L_u^{\mathrm{sym}}) := (L_u - \lambda_\bullet)^{-1}(L^2_+) \subseteq H^{1/2}_+ \tag{22}$$

for some $\lambda_\bullet \in \Lambda_{u_0} \cap \mathbb{R}$. The map

$$C_u^{\mathrm{sym}} := (L_u^{\mathrm{sym}} - \lambda_\bullet)^{-1} : L^2_+ \to L^2_+ \tag{23}$$

is a composition of the following bounded linear maps:

$$L^2_+ \hookrightarrow H^{-1/2}_+ \xrightarrow{(L_u - \lambda_\bullet)^{-1}} H^{1/2}_+ \hookrightarrow L^2_+. \tag{24}$$

Hence, C_u^{sym} is bounded and compact. Since u is real-valued, $\langle L_u f | g \rangle = \langle f | L_u g \rangle$ for any $f, g \in \mathrm{Dom}(L_u^{\mathrm{sym}})$. This implies that C_u^{sym} is symmetric, and hence, selfadjoint. In particular, we obtain that L_u^{sym} is a selfadjoint operator in L^2_+ with domain $\mathrm{Dom}(L_u^{\mathrm{sym}})$ and compact resolvent. Now, we can apply the Hilbert-Schmidt theorem to (23) to obtain the following specification of Theorem 2.1 in the case when $u \in H_{r,0}^{-1/2,\sqrt{\log}}$.

Theorem 2.2. *Assume that $u \in H_{r,0}^{-1/2,\sqrt{\log}}$. Then, the following statements hold.*

(i) *The operator L_u^{sym} defined by (21) and (22) is selfadjoint on L^2_+ with domain $\mathrm{Dom}(L_u^{\mathrm{sym}})$ dense in $H^{1/2}_+$ (and L^2_+).*

(ii) *The operator L_u^{sym} has a compact resolvent and a discrete spectrum*

$$\mathrm{Spec}(L_u^{\mathrm{sym}}) = \{\lambda_0 \le \lambda_1 \le \cdots \le \lambda_n \le \lambda_{n+1} \le \cdots\} \tag{25}$$

that consists of infinitely many simple (real) eigenvalues such that $\lambda_n \to \infty$ *as* $n \to \infty$ *and*

$$\lambda_{n+1} \geq 1 + \lambda_n, \quad n \geq 0. \tag{26}$$

The corresponding normalized eigenfunctions $f_n \in H_+^{1/2}$ $(n \geq 0)$ *form an orthonormal basis in* L_+^2.

(iii) *The operators* L_u^{sym} *and* L_u *(cf. Theorem 2.1) have the same eigenvalues and root spaces. In particular, the eigenvalues* λ_n $(n \geq 0)$ *of* L_u *(when ordered as in (25)) are simple and depend real analytically on the potential* $u \in H_{r,0}^{-1/2,\sqrt{\log}}$.

(iv) *Take* $u_0 \in L_{r,0}^2$, $u \in B_{-1/2,\sqrt{\log}}^{r,0}(u_0)$, *and* $\lambda_\bullet \in \Lambda_{u_0} \cap \mathbb{R}$. *Then, there exist constants* $0 < c < C$ *such that for any* $f \in H_+^{1/2}$,

$$c\|f\|_{1/2}^2 \leq \big\langle (L_u - \lambda_\bullet) f \big| f \big\rangle \leq C(1 + |\lambda_\bullet|)\|f\|_{1/2}^2. \tag{27}$$

The constants in (27) can be chosen uniform in $u \in B_{-1/2,\sqrt{\log}}^{r,0}(u_0)$ *and* $\lambda_\bullet \in \Lambda_{u_0} \cap \mathbb{R}$.

Proof of Theorem 2.2. Assume that $u \in H_{r,0}^{-1/2,\sqrt{\log}}$. The fact that L_u^{sym} is selfadjoint is already proved. The density of the domain $\text{Dom}(L_u^{\text{sym}})$ in $H_+^{1/2}$ follows from (24) since L_+^2 is dense in $H_+^{-1/2}$ and $(L_u - \lambda_\bullet)^{-1} : H_+^{-1/2} \to H_+^{1/2}$ is a linear isomorphism. This proves item (i).

In order to prove (ii), recall that $C_u^{\text{sym}} \equiv (L_u^{\text{sym}} - \lambda_\bullet)^{-1} : L_+^2 \to L_+^2$ is compact and symmetric with respect to the scalar product $\langle \cdot, \cdot \rangle$ on L_+^2. Hence, we can apply the Hilbert-Schmidt theorem to conclude that there exists an orthonormal basis of eigenfunctions of C_u^{sym} in L_+^2. Since the kernel of C_u^{sym} is trivial, zero is not an eigenvalue of C_u^{sym}. Hence, there are infinitely many real eigenvalues μ_n $(n \geq 0)$ of C_u^{sym} that converge to zero in $\mathbb{R}$. In view of (23), we then conclude that $\mu_n = 1/(\lambda_n - \lambda_\bullet)$, where λ_n $(n \geq 0)$ is the spectrum of L_u^{sym}. The fact that the spectrum of L_u^{sym} is bounded below follows from the fact that any eigenfunction of L_u^{sym} is also an eigenfunction of L_u with the same eigenvalue. This follows directly from the definition (21) and the inclusion $\text{Dom}(L_u^{\text{sym}}) \subseteq H_+^{1/2} \equiv \text{Dom}(L_u)$. The simplicity of the spectrum of L_u^{sym} and the inequality (26) can be obtained from the max-min principle in the same way as in [6, Proposition 2.2].

Let us now prove item (iii). We already mentioned that any eigenfunction of L_u^{sym} is an eigenfunction of L_u with the same eigenvalue. Let $\lambda \in \mathbb{C}$ be an eigenvalue of L_u and let $V_\lambda \subseteq H_+^{1/2}$ be its (finite dimensional) root space. Since the root space is an invariant subspace of L_u and since the operator $L_u|_{V_\lambda} : V_\lambda \to V_\lambda$ is symmetric with respect to the restriction of the scalar product $\langle \cdot | \cdot \rangle$ to V_λ, we conclude that λ is real and V_λ consists of eigenvectors of L_u with eigenvalue λ. The same argument shows that the eigenspaces V_λ and V_μ of L_u corresponding to different eigenvalues $\lambda \neq \mu$ are orthogonal in L_+^2. Hence, if $\lambda \in \mathbb{R}$ is an eigenvalue of L_u that is not an eigenvalue of L_u^{sym} then its eigenfunction f is orthogonal to the eigenfunctions f_n $(n \geq 0)$ of L_u^{sym}, that contradicts the fact that f_n $(n \geq 0)$ is an orthonormal basis in L_+^2. Hence, λ is an eigenvalue of L_u^{sym}. A similar argument also shows that the eigenspaces V_λ of L_u are one dimensional. This proves the first statement in (iii). The analytic dependence of $\lambda_n \equiv \lambda_n(u)$ with $n \geq 0$ on $u \in H^{-1/2,\sqrt{\log}}_{r,0}$ then follows from Theorem 2.1 (ii), the simplicity of the eigenvalue λ_n, and the properties of the Riesz's projector.

(iv) Choose u, u_0, and $\lambda_\bullet$ as in item (iv). As in (17), we have

$$L_u - \lambda_\bullet = R(L_{u_0} - \lambda_\bullet) = (L_{u_0} - \lambda_\bullet)\widetilde{R}, \tag{28}$$

where

$$R := I - T_{\tilde{u}}(L_{u_0} - \lambda_\bullet)^{-1} \quad \text{and} \quad \widetilde{R} := I - (L_{u_0} - \lambda_\bullet)^{-1} T_{\tilde{u}} \tag{29}$$

and $\tilde{u} \equiv u - u_0$. It follows from (16) and (18) that

$$\|T_{\tilde{u}}(L_{u_0} - \lambda_\bullet)^{-1}\|_{H^{-1/2}_+ \to H^{-1/2}_+} < 1/2, \quad \|(L_{u_0} - \lambda_\bullet)^{-1} T_{\tilde{u}}\|_{H^{1/2}_+ \to H^{1/2}_+} < 1/2,$$

uniformly on the choice of $u \in B_{-1/2,\sqrt{\log}}$ and $\lambda_\bullet \in \Lambda_{u_0} \cap \mathbb{R}$. This implies that the operators $R : H^{-1/2}_+ \to H^{-1/2}_+$ and $\widetilde{R} : H^{1/2}_+ \to H^{1/2}_+$ are linear isomorphisms that have well-defined (as convergent power series) square roots

$$\sqrt{R} : H^{-1/2}_+ \to H^{-1/2}_+ \quad \text{and} \quad \sqrt{\widetilde{R}} : H^{1/2}_+ \to H^{1/2}_+$$

that are also linear isomorphisms. Since the potentials u, u_0 and the constant $\lambda_\bullet$ are real, we obtain from (29) that $\langle Rf | g \rangle = \langle f | \widetilde{R} g \rangle$ for any $f \in H^{-1/2}_+$ and $g \in H^{1/2}_+$. This, together with the second equality in (28) and the definition of the square roots as convergent power series implies that

$$\big\langle \sqrt{R} f \big| g \big\rangle = \big\langle f \big| \sqrt{\widetilde{R}} g \big\rangle \quad \text{and} \quad \sqrt{R}(L_{u_0} - \lambda_\bullet) = (L_{u_0} - \lambda_\bullet)\sqrt{\widetilde{R}}.$$

Then, for any $f \in H^{1/2}_+$,

$$\begin{aligned} \big\langle (L_u - \lambda_\bullet) f \big| f \big\rangle &= \big\langle R(L_{u_0} - \lambda_\bullet) f \big| f \big\rangle = \big\langle \sqrt{R}(L_{u_0} - \lambda_\bullet) f \big| \sqrt{\widetilde{R}} f \big\rangle \\ &= \big\langle (L_{u_0} - \lambda_\bullet)\sqrt{\widetilde{R}} f \big| \sqrt{\widetilde{R}} f \big\rangle. \end{aligned} \tag{30}$$

Item (iv) now follows from (30) and Lemma 2.1. ∎

Let us now take $u \in H^{-1/2,\sqrt{\log}}_{r,0}$ and choose $u_0 \in L^2_{r,0}$ and $\lambda_\bullet \in \Lambda_{u_0}$ as in Theorem 2.2 (iv). Following [6], we consider the nth *gap*

$$\gamma_n(u) := \lambda_n(u) - \lambda_{n-1}(u) - 1 \geq 0, \quad n \geq 1,$$

that is well defined by Theorem 2.2 (ii) and non-negative in view of (26). We have

$$0 \leq \sum_{k=1}^{n} \gamma_k(u) = \lambda_n(u) - \lambda_0(u) - n, \tag{31}$$

and hence,

$$\lambda_n(u) \geq n + \lambda_0(u), \quad n \geq 1. \tag{32}$$

It follows from the estimate (27) that $\lambda_0(u) > \lambda_\bullet$ which implies that

$$\lambda_0(u) \geq -\big(1 + \eta(\|u_0\|)\big) \tag{33}$$

in view of the arbitrariness of the choice of $\lambda_\bullet \in \Lambda_{u_0}$. Hence, for a given $\lambda_\bullet \in \Lambda_{u_0} \cap \mathbb{R}$, we have that

$$\lambda_0(u) - \lambda_\bullet \geq -\big(\lambda_\bullet + (1 + \eta(\|u_0\|))\big) > 0 \tag{34}$$

uniformly in $u \in B^{r,0}_{-1/2,\sqrt{\log}}(u_0)$. Theorem 2.1 allows us to define the meromorphic function (cf. [6])

$$\mathcal{H}_\lambda(u) := \big\langle (L_u + \lambda)^{-1} 1 \,\big|\, 1 \big\rangle$$

with poles at $\{-\lambda_n(u) \,|\, n \geq 0\}$. By arguing as in the proof of [6, Proposition 3.1], one sees that

$$\mathcal{H}_\lambda(u) = \frac{1}{\lambda_0 + \lambda} \prod_{n=1}^{\infty} \Big(1 - \frac{\gamma_n}{\lambda_n + \lambda}\Big),$$

where the infinite product converges absolutely and we set $\gamma_n \equiv \gamma_n(u)$, $n \geq 1$, and $\lambda_n \equiv \lambda_n(u)$, $n \geq 0$, for simplicity of notation. The arguments in the proof of [6, Proposition 3.1] also show that one has the trace formula

$$\sum_{n=1}^{\infty} \gamma_n(u) = -\lambda_0(u) \geq 0, \tag{35}$$

where the sequence $(\gamma_n(u))_{n\geq 1}$ is absolutely summable. As a consequence from (31) and (35), we obtain that $\lambda_n - n = -\sum_{k>n} \gamma_k$, and hence, $\lambda_n(u) \leq n$. By combining this with (32) and (33), we then conclude that

$$n - (1 + \eta(\|u_0\|)) \leq n + \lambda_0(u) \leq \lambda_n(u) \leq n, \quad n \geq 0, \tag{36}$$

uniformly in $u \in B^{r,0}_{-1/2,\sqrt{\log}}(u_0)$. Note that the estimate

$$n + \lambda_0(u) \leq \lambda_n(u) \leq n, \quad n \geq 0, \tag{37}$$

holds for any $u \in H^{-1/2,\sqrt{\log}}_{r,0}$. Further, note that the statements of Lemmas 2.5 and 2.7 in [6] are purely algebraic in nature and continue to hold for $u \in H^{-1/2,\sqrt{\log}}_{r,0}$ since, by Theorem 2.2, all the quantities involved are well defined. In particular, we obtain that

$$\langle f_0(u)|1\rangle \neq 0 \quad \text{and} \quad \langle f_n(u)|Sf_{n-1}(u)\rangle \neq 0, \quad n \geq 1,$$

where $f_n(u)$ $(n \geq 0)$ is an orthonormal basis of eigenfunctions of L_u in L^2_+ (see Theorem 2.2 (ii)). This allows us to choose the orthonormal basis $f_n(u)$ $(n \geq 0)$ in a unique way by imposing the conditions (cf. [6, Definition 2.8])

$$\langle f_0(u)|1\rangle > 0 \quad \text{and} \quad \langle f_n(u)|Sf_{n-1}(u)\rangle > 0, \quad n \geq 1. \tag{38}$$

In what follows, we will assume that $f_n(u)$, $n \geq 0$, denotes this particular orthonormal basis. Note that Theorem 2.1 (ii), the simplicity of the eigenvalues of L_u, and the properties of the Riesz's projector imply that for any given $n \geq 0$ the map

$$f_n : H^{-1/2,\sqrt{\log}}_{r,0} \to H^{-1/2}_+, \quad u \mapsto f_n(u), \tag{39}$$

is real-analytic[1].

As above, we fix $u_0 \in L^2_{r,0}$ and choose $\lambda_\bullet \in \Lambda_{u_0} \cap \mathbb{R}$. By Theorem 2.2 (iv), there exist constants $0 < c < C$ such that inequality (27) holds uniformly in $u \in B^{r,0}_{-1/2,\sqrt{\log}}(u_0)$. In particular, this implies that the sesquilinear form

$$\mathcal{Q}_{u,\lambda_\bullet} : H^{1/2}_+ \times H^{1/2}_+ \to \mathbb{C}, \quad (f,g) \mapsto \big\langle (L_u - \lambda_\bullet) f \big| g \big\rangle, \tag{40}$$

gives an equivalent Hilbert structure in $H^{1/2}_+$. Since the system of eigenfunction

$$\tilde{f}_n := f_n / \sqrt{\lambda_n - \lambda_\bullet}, \quad n \geq 0, \tag{41}$$

is complete in $H^{1/2}_+$ and orthonormal with respect to (40), we conclude that it is a basis in $H^{1/2}_+$. (Recall from (34) that $\lambda_n > \lambda_\bullet$.) By the Parseval's identity, for any $f \in H^{1/2}_+$,

$$\mathcal{Q}_{u,\lambda_\bullet}(f,f) = \sum_{n=0}^{\infty} \big|\mathcal{Q}_{u,\lambda_\bullet}(f,\tilde{f}_n)\big|^2 = \sum_{n=0}^{\infty} (\lambda_n - \lambda_\bullet) |\langle f | f_n \rangle|^2 .$$

This, together with (27), implies that

$$c\|f\|^2_{1/2} \leq \sum_{n=0}^{\infty} (\lambda_n - \lambda_\bullet) \, \big|\langle f | f_n \rangle\big|^2 \leq C(1 + |\lambda_\bullet|)\|f\|^2_{1/2} .$$

By combining this with (36), we then see that there exist constants $0 < \varkappa_1 < 1$ independent of the choice of $u \in B^{r,0}_{-1/2,\sqrt{\log}}(u_0)$ such that, for any $f \in H^{1/2}_+$,

$$\varkappa_1^2 \sum_{n=0}^{\infty} (\langle n \rangle + 1) |\hat{f}(n)|^2 \leq \sum_{n=0}^{\infty} (\langle n \rangle + 1) |\langle f | f_n \rangle|^2 \leq \frac{1}{\varkappa_1^2} \sum_{n=0}^{\infty} (\langle n \rangle + 1) |\hat{f}(n)|^2 . \tag{42}$$

Let us now consider the Fourier transform corresponding to the orthonormal basis f_n $(n \geq 0)$,

$$K_{u;0} : L^2_+ \to \ell^2_{\geq 0}, \quad f \mapsto \big(\langle f | f_n \rangle\big)_{n \geq 0}, \tag{43}$$

together with its restriction to $H^{1/2}_+$,

$$K_{u;1/2} := K_{u;0}\big|_{H^{1/2}_+} : H^{1/2}_+ \to \mathfrak{h}^{1/2}_{\geq 0} . \tag{44}$$

Note that the image of (44) is dense in $\mathfrak{h}^{1/2}_{\geq 0}$ since it contains all finite sequences $c_n \in \mathbb{C}$ $(0 \leq n \leq N)$ with $N \geq 0$. This, together with (42), implies that (44) is a linear isomorphism.

---

[1]Here, we ignore the complex structure on $H^{-1/2}_+$ and consider the space as real.

Remark 2.1. The argument above also shows that for any $u \in H^{-1/2,\sqrt{\log}}_{r,0}$ the orthonormal basis $f_n \in H^{1/2}_+$ $(n \geq 0)$ in L^2 gives a basis in $H^{1/2}_+$ such that for any $f \in H^{1/2}_+$ the Fourier series $f = \sum_{n\geq 0}\langle f|f_n\rangle f_n$ converges in $H^{1/2}_+$ and $(\langle f|f_n\rangle)_{n\geq 0} \in \mathfrak{h}^{1/2}_{\geq 0}$. Since (44) is an isomorphism we also see that for any given $(x_n)_{n\geq 0} \in \mathfrak{h}^{1/2}_{\geq 0}$ there exists $f \in H^{1/2}_+$ such that $\langle f|f_n\rangle = x_n$, $n \geq 0$.

We then interpolate between the maps (43) and (44) as well as their inverses $K^{-1}_{u;0}$: $\ell^2_{\geq 0} \to L^2_+$ and $K^{-1}_{u;1/2} : \mathfrak{h}^{1/2}_{\geq 0} \to H^{1/2}_+$ to conclude (cf. [17, Example 3, Appendix to Section IX.4]) that for any $u \in B^{r,0}_{-1/2,\sqrt{\log}}(u_0)$, $0 \leq \theta \leq 1$, and for any $f \in H^{1/2}_+$,

$$\varkappa_1^2 \sum_{n=0}^{\infty}(\langle n\rangle + 1)^{\theta}\,|\hat f(n)|^2 \leq \sum_{n=0}^{\infty}(\langle n\rangle + 1)^{\theta}\,|\langle f|f_n\rangle|^2 \leq \frac{1}{\varkappa_1^2}\sum_{n=0}^{\infty}(\langle n\rangle + 1)^{\theta}\,|\hat f(n)|^2.$$

By integrating these inequalities with respect to θ on the interval $0 \leq \theta \leq 1$, we obtain that for any $f \in H^{1/2}_+$,

$$\varkappa_1^2\,\|f\|^2_{1/2,1/\sqrt{\log}} \leq \sum_{n=0}^{\infty}\frac{\langle n\rangle}{\log(\langle n\rangle + 1)}\,\big|\langle f|f_n\rangle\big|^2 \leq \frac{1}{\varkappa_1^2}\,\|f\|^2_{1/2,1/\sqrt{\log}} \tag{45}$$

with $0 < \varkappa_1 < 1$ independent of the choice of $u \in B^{r,0}_{-1/2,\sqrt{\log}}(u_0)$. This inequality implies that the map

$$K_{u;1/2,1/\sqrt{\log}} : H^{1/2,1/\sqrt{\log}}_+ \to \mathfrak{h}^{1/2,1/\sqrt{\log}}_{\geq 0}, \quad f \mapsto \big(\langle f|f_n\rangle\big)_{n\geq 0}, \tag{46}$$

is a linear isomorphism. Hence, the map conjugate to (46) with respect to the L^2_+- and the $\ell^2_{\geq}$-pairing,

$$K^*_{u;1/2,1/\sqrt{\log}} : \mathfrak{h}^{-1/2,\sqrt{\log}}_{\geq 0} \to H^{-1/2,\sqrt{\log}}_+,$$

and its inverse

$$K_{u;-1/2,\sqrt{\log}} := \big(K^*_{u;1/2,1/\sqrt{\log}}\big)^{-1} : H^{-1/2,\sqrt{\log}}_+ \to \mathfrak{h}^{-1/2,\sqrt{\log}}_{\geq 0} \tag{47a}$$

are also linear isomorphisms. It is a straightforward task to see that

$$K_{u;-1/2,\sqrt{\log}}\,f = \big(\langle f|f_n\rangle\big)_{n\geq 0} \quad \forall f \in H^{-1/2,\sqrt{\log}}_+, \tag{47b}$$

and that, in view of (45),

$$\varkappa_1\|f\|_{-1/2,\sqrt{\log}} \leq \big\|K_{u;-1/2,\sqrt{\log}}\,f\big\|_{\mathfrak{h}^{-1/2,\sqrt{\log}}_{\geq 0}} \leq \frac{1}{\varkappa_1}\,\|f\|_{-1/2,\sqrt{\log}} \tag{48}$$

uniformly in $f \in H^{-1/2,\sqrt{\log}}_+$ and $u \in B^{r,0}_{-1/2,\sqrt{\log}}(u_0)$. Let

$$\widetilde{\mathrm{Dom}}(L_u) := (L_u - \lambda_\bullet)^{-1}\big(H^{-1/2,\sqrt{\log}}_+\big) \subseteq H^{1/2}_+$$

be the domain of the operator L_u in $H_+^{-1/2,\sqrt{\log}}$. For any $u \in H_{r,0}^{-1/2,\sqrt{\log}}$ and $\lambda \in \mathbb{R}$, we have the following commutative diagram:

$$
\begin{tikzcd}
H_+^{1/2} \arrow[d, "L_u-\lambda"] & \widetilde{\mathrm{Dom}}(L_u) \arrow[l, hook'] \arrow[r, "K_u^{(1)}"] \arrow[d, "L_u-\lambda"] & \mathfrak{h}_{\geq 0}^{1/2,\sqrt{\log}} \arrow[d, "\ell_u-\lambda"] \\
H_+^{-1/2} & H_+^{-1/2,\sqrt{\log}} \arrow[l, hook'] \arrow[r, "K_u^{(2)}"] \arrow[ur, "D_{u,\lambda}"] & \mathfrak{h}_{\geq 0}^{-1/2,\sqrt{\log}}
\end{tikzcd}
\tag{49}
$$

where $\ell_u : \mathfrak{h}_{\geq 0}^{1/2,\sqrt{\log}} \to \mathfrak{h}_{\geq 0}^{-1/2,\sqrt{\log}}$ is the multiplication $(z_n)_{n\geq 0} \mapsto \big(\lambda_n z_n\big)_{n\geq 0}$, and the maps $K_u^{(1)}$ and $K_u^{(2)}$ stand for $K_{u;1/2}|_{\widetilde{\mathrm{Dom}}(L_u)}$ and $K_{u;-1/2,\sqrt{\log}}$ (see (47a), (47b)). For λ in the resolvent set of L_u, there is a well-defined "diagonal" map (cf. (36))

$$D_{u,\lambda} := (\ell_u - \lambda)^{-1} K_{u;-1/2,\sqrt{\log}}. \tag{50}$$

The operator $D_{u;\lambda}$ will play an important role in our study of the Birkhoff map. The map $K_{u;-1/2,\sqrt{\log}}$ as well the maps $\ell_u - \lambda$, and $D_{u;\lambda}$ (for λ in the resolvent set of L_u) are linear isomorphisms.

Remark 2.2. By Corollary 5.2, the space $\widetilde{\mathrm{Dom}}(L_u)$ does *not* coincide with $H_+^{1/2,\sqrt{\log}}$. This is in contrast with the case of the scale $H_{r,0}^s$, $s > -1/2$ (cf. [10, Lemma 3.10]).

We will need an (improved) explicit formula for the constant appearing on the left-hand side of (48). To this end, we fix $u \in H_{r,0}^{-1/2,\sqrt{\log}}$ and consider the operator L_u on $H_+^{-1/2}$ with domain $H_+^{1/2}$ (cf. Theorem 2.1). It follows from Remark 2.1 that, for any $f \in H_+^{1/2}$,

$$\big\langle (L_u - \lambda_0) f \,\big|\, f \big\rangle = \sum_{n\geq 0} (\lambda_n - \lambda_0) |\langle f | f_n \rangle|^2 \geq 0,$$

where $\lambda_0 \equiv \lambda_0(u)$ is the first eigenvalue of L_u. This together with Corollary A.1 then implies that

$$
\begin{aligned}
\|f\|^2 &\leq \big\langle (L_u - \lambda_0 + 1) f \,\big|\, f \big\rangle = \langle -i\partial_x f | f \rangle - \big\langle \Pi(uf) \big| f \big\rangle + (-\lambda_0 + 1)\|f\|^2 \\
&\leq \big(2 - \lambda_0 + K_0 \|u\|_{-1/2,\sqrt{\log}}\big) \|f\|_{1/2}^2, \quad f \in H_+^{1/2}.
\end{aligned}
$$

Hence, for any $f \in H_+^{1/2}$,

$$\|f\|^2 \leq \big\langle (L_u - \lambda_0 + 1) f \,\big|\, f \big\rangle \leq M_u \|f\|_{1/2}^2,$$

where

$$M_u := 2 - \lambda_0 + K_0 \|u\|_{-1/2,\sqrt{\log}}. \tag{51}$$

This inequality together with (37) implies that there exists a constant $C_{\lambda_0} \geq 1$ (chosen uniformly on bounded sets of λ_0 in $\mathbb{R}_{\leq 0}$) such that, for any $f \in H_+^{1/2}$,

$$\|K_{u;1/2} f\|_{h_{\geq 0}^{1/2}}^2 \leq C_{\lambda_0} M_u \|f\|_{1/2}^2, \tag{52}$$

where $K_{u;1/2} : H_+^{1/2} \to \mathfrak{h}_{\geq 0}^{1/2}$ is the linear isomorphism (44). As above, we then interpolate between the maps $K_{u;1/2} : H_+^{1/2} \to \mathfrak{h}_{\geq 0}^{1/2}$ and $K_{u;0} : L_+^2 \to \ell_{\geq 0}^2$ to obtain that, for any $0 \leq \theta \leq 1$ and $f \in H_+^{1/2}$,

$$\sum_{n\geq 0} (\langle n\rangle + 1)^\theta |\langle f | f_n\rangle|^2 \leq (C_{\lambda_0} M_u)^\theta \sum_{n\geq 1} (\langle n\rangle + 1)^\theta |\hat{f}(n)|^2,$$

where, in order to accommodate the slight change of the weights, we choose $C_{\lambda_0} \geq 1$ larger if necessary. By integrating this inequality with respect to θ on the interval $0 \leq \theta \leq 1$, we conclude that for, any $f \in H_+^{1/2}$,

$$\sum_{n\geq 0} \frac{\langle n\rangle}{\log(\langle n\rangle + 1)} |\langle f | f_n\rangle|^2 \leq C_{\lambda_0} M_u \sum_{n\geq 1} \frac{\langle n\rangle}{\log(\langle n\rangle + 1)} |\hat{f}(n)|^2.$$

This implies that, for any $f \in H_+^{1/2,1/\sqrt{\log}}$,

$$\| K_{u;1/2,1/\sqrt{\log}} f \|^2_{\mathfrak{h}_{\geq 0}^{1/2,1/\sqrt{\log}}} \leq C_{\lambda_0} M_u \| f \|^2_{1/2,1/\sqrt{\log}}.$$

We then argue by duality (as in the proof of (48)) to conclude that

$$\| f \|^2_{-1/2,\sqrt{\log}} \leq C_{\lambda_0} M_u \| K_{u;-1/2,\sqrt{\log}} f \|^2_{\mathfrak{h}_{\geq 0}^{-1/2,\sqrt{\log}}}, \tag{53}$$

where $K_{u;-1/2,\sqrt{\log}} : H_{r,0}^{-1/2,\sqrt{\log}} \to \mathfrak{h}_{\geq 0}^{-1/2,\sqrt{\log}}$ is the linear isomorphism (47a), (47b). Further, we set $f \equiv \Pi u$ in (53) and use that $\langle \Pi u | f_n\rangle = -\langle L_u 1 | f_n\rangle = -\lambda_n \langle 1 | f_n\rangle$, $n \geq 0$, to conclude from (37) that

$$\frac{1}{2} \|u\|^2_{-1/2,\sqrt{\log}} \leq C_{\lambda_0} M_u \left\| \big(\langle 1 | f_n(u)\rangle\big)_{n\geq 0} \right\|^2_{\mathfrak{h}_{\geq 0}^{1/2,\sqrt{\log}}} \tag{54}$$

with (possibly different) $C_{\lambda_0} \geq 1$ chosen uniformly on bounded sets of λ_0 in $\mathbb{R}_{\leq 0}$. In view of (51), the estimate (54) can be written in the form

$$-\frac{1}{2} \|u\|^2_{-1/2,\sqrt{\log}} + A \|u\|_{-1/2,\sqrt{\log}} + B \geq 0,$$

where

$$A := K_0 C_{\lambda_0} \left\| \big(\langle 1 | f_n\rangle\big)_{n\geq 0} \right\|^2_{\mathfrak{h}_{\geq 0}^{1/2,\sqrt{\log}}}, \quad B := (2 - \lambda_0) C_{\lambda_0} \left\| \big(\langle 1 | f_n\rangle\big)_{n\geq 0} \right\|^2_{\mathfrak{h}_{\geq 0}^{1/2,\sqrt{\log}}}.$$

This implies that $\|u\|_{-1/2,\sqrt{\log}}$ is bounded by the two roots of the quadratic polynomial $-\frac{1}{2}z^2 + Az + B = 0$. Hence, for any given $R > 0$, we can choose the constant $C_{\lambda_0} \geq 1$ and a constant $C_R > 0$ such that, for any $u \in H_{r,0}^{-1/2,\sqrt{\log}}$ that satisfies

$$\max\Big(|\lambda_0(u)|, \left\| \big(\langle 1 | f_n(u)\rangle\big)_{n\geq 0} \right\|_{\mathfrak{h}_{\geq 0}^{1/2,\sqrt{\log}}} \Big) \leq R, \tag{55a}$$

we have that $C_{\lambda_0} M_u \le C_R$, and hence, by (50) and (53),

$$\big\| (D_{u,\lambda_0(u)-1})^{-1} \big\|_{\mathfrak{h}_{\ge 0}^{1/2,\sqrt{\log}} \to H_+^{-1/2,\sqrt{\log}}} < C_R. \tag{55b}$$

Summarizing the above, we obtain the main result in this section.

Proposition 2.1. *For any $u \in H_{r,0}^{-1/2,\sqrt{\log}}$, we have that $(\langle 1 | f_n(u)\rangle)_{n\ge 0} \in \mathfrak{h}_{\ge 0}^{1/2,\sqrt{\log}}$ and*

$$D_{u,\lambda_0(u)-1}\big(- \Pi u - \lambda_0(u) + 1\big) = \big(\langle 1 | f_n(u)\rangle\big)_{n\ge 0}, \tag{56}$$

where $D_{u,\lambda} : H_+^{-1/2,\sqrt{\log}} \to \mathfrak{h}_{\ge 0}^{1/2,\sqrt{\log}}$ is given by (50). Moreover, one has

(i) *for any $v \in H_{r,0}^{-1/2,\sqrt{\log}}$, there exist constant $C \equiv C_v > 0$ and an open neighborhood $U(v)$ of v in $H_{r,0}^{-1/2,\sqrt{\log}}$ such that*

$$\big\| D_{u,\lambda_0(u)-1} \big\|_{H_+^{-1/2,\sqrt{\log}} \to \mathfrak{h}_{\ge 0}^{1/2,\sqrt{\log}}} < C \tag{57}$$

for any $u \in U(v)$[2]*,*

(ii) *for any $R > 0$, there exists a constant $C_R > 0$ such that inequality (55b) holds for any $u \in H_{r,0}^{-1/2,\sqrt{\log}}$ that satisfies (55a).*

The following remark is needed for the proof of the properness of the Birkhoff map and its inverse.

Remark 2.3. Since for any $u \in H_{r,0}^{-1/2,\sqrt{\log}}$ the Fourier transform (43) is an isomorphism and since $K_{u;-1/2,\sqrt{\log}}|_{L^2_+} \equiv K_{u;0}$, it follows from (50) and (37) that for any set U in $H_{r,0}^{-1/2,\sqrt{\log}}$ such that $\lambda_0(u)$ is bounded uniformly in U there exists $C > 0$ such that

$$\big\| D_{u,\lambda_0(u)-1} \big\|_{L^2_+ \to \mathfrak{h}^1_{\ge 0}}, \ \big\| (D_{u,\lambda_0(u)-1})^{-1} \big\|_{\mathfrak{h}^1_{\ge 0} \to L^2_+} < C$$

for any $u \in U$.

Remark 2.4. The identity (56) and the estimates in Proposition 2.1 and Remark 2.3 can be interpreted as a "quasi-linearity" of the pre-Birkhoff map $u \mapsto (\langle 1 | f_n(u)\rangle)_{n\ge 0}$, $H_{r,0}^{-1/2,\sqrt{\log}} \to \mathfrak{h}_{\ge 0}^{1/2,\sqrt{\log}}$.

Proof of Proposition 2.1. For a given $u \in H_{r,0}^{-1/2,\sqrt{\log}}$, we set $\lambda \equiv \lambda_0(u) - 1$ in the diagram (49) and note that $\lambda_0(u) - 1$ is in the resolvent set of L_u. As above, we then choose $u_0 \in L^2_{r,0}$ such that $u \in B^{r,0}_{-1/2,\sqrt{\log}}(u_0)$. Let us first prove that $(\langle 1 | f_n(u)\rangle)_{n\ge 0} \in \mathfrak{h}_{\ge 0}^{1/2,\sqrt{\log}}$. To this end, note that $L_u 1 = -\Pi u \in H_+^{-1/2,\sqrt{\log}}$, and hence,

$$1 \in \widetilde{\mathrm{Dom}}(L_u).$$

[2]In fact, $U(v)$ can be taken to be an open ball in $H_{r,0}^{-1/2,\sqrt{\log}}$ of radius $1/(8K_0)$, where $K_0 > 0$ is the constant in Corollary A.1.

We then obtain from the commutative diagram (49) that

$$\big(\langle 1|f_n(u)\rangle\big)_{n\ge 0} = K_{u;1/2}\,1 = D_{u,\lambda_0(u)-1}\big(-\Pi u - \lambda_0(u) + 1\big) \in \mathfrak{h}_{\ge 0}^{1/2,\sqrt{\log}}.$$

This proves (56) and the fact that $(\langle 1|f_n(u)\rangle)_{n\ge 0} \in \mathfrak{h}_{\ge 0}^{1/2,\sqrt{\log}}$. Let us now prove (57). Since $D_{u,\lambda_0(u)-1} = (\ell_u - \lambda_0(u) + 1)^{-1} K_{u;-1/2,\sqrt{\log}}$, we conclude from (48), (37), and $\lambda_0(u) \le 0$, that there exists $C > 0$ such that, for any $f \in H_+^{-1/2,\sqrt{\log}}$,

$$\big\| D_{u,\lambda_0(u)-1} f \big\|_{\mathfrak{h}_{\ge 0}^{1/2,\sqrt{\log}}} \le C\, \| f \|_{H_+^{-1/2,\sqrt{\log}}}$$

uniformly in $u \in B^{r,0}_{-1/2,\sqrt{\log}}(u_0)$. This proves (i). Item (ii) is already proved. ∎

3. The Birkhoff map

In this section, we extend the Birkhoff map (5) for $u \in H_{r,0}^{-1/2,\sqrt{\log}}$ and prove Theorem 1.1 stated in the Introduction. For simplicity of notation, we will identify the (real) space $\mathfrak{h}_{r,0}^{-1/2,\sqrt{\log}}$ with the space $\mathfrak{h}_{+}^{0,\sqrt{\log}}$.

For a given $u \in H_{r,0}^{-1/2,\sqrt{\log}}$, consider the norming constants (cf. [6, Corollary 3.4])

$$\kappa_0(u) := \prod_{p\ge 1}\Big(1 - \frac{\gamma_p(u)}{\lambda_p(u) - \lambda_0(u)}\Big) \tag{58}$$

and

$$\kappa_n(u) := \frac{1}{\lambda_n(u) - \lambda_0(u)} \prod_{1\le p\ne n}\Big(1 - \frac{\gamma_p(u)}{\lambda_p(u) - \lambda_n(u)}\Big), \quad n \ge 1. \tag{59}$$

The infinite products converge absolutely in view of the absolute convergence in (35) and the fact that $|\lambda_p(u) - \lambda_n(u)| \ge 1$ for $p \ne n$ and $n, p \ge 1$ (cf. (26)). Note that for any $u \in H_{r,0}^{-1/2,\sqrt{\log}}$,

$$\kappa_n(u) > 0, \quad n \ge 0. \tag{60}$$

This follows since the infinite products converge and since by (26),

$$1 - \frac{\gamma_p(u)}{\lambda_p(u) - \lambda_n(u)} = \frac{\lambda_{p-1}(u) - \lambda_n(u) + 1}{\lambda_p(u) - \lambda_n(u)} > 0$$

for any $n, p \ge 0$, $p \ne n$. The following lemma is proved in Appendix A.

Lemma 3.1. *For any $n \ge 0$, the norming constant $\kappa_n(u)$ is well defined and depends continuously on the potential $u \in H_{r,0}^{-1/2,\sqrt{\log}}$. For any $v \in H_{r,0}^{-1/2,\sqrt{\log}}$, there exist an open neighborhood $U(v)$ of v in $H_{r,0}^{-1/2,\sqrt{\log}}$ and constants $0 < c < C$ such that*

$$c \le n\kappa_n(u) \le C \tag{61}$$

for any $u \in U(v)$ and $n \ge 1$.

Remark 3.1. The proof of Lemma 3.1 shows that the following version of the lemma holds: let U be a set in $H^{-1/2,\sqrt{\log}}_{r,0}$ such that the image of the map $U \to \ell^1_+$, $u \mapsto \big(\gamma_n(u)\big)_{n\ge 1}$, is a pre-compact set in ℓ^1_+. Then, there exist constants $0<c<C$ such that the inequality (61) holds for any $u\in U$ and $n\ge 1$. Moreover, $\kappa_0(u)$ is bounded uniformly for $u\in U$.

We can now extend the *Birkhoff map* (5) for $u\in H^{-1/2,\sqrt{\log}}_{r,0}$ by setting (cf. [6, formula (4.1)])

$$\Phi(u) := \big(\Phi_n(u)\big)_{n\ge 1}, \quad \Phi_n(u) := \frac{\langle 1|f_n(u)\rangle}{\sqrt{\kappa_n(u)}}, \quad n\ge 1. \tag{62}$$

Recall from [6, Corollary 3.4] that, for $u\in L^2_{r,0}$, we have

$$\big|\langle 1|f_0(u)\rangle\big|^2 = \kappa_0(u) \quad \text{and} \quad \big|\langle 1|f_n(u)\rangle\big|^2 = \gamma_n(u)\kappa_n(u), \quad n\ge 1. \tag{63}$$

Since the quantities $\gamma_n(u)$, $\kappa_n(u)$, and $\langle 1|f_n(u)\rangle$, are well defined and depend continuously on $u\in H^{-1/2,\sqrt{\log}}_{r,0}$ (cf. Theorem 2.2 (iii), Lemma 3.1, and (39)), the relations (63) continue to hold for $u\in H^{-1/2,\sqrt{\log}}_{r,0}$. In particular, we obtain from (62) and (63) that

$$\gamma_n(u) = \big|\Phi_n(u)\big|^2, \quad n\ge 1 \tag{64}$$

for $u\in H^{-1/2,\sqrt{\log}}_{r,0}$.

We are now ready to prove Theorem 1.1 stated in the Introduction. We have the following theorem.

Theorem 3.1. *The formula* (62) *defines a map*

$$\Phi:\quad H^{-1/2,\sqrt{\log}}_{r,0} \to \mathfrak{h}^{0,\sqrt{\log}}_+, \tag{65}$$

which is a homeomorphism.

Proof of Theorem 3.1. We prove the theorem in several steps.

(a) *The Birkhoff map* (65) *is well defined and injective.* The fact that (65) is well defined follows from the first statement of Proposition 2.1 and Lemma 3.1. The injectivity of (65) follows from the explicit formulas in [6, Lemma 4.2] and the arguments in [6, Proposition 4.3]. Note that all quantities appearing in [6, Proposition 4.3] are well defined for $u\in H^{-1/2,\sqrt{\log}}_{r,0}$. Denote by $\mathrm{Image}(\Phi)\subseteq \mathfrak{h}^{0,\sqrt{\log}}_+$ the image of (65) and consider the inverse map,

$$\Phi^{-1}:\mathrm{Image}\,\big(\Phi\big)\to H^{-1/2,\sqrt{\log}}_{r,0}. \tag{66}$$

(b) *The image of any pre-compact set with respect to the Birkhoff map* (65)*, and its inverse* (66)*, is pre-compact.* Let us first consider the case of the map (65). We will follow the arguments in the proof of [10, Proposition 2 (iii)]. Let K be a pre-compact set in $H^{-1/2,\sqrt{\log}}_{r,0}$. Without loss of generality, we will assume $K\subseteq U(v)$ where $v\in H^{-1/2,\sqrt{\log}}_{r,0}$

and $U(v)$ is an open neighborhood of v in $H^{-1/2,\sqrt{\log}}_{r,0}$ such that the statement of Proposition 2.1 (i), Remark 2.3, and Lemma 3.1 hold. Then, in view of (56) and (62), there exist a constant $C \equiv C_v > 0$ and a linear map $J_u : H^{-1/2,\sqrt{\log}}_+ \to \mathfrak{h}^{0,\sqrt{\log}}_+$ such that $J_u|_{L^2_+} : L^2_+ \to \mathfrak{h}^{1/2}_+$ and for any $u \in U(v)$,

$$\Phi(u) = J_u(\Pi u - \lambda_0(u) + 1) \tag{67}$$

and

$$\|J_u\|_{H^{-1/2,\sqrt{\log}}_+ \to \mathfrak{h}^{0,\sqrt{\log}}_+} \leq C, \quad \|J_u\|_{L^2_+ \to \mathfrak{h}^{1/2}_+} \leq C. \tag{68}$$

For any integer $N \geq 0$, consider the projections

$$\Pi_{\geq N} : H^{-1/2,\sqrt{\log}}_+ \to H^{-1/2,\sqrt{\log}}_+, \quad f \mapsto \sum_{n \geq N} \hat{f}(n)\, e^{\mathrm{inx}}$$

and

$$\Pi_{<N} : H^{-1/2,\sqrt{\log}}_+ \to H^{-1/2,\sqrt{\log}}_+, \quad f \mapsto \sum_{0 \leq n < N} \hat{f}(n)\, e^{\mathrm{inx}}$$

as well as the projections $\pi_{\geq N} : \mathfrak{h}^{0,\sqrt{\log}}_+ \to \mathfrak{h}^{0,\sqrt{\log}}_+$ and $\pi_{<N} : \mathfrak{h}^{0,\sqrt{\log}}_+ \to \mathfrak{h}^{0,\sqrt{\log}}_+$ defined in a similar way. Now, take $\varepsilon > 0$. By Lemma 3.2 below, there exists an integer $N_\varepsilon \geq 1$ and $R_\varepsilon > 0$ such that

$$\big\|\Pi_{\geq N_\varepsilon} u\big\|_{-1/2,\sqrt{\log}} \leq \varepsilon/(2C), \quad \big\|\Pi_{<N_\varepsilon}\big(u - \lambda_0(u) + 1\big)\big\|_{L^2_+} \leq R_\varepsilon \tag{69}$$

for any $u \in K$. (Note that $\lambda_0(u)$ is uniformly bounded on K since λ_0 depends continuously on $u \in H^{-1/2,\sqrt{\log}}_{r,0}$.) Then, by (67), for any $u \in K$,

$$\Phi(u) = J_u\big(\Pi_{<N_\varepsilon}(u - \lambda_0(u) + 1)\big) + J_u(\Pi_{\geq N_\varepsilon} u).$$

This implies that

$$\Phi(K) = \mathcal{I}_1 + \mathcal{I}_2, \tag{70}$$

where

$$\mathcal{I}_1 := \big\{J_u(\Pi_{<N_\varepsilon}(u - \lambda_0(u) + 1)) \mid u \in K\big\}, \quad \mathcal{I}_2 := \big\{J_u(\Pi_{\geq N_\varepsilon} u) \mid u \in K\big\},$$

and $\Phi(K)$ denotes the set $\{\Phi(u) \mid u \in K\}$. It follows from the second inequality in (68) and the second inequality in (69) that the set $\mathcal{I}_1$ is bounded in $\mathfrak{h}^{1/2}_+$, and hence, it is a *compact* set in $\mathfrak{h}^{0,\sqrt{\log}}_+$. Moreover, by the first inequality in (68) and the first inequality in (69), the set $\mathcal{I}_2$ is contained inside a centered at zero open ball of radius $\varepsilon/2$ in $\mathfrak{h}^{0,\sqrt{\log}}_+$. By applying Lemma 3.2 to the compact set $\mathcal{I}_1$ in $\mathfrak{h}^{0,\sqrt{\log}}_+$ and by taking $N_\varepsilon \geq 1$ and $R_\varepsilon > 0$ larger if necessary, we obtain from (70) that

$$\big\|\pi_{\geq N_\varepsilon} \Phi(K)\big\|_{\mathfrak{h}^{0,\sqrt{\log}}_+} \leq \big\|\pi_{\geq N_\varepsilon} \mathcal{I}_1\big\|_{\mathfrak{h}^{0,\sqrt{\log}}_+} + \big\|\pi_{\geq N_\varepsilon} \mathcal{I}_2\big\|_{\mathfrak{h}^{0,\sqrt{\log}}_+} \leq \varepsilon$$

and

$$\left\|\pi_{<N_\varepsilon}\Phi(K)\right\|_{\ell^2_+} \le R_\varepsilon.$$

This and Lemma 3.2 then imply that $\Phi(K)$ is a pre-compact set in $\mathfrak{h}_+^{0,\sqrt{\log}}$.

Let us now prove that the image of any pre-compact set with respect to (66) is pre-compact. Take a pre-compact set K in $\mathfrak{h}_+^{0,\sqrt{\log}}$. For any $u \in \Phi^{-1}(K)$, we have that $\Phi(u) \in K$, and hence, by (64), $\{(\gamma_n(u))_{n\ge1} \mid u \in \Phi^{-1}(K)\}$ is a pre-compact set in ℓ^1_+. By Remark 3.1, there exist constants $0 < c < C$ such that the inequality (61) holds for any $u \in \Phi^{-1}(K)$ and $n \ge 1$. It then follows from (62) and the trace formula (35) that there exists a constant $R > 0$ such that the condition (55a) holds for any $u \in \Phi^{-1}(K)$. By Proposition 2.1, we then conclude that there exists a constant $C_R > 0$ such that

$$-\Pi u - \lambda_0(u) + 1 = (D_{u,\lambda_0(u)-1})^{-1}\big(\langle 1|f_n(u)\rangle\big)_{n\ge0},$$

where

$$\left\|(D_{u,\lambda_0(u)-1})^{-1}\right\|_{\mathfrak{h}_{\ge0}^{1/2,\sqrt{\log}}\to H_+^{1/2,\sqrt{\log}}} \le C_R$$

for any $u \in \Phi^{-1}(K)$. This and the second inequality in Remark 2.3 imply that there exist a constant $C_K > 0$ and a linear map $Q_u : \mathfrak{h}_{\ge0}^{0,\sqrt{\log}} \to H_+^{-1/2,\sqrt{\log}}$ such that $Q_u|_{\mathfrak{h}_{\ge0}^{1/2}} : \mathfrak{h}_{\ge0}^{1/2} \to L^2_+$ and for any $u \in \Phi^{-1}(K)$,

$$\Pi u - \lambda_0(u) + 1 = Q_u\big(\langle 1|f_0(u)\rangle, \Phi(u)\big) \tag{71}$$

and

$$\|Q_u\|_{\mathfrak{h}_{\ge0}^{0,\sqrt{\log}}\to H_+^{-1/2,\sqrt{\log}}} \le C_K, \quad \|Q_u\|_{\mathfrak{h}_{\ge0}^{1/2}\to L^2_+} \le C_K. \tag{72}$$

Note in addition that by Remark 3.1 the quantity $|\langle 1|f_0(u)\rangle|^2 = \kappa_0(u)$ is bounded uniformly for $u \in \Phi^{-1}(K)$. The pre-compactness of $\Phi^{-1}(K)$ then follows from (71), (72), and Lemma 3.2, in exactly the same way as in the proof of the first part of (b).

(c) *The Birkhoff map* (65) *is continuous and onto.* Since for any given $n \ge 1$ the map (39) is continuous, we obtain from Lemma 3.1 and (62) that for any given $n \ge 1$ the component map

$$\Phi_n : H_{r,0}^{-1/2,\sqrt{\log}} \to \mathbb{C}$$

is continuous. Now, we take a sequence $(u_k)_{k\ge1}$ in $H_{r,0}^{-1/2,\sqrt{\log}}$ that converges to u in $H_{r,0}^{-1/2,\sqrt{\log}}$. Since the set $\{u_k \mid k \ge 1\}$ is pre-compact in $H_{r,0}^{-1/2,\sqrt{\log}}$, we conclude from (b) that $\{\Phi(u_k) \mid k \ge 1\}$ is pre-compact in $\mathfrak{h}_+^{0,\sqrt{\log}}$. This implies that any subsequence of $(\Phi(u_k))_{k\ge1}$ has a convergent subsequence. Since $\Phi_n(u_k) \to \Phi_n(u)$ as $k \to \infty$, we then conclude that $\Phi(u_k) \to \Phi(u)$ as $k \to \infty$ in $\mathfrak{h}_+^{0,\sqrt{\log}}$. The ontoness of the Birkhoff map then follows since (65) is continuous, proper, and has a dense image in $\mathfrak{h}_+^{0,\sqrt{\log}}$.

(d) *The map* $\Phi^{-1} : \mathfrak{h}_+^{0,\sqrt{\log}} \to H_{r,0}^{-1/2,\sqrt{\log}}$ *is continuous.* This statement follows from the arguments in (c) and the fact that the components of $\Phi^{-1} : \mathfrak{h}_+^{0,\sqrt{\log}} \to H_{r,0}^{-1/2,\sqrt{\log}}$ are continuous. The latter follows easily from [6, Lemma 4.2] and Cauchy's formula. ∎

In the proof of Theorem 3.1, we use the following characterization of pre-compact sets in $H_+^{-1/2,\sqrt{\log}}$ (and $\mathfrak{h}_+^{0,\sqrt{\log}}$). The proof follows easily from Cantor's diagonalization process.

Lemma 3.2. *A set K is pre-compact in $H_+^{-1/2,\sqrt{\log}}$ if and only if for any $\varepsilon > 0$ there exist an integer $N_\varepsilon \geq 0$ and $R_\varepsilon > 0$ such that, for any $u \in K$,*

$$\sum_{n \geq N_\varepsilon} \frac{\log(\langle n\rangle + 1)}{\langle n\rangle} |z_n|^2 \leq \varepsilon^2, \qquad \sum_{0 \leq n < N_\varepsilon} |z_n|^2 \leq R_\varepsilon^2,$$

where $z_n \equiv \hat{u}(n)$, $n \geq 0$.

A similar condition (that involves the weight $\log(\langle n\rangle + 1)$ instead of $\frac{\log(\langle n\rangle+1)}{\langle n\rangle}$) characterizes the pre-compact sets in $\mathfrak{h}_+^{0,\sqrt{\log}}$.

Corollary 1.1 follows from the arguments in [10, Section 5] (see also [11, Section 4]).

4. The lack of weak continuity of the flow map and of the Birkhoff map

In this section, we prove Proposition 1.1. For this, we revisit the counterexample to well-posedness in $H_{r,0}^{-1/2}(\mathbb{T})$ constructed in [10], of which we recall the setting.

We consider potentials of the form

$$u_{0,q}(x) = v_q(\mathrm{e}^{ix}) + \overline{v_q(\mathrm{e}^{ix})},$$

where v is the following Hardy function, defined in the unit disk by

$$v_q(z) = \frac{\varepsilon q z}{1 - qz}, \qquad 0 < \varepsilon < q < 1, \quad |z| < 1.$$

Note that

$$\|u_{0,q}\|^2_{-1/2,\sqrt{\log}} = 2\varepsilon^2 \sum_{n=1}^{\infty} n^{-1} \log(1+n) q^{2n} \sim \varepsilon^2 (\log(1-q))^2 \tag{73}$$

as q tends to 1. We choose

$$\varepsilon = \frac{\beta}{|\log(1-q)|}, \tag{74}$$

where $\beta > 0$ is a positive parameter which will be fixed later. Therefore, we have $\|u_{0,q}\|_{-1/2,\sqrt{\log}} \to \beta$ as $q \to 1$, and $u_{0,q}$ tends weakly to 0 in $H_{r,0}^{-1/2,\sqrt{\log}}$. The study of the Lax operator $L_{u_{0,q}}$ reduces to the study of a first order linear differential equation in the complex domain, which is processed in [10]. From this analysis, we infer that $-\mu$

is a negative eigenvalue of $L_{u_{0,q}}$ if and only if $F(\mu,q)=0$, where

$$
\begin{aligned}
F(\mu,q) &= F_+(\mu,q) - F_-(\mu,q),\\
F_+(\mu,q) &:= \int_0^q \frac{\mu t^{\varepsilon+\mu}(1-qt)^\varepsilon}{t(q-t)^\varepsilon}\,dt,\\
F_-(\mu,q) &:= \int_0^q \frac{\varepsilon q t^{\varepsilon+\mu}(1-qt)^\varepsilon}{(q-t)^\varepsilon(1-qt)}\,dt,
\end{aligned}
$$

where we recall that ε is given by (74). Moreover, $F(\mu,q)>0$ for $\mu=\varepsilon q^2/(1-q^2)$, and as $q\to 1$, for every fixed $\mu>0$,

$$
F_+(\mu,q)\to 1,\quad F_-(\mu,q)\sim -\varepsilon\log(1-q^2)\to\beta.
$$

Consequently, $F(\mu,q)\to 1-\beta$ as $q\to 1$. Let us now choose $\beta>1$. Then, we infer that $F(\mu,q)$ must vanish for some μ_q tending to $+\infty$ as q tends to 1. Furthermore, since $\partial_\mu F(\mu,q)>0$ if $F(\mu,q)=0$, we know that such a zero μ_q is unique. We conclude that $L_{u_{0,q}}$ has a unique negative eigenvalue $\lambda_0(u_{0,q})=-\mu_q$, and that this eigenvalue tends to $-\infty$. Consequently,

$$
\gamma_1(u_{0,q})=\lambda_1(u_{0,q})-\lambda_0(u_{0,q})\to+\infty,
$$

and therefore, the function γ_1 is not weakly continuous on $H^{-1/2,\sqrt{\log}}_{r,0}$. A fortiori, $\Phi: H^{-1/2,\sqrt{\log}}_{r,0}\to\mathfrak{h}^{0,\sqrt{\log}}_+$ is not weakly continuous.

Finally, we prove that the flow map is not weakly continuous in the same way than in [10]. Denote by u_q the Benjamin–Ono solution with the initial datum $u_{0,q}$. Then, it is proved in [10] that the function

$$
\xi_q(t)=\langle u_q(t)|\mathrm{e}^{ix}\rangle
$$

is bounded and satisfies, for every finite interval I,

$$
\left|\int_I \xi_q(t)\mathrm{e}^{-it(1-2\mu_q)}\,dt\right| = \sqrt{2}|I| + O\Big(\frac{1}{\mu_q}\Big).
$$

Hence, $\xi_q(t)$ cannot tend to 0 on any time interval of positive length. This completes the proof of Proposition 1.1.

5. The convolution in log-spaces

In this section, we discuss basic properties of the convolution in the spaces with logarithmic weights and prove Proposition 1.2 formulated in the Introduction.

We will first prove the following auxiliary lemma.

Lemma 5.1. *There exist* $(x_n)_{n\in\mathbb{Z}} \in \mathfrak{h}_{r,0}^{-1/2,\sqrt{\log}}$ *and* $(y_n)_{n\in\mathbb{Z}} \in \mathfrak{h}_{r,0}^{1/2,\sqrt{\log}}$ *such that the sequence* $z_n := \sum_{k\geq 0, k\neq n} x_k y_{n-k}$, $n \geq 1$, *does not belong to* $\mathfrak{h}_{+}^{-1/2,\sqrt{\log}}$.

Proof of Lemma 5.1. Assume that $(x_n)_{n\in\mathbb{Z}} \in \mathfrak{h}_{r,0}^{-1/2,\sqrt{\log}}$, $(y_n)_{n\in\mathbb{Z}} \in \mathfrak{h}_{r,0}^{1/2,\sqrt{\log}}$, and $z_n := \sum_{k\geq 0, k\neq n} x_k y_{n-k}$ for $n \geq 1$. Then, we can write

$$x_n = \frac{\sqrt{\langle n\rangle}}{\sqrt{\log(\langle n\rangle + 1)}} a_n, \quad y_n = \frac{1}{\sqrt{\langle n\rangle}\sqrt{\log(\langle n\rangle + 1)}} b_n, \quad z_n = \frac{\sqrt{\langle n\rangle}}{\sqrt{\log(\langle n\rangle + 1)}} c_n,$$

where $a := (a_n)_{n\in\mathbb{Z}} \in \ell^2_{r,0}$, $b := (b_n)_{n\in\mathbb{Z}} \in \ell^2_{r,0}$ and $c := (c_n)_{n\geq 1}$ is a complex-valued sequence. Since $z_n = \sum_{k\geq 0, k\neq n} x_k y_{n-k}$ we obtain that, for $n \geq 1$,

$$c_n = \sum_{k\geq 0, k\neq n} a_k \frac{b_{n-k}}{\sqrt{\langle n-k\rangle \log(\langle n-k\rangle + 1)}} B_{n,k}, \tag{75}$$

where

$$B_{n,k} := \left(\frac{\langle k\rangle \log(\langle n\rangle + 1)}{\langle n\rangle \log(\langle k\rangle + 1)}\right)^{1/2}. \tag{76}$$

The lemma will follow once we construct $(a_n)_{n\in\mathbb{Z}}, (b_n)_{n\in\mathbb{Z}} \in \ell^2_{r,0}$ such that $(c_n)_{n\geq 1} \notin \ell^2_+$. Assume that the elements of the sequences $a, b \in \ell^2_{r,0}$ are chosen real-valued and non-negative,

$$a_n \geq 0, \quad b_n \geq 0, \quad n \in \mathbb{Z}.$$

Note that the sequence $(\langle k\rangle / \log(\langle k\rangle + 1))_{k\geq 1}$ is monotone increasing. This together with (76) implies that there exists a constant $C > 0$ such that for any $n \geq 1$ and $n/2 < k < n$ we have that

$$B_{n,k} \geq B_{n,[n/2]} \geq C > 0,$$

where $[n/2]$ denotes the integer part of $n/2$. We then obtain from (75) that

$$\begin{aligned} \|c\|^2_{\ell^2_+} &\geq \sum_{n\geq 1} \Big(\sum_{n/2<k<n} a_k \frac{b_{n-k}}{\sqrt{\langle n-k\rangle \log(\langle n-k\rangle + 1)}} B_{n,k} \Big)^2 \\ &\geq C \sum_{n\geq 1} \Big(\sum_{n/2<k<n} a_k \frac{b_{n-k}}{\sqrt{\langle n-k\rangle \log(\langle n-k\rangle + 1)}} \Big)^2. \end{aligned} \tag{77}$$

Now, assume that the sequence $a \in \ell^2_{r,0}$ is chosen so that $(a_n)_{n\geq 1}$ is monotone decreasing. Then, in view of (77),

$$\|c\|^2_{\ell^2_+} \geq C \sum_{n\geq 1} a_n^2 \Big(\sum_{0<l<n/2} \frac{b_l}{\sqrt{\langle l\rangle \log(\langle l\rangle + 1)}} \Big)^2, \tag{78}$$

where we passed to the index $l := n - k$ in the internal sum. By choosing $b_0 := 0$ and

$$b_l := \frac{1}{\sqrt{\langle l\rangle \log(\langle l\rangle + 1)} (\log(\log(\langle l\rangle + 1)))^{3/4}}, \quad |l| \geq 1,$$

we see that $b \in \ell^2_{r,0}$ and by the integral test's estimate

$$\sum_{0<l<n/2} \frac{b_l}{\sqrt{\langle l\rangle \log(\langle l\rangle+1)}} = \sum_{0<l<n/2} \frac{1}{\langle l\rangle \log(\langle l\rangle+1)(\log(\log(\langle l\rangle+1)))^{3/4}} \\ \geq C_1(\log(\log(\langle n\rangle+1)))^{1/4} \tag{79}$$

for some positive constant $C_1 > 0$ independent of $n \geq 1$. Hence, by (78) and (79),

$$\|c\|^2_{\ell^2_+} \geq C_2 \sum_{n\geq 1}((\log(\log(\langle n\rangle+1)))^{1/4} a_n)^2 \tag{80}$$

for some constant $C_2 > 0$ independent of $n \geq 1$. If we now choose $a_0 := 0$ and

$$a_n := b_n \equiv \frac{1}{\sqrt{\langle n\rangle \log(\langle n\rangle+1)\big(\log(\log(\langle n\rangle+1))\big)^{3/4}}}, \quad |n| \geq 1, \tag{81}$$

we obtain that $a \in \ell^2_{r,0}$ and the series on the right-hand side of (80) diverges by the integral test. This completes the proof of the lemma. ■

For the proof of Proposition 1.2, we will need the following variant of Lemma 5.1. For $s > -1/2$, consider the quadratic form

$$\mathfrak{h}^s_{r,0} \to \mathfrak{h}^s_+, \quad x \mapsto Q(x) := \left(\frac{1}{\sqrt{n}} \sum_{k\geq 0, k\neq n} x_k \frac{x_{n-k}}{n-k}\right)_{n\geq 1}. \tag{82}$$

The quadratic form (82) is well defined and bounded by Lemma 3.1 in [8].

Lemma 5.2. *There exists $x \in \mathfrak{h}^{-1/2,\sqrt{\log}}_{r,0}$ such that $Q(x) \in \ell^2_+$ but $Q(x)$ does not belong to $\mathfrak{h}^{0,\sqrt{\log}}_+$.*

Proof of Lemma 5.2. We set $x_0 := 0$ and

$$x_n := \frac{\sqrt{\langle n\rangle}}{\sqrt{\log(\langle n\rangle+1)}}, \quad a_n = \frac{1}{\log(\langle n\rangle+1)(\log(\log(\langle n\rangle+1)))^{3/4}}, \quad |n| \geq 1,$$

where $(a_n)_{n\in\mathbb{Z}}$ is the sequence (81) from the proof of Lemma 5.1. The fact that the sequence $x := (x_n)_{n\in\mathbb{Z}}$ satisfies the conditions of the lemma follows easily from the proof of Lemma 5.1. ■

Corollary 5.1. *The quadratic form (82) cannot be extended to a bounded quadratic form $Q : \mathfrak{h}^{-1/2,\sqrt{\log}}_{r,0} \to \mathfrak{h}^{0,\sqrt{\log}}_+$.*

On a side note, let us also mention that the arguments in the proof of Lemma 5.1 above imply that in contrast to the boundedness of the maps (13) we have the following corollary.

Corollary 5.2. *There exist $u \in H_{r,0}^{-1/2,\sqrt{\log}}$ and $f \in H_+^{1/2,\sqrt{\log}}$ such that $T_u f \in H_+^{-1/2}$ but $T_u f \notin H_+^{-1/2,\sqrt{\log}}$.*

Let us now compute the second differential of the Birkhoff map (5) at $u = 0$. For simplicity of notation, we identify the (real) space $\mathfrak{h}_{r,0}^{\beta}$ with $\mathfrak{h}_+^{\beta}$, $\beta \in \mathbb{R}$, and write

$$\Phi : H_{r,0}^s \to \mathfrak{h}_+^{\frac{1}{2}+s}, \ u \mapsto \big(\Phi_n(u)\big)_{n\geq 1}, \quad s > -1/2. \tag{83}$$

Recall from [6] and [8, formula (93)] that the differential $d_0\Phi : H_{r,0}^s \to \mathfrak{h}_+^{\frac{1}{2}+s}$ of (83) at $u = 0$ coincides with the weighted Fourier transform $\xi \mapsto (-\frac{\widehat{\xi}(n)}{\sqrt{n}})_{n\geq 1}$. For the second differential $d_0^2\Phi$ of (83) at $u = 0$, we have the following lemma.

Lemma 5.3. *For $s > -1/2$ and for any $\xi \in H_{r,0}^s$,*

$$d_0^2\Phi(\xi) = \Big(-\frac{1}{\sqrt{n}} \sum_{k\geq 0, k\neq n} \widehat{\xi}(-k) \frac{\widehat{\xi}(k-n)}{k-n} \Big)_{n\geq 1}. \tag{84}$$

Proof of Lemma 5.3. We will follow the framework developed in [8, Section 4]. Assume that $s > -1/2$. By [8, formula (58)], for any u in an open neighborhood U of zero in $H_{r,0}^s$,

$$\Phi_n(u) = -\sqrt{n}\, \frac{a_n(u)}{\sqrt[+]{n\kappa_n(u)}}\, \Psi_n(u), \quad n \geq 1, \tag{85}$$

where $\Psi_n(u) : U \to \mathbb{C}$, $\kappa_n : U \to \mathbb{R}$, and $a_n : U \to \mathbb{C}$, are analytic maps (cf. [8, Proposition 3.1], [8, Lemmas 4.1 and 4.3]) and $\sqrt[+]{\cdot}$ denotes the branch of the square root defined by $\sqrt[+]{1} = 1$. Here, $\Psi_n(u) := \langle h_n(u), 1\rangle$ is the nth component of the pre-Birkhof map studied in [8, Section 3], $h_n(u) := P_n(u)e_n$, where $P_n \equiv P_n(u)$ is the Riesz projector onto the n-th eigenspace of the Lax operator $L_u \equiv D - T_u$, and $e_n := e^{\mathrm{inx}}$, $n \geq 0$ (cf. [8, formula (19)]). Since $h_n(0) = e^{\mathrm{inx}}$, $n \geq 0$, we conclude that $\Psi_n(0) = 0$, $n \geq 1$. This together with (26) and (29) in [8] implies that for any $\xi \in H_{r,0}^s$ and $n \geq 1$,

$$\Psi_n(0) = 0, \quad d_0\Psi_n(\xi) = \frac{\widehat{\xi}(-n)}{n}, \quad d_0^2\Psi_n(\xi) = -\frac{1}{n} \sum_{k\geq 0, k\neq n} \widehat{\xi}(-k) \frac{\widehat{\xi}(k-n)}{k-n}. \tag{86}$$

The norming constants $\kappa_n(u)$, $n \geq 0$, are given by the product representation (34) in [8], $\kappa_n(u) > 0$ for $u \in U$, and (see [6, Remark 5.2], [7, Corollary 6 (iv)])

$$\sqrt[+]{n\kappa_n(0)} = 1, \quad d_0\kappa_n = 0, \quad n \geq 0. \tag{87}$$

We will also need the norming constants $\mu_n(u) > 0$, $n \geq 1$, $u \in U$, given by the product representation (see, e.g., [6], [8, formula (35)])

$$\mu_n := \Big(1 - \frac{\gamma_n}{\lambda_n - \lambda_0}\Big) \prod_{k\geq 1, k\neq n} \Big(1 - \gamma_n \frac{\gamma_k}{(\lambda_{k-1} - \lambda_{n-1})(\lambda_k - \lambda_n)}\Big), \tag{88}$$

where $\gamma_n \equiv \gamma(u) := \lambda_n(u) - \lambda_{n-1}(u) - 1 \geq 0$ are the spectral gaps and $\lambda_n \equiv \lambda_n(u)$, $n \geq 0$, are the eigenvalues of the Lax operator L_u. Since the product (88) converges absolutely and locally uniformly on U (cf. [7, Theorem 3]) we can differentiate it term by term to conclude from $\lambda_n(0) = n$, $d_0\gamma_n = 0$, $n \geq 1$ (see [6, Remark 5.2]) that

$$\mu_n(0) = 1, \quad d_0\mu_n = 0, \quad n \geq 1. \tag{89}$$

Let us now turn our attention to the quantities $a_n(u)$, $n \geq 1$, defined recursively for $u \in U$ by

$$a_0(u) := \frac{\sqrt[+]{\kappa_0(u)}}{\langle h_0(u), 1\rangle}, \quad a_n(u) = \frac{\nu_n(u)}{\sqrt[+]{\mu_n(u)}} a_{n-1}(u), \quad n \geq 1, \tag{90}$$

where

$$\nu_n(u) := 1 + \frac{\delta_n(u)}{\alpha_n(u)}, \quad \alpha_n(u) := \langle P_n e_n | e_n\rangle, \quad \beta_n(u) := \langle P_n S P_{n-1} e_{n-1} | e_n\rangle, \tag{91}$$

$$\delta_n(u) := \beta_n(u) - \alpha_n(u), \tag{92}$$

and $S : H^{s+1}_+ \to H^{s+1}_+$ is the shift operator (cf. [8, Section 4]). Since $\alpha_n(0) = \beta_n(0) = \langle e_n | e_n\rangle = 1$, we conclude from (92) that $\delta_n(0) = 0$, $n \geq 1$. By combining this with the first formula in (91), we obtain that

$$\nu_n(0) = 1, \quad d_0\nu_n = d_0\delta_n, \quad n \geq 1. \tag{93}$$

It follows from (89), (93), and (90) that

$$a_n(0) = 1, \quad n \geq 0. \tag{94}$$

In order to compute the differential $d_0 a_0$, consider the Taylor's expansion of $\Psi_0(u) := \langle h_0(u), 1\rangle$ for $u \in U$ at zero

$$\begin{aligned}
\Psi_0(u) \equiv \langle P_0(u)1, 1\rangle &= -\frac{1}{2\pi i}\oint_{\partial D_0} \langle (L_u - \lambda)^{-1}1, 1\rangle \, d\lambda \\
&= \frac{1}{2\pi i}\sum_{m\geq 1}\oint_{\partial D_0} \big\langle [T_u(D-\lambda)^{-1}]^m 1, 1\big\rangle \frac{d\lambda}{\lambda} \\
&= -\frac{1}{2\pi i}\oint_{\partial D_0} \frac{\langle u, 1\rangle}{\lambda^2} \, d\lambda + \cdots ,
\end{aligned} \tag{95}$$

where $\cdots$ stands for terms of order ≥ 2 in u and ∂D_0 is the counterclockwise oriented boundary of the centered at zero closed disk of radius $1/3$ in $\mathbb{C}$ and the neighborhood U is chosen as in [8, Proposition 2.2]. Since the integral in (95) vanishes, we conclude that

$$\Psi_0(0) = \langle P_0(0)1, 1\rangle = 1, \quad d_0\Psi_0 = 0.$$

By combining this with (87), we obtain from the first formula in (90) that

$$d_0 a_0 = 0. \tag{96}$$

It follows from (89), (93), (94), and the second formula in (90) that

$$d_0 a_n = d_0 \delta_n + d_0 a_{n-1}, \quad n \geq 1.$$

Hence, we conclude from (96) that

$$d_0 a_n = \sum_{1 \leq k \leq n} d_0 \delta_k, \quad n \geq 1. \tag{97}$$

In order to compute $d_0\delta_n$, $n \geq 1$, we argue as follows. Recall from [8, Section 5] that the Taylor's expansion of $\delta_n(u)$ for $u \in U$ at zero is given by [8, formula (69)]. This implies that

$$d_0 \delta_n(u) = -\sum_{k \geq 0} \sigma_{n,k} C_{u,n}(k), \quad n \geq 1, \tag{98}$$

where

$$C_{u,n}(k) := \frac{1}{2\pi i} \oint_{\partial D_{n-1}} \frac{\hat{u}\big(k-(n-1)\big)}{(n-1)-\lambda} \frac{d\lambda}{k-\lambda} = \begin{cases} \frac{\hat{u}(k-(n-1))}{k-(n-1)}, & k \neq n-1, \\ 0, & k = n-1, \end{cases} \tag{99}$$

and $\sigma_{n,k}$ is the term of order zero in u in the expansion of $\langle P_n(u) S e_k | e_n\rangle$ for $u \in U$ at zero

$$\begin{aligned} \langle P_n(u) S e_k | e_n\rangle &= -\frac{1}{2\pi i} \sum_{r \geq 0} \oint_{\partial D_n} \big\langle (D-\lambda)^{-1} [T_u (D-\lambda)^{-1}]^r S e_k \big| e_n \big\rangle \, d\lambda \\ &= -\frac{1}{2\pi i} \oint_{\partial D_n} \frac{\langle e_{k+1} | e_n\rangle}{(k+1)-\lambda} \, d\lambda + \cdots, \quad n \geq 1, \end{aligned}$$

where $\cdots$ stands for terms of order ≥ 1 in u and ∂D_n is the counterclockwise oriented boundary of the centered at n closed disk of radius $1/3$ in $\mathbb{C}$. This implies that

$$\sigma_{n,k} = -\frac{1}{2\pi i} \oint_{\partial D_n} \frac{\langle e_{k+1} | e_n\rangle}{(k+1)-\lambda} \, d\lambda = \delta_{k,n-1}.$$

By combining this with (98) and (99), we obtain that $d_0\delta_n = 0$, $n \geq 1$. Hence, by (97),

$$d_0 a_n = 0, \quad n \geq 1. \tag{100}$$

Finally, the expression (84) for the second differential of (83) follows from the product rule applied twice to (85) together with (87), (94), and (100). ∎

Proof of Proposition 1.2. The proposition follows directly from Corollary 5.1, Lemma 5.3. In fact, take $s > -1/2$ and assume that the Birkhoff map (83) extends to a C^2-map

$$\Phi : H_{r,0}^{-1/2,\sqrt{\log}} \to \mathfrak{h}_+^{0,\sqrt{\log}}.$$

Then, its second differential at zero $d_0^2\Phi : H_{r,0}^{-1/2,\sqrt{\log}} \to \mathfrak{h}_+^{0,\sqrt{\log}}$ is a bounded extension of the second differential (84) of the map (83). Since this contradicts Corollary 5.1, we conclude that the map (83) cannot be extended to a C^2-map and, in particular, to an analytic map. ∎

A. Auxiliary results

In this appendix, we provide the proofs of several technical results used in the main body of the paper. We start with the following lemma on the pointwise multiplication of functions in $H^{1/2}_c$.

Lemma A.1. *For any $u, v \in H^{1/2}_c$, we have that $uv \in H^{1/2,1/\sqrt{\log}}_c$ and the map*

$$H^{1/2}_c \times H^{1/2}_c \to H^{1/2,1/\sqrt{\log}}_c, \quad (u,v) \mapsto uv$$

is bounded.

Proof of Lemma A.1. The lemma easily follows by using the dyadic decomposition of functions (see, e.g., [1, Chapter II]). Below, we give the proof for the reader's convenience. For $f \in \mathcal{D}'(\mathbb{T})$, we set

$$f_{-1} := \hat{f}(0), \quad f_n := \sum_{2^{n-1}<|k|<2^{n+1}} \varphi\big(k/2^n\big)\hat{f}(k)\, e^{ikx}, \quad n \geq 0,$$

where $\varphi(\xi) := \psi(\xi/2) - \psi(\xi)$, $\psi \in C^\infty_c(\mathbb{R})$ has non-negative values, $\psi(\xi) = 1$ for $|\xi| \leq 1/2$, and $\psi(\xi) = 0$ for $|\xi| \geq 1$. Then, the functions $\psi(\xi)$ and $\varphi(\xi/2^n)$, $n \geq 0$, provide a partition of unity of $\mathbb{R}$. As in the case on the line one then sees that $f \in H^s_c$, $s \in \mathbb{R}$, if and only if $(2^{ns}\,\|f_n\|)_{n\geq -1} \in \ell^2_{\geq -1}$. The norm on H^s_c and the norm $\|f\|_s := (\sum_{n\geq -1} 2^{2ns}\,\|f_n\|^2)^{1/2}$ are equivalent. Similarly, $f \in H^{s,1/\sqrt{\log}}_c$ with $s \in \mathbb{R}$ if and only if $(\frac{2^{ns}}{\sqrt{\langle n\rangle}}\,\|f_n\|)_{n\geq -1} \in \ell^2_{\geq -1}$, and the corresponding norms are equivalent. For $u, v \in H^{1/2}_c$, we write

$$uv = \sum_{m,n\geq -1} u_n v_m = \sum_{n\geq -1} (S_n u) v_n + \sum_{m\geq -1} u_m (S_{m+1} v), \tag{101}$$

where $S_n f := \sum_{-1\leq k\leq n-1} f_k$. We have

$$\begin{aligned}\|S_n u\|_{L^\infty} &\leq \sum_{|k|\leq 2^n} |\hat{u}(k)| \leq C_1 \Big(\sum_{|k|\leq 2^n} \frac{1}{k}\Big)^{1/2} \|u\|_{1/2} \\ &\leq C_2 \sqrt{\langle n\rangle}\, \|u\|_{1/2}\end{aligned}$$

with constants $C_1, C_2 > 0$ independent of $n \geq -1$. Hence,

$$\|(S_n u) v_n\| \leq \|S_n u\|_{L^\infty} \|v_n\| \leq C_2\, \|u\|_{1/2} \sqrt{\langle n\rangle}\, \|v_n\|$$

and from the dyadic characterization of $H^{1/2}_c$,

$$\frac{2^{n/2}}{\sqrt{\langle n\rangle}}\, \|(S_n u) v_n\| \leq C_3\, \|u\|_{1/2}\, c_n,$$

where $\sum_{n\geq -1} c_n^2 = 1$ and $C_3 > 0$ is independent of $n \geq -1$. By arguing as in the proof of [1, Lemma 2.1], we then conclude that the first sum on the right-hand side of (101) belongs to $H_c^{1/2,1/\sqrt{\log}}$ and

$$\Big\| \sum_{n\geq -1} (S_n u) v_n \Big\|_{1/2,1/\sqrt{\log}} \leq C\, \|u\|_{1/2} \|v\|_{1/2}$$

with a constant $C > 0$ independent of the choice of $u, v \in H_c^{1/2}$. The second sum on the right-hand side of (101) is treated in the same way. This completes the proof of the lemma. ∎

As a corollary from Lemma A.1, we obtain the following corollary.

Corollary A.1. *For any $u \in H_c^{-1/2,\sqrt{\log}}$ and $v \in H_c^{1/2}$, we have that $uv \in H_c^{-1/2}$ and the map*

$$H_c^{-1/2,\sqrt{\log}} \times H_c^{1/2} \to H_c^{-1/2}, \quad (u,v) \mapsto uv$$

is bounded. In particular, there exists a positive constant $K_0 > 0$ such that $\|uv\|_{-1/2} \leq K_0 \|u\|_{-1/2,\sqrt{\log}} \|v\|_{1/2}$ for any $u \in H_c^{-1/2,\sqrt{\log}}$ and $v \in H_c^{1/2}$.

The corollary follows easily by duality form Lemma A.1.

Proof of Lemma 3.1. Recall from (35) that, for any $u \in H_{r,0}^{-1/2,\sqrt{\log}}$,

$$\sum_{p=1}^{\infty} \gamma_p(u) = -\lambda_0(u), \tag{102}$$

where $\lambda_0(u)$ and $\gamma_p(u) \geq 0$, $p \geq 1$, depend continuously on $u \in H_{r,0}^{-1/2,\sqrt{\log}}$ (Theorem 2.2 (iii)). Then, by Dini's theorem (see, e.g., [19, Theorem 8, Chapter 4]), the series in (102) converges uniformly on compact sets of u's in $H_{r,0}^{-1/2,\sqrt{\log}}$. The continuity of (58) and (59) then follows from the uniform convergence of the infinite products and the continuity of the quantities involved. Let us now prove (61). For any $u \in H_{r,0}^{-1/2,\sqrt{\log}}$ and $n \geq 1$, we have

$$\begin{aligned} \sum_{p\geq 1, p\neq n} \frac{\gamma_p(u)}{|\lambda_p(u) - \lambda_n(u)|} &\leq \sum_{|n-p|>\frac{n}{2}} \frac{\gamma_p(u)}{|p-n|} + \sum_{|n-p|\leq \frac{n}{2}, p\neq n} \frac{\gamma_p(u)}{|p-n|} \\ &\leq \frac{2}{n}(-\lambda_0(u)) + \sum_{p\geq \frac{n}{2}} \gamma_p(u), \end{aligned} \tag{103}$$

where we use that $|\lambda_p - \lambda_n| \geq |p-n|$ by (26). Let us now pick $v \in H_{r,0}^{-1/2,\sqrt{\log}}$ and $\varepsilon > 0$. Then, we can choose $n_0 \geq 1$ so that $\frac{2}{n_0}(-\lambda_0(v)) \leq \varepsilon/4$ and $\sum_{p\geq \frac{n_0}{2}} \gamma_p(v) \leq \varepsilon/4$. By the continuity of $\lambda_0(u)$ and $\sum_{p\geq \frac{n_0}{2}} \gamma_p(u)$ with respect to $u \in H_{r,0}^{-1/2,\sqrt{\log}}$, we then obtain that there exists an open neighborhood $U(v)$ of v in $H^{-1/2,\sqrt{\log}}_{r,0}$ such that the expression on the right-hand side of (103) is bounded above by ε uniformly in $u \in U(v)$ and $n \geq n_0$. This proves that

$$\sum_{p\geq 0, p\neq n} \frac{\gamma_p(u)}{|\lambda_p(u) - \lambda_n(u)|} \leq \varepsilon \tag{104}$$

for any $u \in U(v)$ and $n \geq n_0$. The existence of the constants $0 < c < C$ and the estimate (61) for $n \geq n_0$ then follows from (104), (59), and the continuous dependence of the eigenvalues on the potential $u \in H^{-1/2,\sqrt{\log}}_{r,0}$.

The case $1 \leq n < n_0$ follows from (60) and the continuous dependence of $\kappa_n(u)$ on $u \in H^{-1/2,\sqrt{\log}}_{r,0}$. ∎

References

[1] S. Alinhac and P. Gérard, *Pseudo-differential operators and the Nash–Moser theorem*. Grad. Stud. Math. 82, American Mathematical Society, Providence, RI, 2007 Zbl 1121.47033 MR 2304160

[2] T. Benjamin, Internal waves of permanent form in fluids of great depth. *J. Fluid Mech.* **29** (1967), no. 3, 559–592 Zbl 0147.46502

[3] T. L. Bock and M. D. Kruskal, A two-parameter Miura transformation of the Benjamin–Ono equation. *Phys. Lett. A* **74** (1979), no. 3-4, 173–176 MR 0591320

[4] R. Davis and A. Acrivos, Solitary internal waves in deep water. *J. Fluid Mech.* **29** (1967), no. 3, 593–607 Zbl 0147.46503

[5] P. Gérard, An explicit formula for the Benjamin–Ono equation. *Tunis. J. Math.* **5** (2023), no. 3, 593–603 Zbl 1531.37066 MR 4662323

[6] P. Gérard and T. Kappeler, On the integrability of the Benjamin–Ono equation on the torus. *Comm. Pure Appl. Math.* **74** (2021), no. 8, 1685–1747 Zbl 1471.35354 MR 4275336

[7] P. Gérard, T. Kappeler, and P. Topalov, On the spectrum of the Lax operator of the Benjamin–Ono equation on the torus. *J. Funct. Anal.* **279** (2020), no. 12, article no. 108762 Zbl 1452.37070 MR 4155287

[8] P. Gérard, T. Kappeler, and P. Topalov, On the analytic Birkhoff normal form of the Benjamin–Ono equation and applications. *Nonlinear Anal.* **216** (2022), article no. 112687 Zbl 1521.37085 MR 4348313

[9] P. Gérard, T. Kappeler, and P. Topalov, On the Benjamin–Ono equation on $\mathbb{T}$ and its periodic and quasiperiodic solutions. *J. Spectr. Theory* **12** (2022), no. 1, 169–193 Zbl 1495.37060 MR 4404811

[10] P. Gérard, T. Kappeler, and P. Topalov, Sharp well-posedness results of the Benjamin–Ono equation in $H^s(\mathbb{T}, \mathbb{R})$ and qualitative properties of its solutions. *Acta Math.* **231** (2023), no. 1, 31–88 Zbl 1533.35061 MR 4652410

[11] T. Kappeler and P. Topalov, Global wellposedness of KdV in $H^{-1}(\mathbb{T}, \mathbb{R})$. *Duke Math. J.* **135** (2006), no. 2, 327–360 Zbl 1106.35081 MR 2267286

[12] R. Killip, T. Laurens, and M. Vişan, Sharp well-posedness for the Benjamin–Ono equation. *Invent. Math.* **236** (2024), no. 3, 999–1054 Zbl 1543.35207 MR 4743514

[13] C. Klein and J.-C. Saut, *Nonlinear dispersive equations—inverse scattering and PDE methods*. Appl. Math. Sci. 209, Springer, Cham, 2021 Zbl 1506.35002 MR 4400881

[14] L. Molinet, Global well-posedness in L^2 for the periodic Benjamin–Ono equation. *Amer. J. Math.* **130** (2008), no. 3, 635–683 MR 2418924

[15] L. Molinet and D. Pilod, The Cauchy problem for the Benjamin–Ono equation in L^2 revisited. *Anal. PDE* **5** (2012), no. 2, 365–395 Zbl 1273.35096 MR 2970711

[16] A. Nakamura, A direct method of calculating periodic wave solutions to nonlinear evolution equations. I. Exact two-periodic wave solution. *J. Phys. Soc. Japan* **47** (1979), no. 5, 1701–1705 Zbl 1334.35006 MR 0552573

[17] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press Harcourt Brace Jovanovich, New York, 1975 Zbl 0308.47002 MR 0493420

[18] J.-C. Saut, Benjamin–Ono and intermediate long wave equations: Modeling, IST and PDE. In *Nonlinear dispersive partial differential equations and inverse scattering*, pp. 95–160, Fields Inst. Commun. 83, Springer, New York, 2019 Zbl 1442.35355 MR 3931835

[19] H. Widom, *Lectures on integral equations*. Van Nostrand Mathematical Studies 17, Van Nostrand Reinhold, New York, 1969 Zbl 0181.12101 MR 0243299

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