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The word problem for some classes of Adian inverse semigroups – II

Muhammad Inam

Abstract. We introduce the notion of a subgraph generated by an R-word r of the Schützenberger graph of a positive word w, $S\Gamma(w)$, where w contains r as its subword. We show that the word problem for a finitely presented Adian inverse semigroup $\mathrm{Inv}\langle X|R\rangle$ is decidable if the subgraphs of $S\Gamma(t)$, for all $t \in X^+$, generated by all the R-words over the presentation $\langle X|R\rangle$, are finite. As a consequence of this result, we show that the word problem is decidable for some classes of one relation Adian inverse semigroups.

1. Introduction

Throughout this paper, X denotes an alphabet. The set $R = \{(u_i, v_i) \mid i \in I\}$, where $u_i, v_i \in X^+$, denotes the set of *positive relations*. The words u_i and v_i are called *R-words*, provided that $(u_i, v_i) \in R$. The pair $\langle X|R\rangle$ is called a *positive presentation*. The semigroup generated by the set X and having a set R of relations is denoted by $\mathrm{Sg}\langle X|R\rangle$, and the group generated by the set X and having a set R of relations is denoted by $\mathrm{Gp}\langle X|R\rangle$. There exists a natural homomorphism $\phi : \mathrm{Sg}\langle X|R\rangle \to \mathrm{Gp}\langle X|R\rangle$.

We can construct two undirected graphs corresponding to a positive presentation. The *left graph* of the presentation $\langle X|R\rangle$ is denoted by $\mathrm{LG}\langle X|R\rangle$. The vertices of $\mathrm{LG}\langle X|R\rangle$ are labeled by the elements of X, and there is an edge corresponding to every relation $(u_i, v_i) \in R$, that connects the first (prefix) letters of u_i and v_i together. Similarly, the *right graph* of the presentation $\langle X|R\rangle$ is denoted by $\mathrm{RG}\langle X|R\rangle$, and it can be obtained by connecting the last (suffix) letters of u_i and v_i together, for every $(u_i, v_i) \in R$. A closed path in $\mathrm{LG}\langle X|R\rangle$ is called a *left cycle* and a closed path in $\mathrm{RG}\langle X|R\rangle$ is called a *right cycle*. Further details about the left and right graphs of a positive presentation $\langle X|R\rangle$ along with some examples can be found in [3]. If for

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some presentation $\langle X|R\rangle$, there is no closed path (cycle) in $\mathrm{LG}\langle X|R\rangle$ and $\mathrm{RG}\langle X|R\rangle$, then the presentation is called a *cycle free* presentation. A cycle free presentation is also called an Adian presentation because these presentations were first studied by S. I. Adian [1]. A semigroup (group) presented by an Adian presentation is called an *Adian semigroup (Adian group)*.

Meakin [8] proved that for a given finite positive presentation $\langle X|R\rangle$, it is undecidable whether $\mathrm{Sg}\langle X|R\rangle$ is an Adian semigroup, whether $\mathrm{Inv}\langle X|R\rangle$ is an Adian inverse semigroup, or whether $\mathrm{Gp}\langle X|R\rangle$ is an Adian group.

A semigroup S is called an *inverse semigroup* if for every element $a \in S$ there exists a unique element $b \in S$ such that $aba = a$ and $bab = b$. The unique element b is denoted by a^{-1}. The idempotents commute in an inverse semigroup, and the product of two idempotents is an idempotent. The *natural partial order* on an inverse semigroup S is defined as $a \leq b$ if and only if $aa^{-1}b = a$, for some $a, b \in S$. A congruence relation σ is defined on S, for $a, b \in S$, by $a\sigma b$ if and only if there exists an element $c \in S$ such that $c \leq a, b$. It turns out that σ is the minimum group congruence relation on S, i.e., S/σ is the maximum group homomorphic image of S. Just like groups and semigroups, an inverse semigroup can also be presented by a set of generators and a set of relations. We denote an inverse semigroup by $\mathrm{Inv}\langle X|R\rangle$, where X is the set of generators, and R is the set of relations. An inverse semigroup presented by an Adian presentation is called an *Adian inverse semigroup*. If $S = \mathrm{Inv}\langle X|R\rangle$, then S/σ is the group $\mathrm{Gp}\langle X|R\rangle$. The set of idempotents of S is denoted by $E(S) = \{e \in S \mid e^2 = e\}$. An inverse semigroup is called *E-unitary* if the inverse image of 1 (the identity element of the group S/σ) is precisely $E(S)$. All these facts about the inverse semigroups along with more details can be found in the text [6].

In [2], Gray proved that the word problem for one relator E-unitary inverse semigroups is undecidable in general.

Adian conjectured [1] that the word problem is decidable for Adian semigroups. The following result was first proved by Adian [1] for only finite Adian presentations. Later Remmers [9] proved the same result by using geometric techniques for any Adian presentation.

Theorem 1.1. *An Adian semigroup embeds in the Adian group with the same presentation.*

Magnus [7] proved that the word problem is decidable for one relator groups. Since Adian semigroups embed in Adian groups, it follows that the word problem is decidable for one relation Adian semigroups as well. However, the question whether the word problem is decidable for one relation Adian inverse semigroups or not, has not been answered yet.

The following theorem is proved in [5].

Theorem 1.2. *Adian inverse semigroups are E-unitary.*

By using Theorem 1.2, we can make the following remark about one relation Adian inverse semigroups.

Remark 1.1. Let $M = \mathrm{Inv}\langle X|u = v\rangle$ be an Adian inverse semigroup. Then the membership problem for the set of idempotents of M, $E(M)$, is decidable, because a word w represents an element of $E(M)$ iff $w/\sigma = 1$ in $\mathrm{Gp}\langle X|u = v\rangle$ (by E-unitary) which is decidable by Magnus.

The following result has been proved in [4].

Theorem 1.3. *Let $M = \mathrm{Inv}\langle X|R\rangle$ be a finitely presented Adian inverse semigroup. Then the Schützenberger graph of w, for all $w \in (X \cup X^{-1})^+$, is finite if and only if the Schützenberger graph of w' is finite, for all $w' \in X^+$.*

In this paper, we prove the following result.

Theorem 1.4. *Let $M = \mathrm{Inv}\langle X|R\rangle$ be a finitely presented Adian inverse semigroup. The Schützenberger graph of every positive word is finite if and only if the subgraphs of $S\Gamma(w)$, for all $w \in X^+$, generated by all the R-words are finite.*

The proof of Theorem 1.4 is given in Section 3 of this paper, where this theorem is labeled as Theorem 3.2. It follows from Theorem 1.3 and 1.4 that in a finitely presented Adian inverse semigroup $M = \mathrm{Inv}\langle X|R\rangle$, if the subgraphs of the Schützenberger graph of w', for all $w' \in X^+$, generated by all the R-words, are finite, then the Schützenberger graph of w, for all $w \in (X \cup X^{-1})^+$, is finite, which implies the decidability of the word problem for M [10].

In this paper, we also study the word problem for one relation Adian inverse semigroups and show that the word problem is decidable for those classes of one relation Adian inverse semigroups that are mentioned in Theorem 1.5 below. We apply Theorem 1.4 in the proof of Theorem 1.5 which shows the usefulness of Theorem 1.4.

Theorem 1.5. *Let $M = \mathrm{Inv}\langle X|u = v\rangle$ be an Adian inverse semigroup, such that no R-word is a subword of the other R-word, and the relation (u, v) is in one of the following forms:*

(1) *No R-word overlaps with itself or with the other R-word.*

(2) *One of the R-words overlaps with itself, and the other R-word neither overlaps with itself nor with the former R-word.*

(3) *Both R-words overlap with themselves, there is no overlap between both the R-words, and at least one of the R-words is not of the form x^n, for some $x \in X^+$ and $n \geq 2$.*

(4) *A prefix of one R-word is a suffix of the other R-word, no suffix of the former R-word is a prefix of the latter R-word, and no R-word overlaps with itself.*

Then the word problem is decidable for M.

The proof of Theorem 1.5 is given in Section 4 of this paper, where this theorem is labeled as Theorem 4.1.

Remark 1.2. In the proof of Theorem 1.5 we show that in any of the cases (1)–(4) all the Schützenberger graphs are finite, and hence the word problem is decidable.

For the convenience of the readers we provide an example of the inverse semigroup presented by the Adian presentation of each type (1)–(4) of Theorem 1.5. These examples mutually exclusive in the sense none of the following Adian inverse semigroups belong to more than one case.

(1) $M = \mathrm{Inv}\langle a, b, c, d \,|\, ab = cd\rangle$ is an example of case (1).

(2) $M = \mathrm{Inv}\langle a, b, c, d \,|\, aba = cd\rangle$ is an example of case (2).

(3) $M = \mathrm{Inv}\langle a, b \,|\, aba = b^2\rangle$ is an example of case (3).

(4) $M = \mathrm{Inv}\langle a, b, c \,|\, ab = ca\rangle$ is an example of case (4).

In general, one relation Adian inverse semigroup $\mathrm{Inv}\langle X \,|\, u = v\rangle$, where no R-word is a subword of the other R-word, may contain some positive words with infinite Schützenberger graphs. For instance, in the Adian inverse semigroup $M = \mathrm{Inv}\langle a, b \,|\, ab = b^2a^2\rangle$, the word a^2b has an infinite Schützenberger graph. Therefore, some extra condition like (1)–(4) of Theorem 1.5 is needed to make all the Schützenberger graphs finite.

In this paper, we study the word problem for a finitely presented Adian inverse semigroup.

Section 2 of this paper provides some fundamental definitions and results of inverse semigroup theory that have been used in this paper. In this section, the notion of Schützenberger graphs has been elaborated and an iterative procedure for constructing these graphs is provided.

Section 3 of this paper mainly consists of the proof of Theorem 1.4. At the beginning of this section we introduce the notion of a subgraph of a Schützenberger graph, generated by an R-word. We provide an iterative procedure for constructing these subgraphs. We also provide some basic structural properties of these subgraphs. We give the definition of S-diagram as it has been used in the proof of Theorem 1.4.

Section 4 of this paper provides some of the applications of Theorem 1.4. In this section we study the word problem for one relation Adian inverse semigroups, and prove Theorem 1.5.

2. Preliminaries

For an alphabet X, the set X^+ denotes the set of all positive words on X, and the set X^* denotes the set $X^+ \cup \{\varepsilon\}$, where ε is the empty word. The set X^+ forms a semigroup under the binary operation of concatenation of words, and is called the free semigroup on X. The empty word ε serves as the identity element under the concatenation of words, and the set X^* denotes the free monoid on X. If ρ denotes the congruence generated by a set relation R, then X^+/ρ is the *semigroup* given by the presentation $\mathrm{Sg}\langle X|R\rangle$. For $w_1, w_2 \in X^*$, we write $w_1 \equiv w_2$ when w_1 and w_2 are identical words, and write $w_1 = w_2$ when $w_1\rho = w_2\rho$ in the monoid $\mathrm{Mon}\langle X|R\rangle := X^*/\rho$, where ρ is the congruence generated by R. If ρ denotes the Vagner congruence on the monoid $(X \cup X^{-1})^*$, then $\mathrm{FIM}(X) := (X \cup X^{-1})^*/\rho$ is the *free inverse monoid* on X. If $R \subseteq (X \cup X^{-1})^* \times (X \cup X^{-1})^*$, and τ denotes the congruence relation generated by $R \cup \rho$, then $M = \mathrm{Inv}\langle X|R\rangle := (X \cup X^{-1})^*/\tau$ is the inverse monoid presented by the set of generators X and the set of relations R.

A *labeled directed graph* over a set X is a directed graph in which the edges are labeled by elements of X. We write (u, x, v) to denote the edge labeled by x with initial vertex u and terminal vertex v. A *path* or *segment* of length n is a sequence of edges

$$\{(v_0, x_1, v_1), (v_1, x_2, v_2), \ldots, (v_{n-1}, x_n, v_n)\}$$

such that the initial vertex of an edge (except the first) equals the terminal vertex of the previous edge. If $v_0 = v_n$, the path is a cycle. We say that the path is labeled by the word $w = x_1x_2\cdots x_n$ and that w can be read in the graph starting at v_0.

An *inverse word graph* over X is a labeled directed graph over $X \cup X^{-1}$ such that the labeling is consistent with an involution, that is, (u, x, v) is an edge from a vertex u to a vertex v if and only if (v, x^{-1}, u) is an edge from v to u. A *birooted inverse word graph* is an inverse word graph Γ with vertices $\alpha, \beta \in V(\Gamma)$ identified as the start and end vertices, respectively. The language $L[A]$ of a birooted inverse word graph $A = (\alpha, \Gamma, \beta)$ is the set of words that label a path from α to β in Γ. In a birooted inverse word graph over a presentation $\langle X|R\rangle$, for each relation $(r, s) \in R$ and two distinct vertices v_1 and v_2, if r and s can be read along two directed paths going from v_1 to v_2, then there is a *region* with boundary given by the pair of paths labeled by r and s starting from v_1 and ending at v_2. Every region is simply connected, and so is homeomorphic to the open disk. In a birooted inverse word graph, for each relation $(r, s) \in R$ and a vertex v, if we can find a segment labeled by one side of the relation r starting from the vertex v, but we do not find a segment labeled by the other side s of the relation starting from v, then the segment labeled by r is called an *unsaturated segment*.

J. B. Stephen [10] introduced the notion of *Schützenberger graphs* as a tool to study the word problem for inverse semigroups. If $M = \mathrm{Inv}\langle X|R\rangle$ is an inverse

semigroup then we may consider the corresponding Cayley graph $\Gamma(M, X)$. The vertices of this graph are labeled by the elements of M and there exists a directed edge labeled by $x \in X \cup X^{-1}$ from the vertex labeled by m_1 to the vertex labeled by m_2 if $m_2 = m_1 x$. The Cayley graph $\Gamma(M, X)$ is not necessarily strongly connected unless M happens to be a group because it may happen that when there is an edge labeled by x from m_1 to m_2, there is no edge labeled by x^{-1} from m_2 to m_1 (so, $m_2 = m_1 x$, but $m_1 \neq m_2 x^{-1}$). The strongly connected components of $\Gamma(M, X)$ are called the *Schützenberger graphs* of M. For any word $u \in (X \cup X^{-1})^*$, the strongly connected component of $\Gamma(M, X)$ that contains the vertex corresponding to u is the *Schützenberger graph of* u and it is denoted by $S\Gamma(M, X, u)$. In [10] it is shown that the vertices of $S\Gamma(M, X, u)$ are precisely those vertices that are labeled by the elements of the $\mathcal{R}$-class of u, i.e., $R_u = \{m \in M \mid mm^{-1} = uu^{-1}\}$.

For any word $u \in (X \cup X^{-1})^*$, it is useful to consider the *Schützenberger automaton* $(uu^{-1}, S\Gamma(M, X, u), u)$ with initial vertex $uu^{-1} \in M$, terminal vertex $u \in M$ and with the Schützenberger graph of u as the underlying graph. The language accepted by this automaton is a subset of $(X \cup X^{-1})^*$ and will be denoted as $L(u)$,

$$L(u) = \{w \in (X \cup X^{-1})^* \mid w \text{ labels a path from } uu^{-1} \text{ to } u \text{ in } S\Gamma(M, X, u)\}.$$

Here, u and w may be regarded both as elements of $(X \cup X^{-1})^*$ and as elements of M. Thus, $L(u)$ may be regarded as a subset of $(X \cup X^{-1})^*$ or as a subset of M.

The following result of Stephen [10] plays a key role in solving the word problem for inverse semigroups.

Theorem 2.1. *Let* $M = \mathrm{Inv}\langle X|R\rangle$ *and let* $u, v \in (X \cup X^{-1})^*$.

(1) $L(u) = \{w \mid w \geq u$ *in the natural partial order on* $M\}$.

(2) *The following are equivalent:*

  (i) $u = v$ *in* M.

  (ii) $L(u) = L(v)$.

  (iii) $u \in L(v)$ *and* $v \in L(u)$.

  (iv) $(uu^{-1}, S\Gamma(M, X, u), u)$ *and* $(vv^{-1}, S\Gamma(M, X, v), v)$ *are isomorphic as automata.*

We briefly describe the iterative procedure described by Stephen [10] for building a Schützenberger graph. Let $\mathrm{Inv}\langle X|R\rangle$ be a presentation of an inverse monoid.

Given a word $u = a_1 a_2 \cdots a_n \in (X \cup X^{-1})^*$, the *linear graph* of u is the birooted inverse word graph $(\alpha_u, \Gamma_u, \beta_u)$ consisting of the set of vertices

$$V((\alpha_u, \Gamma_u, \beta_u)) = \{\alpha_u, \beta_u, \gamma_1, \ldots, \gamma_{n-1}\}$$

and edges

$$(\alpha_u, a_1, \gamma_1), (\gamma_1, a_2, \gamma_2), \ldots, (\gamma_{n-2}, a_{n-1}, \gamma_{n-1}), (\gamma_{n-1}, a_n, \beta_u),$$

together with the corresponding inverse edges.

Let (α, Γ, β) be a birooted inverse word graph over $X \cup X^{-1}$. The following operations may be used to obtain a new birooted inverse word graph $(\alpha', \Gamma', \beta')$:

• *Determination* or *folding:* Let (α, Γ, β) be a birooted inverse word graph with vertices v, v_1, v_2, with $v_1 \neq v_2$, and edges (v, x, v_1) and (v, x, v_2) for some $x \in X \cup X^{-1}$.

Then we obtain a new birooted inverse word graph $(\alpha', \Gamma', \beta')$ via taking the quotient of (α, Γ, β) by the equivalence relation that identifies the vertices v_1 and v_2 and the two edges. In other words, edges with the same label coming out of a vertex are folded together to become one edge.

• *Elementary $\mathcal{P}$-expansion:* Let $r = s$ be a relation in R and r can be read from v_1 to v_2 in Γ, but s cannot be read from v_1 to v_2 in Γ. Then we define $(\alpha', \Gamma', \beta')$ to be the quotient of $\Gamma \cup (\alpha_s, \Gamma_s, \beta_s)$ by the equivalence relation that identifies vertices v_1 and α_s and vertices v_2 and β_s. In other words, we "sew" on a linear graph for s from v_1 to v_2 to complete the other half of the relation $r = s$.

An inverse word graph is *deterministic* if no folding can be performed and is *closed* if it is deterministic and no elementary expansion can be performed over a presentation $\langle X | R \rangle$. Note that given a finite inverse word graph it is always possible to produce a determinized form of the graph because determination reduces the number of vertices. So, the process of determination must stop after finitely many steps. We also observe that the process of folding is confluent [10] .

If $(\alpha_1, \Gamma_1, \beta_1)$ is obtained from (α, Γ, β) by an elementary $\mathcal{P}$-expansion, and $(\alpha_2, \Gamma_2, \beta_2)$ is the determinized form of $(\alpha_1, \Gamma_1, \beta_1)$, then we write $(\alpha, \Gamma, \beta) \Rightarrow (\alpha_2, \Gamma_2, \beta_2)$ and say that $(\alpha_2, \Gamma_2, \beta_2)$ is obtained from (α, Γ, β) by a $\mathcal{P}$-*expansion*. The reflexive and transitive closure of $\Rightarrow$ is denoted by $\Rightarrow^*$.

For $u \in (X \cup X^{-1})^*$, an *approximate graph* of $(uu^{-1}, S\Gamma(u), u)$ is a birooted inverse word graph $A = (\alpha, \Gamma, \beta)$ such that $u \in L[A]$ and $y \geq u$ holds in M for all $y \in L[A]$. Stephen showed in [10] that the linear automaton of u is an approximate graph of $(uu^{-1}, S\Gamma(u), u)$. He also proved the following.

Theorem 2.2. *Let $u \in (X \cup X^{-1})^*$ and let (α, Γ, β) be an approximate graph of $(uu^{-1}, S\Gamma(u), u)$. If $(\alpha, \Gamma, \beta) \Rightarrow^* (\alpha', \Gamma', \beta')$ and $(\alpha', \Gamma', \beta')$ is closed, then $(\alpha', \Gamma', \beta')$ is the Schützenberger automaton for u.*

In [10], Stephen showed that the class of all birooted inverse word graphs over $X \cup X^{-1}$ is a co-complete category and that the directed system of all finite $\mathcal{P}$-expansions of a linear graph of u has a direct limit. Since the directed system includes

all possible $\mathcal{P}$-expansions, this limit must be closed. Therefore, by Theorem 2.2, the Schützenberger graph of u is the direct limit.

An elementary $\mathcal{P}$-expansion is said to be *relative to* a birooted inverse word graph (α, Γ, β) if it can be applied to it.

Full $\mathcal{P}$-expansion (a generalization of the concept of $\mathcal{P}$-expansion): A full $\mathcal{P}$-expansion of a finite birooted inverse word graph (α, Γ, β) over a finite (infinite) presentation $\langle X|R\rangle$ is obtained in the following way:

• Form the graph $(\alpha', \Gamma', \beta')$, which is obtained from (α, Γ, β) by performing all possible elementary $\mathcal{P}$-expansions of (α, Γ, β), relative to (α, Γ, β). We emphasize that an elementary $\mathcal{P}$-expansion may introduce a path labeled by one side of a relation in R, but we do not perform an elementary $\mathcal{P}$-expansion that could not be done to (α, Γ, β) when we do a full $\mathcal{P}$-expansion.

• Find the determinized form $(\alpha_1, \Gamma_1, \beta_1)$, of $(\alpha', \Gamma', \beta')$.

The birooted inverse word graph $(\alpha_1, \Gamma_1, \beta_1)$ is called the full $\mathcal{P}$-expansion of (α, Γ, β). We denote this relationship by $(\alpha, \Gamma, \beta) \Rightarrow_f (\alpha_1, \Gamma_1, \beta_1)$. If $(\alpha_n, \Gamma_n, \beta_n)$ is obtained from (α, Γ, β) by a sequence of full $\mathcal{P}$-expansions then we denote this by $(\alpha, \Gamma, \beta) \Rightarrow_f^* (\alpha_n, \Gamma_n, \beta_n)$.

For any word $w \in (X \cup X^{-1})^*$, $(\alpha_0, \Gamma_0(w), \beta_0)$ denotes the linear graph of w. We apply full $\mathcal{P}$-expansion to $(\alpha_i, \Gamma_i(w), \beta_i)$ to obtain $(\alpha_{i+1}, \Gamma_{i+1}(w), \beta_{i+1})$, for $i \in \mathbb{N} \cup \{0\}$. This recursive process generates a sequence of birooted approximate graphs $\{(\alpha_i, \Gamma_i(w), \beta_i) \mid i \in \mathbb{N} \cup \{0\}\}$. This sequence of birooted approximate graphs converges to the Schützenberger graph of w. In a finitely presented inverse semigroup $\mathrm{Inv}\langle X|R\rangle$, there exists a graph morphism

$$\phi_i : (\alpha_i, \Gamma_i(w), \beta_i) \to (\alpha_{i+1}, \Gamma_{i+1}(w), \beta_{i+1}),$$

for any $w \in (X \cup X^{-1})^*$, and $i \in \mathbb{N} \cup \{0\}$. If $\mathrm{Inv}\langle X|R\rangle$ happens to be a finitely presented Adian inverse semigroup, and $w \in X^+$, then it has been proved in [4, Proposition 3] that the birooted graph $(\alpha_i, \Gamma_i(w), \beta_i)$ embeds in $(\alpha_{i+1}, \Gamma_{i+1}(w), \beta_{i+1})$, for all $i \in \mathbb{N} \cup \{0\}$. This embedding is as an induced subgraph embedding. Those regions which appear in $(\alpha_i, \Gamma_i(w), \beta_i)$ as a consequence of application of full $\mathcal{P}$- expansion on $(\alpha_{i-1}, \Gamma_{i-1}(w), \beta_{i-1})$, are called the *$i$-th generation regions*, for all $i \in \mathbb{N}$.

3. Subgraphs generated by an R-word of a Schützenberger graph

For a finitely presented Adian inverse semigroup $\mathrm{Inv}\langle X|R\rangle$ and a positive word $w \in X^+$ that contains an R-word r as its subword, a birooted inverse word subgraph of $S\Gamma(w)$ can be generated by r by using an iterative procedure similar to the Stephen's full $\mathcal{P}$-expansion. The word w can contain the R-word r only a finite number of times

as its subword. Therefore, we can label each occurrence of r by a number $i \in \mathbb{N}$, starting from the initial letter of w and going along w up to the terminal letter of w. So, $r_w(i)$ denotes the i-th occurrence of r in w, and $\Delta(r_w(i))$ denotes the birooted inverse word subgraph of $S\Gamma(w)$ generated by the i-th occurrence of r in w. We construct $\Delta(r_w(i))$ by applying elementary $\mathcal{P}$-expansions successively in the following manner.

We construct the linear graph of w, and denote it by $(\alpha_0, \Delta_0(r_w(i)), \beta_0)$, where the underlying graph is denoted by $\Delta_0(r_w(i))$, and α_0 and β_0 are the initial and terminal vertices of the underlying linear graph.

At the first step, we only apply the elementary $\mathcal{P}$-expansion on the i-th segment labeled by r by sewing on a segment labeled by s from the initial vertex of the segment r to the terminal vertex of the segment r, for some $(r, s) \in R$. We denote the resulting graph by $(\alpha_1, \Delta_1(r_w(i)), \beta_1)$, where the underlying graph is denoted by $\Delta_1(r_w(i))$, and α_1 and β_1 are the initial and terminal vertices of the underlying graph. The initial and the terminal vertices remain the same as before, that is, $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$. This creates a first generation region whose one side is labeled by r, and it lies on the linear automaton of w. In $(\alpha_1, \Delta_1(r_w(i)), \beta_1)$, if we cannot find any unsaturated segment labeled by some R-word that either starts from or terminates at an interior vertex of the segment s that was sewn on at the first step, then we cannot continue further. In this case, the graph $\Delta_1(r_w(i))$ is the subgraph of $S\Gamma(w)$ generated by the i-th occurrence of the R-word r. Otherwise, there will be some unsaturated segments labeled by some R-words r_j, for $1 \leq j \leq n_1$, that either start from or terminate at an interior vertex of the segment s that was sewn on at the first step. These new unsaturated segments share an edge with the segment s that was sewn on at the first step.

At the second step, we only apply elementary $\mathcal{P}$-expansion on all these new unsaturated segments labeled by r_j's by sewing on segments labeled by s_j's from the initial vertices to the terminal vertices of the corresponding segments r_i's, where $\{(r_j, s_j), 1 \leq j \leq n_1\} \subseteq R$. This step creates the second generation regions of $\Delta(r_w(i))$. We denote the resulting graph by $(\alpha_2, \Delta_2(r_w(i)), \beta_2)$, where the underlying graph is denoted by $\Delta_2(r_w(i))$, and α_2, and β_2 are the initial and terminal vertices of the underlying graph. All these second generation regions share an edge with the first generation region created at the first step. If we cannot find any unsaturated segment labeled by an R-word that either starts from an interior vertex of the segment s_j or terminates an in interior vertex of the segment s_j, for some $j \in \{1, 2, \ldots, n_1\}$, where the segments s_j's were sewn on at the second step, then we cannot proceed further. In this case, the graph $\Delta_2(r_w(i))$ is the subgraph of $S\Gamma(w)$ generated by the i-th occurrence of the R-word r. Otherwise, there will be some unsaturated segments labeled by some R-words u_j's, for $1 \leq j \leq n_2$, that either start from an interior vertex of the segment s_j or terminate at an interior vertex of s_j, for some $j \in \{1, 2, \ldots, n_1\}$, where the segments s_j's were sewn on at the second step. These new unsaturated segments

u_j's share an edge with some of the segments s_j's that were sewn on at the second step.

At the third step, we only apply elementary $\mathcal{P}$-expansion on all these new unsaturated segments u_j's by sewing on segments labeled by v_j's from the initial vertices to the terminal vertices of the corresponding segments u_j's, where $\{(u_j, v_j), 1 \leq j \leq n_2\} \subseteq R$. We denote the resulting graph by $(\alpha_3, \Delta_3(r_w(i)), \beta_3)$, where the underlying graph is denoted by $\Delta_3(r_w(i))$, and α_3, and β_3 are the initial and terminal vertices of the underlying graph. This step forms all the third generation regions of $\Delta(r_w(i))$. Note that all the third generation regions share an edge with a second generation region. If we cannot find any unsaturated segment labeled by an R-word that either starts from or terminates at an interior vertex of the segment v_j, for some $j \in \{1, 2, \ldots, n_2\}$, where the segments v_j's were sewn on at the third step, then we cannot proceed further. In this case, the graph $\Delta_3(r_w(i))$ is the subgraph of $S\Gamma(w)$ generated by the i-th occurrence of the R-word r. Otherwise, there will be some unsaturated segments labeled by some R-words that either start from or terminate at an interior vertex of the segment v_j, for some $j \in \{1, 2, \ldots, n_2\}$, where the segments v_j's were sewn on at the third step. So, we continue the process of successive applications of elementary $\mathcal{P}$-expansions in this manner, and we obtain a sequence of approximate graphs $\{(\alpha_n, \Delta_n(r_w(i)), \beta_n) \mid n \in \mathbb{N}\}$ that converges to the subgraph $\Delta(r_w(i))$ of $S\Gamma(w)$. If the sequence $\{(\alpha_n, \Delta_n(r_w(i)), \beta_n) \mid n \in \mathbb{N}\}$ stabilizes after a finite number of terms, then $\Delta(r_w(i))$ is finite. Otherwise, $\Delta(r_w(i))$ is infinite.

The construction of $\Delta(r_w(i))$ can be summarized in the following way:

(1) We construct the linear graph of w, and denote it by $(\alpha_0, \Delta_0(r_w(i)), \beta_0)$, where the underlying graph is denoted by $\Delta_0(r_w(i))$, and α_0 and β_0 are the initial and terminal vertices of the underlying graph.

(2) We only apply the elementary $\mathcal{P}$-expansion on the i-th segment labeled by r. We denote the resulting graph by $(\alpha_1, \Delta_1(r_w(i)), \beta_1)$, where the underlying graph is denoted by $\Delta_1(r_w(i))$, and α_1 and β_1 are the initial and terminal vertices of the underlying graph. Here $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$.

(3) For $n \geq 1$, to obtain $(\alpha_{n+1}, \Delta_{n+1}(r_w(i)), \beta_{n+1})$ from $(\alpha_n, \Delta_n(r_w(i)), \beta_n)$, we only apply elementary $\mathcal{P}$-expansion on those unsaturated segments labeled by some R-words of $\Delta_n(r_w(i))$ that either start from or terminate at an interior vertex of the segments labeled by some R-words and were sewn on at the n-th iterative step. Here $\alpha_n = \alpha_{n+1}$ and $\beta_n = \beta_{n+1}$.

Consequently, we obtain a sequence of approximate graphs $\{(\alpha_n, \Delta_n(r_w(i)), \beta_n) \mid n \in \mathbb{N} \cup \{0\}\}$, where $\alpha_0 = \alpha_n$ and $\beta_0 = \beta_n$, for all $n \in \mathbb{N}$. This sequence converges to the subgraph $\Delta(r_w(i))$ of $S\Gamma(w)$. The subgraph $\Delta(r_w(i))$ is an induced subgraph of $S\Gamma(w)$.

In the following example, we consider an inverse semigroup from the family of Baumslag–Solitar inverse semigroups, and a positive word w that contains some of the R-words as its subword to demonstrate the construction of the subgraphs of $S\Gamma(w)$ generated by an R-word. It has already been proved in [4] that all the Schützenberger graphs are finite in a Baumslag–Solitar inverse semigroup.

Example 3.1. We consider the inverse semigroup $\mathrm{Inv}\langle a, b | ab = ba\rangle$ and the word $w \equiv a^2b^2a^2b^2 \in \{a, b\}^+$. The word w contains both the R-words ab and ba as its subwords. The R-word ab occurs twice and the R-word ba occurs once in w. We construct the subgraph of $S\Gamma(w)$ generated by the first occurrence of the R-word ab.

First, we construct the linear graph of w, $(\alpha_0, \Delta_0(ab_w(1)), \beta_0)$ (see Figure 1). We apply the elementary $\mathcal{P}$-expansion only at the first subsegment labeled by ab of w by sewing on a path labeled by ba from the initial vertex of ab to the terminal vertex of ab, and we denote the resulting graph by $(\alpha_1, \Delta_1(ab_w(1)), \beta_1)$ (see Figure 2). In $\Delta_1(ab_w(1))$, we can find two unsaturated segments labeled by ab, one of which terminates at an interior vertex of the segment ba, and the other one starts from an interior vertex of the segment ba, where the segment labeled by ba was sewn at the previous step. We apply elementary $\mathcal{P}$-expansion only on both of these unsaturated segments by sewing on segments labeled by ba from the initial vertices to the terminal vertices of these unsaturated segments. We denote the resulting graph by $(\alpha_2, \Delta_2(ab_w(1)), \beta_2)$ (see Figure 3). In $\Delta_2(ab_w(1))$, we can find only one unsaturated segment labeled by ab that starts from an interior vertex of a segment labeled by ba that was sewn on at the previous step. So, we apply elementary $\mathcal{P}$-expansion only on this unsaturated segment labeled by sewing on a path labeled by ba from the initial vertex to the terminal vertex of the segment labeled by ab. We denote the resulting graph by $(\alpha_3, \Delta_3(ab_w(1)), \beta_3)$ (see Figure 4). In $\Delta_3(ab_w(1))$, we cannot find any unsaturated segments labeled by an R-word that either starts from or terminates at an interior vertex of the segment ba that was sewn on at the previous step. Therefore, we cannot continue further. Hence $\Delta_3(ab_w(1))$ is the subgraph of $S\Gamma(w)$ generated by the first occurrence of the R-word ab in w, which is denoted by $\Delta(ab_w(1))$.

Similarly, we can construct $\Delta(ab_w(2))$ (see Figure 5) and $\Delta(ba_w(1))$ (see Figure 6).

The complete graph of $S\Gamma(w)$ can be constructed by five successive applications of Stephen's full $\mathcal{P}$-expansion (see Figure 7).

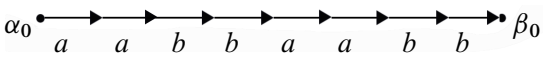

Figure 1. $(\alpha_0, \Delta_0(ab_w(1)), \beta_0)$

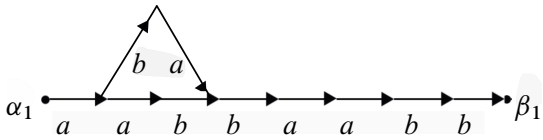

Figure 2. $(\alpha_1, \Delta_1(ab_w(1)), \beta_1)$

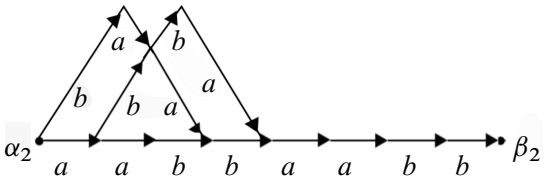

Figure 3. $(\alpha_2, \Delta_2(ab_w(1)), \beta_2)$

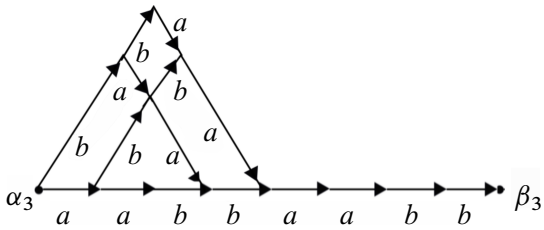

Figure 4. $(\alpha_3, \Delta_3(ab_w(1)), \beta_3)$

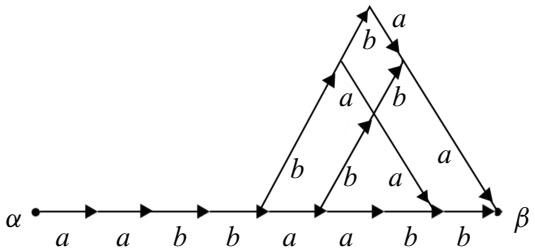

Figure 5. $\Delta(ab_w(2))$

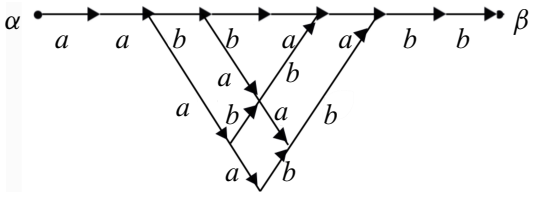

Figure 6. $\Delta(ba_w(1))$

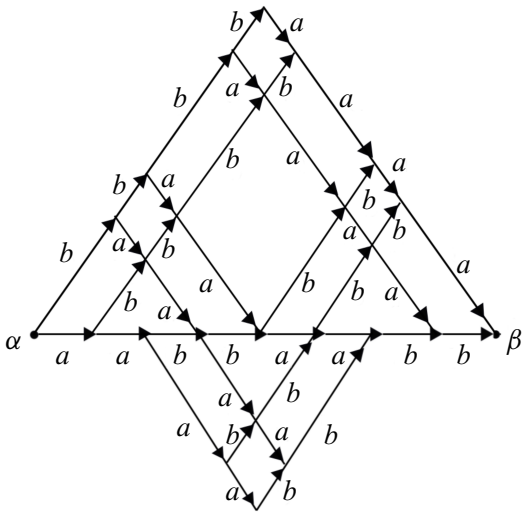

Figure 7. $S\Gamma(w)$

The following lemma is proved in [4] and it implies that we only use elementary $\mathcal{P}$-expansions, and no foldings in the construction of $S\Gamma(w)$, for some $w \in X^+$.

Lemma 3.1. *Let $M = \mathrm{Inv}\langle X, R\rangle$ be an Adian inverse semigroup and $w \in X^+$. Then no two edges fold together in Stephen's process of constructing approximations of the Schützenberger graph of w.*

The following lemma and proposition are also proved in [4] and they will be used latter in the proof of Theorem 3.2.

Lemma 3.2. *Let $M = \mathrm{Inv}\langle X|R\rangle$ be an Adian inverse semigroups and $w \in (X \cup X^{-1})^+$. Then the Schützenberger graph of w contains no directed cycle of edges.*

Proposition 3.1. *Let $M = \mathrm{Inv}\langle X|R\rangle$ be an Adian inverse semigroup and $w \in X^+$. Then:*

(i) $\psi_n : (\alpha_n, \Gamma_n(w), \beta_n) \to (\alpha_{n+1}, \Gamma_{n+1}(w), \beta_{n+1})$ *is an embedding for all* $n \in \mathbb{N}$.

(ii) $\Gamma(w)$ *has exactly one source vertex α and exactly one sink vertex β, where $(\alpha, \Gamma(w), \beta)$ is the Schützenberger automaton of w.*

(iii) *For every $v \neq \alpha$ in $(\alpha, \gamma(w), \beta)$ there exists a positively labeled path in $\Gamma(w)$ from α to v. For every $v \neq \beta$ in $(\alpha, \gamma(w), \beta)$ there exists a positively labeled path in $\Gamma(w)$ from v to β.*

(iv) *Every positively labeled path in $\Gamma(w)$ can be extended to a positively labeled transversal from α to β.*

For any semigroup $S = \mathrm{Sg}\langle X|R\rangle$, and two words $p, q \in X^+$, $p \to q$ denotes that q is obtained from p by replacing a subword r by s, for some $(r, s) \in R$. In this case, we can also obtain p from q (that is, $q \to p$) by replacing the subword s by r, for some $(r, s) \in R$. Any two words $u, v \in X^+$ are equal in the semigroup $S = \mathrm{Sg}\langle X|R\rangle$ if and only if there exists a transition sequence from u to v,

$$u \equiv w_0 \to w_1 \to w_2 \to \cdots \to w_n \equiv v, \quad \text{for some } n \geq 0.$$

The above derivation sequence is called a *regular derivation sequence of length n* for the pair (u, v) over the presentation $\mathrm{Sg}\langle X|R\rangle$.

A *semigroup diagram* or *S-diagram* over a semigroup presentation $\mathrm{Sg}\langle X|R\rangle$ for the pair of positive words (u, v) is a finite planar diagram $D \subseteq \mathbb{R}^2$, that satisfies the following properties:

- The diagram D is connected and simply connected.
- Each edge is directed and labeled by a letter of the alphabet X.
- Each region of D is labeled by the word rs^{-1} for some defining relation $(r, s) \in R$.
- There is a distinguished vertex α on the boundary of D such that the boundary of D starting at α is labeled by uv^{-1}. α is a source in D (i.e., there is no edge in D with terminal vertex α).
- There are no interior sources or sinks in D.

In [9], Remmers proved an analogue of Van Kampen's lemma for semigroups to address the word problem for semigroups.

Theorem 3.1. *Let $S = \mathrm{Sg}\langle X|R\rangle$ be a semigroup and $u, v \in X^+$. Then there exists a regular derivation sequence of length n for the pair (u, v) over the presentation $\mathrm{Sg}\langle X|R\rangle$ for the pair (u, v) having exactly n regions.*

A sub-diagram Π' of an S-diagram Π is called a *simple component* of Π if it is a maximal sub-diagram whose boundary is labeled by a simple closed curve.

The proof of the following lemma can be found in [4] and it will be used in the proof of Theorem 3.2. This lemma ensures the embeddability of an S-diagram over an Adian presentation $\mathrm{Sg}\langle X|R\rangle$, into a Schützenberger graph over same presentation $\mathrm{Inv}\langle X|R\rangle$.

Lemma 3.3. *Let $M = \langle X|R\rangle$ be an Adian inverse semigroup, let $w \in (X \cup X^{-1})^*$ and let $w_1, w_2 \in X^+$ label two co-terminal paths in $S\Gamma(w)$. Then there exists an S-diagram corresponding to the pair of words (w_1, w_2) that embeds in $S\Gamma(w)$.*

Theorem 3.2. *Let $M = \mathrm{Inv}\langle X|R\rangle$ be a finitely presented Adian inverse semigroup. The Schützenberger graph of every positive word is finite if and only if the subgraphs of $S\Gamma(w)$, for all $w \in X^+$, generated by all the R-words are finite.*

Idea of the proof: We assume that the subgraphs of $S\Gamma(t)$, for all $t \in X^+$, generated by all the R-words are finite and we let w be an arbitrary positive word, $w \in X^+$. We use induction on the number of segments labeled by the R-words of the linear automaton of w, $(\alpha_0, \Gamma_0(w), \beta_0)$, to show that $S\Gamma(w)$ is finite. The construction of $S\Gamma(w)$ only involves the applications of elementary $\mathcal{P}$-expansions and no foldings by Lemma 3.1. We factorize w as $w \equiv w_1 w_2$, such that w_1 and w_2 contain fewer segments labeled by the R-words than w. So, by induction hypothesis, $S\Gamma(w_1)$ and $S\Gamma(w_2)$ are finite. To speedup the Stephen's process of successive applications of elementary $\mathcal{P}$-expansions we sew on $S\Gamma(w_1)$ and $S\Gamma(w_2)$ to the corresponding segments of $(\alpha_0, \Gamma_0(w), \beta_0)$, and denote the resulting finite inverse word graph by S_0. We observe that in S_0 there is only one vertex (labeled by γ) that is common to $S\Gamma(w_1)$ and $S\Gamma(w_2)$. Since $S\Gamma(w_1)$ and $S\Gamma(w_2)$ are $\mathcal{P}$-complete, therefore we can only find some unsaturated segments passing through γ. To speedup Stephen's processes of successive applications of $\mathcal{P}$-expansions we sew on the finite subgraphs generated by the R-words that label unsaturated segments passing through γ, to the corresponding transversals of S_0, and obtain a finite inverse word graph, S_1. Finally, we show that S_1 is $\mathcal{P}$-complete. So, S_1 is $S\Gamma(w)$.

Proof. The direct statement is obvious. So, we prove the converse statement only. Let w be an arbitrary word, $w \in X^+$. If w does not contain any R-word as its subword, then our claim follows immediately. If w contains only one R-word as its subword, such that the R-word occurs only once in w, then the finiteness of $S\Gamma(w)$ follows from the hypothesis of the theorem.

We assume that the statement of the theorem is true for any positive word that contains less than n number of (not necessarily distinct) R-words as its subword, where $n \in \mathbb{N}$ and $n \geq 2$. We assume that w contains n number of (not necessarily distinct) R-words as its subwords. We show that $S\Gamma(w)$ is finite.

We factorize w as $w \equiv w_1 w_2$, such that both w_1 and w_2 contain fewer than n R-words as their subwords. Such a factorization of w is possible, because w_1 can be considered any proper prefix of w that can be read along a segment labeled by w, starting from the initial vertex and terminating at an interior vertex η of a subsegment, labeled by an R-word, of w. The suffix w_2 can be read from η to the terminal vertex of the segment labeled by w.

The Schützenberger graphs of w_1 and w_2 are finite by the induction hypothesis. We construct the linear automaton of w, $(\alpha_0, \Gamma_0(w), \beta_0)$. We sew on $S\Gamma(w_1)$ and $S\Gamma(w_2)$ to $(\alpha_0, \Gamma_0(w), \beta_0)$ at the segments labeled by w_1 and w_2, respectively. We denote the resulting graph by S_0 (see Figure 8).

We observe that in S_0 there is only one vertex (labeled by γ) common to $S\Gamma(w_1)$ and $S\Gamma(w_2)$. The vertex γ is the terminal vertex of $S\Gamma(w_1)$, and the initial vertex of $S\Gamma(w_2)$. Therefore, every transversal (a positively labeled path from the initial ver-

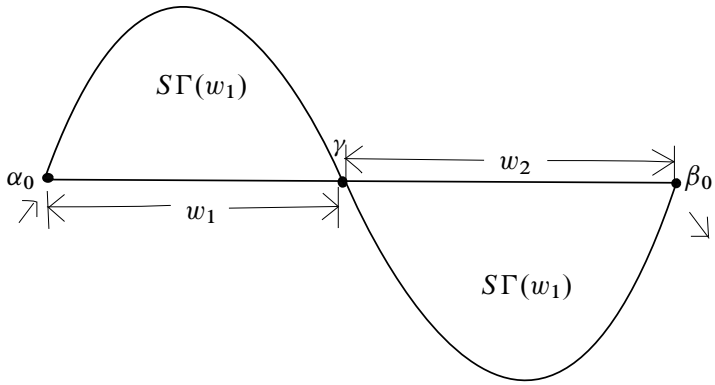

Figure 8. S_0 with $S\Gamma(w_1)$ and $S\Gamma(w_2)$ attached to the segments w_1 and w_2

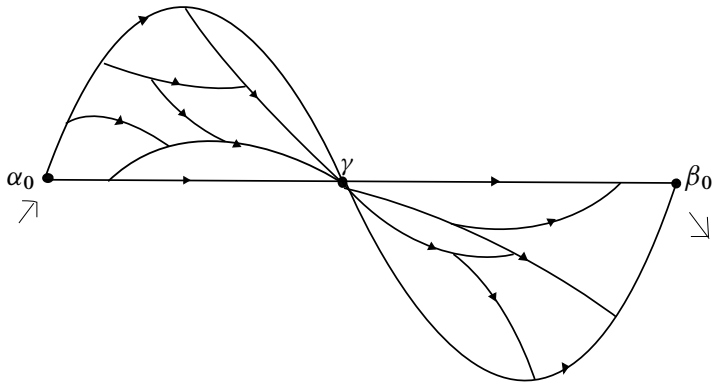

Figure 9. S_0 with some transversals highlighted

tex α_0 to the terminal vertex β_0) of S_0 passes through γ (see Figure 9). We know that $S\Gamma(w_1)$ and $S\Gamma(w_2)$ are $\mathcal{P}$-complete. So, in S_0 we can only find some unsaturated segments labeled by some R-words starting from a vertex of $S\Gamma(w_1)$, passing through γ, and terminating at a vertex of $S\Gamma(w_2)$.

By Lemma 3.2, the Schützenberger graphs $S\Gamma(w_1)$ and $S\Gamma(w_2)$ do not contain any positively labeled cycle (closed path). Therefore, there are only a finite number of transversals in S_0, as it is a finite inverse word graph. We consider those transversals of S_0 which contain an unsaturated segment labeled by an R-word passing through γ. We assume that these transversals are labeled by $t_1, t_2, \ldots, t_m$, for some $m \in \mathbb{N}$.

By hypothesis, the subgraphs of $S\Gamma(t_i)$, for $1 \leq i \leq m$, generated by all the R-words are finite. It is possible that a transversal t_i, for some $i \in \{1, 2, \ldots, m\}$, may contain more than one unsaturated segment passing through γ, because the R-words can overlap with themselves or with the other R-words. We generate the subgraphs $\Delta(r_{t_i})$

of $S\Gamma(t_i)$ by the R-words r_{t_i}'s labeling an unsaturated segment passing through γ, and denote them simply by Δ_j, where $1 \leq j \leq l$ for some $l \geq m$.

We sew on the finite subgraphs $\Delta_1, \Delta_2, \ldots, \Delta_l$ to S_0 at the corresponding transversals t_i's (see Figure 10). Consequently, we obtain a finite graph. We denote this new graph by S_1. No two edges fold together in S_1 by Lemma 3.1. Finally, we show that S_1 is $\mathcal{P}$-complete.

To prove the $\mathcal{P}$-completeness of S_1 we need to show that S_1 does not contain any unsaturated segment, where elementary $\mathcal{P}$-expansions can be applied. So, we prove the following two claims.

(1) Every segment labeled by an R-word that either starts from a vertex of S_0 and terminates at a vertex of Δ_i, or vice versa, for some $i \in \{1, 2, \ldots, l\}$, is contained in Δ_j, for some j $(1 \leq j \neq i \leq l)$.

(2) Every segment labeled by an R-word that starts from a vertex of Δ_i and terminates at a vertex of Δ_j, for $1 \leq i \neq j \leq l$, is contained in Δ_k for some $k \neq i, j$.

From claims (1) and (2), it follows that every segment of S_1 that is labeled by an R-word, is either entirely contained in S_0 or in Δ_i, for some $i \in \{1, 2, \ldots, l\}$. Hence, S_1 does not contain any unsaturated segment labeled by an R-word.

Proof of (1): Without loss of generality, we assume that we can find a segment labeled by an R-word that starts from a vertex γ_0 of S_0, passes through a vertex γ_1 of the transversal t_i, and terminates at a vertex γ_2 of Δ_i, for some $1 \leq i \leq l$. Here γ_1 is the first vertex that is common between the transversal t_i and the segment labeled by the R-word from γ_0 to γ_2. A dual argument can be used to show that every segment labeled by an R-word, that starts from a vertex of Δ_i and terminates at a vertex of S_0, is contained in Δ_j, for $1 \leq i \neq j \leq l$.

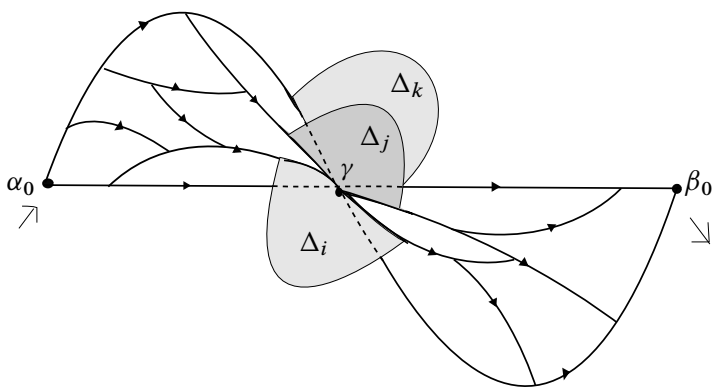

Figure 10. S_0 with some Δ_i's attached

The vertex γ_1 lies either before the vertex γ, or at the vertex γ. If the vertex γ_1 lies after the vertex γ, then by Proposition 3.1 (iii), we can find a positively labeled segment from α_0 to γ_0. We can extend this path to a transversal of S_0 by extending this path to γ_1, and proceeding along the transversal t_i up to β. This transversal does not pass through γ, which contradicts the fact that every transversal of S_0 passes through γ.

If the subsegment from γ_1 to γ_2 of the segment labeled by the R-word lies on the transversal t_i, then by Proposition 3.1 (iii), we can find a positively labeled segment from α_0 to γ_0. We can extend this segment to a transversal of S_0 by proceeding along the segment from γ_0 to γ_2, and further proceeding along the transversal t_i. We denote this transversal by t. If γ_2 lies before or at γ, then the segment from γ_0 to γ_2 lies in $S\Gamma(w_1)$. In this case, the segment from γ_0 to γ_2 cannot be unsaturated, as $S\Gamma(w_1)$ is $\mathcal{P}$-complete. If γ_2 lies after γ, that is, γ_1 lies at γ, or γ lies between the vertices γ_1 and γ_2, then the transversal t contains the segment from γ_0 to γ_2, that is labeled by an R-word, and passes through γ. So, the transversal t is among one of the transversals $t_1, t_2, \ldots, t_m$. Hence, $t \equiv t_j$ where $1 \leq i \neq j \leq m$. The subgraph Δ_j generated by an R-word, that labels the segment from γ_0 to γ_2, and sewed onto the transversal t_j, contains the segment from γ_0 to γ_2. So, our claim holds in this case.

If the vertex γ_2 does not lie on the transversal t_i, then by using Proposition 3.1 (iii), we can find a positively labeled segment in Δ_i that starts from γ_1, passes through γ_2 and terminates at a vertex of the transversal t_i. We assume that this segment is labeled by p', for some $p' \in X^+$. We assume that the subsegment of t_i, from γ_1 to the terminal vertex of p', is labeled by p, for some $p \in X^+$. By Lemma 3.3, there exists an S-diagram over the semigroup presentation $\mathrm{Sg}\langle X|R\rangle$ whose boundary is labeled by the pair of words (p, p'). Again, by using Proposition 3.1 (iv), we can find a transversal s of S_0 that passes through the subsegment from γ_0 to γ_1 of the segment labeled by the R-word, and the segment p. Clearly, the S-diagram corresponding to (p, p') embeds in $S\Gamma(s)$. That is because there exists a regular derivation sequence from p to p' corresponding to this S-diagram, the terms of this regular derivation sequence label the transversals of the S-diagram corresponding to (p, p'), p is a subword of s, and $S\Gamma(s)$ is $\mathcal{P}$-complete. So $S\Gamma(s)$ contains every region of the S-diagram corresponding to (p, p'). If the transversal s does not contain an unsaturated segment labeled by an R-word, then $S\Gamma(s)$ embeds in S_0. This implies that the vertex γ_2 lies in S_0, which is a contradiction. Hence, s contains an unsaturated segment labeled by an R-word passing through γ. Therefore, $s \equiv t_j$, for some $1 \leq i \neq j \leq m$. The segment from γ_0 to γ_2, labeled by the R-word, is contained in some Δ_j, that was sewn onto the transversal t_j and contains the S-diagram with boundary labels (p, p').

Proof of (2): We assume that there exists a segment labeled by an R-word that starts from a vertex γ_i of Δ_i and terminates at a vertex γ_j of Δ_j for $i \neq j$. If this segment from γ_i to γ_j lies in S_0, such that it does not pass through the vertex γ, then

this segment either lies in $S\Gamma(w_1)$ or in $S\Gamma(w_2)$. In this case, the segment from γ_i to γ_j cannot be unsaturated, as $S\Gamma(w_1)$ and $S\Gamma(w_2)$ both are $\mathcal{P}$-complete.

If the segment from γ_i to γ_j lies in S_0, and passes through γ, then by Proposition 3.1 (iv) we can extend this segment to a transversal labeled by t, for some $t \in X^+$ of S_0. The transversal t contains a segment labeled by an R-word passing through γ, therefore, $t \equiv t_k$, for some $k \in \{1, 2, \ldots, m\}$. So, the claim is true in this case, because the segment from γ_i to γ_j is contained in Δ_k (where Δ_k is the subgroup of $S\Gamma(t_k)$, generated by the R-word that labels the segment from γ_i to γ_j).

Now we assume that the segment from γ_i to γ_j does not lie in S_0, where the vertices γ_i and γ_j can lie in S_0. We can find a vertex γ_{t_i} on the transversal t_i, and a vertex γ_{t_j} on the transversal t_j, such that the segments from γ_{t_i} to γ_i, and from γ_j to γ_{t_j} are the shortest non-negatively labeled segments from S_0 to γ_i and from γ_j to S_0. If γ_i or γ_j lies in S_0, then γ_i (γ_j) and γ_{t_i} (γ_{t_j}) represents the same vertex, otherwise, there will be positively labeled segment from γ_{t_i} to γ_i (γ_{t_j} to γ_j). Obviously, none of the edges of the segments from γ_{t_i} to γ_i, and from γ_j to γ_{t_j} lie in S_0, otherwise they will not be the shortest ones. We consider the positively labeled segment starting from γ_{t_i}, passing through the segment from γ_i to γ_j labeled by an R-word, and terminating at γ_{t_j}. We assume that this segment is labeled by p', for some $p' \in X^+$. The vertex γ_{t_i} lies in $S\Gamma(w_1)$ and the vertex γ_{t_j} lies in $S\Gamma(w_2)$. So, S_0 contains a positively labeled segment from γ_{t_i} to γ_{t_j}, passing through γ. We assume that the segment from γ_{t_i} to γ_{t_j}, of S_0, is labeled by p, for some $p \in X^+$. By Lemma 3.3, there exists an S-diagram over the semigroup presentation $\mathrm{Sg}\langle X|R\rangle$ with boundary labeled by the pair of words (p, p') that embeds in $S\Gamma(w)$. We denote this S-diagram by Π.

The segment labeled by p contains a subsegment labeled by an R-word passing through γ. Otherwise, the S-diagram Π embeds in S_0. This leads us to a contradictory conclusion that the segment from γ_i to γ_j, labeled by an R-word, lies in S_0.

The S-diagram Π consists of only one simple component. If this S-diagram consists of more than one simple component, then there exists a vertex δ between γ_{t_i} and γ_{t_j}, where the segments p and p' intersect with each other. We already observed that p contains a subsegment labeled by an R-word passing through γ. If δ lies anywhere between the initial and terminal vertex of the subsegment of p, that is labeled by an R-word and passes through γ, then it means that this subsegment of p does not play any role in construction of the S-diagram Π. So, the S-diagram Π embeds in S_0. This leads us to a contradiction that the segment from γ_i to γ_j, labeled by an R-word, lies in S_0.

Now we show that δ cannot lie before or at the initial vertex of the subsegment of p, that is labeled by an R-word and passes through γ. A similar argument can be used to show that δ cannot lie at the terminal vertex or after the terminal vertex of the subsegment of p, that is labeled by an R-word and passes through γ.

If δ lies anywhere before the initial vertex or at the initial vertex of the subsegment of p, that is labeled by an R-word and passes through γ, then p can be factorized as $p \equiv p_1 p_2$, where p_1 labels the segment from γ_{t_i} to δ, and p_2 labels the segment from δ to γ_{t_j}. The segment p_1 lies in $S\Gamma(w_1)$. But, $S\Gamma(w_1)$ is $\mathcal{P}$-complete. Therefore, the subsegment of p' from γ_{t_i} to δ, also lies in $S\Gamma(w_1)$, hence in S_0. This contradicts the choice of the vertex γ_{t_i}.

By Proposition 3.1 (iv), the segment p can be extended to a transversal t of S_0. Since this transversal t contains a segment labeled by an R-word passing through γ, $t \equiv t_k$, for some $k \in \{1, 2, \ldots, m\}$. The S-diagram Π consists of only one simple component. Therefore, it can be generated by some successive applications of elementary $\mathcal{P}$-expansions starting from the subsegment of p, that is labeled by an R-word and passes through γ. The subgraph Δ_k generated by the subsegment of p, that is labeled by an R-word and passes through γ, is the largest subgraph of $S\Gamma(w)$ that can be generated by successive applications of elementary $\mathcal{P}$-expansions. So, the S-diagram Π is contained in Δ_k. This implies that the segment from γ_i to γ_j labeled by an R-word is also contained in Δ_k. ∎

4. The word problem for some classes of one relation Adian inverse semigroups

In this paper, we only focus on those one relation Adian inverse semigroups, in which no R-word is a subword of the other R-word.

For any word $w \in X^+$, a prefix (suffix) u of w is said to be a *proper* prefix (suffix) of w if the length of u is less than the length of w (that is, $|u| < |w|$). For any two words $u, v \in X^+$, if a proper prefix of u is a proper suffix of v, we say that u and v *overlap* with each other. If a proper prefix of u is also a proper suffix of u, we say that u overlaps with itself.

Lemma 4.1. *Let $M = \mathrm{Inv}\langle X | u = v\rangle$ be an Adian inverse semigroup. If none of the R-words is a subword of the other R-word, and no R-word overlaps with itself or with the other R-word, then $S\Gamma(t)$, for all $t \in X^+$, is finite.*

Proof. Let t be a positive word. If t does not contain any R-word as its subword, then there is nothing to prove, because $S\Gamma(t)$ is finite in this case.

We assume that t contains n number of (not necessarily distinct) R-words, for some $n \in \mathbb{N}$. We construct the linear graph of t, denoted by $(\alpha_0, \Gamma_0(t), \beta_0)$. We apply full $\mathcal{P}$-expansion on $(\alpha_0, \Gamma_0(t), \beta_0)$ and obtain $(\alpha_1, \Gamma_1(t), \beta_1)$. Since none of the R-word is subword of the other R-word, we cannot find any unsaturated subsegment labeled by an R-word of the newly attached segments of $(\alpha_1, \Gamma_1(t), \beta_1)$. We know from [4, Proposition 3 (iii)], that $(\alpha_1, \Gamma_1(t), \beta_1)$ is a directional graph in the sense

that every vertex of $(\alpha_1, \Gamma_1(t), \beta_1)$ lies on a transversal from α_1 to β_1. We cannot find an unsaturated segment labeled by an R-word that either starts from a vertex which lies before the initial vertex of a newly attached segment and terminates at an interior vertex of the newly attached segment, or that starts from an interior vertex of a newly attached segment and terminates at a vertex which lies after the terminal vertex of the newly attached segment, as none of the R-words overlap with themselves or with the other R-word. So, $(\alpha_1, \Gamma_1(t), \beta_1)$ cannot be expanded further by applying full $\mathcal{P}$-expansion. Hence, the underlying finite graph of $(\alpha_1, \Gamma_1(t), \beta_1)$ is $S\Gamma(t)$. ∎

The one relation Adian presentations with no R-word being a subword of the other R-word, as well as some overlap between the R-words, can be distributed into four classes.

Proposition 4.1. *Let $\langle X|u = v\rangle$ be a positive presentation, such that no R-word is a subword of the other R-word. There are four different types of overlaps possible in the presentation $\langle X|u = v\rangle$.*

(1) *One of the R-words has same prefix and suffix. The other R-word does not overlap with itself and there is no overlap between the two R-words.*

(2) *Both the R-words overlap with themselves and they do not overlap with each other.*

(3) *A prefix of an R-word is suffix of the other R-word, and no suffix of the former R-word is a prefix of the latter R-word.*

(4) *A prefix of one R-word is a suffix of the other R-word, and a suffix of the former R-word is a prefix of the later R-word.*

Proof. There are only two R-words u and v in the presentation $\langle X|u = v\rangle$. So, there can be only two possibilities of an R-word overlapping with itself. That is, either one of the R-words overlaps with itself or both of the R-words overlap with themselves. Both of these possibilities are considered in cases (1) and (2) of the above statement. Obviously, cases (1) and (2) cannot occur simultaneously.

There are only two possibilities of an R-word overlapping with the other R-word in an Adian presentation of the form $\langle X|u = v\rangle$. That is, either a prefix of an R-word is a suffix of the other R-word, and no suffix of the former R-word is a prefix of the latter R-word, or a prefix of one R-word is the suffix of the other R-word and a suffix of the former R-word is the prefix of the latter R-word. Both of these possibilities are considered in cases (3) and (4) of the above statement. Obviously, cases (3) and (4) cannot occur simultaneously.

In case (3), it is also possible that one or both of the R-words overlap with themselves as well. But, these possibilities are considered as sub-cases of (3). Also in

case (4), it is possible that one or both of the R-words overlap with themselves. But, these possibilities are considered as sub-cases of (4). ∎

We say that an Adian inverse semigroup $M = \langle X | u = v \rangle$ is of type (1), when the presentation $\langle X | u = v \rangle$ is of type (1) of Proposition 4.1.

Lemma 4.2. *Let $M = \mathrm{Inv}\langle X | u = v \rangle$ be an Adian inverse semigroup of type* (1). *Then every subgraph of $S\Gamma(w)$, for all $w \in X^+$, generated by an R-word, is finite.*

Proof. Without loss of generality, we assume that u overlaps with itself and v neither overlaps with itself nor with u. Obviously, for any $w \in X^+$ that does not contain any R-word as its subword, $S\Gamma(w)$ is finite. So, we consider an arbitrary positive word w, such that w contains some R-words as its subwords. We construct the linear graph of w, $(\alpha_0, \Gamma_0(w), \beta_0)$.

If w contains u as its subword, we apply the elementary $\mathcal{P}$-expansion on the segment labeled by u of $(\alpha_0, \Gamma_0(w), \beta_0)$ by sewing on a path labeled by v from the initial vertex to the terminal vertex of the segment labeled by u. Since v neither overlaps with itself nor with u, the first application of elementary $\mathcal{P}$-expansion does not create any new unsaturated segment labeled by an R-word. So, the subgraph of $S\Gamma(w)$ generated by u stops to grow at the first iterative step. Hence, the subgraph of $S\Gamma(w)$ generated by u is finite.

If w contains v as its subword, we apply elementary $\mathcal{P}$-expansion on the segment labeled by v of $(\alpha_0, \Gamma_0(w), \beta_0)$ by sewing on a path labeled by u from the initial vertex to the terminal vertex of the segment labeled by v. Since u overlaps with itself, we may find a finite number of unsaturated segments labeled by u starting from a vertex of $(\alpha_0, \Gamma_0(w), \beta_0)$ that lies before the initial vertex of the segment labeled by v and terminating at an interior vertex of the segment labeled by u or starting from an interior vertex of the segment labeled by u and terminating at a vertex of $(\alpha_0, \Gamma_0(w), \beta_0)$ that lies after the terminal vertex of the segment labeled by v. We apply elementary $\mathcal{P}$-expansions on all of these new unsaturated segments by sewing on segments labeled by v. Since v does not overlap with itself or with u, the second applications of elementary $\mathcal{P}$-expansions do not create any new unsaturated segments labeled by some R-words. Hence, the subgraph of $S\Gamma(w)$ generated by v remains finite. ∎

We say that an Adian inverse semigroup $M = \langle X | u = v \rangle$ is of type (2) when the presentation $\langle X | u = v \rangle$ is of type (2) of Proposition 4.1.

Proposition 4.2. *Let $M = \mathrm{Inv}\langle X | u = v \rangle$ be an Adian inverse semigroup of type* (2). *Then all the segments sewn on in an iterative step of the construction of a subgraph of $S\Gamma(w)$, for all $w \in X^+$, generated by an R-word, are labeled by the same R-word.*

Proof. If $w \in X^+$ does not contain any R-word as its subword, then $S\Gamma(w)$ is finite. So, there is nothing to prove in this case.

Let $w \in X^+$ be an arbitrary word that contains some R-words as its subwords. To prove the above statement we use induction on the number of iterative steps involved in the construction of a subgraph of $S\Gamma(w)$, generated by an R-word. Without loss of generality, we assume that v is a subword of w. We construct the linear graph of w, $(\alpha_0, \Gamma_0(w), \beta_0)$. At the first iterative step, we sew on a segment labeled by u from the initial vertex to the terminal vertex of the segment labeled by v of $(\alpha_0, \Gamma_0(w), \beta_0)$. So, the above statement is obviously true for the first iterative step.

We assume that the above statement is true for some $n \in \mathbb{N}$, where $n \geq 2$. That is, all the segments that are sewn on at the n-th iterative step are labeled by the same R-word. Since both the R-words overlap with themselves and there is no overlap between them, if some of the segments sewn on at the n-th iterative step create some new unsaturated segments, they must be labeled by the same R-word that labels all the segments sewn on at the n-th iterative step. So, all the segments sewn on at the $(n+1)$-st iterative step, are labeled by the other R-word. ∎

We can distribute the Adian presentations of type (2) into the following two categories.

(a) No proper subword of one R-word is a proper subword of the other R-word.

(b) A proper subword of one R-word is a proper subword of the other R-word.

For instance, the presentation $\langle a, b, c | aba = c^2 \rangle$ is of type (2a), and the presentation $\langle a, b | aba = b^3 \rangle$ is of type (2b) as b is common between both the R-words. The presentation $\langle a, b, c | ababa = cbabc \rangle$ is also of type (2b).

Proposition 4.3. *Let $M = \mathrm{Inv}\langle X | u = v \rangle$ be an Adian inverse semigroup of type* (2a). *If every third generation region of a subgraph of $S\Gamma(w)$, for all $w \in X^+$, generated by an R-word uses an edge of the linear automaton of w, then every higher generation region of the subgraph uses an edge of the linear automaton of w.*

Proof. If $w \in X^+$ does not contain any R-word as its subword, then $S\Gamma(w)$ is finite. So, there is nothing to prove in this case.

Let $w \in X^+$ be an arbitrary word that contains some R-words as its subwords. We apply induction on the number of iterative steps involved in the construction of a subgraph of $S\Gamma(w)$, generated by an R-word.

We assume that every third generation region of the subgraph of $S\Gamma(w)$, generated by an R-word, uses an edge of the linear graph of w, denoted by $(\alpha_0, \Gamma_0(w), \beta_0)$. For the base case of our induction, we show that every fourth generation region of the subgraph uses an edge of $(\alpha_0, \Gamma_0(w), \beta_0)$.

Since every third generation region uses an edge of $(\alpha_0, \Gamma_0(w), \beta_0)$, either the initial vertex or the terminal vertex of all the segments sewed on at the third iterative

step, lies on $(\alpha_0, \Gamma_0(w), \beta_0)$. Without loss of generality, we assume that the initial vertex of a segment, labeled by an R-word, and sewed on at the third iterative step, lies on $(\alpha_0, \Gamma_0(w), \beta_0)$. We denote this segment by s_3. Since both the R-words overlap with themselves and there is no overlap between them, if the third iterative step creates some new unsaturated segments, then they are labeled by the same R-word as s_3.

The terminal vertex of s_3 lies on a segment that was sewn on at the second iterative step and labeled by the other R-word. If s_3 creates some new unsaturated segments, then these new unsaturated segments either start from a vertex of $(\alpha_0, \Gamma_0(w), \beta_0)$ which lies before the initial vertex of s_3 and terminates at an interior vertex of s_3, or they start from an interior vertex of s_3 and passes through a segment labeled by the other R-word and sewed on at the second iterative step. The latter case is not possible, as no proper subword of one R-word is a proper subword of the other R-word. Therefore, if a fourth generation region exists, then it also uses an edge of $(\alpha_0, \Gamma_0(w), \beta_0)$.

We assume that every k-th generation region uses an edge of $(\alpha_0, \Gamma_0(w), \beta_0)$, for some $k > 4$. Then either the initial vertex or the terminal vertex of every segment that is sewed on at the k-th iterative step lies at $(\alpha_0, \Gamma_0(w), \beta_0)$. Without loss of generality, we assume that the initial vertex of a segment labeled by an R-word and sewed on at the k-th iterative step lies on $(\alpha_0, \Gamma_0(w), \beta_0)$. We denote this segment by s_k. Since both the R-words overlap with themselves and there is no overlap between both of them, if the k-th iterative step creates some new unsaturated segments, then they will be labeled by the same R-word as s_k. Moreover, these new unsaturated segments either start from a vertex of $(\alpha_0, \Gamma_0(w), \beta_0)$ which lies before the initial vertex of s_k and terminates at an interior vertex of s_k or they start from an interior vertex of s_k and pass through the segment labeled by the other R-word that was sewed on at the $(k-1)$-st iterative step. The latter case is not possible as no proper subword of one R-word is a proper subword of the other R-word. Therefore, if the $(k+1)$-st generation region exists, then it also uses an edge of $(\alpha_0, \Gamma_0(w), \beta_0)$. ∎

We call a vertex δ of $S\Gamma(w)$, for some $w \in X^+$, to be a *special vertex*, if δ is the terminal vertex of a segment labeled by an R-word and the initial vertex of a segment labeled by the same R-word.

Proposition 4.4. *If a word $t \in X^+$ overlaps with itself, then t is in one of the following forms.*

(1) $t \equiv x^n$, *where* $x \in X^+$, *and* $n \geq 2$.

(2) $t \equiv (xs)^n x$, *for some* $x, s \in X^+$, *and* $n \geq 1$. *For* $n = 1$, x *is the maximal prefix of* t *that is also a suffix of* t. *For* $n \geq 2$, $(xs)^{n-1}x$ *is the maximal prefix of* t *that is also a suffix of* t.

Proof. If $t \equiv x^n$, for some $x \in X^+$ and $n \geq 2$, then clearly t overlaps with itself. If $t \not\equiv x^n$, for some $x \in X^+$, then there are two possibilities.

(1) The maximal prefix $x \in X^+$ of t, that is also a suffix of t, does not overlap with itself in t. In this case, t is of the form $t \equiv xsx \equiv (xs)^1 x$, for some $x, s \in X^+$. Here $s \not\equiv \varepsilon$. Otherwise, $t \equiv x^2$, for some $x \in X^+$.

It is worth noticing that the prefix word x can possibly overlap with itself. That is, x can have a same prefix and suffix, but inside t, the prefix x and the suffix x are separated from each other by a nonempty word s.

(2) The maximal prefix $y_0 \in X^+$ of t, that is also a suffix of t, overlaps with itself in t. In this case, since y_0 overlaps with itself in t, a prefix of y_0 is also a suffix of y_0. Let $y_1 \in X^+$ be the maximal prefix of y_0 that is also a suffix of y_0. There are two possibilities.

(2.1) If y_1 does not overlap with itself in y_0, then $y_0 \equiv y_1 s y_1$, for some $y_1, s \in X^+$ (just like case (1) above). In this case, $t \equiv (y_1 s y_1) s y_1 \equiv (y_1 s)^2 y_1$. Here $s \not\equiv \varepsilon$. Otherwise, $t \equiv y_1^3$, for some $y_1 \in X^+$.

(2.2) If y_1 overlaps with itself in y_0, then a prefix of y_1 is also a suffix of y_1. Let y_2 be the maximal prefix of y_1 that is also a suffix of y_1. Again, there are two possibilities.

(2.2.1) If y_2 does not overlap with itself in y_1, then $y_1 \equiv y_2 s y_2$, for some $y_2, s \in X^+$ (just like case (1) above). In this case, $y_0 \equiv (y_2 s y_2) s y_2 \equiv (y_2 s)^2 y_2$, and $t \equiv (y_2 s)^3 y_2$, for some $y_2, s \in X^+$. Clearly $s \not\equiv \varepsilon$. Otherwise, $t \equiv y_2^4$, for some $y_2 \in X^+$.

(2.2.2) If y_2 overlaps with itself in y_1, then a prefix of y_2 is also a suffix of y_2. Let y_3 be the maximal prefix of y_2 that is also a suffix of y_2.

If y_3 does not overlap with itself in y_2, then $y_2 \equiv y_3 s y_3$, for some $y_3, s \in X^+$. In this case, $y_1 \equiv (y_3 s)^2 y_3$ which implies that $y_0 \equiv (y_3 s)^3 y_3$, and consequently $t \equiv (y_3 s)^4 y_3$, for some $y_3, s \in X^+$. Clearly $s \not\equiv \varepsilon$. Otherwise, $t \equiv y_3^5$, for some $y_3 \in X^+$.

If y_3 overlaps with itself in y_2, then by using the same argument we can show that $t \equiv (y_4 s)^5 y_4$, for some $y_4, s \in X^+$.

We observe that y_4 is a proper prefix of y_3, y_3 is a proper prefix of y_2, y_2 is a proper prefix of y_1, y_1 is a proper prefix of y_0, and y_0 is a proper prefix of t. Hence, this process eventually terminates after n (a finite) number of steps and $t \equiv (zs)^n z$, for some $z, s \in X^+$. ∎

Proposition 4.5. *Let $M = \mathrm{Inv}\langle X | u = v\rangle$ be an Adian inverse semigroup of type* (2a). *A segment of $S\Gamma(w)$, for some $w \in X^+$, labeled by u^2 contains a subsegment labeled by u passing through the special vertex of the segment u^2 if and only if $u \equiv x^n$, for some $x \in X^+$ and $n \geq 2$.*

Proof. The converse of the above statement is obvious. Therefore, we only prove the direct statement.

Let a segment of $S\Gamma(w)$, for some $w \in X^+$, labeled by u^2 contain a subsegment labeled by u passing through the special vertex, and $u \not\equiv x^n$, for any $x \in X^+$ and $n \geq 2$. Then by Proposition 4.4, $u \equiv (xs)^m x$, for any $x, s \in X^+$ and $m \geq 1$. If $m = 1$ then x is the maximal prefix of u that is also a suffix of u. Otherwise, $(xs)^{m-1}x$ is the maximal prefix of u that is also a suffix of u.

Case 1: If $m = 1$, then $u \equiv xsx$, for some $x, s \in X^+$, where x is the maximal prefix of u that is also a suffix of u. By hypothesis there exists a subsegment of the segment u^2, that is labeled by u, and passing through the special vertex of the segment u^2. If u can be read starting from the initial vertex of x (where x is the suffix of the first u in the segment u^2), then $x \equiv s$. This implies that $u \equiv x^3$, which contradicts our assumption that $u \not\equiv x^n$, for some $x \in X^+$ and $n \geq 2$.

If u can be read starting from a vertex that lies before the initial vertex of x (where x is the suffix of the first u in the segment u^2), then a longer prefix of u will be a suffix of u. This contradicts the maximality of x.

If u can be read starting from a vertex that lies between the initial and terminal vertices of x (where x is the suffix of the first u in the segment u^2), then again a longer prefix of u will be a suffix of u. This contradicts the maximality of x.

Case 2: If $m > 1$, then $u \equiv (xs)^m x$, for some $x, s \in X^+$, where $(xs)^{m-1}x$ is the maximal prefix of u that is also a suffix of u. By hypothesis, there exists a subsegment of the segment u^2, that is labeled by u, and passing through the special vertex of the segment u^2. If this subsegment starts from the initial vertex of y (where $y \equiv (xs)^{m-1}x$ a suffix of the first u in the segment u^2), this is only possible when $x \equiv s$. This implies that $u \equiv x^{2m+1}$, for some $x \in X^+$, which contradicts our hypothesis that u is not a proper power a positive word.

If the subsegment labeled by u starts from a vertex that lies before the initial vertex of y, then a longer prefix of u will be a suffix of u, which contradicts the maximality of $(xs)^{m-1}x$.

Similarly, if the subsegment labeled by u starts from a vertex that lies between the initial and terminal vertices of y, then a longer prefix of u will be a suffix of u, which contradicts the maximality of $(xs)^{m-1}x$. ∎

We call a third generation region of a subgraph of $S\Gamma(w)$, for some $w \in X^+$, generated by an R-word, that does not use an edge of the linear automaton of w, to be a *special region*.

Proposition 4.6. *Let $M = \mathrm{Inv}\langle X | u = v\rangle$ be an Adian inverse semigroup of type* (2a). *There exists a word $w \in X^+$ such that there is a subgraph of $S\Gamma(w)$ generated by an R-word containing a special region if an only if $u \equiv x^n$ and $v \equiv y^m$, for some $x, y \in X^+$ and $n, m \geq 2$.*

Proof. The converse of the above proposition is easy to see. Let $u \equiv x^n$ and $v \equiv y^m$, for some $x, y \in X^+$ and $n, m \geq 2$. Then $S\Gamma(y^{p_1}uy^{q_1})$, where $p_1 + q_1 = n$, and $S\Gamma(x^{p_2}vx^{q_2})$, where $p_2 + q_2 = m$, contain a third generation region that does not use an edge of linear automatons of $y^{p_1}uy^{q_1}$ and $x^{p_2}vx^{q_2}$, respectively.

Now we prove the direct statement. We assume that a subgraph of $S\Gamma(w)$, for some $w \in X^+$, generated by an R-word, contains a special region. Without loss of generality, we assume that one side of this special region that lies on the boundary of second generation regions of the subgraph, is labeled by u. Since $\langle X|u = v\rangle$ is a presentation of type (2a), it follows that there is no letter from X that belongs both to u and v, hence one side of the special region labeled by u lies on a segment labeled by u^2 of the subgraph of $S\Gamma(w)$, generated by an R-word (see Figure 11). The segment labeled by u^2 contains a special vertex and the one side of the special region labeled by u is passing through this special vertex. So, by Proposition 4.5, $u \equiv x^n$, for some $x \in X^+$ and $n \geq 2$. The segment labeled by u^2 can only exist, if there is a segment labeled by v^2 after the first iterative step. The segment labeled by v^2 contains a special vertex that lies on the segment labeled by an R-word, and sewed on at the first iterative step. Since $\langle X|u = v\rangle$ is an Adian presentation of type (2a), the segment sewed on at the first iterative step is labeled by v. The segment labeled by v^2 contains a segment labeled by v that is passing through the special vertex. By Proposition 4.5, $v \equiv y^m$, for some $y \in X^+$ and $m \geq 2$. ∎

An Adian inverse semigroup $M = \mathrm{Inv}\langle X|u = v\rangle$ is of type (3) when the presentation $\langle X|u = v\rangle$ is of type (3) of Proposition 4.1.

Proposition 4.7. *Let $M = \mathrm{Inv}\langle X|u = v\rangle$ be an Adian inverse semigroup of type* (3)*, such that no R-word overlaps with itself. Then there is no third generation region in a subgraph of w generated by an R-word, for all $w \in X^+$, that does not use an edge of the linear automaton of w.*

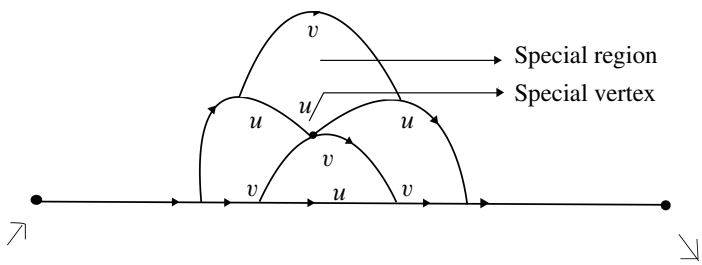

Figure 11. Special region of a subgraph of $S\Gamma(w)$ generated by an R-word

Proof. Without loss of generality, we assume that $u \equiv xy$ and $v = yz$, where $x, y, z \in X^+$, and y is the maximal prefix of v that is also a suffix of u. We also assume that u is a subword of w, for some $w \in X^+$. We show that the subgraph of $S\Gamma(w)$ generated by u, cannot contain a third generation region that does not use an edge of the linear automaton of w. Since w is any arbitrary positive word, the above proposition is true for any positive word that contains u as its subword.

We construct the linear graph of w. We sew on a segment labeled v from the initial vertex to the terminal vertex of the segment labeled by u. This creates the first generation region of the subgraph of $S\Gamma(w)$ generated by u. Since u and v do not overlap with themselves, and no prefix of u is a suffix of v, the second generation region of the subgraph generated by u exists only if the word u is followed by x in w. So, we assume that the subword u is followed by x in w. Now, we can find an unsaturated segment labeled by u starting from the initial vertex of x and terminating at an interior vertex of the segment labeled by v that was sewn on at the first step. We sew on a segment labeled v from the initial vertex to the terminal vertex of this unsaturated segment, and create the second generation region of the subgraph generated by u.

If we assume that the subgraph of $S\Gamma(w)$ generated by u contains a third generation region that does not use an edge of the linear automaton of w, then there exists an unsaturated segment labeled by an R-word that starts from an interior vertex of the segment labeled by v (that was sewn at the second step) and terminates at a vertex of the segment labeled by v (that was sewn at the first step). If this unsaturated segment is labeled by v, then this implies that v overlaps with itself, which is a contradiction to the fact that no R-word overlaps with itself. If this unsaturated segment is labeled by u, then it implies that a suffix of v is also a prefix of u, which is also a contradiction.

A dual argument can be used to show that a subgraph of $S\Gamma(w)$, for all $w \in X^+$, generated by v cannot contain a third generation region that does not use an edge of the linear automaton of w. ∎

Proposition 4.8. *Let $M = \mathrm{Inv}\langle X | u = v\rangle$ be an Adian inverse semigroup of type* (3)*, such that no R-word overlaps with itself. Then every region of a subgraph of $S\Gamma(w)$, for all $w \in X^+$, generated by an R-word uses an edge of the linear automaton of w.*

Proof. Without loss of generality, we assume that $u \equiv xy$ and $v = yz$, where $x, y, z \in X^+$, and y is the maximal prefix of v that is also a suffix of u. Let w be an arbitrary positive word that contains u as its subword. We prove the above statement for a subgraph of $S\Gamma(w)$ generated by u, a similar argument can be used to prove the above statement for a subgraph generated by v.

We use mathematical induction on the number of iterative steps involved in the construction of the subgraph generated by u. We have already seen in Proposition 4.7 that every region of the subgraph generated by u up to the third generation uses an

edge of the linear automaton of w. Which proves the base case for the mathematical induction (that is, every third generation region of the subgraph generated by u uses an edged of the linear automaton of w).

We assume that every region of the subgraph generated by u up to k-th iterative step uses an edge of the linear automaton of w, for some $k \geq 3$. We show that the $(k+1)$-generation region of the subgraph generated by u also uses an edge of linear automaton of w.

We observe in the proof of Proposition 4.7 that we only sew on a segment labeled by v in the first three iterative steps involved in the construction of the subgraph generated by u. So if we continue in the same manner up to the k-th iterative step, then the segment labeled by v is sewn on at the k-th iterative step to form the k-th generation region of the subgraph generated by u. The initial vertex of this segment lies at the linear automaton of w and the terminal vertex of this segment lies on the segment labeled by v sewed on at the $(k-1)$-st iterative step.

We cannot find an unsaturated segment labeled by an R-word that starts from an interior vertex of the segment labeled by v (sewed on at the k-th iterative step) and terminating anywhere after passing through the terminal vertex of this segment labeled by v. Because if such a segment exists and it is labeled by v, then it means that v overlaps with itself. This contradicts the fact that no R-word overlaps with itself. If such a segment is labeled by u then it is only possible when a prefix of u is also a suffix of v, which again a contradiction. So if the k-th iterative step creates a new unsaturated segment, then this segment will start from a vertex of the linear automaton of w and after passing through the initial vertex of the segment v (sewed on at the k-th iterative step) terminates at an interior vertex of this segment v. Hence the $(k+1)$-st generation region also uses an edge of the linear automaton of w. ∎

Theorem 4.1. *Let $M = \mathrm{Inv}\langle X | u = v\rangle$ be an Adian inverse semigroup, such that no R-word is a subword of the other R-word, and the relation (u, v) is in one of the following forms:*

(1) *No R-word overlaps with itself or with the other R-word.*

(2) *One of the R-words overlaps with itself, and the other R-word neither overlaps with itself nor with the former R-word.*

(3) *Both R-words overlap with themselves, there is no overlap between both the R-words, and at least one of the R-words is not of the form x^n, for some $x \in X^+$ and $n \geq 2$.*

(4) *A prefix of one R-word is a suffix of the other R-word, no suffix of the former R-word is a prefix of the latter R-word, and no R-word overlaps with itself.*

Then the word problem is decidable for M.

Proof. If the relation (u, v) is of the form (1), then, by Lemma 4.1, $S\Gamma(w)$, for all $w \in X^+$, is finite. Therefore, by Theorem 1.3, $S\Gamma(w)$, for all $w \in (X \cup X^{-1})^*$, is finite, which implies that the word problem is decidable for M.

If the relation (u, v) is of the form (2), then, by Lemma 4.2, every subgraph of $S\Gamma(w)$, for all $w \in X^+$, generated by an R-word is finite. Therefore, by Theorem 3.2, $S\Gamma(w)$, for all $w \in X^+$, is finite. So, by Theorem 1.3, $S\Gamma(w)$, for all $w \in (X \cup X^{-1})^*$, is finite. Hence, the word problem is decidable for M.

If the relation (u, v) is of the form (3), then by Proposition 4.6, none of the subgraphs of $S\Gamma(w)$, for all $w \in X^+$, generated by an R-word, contains a special region. Therefore, for any arbitrary $w \in X^+$, every region of a subgraph of $S\Gamma(w)$, generated by an R-word, uses an edge of the linear automaton of w. Since w is a word of finite length, every subgraph of $S\Gamma(w)$, generated by an R-word, is finite. Since $w \in X^+$ is arbitrary, every subgraph of $S\Gamma(w)$, for all $w \in X^+$, is finite. It follows by Theorem 3.2 that $S\Gamma(w)$, for all $w \in X^+$, is finite. By Theorem 1.3, $S\Gamma(w)$, for all $w \in (X \cup X^{-1})^*$, is finite. So, the word problem is decidable for M.

If the relation (u, v) is of the form (4), then by Proposition 4.8 every region of a subgraph of $S\Gamma(w)$, for all $w \in X^+$, generated by an R-word, uses an edge of the linear automaton of w. Since w is a word of finite length, every subgraph of $S\Gamma(w)$, generated by an R-word, is finite. Since $w \in X^+$ is arbitrary, every subgraph of $S\Gamma(w)$, for all $w \in X^+$, is finite. It follows by Theorem 3.2 that $S\Gamma(w)$, for all $w \in X^+$, is finite. By Theorem 1.3, $S\Gamma(w)$, for all $w \in (X \cup X^{-1})^*$, is finite. So, the word problem is decidable for M. ∎

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Muhammad Inam
Department Of Mathematics and Computer Science, Southern Arkansas University, Magnolia, AR 71753, USA; minam@saumag.edu