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# A sphericity criterion for strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$ via invariant curves

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**Abstract.** We prove that if every chain on a strictly pseudoconvex hypersurface $M$ in $\mathbb{C}^2$ coincides with the boundary of a stationary disc, then $M$ is locally spherical.

## 1. Introduction

Every strictly pseudoconvex hypersurface $M \subset \mathbb{C}^2$ bounding a domain $\Omega \subset \mathbb{C}^2$ carries two natural, biholomorphically invariant families of real curves: the so-called *chains* and boundaries of *stationary discs*. These come from very different types of geometrical constructions. Chains have been introduced by Chern and Moser [5] as the CR geometry analogue of geodesics in Riemannian geometry. Stationary discs, on the other hand, are the solutions to the Euler–Lagrange equations of Kobayashi extremal discs in $\Omega$. If $M$ is in addition real-analytic, it carries a third natural biholomorphically invariant family of real curves: traces of Segre varieties. A theorem of Faran [6] shows that if the traces of Segre varieties agree with the chains, then $M$ is locally spherical. In a former paper [1], we showed that if the traces of Segre varieties agree with the traces of stationary discs, then $M$ is also locally spherical.

In this paper, we address the remaining question: if the traces of stationary discs coincide with chains, is $M$ also necessarily spherical? We have been asked this repeatedly when presenting the results in [1], and it turns out that the answer is also yes. The natural setting for this question is for sufficiently smooth hypersurfaces.

**Theorem 1.1.** *Assume that $M$ is a strictly pseudoconvex hypersurface of class $\mathcal{C}^{12}$ in $\mathbb{C}^2$. If the chains of $M$ are boundaries of stationary discs, then $M$ is locally spherical.*

We remark that $M$ in Theorem 1.1 is not assumed to be closed (so the theorem is a local result). In order to prove this theorem, we cannot utilize the cited results. Instead, we rely on Fefferman's characterization of chains as projections of light rays of an associated Lorentz metric and analyzing its Hamiltonian. We construct a special family of chains centered at the origin and show that if each of the members of this family is the trace of a stationary disc, then the origin is an umbilical point.

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The organization of this paper is to review the basics in Section 2, summarize facts about the chains in Section 3, and give the proof of a (slightly sharpened) version of the theorem in Section 4.

## 2. Preliminaries

### 2.1. Intrinsic geometry of strictly pseudoconvex hypersurfaces

In this section, we give a quick review of the basic local biholomorphic equivalence theory for strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$. We thank one of the referees for the suggestion to include some background material in this paper, and hope the reader will enjoy it.

A hypersurface $M \subset \mathbb{C}^2$ inherits the complex structure of $\mathbb{C}^2$ on its complex tangent spaces $T_p^c M = T_p M \cap i T_p M$. The complex tangent bundle $T^c M$ is defined by the vanishing of a not uniquely determined 1-form $\theta$, which we usually call *characteristic form*; the annihilator $(T^c M)^\perp = T^0 M = \mathcal{N} M \subset T^* M$ is called the *characteristic* or *conormal* bundle of $M$. If $d\theta$ induces a hermitian inner product on $T^c M$ by $h(X,Y) = i\theta([X,\bar{Y}]) = -i\, d\theta(X,\bar{Y})$, called the Levi form of $M$ with respect to $\theta$, then we say that $M$ is *strictly pseudoconvex*.

Thus, geometrically speaking, a strictly pseudoconvex hypersurface $M \subset \mathbb{C}^2$ can be thought of as a 3-dimensional manifold with a contact structure with some additional compatibility conditions. The choice of a contact form gives rise to a *pseudohermitian structure* $(M,\theta)$.

The ambivalence in choosing a contact form gives rise to fascinating mathematics which mixes aspects of complex, conformal, contact, and symplectic geometry. The *equivalence problem* for strictly pseudoconvex hypersurfaces in $\mathbb{C}^2$ was solved in a series of groundbreaking papers by E. Cartan [3, 4] applying his method of equivalence; later, Tanaka [16] and Chern–Moser [5] gave solutions of the problem for strictly pseudoconvex and Levi-nondegenerate hypersurfaces in higher dimensions. For the convenience of the reader, we recall some of the necessary background, with a view towards the Fefferman construction of chains which we are going to use.

If we start with an arbitrary characteristic form $\theta$, we can consider a real line bundle $E$ over $M$ consisting of the multiples $u\theta$, where $u > 0$. The form $\omega = u\theta$ is intrinsically defined, and (on $E$) we have

$$d\omega = i g_{1\bar{1}}\, \omega^1 \wedge \bar{\omega}^1 + \omega \wedge \varphi,$$

with a real one form $\varphi$; the forms $\omega$, $\omega^1$, $\bar{\omega}^1$, and $\varphi$ span $\mathbb{C}T^*E$. Since we assume that $M$ is strictly pseudoconvex, we have $g_{1\bar{1}} \neq 0$, and we will assume for simplicity that $g_{1\bar{1}} = 1$; the more general case where $g_{1\bar{1}}$ is not assumed to be constant follows in a similar but more involved way. Since this short review is only meant to recall how the main computations work, we opted to keep the simple variant. We follow the notation of Chern–Moser [5], so that the reader can pick up the necessary modifications in that source easily.

Now any other frame of $\mathbb{C}T^*E$ satisfying the condition above is given by

$$\begin{pmatrix} \tilde{\omega} \\ \tilde{\omega}^1 \\ \bar{\tilde{\omega}}^1 \\ \tilde{\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda & \mu & 0 & 0 \\ \bar{\lambda} & 0 & \bar{\mu} & 0 \\ s & i\mu\bar{\lambda} & -i\bar{\mu}\lambda & 1 \end{pmatrix} \begin{pmatrix} \omega \\ \omega^1 \\ \bar{\omega}^1 \\ \varphi \end{pmatrix},$$

where $|\mu|^2 = 1$. The group of matrices of this form is denoted by $G_1$, and we can form the principal $G_1$-bundle $Y$ over $E$; thus $s$, $\lambda$, and $\mu$ are fiber coordinates. One has the integrability condition

$$d\omega^1 = \omega^1 \wedge \alpha + \omega \wedge \beta,$$

for every frame as above, with some not uniquely determined forms $\alpha$ and $\beta$.

One then obtains a uniquely determined frame $\omega, \omega^1, \bar{\omega}^1, \varphi, \alpha, \beta, \psi$ of $\mathbb{C}T^*Y$ satisfying a number of identities:

**Theorem 2.1** ([3]). *There exists a unique frame $\omega, \omega^1, \bar{\omega}^1, \varphi, \alpha, \beta, \psi$ of $\mathbb{C}T^*Y$ and invariantly defined functions $Q$ and $R$ such that*

$$\begin{aligned} d\omega &= i\omega^1 \wedge \bar{\omega}^1 + \omega \wedge \varphi, \\ d\omega^1 &= \omega^1 \wedge \alpha + \omega \wedge \beta, \\ \varphi &= \alpha + \bar{\alpha}, \\ d\varphi &= i\omega^1 \wedge \bar{\beta}, -i\bar{\omega}^1 \wedge \beta + \omega \wedge \psi, \\ d\alpha &= i\bar{\omega}^1 \wedge \beta + 2i\omega^1 \wedge \bar{\beta} - \frac{\psi}{2} \wedge \omega, \\ d\beta &= \bar{\alpha} \wedge \beta - \frac{\psi}{2} \wedge \omega^1 + Q\bar{\omega}^1 \wedge \omega, \\ d\psi &= \varphi \wedge \psi + 2i\beta \wedge \bar{\beta} + (R\omega^1 + \bar{R}\bar{\omega}^1) \wedge \omega. \end{aligned}$$

**Definition 2.2.** A curve $\gamma$ is called a *chain* if it solves the system of ODEs

$$\omega^1 = \beta = 0.$$

We note that the equations for the real forms $\omega$, $\varphi$, and $\psi$ along a chain simply read

$$d\omega = \omega \wedge \varphi, \quad d\varphi = \omega \wedge \psi, \quad d\psi = \varphi \wedge \psi.$$

One can use these equations to therefore introduce a canonical parameter along the chain defined up to a linear fractional map.

We now give, as promised, the details for the construction of the canonical forms in Theorem 2.1, basically to set the stage. We also point the reader to the book [13] by Jacobowitz.

First, one observes that exterior differentiation of the frame conditions

$$\begin{aligned} d\omega &= i\omega^1 \wedge \bar{\omega}^1 + \omega \wedge \varphi, \\ d\omega^1 &= \omega^1 \wedge \alpha + \omega \wedge \beta, \end{aligned} \tag{2.1}$$

yields

$$
\begin{aligned}
0 &= i\,d\omega^1 \wedge \bar\omega^1 - i\omega^1 \wedge d\bar\omega^1 + d\omega \wedge \varphi - \omega \wedge d\varphi \\
&= i(\omega^1 \wedge \alpha + \omega \wedge \beta) \wedge \bar\omega^1 - i\omega^1 \wedge (\bar\omega^1 \wedge \bar\alpha + \omega \wedge \bar\beta) \\
&\quad + (i\omega^1 \wedge \bar\omega^1 + \omega \wedge \varphi) \wedge \varphi - \omega \wedge d\varphi \\
&= i(-\alpha - \bar\alpha + \varphi) \wedge \omega^1 \wedge \bar\omega^1 + (-d\varphi + i\beta \wedge \bar\omega^1 - i\bar\beta \wedge \omega^1) \wedge \omega, \\
0 &= d\omega^1 \wedge \alpha - \omega^1 \wedge d\alpha + d\omega \wedge \beta - \omega \wedge d\beta \\
&= \omega \wedge \beta \wedge \alpha - \omega^1 \wedge d\alpha + (i\omega^1 \wedge \bar\omega^1 + \omega \wedge \varphi) \wedge \beta - \omega \wedge d\beta \\
&= (-d\alpha + i\bar\omega^1 \wedge \beta) \wedge \omega^1 + (\beta \wedge \alpha + \varphi \wedge \beta - d\beta) \wedge \omega.
\end{aligned} \tag{2.2}
$$

It follows from the first equation in (2.2) that

$$-\alpha - \bar\alpha + \varphi = A\omega^1 + \bar A\bar\omega^1 + C\omega, \quad \text{with } C = \bar C,$$

i.e., with the choice $\tilde\alpha = \alpha + A\omega^1 + \frac{C}{2}\omega$ we have $\varphi = \tilde\alpha + \bar{\tilde\alpha}$. It is easy to see that $\varphi$ with this property are unique up to multiples of $\omega$, and for such a choice of $\varphi$, we have

$$-d\varphi + i\beta \wedge \bar\omega^1 - i\bar\beta \wedge \omega^1 = -\omega \wedge \psi, \quad \text{for a real 1-form } \psi.$$

Summarizing, we have imposed the following restrictions:

$$
\begin{aligned}
d\omega &= i\omega^1 \wedge \bar\omega^1 + \omega \wedge \varphi, \\
d\omega^1 &= \omega^1 \wedge \alpha + \omega \wedge \beta, \\
\varphi &= \alpha + \bar\alpha, \\
d\varphi &= i\omega^1 \wedge \bar\beta, -i\bar\omega^1 \wedge \beta + \omega \wedge \psi,
\end{aligned}
$$

and after simplification of the second equation in (2.2), we have the equation

$$(d\alpha - i\bar\omega^1 \wedge \beta) \wedge \omega^1 + (d\beta - \bar\alpha \wedge \beta) \wedge \omega = 0. \tag{2.3}$$

The forms $\alpha$, $\beta$, and $\psi$ satisfying these identities are uniquely determined up to a change

$$
\begin{aligned}
\tilde\alpha &= \alpha + D\omega, \\
\tilde\beta &= \beta + D\omega^1 + E\omega, \\
\tilde\psi &= \psi + G\omega + i(\bar E\omega^1 - E\bar\omega^1),
\end{aligned} \tag{2.4}
$$

where $D$ is purely imaginary and $G$ is real.

One can then check that the form

$$\Phi = d\alpha - i\bar\omega^1 \wedge \beta - 2i\omega^1 \wedge \bar\beta$$

satisfies $\Phi = -\bar\Phi$ modulo $\omega$:

$$
\begin{aligned}
\Phi + \bar\Phi &= d\alpha - i\bar\omega^1 \wedge \beta - 2i\omega^1 \wedge \bar\beta + d\bar\alpha + i\omega^1 \wedge \bar\beta + 2i\bar\omega^1 \wedge \beta \\
&= d\varphi - i\omega^1 \wedge \bar\beta + i\bar\omega^1 \wedge \beta = \omega \wedge \psi.
\end{aligned}
$$

Since in addition $\Phi \wedge \omega^1 = 0$ modulo $\omega$ by (2.3), we have

$$d\alpha - i\bar\omega^1 \wedge \beta - 2i\omega^1 \wedge \bar\beta \cong S\omega^1 \wedge \bar\omega^1 \mod \omega,$$

and the real number $S$ transforms in the following way under a change of frame as above (computing modulo $\omega$):

$$\begin{aligned}\tilde{S}\omega^1 \wedge \bar{\omega}^1 &= d\tilde{\alpha} - i\bar{\omega}^1 \wedge \tilde{\beta} - 2i\omega^1 \wedge \bar{\tilde{\beta}} \\ &= d(\alpha + D\omega) - i\bar{\omega}^1 \wedge (\beta + D\omega^1) - 2i\omega^1 \wedge (\bar{\beta} - D\bar{\omega}^1) \\ &= d\alpha + D \wedge d\omega - i\bar{\omega}^1 \wedge \beta + 3iD\omega^1 \wedge \bar{\omega}^1 - 2i\omega^1 \wedge \bar{\beta} \\ &= (S + 4iD)\omega^1 \wedge \bar{\omega}^1.\end{aligned}$$

The condition $S = 0$ therefore is possible for a certain $D$, and fixes the form $\alpha$ uniquely. We proceed to calculate with

$$\Phi = d\alpha - i\bar{\omega}^1 \wedge \beta - 2i\omega^1 \wedge \bar{\beta} = \lambda \wedge \omega$$

for some 1-form $\lambda$, and we note right away that

$$(\lambda + \bar{\lambda}) \wedge \omega = \Phi + \bar{\Phi} = \omega \wedge \psi,$$

in other words, $\lambda + \bar{\lambda} = -\psi$ modulo $\omega$. Plugging

$$d\alpha - i\bar{\omega}^1 \wedge \beta - 2i\omega^1 \wedge \bar{\beta} = \lambda \wedge \omega$$

into (2.3), we obtain

$$0 = (d\alpha - i\bar{\omega}^1 \wedge \beta) \wedge \omega^1 + (d\beta - \bar{\alpha} \wedge \beta) \wedge \omega = (d\beta - \bar{\alpha} \wedge \beta + \omega^1 \wedge \lambda) \wedge \omega,$$

so that

$$d\beta - \bar{\alpha} \wedge \beta + \omega^1 \wedge \lambda = \mu \wedge \omega$$

for some other 1-form $\mu$.

We next take the derivative of

$$d\alpha - i\bar{\omega}^1 \wedge \beta - 2i\omega^1 \wedge \bar{\beta} = \lambda \wedge \omega,$$

which modulo $\omega$ yields

$$\begin{aligned}0 &= id\bar{\omega}^1 \wedge \beta - i\bar{\omega}^1 \wedge d\beta + 2id\omega^1 \wedge \bar{\beta} - 2i\omega^1 \wedge d\bar{\beta} - \lambda \wedge d\omega \\ &= i\bar{\omega}^1 \wedge (\bar{\alpha} \wedge \beta - d\beta) + 2i\omega^1 \wedge (\alpha \wedge \bar{\beta} - d\bar{\beta}) - i\lambda \wedge \omega^1 \wedge \bar{\omega}^1 \\ &= -i\bar{\omega}^1 \wedge (\omega^1 \wedge \lambda) - 2i\omega^1 \wedge (\bar{\omega}^1 \wedge \bar{\lambda}) - i\lambda \wedge \omega^1 \wedge \bar{\omega}^1 \\ &= 2i\bar{\omega}^1 \wedge \omega^1 \wedge \bar{\lambda}.\end{aligned}$$

It follows that we can write $\lambda = -\frac{\psi}{2} + V\omega^1 - \bar{V}\bar{\omega}^1 + a\omega$, and we have a complete expression for $\lambda$, and therefore, for

$$\Phi = d\alpha - i\bar{\omega}^1 \wedge \beta - 2i\omega^1 \wedge \bar{\beta} = \lambda \wedge \omega = -\frac{\psi}{2} \wedge \omega + V\omega^1 \wedge \omega - \bar{V}\bar{\omega}^1 \wedge \omega,$$

and the effect of the frame change

$$\begin{aligned}\tilde{\beta} &= \beta + E\omega, \\ \tilde{\psi} &= \psi + G\omega + i(\bar{E}\omega^1 - E\bar{\omega}^1),\end{aligned}$$

yields

$$\tilde{V} = V - \frac{3i}{2}\bar{E}.$$

We can thus choose $\beta$ uniquely requiring that $V = 0$ and now also have

$$d\alpha = i\bar{\omega}^1 \wedge \beta + 2i\omega^1 \wedge \bar{\beta} - \frac{\psi}{2} \wedge \omega. \tag{2.5}$$

Substituting (2.5) back into (2.3) yields

$$(d\alpha - i\bar{\omega}^1 \wedge \beta) \wedge \omega^1 + (d\beta - \bar{\alpha} \wedge \beta) \wedge \omega = \left(d\beta - \bar{\alpha} \wedge \beta + \frac{\psi}{2} \wedge \omega^1\right) \wedge \omega,$$

so that

$$d\beta - \bar{\alpha} \wedge \beta + \frac{\psi}{2} \wedge \omega^1 = \nu \wedge \omega. \tag{2.6}$$

Taking the derivative of $d\varphi = i\omega^1 \wedge \bar{\beta} - i\bar{\omega}^1 \wedge \beta + \omega \wedge \psi$ and using (2.1) and (2.6), we obtain

$$\left(d\psi - \varphi \wedge \psi - 2i\beta \wedge \bar{\beta} + i\omega^1 \wedge \bar{\nu} - i\bar{\omega}^1 \wedge \nu\right) \wedge \omega = 0.$$

We can therefore write

$$\Psi = d\psi - \varphi \wedge \psi - 2i\beta \wedge \bar{\beta} = i\bar{\omega}^1 \wedge \nu - i\omega^1 \wedge \bar{\nu} + \varrho \wedge \omega. \tag{2.7}$$

In the next (and last) step, we take the exterior derivative of (2.6), obtaining after using (2.5), (2.6), (2.7), and (2.1), computing modulo $\omega$:

$$\begin{aligned} 0 &= -d\bar{\alpha} \wedge \beta + \bar{\alpha} \wedge d\beta + \frac{d\psi}{2} \wedge \omega^1 - \frac{\psi}{2} \wedge d\omega^1 - d\nu \wedge \omega + \nu \wedge d\omega \\ &= -i\omega^1 \wedge \bar{\beta} \wedge \beta - \bar{\alpha} \wedge \frac{\psi}{2} \wedge \omega^1 + \frac{1}{2}(\varphi \wedge \psi + 2i\beta \wedge \bar{\beta} + i\bar{\omega}^1 \wedge \nu) \wedge \omega^1 \\ &\quad - \frac{\psi}{2} \wedge \omega^1 \wedge \alpha + \nu \wedge (i\omega^1 \wedge \bar{\omega}^1) \\ &= -\frac{3i}{2}\nu \wedge \omega^1 \wedge \bar{\omega}^1 + \frac{1}{2}\underbrace{(\varphi - \alpha - \bar{\alpha})}_{=0} \wedge \psi \wedge \omega^1. \end{aligned}$$

Since $\nu$ is only defined modulo $\omega$, we can therefore write $\nu = P\omega^1 + Q\bar{\omega}^1$. It turns out that the exterior differentiation of (2.5), using the expressions for $d\beta$ and $d\psi$ already obtained, implies that $P = \bar{P}$ is real. Hence from (2.7) we have

$$\Psi = i\bar{\omega}^1 \wedge \nu - i\omega^1 \wedge \bar{\nu} + \varrho \wedge \omega = 2iP\bar{\omega}^1 \wedge \omega^1 + \varrho \wedge \omega.$$

The last free parameter $G$ in the frame change $\tilde{\psi} = \psi + G\omega$ transforms $\tilde{P} = P + G$, and we finally have an invariantly defined frame as we wanted by requiring $P = 0$. Note that with this choice we have

$$d\beta = \bar{\alpha} \wedge \beta - \frac{\psi}{2} \wedge \omega^1 + Q\bar{\omega}^1 \wedge \omega. \tag{2.8}$$

We can also differentiate the equation for $\Psi$ above, and obtain from it that

$$\rho \wedge \omega^1 \wedge \bar{\omega}^1 = 0,$$

so that $\rho$ is a linear combination $R\omega^1 + S\bar{\omega}^1$, which by reality of $\Psi$ also implies $S = \bar{R}$. Hence we can write

$$d\psi = \varphi \wedge \psi + 2i\beta \wedge \bar{\beta} + (R\omega^1 + \bar{R}\bar{\omega}^1) \wedge \omega. \tag{2.9}$$

### 2.2. The canonical connection

Theorem 2.1 has a neat description in terms of a canonical connection for $Y$, which we recall again using the notation of [5]. One sets

$$h = \begin{pmatrix} 0 & 0 & -i/2 \\ 0 & 1 & 0 \\ i/2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \pi = \begin{pmatrix} -(2\alpha + \bar\alpha)/3 & \omega^1 & 2\omega \\ -i\bar\beta & (\alpha - \bar\alpha)/3 & 2i\bar\omega^1 \\ -\psi/4 & \beta/2 & (2\bar\alpha + \alpha)/3 \end{pmatrix},$$

so that $\pi h + h\pi^* = 0$ and $\operatorname{Tr}\pi = 0$; in other words, $\pi$ is $\mathfrak{su}(2,1)$-valued (with the hermitian form given by $h$). It turns out that the equations of Theorem 2.1 are equivalent to

$$d\pi - \pi \wedge \pi = \begin{pmatrix} 0 & 0 & 0 \\ -iQ\bar\omega^1 \wedge \omega & 0 & 0 \\ -\frac{1}{4}(R\omega^1 + \bar R\bar\omega^1) \wedge \omega & \frac{1}{2}\bar Q\omega^1 \wedge \omega & 0 \end{pmatrix}.$$

### 2.3. The Chern–Moser normal form and chains

It is well known that the group of germs of biholomorphisms $G = \operatorname{Aut}(\mathbb{H}^2, 0)$ of the Heisenberg hypersurface $\mathbb{H}^2 \subset \mathbb{C}^2_{z_2,z_1}$ (defined by $\operatorname{Re} z_1 = |z_2|^2$) fixing the origin are explicitly given by

$$\begin{aligned} H(z_1,z_2) &= (g(z_1,z_2), f(z_1,z_2)) \\ &= \Big(\frac{|\lambda|^2 z_1}{1 + 2\bar a z_2 + (|a|^2 + it)z_1}, \frac{\lambda(z_2 + a z_1)}{1 + 2\bar a z_2 + (|a|^2 + it)z_1}\Big). \end{aligned} \tag{2.10}$$

They are therefore uniquely determined by the derivatives $(f_{z_2}(0), f_{z_1}(0), \operatorname{Im} g_{z_1^2}(0)) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{R}$. Using this, we identify $G$ with $\mathbb{C}^* \times \mathbb{C} \times \mathbb{R}$.

Next, we recall the celebrated Chern–Moser theorem [5]. If we consider a germ of a strictly pseudoconvex real-analytic hypersurface $(M,p) \subset (\mathbb{C}^2, p)$, then after an affine change of coordinates, $p = 0$ and $M$ is given near $p$ by an equation of the form

$$\operatorname{Re} z_1 = |z_2|^2 + \varphi(z_2, \bar z_2, \operatorname{Im} z_1) = |z_2|^2 + \sum_{\alpha,\bar\beta} \varphi_{\alpha,\bar\beta}(\operatorname{Im} z_1)\, {z_2}^\alpha\, \bar z_2^{\bar\beta}.$$

The Chern–Moser normal form imposes conditions on the $\varphi_{\alpha,\bar\beta}$ which make this coordinate choice unique up to a parameter $\Lambda \in G$ in the isotropy group of $\mathbb{H}_2$:

**Theorem 2.3** (Chern–Moser [5], $n = 2$). *Let $(M,p)$ be a real-analytic hypersurface. Then there exists a holomorphic choice of holomorphic coordinates $(z,w)$ in which $p = 0$ and the equation of $M$ satisfies the* normalization conditions

$$\varphi_{\alpha,\bar\beta}(\operatorname{Im} z_1) = 0, \quad \textit{if } \min(\alpha,\bar\beta) \le 1 \textit{ or } (\alpha,\beta) \in \{(1,1),(2,2),(3,3),(2,3),(3,2)\}.$$

*Any other choice $(\tilde z, \tilde w)$ of holomorphic coordinates in which the defining equation of $M$ takes this form is given by $(\tilde z, \tilde w) = H_\Lambda(z,w)$ for some $\Lambda \in G$, with $H_\Lambda$ uniquely determined by the requirement that it agrees with the map in (2.10) up to order two.*

The lowest order term in the defining equation $\varphi$ of $M$ in normal coordinates which is not necessarily vanishing is therefore of the form $A_p z_2^2 \bar{z}_2^4 + \bar{A}_p z_2^4 \bar{z}_2^2$. The number $A$ transforms nicely under changes of normal coordinates, and is called *Cartan's cubic tensor*. It already appears in Cartan's early work [3, 4], and it being 0 is a biholomorphic invariant; points where $A_p = 0$ are called *umbilical points*. For ease of notation later on, we always normalize $A_p$ to be real. Vanishing of $A_p$ on an open subset is equivalent to local sphericity:

**Theorem 2.4** (Cartan's umbilical tensor [3, 4]). *Let $(M, p) \subset \mathbb{C}^2$ be a germ of a smooth strictly pseudoconvex hypersurface. If $A_q = 0$ for $q$ in a neighborhood of $p$ in $M$, then $M$ is near $p$ CR-equivalent to $\mathbb{H}^2$.*

The theorem actually hods true for a $\mathcal{C}^6$ hypersurface, for which we can still define the cubic tensor, and can even (due to recent not yet published work of Kossovskiy) be formulated in lower regularity.

The quantity $A_p$ can be thought of as a form of intrinsic curvature, and in a similar vein, Chern and Moser used their normal form to introduce the notion of *chains* as replacements for geodesics in Riemannian geometry. For each $\Lambda = (\lambda, a, t) \in G$, we obtain a parametrized curve (defined for $|s|$ small enough)

$$\gamma(s) = H_\Lambda(is, 0).$$

If one disregards the parametrization of $\gamma$, then it turns out that the condition $\varphi_{2,\bar{3}} = \varphi_{3,\bar{2}} = 0$ is a *second order ODE* whose solution is unique given $a$ (which one thinks of as a vector transverse to the complex tangent space $T_0^c M$). The rest of the data in $\Lambda$ geometrically correspond to a choice of frame of $T_0^c M$ and a choice of parametrization of $\gamma$ amongst a family of projectively equivalent ones. The second order differential equations for chains are not easy to compute from a defining equation of $M$. For boundaries of strictly pseudoconvex domains, the best way to get a computational handle on chains for our problem turned out to be their interpretation as projections of light rays of an associated Lorentz metric introduced by Fefferman [7] which we discuss in the next section.

## 2.4. Chains and the Fefferman Hamiltonian

We will now recall the Fefferman metric for a strictly pseudoconvex hypersurface $M = \{\rho = 0\} \subset \mathbb{C}^2$. We write $z_j = x_j + iy_j$, $j = 1, 2$, and we assume that $(y_1, x_2, y_2)$ are local coordinates on $M$ near the origin, which we assume to be defined by

$$\rho(x_1, y_1, x_2, y_2) = x_1 - (x_2^2 + y_2^2) - \varphi(y_1, x_2, y_2),$$

where $\varphi$ vanishes to order at least 3.

The constructive appeal of Fefferman's metric is based on the fact that for the complex Monge–Ampère operator

$$J(\rho) = \det \begin{pmatrix} \rho & \rho_{\bar{z}_1} & \rho_{\bar{z}_2} \\ \rho_{z_1} & \rho_{z_1\bar{z}_1} & \rho_{z_1\bar{z}_2} \\ \rho_{z_2} & \rho_{z_2\bar{z}_1} & \rho_{z_2\bar{z}_2} \end{pmatrix},$$

one can construct approximate solutions $\rho^{(k)}$ to the equation $J(\rho^{(k)}) = 1 + O(\rho^{k+1})$ in an iterative way, in this particular case by

$$\rho^{(1)} = \frac{\rho}{\sqrt[3]{J(\rho)}} \quad \text{and} \quad \rho^{(2)} = \rho^{(1)}\Big(\frac{5 - J(\rho^{(1)})}{4}\Big).$$

The Fefferman metric is defined on a circle bundle over $M$. We denote by $(x_0, y_1, x_2, y_2)$ the coordinates on $\mathbb{S}^1 \times M$. The conjugate momenta will be denoted by $p_{x_0}$, $p_{y_1}$, $p_{x_2}$ and $p_{y_2}$. There is a lot of flexibility in which metric is actually used, because the light rays of conformally equivalent Lorentz metrics are the same. The one defined in [7] is

$$ds^2 = -\frac{i}{3}\left(\partial\rho^{(2)} - \bar\partial\rho^{(2)}\right)dx_0 + \sum_{j,k=1}^{2} \frac{\partial^2\rho^{(2)}}{\partial z_j \bar z_k}\, dz_j\, d\bar z_k.$$

Setting

$$\begin{aligned}
\Phi &= J(\rho),\\
A &= \begin{pmatrix} 0 & i\rho_{\bar z_1} & i\rho_{\bar z_2} \\ -i\rho_{z_1} & 3\rho_{z_1\bar z_1} & 3\rho_{z_1\bar z_2} \\ -i\rho_{z_2} & 3\rho_{z_2\bar z_1} & 3\rho_{z_2\bar z_2} \end{pmatrix},\\
P &= (p_{x_0}, i p_{y_1}, p_{x_2} + i p_{y_2}),\\
\bar\partial\Phi &= (0, \Phi_{\bar z_1}, \Phi_{\bar z_2}),\\
\tilde\Phi &= \Big(3\Phi_{j\bar k} - \frac{5}{\Phi}\,\Phi_j\,\Phi_{\bar k}\Big)_{j,k},
\end{aligned}$$

the Hamiltonian of Fefferman's metric is now given by

$$H = PA^{-1}P^* - \frac{2p_{x_0}}{\Phi}\,\mathrm{Im}(\bar\partial\Phi\cdot A^{-1}\cdot P^*) - \frac{p_{x_0}^2}{2\Phi}\,\mathrm{Tr}\left(\tilde\Phi A^{-1}\right), \tag{2.11}$$

where $\mathrm{Tr}(\tilde\Phi A^{-1})$ stands for the trace of the matrix $\tilde\Phi A^{-1}$. Note that the formula of the Hamiltonian in p. 410 of [7] contains a minor sign mistake, see [8]. Now, writing $x = (x_0, y_1, x_2, y_2)$, and $p = (p_{x_0}, p_{y_1}, p_{x_2}, p_{y_2})$, chains are the projections on $M$ of the solutions of the Hamiltonian system

$$H(x,p) = 0, \quad x' = H_p(x,p), \quad p' = -H_x(x,p). \tag{2.12}$$

We are now ready to discuss a basic example.

**Example 2.5.** In the case of the sphere $2\,\mathrm{Re}\, z_1 = |z_2|^2$,

$$A^{-1} = \begin{pmatrix} 0 & i & 0 \\ -i & -|z_2|^2/3 & -z_2/3 \\ 0 & -\bar z_2/3 & -1/3 \end{pmatrix}.$$

For convenience, and since light rays for $H$ or $3H$ are the same, we consider

$$A^{-1} = \begin{pmatrix} 0 & 3i & 0 \\ -3i & -|z_2|^2 & -z_2 \\ 0 & -\bar z_2 & -1 \end{pmatrix}.$$

In that case, the Fefferman Hamiltonian is given by

(2.13) $$H = PA^{-1}P^* = 6p_{x_0}p_{y_1} - |z_2|^2 p_{y_1}^2 + 2y_2 p_{y_1} p_{x_2} - 2x_2 p_{y_1} p_{y_2} - p_{x_2}^2 - p_{y_2}^2.$$

Now, we seek solution curves $(x_0(t), y_1(t), x_2(t), y_2(t), p_{x_0}(t), p_{y_1}(t), p_{x_2}(t), p_{y_2}(t))$ to the Hamiltonian system (2.12), which written out is given by

$$\begin{cases} 0 = 6p_{x_0}p_{y_1} - |z_2|^2 p_{y_1}^2 + 2y_2 p_{y_1} p_{x_2} - 2x_2 p_{y_1} p_{y_2} - p_{x_2}^2 - p_{y_2}^2, \\ x_0' = 6p_{y_1}, \\ y_1' = 6p_{x_0} - 2|z_2|^2 p_{y_1} + 2y_2 p_{x_2} - 2x_2 p_{y_2}, \\ x_2' = 2y_2 p_{y_1} - 2p_{x_2}, \\ y_2' = -2x_2 p_{y_1} - 2p_{y_2}, \\ p_{x_0}' = 0, \\ p_{y_1}' = 0, \\ p_{x_2}' = 2x_2 p_{y_1}^2 + 2p_{y_1} p_{y_2}, \\ p_{y_2}' = 2y_2 p_{y_1}^2 - 2p_{y_1} p_{x_2}. \end{cases}$$

To solve this system, note that we have $p_{y_1} = c/4$ for some $c$, so that we get $x_2'' = cy_2'$ and $y_2'' = -cx_2'$, and therefore $z_2 = c_1 e^{-ict} + c_2$ for some $c_1, c_2 \in \mathbb{C}$. It remains to determine $y_1$. Next we note that the quantity $y_2 p_{x_2} - x_2 p_{y_2}$ is conserved, and solving one sees that the chains are the curves of the form

$$\begin{aligned} z_1(t) &= \frac{1}{2}|c_1 e^{-ict} + c_2|^2 + i\,(\tilde{c_1} t + \tilde{c_2}\cos(ct) + \tilde{c_3}\sin(ct) + \tilde{c_4}), \\ z_2(t) &= c_1 e^{-ict} + c_2, \end{aligned}$$

where $\tilde{c_j} \in \mathbb{R}$ and $c_j \in \mathbb{C}$.

We will now assign weights to all variables in the following way. The usual anisotropic scaling on $\mathbb{C}^2$,

$$\Lambda_\delta : (z_1, z_2) \mapsto (\delta z_1, \delta^2 z_2), \quad \delta > 0,$$

lifts to the cotangent bundle as

$$\tilde{\Lambda}_\delta : (z_1, z_2, p_{z_1}, p_{z_2}) \mapsto (\delta z_1, \delta^2 z_2, \delta^{-1} p_{z_1}, \delta^{-2} p_{z_2}).$$

This leads to assign the respective natural weights 1, 2, −1, −2 and −2 to the variables $z_1$, $z_2$, $p_{y_1}$, $p_{x_2}$ and $p_{y_2}$. The variables $x_0$ and $p_{x_0}$ both carry a weight 0. However, with this convention, the Hamiltonian for the sphere (2.13) is homogeneous of degree −2. It will be more convenient for us if the Hamiltonian (2.13) is homogeneous of degree 2, and so we shift the weights of the momenta by 2. To summarize, we assign the following weights:

(2.14) $$\begin{aligned} &\operatorname{wt} x_0 = 0, \quad \operatorname{wt} y_1 = 2, \quad \operatorname{wt} x_2 = \operatorname{wt} y_2 = \operatorname{wt} z_2 = 1, \\ &\operatorname{wt} p_{x_0} = 2, \quad \operatorname{wt} p_{y_1} = 0, \quad \operatorname{wt} p_{x_2} = \operatorname{wt} p_{y_2} = 1. \end{aligned}$$

### 2.5. Stationary discs

We recall that a holomorphic disc $f = (g, h)$, with $g, h \in \mathcal{O}(\Delta) \cap C(\overline{\Delta})$ is said to be *attached* to $M$ if $f(b\Delta) \subset M$. We recall that the complex tangent space of $M$ at $p$ is given by

$$T^c_p M = T_p M \cap i T_p M,$$

and denote the conormal bundle of $M$ by $\mathcal{N} M \subset T^* M$, defined by

$$\mathcal{N}_p M = \left\{\theta_p \in T^*_p M : \theta_p(X_p) = 0,\ X_p \in T^c_p M\right\}.$$

An attached disc $f = (g, h)$ is said to be *stationary* if it has a lift $(f, \tilde{f})$ attached to $\mathcal{N} M$ which is holomorphic up to a pole of order at most 1 in $\Delta$. If $M = \partial\Omega$ is the boundary of a strictly pseudoconvex domain, stationarity is related to the Euler–Lagrange equations for extremal discs for the Kobayashi metric.

In terms of equations, we can use the fact that $\mathcal{N}_p M$ is spanned by

$$\varrho_z = \left(\frac{\partial\rho}{\partial z_1}, \frac{\partial\rho}{\partial z_2}\right)$$

to express the fact that $f$ is stationary in the following form: $f$ is stationary if and only if there exists a real-valued positive function $a$ on $b\Delta$ such that the map $\tilde{f} = (\tilde{g}, \tilde{h})$ defined by

$$\tilde{g}(\zeta) = \zeta a(\zeta) \frac{\partial\rho}{\partial z_1}\left(f(\zeta), \overline{f(\zeta)}\right) \quad \text{and} \quad \tilde{h}(\zeta) = \zeta a(\zeta) \frac{\partial\rho}{\partial z_2}\left(f(\zeta), \overline{f(\zeta)}\right), \tag{2.15}$$

for $\zeta \in b\Delta$, extends holomorphically to $\Delta$. To deal with this extension property, we will use the well-known fact (see [1] for a proof) that a continuous function $\varphi : b\Omega \to \mathbb{C}$ defined on the smooth boundary of a simply connected domain $\Omega$ extends holomorphically to $\Omega$ if and only if it satisfies the *moment conditions*

$$\int_{b\Omega} \zeta^m \varphi(\zeta)\, d\zeta = 0 \quad \text{for all } m \geq 0. \tag{2.16}$$

It turns out that if $M$ is strictly pseudoconvex, then its conormal bundle is actually totally real [17], and so the attachment of stationary discs turns into a standard Riemann-Hilbert problem [9–11, 14]. In the case of the model hypersurface $2\,\mathrm{Re}\, z_1 = |z_2|^2$, a typical stationary disc $f$ passing through 0 at 1 (i.e. $f(1) = 0$) is $f(\zeta) = (1 - \zeta, 1 - \zeta)$ and its lifts are given by $(1 - \zeta, 1 - \zeta, a\zeta, a(\zeta - 1))$, $a \in \mathbb{R}$.

The boundary traces of stationary discs are preserved under (local) CR diffeomorphisms in the following sense. In the case of a strictly pseudoconvex hypersurface $M$, every CR function on $M$ extends to the pseudoconvex side of $M$. Since the components of a CR map $H$ are CR functions, the map actually extends as a holomorphic map to the pseudoconvex side of $M$. Therefore, for a small enough stationary disc attached to $M$, the disc $H \circ f$ is attached to $H(M)$ and is stationary (this is obvious from the characterization as lifts, or one can use the defining equation $\tilde{\rho} = \rho \circ H^{-1}$ in (2.15)).

## 3. The Fefferman Hamiltonian in normal form

### 3.1. The model case

Consider a strictly pseudoconvex hypersurface of the form

$$M = \{2\,\mathrm{Re}\, z_1 = Q(z_2, \overline{z_2})\} \subset \mathbb{C}^2.$$

Such hypersurfaces (whose defining equations do not depend on $\mathrm{Im}\, z_1$) are called *rigid*; the 1-parameter group of transformations $z_1 \mapsto z_1 + it$, $t \in \mathbb{R}$, yields a cyclic variable for the Hamiltonian (2.11).

As before, we write $z_j = x_j + iy_j$, $j = 1, 2$ and use $(x_0, y_1, x_2, y_2)$ as variables on $\mathbb{S}^1 \times M$. We also write $z_0 = e^{i\theta}$ and $x_0 = \theta$. We now consider the defining equation

$$\rho = 2\,\mathrm{Re}\, z_1 - Q(z_2, \overline{z_2}).$$

We have

$$\Phi = \det \begin{pmatrix} \rho & 1 & -Q_{\bar z_2} \\ 1 & 0 & 0 \\ -Q_{z_2} & 0 & -Q_{z_2\bar z_2} \end{pmatrix} = Q_{z_2\bar z_2}$$

and

$$A^{-1} = \begin{pmatrix} 0 & i & -iQ_{\bar z_2} \\ -i & 0 & 0 \\ iQ_{z_2} & 0 & -3Q_{z_2\bar z_2} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & i & 0 \\ -i & -\frac{|Q_{\bar z_2}|^2}{3Q_{z_2\bar z_2}} & -\frac{Q_{\bar z_2}}{3Q_{z_2\bar z_2}} \\ 0 & -\frac{Q_{z_2}}{3Q_{z_2\bar z_2}} & -\frac{1}{3Q_{z_2\bar z_2}} \end{pmatrix}.$$

Moreover, following (2.11), we have

$$\sum_{l\ge 1, k\ge 0} \Phi_{\bar z_l} A^{lk}(p_{x_k} - ip_{y_k}) = \sum_{k\ge 0} \Phi_{\bar z_2} A^{2k}(p_{x_k} - ip_{y_k}),$$

and since $Q$ is independent of $z_1$, we also have

$$\begin{aligned} \tilde\Phi A^{-1} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{(n+1)}{\Phi}\Phi_{z_1\bar z_1} - \frac{2n+1}{\Phi^2}\Phi_{z_1}\Phi_{\bar z_1} & \frac{(n+1)}{\Phi}\Phi_{z_1\bar z_2} - \frac{2n+1}{\Phi^2}\Phi_{z_1}\Phi_{\bar z_2} \\ 0 & \frac{(n+1)}{\Phi}\Phi_{z_2\bar z_1} - \frac{2n+1}{\Phi^2}\Phi_{z_2}\Phi_{\bar z_1} & \frac{(n+1)}{\Phi}\Phi_{z_2\bar z_2} - \frac{2n+1}{\Phi^2}\Phi_{z_2}\Phi_{\bar z_2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{3}{\Phi}\Phi_{z_2\bar z_2} - \frac{5}{\Phi^2}\Phi_{z_2}\Phi_{\bar z_2} \end{pmatrix}. \end{aligned}$$

Thus the Fefferman Hamiltonian is computed to be

$$\begin{aligned} H &= PA^{-1}P^* - \frac{2}{3}\frac{p_{x_0}}{(Q_{z_2\bar z_2})^2}\,\mathrm{Im}\left(Q_{z_2\bar z_2^2}(iQ_{z_2}p_{y_1} - p_{x_2} + ip_{y_2})\right) \\ &\quad + \frac{p_{x_0}^2}{6(Q_{z_2\bar z_2})^2}\left(3Q_{z_2^2\bar z_2^2} - 5\frac{Q_{z_2^2\bar z_2}Q_{z_2\bar z_2^2}}{Q_{z_2\bar z_2}}\right). \end{aligned} \tag{3.1}$$

Using the notations

$$p_{z_2} = p_{x_2} + i\,p_{y_2} \quad \text{and} \quad p_{\bar z_2} = \overline{p_{z_2}} = p_{x_2} - i\,p_{y_2},$$

the expression involving $A$ in (3.1) is explicitly given by

$$\begin{aligned} PA^{-1}P^* &= (p_{x_0}, i\,p_{y_1}, p_{z_2}) \begin{pmatrix} 0 & i & 0 \\ -i & -\frac{|Q_{\bar z_2}|^2}{3Q_{z_2\bar z_2}} & -\frac{Q_{\bar z_2}}{3Q_{z_2\bar z_2}} \\ 0 & -\frac{Q_{z_2}}{3Q_{z_2\bar z_2}} & -\frac{1}{3Q_{z_2\bar z_2}} \end{pmatrix} \begin{pmatrix} p_{x_0} \\ -i\,p_{y_1} \\ p_{\bar z_2} \end{pmatrix} \\ &= 2p_{x_0}p_{y_1} - \frac{|Q_{\bar z_2}|^2}{3Q_{z_2\bar z_2}}\,p_{y_1}^2 + i\,\frac{Q_{z_2}}{3Q_{z_2\bar z_2}}\,p_{y_1}p_{z_2} - i\,\frac{Q_{\bar z_2}}{3Q_{z_2\bar z_2}}\,p_{y_1}p_{\bar z_2} - \frac{1}{3Q_{z_2\bar z_2}}\,|p_{z_2}|^2. \end{aligned}$$

Adding the rest of the Hamiltonian, we get

$$\begin{aligned} H &= \Big(\frac{1}{2}\,\frac{Q_{z_2^2\bar z_2^2}}{Q_{z_2\bar z_2}^2} - \frac{5}{6}\,\frac{Q_{z_2^2\bar z_2}Q_{z_2\bar z_2^2}}{Q_{z_2\bar z_2}^3}\Big)\,p_{x_0}^2 + \Big(2 - \frac{Q_{z_2\bar z_2^2}Q_{z_2} + Q_{z_2^2\bar z_2}Q_{\bar z_2}}{3Q_{z_2\bar z_2}^2}\Big)\,p_{x_0}p_{y_1} \\ &\quad + i\,\frac{Q_{z_2^2\bar z_2}}{3Q_{z_2\bar z_2}^2}\,p_{x_0}p_{z_2} - i\,\frac{Q_{z_2\bar z_2^2}}{3Q_{z_2\bar z_2}^2}\,p_{x_0}p_{\bar z_2} - \frac{|Q_{\bar z_2}|^2}{3Q_{z_2\bar z_2}}\,p_{y_1}^2 + i\,\frac{Q_{z_2}}{3Q_{z_2\bar z_2}}\,p_{y_1}p_{z_2} \\ &\quad - i\,\frac{Q_{\bar z_2}}{3Q_{z_2\bar z_2}}\,p_{y_1}p_{\bar z_2} - \frac{1}{3Q_{z_2\bar z_2}}\,|p_{z_2}|^2 \\ &= P\begin{pmatrix} \frac{1}{2}\frac{Q_{z_2^2\bar z_2^2}}{Q_{z_2\bar z_2}^2} - \frac{5}{6}\frac{Q_{z_2^2\bar z_2}Q_{z_2\bar z_2^2}}{Q_{z_2\bar z_2}^3} & i\Big(1 - \frac{Q_{z_2\bar z_2^2}Q_{z_2}}{3Q_{z_2\bar z_2}^2}\Big) & -i\,\frac{Q_{z_2\bar z_2^2}}{3Q_{z_2\bar z_2}^2} \\ -i\Big(1 - \frac{Q_{z_2^2\bar z_2}Q_{\bar z_2}}{3Q_{z_2\bar z_2}^2}\Big) & -\frac{|Q_{\bar z_2}|^2}{3Q_{z_2\bar z_2}} & -\frac{Q_{\bar z_2}}{3Q_{z_2\bar z_2}} \\ i\,\frac{Q_{z_2^2\bar z_2}}{3Q_{z_2\bar z_2}^2} & -\frac{Q_{z_2}}{3Q_{z_2\bar z_2}} & -\frac{1}{3Q_{z_2\bar z_2}} \end{pmatrix} P^*. \end{aligned}$$

As a particular case, we consider

$$Q(z_2, \bar z_2) = |z_2|^2 + a\,z_2^2\,\bar z_2^4 + \bar a\,z_2^4\,\bar z_2^2 = |z|^2 + 2a\,|z_2|^4\,\mathrm{Re}\,z_2^2,$$

where we assume that $a \in \mathbb{R}$ from now on. We can explicitly compute that

$$\begin{aligned} H = &-\frac{2}{3d^2}\,2p_{x_0}p_{y_1}\left(8a\,z_2\,\bar z_2\,(5a\,z_2^5\,\bar z_2 + 14a\,z_2^3\,\bar z_2^3 + 5a\,z_2\,\bar z_2^5 + 2\,\bar z_2^2 + 2z_2^2) - 3d^2\right) \\ &- \frac{4ia}{d^3}\,p_{x_0}\left(z_2\,(3\,\bar z_2^2 + z_2^2)\,p_{\bar z_2} - (\bar z_2^3 + 3\,z_2^2\,\bar z_2)\,p_{z_2}\right) \\ &- \frac{1}{3d}\,|z_2\,p_{y_1}\,(2a\,z_2\,\bar z_2\,(2\,\bar z_2^2 + z_2^2) + 1) - i\,p_{z_2}|^2 \\ &- \frac{4a}{3d^3}\,p_{x_0}^2\left(40a\,z_2\,\bar z_2\,(10\,z_2^2\,\bar z_2^2 + 3\,\bar z_2^4 + 3\,z_2^4) - 9d\,(\bar z_2^2 + z_2^2)\right), \end{aligned}$$

where

$$d = Q_{z_2\bar z_2} = 8a\,z_2^3\,\bar z_2 + 8a\,z_2\,\bar z_2^3 + 1.$$

If we truncate this expression at order 6, discarding terms which are quadratic or higher order in $a$, we obtain

$$\begin{aligned} H_0 &= 24a\,(\operatorname{Re} z_2^2)\,p_{x_0}^2 + \Big(2 - \frac{64a}{3}\,\big(|z_2|^2 \operatorname{Re} z_2^2\big)\Big)\,p_{x_0}p_{y_1} + \frac{ia}{3}\,\big(24z_2^2\bar z_2 + 8\bar z_2^3\big)\,p_{x_0}p_{z_2} \\ &\quad - \frac{ia}{3}\,\big(8z_2^3 + 24z_2\bar z_2^2\big)\,p_{x_0}p_{\bar z_2} + \frac{1}{3}\,\big(-|z_2|^2 + 4a\,|z_2|^4 \operatorname{Re} z_2^2\big)\,p_{y_1}^2 \\ &\quad + \frac{i}{3}\,\big(\bar z_2 - a\,z_2\,(4|z_2|^4 + 6\bar z_2^4)\big)\,p_{y_1}p_{z_2} - \frac{i}{3}\,\big(z_2 - a\,\bar z_2\,(4|z_2|^4 + 6z_2^4)\big)\,p_{y_1}p_{\bar z_2} \\ &\quad - \frac{1}{3}\,\big(1 - 16a\,|z_2|^2 \operatorname{Re} z_2^2\big)\,|p_{z_2}|^2. \end{aligned}$$

We are going to use this as our *model Hamiltonian*. The system of ODEs associated to $H_0$ can now be obtained as in Example 2.5, with the only difference that

$$z_2' = 2\,\frac{\partial H_0}{\partial p_{\bar z_2}}, \quad \bar z_2' = 2\,\frac{\partial H_0}{\partial p_{z_2}}, \quad p_{z_2}' = -2\,\frac{\partial H_0}{\partial \bar z_2} \quad \text{and} \quad p_{\bar z_2}' = -2\,\frac{\partial H_0}{\partial z_2},$$

as follows:

$$\begin{aligned} 0 &= H_0(z_2(t), p_{x_0}(t), p_{y_1}(t), p_{z_2}(t)), \\ x_0' &= 48a\,(\operatorname{Re} z_2)\,p_{x_0} + \Big(2 - \frac{64a}{3}\,\big(|z_2|^2 \operatorname{Re} z_2\big)\Big)p_{y_1} \\ &\quad + \frac{ia}{3}\,(24z_2^2\bar z_2 + 8\bar z_2^3)\,p_{z_2} - \frac{ia}{3}\,(8z_2^3 + 24z_2\bar z_2^2)\,p_{\bar z_2}, \\ y_1' &= \Big(2 - \frac{64a}{3}\,(|z_2|^2 \operatorname{Re} z_2^2)\Big)\,p_{x_0} + \frac{2}{3}\,\big(-|z_2|^2 + 2a\,|z_2|^4 \operatorname{Re} z_2^2\big)\,p_{y_1} \\ &\quad + \frac{i}{3}\,\big(\bar z_2 - a\,z_2\,(4|z_2|^4 + 6\bar z_2^4)\big)\,p_{z_2} - \frac{i}{3}\,\big(z_2 + a\,\bar z_2\,(4|z_2|^4 + 6z_2^4)\big)\,p_{\bar z_2}, \\ z_2' &= -\frac{16ia}{3}\,\big(z_2^3 + 3|z_2|^2\bar z_2\big)\,p_{x_0} - \frac{2i}{3}\,\big(z_2 - a\,|z|^2\,(6z_2^3 - 4z_2\bar z_2^2)\big)\,p_{y_1} \\ &\quad - \frac{2}{3}\,\big(1 - 16a\,|z_2|^2 \operatorname{Re} z_2^2\big)\,p_{z_2}, \\ p_{x_0}' &= 0, \\ p_{y_1}' &= 0, \\ p_{z_2}' &= -48a\,\bar z_2\,p_{x_0}^2 + \frac{64a}{3}\,\big(z_2^3 + 3z_2\bar z_2^2\big)\,p_{x_0}p_{y_1} - \frac{96ai}{3}\,(\operatorname{Re} z_2^2)\,p_{x_0}p_{z_2} \\ &\quad + \frac{96ai}{3}\,|z_2|^2\,p_{x_0}p_{\bar z_2} + \frac{2}{3}\,\big(z_2 - 4a\,|z_2|^2\,(z_2^3 + 2z_2\bar z_2^2)\big)\,p_{y_1}^2 \\ &\quad + \frac{i}{3}\,(-2 + 16a\,|z_2|^2\,(z_2^2 + 3\bar z_2^2))\,p_{y_1}p_{z_2} - \frac{12ai}{3}\,(z_2^4 + 2z_2^2\bar z_2^2)\,p_{y_1}p_{\bar z_2} \\ &\quad - \frac{16a}{3}\,(z_2^3 + 3z_2\bar z_2^2)\,|p_{z_2}|^2. \end{aligned} \tag{3.2}$$

### 3.2. The general case: weighted Taylor expansion of the Fefferman Hamiltonian

Consider a strictly pseudoconvex hypersurface $M \subset \mathbb{C}^2$, locally written in Chern–Moser normal form as $\rho = 0$, with

$$\rho = 2\operatorname{Re} z_1 - (|z_2|^2 + 2a|z_2|^4 \operatorname{Re} z_2^2 + (\operatorname{Im} z_1)\cdot \eta(\operatorname{Im} z_1, z_2, \bar{z}_2) + \delta(z_2, \bar{z}_2)),$$

where $a \in \mathbb{R}$, and $\eta$ and $\delta$ are functions of weighted orders $O(6)$ and $O(7)$ respectively. In the following, we give weighted degrees to all the monomials in the variables $x_0, y_1, z_2$ and the conjugate momenta $p_{x_0}, p_{y_1}, p_{z_2}$ according to the weight assignments (2.14).

**Lemma 3.1.** *Let $H$ and $H_0$ be the Fefferman Hamiltonians associated respectively to $\rho$ and $\rho_0 = 2\operatorname{Re} z_1 - (|z_2|^2 + a\, z_2^2 \bar{z}_2^4 + a\, z_2^4 \bar{z}_2^2)$. We then have*

$$H = H_0 + O(7).$$

*Proof.* We denote by $Q_k$ a generic homogeneous polynomial of weighted order $k$. We emphasize that, in the below, the polynomials $Q_k$ are not necessarily the same. Moreover, throughout the computations, each $Q_k$, $k \le 6$, comes directly from differentiating or multiplying terms in $\rho_0$. We then write

$$\rho = \rho_0 + O(7) = 2\operatorname{Re} z_1 - |z_2|^2 - 2a\,|z_2|^4 \operatorname{Re} z_2^2 + O(7).$$

The expression of the Hamiltonian $H$ associated to $\rho$ is given by (2.11). Since all terms in $H$ involve the matrix $A^{-1}$, we first focus on computing the order in its entries. Computing explicitly the inverse of $A$, we get

$$A^{-1} = \begin{pmatrix} 0 & i\rho_{\bar{z}_1} & i\rho_{\bar{z}_2} \\ -i\rho_{z_1} & 3\rho_{z_1\bar{z}_1} & 3\rho_{z_1\bar{z}_2} \\ -i\rho_{z_2} & 3\rho_{z_2\bar{z}_1} & 3\rho_{z_2\bar{z}_2} \end{pmatrix}^{-1}$$

$$= \frac{3}{\det A}\begin{pmatrix} 3(\rho_{z_1\bar{z}_1}\rho_{z_2\bar{z}_2} - \rho_{z_1\bar{z}_2}\rho_{z_2\bar{z}_1}) & -i(\rho_{\bar{z}_1}\rho_{z_2\bar{z}_2} - \rho_{\bar{z}_2}\rho_{z_1\bar{z}_2}) & i(\rho_{\bar{z}_1}\rho_{z_1\bar{z}_2} - \rho_{\bar{z}_2}\rho_{z_1\bar{z}_1}) \\ i(\rho_{z_1}\rho_{z_2\bar{z}_2} - \rho_{z_2}\rho_{z_1\bar{z}_2}) & \rho_{z_2}\rho_{\bar{z}_2}/3 & -\rho_{z_2}\rho_{\bar{z}_1}/3 \\ -i(\rho_{z_1}\rho_{z_2\bar{z}_1} - \rho_{z_2}\rho_{z_1\bar{z}_1}) & -\rho_{z_1}\rho_{\bar{z}_2}/3 & \rho_{z_1}\rho_{\bar{z}_1}/3 \end{pmatrix}.$$

A careful bookkeeping of the weighted order of the entries of the matrix above, as well as the tracking of the contribution of $\rho_0$ alone, lead to

$$A^{-1} = \frac{3}{\det A}\begin{pmatrix} O(4) & Q_0 + Q_4 + O(5) & O(5) \\ Q_0 + Q_4 + O(5) & Q_2 + Q_6 + O(7) & Q_1 + Q_5 + O(6) \\ O(5) & Q_1 + Q_5 + O(6) & Q_0 + O(6) \end{pmatrix}. \tag{3.3}$$

Moreover, we have

$$\frac{1}{\det A} = \frac{1}{3\rho_{z_2\bar{z}_2} + O(5)} = -\frac{1}{3} + Q_4 + O(5).$$

We can now investigate the first term $PA^{-1}P^*$ in the Hamiltonian $H$. It follows from (3.3) and the order of the components of $P = (p_{x_0}, i p_{y_1}, p_{x_2} + i p_{y_2})$ (see (2.14)) that

$$PA^{-1}P^* = Q_2 + Q_6 + O(7).$$

We now consider the term

$$\frac{2p_{x_0}}{\Phi}\operatorname{Im}(\bar{\partial}\Phi \cdot A^{-1} \cdot P^*).$$

By a very similar computation, we obtain

$$\bar\partial\Phi = (0, \Phi_{\bar z_1}, \Phi_{\bar z_2}) = (0, O(4), Q_3 + O(4)),$$

and thus

$$\frac{2p_{x_0}}{\Phi}\,\mathrm{Im}(\bar\partial\Phi \cdot A^{-1} \cdot P^*) = Q_6 + O(7).$$

Finally, we also have

$$\frac{p_{x_0}^2}{2\Phi}\,\mathrm{Tr}(\tilde\Phi A^{-1}) = Q_6 + O(7).$$

This proves the lemma. ■

## 4. Proof of the main theorem

Let us reformulate our main theorem in the way we will prove it.

**Theorem 4.1.** *Let $M \subset (\mathbb{C}^2, 0)$ be a strictly pseudoconvex hypersurface of class $\mathcal{C}^{12}$ with local defining equation of the form*

$$\rho = 2\,\mathrm{Re}\, z_1 - \big(|z_2|^2 + 2a\,|z_2|^4\,\mathrm{Re}\, z_2^2 + (\mathrm{Im}\, z_1)\cdot\eta(\mathrm{Im}\, z_1, z_2, \bar z_2) + \delta(z_2, \bar z_2)\big),$$

*where $\eta$ and $\delta$ are of weighted order $O(6)$ and $O(7)$, respectively. If every chain for $M$ for a family of starting conditions as in Lemma 4.2 is the boundary of a stationary disc, then $a = 0$.*

In order to prove Theorem 4.1, we compute an asymptotic expansion of a family of chains which would come from circles in the case of the sphere, and find that the moment conditions for the members of the family yield an obstruction to umbilicity in the fourth order term of that expansion. The details are as follows.

### 4.1. Computations of the orbits

We will use a special family of solutions of the Hamiltonian system associated to $\rho$ depending on a small real parameter $s > 0$. This will be achieved by making a suitable choice of initial conditions imposed in order to reproduce the circular orbits in the case of the sphere $\{2\,\mathrm{Re}\, z_1 = |z_2|^2\}$.

**Lemma 4.2.** *Let $x_0(\cdot, 0)$, $\varphi$, $\psi$ and $\xi$ be four functions in $s$ of class $\mathcal{C}^7$. Then there exists a family of initial conditions of the form*

$$\begin{aligned} y_1(s,0) &= s^2\varphi(s),\\ z_2(s,0) &= s + s^5\psi(s),\\ p_{x_0}(s,0) &= -\frac{1}{2}s^2 + s^6\chi(s),\\ p_{y_1}(s,0) &= -\frac{3}{4},\\ p_{z_2}(s,0) &= -\frac{3i}{4}s + s^5\xi(s), \end{aligned} \tag{4.1}$$

*for some function $\chi$ of class $\mathscr{C}^7$, such that*

$$H(x_0(s,0), y_1(s,0), z_2(s,0), p_{x_0}(s,0), p_{y_1}(s,0), p_{z_2}(s,0)) = 0, \tag{4.2}$$

*where $H$ is the Fefferman Hamiltonian associated to $\rho$.*

*Proof.* Substituting the initial conditions (4.1) into the formula of the Hamiltonian and using Lemma 3.1, we get

$$\begin{aligned}
&(24a\,s^2 + O(s^6))\Big(\frac{1}{4}s^4 - s^8\chi + s^{12}\chi^2\Big) \\
&\quad + \Big(2 - \frac{64}{3}a\,s^4 + O(s^8)\Big)\Big(\frac{3}{8}s^2 - \frac{3}{4}s^6\chi\Big) + (16a\,s^4 + O(s^8))\Big(-\frac{1}{2}s^2 + s^6\chi\Big) \\
&\quad + \frac{3}{16}(-s^2 + O(s^6)) - \frac{3}{8}(s^2 + O(s^6)) - \frac{3}{16}s^2 + O(s^6) = 0,
\end{aligned}$$

where the $O(\cdot)$ terms in the expression above may depend on $s$, $\alpha$, $x_0(\cdot,0)$, $\varphi$, $\psi$ and $\xi$, but not on $\chi$. Developing the products explicitly, we note that the $s^2$ terms (coming only from the spherical part of the Hamiltonian) simplify, while the next lowest order terms are $O(s^6)$, giving the following:

$$s^6\Big(\Big(-\frac{3}{2} + O(s)\Big)\chi + O(s^6)\chi^2 + O(1)\Big) = 0,$$

where once again the $O(\cdot)$ terms do not depend on $\chi$. Applying the implicit function theorem to the expression inside the parenthesis, we conclude that for any choice of $x_0(\cdot,0)$, $\varphi$, $\psi$, and $\xi$, there exists locally a unique function $\chi(s)$ of class $\mathscr{C}^7$ such that equation (4.2) is satisfied. ■

This choice of initial conditions then provides the following family of solutions of the Hamiltonian system associated to $\rho$:

$$\begin{aligned}
x_0(s,t) &= x_0^0(t) + \cdots, \\
y_1(s,t) &= s^2\,y_1^2(t) + \cdots, \\
z_2(s,t) &= s\,z_2^1(t) + s^2\,z_2^2(t) + \cdots, \\
p_{x_0}(s,t) &= -\frac{1}{2}s^2 + s^6\chi(s) + \cdots, \\
p_{y_1}(s,t) &= -\frac{3}{4} + \cdots, \\
p_{z_2}(s,t) &= s\,p_{z_2}^1(t) + s^2\,p_{z_2}^2(t) + \cdots.
\end{aligned} \tag{4.3}$$

**Remark 4.3.** For our later computations, the most important components of the solutions are $y_1(s,t)$ and $z_2(s,t)$. Due to the expression of the defining function, the terms involving $y_1(s,t)$ appear to high order, and for this reason, it is enough to know that $y_1(s,t)$ is of order $O(s^2)$. As for $z_2(s,t)$, we need to know more precisely its asymptotic behavior, beyond the fact that its order is $O(s)$.

**Lemma 4.4.** *We have*

$$z_2(s,t) = s\,e^{it} - \frac{4}{3}\,s^5 a\,e^{3it} + O(s^6).$$

*Proof.* According to Lemma 3.1, solving the Hamiltonian system associated to $\rho$ up to order $s^5$ is equivalent to solving the system (3.2). We will proceed iteratively by expanding in powers of $s$. The terms of order $s$ in the equations for $z_2'$ and $p_{z_2}'$ give the system

$$z_2^{1\prime}(t) = \frac{i}{2}\,z_2^1(t) - \frac{2}{3}\,p_{z_2}^1(t),$$
$$p_{z_2}^{1\,\prime}(t) = \frac{3}{8}\,z_2^1(t) + \frac{i}{2}\,p_{z_2}^1(t).$$

Using the initial conditions $z_2^1(0) = 1$ and $p_{z_2}^1(0) = -3i/4$, we get $z_2^1(t) = e^{it}$ and $p_{z_2}^1(t) = -3i\,e^{it}/4$. Now, the terms in $s^2$ lead to

$$z_2^{2\prime}(t) = \frac{i}{2}\,z_2^2(t) - \frac{2}{3}\,p_{z_2}^2(t),$$
$$p_{z_2}^{2\,\prime}(t) = \frac{3}{8}\,z_2^2(t) + \frac{i}{2}\,p_{z_2}^2(t).$$

If $p_{y_1}^1 = 0, z_2^2(0) = 0$ and $p_{z_2}^2(0)$, then $z_2^2(t) \equiv 0$ and $p_{z_2}^2(t) \equiv 0$. Similarly, we get that $z_2^j(t) \equiv 0$ and $p_{z_2}^j(t) \equiv 0$ for $j = 3, 4$. Finally, computing the terms of order $s^5$, we obtain

$$z_2^{5\prime}(t) = \frac{i}{2}\,z_2^5(t) - \frac{2}{3}\,p_{z_2}^5(t) - \frac{13i}{3}\,a\,e^{3it} + 2i\,a\,e^{-it},$$
$$p_{z_2}^{5\,\prime}(t) = \frac{3}{8}\,z_2^5(t) + \frac{i}{2}\,p_{z_2}^5(t) + \frac{17}{4}\,a\,e^{3it} + \frac{9}{2}\,a\,e^{-it}.$$

A particular solution is given by

$$z_2^5(s,t) = -\frac{4}{3}\,a\,e^{3it} \quad \text{and} \quad p_{z_2}^5(s,t) = -\frac{3i}{2}\,a\,e^{3it} + 3i\,a\,e^{-it}.$$

This concludes the proof of the lemma. ∎

### 4.2. Enforcing stationarity

We consider the $y_1(s,t)$ and $z_2(s,t)$ components of the family of solutions (4.3) of the Hamiltonian system associated to $\rho$. Assume that there exists a family of stationary discs $f_s = (g_s, h_s)$ such that $f_s(b\Delta)$ coincides with the image of the chain $(z_1(s,\cdot), z_2(s,\cdot))$, where the real part of $z_1(s,\cdot)$ is determined by $\rho$. In particular, note this implies that the chain $(z_1(s,\cdot), z_2(s,\cdot))$ is periodic. We denote by $T_s$ its period. We may assume that the projection on the second coordinate $\pi_2\colon f_s(\Delta) \to \mathbb{C}_{z_2}$ is injective and that $0 \in \pi_2(f_s(\Delta)) = h_s(\Delta)$. Then we can take as $h_s$ the unique Riemann map $\Delta \to h_s(\Delta)$ such that $h_s(0) = 0$ and $h_s'(0) > 0$.

By definition, $f_s = (g_s, h_s)$ is stationary if and only if there exists a continuous function $a_s\colon b\Delta \to \mathbb{R}^+$ and functions $\tilde g_s, \tilde h_s \in \mathcal{O}(\Delta) \cap C(\overline{\Delta})$ satisfying

$$\begin{aligned} \tilde g_s(\zeta) &= \zeta\, a_s(\zeta)\, \frac{\partial\rho}{\partial z_1}\big(g_s(\zeta), h_s(\zeta), \overline{g_s(\zeta)}, \overline{h_s(\zeta)}\big), \\ \tilde h_s(\zeta) &= \zeta\, a_s(\zeta)\, \frac{\partial\rho}{\partial z_2}\big(g_s(\zeta), h_s(\zeta), \overline{g_s(\zeta)}, \overline{h_s(\zeta)}\big), \end{aligned} \tag{4.4}$$

for all $\zeta \in b\Delta$. Define now

$$\Omega_s := h_s(\Delta) \subset \mathbb{C} \quad \text{and} \quad S_s := b\Omega_s.$$

We also write $h_s^{-1}(z) = ze^{\varphi_s(z)}$ for a certain holomorphic function $\varphi_s(z)$. Evaluating Equations (4.4) for $\zeta = h_s^{-1}(z)$, we obtain

$$\begin{aligned} \tilde g_s(h_s^{-1}(z)) &= z\, e^{\varphi_s(z)}\, a_s(h_s^{-1}(z))\, \frac{\partial\rho}{\partial z_1}\big(g_s(h_s^{-1}(z)), z, \overline{g_s(h_s^{-1}(z))}, \bar z\big), \\ \tilde h_s(h_s^{-1}(z)) &= z\, e^{\varphi_s(z)}\, a_s(h_s^{-1}(z))\, \frac{\partial\rho}{\partial z_2}\big(g_s(h_s^{-1}(z)), z, \overline{g_s(h_s^{-1}(z))}, \bar z\big), \end{aligned}$$

for all $z \in S_s$. We then set

$$b_s(z) := a_s(h_s^{-1}(z)), \quad G_s(z) := e^{-\varphi_s(z)}\, \tilde g_s(h_s^{-1}(z)) \quad \text{and} \quad H_s(z) := e^{-\varphi_s(z)}\, \tilde h_s(h_s^{-1}(z)).$$

Furthermore, if we write each disc $f_s(\Delta)$ as a graph $\{z_1 = w_s(z)\}$ over its projection $\Omega_s$, we have $w_s(z) = g_s(h_s^{-1}(z))$. Thus we can rewrite the previous system as

$$\begin{aligned} G_s(z) &= z\, b_s(z)\, \frac{\partial\rho}{\partial z_1}\big(w_s(z), z, \overline{w_s(z)}, \bar z\big), \\ H_s(z) &= z\, b_s(z)\, \frac{\partial\rho}{\partial z_2}\big(w_s(z), z, \overline{w_s(z)}, \bar z\big), \end{aligned} \tag{4.5}$$

for $z \in S_s$. In order to apply the moment conditions (2.16) to the functions $G_s$ and $H_s$, we need to find an adapted parametrization of the curve $S_s$. We first consider the scaling $\Lambda_s\colon \mathbb{C} \to \mathbb{C}$ defined by $\Lambda_s(z) = z/s$ and define $\widetilde\Omega_s := \Lambda_s(\Omega_s)$ and $\widetilde S_s := \Lambda_s(S_s)$. Note that, with this change of variables, the moment conditions (2.16) applied to $G_s(z)$ and $H_s(z)$ become

$$\int_{\widetilde S_s} z^m\, G_s(sz)\, dz = \int_{\widetilde S_s} z^m\, H_s(sz)\, dz = 0, \tag{4.6}$$

for all $m \geq 0$. We may now set an adapted parametrization of $\widetilde S_s$. Since the image $f_s(b\Delta)$ coincides with the image of the chain $(z_1(s,\cdot), z_2(s,\cdot))$, we consider the parametrization of $\widetilde S_s$ given by

$$[0, T_s] \ni t \;\mapsto\; \hat z_2(s,t) := \frac{z_2(s,t)}{s} \in \widetilde S_s. \tag{4.7}$$

According to Lemma 4.4, we can write $\hat z_2(s,t) = r(s,t)\, e^{it}$, with

$$r(s,t) = 1 + k(t)s^4 + O(s^5), \quad \text{where} \quad k(t) = -\frac{4}{3}\, a\, e^{2it}.$$

Note that $r(s,t)$ is not necessarily real valued. Moreover, due to standards results on ODEs (see, for instance, Theorem 4.1 in [12]), the parametrization $\hat{z}_2(s,t)$ is of class $\mathcal{C}^7$ in both variables. A straightforward computation leads to the following useful lemma.

**Lemma 4.5.** *For $j \geq 1$,*

$$\begin{aligned} r(s,t)^j &= 1 + jk(t)s^4 + O(s^5), \\ r(s,t)^j\, \overline{r(s,t)} &= 1 + (jk(t) + \overline{k(t)})s^4 + O(s^5), \\ \frac{\partial r}{\partial t}(s,t) &= \frac{dk}{dt}(t)\, s^4 + O(s^5). \end{aligned}$$

Moreover, in order to apply Fourier analysis later on, we need to understand the behavior of the period $T_s$ of $\hat{z}_2(s,\cdot)$ as $s \to 0$.

**Lemma 4.6.** *The function $[0,\varepsilon] \ni s \mapsto T_s \in \mathbb{R}$ is of class $\mathcal{C}^7$. Furthermore, we have*

$$T_s = 2\pi + O(s^5).$$

*Proof.* The function $(s,t) \mapsto \hat{z}_2(s,t)$ of class $\mathcal{C}^7$ and, as $s \to 0$, converges uniformly to $t \mapsto e^{it}$ on any fixed neighborhood of $[0,2\pi]$. Since, by assumption, $\hat{z}_2(s,t)$ parametrizes a simple closed curve on $[0,T_s]$, the period $T_s$ tends to $2\pi$ as $s \to 0$.

By the $C^k$ smoothness of $\hat{z}_2(s,t)$, we have $|\hat{z}_2'(s,t)| \to |\hat{z}_2'(0,t)| = 1$ uniformly as $s \to 0$. Then the period $T_s$ must satisfy $\int_0^{T_s} |\hat{z}_2'(s,t)|dt = \text{length}(\widetilde{S}_s) \to 2\pi$ as $s \to 0$. This is only possible if $T_s \to 2\pi$ as $s \to 0$, since any sequence $s_n \to 0$ such that $|T_{s_n} - 2\pi| > \varepsilon > 0$ would lead to a contradiction by taking the limit as $n \to \infty$.

To prove the smoothness of $T_s$, we will use the fact that the function $\psi$ appearing in Lemma 4.2 is real-valued (since $a \in \mathbb{R}$), which implies $\operatorname{Im} \hat{z}_2(s,0) = 0$ for $s \in [0,\varepsilon]$. Consider the function $\iota(s,t) := \operatorname{Im} \hat{z}_2(s,t)$. At $s = 0$ and $t = 2\pi$, we have

$$\iota(0,2\pi) = \operatorname{Im} \hat{z}_2(0,2\pi) = 0 \quad \text{and} \quad \frac{\partial \iota}{\partial t}(0,2\pi) = \operatorname{Im} \frac{\partial}{\partial t}(\hat{z}_2(0,2\pi)) = 1.$$

By the implicit function theorem, there exists a function $\kappa\colon [0,\varepsilon] \to \mathbb{R}$, of the same smoothness as $\iota$, such that $\kappa(0) = 2\pi$ and $\iota(s,\kappa(s)) = 0$. We claim that $T_s = \kappa(s)$. Indeed, by the implicit function theorem, $t = \kappa(s)$ represent the unique time in a neighborhood of $t = 2\pi$ at which the curve $\hat{z}_2(s,t)$ crosses the line $\operatorname{Im} z = 0$. Since the period of $\hat{z}_2(s,\cdot)$ approaches $2\pi$ as $s \to 0$, and $\operatorname{Im} \hat{z}_2(s,0) = 0$, we necessarily have $\hat{z}_2(s,\kappa(s)) = \hat{z}_2(s,0)$, which means that $T_s = \kappa(s)$ is of class $\mathcal{C}^7$ near $s = 0$.

We now turn to the asymptotic expression of $T_s$. By Lemma 4.4,

$$\hat{z}_2(s,t) = e^{it} - \frac{4}{3} s^4 a\, e^{3it} + O(s^5),$$

so $\hat{z}_2(s,t)$ can be seen as a small perturbation of the unit circle parametrized by $t \mapsto e^{it}$.

Denoting the velocity vector of $\hat{z}_2$ by $\hat{z}_2' = \partial \hat{z}_2/\partial t$, we have

$$\hat{z}_2'(s,t) = i e^{it} + O(s^4).$$

On the one hand, from the expression of $\hat{z}_2(s,t)$, we have

$$\hat{z}_2(s,2\pi) = \hat{z}_2(s,0) + O(s^5).$$

On the other hand, we can write

$$\hat{z}_2(s,2\pi) = \hat{z}_2(s,T_s) + \int_{T_s}^{2\pi} \hat{z}_2'(s,t)\,dt$$

and since $\hat{z}_2(s,T_s) = \hat{z}_2(s,0)$, putting together the expressions above we get

$$O(s^5) = \int_{T_s}^{2\pi} \hat{z}_2'(s,t)dt = \int_{T_s}^{2\pi} (ie^{it} + O(s^4))\,dt.$$

For $s$ small enough, we may suppose that $|ie^{it} + O(s^4)| \geq 1/2$, and moreover, that

$$|\arg(ie^{it} + O(s^4)) - \pi/2| < \pi/4$$

for $t \in [T_s, 2\pi]$, due to the fact that $T_s$ is close to $2\pi$. It follows that

$$\operatorname{Im}(ie^{it} + O(s^4)) \geq \frac{\sqrt{2}}{4}$$

for $t \in [T_s, 2\pi]$, and thus

$$O(s^5) = \int_{T_s}^{2\pi} \operatorname{Im}(ie^{it} + O(s^4))\,dt \geq \frac{\sqrt{2}}{4}|T_s - 2\pi|,$$

so that $|T_s - 2\pi| = O(s^5)$. ∎

In view of equation (4.6), we now define

$$c(s,t) := b_s(s\hat{z}_2(s,t)) = a_s(h_s^{-1}(s\hat{z}_2(s,t)) = a_s(h_s^{-1}(z_2(s,t)).$$

**Lemma 4.7.** *There is a choice of $a_s$ such that the function $c(s,t)$ is of class $\mathscr{C}^4$ in a neighborhood of $\{0\} \times [0,2\pi]$ and satisfies $\int_0^{2\pi} c(s,t)\,dt = 1$ for all $s > 0$ small enough.*

The proof is an adaptation of both proofs of Lemma 3.3 and Lemma 3.4 in [1]. The main differences come from the facts that, in the present paper, the parametrization intervals depend on $s$, and the first component of the discs we consider is not constant.

*Proof.* We first show that $h_s^{-1}(z_2(s,t))$ is of class $\mathscr{C}^5$ in both variables $s$ and $t$. In order to extend the parametrization (4.7) to a uniform domain, namely the unit disc, we consider

$$[0,2\pi] \ni t \;\mapsto\; \hat{z}_2\left(s, \frac{T_s}{2\pi}t\right) \in \widetilde{S}_s.$$

According to Lemma 4.6, this map is of class $\mathscr{C}^7$ in both variables $s$ and $t$. Moreover, its form allows us to extend it to the interior of the unit disc, and, thus, to obtain a family of diffeomorphisms $\Gamma_s = \Gamma(s,\cdot)\colon \overline{\Delta} \to \overline{\widetilde{\Omega}_s}$ of class $\mathscr{C}^7$ in both variables $s$ and $z$. It follows from Corollary 9.4 in [2] and Lemma 2.1 in [1] that the function $(s,t) \mapsto h_s^{-1}(s\hat{z}_2(s, T_s t/(2\pi)))$, and so $(s,t) \mapsto h_s^{-1}(z_2(s,t))$, are of class $\mathscr{C}^5$.

We now define $\widehat{a}_s$ by

$$\frac{1}{\widehat{a}_s(\zeta)} = \zeta\, \partial\rho(f_s(\zeta)) \cdot f_s'(\zeta),$$

for $\zeta \in b\Delta$, where $\cdot$ denotes the dot product in $\mathbb{C}^2$. According to Pang [15], since $f_s$ is stationary and satisfies (4.4) for a continuous function $a_s$, then $a_s$ is a positive multiple of $\widehat{a}_s$, where the multiple may be any function of $s$. We now show that the function $(s,t) \mapsto \widehat{a_s}(h_s^{-1}(z_2(s,t))$ is of class $\mathcal{C}^4$. Note first that the map

$$\partial\rho(f_s(h_s^{-1}(z_2(s,t)e^{it})) = \partial\rho\big(w_s(z_2(s,t)), z_2(s,t), \overline{w_s(z_2(s,t))}, \overline{z_2(s,t)}\big)$$

is of class $\mathcal{C}^7$ in both variables. To study the smoothness of $f_s'(h_s^{-1}(z_2(s,t))$, note that by the chain rule, we have

$$\frac{d}{dt}(z_1(s,t), z_2(s,t))) = \frac{d}{dt} f_s(h_s^{-1}(z_2(s,t)) = f_s'(h_s^{-1}(z_2(s,t)) \cdot \frac{d}{dt} h_s^{-1}(z_2(s,t)),$$

and so,

$$f_s'(h_s^{-1}(z_2(s,t)) = \left( \frac{\frac{d}{dt}(z_1(s,t)))}{\frac{d}{dt} h_s^{-1}(z_2(s,t))}, \frac{\frac{d}{dt}(z_2(s,t)))}{\frac{d}{dt} h_s^{-1}(z_2(s,t))} \right).$$

Following the proof of Lemma 3.3 in [1], we have $h_s^{-1}(z_2(s,t)) = e^{it} + O(s)$, and since $h_s^{-1}(z_2(s,t))$ is of class $\mathcal{C}^5$, so is the map $f_s'(h_s^{-1}(z_2(s,t))$. This shows that the function $(s,t) \mapsto \widehat{a_s}(h_s^{-1}(z_2(s,t))$ is of class $\mathcal{C}^4$.

Finally, with the same proof of Lemma 3.4 in [1], we get

$$\frac{1}{\widehat{a_s}(h_s^{-1}(z_2(s,t))} = s^2 + O(s^3).$$

The function $a_s$ we seek can be obtained by rescaling $\widehat{a_s}$ to ensure $\int_0^{2\pi} c(t,s)\,dt = 1$ for all $s > 0$ small enough. ■

We are now in a position to apply the moment conditions (4.6) to the system (4.5). We start with the function $G_s$:

$$\int_{\widetilde{S}_s} z^m \left( s\, z\, b_s(sz)\, \frac{\partial\rho}{\partial z_1}(w_s(sz), sz, \overline{w_s(sz)}, s\bar{z}) \right) dz = 0,$$

for all $m \geq 0$, that is, using the form of the defining function $\rho$,

$$\int_{\widetilde{S}_s} z^j b_s(sz) \left( 1 + \frac{i}{2}\eta(\operatorname{Im} w_s(sz), sz, s\bar{z}) - (\operatorname{Im} w_s(sz)) \cdot \frac{\partial\eta}{\partial z_1}(\operatorname{Im} w_s(sz), sz, s\bar{z}) \right) dz = 0$$

for all $j \geq 1$. We use the parametrization of $\widetilde{S}_s$ given by (4.7). With this parametrization, as observed in Remark 4.3, $s\hat{z}_2$ is of order $O(s)$ and $\operatorname{Im} w_s(s\hat{z}_2) = y_1$ of order $O(s^2)$. Since $\eta$ is of weighted order $O(6)$, the first term involving $\eta$ in the above integral is of order $O(s^6)$, while the second one is of order $O(s^7)$. Accordingly, we obtain, for $j \geq 1$,

$$\int_0^{T_s} r^j\, e^{i(j+1)t}\, b_s(sre^{it})\,(1 + O(s^6)) \left( \frac{\partial r}{\partial t} + ir \right) dt = 0.$$

Using Lemma 4.5, we have

$$\int_0^{T_s} e^{i(j+1)t}\, c(s,t)\big(1 + jk(t)s^4 + O(s^5)\big)\Big(i + \Big(\frac{dk}{dt}(t) + ik(t)\Big)s^4 + O(s^5)\Big)\, dt = 0.$$

Developing the product leads to

$$\int_0^{T_s} e^{i(j+1)t}\, c(s,t)\Big(1 - \frac{4}{3}\, ja\, e^{2it} s^4 + O(s^5)\Big)\big(i - 4ia\, e^{2it} s^4 + O(s^5)\big)\, dt$$
$$= i\int_0^{T_s} e^{i(j+1)t}\, c(s,t)\Big(1 - \frac{4}{3}\,(j+3)a\, e^{2it} s^4 + O(s^5)\Big)\, dt = 0.$$

In order to apply Fourier analysis, we apply the change of variables $t \mapsto \frac{2\pi}{T_s} t$ and, using Lemma 4.6, we obtain

$$\int_0^{2\pi} e^{i(j+1)\frac{T_s}{2\pi}t}\, c\Big(s, \frac{T_s}{2\pi}t\Big)\Big(1 - \frac{4}{3}(j+3)a\, e^{2i\,\frac{T_s}{\pi}t} s^4 + O(s^5)\Big)\, dt$$
$$= i\int_0^{2\pi} e^{i(j+1)t}\, c(s,t)\Big(1 - \frac{4}{3}(j+3)a\, e^{2it} s^4 + O(s^5)\Big)\, dt = 0.$$

We may then expand $c(s,\cdot)$ in its Fourier series,

$$c(s,t) = \sum_{k=-\infty}^{+\infty} \gamma_k(s)\, e^{ikt},$$

where $\gamma_{-k} = \bar{\gamma}_k$ for all $k \in \mathbb{Z}$ and, by Lemma 4.7, $\gamma_k$ is $\mathcal{C}^4$ and satisfies $\gamma_0(s) \equiv 1$. Inserting the Fourier expansion of $c(s,t)$ in

$$\int_0^{2\pi} e^{i(j+1)t}\, c(s,t)\Big(1 - \frac{4}{3}(j+3)a\, e^{2it} s^4 + O(s^5)\Big)\, dt = 0,$$

we deduce that

$$\bar{\gamma}_{j+1}(s) = O(s^4), \tag{4.8}$$

for all $j \geq 1$. Taking the fourth derivative with respect to $s$, we get

$$\sum_{\ell=0}^{4}\binom{4}{\ell}\Big(\frac{d^\ell \bar{\gamma}_{j+1}}{ds^\ell}(s)\,\delta_4^\ell - \frac{4!}{\ell!}\Big(\frac{4}{3}(j+3)a\,\frac{d^\ell \bar{\gamma}_{j+3}}{ds^\ell}(s)\Big)s^\ell + O(s^{\ell+1})\Big) = 0,$$

where $\delta_4^\ell$ is the Kronecker symbol, which for $j = 1$ leads to

$$\frac{d^4 \bar{\gamma}_2}{ds^4}(s) - 4!\,\frac{16}{3}\, a\,\bar{\gamma}_4(s) = O(s),$$

implying that

$$\frac{d^4 \bar{\gamma}_2}{ds^4}(s) = O(s). \tag{4.9}$$

We now apply the moment conditions (4.6) to the function $H_s$ in (4.5):

$$\int_{\widetilde{S}_s} z^m \left( s\, z\, b_s(sz)\, \frac{\partial \rho}{\partial z_2}(w_s(sz), sz, \overline{w_s(sz)}, s\,\bar{z}) \right) dz = 0,$$

for all $m \geq 0$. Due to the form of the defining equation $\rho$, we get

$$\int_{\widetilde{S}_s} z^j\, b_s(sz) \Big( s\,\bar{z} + 2a\, s^5\, z\, \bar{z}^4 + 4a\, s^5\, z^3\, \bar{z}^2 + (\operatorname{Im} w_s(sz)) \cdot \frac{\partial \eta}{\partial z_2}(\operatorname{Im} w_s(sz), sz, s\,\bar{z}) + \frac{\partial \delta}{\partial z_2}(sz, s\,\bar{z}) \Big) \Big) dz = 0,$$

for all $j \geq 1$. We once again parametrize $\widetilde{S}_s$ by (4.7). With this parametrization, the term involving $\eta$ in the above integral is of order $O(s^7)$, and the one involving $\delta$ is of order $O(s^6)$. We then obtain

$$\int_0^{T_s} e^{i(j+1)t}\, c(s,t) \left( r^j\, \bar{r}\, e^{-it} s + r^j\, (2a\, r \bar{r}^4\, e^{-3it} + 4a\, r^3\, \bar{r}^2\, e^{it})\, s^5 + r^j\, O(s^6) \right) \cdot \left( \frac{\partial r}{\partial t} + i\, r \right) dt = 0.$$

Using once again Lemma 4.5 and dividing by $i\,s$ gives

$$\int_0^{T_s} e^{i(j+1)t}\, c(s,t) \left( e^{-it} + s^4 \left( \frac{2}{3}\, a\, e^{-3it} + \left( -\frac{4}{3}\, j + 4 \right) a\, e^{it} \right) + O(s^5) \right) \cdot \left( 1 - 4a e^{2it} s^4 + O(s^5) \right) dt = 0,$$

and, after developing the product, and applying as above the change of variables $t \mapsto \frac{2\pi}{T_s} t$, we obtain for $j \geq 1$,

$$\int_0^{2\pi} c(s,t) \left( e^{ijt} + \frac{2}{3} \left( a\, e^{i(j-2)t} - \frac{4}{3}\, j\, a\, e^{i(j+2)t} \right) s^4 + O(s^5) \right) dt = 0.$$

Once again, we integrate from 0 to $2\pi$, insert the Fourier expansion of $c(s,t)$, and differentiate four times with respect to $s$, and obtain

$$\sum_{\ell=0}^{4} \binom{4}{\ell} \left( \frac{d^\ell \bar{\gamma}_j}{ds^\ell}(s)\, \delta_4^\ell + \frac{4!}{\ell!} \left( \frac{2}{3}\, a\, \frac{d^\ell \bar{\gamma}_{j-2}}{ds^\ell}(s) - \frac{4}{3}\, j\, a\, \frac{d^\ell \bar{\gamma}_{j+2}}{ds^\ell}(s) \right) s^\ell + O(s^{\ell+1}) \right) = 0.$$

For $j = 2$, this implies

$$\frac{d^4 \bar{\gamma}_2}{ds^4}(s) - 4! \left( \frac{2}{3}\, a\, \bar{\gamma}_0(s) - \frac{8}{3}\, a\, \bar{\gamma}_4(s) \right) = O(s).$$

Using (4.8) and (4.9), we then deduce that

$$\frac{2}{3}\, a\, \bar{\gamma}_0(s) = \frac{2}{3}\, a = 0.$$

This concludes the proof of Theorem 4.1.

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