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Graded homotopy classification of Leavitt path algebras over finite primitive graphs

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Abstract. We show that the graded Grothendieck group classifies unital Leavitt path algebras of primitive graphs up to graded homotopy equivalence. To this end, we further develop classification techniques for Leavitt path algebras by means of (graded, bivariant) algebraic K-theory.

1. Introduction

Let ℓ be a commutative unital ring. Given a graph E, we will consider its Leavitt path ℓ-algebra $L(E)$ (see Definition 2.5 in [24]), which carries a natural grading over $\mathbb{Z}$ (Proposition 4.7 in [24]). Its graded Grothendieck group $K_0^{\mathrm{gr}}(L(E))$ is the group completion of the monoid of isomorphism classes of $\mathbb{Z}$-graded finitely generated projective modules. This group carries an action from the infinite cyclic group $C_\infty = \langle \sigma \rangle$, and is moreover a pointed preordered $\mathbb{Z}[\sigma]$-module; see Section 2.8 for a precise defintion of these terms. This paper is mainly concerned with the graded classification question for Leavitt path algebras.

Conjecture 1.1 (Conjecture 1 in [18]). *Assume that ℓ is a field. Let E and F be two finite graphs. If there exists a pointed, preordered $\mathbb{Z}[\sigma]$-module isomorphism $K_0^{\mathrm{gr}}(L(E)) \xrightarrow{\sim} K_0^{\mathrm{gr}}(L(F))$, then the algebras $L(E)$ and $L(F)$ are isomorphic as graded algebras.*

In line with recent advances in the (ungraded) classification question for purely infinite simple Leavitt path algebras [11–13], we investigate the notion of graded classification up to polynomial homotopy. Before stating our main result, we recall the relevant definitions and provide some motivation. An elementary (graded, polynomial) homotopy between graded algebra homomorpshisms $f, g\colon A \to B$ is a graded homomorphism $h\colon A \to B[t]$ such that $\mathrm{ev}_0 \circ h = f$ and $\mathrm{ev}_1 \circ h = g$; here the indeterminate t is set to be homogeneous of degree zero. Two graded algebra maps are homotopic if they are connected via finitely many elementary homotopies. Homotopy equivalences are then defined to be homomorphisms which have an inverse up to this notion of homotopy.

It can be shown that K_0^{gr} maps graded homotopy equivalences to isomorphisms (see Remark 2.12). Hence, a positive answer to Conjecture 1.1 would imply that two Leavitt

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path algebras are graded isomorphic if and only if they are graded homotopy equivalent. One could thus first aim at deciding whether the graded Grothendieck group classifies Leavitt path algebras up to homotopy equivalence. In the ungraded, purely infinite simple case this program was carried out in [12] and [11]. The main theorem of this paper provides a similar result in the graded context.

Main Theorem (Theorem 8.1). *Let E and F be two finite, primitive graphs. Assume that ℓ is a field. If there exists a pointed, preordered $\mathbb{Z}[\sigma]$-module isomorphism $K_0^{\mathrm{gr}}(L(E)) \xrightarrow{\sim} K_0^{\mathrm{gr}}(L(F))$, then the algebras $L(E)$ and $L(F)$ are graded homotopy equivalent.*

As a consequence, the primitive case of Conjecture 1.1 is equivalent to the following.

Conjecture (Conjecture 8.2). *Let ℓ be a field. If E and F are finite primitive graphs, then $L_\ell(E)$ and $L_\ell(F)$ are graded isomorphic if and only if they are unitally graded homotopy equivalent.*

We point out that despite the resemblance of the Main Theorem above with ungraded homotopy classification results, significant technical work is needed to obtain similar conclusions. The hypothesis that graphs be *primitive* (see Definition 7.2) stems from adapting the techniques of [11]; put succinctly, we need a family of idempotents of $L(E)$ arising from edges to be full (Proposition 7.4). Such graphs are in particular *essential*, meaning that they have no sinks or sources. This allows us to interpret their associated Leavitt path algebras as corner skew Laurent polynomial rings. The latter are characterized as the $\mathbb{Z}$-graded rings with a homogeneous left invertible element of degree 1 (Lemma 2.4 in [4]). These ideas are due to Ara and Pardo and go back to [6].

The main tool used in this article is graded bivariant algebraic K-theory ([17]). This is a functor $j\colon \mathsf{Alg}_\ell{}^{\mathrm{gr}} \to kk^{\mathrm{gr}}$ from the category of graded ℓ-algebras to a triangulated category which is universal in a specific sense; see Section 3 for a brief recollection of its main properties. Here ℓ is viewed as a graded algebra with trivial grading. The functor j is the identity on objects; hence we shall omit it from our notation. Writing $[L(E), L(F)]$ for the set of graded algebra homormophisms between two Leavitt path algebras up to homotopy, our objective will be to understand the canonical map

$$[L(E), L(F)] \to kk^{\mathrm{gr}}(L(E), L(F)). \tag{1.1}$$

In Corollary 11.11 of [9], it is shown that homomorphisms between two Leavitt path algebras $L(E)$ and $L(F)$ in kk^{gr} fit into an exact sequence involving their graded K-theory groups. Concretely, let A_E be the adjacency matrix of E and I the identity matrix on its set of vertices. Write $\mathrm{BF}_{\mathrm{gr}}(E) = \operatorname{coker}(I - \sigma A_E^t)$ for the *Bowen–Franks module* of E and $\mathrm{BF}_{\mathrm{gr}}^\vee(E) = \operatorname{coker}(I^t - \sigma A_E)$ for its so-called dual. Abbreviating $\otimes = \otimes_{\mathbb{Z}[\sigma]}$ and $\hom = \hom_{\mathbb{Z}[\sigma]}$, there is a diagram of $\mathbb{Z}[\sigma]$-modules with exact top-row as follows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{BF}_{\mathrm{gr}}^\vee(E) \otimes K_1^{\mathrm{gr}}(L(F)) \arrow[r, "d"] & kk^{\mathrm{gr}}(L(E), L(F)) \arrow[r] & \hom(\mathrm{BF}_{\mathrm{gr}}(E), K_0^{\mathrm{gr}}(L(F))) \arrow[r] & 0 \\
 & & {[L(E), L(F)]} \arrow[u, "\bar{j}"] \arrow[ur, bend right, "K_0^{\mathrm{gr}}"'] & &
\end{tikzcd}$$

The first part of this article is devoted to Poincaré duality for Leavitt path algebras, which is used in Lemma 13.1 of [11] to effectively compute the map d in the exact sequence above. Let us recall the relevant terminology. In this paper, a graph is a pair of source and range functions $s, r\colon E^1 \to E^0$ from a set of edges E^1 to a set of vertices E^0. The dual graph E_t of E has the same sets of vertices and edges with the functions r and s interchanged one for the other; informally, we revert the direction of all arrows. The suspension in the triangulated structure of kk^{gr} is represented by tensoring by the trivially graded algebra $\Omega = t(1-t)\ell[t]$; we shall write $\Omega L(E_t)$ for $\Omega \otimes_\ell L(E_t)$.

Theorem (Theorem 6.1). *If E is a finite essential graph, then the functor $-\otimes_\ell L(E)$ is left adjoint to $-\otimes \Omega L(E_t)$ as endofunctors of the graded bivariant K-theory category kk^{gr}. Thus, for each pair of graded algebras R and S, there are isomorphisms*

$$kk^{\mathrm{gr}}(R \otimes_\ell L(E), S) \cong kk^{\mathrm{gr}}(R, S \otimes_\ell \Omega L(E_t))$$

which are natural in both R and S.

The proof of Poincaré duality given in [11] involves a specific homomorphism from $L(E)$ to the suspension algebra Σ_X for a suitable set X. The latter algebra is a quotient of Karoubi's cone Γ_X by the ideal of X-indexed matrices M_X. In our context, we want this homomorphism to preserve the gradings; in particular, we have to equip Σ_X with a grading to begin with. Further, in ungraded algebraic bivariant K-theory tensoring by Σ_X plays the role of the inverse for the suspension functor. In Section 4 we generalize the notion of infinity sum-rings and, for a suitable notion of graded infinite set X, we define a graded analogue Γ_X^{gr} of Karoubi's cone and produce a quotient algebra Σ_X^{gr} which plays the role of the suspension in kk^{gr}.

Another key ingredient of the proof of Poincaré duality is the relationship between a degree zero unit element $u_1 \in L(E) \otimes L(E_t)$ and the class in kk^{gr} represented by the algebra homomorphism from $\mathcal{S} := \ker(\ell[t, t^{-1}] \xrightarrow{\mathrm{ev}_1} \ell)$ to $L(E) \otimes L(E_t)$ mapping $t \mapsto u_1$. In this direction, we prove the following.

Theorem (Theorem 5.6). *Let A be a unital, strongly graded algebra, $p \in A_0$ an idempotent and u a unit in pA_0p. Consider the map $\phi\colon \mathcal{S} \to A$ given by $1 \mapsto p$, $t \mapsto u$. There is a chain of isomorphisms*

$$kk^{\mathrm{gr}}(\mathcal{S}, A) \cong kk(\mathcal{S}, A_0) \simeq KH_1(A_0),$$

mapping $j(\phi)$ to $[1-p+u]$.

Here KH is Weibel's homotopy K-theory [27]. Finally, in Section 3 we also record some results that we use regarding left and right boundary maps of a triangle in kk^{gr} and their compatibility with tensor products. We remark that these statements are not found in the literature, even in the ungraded case.

With a graded version of Poincaré duality in place, and the fact that K_0^{gr} is a full functor when restricted to Leavitt path algebras ([8, 25]), we are able to study the image of (1.1) (Lemmas 7.19 and 7.20). This relies on a procedure to deform a unital graded algebra homomorphism $L(E) \to L(F)$ using a given element of $K_1^{\mathrm{gr}}(L(F))$, adapted from the ungraded setting in Section 7. Studying the extent to which (1.1) is injective necessitates a study of $K_1^{\mathrm{gr}}(L(F))$. By a result of Hazrat (see Theorem 3.15 in [19]), together with Dade's theorem (Theorem 2.8 in [16]), if F is a graph with no sinks, then one has canonical isomorphisms $K_*^{\mathrm{gr}}(L(F)) \cong K_*(L(F)_0)$. Further, it is known that $L(F)_0$ is an ultramatricial algebra, see the proof of Theorem 5.3 in [5]. Using these results, in Lemma 3.6 of [6], Ara and Pardo give an explicit description of the shift action on $K_0(L(F)_0) \cong K_0^{\mathrm{gr}}(L(F))$. We extend the characterization above to $K_1(R_0) \cong K_1^{\mathrm{gr}}(R)$ for any strongly graded, corner skew Laurent polynomial ring R such that R_0 is ultramatricial.

Theorem (Corollary 7.16). *Let (R, t_+, t_-) be a strongly graded, corner skew Laurent polynomial ring. Assume that ℓ is a field and R_0 a unital ultramatricial algebra. Writing $\alpha\colon R_0 \to R_0$ for the homomorphism given by $x \mapsto t_+xt_-$, the following diagram is commutative*:

$$\begin{tikzcd} K_1^{\mathrm{gr}}(R) \arrow[r, "\sigma"] & K_1^{\mathrm{gr}}(R) \\ K_1(R_0) \arrow[u, "\sim"] \arrow[r, "K_1(\alpha)"] & K_1(R_0). \arrow[u, "\sim"'] \end{tikzcd}$$

The proof involves the observation that $K_1(R_0)$ agrees with the abelianization of the unit group of R_0, recorded as Proposition 7.8. We also need an alternative description of the K_1 of an ultramatricial algebra, akin to Cortiñas and Montero's characterization of Karoubi and Villamayor's KV_1 group for purely infinite simple rings (Proposition 2.8 in [12]).

Theorem (Proposition 7.10). *Assume that ℓ is a field. If R is a unital ultramatricial algebra, then*

$$K_1(R) = R^\times / \{u(1) : u \in (R[t])^\times, u(0) = 1\}.$$

The theorem above allows us to deduce injectivity of (1.1) up to a relaxed notion of homotopy. We say that two graded algebra maps $f, g\colon A \to B$ are *ad-homotopic* if there exists a degree zero unit $u \in B_0^\times$ such that f is homotopic to the conjugation of g by u. A particular case of Theorem 7.25 is the following.

Theorem. *Let E and F be two primitive graphs. Assume that ℓ is a field. Two unital graded homomorphisms $f, g\colon L(E) \to L(F)$ satisfy $j(f) = j(g)$ if and only if they are ad-homotopic.*

With all of this in place, we prove our Main Theorem in Section 8 as Theorem 8.1.

2. Preliminaries

For the rest of the article, we fix an abelian group G and a commutative unital ring ℓ. The adjective graded will always mean G-graded. Recall that a *graded algebra* is an ℓ-algebra R together with an abelian group decomposition $R = \bigoplus_{g \in G} R_g$ satisfying $R_g R_h \subset R_{gh}$ for each $g, h \in G$. The projection of $x \in R$ to R_g will be denoted x_g. If $x \in R_g$, then we say that x is a *homogeneous element of degree* g and write $|x| = g$. The base ring ℓ is viewed as a graded algebra by equipping it with the trivial grading, that is, we set $|\lambda| = 1_G$ for all $\lambda \in \ell$. A graded algebra homomorphism $f\colon R \to S$ is an algebra homomorphism satisfying $f(R_g) \subset S_g$ for each $g \in G$. We shall denote the category of graded algebras by $\mathsf{Alg}_\ell{}^{\mathrm{gr}}$.

2.1. Tensor products

Given two graded algebras R and S, their tensor product has a canonical grading given by

$$(R \otimes_\ell S)_d = \bigoplus_{gh=d} R_g \otimes S_h.$$

We equip the ring $\ell[t]$ of polynomials with the trivial grading, and set $R[t] := R \otimes_\ell \ell[t]$ with the tensor product grading for any graded algebra R. In other words, we set $|t| = 1_G$.

We will often omit the tensor product symbol and use juxtaposition in its place, especially when one of the algebras carries the trivial grading. For example, we define the *path* and *loop* algebras respectively as

$$P = \ker(\ell[t] \xrightarrow{\mathrm{ev}_1} \ell) \quad \text{and} \quad \Omega = \ker(P \xrightarrow{\mathrm{ev}_0} \ell),$$

and write

$$PR = P \otimes_\ell R \quad \text{and} \quad \Omega R = \Omega \otimes_\ell R$$

for any graded algebra R.

2.2. Graded sets, matricial stability and G-stability

A *graded set* will mean a pair (X, d) where X is a set and $d\colon X \to G$ is a function. When understood from context, we shall write $|\cdot|$ instead of d. An element $x \in X$ is said to have *degree* $d(x) \in G$, and the degree $g \in G$ component of X is $X_g := d^{-1}(g)$. A morphism of graded sets $f\colon (X, d) \to (Y, d')$ is a function $f\colon X \to Y$ such that $d' \circ f = d$. If X is a set, we write $|X|$ for the graded set given by the constant function with value 1_G.

From a graded set X, one can form a graded algebra of X*-indexed matrices*, denoted by M_X. As an algebra, it is the free ℓ-module with basis $\{\varepsilon_{x,y} : x, y \in X\}$ with product $\varepsilon_{x,y} \cdot \varepsilon_{w,z} = \delta_{y,w}\varepsilon_{x,z}$. Its grading is induced by the assignment $|\varepsilon_{x,y}| = |x||y|^{-1}$. If R is a graded algebra, we set $M_X R := M_X \otimes_\ell R$.

Definition 2.1. We put M_∞ for $M_{|\mathbb{N}|}$ and $M_n = M_{|\{1,\dots,n\}|}$ for each $n \in \mathbb{N}$.

Any morphism $f\colon X \to Y$ of graded sets with injective underlying function gives rise to a graded algebra map $Mf\colon M_X \to M_Y$ mapping $\varepsilon_{x,y}$ to $\varepsilon_{f(x),f(y)}$.

Definition 2.2. Let $F\colon \mathsf{Alg}_\ell^{\mathrm{gr}} \to \mathsf{C}$ be a functor. We say that F is *matricially stable* if for every pair of sets X, Y of cardinality less or equal than $\beth = \max\{\aleph_0, |G|\}$ and any graded algebra R, the functor F maps the inclusion $M_{|X|}R \to M_{|X\sqcup Y|}R$ to an isomorphism. If F moreover maps inclusions $M_X R \to M_{X\sqcup Y} R$ to isomorphisms for every pair of graded sets of cardinality less or equal than $\beth$, we say that it is G*-stable*.

The following proposition is implied by Proposición 3.3.8 in [7].

Proposition 2.3. *Let X be a graded set and A a graded algebra. If $x \in X$, then*

$$\iota_x : a \mapsto \varepsilon_{x,x} \otimes a \in M_X A$$

is a graded homomorphism. Moreover, if $y \in X$ is such that $d(x) = d(y)$, then any G-stable functor F satisfies $F(\iota_x) = F(\iota_y)$.

Unless specified otherwise, we view G as a graded set via $\mathrm{id}_G\colon G \to G$. If A is a graded algebra, we write ι_A for $\iota_{1_G}\colon A \to M_G A$.

2.3. Extensions

An *extension* of graded algebras is an exact sequence

$$K \xrightarrow{i} E \xrightarrow{p} Q$$

such that $p = \mathrm{coker}(i)$, $i = \ker(p)$, and p admits a linear section $s\colon Q \to E$.

Remark 2.4. The section in the definition of extension is not required to preserve the grading. This is justified by the fact that the existence of an ℓ-linear section s guarantees the existence of a section

$$\hat{s}(m) = \sum_{d \in G} s(m_d)_d.$$

which preserves the grading.

Example 2.5. The *loop extension* of a graded algebra R is $\Omega R \hookrightarrow PR \xrightarrow{\mathrm{ev}_1} R$.

Example 2.6. Let

$$\Gamma = \big\{ f\colon \mathbb{N} \times \mathbb{N} \to \ell : |\operatorname{im} f| < \infty \text{ and } (\exists N \geq 1) \\ \text{such that } |\operatorname{supp}(x,-)|, |\operatorname{supp}(-,x)| \leq N \ (\forall x \in X) \big\}$$

be Karoubi's cone ring, equipped with pointwise addition and the convolution product. We view Γ as a graded algebra via the trivial grading. Observe that it contains M_∞ as an ideal; put $\Sigma = \Gamma / M_\infty$ for the suspension ring. The extension

$$M_\infty R \to \Gamma R \to \Sigma R$$

is the cone extension of R (see Section 4.7 in [14]). We will deal with generalizations of this extension in Section 4.

2.4. Homotopy invariance

A *graded elementary* (*polynomial*) *homotopy* between graded algebra homomorphisms $f, g\colon A \to B$ is a graded map $h\colon A \to B[t]$ such that $\mathrm{ev}_0 \circ h = f$, $\mathrm{ev}_1 \circ h = g$. We say that f and g are *graded homotopic* if there exists a sequence of graded elementary homotopies $h_1, \ldots, h_n\colon A \to B[t]$ such that $\mathrm{ev}_0 \circ h_1 = f$, $\mathrm{ev}_1 \circ h_n = g$ and $\mathrm{ev}_1 \circ h_j = \mathrm{ev}_0 \circ h_{j+1}$ for all j. This is an equivalence relation which will be denoted $\sim$. Two graded (unital) algebras R and S are (*unitally*) *graded homotopy equivalent* if there exist (unital) maps $f\colon R \to S$ and $g\colon S \to R$ such that $fg \sim 1_S$ and $gf \sim 1_R$.

Definition 2.7. We say that a functor $F\colon \mathsf{Alg}_\ell{}^{\mathrm{gr}} \to \mathsf{C}$ is graded homotopy invariant if it maps the inclusion $A \subset A[t]$ for each graded algebra A to an isomorphism. Equivalently, a functor is graded homotopy invariant if $f \sim g$ implies $F(f) = F(g)$.

We will also need two more notions of graded homotopy. The first one involves matricial stabilization. We say that two graded homomorphisms $f, g\colon A \to B$ are graded M_2-homotopic if the maps $\iota_1 \circ f, \iota_1 \circ g\colon A \to M_2 B$ are graded homotopic. As above, this also induces an equivalence relation denoted by $\sim_{M_2}$. Likewise, we have a notion of *graded M_2-homotopy equivalence*; we say that two algebras are *graded M_2-homotopy equivalent* if there exists a graded M_2-homotopy equivalence between them. For the second notion, we first need a definition.

Definition 2.8. Let C be a graded unital algebra and $A, B \subset C$ two subalgebras with inclusion maps $\mathrm{inc}_A\colon A \to C$ and $\mathrm{inc}_B\colon B \to C$. Given two homogeneous elements $u, v \in C$ such that $|u||v| = 1$, $avua' = aa'$ for each $a, a' \in A$, and $uAv \subset B$, we define the graded homomorphism

$$\mathrm{ad}(u, v) : A \to B, \quad a \mapsto uav.$$

If u is a unit, we abbreviate $\mathrm{ad}(u) := \mathrm{ad}(u, u^{-1})$.

We say that two unital algebra maps $f, g\colon R \to S$ are *graded* ad-*homotopic*, denoted $f \sim_{\mathrm{ad}} g$, if there exists a unit $u \in S_{1_G}$ such that $\mathrm{ad}(u) \circ f \sim g$. When this happens, we will write $f \sim_u g$. The proof of the following lemma is implied by Proposition 2.10 below.

Lemma 2.9. *If two unital graded algebra maps are graded* ad-*homotopic, they are graded M_2-homotopic.*

Proposition 2.10. *Let C be a graded unital algebra and $A, B \subset C$ two subalgebras with inclusion maps* $\mathrm{inc}_A\colon A \to C$ *and* $\mathrm{inc}_B\colon B \to C$. *Given $u, v \in C$ in the situation of Definition 2.8, we have that*

(i) *any G-stable functor F satisfies* $F(\mathrm{inc}_B\, \mathrm{ad}(u, v)) = F(\mathrm{inc}_A)$;

(ii) *if $B = A$, $uA, Av \subset A$, and $|u| = 1_G$, then* $\mathrm{ad}(u, v) \sim_{M_2} \mathrm{id}_A$.

In particular, by (ii)*, any graded homotopy invariant, matricially stable functor F satisfies* $F(\mathrm{ad}(u, v)) = \mathrm{id}_{F(A)}$.

Proof. For the proof of (i), we adapt Proposition 2.2.6 in [10]. Put $d := |u|$ and note that $|v| = d^{-1}$. The assignment $1 \mapsto 1$, $2 \mapsto d$ yields a grading on M_2; in the rest of the proof, we will consider the latter algebra equipped with this particular grading.

Put $U = \varepsilon_{1,1}\, u + \varepsilon_{2,2}\, 1_C$ and $V = \varepsilon_{1,1}\, v + \varepsilon_{2,2}\, 1_C$. A straightforward verification shows that we have a well-defined graded algebra homomorphism

$$\phi : M_2 A \to M_2 C, \quad x \mapsto UxV.$$

For each $k \in \{1, 2\}$, write $\iota_k^R\colon R \to M_2 R$ for the corner inclusions of each algebra R. Observe that

$$\phi\, \iota_1^A = \iota_1^C\, \mathrm{inc}_B\, \mathrm{ad}(u, v) \quad \text{and} \quad \phi\, \iota_2^A = M_2(\mathrm{inc}_A)\, \iota_2^A.$$

Applying F, we obtain that

$$F(\phi) F(\iota_2^A) = F(M_2(\mathrm{inc}_A)) F(\iota_2^A).$$

Since $F(\iota_2^A)$ is an isomorphism by G-stability, it follows that $F(\phi) = F(M_2(\mathrm{inc}_A))$. Hence

$$F(\iota_1^C) F(\mathrm{inc}_B\, \mathrm{ad}(u, v)) = F(\phi\, \iota_1^A) = F(M_2(\mathrm{inc}_A)\iota_1^A) = F(\iota_1^C) F(\mathrm{inc}_A).$$

Once again by G-stability, we know that $F(\iota_1^C)$ is an isomorphism and thus

$$F(\mathrm{inc}_B\,\mathrm{ad}(u,v)) = F(\mathrm{inc}_A).$$

This concludes the proof of (i).

Now suppose that $A = B$, $uA, Av \subset A$ and $|u| = 1_G$. Write $\iota_k = \iota_k^A$. The hypotheses imply in particular that ϕ can be corestricted to a homomorphism $\psi\colon M_2A \to M_2A$ satisfying $\psi\iota_1 = \iota_1\,\mathrm{ad}(u,v)$ and $\psi\iota_2 = \iota_2$. They also say that the grading on M_2A is the standard one and that the homotopy given in Lemma 2.1 of [13] is a graded homotopy between ι_1 and ι_2. This implies that $\iota_1\,\mathrm{ad}(u,v) = \psi\iota_1 \sim \psi\iota_2 = \iota_2 \sim \iota_1$, proving (ii). ■

2.5. Graded K-theory

Given a unital graded algebra R, its *graded K-theory* $K_*^{\mathrm{gr}}(R)$ is the K-theory of the exact category of graded, finitely generated projective R-modules. In Section 3.3 of [9], a homotopy invariant version KH^{gr} of graded K-theory is introduced. It comes equipped with a canonical comparison map $K_*^{\mathrm{gr}}(R) \to KH_*^{\mathrm{gr}}(R)$ for any graded algebra R. Given a graded left R-module M, its *shift* by $g \in G$ is the module $M[g] := M$ with the grading $M[g]_h = M_{gh}$. Note that the shift of modules induces an action of $\mathbb{Z}[G]$ on graded (homotopy) K-theory. In particular, the graded (homotopy) K-theory of a $\mathbb{Z}$-graded algebra is a $\mathbb{Z}[\sigma]$-module.

2.6. Strongly graded rings

A graded unital ring R is *strongly graded* if $R_g R_h = R_{gh}$ for each $g, h \in G$. A theorem of Dade (Theorem 2.8 in [16]) says that R is strongly graded if and only if the functor

$$R \otimes_{R_{1_G}} - : \mathsf{Proj}_{\mathsf{fg}}(R_{1_G}) \to \mathsf{Gr}-\mathsf{Proj}_{\mathsf{fg}}(R)$$

is an equivalence of categories with inverse

$$(-)_{1_G} : \mathsf{Gr}-\mathsf{Proj}_{\mathsf{fg}}(R) \to \mathsf{Proj}_{\mathsf{fg}}(R_{1_G}).$$

In particular, one has canonical isomorphisms

$$K_*(R_{1_G}) \to K_*^{\mathrm{gr}}(R) \quad \text{and} \quad KH_*(R_{1_G}) \to KH_*^{\mathrm{gr}}(R). \tag{2.1}$$

for every strongly graded algebra R.

Example 2.11. By Example 1.1.16 in [18], if A is a strongly graded unital algebra then so is $B \otimes_\ell A$ for any graded unital algebra B.

2.7. Graphs, their Leavitt path algebras and Bowen–Franks modules

A (finite, directed) *graph* is a tuple $E = (E^0, E^1, r, s)$ consisting of two finite sets E^0 of *vertices* and E^1 of *edges* together with *range* and *source* functions $r, s\colon E^1 \to E^0$. We say that $v \in E^0$ is a *sink* if $s^{-1}(v) = \emptyset$, *regular* if it is not a sink, and a *source* if $r^{-1}(v) = \emptyset$. The sets of sinks, regular vertices and sources are denoted by $\mathrm{sink}(E)$, $\mathrm{reg}(E)$ and $\mathrm{sour}(E)$ respectively. A graph E is *regular* if $E^0 = \mathrm{reg}(E)$ and *essential* if it is regular and $\mathrm{sour}(E) = \emptyset$.

To a graph E, we will associate its *Leavitt path ℓ-algebra* $L(E)$. The latter is a quotient of the free algebra on the set $\{v, e, e^* : v \in E^0, e \in E^1\}$ by the relations:

$$\begin{aligned}
&\text{(V)} && vw = \delta_{v,w} v, && (v, w \in E^0),\\
&\text{(E1)} && s(e)e = er(e) = e, && (e \in E^1),\\
&\text{(E2)} && r(e)e^* = e^* s(e) = e^*, && (e \in E^1),\\
&\text{(CK1)} && f^* e = \delta_{f,e}\, r(e), && (f, e \in E^1),\\
&\text{(CK2)} && v = \sum_{e \in s^{-1}(v)} ee^*, && (v \in \operatorname{reg}(E)).
\end{aligned}$$

The *standard grading* of $L(E)$ over $\mathbb{Z}$ is given by extension of the rule $|v| = 0$, $|e| = 1$, $|e^*| = -1$ for each $v \in E^0$, $e \in E^1$. The *Cohn algebra* of E is the one obtained similarly dividing by all of the relations above except (CK2). One has a canonical surjection $C(E) \to L(E)$. Writing $q_v = v - \sum_{e \in s^{-1}(v)} ee^*$ for each $v \in \operatorname{reg}(E)$ and $\mathcal{K}(E) := \langle q_v : v \in \operatorname{reg}(E)\rangle$, we have an exact sequence

$$(\mathscr{C}_E) \qquad \mathcal{K}(E) \to C(E) \to L(E).$$

We shall refer to the exact sequence above as the *Cohn extension* of $L(E)$. By Proposition 1.5.11 in [1], it is always ℓ-linearly split.

The (reduced) adjacency matrix $A_E \in \mathbb{N}_0 \in \mathbb{Z}^{\operatorname{reg}(E) \times E^0}$ of a graph E is the one given by

$$(A_E)_{v,w} = \#\{e \in E^1 : s(e) = v, r(e) = w\}.$$

Writing I for the $\operatorname{reg}(E) \times E^0$-indexed matrix given by $I_{v,w} = \delta_{v,w}$, the *Bowen–Franks module* of E is defined as

$$\mathrm{BF}_{\mathrm{gr}}(E) := \operatorname{coker}(I - \sigma A_E^t) = \frac{\mathbb{Z}[\sigma]^{E^0}}{\langle v - \sigma \sum_{e \in s^{-1}(v)} r(e) : v \in \operatorname{reg}(E)\rangle}.$$

By Theorem 5.3 in [9], the (homotopy) graded K-theory of $L(E)$ can be computed in terms of this group as

$$K_*^{\mathrm{gr}}(L(E)) = \mathrm{BF}_{\mathrm{gr}}(E) \otimes_{\mathbb{Z}} K_*(\ell) \quad \text{and} \quad KH_*^{\mathrm{gr}}(L(E)) = \mathrm{BF}_{\mathrm{gr}}(E) \otimes_{\mathbb{Z}} KH_*(\ell).$$

Remark 2.12. By Theorem 3.9, Equation (3.9) and Theorem 5.3 in [9], if E is a finite graph, then the comparison map $K_*^{\mathrm{gr}}(L(E)) \to KH_*^{\mathrm{gr}}(L(E))$ can be identified with tensoring the canonical comparison map $K_*(\ell) \to KH_*(\ell)$ by $\mathrm{BF}_{\mathrm{gr}}(E)$. In particular, if ℓ is a field, a PID, or more generally a regular noetherian ring, then $K_*^{\mathrm{gr}}(L(E)) = KH_*^{\mathrm{gr}}(L(E))$.

In the case in which ℓ is a field, as a corollary of the above we obtain that any graded homotopy equivalence $L(E) \to L(F)$ induces an isomorphism at the level of K_0^{gr}. If Conjecture 1.1 is true, then this would entail that the algebras $L(E)$ and $L(F)$ are graded isomorphic. In other words, Conjecture 1.1 would imply that two Leavitt path algebras are graded isomorphic if and only if they are graded homotopy equivalent.

Remark 2.13. If E is a finite graph, then $L(E)$ endowed with its standard $\mathbb{Z}$-grading is strongly graded if and only if the underlying graph has no sinks (Theorem 3.15 in [19]).

2.8. Pointed preordered modules

A *pointed preordered module* is a tuple (M, M_+, u) where M is a $\mathbb{Z}[\sigma]$-module together with a distinguished submonoid M_+ such that $\mathbb{N}_0[\sigma]M_+ \subset M_+$ and a distinguished element $u \in M_+$ that is an order unit; this means that for every $m \in M$ there exists $x \in \mathbb{N}_0[\sigma]$ satisfying $x \cdot u - m \in M_+$. A pointed preordered module map $f: (M, M_+, u) \to (N, N_+, v)$ is a $\mathbb{Z}[\sigma]$-linear map $f: M \to N$ such that $f(M_+) \subset N_+$ and $f(u) = v$.

Example 2.14. If R is a graded ring, then its graded Grothendieck group together with the submonoid $K_0^{\mathrm{gr}}(L(E))_+$ of classes of projective modules and the class $[R]$ form a pointed preordered module.

Example 2.15. The Bowen–Franks module of a graph E can be made into a pointed preordered module by setting $\mathrm{BF}_{\mathrm{gr}}(E)_+ = \langle \sum_{v\in E^0} x_v \cdot v : x_v \in \mathbb{N}_0[\sigma]\rangle$ and $1_E := \sum_{v\in E^0}[v]$.

Remark 2.16. There is a canonical pointed preordered module map

$$\mathrm{can} : \mathrm{BF}_{\mathrm{gr}}(E) \to K_0^{\mathrm{gr}}(L(E)), \quad [v] \mapsto [vL(E)]$$

which, by Corollary 5.4 in [9], is an isomorphism whenever $K_0(\ell) \cong \mathbb{Z}$.

3. The category kk^{gr} and its triangulated structure

Recall that an *excisive homology theory* for graded algebras is a functor $H: \mathsf{Alg}_\ell{}^{\mathrm{gr}} \to \mathcal{T}$ taking values in a triangulated category $\mathcal{T}$, such that for each extension

$$(\mathcal{E}) \qquad K \xrightarrow{i} E \xrightarrow{p} Q$$

there exists a triangle

$$H(Q)[+1] \xrightarrow{\partial^H_{\mathcal{E}}} H(K) \xrightarrow{H(i)} H(E) \xrightarrow{H(p)} H(Q).$$

The maps $\partial^H_{\mathcal{E}}$ are called the (*left*) *boundary map* of H associated to an extension $\mathcal{E}$, and they are required to satisfy some compatibility conditions (see Section 6.6 of [14]).

A morphism of excisive homology theories (F, ϕ) from H to $H': \mathsf{Alg}_\ell{}^{\mathrm{gr}} \to \mathcal{T}'$ consists of a triangulated functor $F: \mathcal{T} \to \mathcal{T}'$ such that $FH = H'$, and a natural transformation $\phi: F(H(-)[+1]) \to H'(-)[+1]$ such that the following diagram commutes for all extensions $\mathcal{E}$:

$$\begin{tikzcd}
F(H(Q)[+1]) \arrow[dr, "F(\partial^H_{\mathcal{E}})"] \arrow[dd, "\phi_Q"'] & \\
& H'(K) \\
H'(Q)[+1]. \arrow[ur, "\partial^{H'}_{\mathcal{E}}"'] &
\end{tikzcd}$$

Graded bivariant algebraic K-theory

$$j : \mathsf{Alg}_\ell{}^{\mathrm{gr}} \to kk^{\mathrm{gr}}$$

is the initial G-stable, matricially stable, graded homotopy invariant excisive homology theory (Theorem 4.2.1 in [17]). We refer the reader to [17] (see also Sections 7 and 8 of [9]) for the construction of the category kk^{gr} and its main properties.

In this section, we shall study properties of boundary maps $\partial_{\mathcal{E}} := \partial_{\mathcal{E}}^{j}$ in kk^{gr}. The construction of kk^{gr} is built off that of the initial homology theory among the matricially stable, graded homotopy invariant theories which are not necessarily G-stable. This functor will be denoted $j' \colon \mathsf{Alg}_{\ell}{}^{\mathrm{gr}} \to kk_{\mathsf{Alg}_{\ell}{}^{\mathrm{gr}}}$. We first give an account on boundary maps in the latter theory.

3.1. Boundary maps in $kk_{\mathsf{Alg}_{\ell}{}^{\mathrm{gr}}}$

The category $kk_{\mathsf{Alg}_{\ell}{}^{\mathrm{gr}}}$ is constructed in the same way algebraic bivariant K-theory is constructed in [14]; we refer the reader to Section 2 of [17] for a detailed explanation on the construction of these categories. Its objects are, as those of kk^{gr}, all graded algebras.

Recall that given an extension ($\mathcal{E}$), its *classifying map* $c_{\mathcal{E}}$ is computed explicitly by considering the tensor algebra map $TQ \to E$ induced by a section $s \colon Q \to E$ of p, and then restricting it to the kernel $JQ := \ker(TQ \to Q)$ of the counit map $T \Rightarrow \mathrm{id}$. Up to graded homotopy, $c_{\mathcal{E}}$ is independent of the chosen section.

The boundary map $\partial'_{\mathcal{E}} := \partial_{\mathcal{E}}^{j'}$ of an extension ($\mathcal{E}$) is given by a zig-zag of classifying maps, that of $\mathcal{E}$ and the one associated to the *loop extension*

$$(\mathcal{L}_Q) \qquad \Omega Q \to PQ \xrightarrow{\mathrm{ev}_1} Q.$$

Namely,

$$\partial'_{\mathcal{E}} := c_{\mathcal{E}} \circ c_{\mathcal{L}_Q}^{-1} = \Omega Q \xleftarrow{c_{\mathcal{L}_Q}} JQ \xrightarrow{c_{\mathcal{E}}} K.$$

Remark 3.1. Note that, by construction, the boundary map of $\mathcal{L}_Q$ is the identity map of ΩQ. As recalled above, if $H \colon \mathsf{Alg}_{\ell}{}^{\mathrm{gr}} \to \mathcal{T}$ is a matricially stable, graded homotopy invariant excisive homology then there is a unique homomorphism $(X, \phi) \colon j' \to H$. Although we will not delve into the construction of X, we nonetheless note that since $\partial'_{\mathcal{L}_Q} = \mathrm{id}_{\Omega Q}$, it follows that $\phi_Q = (\partial_{\mathcal{L}_Q}^{H})^{-1}$ for all Q.

3.2. Boundary maps in kk^{gr}

The category kk^{gr} is built in terms of $kk_{\mathsf{Alg}_{\ell}{}^{\mathrm{gr}}}$. Given $A, B \in \mathsf{Alg}_{\ell}{}^{\mathrm{gr}}$, by definition,

$$kk^{\mathrm{gr}}(A, B) = kk_{\mathsf{Alg}_{\ell}{}^{\mathrm{gr}}}(M_G A, M_G B).$$

If $f \colon A \to B$ is a graded algebra homomorphism, then $j(f) = j'(M_G f)$. The boundary map of an extension ($\mathcal{E}$) is, thus, an element $\partial_{\mathcal{E}} \in kk^{\mathrm{gr}}(\Omega Q, K) = kk_{\mathsf{Alg}_{\ell}{}^{\mathrm{gr}}}(M_G \Omega Q, M_G K)$. To construct it, we consider the extension

$$(M_G \mathcal{E}) \qquad M_G K \xrightarrow{M_G i} M_G E \xrightarrow{M_G p} M_G Q.$$

Its boundary map in $kk_{\mathsf{Alg}_{\ell}{}^{\mathrm{gr}}}$ is an arrow $\partial'_{M_G \mathcal{E}} \colon \Omega M_G Q \to M_G K$. To define $\partial_{\mathcal{E}}$, we precompose the latter by the flip map $\tau_Q \colon M_G \Omega Q \to \Omega M_G Q$:

$$\partial_{\mathcal{E}} := \partial'_{M_G \mathcal{E}} \circ \tau_Q = c_{M_G \mathcal{E}} \circ c_{\mathcal{L}_{M_G Q}}^{-1} \circ \tau_Q. \tag{3.1}$$

Our first computation will concern the loop extension.

Lemma 3.2. *If A is a graded algebra, then $\partial_{\mathcal{L}_A} = \mathrm{id}_{\Omega A}$.*

Proof. Put $\mathcal{E} = M_G \mathcal{L}_A$ and $\mathcal{D} = \mathcal{L}_{M_G A}$. Following the definition of (3.1), we obtain the following equality in $kk_{\mathsf{Alg}_\ell{}^{\mathrm{gr}}}$:

$$\partial_{\mathcal{L}_A} = \partial'_{M_G \mathcal{L}_A} \circ \tau_A = c_{\mathcal{E}} \circ c_{\mathcal{D}}^{-1} \circ \tau_A.$$

Consider the flip map $\tau'_A : M_G PA \to PM_G A$. It fits in a morphism of extensions from $\mathcal{E}$ to $\mathcal{D}$:

$$\begin{tikzcd}
M_G \Omega A \arrow[r] \arrow[d, "\tau_A"] & M_G PA \arrow[r] \arrow[d, "\tau'_A"] & M_G A \arrow[d, equal] \\
\Omega M_G A \arrow[r] & PM_G A \arrow[r] & M_G A.
\end{tikzcd}$$

By the graded version of Proposition 4.4.2 in [14] (see [17], p. 205–206), in $kk_{\mathsf{Alg}_\ell{}^{\mathrm{gr}}}$ we have the equality $\tau_A \circ c_{\mathcal{E}} = c_{\mathcal{D}}$. Hence $c_{\mathcal{E}}$ is an isomorphism and $c_{\mathcal{D}}^{-1} = c_{\mathcal{E}}^{-1} \tau_A^{-1}$; this concludes the proof. ∎

Remark 3.3. Let $H : \mathsf{Alg}_\ell{}^{\mathrm{gr}} \to \mathcal{T}$ be a G-stable, matricially stable, graded homotopy invariant excisive homology theory. We recall the construction of the unique map $j \to H$ from Section 4.2 of [17].

Given the unique map $(X', \phi') : j' \to H$, one defines $X(A) = X'(A)$ on objects $A \in kk^{\mathrm{gr}}$ and $X(\alpha) = X'(\iota_B)^{-1} X'(\alpha) X'(\iota_A)$ on morphisms $\alpha : A \to B$. The natural transformation $\iota : \mathrm{id} \to M_G(-)$ induces, for any extension ($\mathcal{E}$), a map of extensions $\mathcal{E} \to M_G \mathcal{E}$. In particular, it follows that

$$\iota_K \circ \partial'_{\mathcal{E}} = \partial'_{M_G \mathcal{E}} \circ \Omega \iota_Q = \partial'_{M_G \mathcal{E}} \circ \tau_Q \circ \iota_{\Omega Q} \quad \text{in } kk_{\mathsf{Alg}_\ell{}^{\mathrm{gr}}},$$

and hence

$$X(\partial_{\mathcal{E}}) = X'(\iota_K)^{-1} X'(\partial'_{M_G \mathcal{E}} \circ \tau_Q) X'(\iota_{\Omega Q}) = X'(\partial'_{\mathcal{E}}).$$

This automatically implies that setting $\phi_Q = \phi'_Q = (\partial^H_{\mathcal{L}_Q})^{-1}$ makes (X, ϕ) into a morphism of excisive homology theories. Note also that, by Lemma 3.2, this is the only possible choice for ϕ; cf. Remark 3.1.

3.3. Ungraded extensions

We will write $j_{kk} : \mathsf{Alg}_\ell \to kk$ for ungraded algebraic bivariant K-theory [14]. There is a canonical map $\mathrm{triv} : kk \to kk^{\mathrm{gr}}$ induced by the trivial grading inclusion $\mathrm{triv} : \mathsf{Alg}_\ell \hookrightarrow \mathsf{Alg}_\ell{}^{\mathrm{gr}}$.

In particular, one may view any extension of ungraded algebras as one of trivially graded ones. Since $j \circ \mathrm{triv} : \mathsf{Alg}_\ell \to kk^{\mathrm{gr}}$ is a an excisive homology theory (for ungraded algebras, i.e., for $G = \{1\}$), there is a unique map $(X, \phi) : j_{kk} \to j \circ \mathrm{triv}$ and $\phi_Q = (\partial_{\mathrm{triv}(\mathcal{L}_Q)})^{-1} = \partial^{-1}_{\mathcal{L}_{\mathrm{triv}(Q)}} = \mathrm{id}_{\Omega Q}$ for each algebra Q. Thus, we have the following.

Theorem 3.4. *Let* $\mathrm{triv} : kk \to kk^{\mathrm{gr}}$ *be the canonical functor induced by the trivial grading functor* $\mathrm{triv} : \mathsf{Alg}_\ell \hookrightarrow \mathsf{Alg}_\ell{}^{\mathrm{gr}}$. *If $\mathcal{E}$ is an extension of ungraded algebras and $\partial^{kk}_{\mathcal{E}}$ its boundary map in kk, then* $\mathrm{triv}(\partial^{kk}_{\mathcal{E}}) = \partial_{\mathcal{E}}$. ∎

We conclude this subsection with a characterization of the boundary map of the cone extension

$$M_\infty \to \Gamma \to \Sigma \tag{$\mathfrak{K}$}$$

in both the graded and ungraded case. First, we need a definition.

Definition 3.5. Let $s_0 = \sum_{i \in \mathbb{N}} \varepsilon_{i+1,i} \in \Gamma$ be the *right shift* and $s := [s_0]$ its class as an element of Σ. Since $s_0^* s_0 = 1$ and $s_0 s_0^* = 1 - \varepsilon_{1,1}$, it follows that s is a unit. Write $L = s^{-1}$ and

$$\xi_L : \ell \to \Omega\Sigma$$

for the morphism in $kk(\ell, \Omega\Sigma)$ corresponding to $[L] \in KH_1(\Sigma)$.

Lemma 3.6. *The following diagram commutes in* kk^{gr}:

$$\begin{tikzcd} \ell \arrow[r, "\mathrm{triv}(\xi_L)"] \arrow[dr, "\mathrm{inc}_1"'] & \Omega\Sigma \arrow[d, "\partial_{\mathfrak{K}}"] \\ & M_\infty. \end{tikzcd}$$

In particular, $\mathrm{triv}(\xi_L)$ *is an isomorphism.*

Proof. Since by Theorem 3.4 the functor triv: $kk \to kk^{\mathrm{gr}}$ is compatible with boundary maps, we may assume that G is the trivial group. The result now follows from the proof of Lemma 11.1 in [11], which in particular says that the boundary $\partial\colon KH_1(\Sigma) \to KH_0(M_\infty)$ maps $[s]$ to $[1 - s_0^* s_0] - [1 - s_0 s_0^*] = -[\varepsilon_{1,1}]$ and thus $\partial([L]) = [\varepsilon_{1,1}]$. ■

3.4. Compatibility with tensor products.

Next we use Lemma 3.2 to prove the compatibility of left boundary maps with tensor products.

Theorem 3.7. *If*

$$K \to E \to Q \tag{$\mathcal{E}$}$$

is an extension and A a graded algebra, then the boundary map of the extension

$$K \otimes A \to E \otimes A \to Q \otimes A \tag{$\mathcal{E} \otimes A$}$$

equals that of $\mathcal{E}$ tensored by A. That is,

$$\partial_{\mathcal{E} \otimes A} = \partial_{\mathcal{E}} \otimes A.$$

Proof. Recall that the functor $- \otimes A\colon kk^{\mathrm{gr}} \to kk^{\mathrm{gr}}$ is defined using the universal property of j as the unique morphism of homology theories from j to $H := j(- \otimes A)$. As noted in Remark 3.3, this entails in particular that $\partial_{\mathcal{E}} \otimes A = \partial_{\mathcal{E}}^H \circ (\partial_{\mathcal{L}_Q}^H)^{-1} = \partial_{\mathcal{E} \otimes A} \circ \partial_{\mathcal{L}_Q \otimes A}$. Since $\mathcal{L}_Q \otimes A = \mathcal{L}_{Q \otimes A}$, it follows from Lemma 3.2 that $\mathcal{L}_Q \otimes A$ is the identity map; this concludes the proof. ■

3.5. Adjoint equivalences between Ω and Σ

As a consequence of Lemma 3.6, the natural transformation

$$\lambda : \Omega\Sigma \Rightarrow \mathrm{id}, \quad \lambda_A := \mathrm{triv}(\xi_L)^{-1} \otimes A, \tag{3.2}$$

is an isomorphism. Put flip: $\Omega\Sigma \cong \Sigma\Omega$ for the permutation of tensor factors. Since flip $\otimes-$ is a natural isomorphism, so is $\gamma := (\mathrm{flip} \otimes -) \circ \lambda^{-1} \colon \mathrm{id} \Rightarrow \Sigma\Omega$. We have thus explicitly constructed pseudoinverses exhibiting the fact that tensoring by Ω and Σ yield inverse equivalences of categories. In particlular, this allows us to see Ω as a left adjoint of Σ by viewing λ as the counit of an adjunction:

Theorem 3.8. *The natural transformation λ of (3.2) is the counit of an adjunction whose unit is $\Theta := \gamma^{-1}\Sigma\Omega \circ \Sigma\lambda^{-1}\Omega \circ \gamma$. Explicitly,*

$$\Theta_B := \Theta_0 \otimes B,$$
$$\textit{where} \quad \Theta_0 = (\mathrm{triv}(\xi_L)^{-1} \circ \mathrm{flip} \otimes \Sigma \otimes \Omega) \circ (\Sigma \otimes \mathrm{triv}(\xi_L) \otimes \Omega) \circ (\mathrm{flip} \circ \mathrm{triv}(\xi_L)).$$

Proof. This follows from the characterization of an adjunction in terms of triangle identities (Remark 4.2.7 in [23]); the reader can view the dual construction of a counit in the proof of Proposition 4.4.5 in [23]. ∎

Similarly, we can use the natural equivalence $\mathfrak{u} := \lambda^{-1} = \mathrm{triv}(\xi_L) \otimes -$ with inverse γ^{-1} to construct an adjunction in which Ω is right adjoint to Σ:

Theorem 3.9. *The natural transformation $\mathfrak{u}$, inverse to (3.2), is the unit of an adjunction whose counit is $\mathfrak{c} := \gamma^{-1} \circ \Sigma\lambda\Omega \circ \Sigma\Omega\gamma$. Explicitly,*

$$\mathfrak{c}_B = \mathfrak{c}_0 \otimes B, \quad \textit{where} \ \ \mathfrak{c}_0 := (\xi_L^{-1} \circ \mathrm{flip}^{-1}) \circ (\Sigma \otimes \xi_L^{-1} \otimes \Omega) \circ (\Sigma \otimes \Omega \otimes \mathrm{flip} \circ \xi_L).$$

From Theorems 3.8 and 3.9 we obtain, for each pair of graded algebras A and B, natural abelian group isomorphisms

$$\mathcal{R}_{A,B} : kk^{\mathrm{gr}}(\Omega A, B) \xrightarrow{\sim} kk^{\mathrm{gr}}(A, \Sigma B), \quad \zeta \mapsto \Sigma\zeta \circ \Theta_A, \tag{3.3}$$
$$\mathcal{L}_{A,B} : kk^{\mathrm{gr}}(A, \Sigma B) \xrightarrow{\sim} kk^{\mathrm{gr}}(\Omega A, B), \quad \zeta \mapsto \lambda_B \circ \Omega\zeta, \tag{3.4}$$

and

$$\mathcal{U}_{A,B} : kk^{\mathrm{gr}}(\Sigma A, B) \xrightarrow{\sim} kk^{\mathrm{gr}}(A, \Omega B), \quad \zeta \mapsto \Omega\zeta \circ \mathfrak{u}_A, \tag{3.5}$$
$$\mathcal{V}_{A,B} : kk^{\mathrm{gr}}(A, \Omega B) \xrightarrow{\sim} kk^{\mathrm{gr}}(\Sigma A, B), \quad \zeta \mapsto \mathfrak{c}_B \circ \Sigma\zeta. \tag{3.6}$$

3.6. Right boundaries

Given an extension ($\mathcal{E}$), its *right boundary map* is defined as

$$\delta_{\mathcal{E}} := -\mathcal{R}_{Q,K}(\partial_{\mathcal{E}}) = -\Sigma\partial_{\mathcal{E}} \circ (\Theta_0 \otimes Q).$$

Remark 3.10. By Theorem 3.7, right boundary maps are compatible with tensoring in the sense that $\delta_{\mathcal{E}\otimes A} = \delta_{\mathcal{E}} \otimes A$.

To conclude the section, we record the following computation, which will be useful to us later on.

Lemma 3.11. *Let A be a graded algebra. The right boundary map of the cone extension*

$$(\mathfrak{K} \otimes A) \qquad M_\infty A \to \Gamma A \to \Sigma A$$

is $\delta_{\mathfrak{K}\otimes A} = -\Sigma \operatorname{inc}_1 \otimes A$.

Proof. In light of Remark 3.10, we may assume that $A = \ell$. In this case, the extension consists of ungraded algebras; by Theorem 3.4, we may thus prove the statement in kk (in other words, we may assume G to be the trivial group). Denote the left and right boundary maps of the cone extension ($\mathfrak{K}$) by ∂ and δ respectively. By Lemma 3.6 and the definition of δ,

$$\delta = -\Sigma\partial \circ \Theta_Q = -\Sigma \operatorname{inc}_1 \circ (\Sigma \otimes \xi_L^{-1}) \circ (\Theta \otimes \Sigma) = -\Sigma \operatorname{inc}_1 \circ \mathcal{R}_{\Sigma,\ell}(\xi_L^{-1}).$$

To conclude we observe that $\xi_L^{-1} = \mathcal{L}_{\Sigma,\ell}(\mathrm{id}_\Sigma) = \mathcal{R}^{-1}_{\Sigma,\ell}(\mathrm{id}_\Sigma)$. ∎

4. Graded infinity-sum algebras, cones and suspensions

A *graded $*$-algebra* is a graded algebra R together with an involution $*\colon R \to R$ such that $R_g^* \subset R_{g^{-1}}$ for each $g \in G$. A graded *sum $*$-algebra* is a graded $*$-algebra R together with homogeneous elements $x, y \in A_{1_G}$ such that

$$x^*x = y^*y = xx^* + yy^* = 1.$$

If $x, y \in A_{1_G}$ make A into a graded sum $*$-algebra, then $y^*x = 0$. This follows from left multiplying by y^* and right multiplying by x in the equality $xx^* + yy^* = 1$. Likewise we have that $x^*y = 0$. As a consequence, the assignment

$$\boxplus : A \times A \to A, \quad a \boxplus b := xax^* + yby^*,$$

is a graded algebra homomorphism. Given graded $*$-algebra homomorphisms $f, g\colon B \to A$, we write $f \boxplus g$ for the $*$-algebra homomorphism $b \mapsto f(b) \boxplus g(b)$.

A graded *infinite-sum* $*$-algebra is a graded sum $*$-algebra A together with a graded homomorphism $(-)^\infty\colon A \to A$ such that $\boxplus \circ (\mathrm{id} \times (-)^\infty) = (-)^\infty$, i.e., such that

$$a \boxplus a^\infty = a^\infty \quad (\forall a \in A).$$

Our motivation for considering such algebras stems from the fact that they possess desirable properties in algebraic bivariant K-theory. Next, we adapt some results from [14] to the graded setting.

Proposition 4.1. *If B is a graded sum $*$-algebra and $f, g\colon A \to B$ are graded algebra homomorphisms, then $j(f) + j(g) \in kk^{\mathrm{gr}}(A, B)$ equals $j(f \boxplus g)$.*

Proof. Since kk^{gr} is an additive category and j is an additive functor (in the sense that it maps finite products to biproducts), it suffices to show that the codiagonal map $\nabla \in kk^{\mathrm{gr}}(B \times B, B)$ is equal to $j(\boxplus)$.

By Proposition 2.3, the inclusions $\iota_1, \iota_2\colon B \to M_2 B$ of B in the top-left and bottom-right corners respectively are mapped to the same arrow in kk^{gr}. Thus, considering the graded homomorphism

$$\epsilon : (b_1, b_2) \in B \times B \mapsto \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \in M_2 B,$$

we have that $\nabla = j(\iota_1)^{-1} \circ j(\epsilon)$. In particular, to prove the proposition it suffices to see that $j(\iota_1 \boxplus) = j(\epsilon)$.

Let $x, y \in B_{1_G}$ be the homogeneous elements that define the graded sum $*$-algebra structure on B. Put

$$Q = \begin{pmatrix} x & y & 0 \\ 0 & 0 & x^* \\ 0 & 0 & y^* \end{pmatrix}$$

for the matrix considered by Wagoner on p. 355 of [26], and set $u = Q^* = Q^{-1}$. Note that u is a unitary element of $M_3 B$ which is homogeneous of degree $1 \in G$. By Lemma 4.8.3 in [14], we have

$$u \begin{pmatrix} b \boxplus b' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} u^* = \begin{pmatrix} b & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.1}$$

In terms of the top-right corner inclusion $\jmath\colon M_2 B \to M_3 B$, equation (4.1) says that $\mathrm{ad}(u) \circ \jmath \circ \iota_1 \circ \boxplus = \jmath \circ \epsilon$. Applying now the functor j and using Proposition 2.10, we see that $j(\jmath) \circ j(\iota_1 \boxplus) = j(\jmath) \circ j(\epsilon)$. To conclude, we note that $j(\jmath)$ is an isomorphism by matricial stability. ∎

Proposition 4.2. *If A is a graded infinite-sum $*$-algebra and $I \lhd A$ an ideal such that $I^\infty \subset I$, then I is kk^{gr}-equivalent to zero.*

Proof. By Proposition 2.10, the matrix u given in the proof of Proposition 4.1 determines a graded $*$-homomorphism $\mathrm{ad}(u)\colon M_3 I \to M_3 I$ representing the identity of $M_3 I$. Hence the same argument as in Proposition 4.1 shows that the restriction $\boxplus'\colon I \times I \to I$ of $\boxplus$ to I is the codiagonal map of I. Since $I^\infty \subset I$, we can also restrict $(-)^\infty$ to a map $(-)^{\infty'}\colon I \to I$ satisfying $(-)^{\infty'} = 1_I \boxplus' (-)^{\infty'}$. It follows that $j((-)^{\infty'}) = j(1_I \boxplus' (-)^{\infty'}) = j(1_I) + j((-)^{\infty'})$; this goes to show that $j(1_I) = 0$ and therefore $j(I) = 0$. ∎

In the ungraded setting, an example of an infinite-sum $*$-algebra is Karoubi's cone Γ_X for any infinite set X (Equation (2.2) in [9], see also Lemma 4.8.2 in [14]). We wish to prove a similar statement for its graded analogue

$$\Gamma_X^\circ = \mathsf{span}_\ell\{f \in \Gamma_X : |x||f(x,y)||y|^{-1} \text{ is constant whenever } f(x,y) \neq 0\}.$$

Notice that this algebra is graded by setting

$$(\Gamma_X^\circ)_g = \mathsf{span}_\ell\{f \in \Gamma_X : |x||f(x,y)||y|^{-1} = g \text{ if } f(x,y) \neq 0.\}.$$

It contains M_X as a homogeneous ideal; the quotient Γ_X° / M_X is denoted by Σ_X°.

A non-empty graded set (X, d) with degree map $d\colon X \to G$ is said to be *graded infinite* if for all $g \in G$ the set $X_g := d^{-1}(g)$ is either empty or infinite.

Proposition 4.3. *If X is graded infinite, there exists a graded bijection $X \sqcup X \xrightarrow{\sim} X$.*

Proof. If X is graded infinite, then each component X_g is either infinite or empty and thus there exist injections $\sigma_g, \tau_g \colon X_g \to X_g$ such that $\sigma_g \sqcup \tau_g \colon X_g \sqcup X_g \to X_g$ is a bijection. It follows that $\sigma = \sqcup_{g \in G} \sigma_g$ and $\tau = \sqcup_{g \in G} \tau_g$ assemble into the desired bijection. ■

Proposition 4.4 (cf. Lemma 4.8.2 in [14]). *If X is graded infinite, then Γ_X° is a graded infinite-sum $*$-algebra.*

Proof. In view of Proposition 4.3, we may consider a graded bijection $X \sqcup X \to X$ induced by graded injections $\sigma, \tau \colon X \to X$, with disjoint image, such that $X = \operatorname{im}(\tau) \sqcup \operatorname{im}(\sigma)$. A direct verification shows that the elements

$$u = \sum_{x \in X} \varepsilon_{\sigma(x),x} \quad \text{and} \quad v \in \sum_{x \in X} \varepsilon_{\tau(x),x}$$

make Γ_X° into a graded sum $*$-algebra.

Next we will show that, for every $x, y \in X$,

$$\tau^n(\sigma(x)) = \tau^m(\sigma(y)) \iff n = m,\ x = y. \tag{4.2}$$

Indeed, suppose without loss of generality that $n = m + k$ for some $k \geq 0$. By injectivity of τ^m, we would have that $\tau^k(\sigma(x)) = \sigma(y)$. Since the images of τ and σ are disjoint, it must be $k = 0$ and hence $n = m$. Finally, the injectivity of σ lets us deduce that $x = y$.

From (4.2) we see that, for each $z \in \Gamma_X^\circ$, there is a well-defined element of Γ_X° given by

$$\begin{aligned} z^\infty &:= \sum_{n \geq 0} v^n u z u^* (v^n)^* = \sum_{n \geq 0, x, y \in X} \varepsilon_{\tau^n(\sigma(x)),x} \cdot z \cdot \varepsilon_{y,\tau^n(\sigma(y))} \\ &= \sum_{n \geq 0, x, y \in X} z(x,y) \cdot \varepsilon_{\tau^n(\sigma(x)),\tau^n(\sigma(y))}. \end{aligned}$$

By definition, $z \mapsto z^\infty$ is an algebra homomorphism and makes Γ_X° into a graded infinite-sum $*$-algebra, as desired. ■

We now apply the definition of graded infinity sum $*$-algebra to a graded analogue of Karoubi's cone and the resulting suspension algebra.

Corollary 4.5. *If X is a graded infinite set, then Γ_X° is kk^{gr}-equivalent to zero.*

Proof. Apply Proposition 4.2 to $I = A = \Gamma_X^\circ$. ■

Note that if (X, d) is any graded set, then

$$\widehat{X} := X \times \mathbb{N}, \quad d(x,n) = d(x),$$

is graded infinite and there is a canonical inclusion $x \in X \mapsto (x, 0) \in \widehat{X}$.

Definition 4.6. Let X be a graded set such that $X_{1_G} \neq \emptyset$. Define

$$\Gamma_X^{\mathrm{gr}} := \Gamma_{\widehat{X}}^\circ, \quad \Sigma_X^{\mathrm{gr}} := \Sigma_{\widehat{X}}^\circ \quad \text{and} \quad M_X^{\mathrm{gr}} := M_{\widehat{X}}.$$

Corollary 4.7. *Let X be a graded set such that $X_{1_G} \neq \emptyset$ and let $x \in X_{1_G}$. The graded inclusion $i_x: k \in \mathbb{N} \mapsto (x,k) \in \widehat{X}$ induces algebra monomorphisms*

$$M_\infty \hookrightarrow M_X^{\mathrm{gr}}, \quad \Sigma \hookrightarrow \Sigma_X^{\mathrm{gr}} \quad \text{and} \quad \Gamma \hookrightarrow \Gamma_X^{\mathrm{gr}}$$

which are kk^{gr}-isomorphisms.

Proof. Since $\Gamma_{\mathbb{N}}^\circ = \Gamma_{\mathbb{N}} = \Gamma$ and $\Sigma_{\mathbb{N}}^\circ = \Sigma_{\mathbb{N}} = \Sigma$, the maps induced by the inclusion i_x yield a diagram of cone extensions (and thus triangles) as follows:

$$\begin{tikzcd} M_\infty \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \Sigma \arrow[d] \\ M_{\widehat{X}} \arrow[r] & \Gamma^\circ_{\widehat{X}} \arrow[r] & \Sigma^\circ_{\widehat{X}}. \end{tikzcd}$$

The leftmost vertical arrow is a kk^{gr}-equivalence by graded matricial stability. The vertical arrow in the middle is a kk^{gr}-equivalence because both its domain and codomain are kk^{gr}-equivalent to zero. It follows, using that kk^{gr} is a triangulated category, that the rightmost vertical arrow is a kk^{gr}-equivalence. ■

4.1. Graded suspensions and deloopings

A family of boundary maps that will be of interest to us come from graded cone extensions. Given a graded set X, we have an extension

$$\Omega\Sigma_X^{\mathrm{gr}} \xrightarrow{\partial_X} M_X^{\mathrm{gr}} \to \Gamma_X^{\mathrm{gr}} \to \Sigma_X^{\mathrm{gr}}.$$

Using that the inclusion $\mathrm{inc}_X^{\mathrm{gr}}: M_X \hookrightarrow M_X^{\mathrm{gr}}$ is a kk^{gr}-isomorphism, we obtain a triangle

$$\Omega\Sigma_X^{\mathrm{gr}} \xrightarrow{(\mathrm{inc}_X^{\mathrm{gr}})^{-1}\circ\partial_X} M_X \to \Gamma_X^{\mathrm{gr}} \to \Sigma_X^{\mathrm{gr}}. \tag{4.3}$$

Since $\Gamma_X^{\mathrm{gr}} = 0$ in kk^{gr} by Corollary 4.5, it follows that the map $(\mathrm{inc}_X^{gr})^{-1} \circ \partial_X$ is an isomorphism. In particular, by matricial stability, for any $x \in X_1$ we have an isomorphism

$$\partial_X^{\mathrm{gr}} := \Omega\Sigma_X^{\mathrm{gr}} \xrightarrow{(\mathrm{inc}_X^{\mathrm{gr}})^{-1}\circ\partial_X} M_X \xrightarrow{j(\iota_x)^{-1}} \ell. \tag{4.4}$$

We may describe ∂_X^{gr} more explicitly, in the same way as for the ungraded cone extension.

Proposition 4.8. *Let X be a graded infinite set such that $X_{1_G} \neq \emptyset$ and $x \in X_{1_G}$. Write L_x for the class of $\sum_{i\geq 1} \varepsilon_{(x,i),(x,i+1)}$ in Σ_X^{gr}, and $\xi_{L_x}: \ell \to \Omega\Sigma$ for the map corresponding to $[L_x] \in KH_1^{\mathrm{gr}}(\Sigma_X^{\mathrm{gr}})$. The following diagram commutes in kk^{gr}:*

$$\begin{tikzcd} \ell \arrow[r, "\xi_{L_x}"] \arrow[dr, "\mathrm{inc}_{(x,1)}"'] & \Omega\Sigma_X^{\mathrm{gr}} \arrow[d, "\partial_X^{\mathrm{gr}}"] \\ & M_X^{\mathrm{gr}}. \end{tikzcd}$$

Proof. The morphism of extensions between the ungraded cone extension ($\mathfrak{K}$) and extension (4.3) given by inclusions, as in Corollary 4.7, extends to a morphism of triangles expressing ∂_X^{gr} in terms of ∂. The conclusion now follows from Lemma 3.6; we leave the details to the reader. ∎

Remark 4.9. If $X = *$ with $|*| = 1_G$, then $\widehat{X}$ is isomorphic to $\mathbb{N}$ with trivial grading. The inclusion $\mathrm{inc}_X^{\mathrm{gr}}$ corresponds to the inclusion $\iota_1\colon \ell \to M_\infty$, and (4.3) to the triangle $\ell \to \Gamma \to \Sigma$. Hence, the boundary (4.4) recovers the isomorphism

$$\Omega\Sigma \xrightarrow{\partial} M_\infty \xrightarrow{\mathrm{inc}_1^{-1}} \ell,$$

and Proposition 4.8 recovers Lemma 3.6 as a particular case.

5. Units as morphisms

The purpose of this section is to represent elements of KH_1 and KH_1^{gr} coming from units as certain arrows in the corresponding bivariant K-theory category. We first give a representation for units in ungraded algebras as maps in kk. Next, we use these results to deduce a representation in kk^{gr} for homogeneous units of degree 1_G of strongly graded rings.

5.1. Non-homogeneous units

Definition 5.1. Let $\mathcal{S} := \ker(\ell[t,t^{-1}] \xrightarrow{\mathrm{ev}_1} \ell)$. Note that a homomorphism $\ell[t,t^{-1}] \to A$ corresponds to the choice of an idempotent $p := \phi(1) \in A$ and a unit in pAp, namely $u := \phi(t)$. We write $\phi_{p,u}$ for such a homomorphism and $\nu_{p,u}$ for its restriction to $\mathcal{S}$. If A is unital and u is a unit in A, we put $\phi_u := \phi_{1,u}$ and $\nu_u := \nu_{1,u}$.

Definition 5.2. Let A be a unital algebra. A unit $u \in A^\times$ determines a class $[u] \in K_1(A)$ which, via the canonical comparison map, determines an element in $KH_1(A)$ which we also call $[u]$. We define $\xi_u\colon \ell \to \Omega$ to be the homomorphism corresponding to $[u] \in KH_1(A)$ via the isomorphism $KH_1(A) \simeq kk(\ell, \Omega A)$. Since $kk(\ell, \Omega(-)) \simeq KH_1(-)$, it follows that for any non-necessarily unital map $f\colon R \to S$ between unital algebras, we have

$$\Omega(j(f)) \circ \xi_u = \xi_{1-f(1)+f(u)}. \tag{5.1}$$

In particular, if A is a unital algebra, then applying (5.1) to the unit $t \in \ell[t,t^{-1}]$ and any map $\phi_{p,u}\colon \ell[t,t^{-1}] \to A$ gives

$$\Omega(j(\phi_{p,u}))\xi_t = \xi_{1+p-u}.$$

By Section 4.10 and the proof of Theorem 7.3.1 in [14], we know that $\nu_L\colon \mathcal{S} \to \Sigma$ is an isomorphism. Upon tensoring by ν_L, we obtain a natural isomorphism $\mathcal{S} \otimes - \cong \Sigma \otimes -$. This allows for the following definition.

Definition 5.3. Put

$$\wp_{A,B} := kk^{\mathrm{gr}}(\mathcal{S}A, B) \xrightarrow{(\nu_L^{-1}\otimes A)^*} kk(\Sigma A, B) \xrightarrow{\mathcal{U}_{A,B}} kk(B, \Omega A), \tag{5.2}$$

$$\wp := \wp_{\ell,\mathcal{S}}(\mathrm{id}_{\mathcal{S}}) = \Omega(\nu_L^{-1})\xi_L. \tag{5.3}$$

The map (5.2) allows us to represent classes of units in KH_1 as classes of algebra homomorphisms in kk:

Theorem 5.4. *Let A be a unital algebra. The natural chain of isomorphisms*

$$kk(\mathcal{S}, A) \xrightarrow{\wp_{\ell,A}} kk(\ell, \Omega A) \simeq KH_1(A)$$

maps $j(\nu_{p,u})$ to the class of the unit $1 - p + u \in A^\times$ in $KH_1(A)$.

Proof. Write $i: \mathcal{S} \to \ell[t, t^{-1}]$ for the inclusion. We aim to show that $\xi_{1-p+u} = \Omega(\phi_{p,u}) \circ \xi_t$ agrees with

$$\Omega(\nu_{p,u})\,\Omega(j(\nu_L))^{-1}\xi_L = \Omega(\phi_{p,u})\,\Omega(i)\,\Omega(j(\nu_L))^{-1}\xi_L.$$

Thus, it suffices to see that

$$\Omega(i)\,\Omega(j(\nu_L))^{-1}\xi_L = \xi_t.$$

We claim that, to conclude, it suffices to see that ξ_t factors through $\Omega(i)$. Indeed, suppose that there exists a map $\zeta: \ell \to \Omega\mathcal{S}$ such that $\xi_t = \Omega(i)\,\zeta$. Then

$$\xi_L = \Omega(j(\phi_L))\,\xi_t = \Omega(j(\phi_L))\,\Omega(i)\,\zeta = \Omega(j(\nu_L))\,\zeta,$$

and composing with $\Omega(i)\,\Omega(j(\nu_L))^{-1}$ on the left to both sides, we obtain the desired equality.

Finally, we have to prove the existence of such a morphism $\zeta: \ell \to \Omega\mathcal{S}$, which amounts to showing that $[t] \in KH_1(\ell[t, t^{-1}])$ lies in the image of $KH_1(i)$. It suffices to do so substituting K_1 for KH_1.

Consider the elements $x = (t - 1)$, $y = (t^{-1} - 1)$ of $\mathcal{S}$. A direct computation shows that $xy + x + y = 0$, which says that in the unitalization U of $\mathcal{S}$ the element $x + 1_U$ is a unit with inverse $y + 1_U$. The map induced by i on K_1 restricts to a map between the units of U and those of the unitalization U' of $\ell[t, t^{-1}]$, which maps $x + 1_U$ to $x + 1_{U'}$. To conclude, we note that the identification of $K_1(\ell[t, t^{-1}])$ with $\ker(K_1(U') \to K_1(\ell))$ maps t to $x + 1_{U'}$. ∎

5.2. Graded units in strongly graded rings

We view $\mathcal{S}$ and $\ell[t, t^{-1}]$ as graded algebras via the trivial grading. A graded homomorphism $\ell[t, t^{-1}] \to \mathcal{S}$ corresponds to a homogeneous idempotent $p \in A_1$ and a unit $u \in pA_1p$. We employ the same notation as in Definition 5.1 for these homomorphisms and their restrictions to $\mathcal{S}$.

Proposition 5.5. *Let C be a trivially graded algebra and A a strongly graded algebra. There is an isomorphism*

$$kk^{\mathrm{gr}}(C, A) \simeq kk(C, A_{1_G})$$

which maps the class of a graded algebra homomorphism $f: C \to A$ to the class of its corestriction $f|: C \to A_{1_g}$.

Proof. This follows directly from the proof of Theorem 6.1.4 in [17] (see also Remark 8.4 in [9]) and the bivariant version of Dade's theorem (Theorem 10.1 in [9]). ∎

From Theorem 5.4 and the proposition above, we obtain the main result of the section.

Theorem 5.6. *Let A be a unital, strongly graded algebra, $p \in A_{1_G}$ an idempotent and u a unit in $pA_{1_G}p$. Consider the map $\phi\colon \mathcal{S} \to A$ given by $1 \mapsto p$, $t \mapsto u$. Under the chain of isomorphisms*

$$kk^{\mathrm{gr}}(\mathcal{S}, A) \cong kk(\mathcal{S}, A_{1_G}) \simeq KH_1(A_{1_G}),$$

the arrow $j(\phi)$ has image $[1 - p + u]$. ■

For the next corollary, we recall from Lemma 9.3 in [9] that given two arrows $\xi \in kk^{\mathrm{gr}}(A, B)$ and $\zeta \in kk^{\mathrm{gr}}(C, D)$, their tensor product is defined as

$$\xi \otimes \zeta := (B \otimes \zeta) \circ (\xi \otimes C) = (\xi \otimes D) \circ (A \otimes \zeta).$$

Corollary 5.7. *Let R and B be unital graded algebras and let $p \in R_{1_G}$ and $q \in B_{1_G}$ be two idempotents. Consider u a unit of R_{1_G} and $\mathrm{inc}_q\colon \ell \to B$ the algebra map sending 1 to q. If R is strongly graded and u is a unit of R_{1_G}, then $\xi_{1\otimes q, u\otimes q} = \xi_u \otimes \mathrm{inc}_q$.*

Proof. The map inc_q defines a natural transformation $\mathrm{id} \Rightarrow - \otimes B$, and thus we have the following commuting diagram:

$$\begin{array}{ccccccc}
\ell & \xrightarrow{\sim} & \Omega\Sigma & \xrightarrow{\nu_L} & \Omega\mathcal{S} & \xrightarrow{\Omega\phi_u} & \Omega R \\
\downarrow{\scriptstyle \mathrm{inc}_q} & & \downarrow{\scriptstyle \Omega\Sigma\otimes\mathrm{inc}_q} & & \downarrow{\scriptstyle \mathcal{S}\otimes\mathrm{inc}_q} & & \downarrow{\scriptstyle \Omega R\otimes\mathrm{inc}_q} \\
B & \xrightarrow{\sim} & \Omega\Sigma \otimes B & \xrightarrow{\nu_L\otimes B} & \Omega\mathcal{S} \otimes B & \xrightarrow{\phi_u\otimes B} & \Omega R \otimes B.
\end{array}$$

By Theorem 5.6, the top row corresponds to ξ_u. Composition by $\Omega R \otimes \mathrm{inc}_q$ corresponds to the morphism $KH_1^{\mathrm{gr}}(R) \to KH_1^{\mathrm{gr}}(R \otimes B)$ induced by $R \otimes \mathrm{inc}_q$; hence, the top row of the diagram followed by the right-most vertical arrow corresponds to $\xi_{(R\otimes\mathrm{inc}_q)(1),(R\otimes\mathrm{inc}_q)(u)} = \xi_{1\otimes q, u\otimes q}$. As the diagram shows, this has to coincide with $\mathrm{inc}_q = \ell \otimes \mathrm{inc}_q$ composed with $\xi_u \otimes B$, which is by definition $\xi_u \otimes \mathrm{inc}_q$. ■

We conclude the section with some results on boundary maps from $K_1^{\mathrm{gr}}$to K_0^{gr}.

Proposition 5.8. *Let $\pi\colon R \to S$ be a surjective, graded $*$-algebra homomorphism; write $I := \ker(\pi)$. Assume that R is strongly graded. Let $u \in S_1$ be a unit. If $\hat{u} \in R_1$ is a partial isometry such that $\pi(\hat{u}) = u$, then the boundary map $\partial\colon K_1^{\mathrm{gr}}(R) \to K_0^{\mathrm{gr}}(I)$ maps $[u]$ to $[1 - \hat{u}^*\hat{u}] - [1 - \hat{u}\hat{u}^*]$.*

Proof. In the case of (hermitian) ungraded K-theory, a more general result is proven in the proof of Lemma 11.1 in [11]; the former implies in particular that $\partial'\colon K_1(R_{1_G}) \to K_0(I_{1_G})$ maps $[u]$ to $[1 - \hat{u}^*\hat{u}] - [1 - \hat{u}\hat{u}^*]$. The conclusion follows from the comparison square

$$\begin{array}{ccc}
K_1(R_{1_G}) & \xrightarrow{\partial'} & K_0(I_{1_G}) \\
\downarrow & & \downarrow \\
K_1^{\mathrm{gr}}(R) & \xrightarrow{\partial} & K_0^{\mathrm{gr}}(I).
\end{array}$$

■

Lemma 5.9. *Let X be a graded set such that $X_1 \neq \emptyset$. For any $x \in X_1$, the algebra map*

$$\theta_X : \mathcal{S} \to \Sigma_X^{\mathrm{gr}}, \quad t \mapsto \sum_{i \geq 1} \varepsilon_{(x,i+1),(x,i)}, \tag{5.4}$$

is a kk^{gr}-equivalence.

Proof. By Corollary 4.7, the map $\Sigma \to \Sigma_X^{\mathrm{gr}}$ induced by the inclusion $\mathbb{N} \subset \{x\} \times \mathbb{N}$ is a kk^{gr}-equivalence. The result is thus implied by the fact that $\nu_L : \mathcal{S} \to \Sigma$ is a kk^{gr}-equivalence. ∎

6. Poincaré duality

This section is devoted to the proof of the graded analogue of Poincaré duality, Theorem 11.2 in [11], and its consequences. Although not needed in the rest of this manuscript, we shall prove the result for any grading on $L(E)$ given by a *weight function* $\omega : E^1 \to G$, that is, the one given by the extension of the rule $|v| = 1_G$, $|e| = \omega(e)$, $|e^*| = \omega(e)^{-1}$ for each $v \in E^0$, $e \in E^1$. We shall write $L_\omega(E)$ to emphasize that $L(E)$ is being considered as a graded algebra with grading induced by ω. Recall that the *dual graph* E_t of a graph E is given by vertex and edge sets

$$E_t^0 = E^0 \quad \text{and} \quad E_t^1 = \{e_t : e \in E^1\},$$

and source and range functions

$$r(e_t) = s(e) \quad \text{and} \quad s(e_t) = r(e) \quad (e \in E^1).$$

Theorem 6.1. *If E is a finite essential graph and $\omega : E^1 \to G$ a weight function, then $- \otimes_\ell L_\omega(E)$ is left adjoint to $- \otimes \Omega L_\omega(E_t)$ as endofuntors of kk^{gr}. Thus, for each $R, S \in \mathsf{Alg}_\ell{}^{\mathrm{gr}}$, there are isomorphisms*

$$kk^{\mathrm{gr}}(R \otimes_\ell L_\omega(E), S) \cong kk^{\mathrm{gr}}(R, S \otimes_\ell \Omega L_\omega(E_t)).$$

natural in both R and S.

Proof. We adapt the proof of Theorem 11.2 in [11] to the present setting, which we will frequently cite in the argument below. Consider $\mathcal{P}_{\geq 1}$ the set of paths of positive length; we endow this set with a grading via the weighted length function $e_1 \cdots e_n \mapsto \omega(e_1) \cdots \omega(e_n)$. From now on, we omit the weight ω from the notation. Given $v \in E^0$, the set of paths starting at v will be denoted $\mathcal{P}^v$. Those ending at v will be denoted $\mathcal{P}_v$. Both are graded sets viewed as subsets of $\mathcal{P}$.

Put $X = \mathcal{P}_{\geq 1} \sqcup \{\bullet\}$ and set $|\bullet| = 1_G$. We shall view $L(E_t) \otimes L(E)$ and $L(E) \otimes L(E_t)$ as graded algebras via the tensor product grading. The morphisms

$$\rho_1(e) = \Big[\sum_{\alpha \in \mathcal{P}_{s(e)}} \varepsilon_{\alpha e, \alpha} \Big] \quad \text{and} \quad \rho_2(e_t) = \Big[\sum_{\alpha \in \mathcal{P}^{r(e)}} \varepsilon_{e\alpha, \alpha} \Big]$$

in loc. cit. can be corestricted to morphisms with codomain Σ_X° which we denote in the same way. Composing with the canonical map $\Sigma_X^\circ \to \Sigma_X^{\mathrm{gr}}$, we obtain a graded homomorphism $\rho\colon L(E_t) \otimes L(E) \to \Sigma_X^{\mathrm{gr}}$. Put

$$\kappa : \Omega L(E_t) \otimes L(E) \to \ell, \quad \kappa := \Omega L(E_t) \otimes L(E) \xrightarrow{\Omega\rho} \Omega\Sigma_X^{\mathrm{gr}} \xrightarrow{\partial_X^{\mathrm{gr}}} \ell.$$

Tensoring by κ on the right yields a natural map

$$(6.1)\quad kk^{\mathrm{gr}}(R, S \otimes \Omega L(E_t)) \to kk^{\mathrm{gr}}(R \otimes L(E), S), \quad \xi \mapsto (S \otimes \kappa) \circ (\xi \otimes L(E)).$$

In the other direction, we consider the elements $u_1 = \sum_{e \in E^1} e \otimes e_t^*$ and $p = \sum_{v \in E^0} v \otimes v$ as in the ungraded case, noting that they lie in the homogeneous component of degree zero of $L(E) \otimes L(E_t)$. Thus $u_1 = u + 1 - p$ is a degree zero unit of $L(E) \otimes L(E_t)$ and we have an induced map $\nu_{u_1}\colon \mathcal{S} \to L(E) \otimes L(E_t)$. By Lemma 5.4, the map $\xi_{u_1} \in kk^{\mathrm{gr}}(\ell, \Omega L(E) \otimes L(E_t))$ associated to $[u_1] \in KH_1^{\mathrm{gr}}(L(E) \otimes L(E_t))$ equals $\wp_{\mathcal{S}, L(E) \otimes L(E_t)}(\nu_{u_1}) = \Omega(\nu_{u_1})\wp$. We now consider the composition

$$\nabla := \ell \xrightarrow{\wp} \Omega\mathcal{S} \xrightarrow{\Omega\nu_{u_1}} \Omega \otimes L(E) \otimes L(E_t) \xrightarrow{\sim} L(E) \otimes \Omega L(E_t).$$

This defines a natural map

$$(6.2)\quad kk^{\mathrm{gr}}(R \otimes L(E), S) \to kk^{\mathrm{gr}}(R, S \otimes \Omega L(E_t)), \quad \xi \mapsto (\xi \otimes \Omega L(E_t)) \circ (R \otimes \nabla).$$

Similar to the ungraded case, to see that the compositions of (6.1) and (6.2) are bijections, it suffices to show that $(\kappa \otimes \Omega L(E_t)) \circ (\Omega L(E_t) \otimes \nabla)$ and $(L(E) \otimes \kappa) \circ (\nabla \otimes L(E))$ are isomorphisms in kk^{gr}. We will indicate how to adapt the argument for the first composition, the other one follows likewise. Define $\zeta\colon \mathcal{S} \otimes L(E_t) \to \Sigma_X^{\mathrm{gr}} \otimes L(E_t)$ to be the restriction of

$$\zeta' : \ell[t, t^{-1}] \otimes L(E_t) \to \Sigma_X^{\mathrm{gr}} \otimes L(E_t), \quad s \otimes 1 \mapsto (\rho_2 \otimes 1)(u), \quad 1 \otimes x \mapsto \rho_1(x) \otimes 1,$$

and consider the following permutations of tensor factors:

$$\begin{aligned}(243) &: \Omega L(E_t) \otimes \Omega\mathcal{S} \xrightarrow{\sim} \Omega \otimes \Omega \otimes \mathcal{S} \otimes L(E_t);\\ (23) &: \Omega \otimes \Omega \otimes \Sigma_X^{\mathrm{gr}} \otimes L(E_t) \xrightarrow{\sim} \Omega\Sigma_X^{\mathrm{gr}} \otimes \Omega L(E_t).\end{aligned}$$

A direct calculation shows that $(\kappa \otimes \Omega L(E_t)) \circ (\Omega L(E_t) \otimes \nabla)$ agrees with the following composition:

$$\Omega L(E_t) \xrightarrow{\Omega L(E_t) \otimes \wp} \Omega L(E_t) \otimes \Omega\mathcal{S} \xrightarrow{(23)\circ(\Omega\otimes\Omega\otimes\zeta)\circ(243)} \Omega\Sigma_X^{\mathrm{gr}} \otimes \Omega L(E_t)$$
$$\xrightarrow{\partial_X^{\mathrm{gr}} \otimes \Omega L(E_t)} \Omega L(E_t).$$

Hence, to see that $(\kappa \otimes \Omega L(E_t)) \circ (\Omega L(E_t) \otimes \nabla)$ is a kk^{gr}-isomorphism it suffices to show that ζ is one. To this end we define, as in the ungraded case, the graded $*$-homomorphism

$$\partial : L(E_t) \to \bigoplus_{v \in E^0} \Sigma_{\mathcal{P}^v}^{\mathrm{gr}}, \quad e_t \mapsto \sum_{\alpha \in \mathcal{P}_{s(e)}^v} \varepsilon_{\alpha e, \alpha}.$$

It lifts to a graded $*$-homomorphism $C(E_t) \to \bigoplus_{v\in E^0} \Gamma^{\circ}_{\widehat{\mathcal{P}}v}$ which restricts to the canonical isomorphism $\mathcal{K}(E_t) \simeq \bigoplus_{v\in E^0} M_{\mathcal{P}^v}$. We thus have maps of triangles

$$\begin{tikzcd}
\ell^{E^0} \arrow[r] \arrow[d, "\sim"] & \ell^{E^0} \arrow[r] \arrow[d, "\sim", "\mathrm{inc}"'] & L(E_t) \arrow[r] \arrow[d, equal] & \Sigma\ell^{E^0} \arrow[d, "\sim"] \\
\mathcal{K}(E_t) \arrow[r] \arrow[d, "\sim"] & C(E_t) \arrow[r] \arrow[d] & L(E_t) \arrow[r] \arrow[d, "\partial"] & \Sigma\mathcal{K}(E_t) \arrow[d, "\sim"] \\
\bigoplus_{v\in E^0} M_{\mathcal{P}^v} \arrow[r] & \bigoplus_{v\in E^0} \Gamma^{\mathrm{gr}}_{\mathcal{P}v} \arrow[r] & \bigoplus_{v\in E^0} \Sigma^{\mathrm{gr}}_{Pv} \arrow[r, "\sim"] & \Sigma \bigoplus_{v\in E^0} M_{\mathcal{P}^v}.
\end{tikzcd} \tag{6.3}$$

Hence, the boundary map $L(E_t) \to \Sigma\mathcal{K}(E)$ corresponds to the kk^{gr}-class of ∂ and have a triangle

$$\ell^{E^0} \xrightarrow{\mathrm{inc}} L(E_t) \xrightarrow{\partial} \bigoplus_{v\in E^0} \Sigma^{\mathrm{gr}}_{\mathcal{P}v}.$$

Tensoring with $\mathcal{S}$ and Σ^{gr}_X respectively, we obtain two distinguished triangles in kk^{gr}. To conclude the proof that ζ is an isomorphism, we will complete ζ to a morphism of triangles as in the following diagram, where both dashed arrows will be isomorphisms:

$$\begin{tikzcd}
\mathcal{S}\otimes\ell^{E^0} \arrow[r, "\mathcal{S}\otimes\mathrm{inc}"] \arrow[d, dashed, "\Upsilon_1"'] & \mathcal{S}\otimes L(E_t) \arrow[r, "\mathcal{S}\otimes\partial"] \arrow[d, "\zeta"] & \mathcal{S}\otimes\bigoplus_{v\in E^0}\Sigma^{\mathrm{gr}}_{\mathcal{P}v} \arrow[d, dashed, "\Upsilon_2"] \\
\Sigma^{\mathrm{gr}}_X\otimes\ell^{E^0} \arrow[r, "\Sigma^{\mathrm{gr}}_X\otimes\mathrm{inc}"] & \Sigma^{\mathrm{gr}}_X\otimes L(E_t) \arrow[r, "\Sigma^{\mathrm{gr}}_X\otimes\partial"] & \Sigma^{\mathrm{gr}}_X\otimes\bigoplus_{v\in E^0}\Sigma^{\mathrm{gr}}_{\mathcal{P}v}.
\end{tikzcd} \tag{6.4}$$

We construct the left-hand arrow first using the map θ_X as defined in (5.4). Set $\Upsilon_1 = \theta_X \otimes \ell^{E^0}$, which is an isomorphism by Lemma 5.9. We shall now see that

$$\zeta(\mathcal{S}\otimes\mathrm{inc}) = (\Sigma^{\circ}_{\widehat{X}}\otimes\mathrm{inc})\circ\Upsilon_1.$$

By additivity, it suffices to see that these compositions agree in each factor $\mathcal{S}\otimes v$. In view of Lemma 5.4, this boils down to checking whether $1-\theta_X(1)\otimes v+\theta_X(t)\otimes v$ and $1-\zeta(1\otimes v)+\zeta(t\otimes v)$ represent the same class in $KH_1^{\mathrm{gr}}(\Sigma^{\mathrm{gr}}_X\otimes L(E_t))$. As in the ungraded setting, this follows from a direct computation using Proposition 5.8 and the fact that, in this particular case, the boundary map $\partial\colon KH_1^{\mathrm{gr}}(\Sigma^{\mathrm{gr}}_X\otimes L(E_t)) \to KH_0^{\mathrm{gr}}(L(E_t))$ is an isomorphism.

Now we turn to defining the dashed right-most arrow. Write

$$\tau := \ell\{t, t^* : t^*t = 1\},$$

where $|t| = 1_G$. Recall that there is an isomorphism $M_\infty \cong \ker(\tau \hookrightarrow \ell[t,t^{-1}])$ mapping $\varepsilon_{1,1}$ to $1-tt^*$. As in the ungraded case, the restriction of ζ' to $\ell[t,t^{-1}]\otimes 1 \subset \ell[t,t^{-1}]\otimes L(E_t)$ can be extended to a graded homomorphism $\hat{\zeta}\colon \tau \to \Sigma^{\mathrm{gr}}_X\otimes L(E_t)$. Put

$$\tau_0 := \ker(\tau \xrightarrow{\mathrm{ev}_1} \ell).$$

Consider now the following morphisms of triangles:

$$\begin{tikzcd}
M_\infty L(E_t) \arrow[r] \arrow[d, equal] & \tau_0 \otimes L(E_t) \arrow[r] \arrow[d] & \mathcal{S} \otimes L(E_t) \arrow[r, "d", "\sim"'] \arrow[d, "i\otimes L(E_t)"] & \Sigma M_\infty L(E_t) \arrow[d, equal] \\
M_\infty L(E_t) \arrow[r] \arrow[d, "\hat{\xi}|"] & \tau \otimes L(E_t) \arrow[r] \arrow[d, "\hat{\xi}"] & \ell[t,t^{-1}] \otimes L(E_t) \arrow[r] \arrow[d, "\zeta'"] & \Sigma M_\infty L(E_t) \arrow[d, "\Sigma(\hat{\xi}|)"] \\
\Sigma_X^{\mathrm{gr}} \mathcal{K}(E_t) \arrow[r] & \Sigma_X^{\mathrm{gr}} C(E_t) \arrow[r] & \Sigma_X^{\mathrm{gr}} L(E_t) \arrow[r] & \Sigma \Sigma_X^{\mathrm{gr}} \mathcal{K}(E_t).
\end{tikzcd}$$

Since τ_0 is trivially graded and by Lemma 7.3.2 in [14], it is kk-equivalent to zero as an ungraded algebra, it follows that $\tau_0 \cong 0$ in kk^{gr}. In particular, the right boundary map d of the top triangle is an isomorphism. Together with (6.3), the diagram above says in particular that we have a commuting diagram as follows:

$$\begin{tikzcd}
\mathcal{S} \otimes L(E_t) \arrow[r, "d", "\sim"'] \arrow[d, "\zeta"] & \Sigma M_\infty L(E_t) \arrow[d, "\Sigma(\hat{\xi}|)"] & \\
\Sigma_X^{\mathrm{gr}} L(E_t) \arrow[r] \arrow[d, "\Sigma_X^{\mathrm{gr}}(\partial)"] & \Sigma \Sigma_X^{\mathrm{gr}} \mathcal{K}(E_t) \arrow[r, "\sim"] & \Sigma_X^{\mathrm{gr}} \Sigma \mathcal{K}(E_t) \arrow[d, "\sim"] \\
\Sigma_X^{\mathrm{gr}} \bigoplus_{v\in E^0} \Sigma_{\mathcal{P}v}^{\mathrm{gr}} \arrow[rr, "\sim"] & & \Sigma_X^{\mathrm{gr}} \Sigma \bigoplus_{v\in E^0} M_{\mathcal{P}v}.
\end{tikzcd}$$

From this we obtain an isomorphism $\mu\colon \Sigma\Sigma_X^{\mathrm{gr}}\mathcal{K}(E_t) \to \Sigma_X^{\mathrm{gr}} \bigoplus_{v\in E^0} \Sigma_{Pv}^{\mathrm{gr}}$ such that

$$\Sigma_X^{\mathrm{gr}}(\partial) \circ \zeta = \mu \circ \Sigma(\hat{\xi}|) \circ d.$$

Similarly, by the same argument as in the ungraded case we obtain a commuting diagram

$$\begin{tikzcd}
L(E_t) \arrow[r, "\partial"] \arrow[d, "\mathrm{inc}_1 \otimes L(E_t)"', "\sim"] & \bigoplus_{v\in E^0} \Sigma_{\mathcal{P}v}^{\mathrm{gr}} \arrow[r, "\bigoplus_v \mathrm{inc}_v", "\sim"'] & \bigoplus_{v\in E^0} \Sigma_X^{\mathrm{gr}} \arrow[d, "\sim"', "\sum_{v\in E^0} \Sigma_X^{\mathrm{gr}} \otimes q_v"] \\
M_\infty L(E_t) \arrow[rr, "\hat{\xi}|"] & & \Sigma_X^{\mathrm{gr}} \mathcal{K}(E_t),
\end{tikzcd}$$

guaranteeing the existence of an isomorphism $\mu'\colon \bigoplus_{v\in E^0} \Sigma_{\mathcal{P}v}^{\mathrm{gr}} \to \Sigma_X^{\mathrm{gr}}\mathcal{K}(E_t)$ such that

$$\mu' \circ \Sigma(\partial) \circ \Sigma(\mathrm{inc}_1 \otimes L(E_t))^{-1} = \Sigma(\hat{\xi}|).$$

Further, as we have an isomorphism $\nu_L\colon \mathcal{S} \xrightarrow{\sim} \Sigma$, it follows that

$$\Big(\nu_L \otimes \bigoplus_{v\in E^0} \Sigma_{\mathcal{P}v}^{\mathrm{gr}}\Big) \circ (\mathcal{S} \otimes \partial) = \Sigma(\partial) \circ (\nu_L \otimes L(E_t))$$

Thus, setting $\mu'' = \nu_L \otimes \bigoplus_{v\in E^0} \Sigma_{\mathcal{P}v}^{\mathrm{gr}}$, we get

$$\Sigma_X^{\mathrm{gr}} \circ \zeta = \mu\mu'\mu'' \circ (\mathcal{S} \otimes \partial) \circ (\nu_L \otimes L(E_t))^{-1} \circ (\Sigma\, \mathrm{inc}_1 \otimes L(E_t))^{-1} \circ d.$$

To conclude, we will see that

$$(\nu_L \otimes L(E_t))^{-1} \circ (\Sigma \operatorname{inc}_1 \otimes L(E_t))^{-1} \circ d = -\operatorname{id}.$$

Indeed, by Lemma 3.11 we have a commuting diagram

$$\begin{tikzcd}
M_\infty L(E_t) \arrow[r] \arrow[d, equal] & \tau_0 L(E_t) \arrow[r] \arrow[d] & \mathcal{S} L(E_t) \arrow[r, "d"] \arrow[d, "\nu_L \otimes L(E_t)"'] & \Sigma M_\infty L(E_t) \arrow[d, equal] \\
M_\infty L(E_t) \arrow[r] & \Gamma L(E_t) \arrow[r] & \Sigma L(E_t) \arrow[r, "\delta"] & \Sigma M_\infty L(E_t),
\end{tikzcd}$$

where $\delta = -(\Sigma \operatorname{inc}_1 \otimes L(E_t))$. Therefore $d = -(\Sigma \operatorname{inc}_1 \otimes L(E_t)) \circ (\nu_L \otimes L(E_t))$ and

$$\Sigma_X^{\mathrm{gr}} \circ \zeta = -\mu\mu'\mu'' \circ (\mathcal{S} \otimes \partial).$$

We may thus complete (6.4) by setting $\Upsilon_2 = -\mu\mu'\mu''$. This finishes the proof. ∎

Corollary 6.2. *Let E and F be finite graphs with E essential. If $f\colon L(E) \to L(F)$ is a graded algebra homomorphism, then the chain of isomorphisms*

$$kk^{\mathrm{gr}}(L(E), L(F)) \xrightarrow{(6.2)} kk^{\mathrm{gr}}(\ell, L(F) \otimes \Omega L(E_t)) \xrightarrow{(5.6)} KH_1((L(F) \otimes L(E_t))_0)$$

maps $j(f)$ to the class of the unit

$$u_f := 1 \otimes 1 - \sum_{v \in E^0} f(v) \otimes v + \sum_{e \in E^1} f(e) \otimes e_t^*. \tag{6.5}$$

Proof. The map in question is given by tensoring $\xi \in kk^{\mathrm{gr}}(L(E), L(F))$ by $\Omega L(E)$, precomposing by

$$\ell \xrightarrow{u_1} \Omega(L(E) \otimes L(E_t)) \xrightarrow{\sim} L(E) \otimes \Omega L(E_t)$$

and postcomposing again with the inverse of the isomorphism $\Omega(L(E) \otimes L(E_t)) \xrightarrow{\sim} L(E) \otimes \Omega L(E_t)$. This coincides with the composition $(\Omega f \otimes L(E_t)) \circ u_1$; that is, the map corresponding to the image of $[u_1] \in KH_1((L(E) \otimes L(E_t))$ under $KH_1(f \otimes L(E_t))$. It remains to note that $u_f = (f \otimes L(E_t))(u_1)$. ∎

Convention 6.3. *For the rest of the article, we will assume that $G = \mathbb{Z}$; in particular, we will use additive notation for the sum of degrees of homogeneous elements.*

6.1. Graded Morita invariance and source elimination

We record some observations on how one can extend Poincaré duality to non-necessarily essential graphs. We first recall the notions of full idempotents and source elimination (Definition 1.2 in [2]).

An idempotent p of a unital ring R is *full* if $RpR = R$, that is, if there exist $n \in \mathbb{N}$ and $x_1, \dots, x_n, y_1, \dots, y_n \in R$ such that

$$\sum_{i \in \mathbb{N}} y_i p x_i = 1.$$

Notice that any unital ring homomorphism maps full idempotents to full idempotents. Let E be a graph and $v \in \operatorname{sour}(E) \setminus \operatorname{sink}(E)$. The *source elimination graph* $E_{\setminus v}$ is given by

$$E^0_{\setminus v} = E^0 \setminus \{v\} \quad \text{and} \quad E^1_{\setminus v} = E^1 \setminus s^{-1}(v), \quad r_{E\setminus v} = r, \quad s_{E\setminus v} = s|_{E^0\setminus\{v\}}.$$

By Lemma 8.3 in [11], the element $p = 1 - v$ is a full homogeneous idempotent of $L(E)$ and the image of the graph inclusion induced map $\operatorname{inc}_v \colon L(E_{\setminus v}) \to L(E)$ is exactly $pL(E)p$. As noted in p. 230 of [20], when ℓ is a field source elimination preserves the (graded) Morita equivalence class of a Leavitt path algebra; this stems from the fact that, by the graded uniqueness theorem, the map inc_v is injective and thus $L(E_{\setminus v}) \cong pL(E)p$. In this direction, we wish to prove that inc_v is a kk^{gr}-isomorphism. This is implied by the result below.

Proposition 6.4. *Let R be a graded algebra. If $p \in R$ a homogeneous full idempotent of degree zero, then the inclusion $pRp \subset R$ is a kk^{gr}-isomorphism.*

Proof. We adapt Lemma 8.12 in [11]. Let $x_1, \dots, x_n, y_1, \dots, y_n \in R$ be such that $1 = y_1 p x_1 + \cdots + y_n p x_n$. Taking degree zero components at both sides of this equality, enlarging n if necessary, we may assume that all x_i, y_j are homogeneous such that $|x_i| = -|y_i|$. Substituting x_i by px_i and x_i^* by $y_i p$ if necessary, we may also assume that $x_i \in pR$ and $y_i \in Rp$. Put $d_i = |y_i|$. For the rest of the proof, we shall consider the grading on M_n given by the assignment $i \mapsto d_i$. Consider the elements

$$c = \sum_{j=1}^{n} \varepsilon_{j,1} x_j \in M_n pR \quad \text{and} \quad r = \sum_{j=1}^{n} \varepsilon_{1,j} y_j \in M_n Rp$$

and notice that these elements are homogeneous and that $|c||r| = 1$ and that $c\varepsilon_{1,1} M_n R \varepsilon_{1,1} r \subset M_n pRp$. Further, sice $rc = \varepsilon_{1,1}$, it follows that $wrcw' = ww'$ for each $w, w' \in \varepsilon_{1,1} M_n R \varepsilon_{1,1}$. We thus have a well-defined graded homomorphism

$$\operatorname{ad}(c,r) : \varepsilon_{1,1} M_n R \varepsilon_{1,1} \to M_n pRp, \quad w \mapsto cwr,$$

and, by Proposition 2.10, upon composing with the inclusion $M_n(\operatorname{inc}_p)\colon M_n pRp \hookrightarrow M_n R$, it coincides in kk^{gr} with the inclusion $\varepsilon_{1,1} M_n R \varepsilon_{1,1} \hookrightarrow M_n R$. Therefore, if we define

$$\phi : R \cong \varepsilon_{1,1} M_n R \varepsilon_{1,1} \xrightarrow{ad(c,r)} M_n pRp,$$

it satisfies $j(M_n(\operatorname{inc}_p)\phi) = j(\iota_1^R)$. In particular, $M_n(\operatorname{inc}_p)\phi$ is a kk^{gr}-isomorphism. A similar argument applied to $cp, rp \in M_n pRp$ says that the composition

$$\phi \operatorname{inc}_p = pRp \cong \varepsilon_{1,1} M_n pRp \varepsilon_{1,1} \xrightarrow{\operatorname{ad}(cp,pr)} M_n pRp$$

agrees in kk^{gr} with ι_1^{pRp}; hence $\phi \operatorname{inc}_p$ is also a kk^{gr}-isomorphism. Finally, this says that ϕ is a kk^{gr}-isomorphism which, in turn, proves that $j(\operatorname{inc}_p) = j(\phi)^{-1} j(\iota_1^{pRp})$ is an isomorphism as desired. ∎

Corollary 6.5. *Assume that ℓ is a field. If E is a graph with at least two vertices and $v \in \operatorname{sour}(E) \setminus \operatorname{sink}(E)$, then the inclusion $L(E_{\setminus v}) \to L(E)$ is a kk^{gr}-isomorphism.*

Remark 6.6. By Theorem 5.3 in [3], two graded unital rings are graded Morita equivalent if and only if there exists a graded set structure on $\mathbb{N}$, say $X = (\mathbb{N}, d)$, $d\colon \mathbb{N} \to G$, such that $M_X S \cong M_\infty R$ as graded algebras. (As per our conventions, here $M_\infty R$ means $M_{\mathbb{N}} R$ where $\mathbb{N}$ is equipped with the trivial grading). In particular, this is a way to show that G-stable functors are Morita invariant.

The results of this section say that given a regular graph E, upon finitely many source eliminations we may find an essential graph F such that we have a kk^{gr}-isomorphism $L(F) \to L(E)$. Theorem 6.1 then implies that tensoring by $L(E)$ is left adjoint to tensoring by $\Omega L(F_t)$.

7. The relationship between kk^{gr}-maps and graded algebra maps

Consider E a finite graph and the Cohn extension ($\mathscr{C}_E$). Write ∂_E and δ_E for its left and right boundary. By Corollary 11.9 in [9], in kk^{gr} we have a triangle

$$\ell^{\mathrm{reg}(E)} \xrightarrow{I-\sigma A_E^t} \ell^{E^0} \xrightarrow{\mathrm{inc}} L(E). \tag{7.1}$$

In particular, for a given graded algebra R, applying $kk^{\mathrm{gr}}(-, R)$ to (7.1) yields an exact sequence

$$\begin{tikzcd} kk^{\mathrm{gr}}(\Sigma^{E^0}, R) \arrow[r, "(\Sigma(I-\sigma A_E^t))^*"] & kk^{\mathrm{gr}}(\Sigma^{\mathrm{reg}(E)}, R) \arrow[d, "\delta_E^*"] & \\ & kk^{\mathrm{gr}}(L(E), R) \arrow[d, "\mathrm{inc}^*"] & \\ & kk^{\mathrm{gr}}(\ell^{E^0}, R) \arrow[r, "(I-\sigma A_E^t)^*"] & kk^{\mathrm{gr}}(\ell^{reg(E)}, R). \end{tikzcd}$$

From this sequence we can obtain, by taking appopriate kernels and cokernels, a short exact sequence involving $kk^{\mathrm{gr}}(L(E), R)$. This is what in the ungraded case is referred to as the universal coefficient theorem (UCT). Recall that the *dual Bowen–Franks module* of E is $\mathrm{BF}^\vee_{\mathrm{gr}}(E) = \mathrm{coker}(I^t - \sigma A_E)$. If E is essential, then A_E is square and $A_E^t = A_{E_t}$; hence $\mathrm{BF}_{\mathrm{gr}}(E_t) = \mathrm{coker}(I - \sigma A^t_{E_t}) = \mathrm{coker}(I^t - \sigma A_E) = \mathrm{BF}^\vee_{\mathrm{gr}}(E)$. With this notation in place, we state a theorem which, in particular, contains a graded version of the UCT.

Theorem 7.1 (UCT). *Let E be a finite graph and R a graded algebra. Put $\otimes = \otimes_{\mathbb{Z}[\sigma]}$ and* $\hom = \hom_{\mathbb{Z}[\sigma]}$. *There is a diagram with exact top-row*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{BF}^\vee_{\mathrm{gr}}(E) \otimes KH_1^{\mathrm{gr}}(R) \arrow[r, "d"] & kk^{\mathrm{gr}}(L(E), R) \arrow[r, "\mathrm{ev}"] & \hom(\mathrm{BF}_{\mathrm{gr}}(E), KH_0^{\mathrm{gr}}(R)) \arrow[r] & 0 \\ & & {[L(E), R]} \arrow[u, "\bar{j}"] \arrow[ur, bend right, "\mathrm{can}^* \circ KH_0^{\mathrm{gr}}"'] & & \end{tikzcd}$$

such that:

(i) *The map* $\bar{j}$ *is the factorization of* j *through the category of graded* ℓ*-algebras with graded homomorphisms up to graded polynomial homotopy.*

(ii) *The map* ev *corresponds to the assignment between hom-sets of the functor*

$$kk^{\mathrm{gr}}(\ell, -) = KH_0^{\mathrm{gr}},$$

followed by precomposition by the canonical map can: $\mathrm{BF}_{\mathrm{gr}}(E) \to KH_0^{\mathrm{gr}}(L(E))$.

(iii) *The map* d *is obtained from the right boundary map* $\delta_E : L(E) \to \Sigma^{\mathrm{reg}(E)}$*, by passing the composition*

$$\mathbb{Z}[\sigma]^{\mathrm{reg}(E)} \otimes_{\mathbb{Z}[\sigma]} KH_1^{\mathrm{gr}}(R) \xrightarrow{\sim} kk^{\mathrm{gr}}(\Sigma^{\mathrm{reg}(E)}, L(F)) \xrightarrow{\delta^*} kk^{\mathrm{gr}}(L(E), L(F))$$

to the quotient module $\mathrm{BF}_{\mathrm{gr}}^{\vee}(E) \otimes_{\mathbb{Z}[\sigma]} KH_1^{\mathrm{gr}}(R)$.

(iv) *If* E *is an essential graph, then for any* $v \in E^0$ *and unit* $z \in R$ *represented by a map* $\xi_z : \ell \to \Omega R$*, the isomorphism* $kk^{\mathrm{gr}}(L(E), R) \cong kk^{\mathrm{gr}}(\ell, \Omega(R \otimes L(E_t)) \cong KH_1^{\mathrm{gr}}(R \otimes L(E_t))$ *maps* $d(v \otimes z)$ *to* $[1 \otimes 1 - 1 \otimes v + z \otimes v]$.

Proof. We prove (iv), the other assertions are proved in the same way as in ungraded case; see, e.g., Corollary 7.20 in [13] and Theorem 12.1 in [11]. Assume that E is essential. Recall that ξ_z corresponds to an arrow $\mathcal{V}_{\ell,R}(\xi_z) : \Sigma \to R$ via the assignment (3.6). Thus, if $p_v : \ell^{E^0} \to \ell$ denotes the projection to the v-th coordinate, then $d(v \otimes \xi_z)$ coincides with the composition

$$L(E) \xrightarrow{\delta} \Sigma^{E^0} \xrightarrow{\Sigma p_v} \Sigma \xrightarrow{\mathcal{V}_{\ell,R}(\xi_z)} R.$$

Recall also that the arrow above is assigned to an element $kk^{\mathrm{gr}}(\ell, \Omega(R \otimes L(E_t))$ in the following way: first, one tensors by $L(E_t)$; next one applies the loop functor Ω and, at last, one precomposes by the arrow $\xi_{u_1} : \ell \to \Omega(L(E) \otimes L(E_t))$ given by the degree zero unit $u_1 \in L(E) \otimes L(E_t)$. Call this element $\eta := \Omega(d(v \otimes \xi_z) \otimes L(E_t)) \circ \xi_{u_1}$.

Notice that, by Remark 3.10 and the definition of (3.6), we have $\delta_E \otimes L(E_t) = \delta_{\mathcal{C}(E) \otimes L(E_t)}$ and $\mathcal{V}_{\ell,R}(\xi_z) \otimes L(E_t) = \mathcal{V}_{\ell, R \otimes L(E_t)}(\xi_z \otimes L(E_t))$. We shall drop the subscripts under $\mathcal{V}$ to ease the notation, and write $\delta = \delta_{\mathcal{C}(E) \otimes L(E_t)}$, $\partial = \partial_{\mathcal{C}(E) \otimes L(E_t)}$, $\widetilde{p_v} = p_v \otimes L(E_t)$, and $\widetilde{\xi_z} = \xi_z \otimes L(E_t)$. Consider now the following diagram:

$$\begin{tikzcd}
\Omega L(E) \otimes L(E_t) \arrow[r, "\Omega\delta"] \arrow[d, equal] & \Omega\Sigma \otimes \ell^{E^0} \otimes L(E_t) \arrow[r, "\Omega\Sigma\widetilde{p_v}"] & \Omega\Sigma \otimes L(E_t) \arrow[r, "\Omega\mathcal{V}(\widetilde{\xi_z})"] & \Omega R \otimes L(E_t) \arrow[d, equal] \\
\Omega(L(E) \otimes L(E_t)) \arrow[r, "\partial"] & \ell^{E^0} \otimes L(E_t) \arrow[r, "\widetilde{p_v}"] \arrow[u, "\xi_L \otimes \ell^{E^0} \otimes L(E_t)"'] & L(E_t) \arrow[r, "\widetilde{\xi_z}"] \arrow[u, "\xi_L \otimes L(E_t)"'] & \Omega R \otimes L(E_t).
\end{tikzcd}$$

Note that the composition of the top row with ξ_{u_1} agrees with η. As in the ungraded case (proof of Lemma 12.3 in [11]), one checks that the boundary of u_1 is the class of $\sum_{v \in E^0} \chi_v \otimes v$, and thus

$$\eta = \xi_z \otimes L(E_t) \circ p_v \circ \Big(\sum_{v \in E^0} \chi_v \otimes v \Big) = \xi_z \otimes \mathrm{inc}_v \,.$$

Finally, by Lemma 5.7, we have $\xi_u \otimes \mathrm{inc}_v = \xi_{1 \otimes v, z \otimes v}$ which corresponds to the class in $KH_1^{\mathrm{gr}}(R \otimes L(E_t))$ of the unit $1 \otimes 1 - 1 \otimes v + z \otimes v$, as desired. ∎

We now want to use the UCT to investigate the relationship between graded algebra maps between Leavitt path algebras and morphisms between them in kk^{gr}. First, we set some conventions.

Definition 7.2. A regular graph E is *primitive* if its adjacency matrix is primitive (Definition 4.5.7 and Theorem 4.5.8 in [22]), that is, if there exists $N \geq 1$ such that $(A_E^N)_{v,w} > 0$ for all $v, w \in E^0$.

Remark 7.3. If E is a primitive graph, then by definition there exists $N \geq 1$ such that there is a path of length N between each pair of vertices. In particular, primitive graphs are essential.

Our interest for primitive graphs stems from the following.

Proposition 7.4. *Assume that ℓ is a field. If E is a primitive graph, then for each $e \in E^1$, the idempotent $ee^* \in L(E)_0$ is full as an element of $L(E)_0$.*

Proof. Recall that if we put

$$L(E)_{0,n} = \operatorname{span}_\ell\{\alpha\beta^* : r(\alpha) = r(\beta), |\alpha| = |\beta| = n\}$$

for each $n \geq 0$, then $L(E)_0 = \bigcup_{n\geq 0} L(E)_{0,n}$. Recall also that, writing $\mathcal{P}_{v,n}$ for the set of paths of length n ending at a vertex v, there are isomorphisms

$$L(E)_{0,n} \cong \bigoplus_{v\in E^0} M_{\mathcal{P}_{v,n}}, \quad \alpha\beta^* \mapsto \varepsilon_{\alpha,\beta} \in M_{\mathcal{P}_{r(\alpha),n}}, \tag{7.2}$$

for each $n \geq 0$.

Since $L(E)$ is an inreasing union of its unital subalgebras $L(E)_{0,n}$, to see that $ee^* \in L(E)_0$ is full it suffices to see that it is so in $L(E)_{0,n}$ for some $n \in \mathbb{N}$. Let $N \geq 1$ be such that there exists a path of length N between every pair of vertices. Observe that

$$ee^* = \sum_{s(\alpha)=r(e),|\alpha|=N} e\alpha(e\alpha)^* = \sum_{v\in E^0} \sum_{s(\alpha)=r(e),|\alpha|=N,r(\alpha)=v} e\alpha(e\alpha)^*.$$

Thus, under the isomorphism (7.2) applied to $n = N + 1$, the idempotent ee^* is mapped to a sum of diagonal matrices

$$\sum_{v\in E^0} \sum_{s(\alpha)=r(e),|\alpha|=N,r(\alpha)=v} \varepsilon_{e\alpha,e\alpha}.$$

Given that matrix rings over a field are simple algebras, to conclude it suffices to prove that the coordinate of element above corresponding to each algebra $M_{\mathcal{P}_{v,N+1}}$ is non-zero. This amounts to showing that for each set $\{e\alpha : r(\alpha) = v, |\alpha| = N\}$ is non-empty, which is implied by the fact that $(A_E^N)_{r(e),v} > 0$ for all $v \in E^0$. ∎

Remark 7.5. In the proof of Proposition 7.4, we only need that for each vertex v in the graph E there exists some $N_v \geq 1$ such that the v-th row of A_E^N has positive entries. However, if E is essential, this condition is equivalent to E being primitive. Indeed, put $N = \max_{u\in E^0} N_u$ and fix $v, w \in E^0$. Since E is essential, inductively we may find a path β of length $N - N_w$ ending at w. By hypothesis we also have a path α of length $N_{s(\beta)}$ from v to $s(\beta)$; hence $\alpha\beta$ is a path of length N from v to w.

Convention 7.6. *From now on, we shall assume that ℓ is a field and all graphs considered are primitive.*

Define

$$kk^{\mathrm{gr}}(L(E),L(F))_1 \\ = \{\xi \in kk^{\mathrm{gr}}(L(E),L(F)) : \mathrm{ev}(\xi) \text{ is an pointed preordered module morphism}\},$$

and write $[L(E),L(F)]_1$ for the set of graded unital algebra homomorphisms $L(E) \to L(F)$ modulo graded polynomial homotopy. Our next objective is to study the map

$$\bar{j} : [L(E),L(F)]_1 \to kk^{\mathrm{gr}}(L(E),L(F))_1. \tag{7.3}$$

To proceed further in the understanding of $kk^{\mathrm{gr}}(L(E),L(F))_1$, we first need to understand the K_1^{gr} group of a Leavitt path algebra. To do this, we first establish some remarks on ultramatricial algebras and corner skew Laurent polynomial rings.

7.1. K_1 of ultramatricial algebras

As pointed out in Remark 2.13, when E is a regular graph, its associated Leavitt path algebra is strongly graded and thus $KH_1^{\mathrm{gr}}(L(E)) = KH_1^{\mathrm{gr}}(L(E)_0)$. If in addition ℓ is a field, then we may replace K for KH. This allows us to compute the graded K-theory of $L(E)$ in terms of its subalgebra of homogeneous elements of degree zero.

The advantage of this passage to $L(E)_0$ is that the latter algebra is *ultramatricial*, that is, it is a countable increasing union of matricial algebras. For this reason, we wish to prove some generalities regarding the first K-theory group of a unital ultramatricial algebra. In what follows we will write G_{ab} for the abelianization of a group. In particular, if R is a unital ring then $K_1(R) = GL(R)_{\mathrm{ab}}$. The field of two elements will be denoted $\mathbb{F}_2$.

The following observation is straightforward from the definition of ultramatricial algebra.

Lemma 7.7. *Let $F, G\colon \mathsf{Alg}_\ell \to \mathsf{Grp}$ be two additive functors which preserve finite products and filtering colimits and let $\eta\colon F \Rightarrow G$ be a natural transformation. The following statements are equivalent:*

(i) *For each unital ultramatricial algebra R, the map η_R is an isomorphism.*

(ii) *For each $n \in \mathbb{N}$, the map $\eta_{M_n(\ell)}$ is an isomorphism.*

Proposition 7.8. *Assume that ℓ is a field different from $\mathbb{F}_2$. If R is a unital ultramatricial ℓ-algebra, then the canonical map $R^\times_{\mathrm{ab}} \to K_1(R)$ is an isomorphism.*

Proof. By Lemma 7.7, it suffices to prove so for $R = M_n(\ell)$ for each $n \in \mathbb{N}$. Notice that $M_n(\ell)^\times = \mathrm{GL}_n(\ell)$ and $[\mathrm{GL}_n(\ell), \mathrm{GL}_n(\ell)] = \mathrm{SL}_n(\ell)$, since $\ell \neq \mathbb{F}_2$.

We know that the (non-unital) inclusion of ℓ in the top-left corner induces an isomorphism in K_1, mapping $\lambda \in \ell^\times = K_1(\ell)$ to the class of $[I_n - \varepsilon_{1,1} + \lambda\varepsilon_{1,1}]$. To conclude, we note that this isomorphism factors as the inverse of the determinant induced map $\det\colon \mathrm{GL}_n(\ell)/\mathrm{SL}_n(\ell) \to \ell^\times$, followed by the comparison map $\mathrm{GL}_n(\ell)/\mathrm{SL}_n(\ell) = M_n(\ell)^\times_{\mathrm{ab}} \to K_1(M_n(\ell))$. ■

Given a functor $F\colon \mathsf{Alg}_\ell \to \mathsf{Grp}$, write

$$F^0(A) = F(\mathrm{ev}_1)(\ker(F(A[t]) \xrightarrow{F(\mathrm{ev}_0)} F(A))).$$

Note that, since $\mathrm{ev}_1\colon A[t] \to A$ is a retraction for any algebra A, the homomorphism $F(\mathrm{ev}_1)$ is a retraction; in particular, it is surjective. Hence, it maps the normal subgroup $\ker(F(\mathrm{ev}_0))$ of $F(A[t])$ to a normal subgroup of $F(A)$. This justifies the fact that

$$\pi_0 F(A) := F(A)/F^0(A)$$

is a group. Further, this assignment can be extended to a functor $\mathsf{Alg}_\ell \to \mathsf{Grp}$ by the universal properties of kernels and images.

In Proposition 2.8 of [12], Cortiñas and Montero show that the Karoubi–Villamayor K_1-group of a purely infinite simple ring R can be computed as

$$\pi_0 R^\times = R^\times/\{u(1) : u \in (R[t])^\times, u(0) = 1\}.$$

In our context, we obtain a similar conclusion for ultramatricial algebras. First, we need a lemma.

Lemma 7.9. *If ℓ is a field, then $(M_n(\ell)^\times)^0 = \mathrm{SL}_n(\ell)$.*

Proof. We prove both inclusions. Since $\mathrm{SL}_n(\ell)$ is generated by elementary matrices, it suffices to show that $I_n + \lambda\varepsilon_{i,j} \in (M_n(\ell)^\times)^0$ for each $\lambda \in \ell^\times$, for which it suffices to consider the elementary matrix

$$I_n + \lambda t\varepsilon_{i,j} \in \mathrm{GL}_n(\ell[t]) = M_n(\ell[t])^\times = (M_n(\ell)[t])^\times.$$

For the converse, let $u \in M_n(\ell)[t] = M_n(\ell[t])$ be an invertible matrix such that $u(0) = I_n$. We have to prove that $u(1) \in \mathrm{SL}_n(\ell)$. Since $\ell[t]$ is an integral domain, a matrix in $M_n(\ell[t])$ is a unit if and only if $\det(u) \in (\ell[t])^\times = \ell^\times$. In particular, $\det(u)$ is constant and thus

$$\det(u(1)) = \det(u)(1) = \det(u)(0) = \det(u(0)) = \det(I_n) = 1. \qquad \blacksquare$$

Proposition 7.10. *Assume that ℓ is a field. If $\ell \neq \mathbb{F}_2$, then the comparison map $R^\times_{ab} \to \pi_0 R^\times \to K_1(R)$ is an isomorphism for every unital ultramatricial algebra R. If $\ell = \mathbb{F}_2$, then $K_1(R) = \pi_0 R^\times = 1$.*

Proof. Since $(M_n(\ell)^\times)^0 = \mathrm{SL}_n(\ell)$ for each $n \in \mathbb{N}$, we have

$$\pi_0(M_n(\ell)^\times) = \mathrm{GL}_n(\ell)/\mathrm{SL}_n(\ell) = \begin{cases} \mathrm{GL}_n(\ell)_{ab} & |\ell| > 2, \\ 1 & |\ell| = 2. \end{cases}$$

This together with Lemma 7.7 prove the first part of the lemma and also that $\pi_0 R^\times = 1$ for each unital ultramatricial algebra R over $\mathbb{F}_2$. It remains to see that $K_1(R)$ is also trivial when $\ell = \mathbb{F}_2$, which follows from matricial stability, since $K_1(M_n(\mathbb{F}_2)) \cong K_1(\mathbb{F}_2) = \mathbb{F}_2^\times = 1$. $\blacksquare$

Definition 7.11. Let E be a finite graph and $\phi\colon L(E) \to R$ a graded unital homomorphism. Write

$$R_\phi := \bigoplus_{e\in E^1} \phi(ee^*)\, R_0\, \phi(ee^*).$$

Lemma 7.12. *Assume that ℓ is a field. If R is a unital ultramatricial algebra and $e \in R$ an idempotent, then eRe is a unital ultramatricial algebra.*

Proof. Let $R = \bigcup_{n\geq 1} R_n$ be an increasing union of unital, matricial subalgebras. Since the union is increasing, we may assume without loss of generality that $1_R, e \in R_1$; consequently, $eRe = \bigcup_{n\geq 1} eR_n e$.

It suffices to show that each algebra $eR_n e$ is matricial, that is, to show that the corner of any idempotent in a matricial algebra is again matricial. Let $A = M_{k_1}(\ell) \times \cdots \times M_{k_N}(\ell)$ be a matricial algebra and $(e_1, \ldots, e_N) \in \mathrm{Idem}(A)$. Since $eAe = \prod_{i=1}^N e_i A_i e_i$, we may prove that any corner of a matrix algebra is again a matrix algebra. In other words, we may assume that $N = 1$. Put $k = k_1$ and $e = e_1$. Since ℓ is a field and an idempotent matrix represents a linear projector on ℓ^k, there is an invertible matrix u such that $u^{-1}eu = \sum_{s=1}^j \varepsilon_{s,s} =: p_j$ for some $j \in \{0, \ldots, k\}$; conjugation by u maps e to p_j and $eM_k(\ell)e$ to $p_j M_k(\ell) p_j$, which is isomorphic to $M_j(\ell)$. ∎

Proposition 7.13. *Let E be a primitive graph and R a strongly graded algebra such that R_0 is ultramatricial. Each graded unital algebra homomorphism $\phi\colon L(E) \to R$ induces an isomorphism*

$$(R_\phi)^\times_{\mathrm{ab}} = \prod_{e\in E^1} (\phi(ee^*)R_0\phi(ee^*))^\times_{\mathrm{ab}} \to K_1(R_0)^{E^1} \cong K_1^{\mathrm{gr}}(R)^{E^1},$$
$$(z_e)_{e\in E^1} \mapsto ([1-\phi(ee^*)+z_e])_{e\in E^1}. \tag{7.4}$$

Proof. It follows from Lemma 7.12 and Proposition 7.8 that $K_1(\phi(ee^*)R_0\phi(ee^*))$ can be computed as $(\phi(ee^*)R_0\phi(ee^*))^\times_{\mathrm{ab}}$ for all $e \in E^1$. By Proposition 7.4, we know that each element ee^* is a full idempotent of $L(E)_0$ and so, since ϕ is a graded unital homomorphism, it follows that $\phi(ee^*)$ is a full idempotent of R_0. Consequently, each corner inclusion $\phi(ee^*)R_0\phi(ee^*) \to R_0$ induces an isomorphism at the level of K_1 groups. This concludes the proof. ∎

7.2. The shift action for corner skew Laurent polynomials

Let R be a *corner skew Laurent polynomial ring*, that is, a $\mathbb{Z}$-graded ring together with elements $t_+ \in R_1$, $t_- \in R_{-1}$ satisfying $t_-t_+ = 1$ (Lemma 2.4 in [4]). Our motivating example is that of the Leavitt path algebra of an essential graph, see p. 210 in [6]; indeed, if one selects one edge e_v with range v for each $v \in E^0$, then the elements

$$t_+ = \sum_{v\in E^0} e_v \quad \text{and} \quad t_l = \sum_{v\in E^0} e_v^*$$

yield a corner skew Laurent polynomial ring structure on $L(E)$.

Hazrat proves in Proposition 1.6.6 of [21] that $p := t_+t_-$ is a full idempotent if and only if R is strongly graded. When this is the case, we know that $K_*^{\mathrm{gr}}(R)$ is naturally isomorphic to $K_*(R_0)$; we wish to understand to what kind of action the shift action translates to when viewed on $K_*(R_0)$.

In the case of the Leavitt path algebra of an essential graph E, Ara and Pardo prove in Lemma 3.6 of [6] that the action on $L(E)_0$ is the one induced by the (non-unital) corner homomorphism

$$\alpha : L(E)_0 \to L(E)_0, \quad x \mapsto t_+ x t_-.$$

The proof relies on the fact that for $L(E)_0$, the Grothendieck group is generated by (1×1)-idempotents. In other words, instead of considering idempotents for all finite matrices, it suffices to do so for the ones of size 1.

Remark 7.14. We remark that in Lemma 3.6 of [6], the shift agrees with the inverse of the map induced by α. This difference is explained by the fact that, if one builds K_0^{gr} for right modules instead of left modules, the isomorphism maps the shift action of σ to that of σ^{-1}.

We will extend this result to K_1, for which we will use that $K_1(L(E)_0)$ is generated by units. Since this is true for any ultramatricial algebra, as noted in Proposition 7.8, we can in fact extend this to a statement on any strongly graded corner skew Laurent polynomial ring whose degree zero subring is ultramatricial. Namely, we prove the following.

Theorem 7.15. *Let (R, t_+, t_-) be a strongly graded, corner skew Laurent polynomial ring. Consider $\alpha\colon R_0 \to R_0$ the homomorphism given by $x \mapsto t_+ x t_-$ and put $p =: \alpha(1)$. For any $x \in R^\times$, write (R_0, x) for the class in K_1 given by the (left) module R_0 together with right multiplication by x. We have the following equality on $K_1(R_0)$:*

$$[(R_1, x)] = [(R_0, 1 + p - \alpha(x)].$$

Proof. Observe that $R_1 = R_0 t_+$ and that right multiplication by t_- yields an isomorphism $R_1 \xrightarrow{\sim} R_0 p$ whose inverse in right multiplication by t_+. It follows that $[(R_1, x)] = [(R_0 p, \alpha(x))]$. Now, the following diagram with exact rows,

$$\begin{tikzcd}
R_0(p-1) \arrow[r, hook] \arrow[d, equal] & R_0 \arrow[r, "-\cdot p"] \arrow[d, "-\cdot(1-p+\alpha(x))"] & R_0 p \arrow[d, "\alpha(x)"] \\
R_0(p-1) \arrow[r, hook] & R_0 \arrow[r, "-\cdot p"] & R_0 p,
\end{tikzcd}$$

says that

$$[(R_0, 1 + p - \alpha(x))] = [(R_0(p-1), 1)] + [(R_0 p, \alpha(x))] = [(R_0 p, \alpha(x))]. \qquad \blacksquare$$

We remark that the following corollary applies to Leavitt path algebras of essential graphs, which is our main interest for this result.

Corollary 7.16. *Let (R, t_+, t_-) be a strongly graded, corner skew Laurent polynomial ring. Assume that ℓ is a field and R_0 is a unital ultramatricial algebra. Writing $\alpha\colon R_0 \to R_0$ for the homomorphism given by $x \mapsto t_+ x t_-$ and* Dade*: $K_1(R_0) \to K_1^{\mathrm{gr}}(R)$ for the isomorphism of (2.1), the following diagram is commutative:*

$$\begin{tikzcd}
K_1^{\mathrm{gr}}(R) \arrow[r, "\sigma"] & K_1^{\mathrm{gr}}(R) \\
K_1(R_0) \arrow[u, "\mathsf{Dade}"] \arrow[r, "K_1(\alpha)"] & K_1(R_0). \arrow[u, "\mathsf{Dade}"']
\end{tikzcd}$$

Proof. By Proposition 7.8 applied to R_0, an element in $K_1(R_0)$ can be represented by the class of the free module R_0 together with the automorphism $\rho_u\colon R_0 \to R_0$ of right multiplying by a unit u; we denote this by (R_0, u).

The map Dade is induced by tensoring by $R \otimes_{R_0} -$. Applying it to (R_0, u) gives $(R \otimes_{R_0} R_0, u)$, which is equal in $K_1^{\mathrm{gr}}(R)$ to the class of (R, u), via the canonical (graded) isomorphism $R \otimes_{R_0} R_0 \cong R$. The shift functor maps (R, u) to $(R[+1], u)$. Finally, the inverse of the map Dade takes the class of the latter to its 0-th component, resulting in $[(R_1, u)]$. This is exactly the action induced on units by α, as shown in Theorem 7.15. ∎

Remark 7.17. Let (R, t_+, t_-) be a corner skew Laurent polynomial ring, and let $f\colon R \to S$ be a unital graded algebra homomorphism. If we put $s_- = f(t_-)$ and $s_+ = f(t_-)$, then (S, s_+, s_-) is a corner skew Laurent polynomial ring. Further, if R is strongly graded, then so is S (see Proposition 1.1.15 (4) in [21]).

7.3. Surjectivity of the map (7.3)

We are now in position to prove that the map (7.3) is surjective. To do this, we first define a certain modification of a graded algebra map $L(E) \to R$ by an element of $K_1(R_0)$, cf. Equation (5.10) in [12] and Equation (13.6) in [15].

Definition 7.18. Let E be a primitive graph. Given $\phi\colon L(E) \to R$ a graded algebra homomorphism, we define the following group epimorphism:

$$\begin{aligned} U : (R_\phi)^\times_{\mathrm{ab}} &\xrightarrow{(7.4)} K_1(S_0)^{E^1} \xrightarrow{s_*} K_1(S_0)^{E^0} \to \mathrm{BF}^\vee_{\mathrm{gr}}(E) \otimes_{\mathbb{Z}[\sigma]} KH_1^{\mathrm{gr}}(S) \\ z &\longmapsto \prod_{e \in E^1} s(e) \otimes (1 - \phi(ee^*) + z_e). \end{aligned} \tag{7.5}$$

Given $z = (z_e)_{e \in E^1} \in (R_\phi)^\times$, we associate to it a graded unital map $\phi_z\colon L(E) \to R$ defined by

$$\begin{aligned} &\phi_z\colon L(E) \to L(F), \quad \phi(e) = z_e \phi(e), \\ &\phi_z(e^*) = \phi(e^*) z_e^{-1}, \quad \phi_u(v) = \phi(v) \ (v \in E^0, e \in E^1). \end{aligned} \tag{7.6}$$

Lemma 7.19. *If $\xi \in kk^{\mathrm{gr}}(L(E), L(F))_1$, then there exist a graded unital algebra map $\phi\colon L(E) \to L(F)$ and $x \in \mathrm{BF}^\vee_{\mathrm{gr}}(E) \otimes_{\mathbb{Z}[\sigma]} KH_1^{\mathrm{gr}}(L(F))$ such that*

$$\bar{j}(\phi) + d(x) = \xi.$$

Proof. By Theorem 6.1 in [8] (see also Theorem 3.2 in [25]), there exists a unital graded algebra map $\phi\colon L(E) \to L(F)$ such that $K_0^{\mathrm{gr}}(\phi) = \mathrm{ev}(\xi)$. Since $K_0^{\mathrm{gr}}(\phi) = \mathrm{ev}\,\bar{j}(\phi)$, this says that $\bar{j}(\phi) - \xi \in \ker(\mathrm{ev}) = \operatorname{im} d$, from which the lemma now follows. ∎

Lemma 7.20. *Let E be a primitive graph and let R be a graded algebra such that R_0 is a unital ultramatricial algebra. If $\phi\colon L(E) \to R$ is a unital graded homomorphism, then for each $z = (z_e)_{e \in E^1} \in R_\phi^\times$, we have*

$$d(U([z])) + j(\phi) = j(\phi_z).$$

Proof. Employing the notation of (6.5), one checks that $u_{\phi_z} = u_\phi \cdot U(z)$. Part (iv) of Theorem 7.1 and Corollary (6.2) imply that $j(\phi_z) = j(\phi) + d(U([z]))$. ∎

Corollary 7.21. *The map* (7.3) *is surjective.*

Proof. Let $\xi \in kk^{\mathrm{gr}}(L(E), L(F))_1$. By Lemma 7.19, there exist $\phi\colon L(E) \to L(F)$ a graded unital algebra homomorphism and $x \in \mathrm{BF}^{\vee}_{\mathrm{gr}}(E) \otimes_{\mathbb{Z}[\sigma]} KH_1^{\mathrm{gr}}(L(F))$ such that $\overline{j}(\phi) + d(x) = \xi$. Since (7.5) is surjective, there exists $z \in L(F)_\phi$ such that $x = [U(z)]$, and thus $\xi = \overline{j}(\phi_z)$ by Lemma 7.20. ∎

7.4. Injectivity of the map (7.3)

Next we analyze the injectivity of $\overline{j}$. We now consider the set $[L(E), L(F)]_{1,M_2}$ of graded unital maps $L(E) \to L(F)$ up to graded M_2-homotopy. As we shall presently see, restricted to this quotient of $[L(E), L(F)]_1$, the map $\overline{j}$ becomes injective.

Lemma 7.22. *With notation as in Lemma 7.20, if $v, u \in R_\phi^\times$ are such that $[v] = [u]$ in $(R_\phi)^\times_{\mathrm{ab}}$, then $\phi_v \sim \phi_u$.*

Proof. By Proposition 7.10, there exist

$$(Z_e)_{e\in E^1} \in \prod_{e\in E^1} (\phi(ee^*)R_0\phi(ee^*))[t])^\times = \prod_{e\in E^1} (\phi(ee^*)(S[t])_0\phi(ee^*)))^\times$$

such that $Z_e(0) = u_e$ and $Z_e(1) = v_e$. If we compose ϕ with the inclusion $i\colon S \to S[t]$ and then consider $h := (i \circ \phi)_Z$, it follows that $ev_i \circ h = \phi_{Z(i)}$; this concludes the proof. ∎

Lemma 7.23. *Let E be a primitive graph and $\phi\colon L(E) \to R$ a graded unital homomorphism. Assume that R_0 is ultramatricial. Given $e, f \in E^1$ such that $r(f) = s(e)$ and $u \in \phi(ee^*)R_0\phi(ee^*)^\times$, we have that*

$$\sigma \cdot [1 - \phi(ee^*) + u] = [1 - \phi(fe(fe)^*) + \phi(f)u\phi(f^*)]$$

in $K_1(R_0)$. In particular, for any other $g \in E^1$ such that $r(g) = s(e)$, we have

$$[1 - \phi(fe(fe)^*) + \phi(f)u\phi(f^*)] = [1 - \phi(ge(ge)^*) + \phi(g)u\phi(g^*)].$$

Proof. For each vertex $v \in E^0 \setminus \{s(e)\}$, let f_v be an edge with range v; their existence is guaranteed by the essentiality hypothesis on E. Set $f_{s(e)} := f$. As pointed out in [6], p. 203, the elements $t_+ = \sum_{v\in E^0} f_v$ and $t_- = t_+^*$ satisfy $t_- t_+ = 1$, yielding a corner skew Laurent polynomial structure on $L(E)$. Further, this gives such a structure on R via the elements $\phi(t_+)$ and $\phi(t_-)$.

As per Theorem 7.15, the action of σ on $K_1(R_0)$ can be described as the one induced by the automorphism

$$\alpha : R_0 \to R_0, \quad x \mapsto \phi(t_+)x\phi(t_-).$$

Hence

$$\begin{aligned}\sigma \cdot [1 - \phi(ee^*) + u] &= [1 - \phi(t_+t_-) + \phi(t_+)(1 - \phi(ee^*) + u)\phi(t_-)]\\ &= [1 - \phi(t_+)(\phi(ee^*) - u)\phi(t_-)].\end{aligned}$$

Since $\phi(ee^*), u \in \phi(ee^*)R_0\phi(ee^*)$, it follows that $f_v e = 0$ unless $v = s(e)$, that is, unless $f_v = f$. Thus

$$[1 - \phi(t_+)(\phi(ee^*) - u)\phi(t_-)] = [1 - \phi(fe(fe)^*) + \phi(f)u\phi(f^*)],$$

concluding the proof. ∎

Lemma 7.24. *Let E be a primitive graph and let R be a graded algebra such that R_0 is an ultramatricial algebra. Let $\phi\colon L(E) \to R$ be a graded algebra map. If the element $z \in \prod_{e\in E^1}(\phi(ee^*)R_0\phi(ee^*))^\times$ is such that $d(U([z])) = 0$, then $\phi \sim_{\mathrm{ad}} \phi_z$.*

Proof. Consider the homomorphism

$$\lambda\colon R_\phi \to R_\phi, \quad a \mapsto \sum_{e\in E^1} \phi(e)a\phi(e^*).$$

As in the ungraded case, writing $B_{e,f} = \delta_{r(e),s(f)}$ and using Lemma 7.23, one checks that the following square is commutative:

$$\begin{tikzcd}
K_1(R_\phi) \arrow[r, "\lambda"] \arrow[d, "\wr"] & K_1(R_\phi) \arrow[d, "\wr"] \\
K_1(R)^{E^1} \arrow[r, "\sigma B"] & K_1(R)^{E^1}.
\end{tikzcd}$$

Writing E_s for the graph with adjacency matrix B, we get a commutative diagram

$$\begin{tikzcd}
K_1(R_\phi) \arrow[r, "1-\lambda"] \arrow[d, "\wr"] & K_1(R_\phi) \arrow[d, "\wr"] & & \\
K_1(R)^{E^1} \arrow[r, "I-\sigma B"] \arrow[d, "s_*"] & K_1(R)^{E^1} \arrow[r] \arrow[d, "s_*"] & \mathrm{BF}^\vee_{\mathrm{gr}}(E_s) \otimes_{\mathbb{Z}} KH_1^{\mathrm{gr}}(R) \arrow[d] & \\
K_1(R)^{E^0} \arrow[r, "I-\sigma A_E"] & K_1(R)^{E^0} \arrow[r] & \mathrm{BF}^\vee_{\mathrm{gr}}(E) \otimes_{\mathbb{Z}} KH_1^{\mathrm{gr}}(R) \arrow[r] & kk^{\mathrm{gr}}(L(E), R).
\end{tikzcd}$$

The rightmost horizontal map is injective by Theorem 7.1. Further, the map induced by s_* at the level of dual Bowen–Franks modules is an isomorphism; its inverse is induced by r^*.

Now, since $\partial(U([z])) = 0$, there exists $[\nu] \in K_1(S_\phi)$ such that $[\nu\lambda(\nu)^{-1}] = [z]$. By Lemma 7.22, this says that $\phi_z \sim \phi_{\nu\lambda(\nu)^{-1}}$. Finally, like in the ungraded case, one checks that $\phi_{\nu\lambda(\nu)^{-1}} = \mathrm{ad}(v) \circ \phi$. ∎

Theorem 7.25. *Assume that ℓ is a field. Let E and F be two primitive graphs. Given two unital graded homomorphisms $f, g\colon L(E) \to L(F)$, the following statements are equivalent*:

(i) $j(f) = j(g)$;

(ii) $f \sim_{\mathrm{ad}} g$;

(iii) $f \sim_{M_2} g$.

Proof. The fact that (ii) implies (iii) is the content of Lemma 2.9. Since j is matricially stable and graded homotopy invariant, it follows that (iii) implies (i); hence, it remains to show that (i) implies (ii). Suppose that $j(f) = j(g)$. In particular, $K_0^{\mathrm{gr}}(f) = \mathrm{ev} \circ j(f)$ agrees with $K_0^{\mathrm{gr}}(g)$ and therefore, by Corollary 3.5 in [8], there exists a homogeneous unit of degree zero $u \in R$ such that $\mathrm{ad}(u) \circ f$ and g agree on $D(E)_1 = \mathrm{span}_\ell\{ee^* : e \in E^1\} = \mathrm{span}_\ell\{ee^*, v : e \in E^1, v \in E^0\}$. Since $\mathrm{ad}(u) \circ f \sim_{\mathrm{ad}} f$, we may without loss of generality assume that f and g agree on $D_1(E)$.

Now, if we put $z_e = g(e)f(e^*)$, for each $e \in E^1$, these are units in each corner $f(ee^*)L(F)_0 f(ee^*)$ with inverse $f(e)g(e^*)$. Using the notation of Lemma 7.20, it follows that $g = f_z$ and $d(U([z])) = j(f_z) - j(f) = j(g) - j(f) = 0$. Thus, we can apply Lemma 7.24 to obtain that $f \sim_{\mathrm{ad}} g$. ∎

Corollary 7.26. *Assume that ℓ is a field. If E and F are two primitive graphs, then the map $\bar{j}\colon [L(E), L(F)]_{1,M_2} \to kk^{\mathrm{gr}}(L(E), L(F))_1$ is bijective.*

Proof. Surjectivity was proven in Corollary 7.21, while injectivity follows from Theorem 7.25. ∎

8. Graded homotopy classification

We conclude with a graded homotopy classification theorem. To simplify its statement, we shall say that two algebras are *unitally graded homotopy equivalent* if there exists a unital graded homotopy equivalence between them whose graded homotopy inverse is also a unital homorphism.

Theorem 8.1. *Let ℓ be a field and E and F two primitive graphs. The following statements are equivalent*:

(i) *The pointed, preordered $\mathbb{Z}[\sigma]$-modules*

$$(\mathrm{BF}_{\mathrm{gr}}(E), \mathrm{BF}_{\mathrm{gr}}(E)_+, 1_E) \quad \text{and} \quad (\mathrm{BF}_{\mathrm{gr}}(F), \mathrm{BF}_{\mathrm{gr}}(E)_+, 1_F)$$

are isomorphic.

(ii) *There exists an isomorphism $\xi \in kk^{\mathrm{gr}}(L(E), L(F))_1$.*

(iii) *The algebras $L(E)$ and $L(F)$ are unitally graded homotopy equivalent.*

Proof. In Theorem 13.1 of [9] it was proved that, in the surjection of Theorem 7.1, one can lift isomorphisms at the level of K_0^{gr} to kk^{gr}-isomorphisms. Since by definition a lifting of a pointed preordered module map lies in $kk^{\mathrm{gr}}(L(E), L(F))_1$; this proves the implication (i) $\Rightarrow$ (ii).

Next assume (ii) and consider the inverse of ξ, noting that $\xi^{-1} \in kk^{\mathrm{gr}}(L(F), L(E))_1$. By Corollary 7.26, there exist unital algebra homomorphisms $f\colon L(E) \to L(F)$ and $g\colon L(F) \to L(E)$ such that $j(f) = \xi$ and $j(g) = \xi^{-1}$. In particular, $j(fg) = \mathrm{id}_{L(F)}$ and $j(gf) = \mathrm{id}_{L(E)}$, and thus Theorem 7.25 says that there exist units $u \in L(F)_0$, $v \in L(E)_0$ such that $fg \sim \mathrm{ad}(u)$ and $gf \sim \mathrm{ad}(v)$. This readily implies that f is a graded homotopy equivalence. We have thus proved that (ii) implies (iii).

Finally, we prove that (iii) implies (i). Let $f: L(E) \to L(F)$ be a unital graded homotopy equivalence. The map $K_0^{\mathrm{gr}}(f)$ is a pointed, preordered module map between the Bowen–Franks modules of E and F, it suffices to see that it is an isomorphism. To prove this, we note that K_0^{gr} agrees with KH_0^{gr} for $L(E)$ and $L(F)$ as per Remark 2.12 and that KH_0^{gr} maps graded homotopy equivalences to isomorphisms. This concludes the proof. ∎

By Theorem 8.1, the primitive case of Conjecture 1.1 is equivalent to the following.

Conjecture 8.2. *Let ℓ be a field. If E and F are primitive graphs, then $L_\ell(E)$ and $L_\ell(F)$ are graded isomorphic if and only if they are unitally graded homotopy equivalent.*

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