Rev. Mat. Iberoam. (Online first)
DOI 10.4171/RMI/1538

Published by EMS Press

Chaos for foliated spaces and pseudogroups

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Abstract. We generalize "sensitivity to initial conditions" to foliated spaces and pseudogroups, offering a definition of Devaney chaos in this setting. In contrast to the case of group actions, where sensitivity follows from the other two conditions of Devaney chaos, we show that this is true only for compact foliated spaces, exhibiting a counterexample in the non-compact case. Finally, we obtain an analogue of the Auslander–Yorke dichotomy for compact foliated spaces and compactly generated pseudogroups.

1. Introduction

There are several definitions of chaos for dynamical systems (Li–Yorke chaos, positive entropy, etc.), but in this article we will consider only Devaney's, first introduced in [13].

Definition 1.1 (Devaney chaos). A continuous map $f: X \to X$ on a metric space (X, d) is *chaotic* if

(i) for all non-empty open $U, V \subset X$, there is $n \geq 0$ such that
$$f^n(U) \cap V \neq \emptyset$$
(f is *topologically transitive*),

(ii) the set of periodic points is dense in X (f has *density of periodic points*), and

(iii) there is $c > 0$ such that, for every $x \in X$ and $r > 0$, there are $y \in B(x, r)$ and $n \geq 0$ satisfying
$$d(f^n(x), f^n(y)) \geq c$$
(f is *sensitive to initial conditions*).

This definition can be readily adapted for group actions $G \curvearrowright X$ by substituting $g \in G$ in place of f^n ($n \in \mathbb{N}$) above. (We will adhere to the convention where $0 \in \mathbb{N}$.) Topological transitivity conveys the indecomposability of the dynamical system, whereas (ii), according to Devaney himself, provides "an element of regularity" ([13], p. 50). Sensitivity to initial conditions, for its part, expresses what is commonly known as the "butterfly

Mathematics Subject Classification 2020: 37B02 (primary); 37B05 (secondary).
Keywords: chaos, sensitivity, foliation, foliated space, Devaney chaos, equicontinuity, pseudogroup, periodic orbits, lamination.

effect”. This rough sketch may lead to the impression that (iii) alone imbues this definition with its chaotic nature; surprisingly, it was proved later that this condition is, in fact, redundant.

Theorem 1.2 ([7]). *If a continuous map $f : X \to X$ on a metric space (X, d) satisfies* (i) *and* (ii)*, then it also satisfies* (iii).

This result was later generalized to topological group and semigroup actions [25, 31]. The reader should bear in mind that these results hold even when the phase space is not compact.

If the local behaviour of f around a point x is not sensitive to initial conditions, then there is an assignment $\varepsilon \mapsto \delta(\varepsilon)$ such that

$$d(x, y) < \delta(\varepsilon) \quad \Longrightarrow \quad d(f^n x, f^n y) < \varepsilon \quad \text{for every } y \in X,\ n \in \mathbb{N},$$

and we say that x is a *point of equicontinuity*. If the set of points of equicontinuity is dense in X, we call the dynamical system *almost equicontinuous*; if every point is of equicontinuity with the same modulus $\varepsilon \mapsto \delta(\varepsilon)$, then the system is *equicontinuous*. This rough opposition between chaos and equicontinuity is rigorously formulated by the Auslander–Yorke dichotomy.

Theorem 1.3 (Auslander–Yorke dichotomy, see Corollary 2 in [6]). *Let X be a compact space and let $f : X \to X$ be a continuous map such that (X, f) is minimal. Then f is either equicontinuous or sensitive to initial conditions.*

After this brief review, we can state the aim of the present paper: to study topological chaos for foliations and their generalization, foliated spaces; as we will see, this requires considering pseudogroups too. Recall that a foliated space is a topological generalization of a foliation where the choice of local transversal models is not restricted to manifolds: they are only required to be Polish spaces (see Section 2.5). The Smale–Williams attractor provides an example of a foliated space that is not a foliation – it is locally homeomorphic to the product of the real line and the Cantor set.

Thus, our work fits into the broader field of topological dynamics for foliated spaces, which has received much attention of late. The most studied foliation dynamics are the equicontinuous, featuring the celebrated tools of the Molino theory for *Riemannian foliations* [28]. Equicontinuity was generalized to foliated spaces and pseudogroups in [3, 24, 28, 30, 33] with varying degrees of generality, the methods of [3] in particular being a main source of inspiration for this paper. Molino’s theory itself has also been generalized in [4, 14, 15], giving rise to the study of *wild* solenoids and Cantor actions, which have a complicated interplay between local and global behavior [2, 22, 23, 26]. There has been some recent work on complex dynamics, concerning Fatou–Julia decompositions for holomorphic foliations [5, 17].

In order to analyze foliated spaces from a dynamical point of view, we regard them as generalized dynamical systems where the leaves play the role of the orbits; as the title of [11] reads, they are dynamical systems “in the absence of time.” We may identify the paper on topological entropy for foliations by Ghys, Langevin, and Walczak [18] as the first study on chaotic foliations, even though the word “chaos” is never mentioned. In fact, it looks like the term “chaotic foliation” has only appeared twice in the literature; its debut was in the context of general relativity, where Churchill was trying to provide a definition of chaos invariant by relativistic reparametrizations of time:

Definition 1.4 (Churchill chaos, [11]). A foliation is chaotic if

(i) there is a dense leaf,

(ii) the set of compact leaves is dense, and

(iii) there are at least two different leaves.

Items (i) and (ii) correspond to items (i)–(ii) in Definition 1.1, whereas (iii) avoids the trivial scenario where the foliation consists of a single leaf. Churchill did not include a foliation-theoretic definition of sensitivity: only foliations by curves arising from a flow were considered.

More recently, Bazaikin, Galaev, and Zhukova have provided the following definition of chaos for foliations:

Definition 1.5 (Bazaikin–Galaev–Zhukova chaos, [9]). A foliation is chaotic if

(i) there is a dense leaf, and

(ii) the set of closed leaves is dense.

They have used this definition to study chaos for Cartan foliations, relating it to conditions in their holonomy pseudogroups and global holonomy groups. Of course, their definition coincides with Churchill's when the ambient manifold is compact and there are at least two different leaves. Again, it does not take into account sensitivity to initial conditions.

Regarding examples of chaotic foliated spaces (according to the definitions above), besides those appearing in [9, 11], the author was involved in the recent study of a hyperbolic version of the cut-and-project method of tiling theory [1]. This yields Delone subsets of $\mathbb{R}$ whose continuous hulls, which are naturally foliated spaces, are chaotic with respect to the natural action by translations.

This discussion motivates the first contribution of the present paper: we introduce a suitable definition of sensitivity to initial conditions for foliated spaces. We phrase this definition in terms of *holonomy pseudogroups*, which have long been used as dynamical models for foliations. A pseudogroup in a topological space X is a collection of homeomorphisms between open subsets of X containing the identity and closed under composition, inversion, restriction, and combination of partial maps (see Section 2.2). H. Nozawa and the author [8] have developed a slightly different dynamical model in order to define sensitivity and Devaney chaos for closed saturated subsets of the Gromov space of pointed colored graphs. These subsets resemble singular foliations by graphs and do not admit a holonomy pseudogroup in the usual sense. On a very related note, Flores and Măntoiu [16] have recently studied the topological dynamics of groupoid actions.

After some preliminary results in Section 2, we discuss our definition of sensitivity for pseudogroups in Section 3.1: showing first why a naive approach fails, we follow the ideas present in [3] in order to arrive at Definition 3.9. We also provide definitions for almost equicontinuity and density of periodic orbits, making use of the latter to define Devaney chaos as follows:

Definition 1.6 (Devaney chaos for pseudogroups). A pseudogroup $\mathcal{G} \curvearrowright X$ is *chaotic* if it is topologically transitive, has density of periodic orbits, and is sensitive to initial conditions.

The next results test whether this new definition of sensitivity constitutes a satisfactory generalization of the original one. We start by examining pseudogroups generated by group actions.

Theorem 1.7. *If G is a finitely generated group acting on a compact Polish space X, then the action is sensitive to initial conditions if and only if the pseudogroup generated by the action is.*

We also show in Section 3.4 that the conditions on G and X are necessary for the result to hold. So, in general, sensitivity of the pseudogroup induced by a group action is strictly stronger than sensitivity of the group action itself.

Theorem 1.8. *There are group actions $G \curvearrowright X$ that are sensitive to initial conditions but such that the pseudogroup generated by the action is not, where either*

- *G is the free group on two generators and $X = \mathbb{T}^2 \times \mathbb{Z}$, where $\mathbb{T}^2$ is the 2-torus, or*
- *G is the free group on countably many generators and $X = \mathbb{T}^2$.*

These actions are constructed using *linked twists*, a family of classical examples of chaotic dynamical systems (see Section 2.6 and the references therein). We will later recycle these counterexamples in the proof of Theorem 1.16.

We continue in Section 3.5 with our three main contributions regarding pseudogroup dynamics. Our first result addresses the following issue: if we are to use pseudogroups as dynamical models for foliated spaces, all our new definitions must be invariant by (Haefliger) equivalences. This is because the holonomy pseudogroup of a foliated space is only well defined up to equivalence (see Sections 2.3 and 2.5).

Theorem 1.9. *Sensitivity to initial conditions, density of periodic orbits, Devaney chaos, and almost equicontinuity are invariant by equivalences of pseudogroups acting on locally compact Polish spaces.*

The corresponding result for equicontinuity was proved in [3]. The main difficulty in Theorem 1.9 is proving the invariance of sensitivity. The reason why this is not trivial is that sensitivity and almost equicontinuity involve a metric, which is a global object; equivalences, however, are made up of local homeomorphisms, so we have to put in some work to construct a global metric using the information carried over by the local maps.

Our next objective will be to study whether Theorems 1.2 and 1.3 extend to the pseudogroup setting. We manage to do so for compactly generated pseudogroups (see Section 2.4 for the definition of compact generation).

Theorem 1.10 (Auslander–Yorke dichotomy for pseudogroups). *Let $\mathcal{G}$ be a compactly generated and topologically transitive pseudogroup acting on a Polish space. Then $\mathcal{G}$ is either sensitive to initial conditions or almost equicontinuous. Moreover, if $\mathcal{G}$ is minimal, then it is either sensitive to initial conditions or equicontinuous.*

Theorem 1.11. *If $\mathcal{G}$ is a compactly generated and topologically transitive pseudogroup acting on a Polish space which has density of periodic orbits, then it is sensitive to initial conditions.*

Even though Theorem 1.2 holds for actions on non-compact spaces, we exhibit in Section 3.6 a non-compactly generated, countably generated pseudogroup that is topologically transitive and has density of periodic orbits, but it is not sensitive to initial conditions. This shows that compact generation cannot be dropped in Theorem 1.11.

At this point, we turn our attention to studying chaos for foliated spaces. By virtue of Theorem 1.9, we can define almost equicontinuity and sensitivity using the holonomy pseudogroup.

Definition 1.12. A foliated space is sensitive to initial conditions or almost equicontinuous if its holonomy pseudogroup is.

Regarding density of periodic orbits and Devaney chaos, we encounter an additional subtlety: it is easy to check that density of periodic orbits for the holonomy pseudogroup implies density of closed leaves, but one might also consider the stronger condition of density of compact leaves. We choose the latter option because the counterexample we exhibit in Theorem 1.16 satisfies this stronger condition. On the other hand, we run into the problem that density of compact leaves cannot be formulated as a equivalence-invariant property of the holonomy pseudogroup (see Example 2.11).

Definition 1.13. A foliated space is *chaotic* if it is topologically transitive, it has a dense set of compact leaves, and it is sensitive to initial conditions.

Note that, by the previous discussion, chaoticity of the foliated space is strictly stronger than chaoticity of the holonomy pseudogroup. We show in Section 4.1 an explicit example of this behavior.

Our next step is to extend Theorems 1.2 and 1.3 for compact foliated spaces, where density of compact leaves and of closed leaves coincide. By the previous discussion, these results follows immediately from Theorems 1.11 and 1.10.

Theorem 1.14. *Let X be a compact Polish foliated space. If X is topologically transitive and has density of compact leaves, then it is sensitive to initial conditions.*

Theorem 1.15. *Let X be a compact and topologically transitive Polish foliated space. Then X is either sensitive to initial conditions or almost equicontinuous. Moreover, if X is minimal, then it is either sensitive to initial conditions or equicontinuous.*

In analogy to the case of pseudogroups, where we need compact generation in Theorem 1.11, compactness cannot be dropped in Theorem 1.14; a simple counterexample with totally disconnected transversals is constructed in Section 4.2. One might wonder whether it is possible to find similar counterexamples among non-compact foliations, perhaps with smooth transversal dynamics. We conclude the paper in Sections 4.3 and 4.4 with the following counterexample.

Theorem 1.16. *There is a foliation by surfaces on a smooth 4-manifold that is topologically transitive and has a dense set of compact leaves, but is not sensitive to initial conditions. This foliation is C^∞ and transversally affine.*

We can offer the following geometrical interpretation of the lack of sensitivity in Theorem 1.16. For this non-compact, smooth 4-manifold M, there is a locally finite foliated atlas (U_i, ϕ_i) (where $\phi_i \colon U_i \to \mathbb{R}^2 \times T_i$ and $T_i \subset \mathbb{R}^2$ are the local transversals) satisfying the following condition: every holonomy transformation is an affine map, and there is a leaf L such that every holonomy transformation h between transversals $T_i \to T_j$ induced by a path in L is an isometry with respect to the Euclidean metric on $T_i, T_j \subset \mathbb{R}^2$.

The results of this paper confirm that our definition of chaos is the right one if we restrict our attention to compactly generated pseudogroups and compact foliated spaces. However, as soon as we drop compactness, it seems to become a strong condition, at least when compared to the case of group actions. Even though it is invariant by equivalences and it is phrased in a way that mirrors other pseudogroup dynamical properties in the literature, it is an open question whether one can find a definition better suited to the non-compact case. The examples in Sections 3.6 and 4 suggest that perhaps one should require only a meager set of equicontinuity points. The author ignores whether this condition could extend the notion of sensitivity to actions of countably generated pseudogroups satisfactorily.

2. Preliminaries

2.1. Metric spaces

In this paper, we consider metric functions $d \colon X \times X \to [0, \infty]$ that may attain an infinite value. A metric on a topological space is said to be *compatible* if the underlying topology agrees with that generated by the open balls. A topological space is *Polish* if it is separable and it admits a compatible complete metric; that is, one where every Cauchy sequence converges. All topological spaces will be implicitly assumed to be Polish.

A *shrinking* of an indexed open covering $\{U_\alpha\}_{\alpha \in A}$ of some topological space is a covering $\{V_\alpha\}_{\alpha \in A}$ with the same index set and such that $\overline{V_\alpha} \subset U_\alpha$ for every $\alpha \in A$. A covering $\{U_\alpha\}_{\alpha \in A}$ of X is *locally finite* if every point $x \in X$ has an open neighborhood W that intersects only finitely many of the sets U_α. We will make use of the following result (see, e.g., p. 227 of [29]):

Lemma 2.1 (Shrinking lemma). *Let X be a Polish space. If $\{U_n\}_{n \in \mathbb{N}}$ is a locally finite countable open covering of X, then $\{U_n\}_{n \in \mathbb{N}}$ admits a shrinking.*

2.2. Partial maps and pseudogroups

Let X and Y be topological spaces. A *partial map* from X to Y is a map $f \colon A \to Y$ with domain a subset $A \subset X$. Given a partial map f, let $\operatorname{dom} f$ and $\operatorname{im} f$ denote the domain and image of f, respectively. We say that a partial map f from X to Y is a *partial homeomorphism* if $\operatorname{dom} f \subset X$ and $\operatorname{im} f \subset Y$ are open and $f \colon \operatorname{dom} f \to \operatorname{im} f$ is a homeomorphism; we denote by $\operatorname{Ph}(X, Y)$ the set of partial homeomorphisms from X to Y. From now on, we use $f(A)$ as shorthand for $f(A \cap \operatorname{dom} f)$, where $f \in \operatorname{Ph}(X, Y)$ and $A \subset X$.

Given $f \in \operatorname{Ph}(X, Y)$ and $g \in \operatorname{Ph}(Y, Z)$, the *composition* $gf \in \operatorname{Ph}(X, Z)$ is defined by

$$\operatorname{dom} gf = f^{-1}(\operatorname{dom} g), \quad (gf)(x) = g(f(x)).$$

Given $f \in \mathrm{Ph}(X, Y)$ and an open set $U \subset \operatorname{dom} f$, the *restriction* $f|_U$ has domain U and image $f(U)$.

Let $\{f_i \mid i \in I\}$ be a family of maps in $\mathrm{Ph}(X, Y)$, and suppose that

$$(f_i)|_{\operatorname{dom} f_i \cap \operatorname{dom} f_j} = (f_j)|_{\operatorname{dom} f_i \cap \operatorname{dom} f_j} \quad \text{for every } i, j \in I. \tag{2.1}$$

Then the *combination* $\bigcup_{i \in I} f_i$ is defined by

$$\operatorname{dom}\Big(\bigcup_{i \in I} f_i\Big) = \bigcup_{i \in I} (\operatorname{dom} f_i), \quad \Big(\bigcup_{i \in I} f_i\Big)(x) = f_i(x) \quad \text{for } x \in \operatorname{dom} f_i.$$

For $f, g \in \mathrm{Ph}(X, Y)$, we say that f *extends* g, or f is an *extension* of g, if

$$\operatorname{dom} g \subset \operatorname{dom} f \quad \text{and} \quad f|_{\operatorname{dom} g} = g.$$

For brevity, we use $\mathrm{Ph}(X)$ to denote the set $\mathrm{Ph}(X, X)$.

Definition 2.2. A subset $\mathcal{G} \subset \mathrm{Ph}(X)$ is a *pseudogroup* if the following conditions are satisfied:

- Group-like axioms:
  - (i) $\mathrm{id}_X \in \mathcal{G}$,
  - (ii) if $f \in \mathcal{G}$, then $f^{-1} \in \mathcal{G}$ (*closure under inversion*), and
  - (iii) if $f, g \in \mathcal{G}$, then $fg \in \mathcal{G}$ (*closure under composition*).
- Sheaf-like axioms:
  - (iv) if $f \in \mathcal{G}$ and $U \subset \operatorname{dom} f$ is open, then $f|_U \in \mathcal{G}$ (*closure under restrictions*), and,
  - (v) if $\{f_i, i \in I\}$ is a family of maps in $\mathcal{G}$ satisfying (2.1), then $\bigcup_{i \in I} f_i \in \mathcal{G}$. (*closure under combinations*).

  The last axiom can be reformulated as follows:
  - (v)′ if $f \in \mathrm{Ph}(X)$ is such that every $x \in \operatorname{dom} f$ has some open neighborhood U_x with $f|_{U_x} \in \mathcal{G}$, then $f \in \mathcal{G}$.

If $\mathcal{G} \subset \mathrm{Ph}(X)$ is a pseudogroup, we say that $\mathcal{G}$ *acts* on X and we denote it by $\mathcal{G} \curvearrowright X$. The $\mathcal{G}$-orbit of a point $x \in X$ is the subset $\mathcal{G}x = \{gx \mid g \in \mathcal{G}\}$. If $\{\mathcal{G}_i\}_{i \in I}$ is a collection of pseudogroups acting on X, then $\bigcap_{i \in I} \mathcal{G}_i \subset \mathrm{Ph}(X)$ is also a pseudogroup. A subset $S \subset \mathcal{G}$ *generates* $\mathcal{G}$ if $\mathcal{G}$ is the smallest pseudogroup containing S; equivalently, $\mathcal{G}$ is the intersection of all the pseudogroups in $\mathrm{Ph}(X)$ that contain S. Let $\mathcal{G}$ be a pseudogroup acting on X, and let U be an open subset of X. Then the *restriction*

$$\mathcal{G}|_U = \{f \in \mathcal{G} \mid \operatorname{dom} f \subset U,\ \operatorname{im} f \subset U\}$$

is a pseudogroup acting on U.

One can find in the literature definitions of pseudogroup that omit Axiom (v) (see, e.g., [21]). The reason is that, by allowing combinations, a pseudogroup might have too many maps to satisfy reasonable dynamical properties (see Definition 3.7 and Lemma 3.8); this motivates the following definition.

Definition 2.3. A *pseudo∗group* is a subset $\mathcal{S} \subset \mathrm{Ph}(X)$ satisfying Axioms (i)–(iv) in Definition 2.2.

This terminology was introduced by S. Matsumoto in [27]. For $S \subset \mathrm{Ph}(X)$, let $\langle S\rangle \subset \mathrm{Ph}(X)$ denote the set consisting of finite compositions and inversions of elements in S, and let $S^* \subset \mathrm{Ph}(X)$ denote the set of partial homeomorphisms obtained from S by composition, inversion, and restriction to open subsets; equivalently,

$$S^* = \{s|_U : s \in \langle S\rangle,\ U \subset \operatorname{dom} s \text{ open}\}$$

and S^* is the smallest pseudo∗group containing S.

Lemma 2.4. *Let $\mathcal{G} \curvearrowright X$ be a pseudogroup and let $S \subset \mathcal{G}$. Then S generates $\mathcal{G}$ if and only if, for every $g \in \mathcal{G}$ and $x \in \operatorname{dom} g$, there is an open neighborhood U of x such that $g|_U \in S^*$.*

Proof. Let $\mathcal{H} \subset \mathrm{Ph}(X)$ denote the set of maps that result from combining families of maps in S^* using Axiom 2.2 (v). In other words, $\mathcal{H}$ is the pseudogroup generated by S. Then S generates $\mathcal{G}$ if and only if $\mathcal{H} = \mathcal{G}$; i.e., every map in $\mathcal{G}$ can be obtained as the combination of a family of maps in S^*, but this follows from the hypothesis and Axiom 2.2 (v)′. ∎

Corollary 2.5. *Let S be a generating set for $\mathcal{G} \curvearrowright X$, let d be a compatible metric on X, let $g \in \mathcal{G}$, and let $K \subset \operatorname{dom} g$ be compact. Then there is $\varepsilon > 0$ such that, for every $x \in K$, the restriction of g to $B_d(x, \varepsilon)$ belongs to S^*.*

2.3. Equivalences

If we are to use pseudogroups to study foliated spaces, the right notion of isomorphism is that of *equivalence*, sometimes also referred to as *Haefliger* or *étale equivalence*.

Definition 2.6. Let $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ be pseudogroups. An *equivalence* $\Phi\colon (X, \mathcal{G}) \to (Y, \mathcal{H})$ is a collection of partial homeomorphisms $\Phi \subset \mathrm{Ph}(X, Y)$ satisfying the following conditions:

(i) $\{\operatorname{dom}\phi \mid \phi \in \Phi\}$ and $\{\operatorname{im}\phi \mid \phi \in \Phi\}$ are open coverings of X and Y, respectively.

(ii) If $\phi \in \Phi$ and U is an open subset of $\operatorname{dom}\phi$, then $\phi|_U \in \Phi$.

(iii) Let $\phi \in \mathrm{Ph}(X, Y)$. If there is an open covering $\{U_i\}_{i\in I}$ of $\operatorname{dom}\phi$ such that $\phi|_{U_i} \in \Phi$ for every $i \in I$, then $\phi \in \Phi$.

(iv) If $g \in \mathcal{G}$, $h \in \mathcal{H}$, and $\phi \in \Phi$, then $h\phi g \in \Phi$.

(v) If $\phi, \psi \in \Phi$, then $\psi^{-1}\phi \in \mathcal{G}$ and $\psi\phi^{-1} \in \mathcal{H}$.

The following properties follow immediately from the definition.

Lemma 2.7. *Let $\Phi\colon (X, \mathcal{G}) \to (Y, \mathcal{H})$ and $\Psi\colon (Y, \mathcal{H}) \to (Z, \mathcal{I})$ be equivalences. Then the* inverse

$$\Phi^{-1} := \{\phi^{-1} \mid \phi \in \Phi\} \subset \mathrm{Ph}(Y, X)$$

and the composition

$$\Psi \circ \Phi := \{\psi \circ \phi \mid \phi \in \Phi,\ \psi \in \Psi\} \subset \mathrm{Ph}(X, Z)$$

are equivalences $(Y, \mathcal{H}) \to (X, \mathcal{G})$ and $(X, \mathcal{G}) \to (Z, \mathcal{I})$, respectively.

Lemma 2.8. *Let $\mathcal{G} \curvearrowright X$ be a pseudogroup and let $U \subset X$ be an open set that meets every $\mathcal{G}$-orbit. Then*

$$\Phi := \{ g \in \mathcal{G} \mid \operatorname{dom} g \subset U \}$$

is an equivalence $\Phi\colon (U, \mathcal{G}|_U) \to (X, \mathcal{G})$; in particular, $\mathcal{G}$ is an equivalence $(X, \mathcal{G}) \to (X, \mathcal{G})$.

Proof. Item (i) in Definition 2.6 follows from the assumption that U meets every $\mathcal{G}$-orbit, whereas (ii)–(v) hold because $\mathcal{G}$ is a pseudogroup. ■

Pseudogroup equivalences are maximal families in the following sense.

Lemma 2.9. *Let Φ, Ψ be equivalences $(X, \mathcal{G}) \to (Y, \mathcal{H})$. If $\Phi \subset \Psi$, then $\Phi = \Psi$.*

Proof. Let $\psi \in \Psi$. Since $\{\operatorname{im} \phi \mid \phi \in \Phi\}$ covers Y, there is a family $\{\phi_i\}_{i \in I} \subset \Phi$ such that $\{\operatorname{im} \phi_i\}_{i \in I}$ covers $\operatorname{im} \psi$. Moreover, since $\phi_i \in \Psi$ by hypothesis, we have $\phi_i^{-1}\psi \subset \mathcal{G}$ by Definition 2.6(v), and then $\psi|_{\operatorname{im} \phi_i} = \phi_i(\phi_i^{-1}\psi)$ belongs to Φ by Definition 2.6(iv). Finally, ψ is the combination of the family $\{\psi|_{\operatorname{im} \phi_i}\}_{i \in I}$, so $\psi \in \Phi$ by Definition 2.6(iii). This shows $\Psi \subset \Phi$ because ψ was chosen arbitrarily. ■

Lemma 2.10. *Let $\mathcal{G} \curvearrowright X$ and $\mathcal{H} \curvearrowright Y$ be pseudogroups, and let $\Sigma \subset \operatorname{Ph}(X, Y)$ be a family of maps such that*

(i) $\bigcup_{\phi \in \Sigma} \operatorname{dom} \phi \subset X$ *meets every $\mathcal{G}$-orbit*;
(ii) $\bigcup_{\phi \in \Sigma} \operatorname{im} \phi \subset Y$ *meets every $\mathcal{H}$-orbit*; *and,*
(iii) *if ϕ, $\psi \in \Sigma$, $g \in \mathcal{G}$, and $h \in \mathcal{H}$, then $\psi^{-1} h \phi \in \mathcal{G}$ and $\psi g \phi^{-1} \in \mathcal{H}$.*

Then there is a unique equivalence $\Phi\colon (X, \mathcal{G}) \to (Y, \mathcal{H})$ containing Σ.

Proof. Let $\Phi \subset \operatorname{Ph}(X, Y)$ consist of the combinations of maps of the form $h\sigma g$, where $g \in \mathcal{G}$, $h \in \mathcal{H}$, and $\sigma \in \Sigma$; then Φ is an equivalence: Axiom 2.6(i) follows from (i) and (ii), Axiom 2.6(ii)–(iv) follow from the definition of Φ, and Axiom 2.6(v) follows from (iii). Finally, Lemma 2.9 yields uniqueness. ■

We will refer to the equivalence given by Lemma 2.10 as the equivalence *generated* by Σ. We say that two pseudogroups are *equivalent* if there is an equivalence from one to the other; this is a reflexive, symmetric, and transitive relation by Lemmas 2.7 and 2.8. The reader should be mindful that equivalence of pseudogroups is a very lax condition, as the next example shows.

Example 2.11 ([20], p. 277). Let $\mathcal{G}$ be the pseudogroup on $\mathbb{R}$ generated by the translation $t \mapsto t + 1$, and let $\mathcal{H}$ be the pseudogroup on $\mathbb{S}^1$ generated by the identity map. Consider the natural projection map $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$. Then

$$\Phi := \{ \pi|_U \mid U \subset \mathbb{R} \text{ open}, \ \pi|_U\colon U \to \phi(U) \text{ is a homeomorphism} \}$$

is an equivalence from $(\mathbb{R}, \mathcal{G})$ to $(\mathbb{S}^1, \mathcal{H})$.

Finally, if X and Y are C^i-manifolds[1] for some $i \in \mathbb{N} \cup \{\infty, \omega\}$, we say that a family $A \subset \operatorname{Ph}(X, Y)$ is C^i if all the maps in A are C^i in the usual sense. In this way we obtain

[1]The notation C^ω means that the manifold or map is analytic

a definition of C^i-pseudogroups and equivalences, and the notion of being a C^i-pseudogroup is then invariant by C^i-equivalences. Similarly, if X and Y are affine manifolds, we can define affine pseudogroups and equivalences, and being an affine pseudogroup is invariant by affine equivalences.

2.4. Compact generation

Definition 2.12. Let $\mathcal{G} \curvearrowright X$ be a pseudogroup. A *system of compact generation* is a triple $(U, F, \widetilde{F})$, where

(i) U is a relatively compact open set of X meeting every $\mathcal{G}$-orbit,

(ii) both $F \subset \mathcal{G}|_U$ and $\widetilde{F} \subset \mathcal{G}$ are finite and symmetric,

(iii) F generates $\mathcal{G}|_U$, and

(iv) there is a bijection $f \mapsto \tilde{f}$ ($f \in F$, $\tilde{f} \in \widetilde{F}$) where $\tilde{f}$ is an extension of f and $\overline{\operatorname{dom} f} \subset \operatorname{dom}(\tilde{f})$ for every $f \in F$.

We say that $\mathcal{G}$ is *compactly generated* if it admits a system of compact generation; note that compact generation implies that X is locally compact. This property is invariant by equivalences [19]. The main family of examples of compactly generated pseudogroups, which moreover gave birth to the definition, consists of holonomy pseudogroups of compact foliated spaces (see next section). As a simpler example, we could mention the pseudogroup generated by the action of a finitely generated group on a compact space.

Contrary to most of the references in the subject, we consider the set of extensions $\widetilde{F}$ as part of the generating set. Note that the symmetry condition of both F and $\widetilde{F}$ is included for simplicity.

From now on, for every map $f \in \langle F \rangle$, $f = f_n \cdots f_1$ with $f_i \in F$, we denote by $\tilde{f} \in \langle \widetilde{F} \rangle$ the composition $\tilde{f}_n \cdots \tilde{f}_1$. For notational convenience, we will assume from now on that $\tilde{f}^{-1} = \widetilde{f^{-1}}$. Properly speaking, the map $\tilde{f}$ depends not only on f, but on the representation $f = f_n \cdots f_1$; we will incur in this slight abuse of notation anyway because this subtlety will be of no relevance to our proofs.

Lemma 2.13. *Let $\mathcal{G}$ be a compactly generated pseudogroup, let $x \in X$, and let S be a generating set. Then there is a system of compact generation $(U, F, \widetilde{F})$ with $x \in U$ and $\widetilde{F}^* \subset S^*$.*

Proof. We begin by showing that we can choose $(U, F, \widetilde{F})$ with $x \in U$. Indeed, let $(V, H, \widetilde{H})$ any system of compact generation and let $g \in \mathcal{G}$ be any map with $x \in \operatorname{dom} g$ and $g(x) \in V$. Let W, W' be relatively compact open neighborhoods of x with $\overline{W} \subset W' \subset \operatorname{dom} g$. Then $(V \cup W, H \cup \{g|_W\}, \widetilde{H} \cup \{g|_{W'}\})$ is a system of compact generation.

Let us show now that we may take $(U, F, \widetilde{F})$ with $\widetilde{F}^* \subset S^*$. Let $(U, H, \widetilde{H})$ satisfy $x \in U$. Write $H = \{f_i\}_{i \in I}$; then, for every $i \in I$, there is a finite open cover $\{V_{i,j}\}_{j \in J_i}$ of $\overline{\operatorname{dom} f_i}$ and a shrinking $\{W_{i,j}\}_{j \in J_i}$ such that $\tilde{f}|_{V_{i,j}} \in S^*$ for every $j \in J_i$ by Corollary 2.5. Then

$$(U, F, \widetilde{F}) := (U, \{\tilde{f}_i|_{W_{i,j} \cap U}\}, \{\tilde{f}_i|_{V_{i,j}}\})$$

is a system of compact generation satisfying the desired conditions. ∎

2.5. Foliated spaces

Let X be a Polish space and let $\mathcal{F}$ be a partition of X. Then $(X, \mathcal{F})$ is a *foliated space of leafwise class* C^k ($k \in \mathbb{N} \cup \{\infty\}$) and dimension $n \in \mathbb{N}$ if X admits an atlas of charts (U_i, ϕ_i), where $\{U_i\}$ is an open covering of X and the maps ϕ_i are homeomorphisms $\phi_i : U_i \to \mathbb{R}^n \times Z_i$ (for Z_i Polish), and with coordinate changes of the form

$$\phi_i \phi_j^{-1}(x_j, z_j) = (x_i(x_j, z_j), z_i(z_j)),$$

where $z_i : \phi_j(U_i \cap U_j) \to Z_i$ is continuous and $x_i : \phi_j(U_i \cap U_j) \to \mathbb{R}^n$ is of class C^k on every plaque. Remember that the *plaques* of the chart (U_i, ϕ_i) are the sets $\phi_i^{-1}(\mathbb{R}^n \times \{z_i\})$. Moreover, we require that the equivalence relation induced by $\mathcal{F}$ coincides with the transitive closure of the relation "being in the same plaque"; it follows that $\mathcal{F}$ partitions X into subsets which, when endowed with an appropriate topology called the *leaf topology*, become connected C^k-manifolds of dimension n: the *leaves* of the foliated space. If $(X, \mathcal{F})$ is a foliation of dimension n, codimension m, and class $C^{k,l}$ (see p. 32 of [10]), then it is an n-dimensional foliated space of leafwise class C^k, the transversal models Z_i are C^l-manifolds of dimension m, and the maps z_i are of class C^l.

The *holonomy pseudogroup* serves as a dynamical model for the foliated space X: let $\{(U_i, \phi_i) \mid i \in I\}$ be a locally finite atlas, let p_i denote the composition of ϕ_i with the projection $\mathbb{R}^n \times Z_i \to Z_i$, and let $Z = \coprod_{i \in I} Z_i$. The transversal components of the change of coordinate maps

$$h_{i,j} : p_j(U_i \cap U_j) \to p_i(U_i \cap U_j), \quad h_{i,j}(z_j) = z_i(z_j),$$

generate a pseudogroup in Z, called the *holonomy pseudogroup* of X. Note that we are only considering holonomy pseudogroups induced by locally finite atlases so that the transversal space Z is Polish and the pseudogroup is countably generated.

The holonomy pseudogroup depends on the choice of atlas $\{(U_i, \phi_i) \mid i \in I\}$, but different choices give rise to equivalent pseudogroups (in the case of foliations of class $C^{k,l}$, C^l-equivalent pseudogroups). Thus, from now on, we restrict ourselves to considering properties of pseudogroups that are invariant by equivalences; this justifies our abuse of language when we talk about "the" holonomy pseudogroup of X.

Lemma 2.14 ([20]). *If X is a compact foliated space, then its holonomy pseudogroup is compactly generated.*

If X is a foliation of codimension m and it admits an atlas such that the transversals have an affine structure and the maps $h_{i,j}$ are all affine, then it is a *transversally affine foliation* and its holonomy pseudogroup is also affine.

A *matchbox manifold* is a compact foliated space admitting an atlas with totally disconnected transversals.

2.6. Toral linked twist maps

Let $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus, whose points we will simply denote as pairs (x, y), where $x, y \in \mathbb{R}^2$. For an interval $A = [a, b]$ with $0 \leq a < b \leq 1$, let H_A be the horizontal closed annulus defined by

$$H_A = \{(x, y) \in \mathbb{T}^2 \mid a \leq y \leq b\},$$

and let V_A be the corresponding vertical closed annulus

$$V_A = \{(x, y) \in \mathbb{T}^2 \mid a \le x \le b\}.$$

For any integer $m > 1$, we have the horizontal and vertical twist maps, defined on H_A and V_A, respectively, by

$$(x, y) \mapsto (x + \phi_m(y), y) \quad \text{and} \quad (x, y) \mapsto (x, y + \phi_m(x)),$$

where

$$\phi_m(t) = \frac{m(t-a)}{(b-a)}$$

is the only affine map satisfying

$$\phi_m(a) = 0 \quad \text{and} \quad \phi_m(b) = m.$$

Note that ϕ_m depends of course on the choice of interval A, but we leave it implicit to avoid cumbersome notation. Toral linked twists can be constructed with more general maps ϕ_m (see, e.g., [13]), but in this paper we restrict our attention to the linear case for the sake of simplicity.

A *toral linked twist* is the composition of horizontal twist maps on a finite number of horizontal annuli with vertical twists on a finite number of vertical annuli. Let $\widehat{H}_1, \dots, \widehat{H}_k$ be a collection of closed intervals in $[0, 1]$ such that every intersection $\widehat{H}_i \cap \widehat{H}_j$ with $i \neq j$ consists of at most one common endpoint, and let $\widehat{V}_1, \dots, \widehat{V}_l$ be another such collection. Let $H_1, \dots, H_k$ be the horizontal closed annuli induced by the intervals $\widehat{H}_1, \dots, \widehat{H}_k$; similarly, let $V_1, \dots, V_l$ be the vertical closed annuli induced by $\widehat{V}_1, \dots, \widehat{V}_l$. Choose two sequences of positive integers

$$m_1, \dots, m_k \quad \text{and} \quad n_1, \dots, n_l,$$

and, for $i = 1, \dots, k$, let h_i denote the horizontal m_i-twist map on H_i,

$$h_i(x, y) = (x + \phi_{m_i}(y), y).$$

Define the vertical n_i-twist maps v_j, $1 \le j \le l$, similarly; see Figure 1 for an illustration.

We combine the horizontal twists h_i into one map T_h and the vertical twists into another map T_v as follows:

$$T_h(x, y) = \begin{cases} h_i(x, y) & \text{if } (x, y) \in H_i \text{ for some } 1 \le i \le k, \\ (x, y) & \text{else;} \end{cases}$$

$$T_v(x, y) = \begin{cases} v_i(x, y) & \text{if } (x, y) \in V_i \text{ for some } 1 \le i \le l, \\ (x, y) & \text{else.} \end{cases}$$

The linked twist map corresponding to our choice of intervals and integers is then $T = T_v \circ T_h$.

We review some of the basic properties that will be of use later. First, note that T is the identity on $\mathbb{T}^2 \setminus M$, where

$$M = H_1 \cup \cdots \cup H_k \cup V_1 \cup \cdots \cup V_l.$$

Also, T is affine (hence smooth) on $\mathbb{T}^2 \setminus \Delta$, where

$$\Delta = \partial H_1 \cup \cdots \cup \partial H_k \cup T_h^{-1}(\partial V_1) \cup \cdots \cup T_h^{-1}(\partial V_l)$$

and ∂ denotes the topological boundary. Finally, we will also employ the following result.

Theorem 2.15 (Theorem A in [12]). *The restriction of the toral linked twist map T to M is topologically transitive and sensitive to initial conditions.*

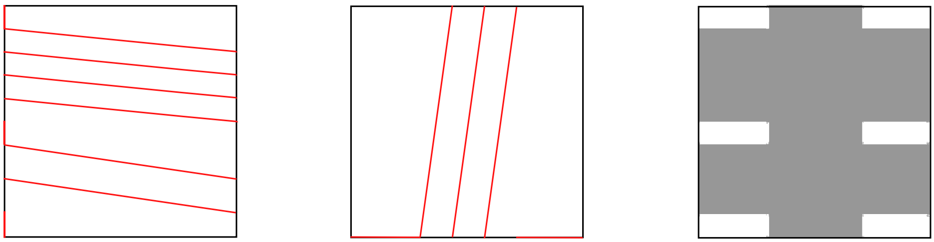

Figure 1. The horizontal (left, T_h) and vertical (middle, T_v) components of a linked twist; the red lines are the images of the circles represented by the vertical (left) and horizontal (middle) boundary segments. They involve two and one intervals, respectively. The shaded area on the right represents M.

2.7. Equicontinuous pseudogroups

Definition 2.16 (Definition 7.1 in [3]). Let X be a topological space. Let $\{(U_i, d_i) \mid i \in I\}$ be a family of metric spaces such that $\{U_i\}$ is an open covering of X and every d_i is a compatible metric on U_i. We say that $\{(U_i, d_i)\}$ is a *cover of X by quasi-locally equal metric spaces* if there is an assignment $\varepsilon \mapsto \delta(\varepsilon)$ such that, for every $i, j \in I$, every point $z \in U_i \cap U_j$ has an open neighborhood $U_{i,j,z} \subset U_i \cap U_j$ satisfying

$$d_i(x, y) < \delta(\varepsilon) \quad \Longrightarrow \quad d_j(x, y) < \varepsilon$$

for every $\varepsilon > 0$ and $x, y \in U_{i,j,z}$. Two such covers $\{(U_i, d_i)\}$ and $\{(V_j, d'_j)\}$ are *equivalent* if their union is again a cover by quasi-locally equal metric spaces; an equivalence class of covers is called a *quasi-local metric space*.

Proposition 2.17 (Theorem 15.1 in [3]). *If X is Hausdorff and paracompact, then every cover by quasi-locally equal metric spaces is equivalent to a metric; that is, equivalent to a cover of the form $\{(X, d)\}$.*

Definition 2.18 (Definition 8.4 in [3]). Let $\mathcal{G}$ be a pseudogroup acting on a Polish space X. We say that $\mathcal{G}$ is *equicontinuous* if there are a cover by quasi-locally equal metric spaces $\{(U_i, d_i) \mid i \in I\}$, a generating pseudo$*$group $\mathcal{S}$, and an assignment $\varepsilon \mapsto \delta(\varepsilon)$ such that

$$d_i(x, y) < \delta(\varepsilon) \quad \Longrightarrow \quad d_j(sx, sy) < \varepsilon$$

for every $i, j \in I$, $s \in \mathcal{S}$, $x, y \in \operatorname{dom} s \cap U_i$ and $sx, sy \in U_j$.

Note that if the above condition is fulfilled with a cover by quasi-locally equal metric spaces, then it is also fulfilled with any other equivalent cover, so we can regard equicontinuity as a property of the quasi-local metric.

By the previous results, Definition 2.18 is equivalent to the following.

Definition 2.19. A pseudogroup $\mathcal{G} \curvearrowright X$ is *equicontinuous* if there exist a generating pseudo∗group S, a compatible metric d, and an assignment $\varepsilon \mapsto \delta(\varepsilon)$ such that

$$d(x, y) < \delta(\varepsilon) \quad \Longrightarrow \quad d(sx, sy) < \varepsilon$$

for every $s \in S$ and $x, y \in \operatorname{dom} s$.

Proposition 2.20 (Lemma 8.8 in [3]). *Being equicontinuous is invariant by equivalences of pseudogroups.*

3. Pseudogroup dynamics

3.1. Sensitivity and chaos

The aim of this section is to introduce our definition of Devaney chaos for pseudogroups; first, we need to obtain suitable analogues of conditions (i)–(iii) in Definition 1.1.

Definition 3.1. A pseudogroup $\mathcal{G} \curvearrowright X$ is *topologically transitive* if, for all non-empty open subsets $U, V \subset X$, there is some $g \in \mathcal{G}$ with

$$g(U) \cap V \neq \emptyset.$$

Definition 3.2. A pseudogroup $\mathcal{G} \curvearrowright X$ is *point transitive* if there is some $x \in X$ such that $\mathcal{G}x$ is dense in X.

Lemma 3.3 (cf. Proposition 1.1 in [32] and Proposition 4.5 in [16]). *Point transitivity implies topological transitivity for every pseudogroup $\mathcal{G} \curvearrowright X$; if X is separable and Baire, then the converse also holds.*

Proof. To show that point transitivity implies topological transitivity, let $\mathcal{G}x$ be a dense orbit and let U, V be non-empty open sets. Then there are $g, h \in \mathcal{G}$ with $g(x) \in U$, $h(x) \in V$, and therefore $hg^{-1}(U) \cap V \neq \emptyset$.

Suppose now that $\mathcal{G}$ is topologically transitive but there is no dense orbit, and let $\{U_n\}_{n \in \mathbb{N}}$ be a countable base for X. For every $x \in X$, there is some $U_{n(x)}$ such that $\mathcal{G}x \cap U_{n(x)} = \emptyset$. For each n, $\mathcal{V}_n = \bigcup_{g \in \mathcal{G}} g(U_n)$ is a dense open set because $\mathcal{G}$ is topologically transitive, so $X \setminus \mathcal{V}_{n(x)}$ is a closed and nowhere dense set containing x. Thus, $X = \bigcup_{n \in \mathbb{N}} X \setminus \mathcal{V}_n$ is a countable union of closed and nowhere dense sets, contradicting the assumption that X was Baire. ∎

Recall that we are working with Polish (hence, Baire) spaces, so topological transitivity and point transitivity coincide.

Regarding density of periodic points, one may be tempted to use the following naive definition: $\mathcal{G} \curvearrowright X$ has density of periodic points if the union of finite $\mathcal{G}$-orbits is dense in X. Unfortunately, Example 2.11 shows that this condition is not invariant by equivalences, so we need to reformulate it as follows.

Definition 3.4. A pseudogroup $\mathcal{G} \curvearrowright X$ has *density of periodic orbits* if there is an open set $U \subset X$ meeting every $\mathcal{G}$-orbit and such that the set of finite $\mathcal{G}|_U$-orbits is dense in U.

Lemma 3.5. *Having density of periodic orbits is invariant by equivalences.*

Proof. Let $\mathcal{G} \curvearrowright X$ be a pseudogroup with density of periodic orbits, let $U \subset X$ be an open set meeting every $\mathcal{G}$-orbit and such that the finite $\mathcal{G}|_U$-orbits are dense, and let $\Phi\colon (X, \mathcal{G}) \to (Y, \mathcal{H})$ be an equivalence of pseudogroups.

Since U is a paracompact space, we can find a subset $\Phi_0 \subset \Phi$ such that $\{\operatorname{dom}\phi \mid \phi \in \Phi_0\}$ is a locally finite family and $\bigcup_{\phi\in\Phi_0} \operatorname{dom}\phi = U$. We claim that

$$V = \bigcup_{\phi\in\Phi_0} \operatorname{im}\phi$$

satisfies the statement in Definition 3.4. Clearly, V meets every $\mathcal{H}$-orbit, so let us prove that finite $\mathcal{H}|_V$-orbits are dense in V.

Let $W \subset V$ be any open set and choose $\phi_0 \in \Phi_0$ with $\operatorname{im}\phi_0 \cap W \neq \emptyset$. $\phi_0^{-1}(W) \subset U$, so there is some $y \in \phi_0^{-1}(W)$ such that $\mathcal{G}|_U(y)$ is finite. Since $\{\operatorname{dom}\phi \mid \phi \in \Phi_0\}$ is a locally finite family and $\mathcal{G}|_U(y)$ is finite, there are only finitely many maps in Φ_0 defined on $\mathcal{G}|_U(y)$, so

$$A_y := \{\phi(z) \mid \phi \in \Phi_0, z \in \mathcal{G}|_U(y)\}$$

is a finite set.

Let us show that $\mathcal{H}|_V(A_y) = A_y$. For every $h\phi(z)$, $h \in \mathcal{H}|_V$, there is some $\psi \in \Phi_0$ with $h\phi(z) \in \operatorname{im}\psi$, and therefore, by Definition 2.6(v),

$$z' := \psi^{-1}h\phi(z) \in \mathcal{G}|_U(z) = \mathcal{G}|_U(y).$$

Hence $h\phi(z) = \psi(z')$ with $z' \in \mathcal{G}|_U(y)$, showing that $\mathcal{H}|_V(A_y) = A_y$.

We have proved that every open set $W \subset V$ meets an $\mathcal{H}|_V$-invariant finite set A_y, so $\mathcal{H}$ has density of periodic orbits. ■

Corollary 3.6. *If $\mathcal{G} \curvearrowright X$ has density of periodic orbits and $W \subset X$ is a relatively compact open set meeting every orbit, then the finite $\mathcal{G}|_W$-orbits are dense in W.*

Proof. Let $U \subset X$ be an open set satisfying the statement of Definition 3.4, and let $F \subset \mathcal{G}$ be a finite set satisfying $\overline{W} \subset \{\operatorname{im} f \mid f \in F\}$ and $\bigcup_{f\in F} \operatorname{dom} f \subset U$. Since W meets every $\mathcal{G}$-orbit, $V := \bigcup_{f\in F} f^{-1}(W)$ also meets every $\mathcal{G}$-orbit; moreover, $V \subset U$, so the set of finite $\mathcal{G}|_V$-orbits is dense in V. Hence $\Phi_0 = \{f|_{f^{-1}(W)} : f \in F\}$ is a finite set satisfying $W = \bigcup_{\phi\in\Phi_0} \operatorname{im}\phi$, and, arguing as in the proof of the previous proposition, we get that the set of finite $\mathcal{G}|_W$-orbits is dense in W. ■

Finally, we come to the definition of sensitivity for pseudogroups. A naive approach would suggest the following definition.

Definition 3.7 (Naive sensitivity). $\mathcal{G} \curvearrowright X$ is sensitive if there is $c > 0$ such that, for every $x \in X$ and $r > 0$, there are $g \in \mathcal{G}$ and $y \in B(x, r)$ with $x, y \in \operatorname{dom} g$ and $d(g(x), g(y)) \geq c$.

However, the following lemma shows that this condition is too weak to model our idea of sensitivity.

Lemma 3.8. *Any topologically transitive pseudogroup $\mathcal{G}$ on a perfect Polish space X satisfies Definition 3.7.*

Proof. Since X is perfect, we can choose two non-empty open sets W_1 and W_2, and $c > 0$, satisfying

$$d(W_1, W_2) \geq 2c.$$

Let $x \in X$ and $r > 0$ be arbitrary, and choose $0 < s < r$ such that $V = B(x,r) \setminus \overline{B(x,s)}$ is non-empty; such s exists because X is perfect. By topological transitivity, there are maps g_1 and g_2 in $\mathcal{G}$ such that

$$g_1(V) \cap W_1 \neq \emptyset \quad \text{and} \quad g_2(V) \cap W_2 \neq \emptyset;$$

by restricting to open subsets if necessary, we may assume $\operatorname{dom} g_1, \operatorname{dom} g_2 \subset V$ and $\operatorname{im} g_i \subset W_i$ for $i = 1, 2$. Let h_i, $i = 1, 2$, be the partial map with domain

$$\operatorname{dom} h_i = \operatorname{dom} g_i \cup B(x,s)$$

defined by

$$h_i|_{\operatorname{dom} g_i} = g_i \quad \text{and} \quad h_i|_{B(x,s)} = \operatorname{id}_{B(x,s)}.$$

Axiom 2.2 (v) ensures that h_i belongs to $\mathcal{G}$. By construction, we have that $x, y_1 \in \operatorname{dom} h_1$, $x, y_2 \in \operatorname{dom} h_2$ and $d(h_1(y_1), h_2(y_2)) \geq 2c$. The triangle inequality now yields that, for some $i \in \{1,2\}$, we have

$$d(h_i(x), h_i(y_i)) = d(x, h_i(y_i)) \geq c. \qquad \blacksquare$$

This argument shows that the problem originates from Axiom 2.2 (v) (closure under combinations). Following the ideas in [3], which in turn can be traced back to previous works (see [21, 27]), we phrase our definition of sensitivity for pseudogroups in terms of generating pseudo∗groups. We also quantify over all compatible metrics in order to make it a topological condition.

Definition 3.9. Given a metric d on X and a generating pseudo∗group $\mathcal{S}$ for a pseudogroup $\mathcal{G} \curvearrowright X$, we say that $\mathcal{G} \curvearrowright X$ is ($\mathcal{S}$,d)*-sensitive to initial conditions* if there is a *sensitivity constant* $c := c(\mathcal{S}, d) > 0$ such that, for every $x \in X$ and $r > 0$, there are $s \in \mathcal{S}$ and $y \in \operatorname{dom} s$ with

$$d(x,y) < r \quad \text{and} \quad d(sx, sy) \geq c.$$

We say that $\mathcal{G} \curvearrowright X$ is *sensitive to initial conditions* if it is ($\mathcal{S}$,d)-sensitive to initial conditions for every choice of $\mathcal{S}$ and d.

Note that c clearly varies with the choice of metric and pseudo∗group: if we fix d and choose a sequence of pseudo∗groups S_n such that all sets $\operatorname{im} s_n$ $(s \in S_n)$ have diameter less than $1/n$, then $c(d, S_n) \downarrow 0$.

In order to provide some intuition for this definition, let us consider the case of a $\mathbb{Z}$-action induced by a homeomorphism f on some topological space X. If the $\mathbb{Z}$-action is sensitive to initial conditions (with respect to the classical definition for group actions), there is $c > 0$ so that, for every $x \in X$ and $\delta > 0$, there are $y \in B(x,r)$ and $z \in \mathbb{Z}$ satisfying $d(f^z(x), f^z(y)) > c$. However, the absolute value of z will diverge as $\delta \to 0$. In the case of the pseudogroup induced by this action, it might happen that, for some generating pseudo$*$group $\mathcal{S}$, the domain of every map $s \in \mathcal{S}$ that agrees with f^z on some open set is so small that it does not contain any such y, and therefore the maps in $\mathcal{S}$ cannot bear witness to the sensitivity of the action. This is precisely why we must quantify over all generating pseudo$*$groups. Coming back to the example at hand, given a covering $\{U_i\}$ of X by open sets, the restrictions $\{f|_{U_i}\}$ are a generating pseudo$*$group for the pseudogroup induced by the action. Note that, by composing maps, the diameters of the domains of the compositions might grow smaller and smaller. In this case, sensitivity means that, for every covering $\{U_i\}$, there is some positive c (which depends on the cover) so that, for every point $x \in X$, we can find points y arbitrarily close to x that have compositions $f|_{U_{i_n}} \circ \cdots \circ f|_{U_{i_1}}$ defined on x and y and satisfying

$$d(f|_{U_{i_n}} \circ \cdots \circ f|_{U_{i_1}}(x), f|_{U_{i_n}} \circ \cdots \circ f|_{U_{i_1}}(y)) > c.$$

Having obtained analogues of Definition 1.1 (i)–(iii), we can introduce our definition of Devaney chaos for pseudogroups.

Definition 3.10. A pseudogroup $\mathcal{G} \curvearrowright X$ is *chaotic* if it is topologically transitive, has density of periodic orbits, and is sensitive to initial conditions.

3.2. Equicontinuous points

In this subsection, we will generalize to the setting of pseudogroups some dynamical notions expressing regularity; we will need them in order to prove the Auslander–Yorke dichotomy.

Definition 3.11. A point $x \in X$ is (S,d)-*equicontinuous* for a generating set S and a metric d on X if there is an assignment $\varepsilon \mapsto \delta(\varepsilon)$ so that

$$d(x,y) < \delta(\varepsilon) \quad \Longrightarrow \quad d(s(x),s(y)) < \varepsilon$$

for every $s \in S^*$ and $y \in X$ with $x, y \in \operatorname{dom} s$. We say that a point x is *equicontinuous* if it is equicontinuous for some choice of S and d.

We will refer to any assignment $\varepsilon \mapsto \delta$ satisfying the above condition as a *modulus of equicontinuity* for (S,d).

Definition 3.12. The pseudogroup $\mathcal{G} \curvearrowright X$ is *almost equicontinuous* if there are S and d so that the set of (S,d)-equicontinuous points is dense in X.

Lemma 3.13. *If $x \in X$ is (S,d)-equicontinuous, then every $y \in \mathcal{G}x$ is (S,d)-equicontinuous (perhaps with a different modulus).*

Proof. Let $y \in \mathcal{G}x$. Since S is a generating set for $\mathcal{G}$, there is some $s \in S^*$ so that $s(x) = y$. For every $\varepsilon > 0$, let $\delta'(\varepsilon) > 0$ be small enough so that

$$B(y,\delta'(\varepsilon)) \subset \operatorname{im} s \quad \text{and} \quad s^{-1}(B(y,\delta'(\varepsilon))) \subset B(x,\delta(\varepsilon)).$$

Then, for every $t \in S^*$, the restrictions of t and tss^{-1} to $\operatorname{dom} t \cap B(y,\delta'(\varepsilon))$ coincide, and thus

$$t(B(y,\delta'(\varepsilon))) = tss^{-1}(B(y,\delta'(\varepsilon))) \subset ts(B(x,\delta(\varepsilon))).$$

Since $ts \in S^*$ and $\delta \mapsto \varepsilon$ is a modulus of equicontinuity for (S, d), we obtain

$$t(B(y, \delta'(\varepsilon))) \subset B(t(y), \varepsilon). \qquad \blacksquare$$

3.3. Dynamics and compact generation

We begin with some preliminary results for compactly generated pseudogroups. The following proposition reveals that, for every system of compact generation $(U, F, \widetilde{F})$ and every point $x \in U$, either the pseudo∗group $\langle \widetilde{F} \rangle$ displays sensitivity to initial conditions on x, or every map in $\langle F \rangle$ defined on x has an extension in $\langle \widetilde{F} \rangle$ whose domain contains a ball of a fixed radius $\rho > 0$.

Proposition 3.14. *Let $\mathcal{G}$ be a compactly generated pseudogroup on X, let d be a compatible metric, let $(U, F, \widetilde{F})$ be a system of compact generation, and let*

$$\sigma := \sigma(U, F, \widetilde{F}) = \sup\{r > 0 \mid B(u, r) \subset \operatorname{dom} \tilde{f},\ \forall f \in F,\ u \in \operatorname{dom} f\} > 0.$$

Then, for every $x \in U$,

(i) *either there is $\rho > 0$ such that $B(x, \rho) \subset \operatorname{dom} \tilde{f}$ for every $f \in \langle F \rangle$ with $\operatorname{dom} f \cap B(x, \rho) \neq \emptyset$;*

(ii) *or for every $r > 0$, there are $y \in B(x, r)$ and $\tilde{f} \in \langle \widetilde{F} \rangle$ satisfying*

$$x, y \in \operatorname{dom} \tilde{f}, \quad d(\tilde{f}(x), \tilde{f}(y)) \geq \sigma/2.$$

Proof. Suppose that (i) does not hold, so that, for every $r > 0$, there are $f \in \langle F \rangle$ and $y, z \in B(x, r)$ satisfying $z \in \operatorname{dom} f \subset \operatorname{dom} \tilde{f}$, $y \notin \operatorname{dom} \tilde{f}$. Let $f = f_n \cdots f_1$, where $f_i \in F$ for $i = 1, \ldots, n$. Let j be the largest index $0 \leq j < n$ satisfying $B(x, r) \subset \operatorname{dom} \tilde{f}_j \cdots \tilde{f}_1$. This means that there is $y \in \operatorname{dom} \tilde{f}_j \cdots \tilde{f}_1$ such that

$$\tilde{f}_j \cdots \tilde{f}_1(y) \notin \operatorname{dom} \tilde{f}_{j+1}.$$

But $z \in \operatorname{dom} \tilde{f}_{j+1} \cdots \tilde{f}_1$, so

$$d(\tilde{f}_j \cdots \tilde{f}_1(y), \tilde{f}_j \cdots \tilde{f}_1(z)) \geq \sigma$$

by the definition of σ. Now the triangle inequality yields either

$$d(\tilde{f}_j \cdots \tilde{f}_1(x), \tilde{f}_j \cdots \tilde{f}_1(y)) \geq \sigma/2, \quad \text{or} \quad d(\tilde{f}_j \cdots \tilde{f}_1(x), \tilde{f}_j \cdots \tilde{f}_1(z)) \geq \sigma/2. \qquad \blacksquare$$

The following result, which will be of use later, is a generalization of the well-known fact that equicontinuity and uniform equicontinuity agree for actions on compact spaces.

Lemma 3.15. *Let $\mathcal{G}$ be a compactly generated pseudogroup. If there are a metric d and a generating pseudo∗group $\mathcal{S}$ such that every point is a point of $(\mathcal{S}, d)$-equicontinuity, then $\mathcal{G}$ is equicontinuous.*

Proof. Let $(U, F, \widetilde{F})$ be a system of compact generation with $\widetilde{F} \subset \mathcal{S}$ (Lemma 2.13). By Proposition 3.14, for every $x \in U$ there is $\rho_x > 0$ such that $U_x := B(x, \rho_x) \subset U$ and

$$U_x \subset \operatorname{dom} \tilde{f} \quad \text{for every } f \in \langle F \rangle \text{ with } \operatorname{dom} f \cap U_x \neq \emptyset.$$

The sets U_x, $x \in U$, form an open cover of U. Since $\mathcal{G}|_U$ is equivalent to $\mathcal{G}$, it must also be compactly generated, so choose some relatively compact open subset $V \subset U$ meeting every orbit, and choose a finite family $\{U_x\}_{x \in I}$ covering V. Shrink every U_x to obtain another finite covering $\{V_x\}_{x \in I}$ of V and such that $\overline{V_x} \subset U_x$ for every $x \in I$.

Let $\varepsilon > 0$ and cover each set $\overline{V_x}$ $(x \in I)$ by finitely many balls $B(z_{x,j}, \delta_{z_{x,j}}(\varepsilon/2))$, where $z_{x,j} \in U_x$ and $\delta_{z_{x,j}}$ is a modulus of $(\mathcal{S}, d)$-equicontinuity at $z_{x,j}$. Let $\delta'(\varepsilon)$ be a Lebesgue number for all these coverings; that is, if u, v are contained in some V_x for some $x \in I$ and $d(u, v) < \delta'(\varepsilon)$, then there is some j such that $u, v \in B(z_{x,j}, \delta_{z_{x,j}}(\varepsilon/2))$.

Let us now show that $\mathcal{G}|_V$ is equicontinuous with respect to the cover by quasi-local metric spaces $(V_x, d)_{x \in I}$ (see Definition 2.16). We first choose $\langle F \rangle|_V$ as a generating pseudo∗group for $\mathcal{G}|_V$, and let $x, y \in I$, $u, v \in V_x$ and $f \in \langle F \rangle$ be such that $fu, fv \in V_y$. Then we need to show that

$$d(u, v) < \delta'(\varepsilon) \quad \Longrightarrow \quad d(fu, fv) < \varepsilon \quad \text{for every } \varepsilon > 0.$$

By our choice of $\delta'(\varepsilon)$, there is some $z := z_{x,j}$ so that $u, v \in B(z, \delta_z(\varepsilon/2))$. Moreover, we have already established that $U_x \subset \operatorname{dom} \tilde{f}$, so $\tilde{f}z$ is well defined. Now we get

$$d(fu, \tilde{f}z),\ d(fv, \tilde{f}z) < \varepsilon/2,$$

so $d(fu, fv) < \varepsilon$ by the triangle inequality.

We have proved that $\mathcal{G}|_V$ is equicontinuous with respect to the cover by quasi-local metric spaces $\{(U_{x_i}, d_{x_i})\}$. Since $\mathcal{G}|_V$ is equivalent to $\mathcal{G}$, the result follows by Proposition 2.20. ∎

Proof of Theorem 1.7. Let us prove that, if the action $G \curvearrowright X$ is sensitive, then the induced pseudogroup $\mathcal{G}$ is sensitive too, the converse implication being trivial. Let $c_G > 0$ be a sensitivity constant for $G \curvearrowright X$, let $H = \{f_1, \dots, f_n\} \subset G$ be a symmetric finite generating set (in the group-theoretic sense), let d be a metric on X, and let $\mathcal{S}$ be a generating pseudo∗group for $\mathcal{G}$. Since $H \subset \mathcal{G}$ and $\mathcal{S}$ generates $\mathcal{G}$, Lemma 2.4 yields a finite sequence of open coverings of X,

$$\widetilde{\mathcal{U}}_i = \{\widetilde{U}_{i,j}\}, \quad i = 1, \dots, n,$$

such that

$$\tilde{f}_{i,j} := (f_i)|_{\widetilde{U}_{i,j}} \in \mathcal{S} \quad \text{for every } i, j;$$

furthermore, we may assume that every $\widetilde{\mathcal{U}}_i$ is finite because X is compact. Let $\mathcal{U}_i = \{U_{i,j}\}$ be a shrinking of $\widetilde{\mathcal{U}}_i$, let $f_{i,j} := (f_i)|_{U_{i,j}}$, and let

$$F = \{f_{i,j}\} \quad \text{and} \quad \widetilde{F} = \{\tilde{f}_{i,j}\}.$$

Then $(X, F, \widetilde{F})$ is a system of compact generation for $\mathcal{G}$.

Let us show that (S, d) is sensitive with constant $c_F := \min\{\sigma/2, c_G\}$, where $\sigma := \sigma(X, F, \widetilde{F})$ is given by Proposition 3.14 (note that its value does not depend on x). If $(X, F, \widetilde{F})$ satisfies Proposition 3.14 (ii) at every point in X, then we are done, so suppose that Proposition 3.14 (i) holds for some $x \in X$ and $\rho > 0$. Since G is sensitive, there are

$$g = f_{i_k} \cdots f_{i_1} \in G$$

and $y \in B(x,\rho)$ with $d(g(x),g(y)) \geq c_G$. Clearly, there is a sequence $j_1,\dots,j_k$ such that $x \in \operatorname{dom} h$, where $h = f_{i_k,j_k} \cdots f_{i_1,j_1}$. But $B(x,\rho) \subset \operatorname{dom} \tilde{h}$ by Proposition 3.14 (i), whence $y \in \operatorname{dom} \tilde{h}$ and

$$d(\tilde{h}(x),\tilde{h}(y)) \geq c_G \geq c_F.$$

This shows that $\mathcal{G}$ is $(\mathcal{S},d)$-sensitive and, since both $\mathcal{S}$ and d were arbitrary, the result follows. ∎

3.4. Sensitive group actions whose induced pseudogroups are not sensitive

In this section, we construct the counterexamples of Theorem 1.8. Let us start by defining a family of linked twists on the 2-torus $\mathbb{T}^2$. Let p_z $(z \in \mathbb{Z})$ denote the following integer-indexed sequence of real numbers:

$$p_z = \begin{cases} 1 - 2^{-1-z} & \text{if } z \geq 1, \\ 2^{z-2} & \text{if } z \leq 0. \end{cases}$$

Let

$$H = \{(x,y) \in \mathbb{T}^2 \mid 1/4 \leq y \leq 3/4\}$$

and

$$V_z = \{(x,y) \in \mathbb{T}^2 \mid p_z \leq x \leq p_{z+1}\} \quad (z \in \mathbb{Z}).$$

Let $T_h \colon \mathbb{T}^2 \to \mathbb{T}^2$ be the horizontal twist defined by

$$T_h(x,y) = \begin{cases} (x + 2(y - \frac{1}{4}), y) & \text{if } (x,y) \in H, \\ (x,y) & \text{else;} \end{cases}$$

and, for $m \in \mathbb{N}$, let $T_{v,m} \colon \mathbb{T}^2 \to \mathbb{T}^2$ be the vertical twist:

$$T_{v,m}(x,y) = \begin{cases} (x, y + 2^{2+|z|}(x - p_z)) & \text{if } (x,y) \in V_z,\ |z| \leq m, \\ (x,y) & \text{else.} \end{cases}$$

Letting $T_m = T_{v,m} \circ T_h$, we obtain a sequence of linked twist maps on $\mathbb{T}^2$.

By Theorem 2.15, T_z is topologically transitive on

$$M_z = H \cup \bigcup_{|z| \leq m} V_z,$$

and is the identity on $\mathbb{T}^2 \setminus M_m$ for every $m \in \mathbb{N}$. Note that, by the definition of the sequence p_z, we have

$$\bigcup_{m \geq 0} M_m = \mathbb{T}^2 \setminus \big(\{0\} \times ([0,1/4) \cup (3/4,1])\big). \tag{3.1}$$

Moreover, T_m is affine on $\mathbb{T}^2 \setminus \Delta_m$, where

$$\Delta_m := \partial H \cup \bigcup_{|z| \leq m} T_h^{-1}(\partial V_z).$$

Lemma 3.16. *Let* $(p/q, r/s) \in \mathbb{T}^2$ *be a point with rational coordinates. Then, for every* $m \geq 0$*, its* T_m*-orbit is contained in the finite set*

$$\left\{\left(\frac{l_1}{d}, \frac{l_2}{d}\right) \mid l_1, l_2 \in \{0, \dots, d-1\}\right\},$$

where $d = \operatorname{lcm}(q, s, 2^{m+2})$.

Proof. Follows from the definition of T_h and $T_{v,m}$. ∎

We are now in position to introduce our examples. We begin by showing that an action $G \curvearrowright X$ with G finitely generated but X non-compact might be sensitive, while the induced pseudogroups is not. Let $X = \mathbb{T}^2 \times \mathbb{Z}$ and let

$$\sigma((x, y), z) = (T_{|z|}(x, y), z), \quad \tau((x, y), z) = ((x, y), z + 1).$$

Proposition 3.17. *The subgroup of* $\operatorname{Homeo}(X)$ *generated by* σ *and* τ *is sensitive as a group action. The pseudogroup generated by* σ *and* τ*, however, is not sensitive to initial conditions* (*in the sense of Definition* 3.9).

Proof. We start by proving that the induced group action is sensitive. Let d be a compatible metric on X that restricts to the standard flat metric d_z on every $\mathbb{T}^2 \times \{z\} \cong \mathbb{T}^2$. Choose $c > 0$ such that

- the action of T_0 on M_0 is c-sensitive (with respect to d_0), and
- $c < 1/8$.

Let $((x, y), z) \in X$, and let $U \times \{z\}$ be a neighborhood of $((x, y), z)$. By (3.1), $\bigcup_z M_{|z|}$ is dense in $\mathbb{T}^2$, so there is $n > 0$ such that

$$\tau^n(U \times \{z\}) \cap (M_{n+z} \times \{n + z\}) \neq \emptyset.$$

Since T_{n+z} is topologically transitive on M_{n+z}, there is $m > 0$ so that

$$\sigma^m \tau^n(U \times \{z\}) \cap (M_0 \times \{n + z\}) \neq \emptyset,$$

and therefore

$$\tau^{-n-z} \sigma^m \tau^n(U \times \{z\}) \cap (M_0 \times \{0\}) \neq \emptyset.$$

Let $\phi = \tau^{-n-z} \sigma^m \tau^n$ for the sake of simplicity. Since the action of T_0 is c-sensitive on M_0, if $\phi((x, y), z) \in M_0 \times \{0\}$, then there are $l > 0$ and $((u, v), z) \in U \times \{z\}$ such that

$$d(\sigma^l \phi((x, y), z), \sigma^l \phi((u, v), z)) \geq c.$$

If $\phi((x, y), z) \notin M_0 \times \{0\}$, then, since the action of T_0 is topologically transitive on M_0, there are $l > 0$ and $((u, v), z) \in U \times \{z\}$ such that

$$\tau^l \phi((u, v), z) \in \left[\frac{3}{8}, \frac{5}{8}\right]^2 \times \{0\}.$$

But $[1/4, 3/4] \subset M_0$ and $\phi((x, y), z) \notin M_0 \times \{0\}$, so

$$d(\sigma^l \phi((x, y), z), \sigma^l \phi((u, v), z)) \geq \frac{1}{8} \geq c.$$

Let us now show that the pseudogroup $\mathcal{G}$ generated by σ and τ is not sensitive. Consider the point $((0,0),0) \in X$. Note that, since $(0,0) \notin M_m$ for every $m \in \mathbb{N}$, σ is the identity on a neighborhood of $((0,0),z)$ for every $z \in \mathbb{Z}$. For each $z \in \mathbb{Z}$, let U_z and O_z be open neighborhoods of $(0,0)$ such that $\overline{O_z} \subset \mathbb{T}^2 \setminus M_{|z|+1}$ and $\overline{U_z} \subset O_z$. Now let

$$U = \bigcup_{z\in\mathbb{Z}} (\mathbb{T}^2 \setminus \overline{U}_z) \times \{z\}, \quad O = \bigcup_{z\in\mathbb{Z}} O_z \times \{z\},$$

and consider the pseudo∗group $\mathcal{S}$ generated by

$$F = \{\sigma|_U, \sigma|_O, \tau|_U, \tau|_O\}.$$

Clearly, $\mathcal{S}$ generates $\mathcal{G}$. Since

$$\mathcal{G}((0,0),0) = \{((0,0),z) \mid z \in \mathbb{Z}\},$$

the only maps in F that are defined on the orbit of $((0,0),0)$ are $\tau|_O$ and $\sigma|_O$, which are isometries (recall that $\sigma|_O = \mathrm{id}|_O$), so we have that every map in $\mathcal{S}$ defined on $((0,0),0)$ is an isometry with respect to d, and therefore $\mathcal{G}$ is not sensitive to initial conditions. ■

We have constructed the first counterexample of Theorem 1.8. We can repurpose this machinery to obtain the second counterexample: an action $F_\omega \curvearrowright \mathbb{T}^2$ that does not satisfy Theorem 1.7, where F_ω is the free group with countably many generators. Define the action by mapping a sequence freely generating F_ω to the sequence T_m, $m \geq 0$. The proof that $F_\omega \curvearrowright \mathbb{T}^2$ is sensitive and the pseudogroup is not sensitive is virtually identical to Proposition 3.17, so we leave the details to the reader.

3.5. Main results

In this section we will prove Theorems 1.9, 1.10, and 1.11, in that order. We begin with the following preliminary result, which follows arguments from Lemma 8.8 and Theorem 15.1 in [3].

Proposition 3.18. *Let $\mathcal{G}$ act on a locally compact and separable metric space (X,d), let $\mathcal{S} \subset \mathcal{G}$ be a generating pseudo∗group, and let $\Phi\colon (X,\mathcal{G}) \to (Y,\mathcal{H})$ be an equivalence. Then there is a generating pseudo∗group $\mathcal{T}$ for $\mathcal{H}$ and a metric d' on Y satisfying the following condition: if there are $x \in X$, $\varepsilon, \delta > 0$ such that*

$$d(x,u) < \delta \quad \Longrightarrow \quad d(sx,su) < \varepsilon$$

for every $u \in X$ and $s \in \mathcal{S}$ with $x,u \in \operatorname{dom} s$, then, for every $y \in \Phi(x)$ there is $\delta_y > 0$ such that

$$d'(y,v) < \delta_y \quad \Longrightarrow \quad d'(ty,tv) < \varepsilon$$

for every $t \in \mathcal{T}$ with $y, v \in \operatorname{dom} t$.

Proof. We begin by proving the following preliminary result.

Claim 1. *There is a subset $\widehat{\Phi}_0 \subset \Phi$ such that*

(a) *$\operatorname{dom}\phi$ and $\operatorname{im}\phi$ are relatively compact for every $\phi \in \widehat{\Phi}_0$,*

(b) *the map $\psi^{-1}\phi$ belongs to $\mathcal{S}$ for every $\phi, \psi \in \widehat{\Phi}_0$, and*

(c) *$\{\operatorname{im}\phi \mid \phi \in \widehat{\Phi}_0\}$ is a locally finite open covering of Y.*

First note that, since Y is a locally compact and separable metric space and Φ is an equivalence, we can find a sequence, $\phi_1, \phi_2, \dots$, in Φ such that

- every ϕ_n has an extension $\tilde\phi_n \in \Phi$ with $\overline{\operatorname{dom}\phi_n} \subset \operatorname{dom}\tilde\phi_n$,
- $\operatorname{dom}\tilde\phi_n$ and $\operatorname{im}\tilde\phi_n$ are relatively compact for every $n \geq 1$, and
- $\{\operatorname{im}\phi_n \mid n \geq 1\}$ and $\{\operatorname{im}\tilde\phi_n \mid n \geq 1\}$ are locally finite open coverings of Y.

We define now by induction on n an increasing sequence of finite subsets $\widehat\Phi_{0,n} \subset \Phi$ ($n \geq 1$) so that $\psi^{-1}\phi$ belongs to $\mathcal{S}$ for all $\phi, \psi \in \widehat\Phi_{0,n}$ and

$$\operatorname{im}\phi_1 \cup \dots \cup \operatorname{im}\phi_n \subset \bigcup_{\phi \in \widehat\Phi_{0,n}} \operatorname{im}\phi.$$

Let $\widehat\Phi_{0,1} = \{\phi_1\}$ and, for $n > 1$, assume that we have defined $\widehat\Phi_{0,n-1}$ satisfying the required properties. Lemma 2.4 yields a finite open covering $\{U_i\}_{i \in I}$ of $\overline{\operatorname{dom}\phi_n}$ such that every U_i is relatively compact and the restriction of $\psi^{-1}\tilde\phi_n$ to every U_i belongs to $\mathcal{S}$ for every $\psi \in \widehat\Phi_{0,n-1}$. Letting

$$\widehat\Phi_{0,n} = \widehat\Phi_{0,n-1} \cup \{\tilde\phi_{n|U_i} \mid i \in I\},$$

we get

$$\operatorname{im}\phi_n \subset \bigcup_{i \in I} \tilde\phi_n(U_i) = \bigcup_{i \in I} \operatorname{im}(\tilde\phi_n|_{U_i}).$$

The induction hypothesis now yields

$$\operatorname{im}\phi_1 \cup \dots \cup \operatorname{im}\phi_n \subset \bigcup_{\phi \in \widehat\Phi_{0,n}} \operatorname{im}\phi.$$

Let us show by cases that $\psi^{-1}\phi$ belongs to $\mathcal{S}$ for all $\phi, \psi \in \widehat\Phi_{0,n}$. If $\phi, \psi \in \widehat\Phi_{0,n-1}$, then it follows from the induction hypothesis, so assume first that $\phi = \tilde\phi_{n|U_i}$ for some $i \in I$. If ψ is also of the form $\tilde\phi_{n|U_j}$ for some $j \in I$, then $\psi^{-1}\phi$ is the identity on its domain, so $\psi^{-1}\phi \in \mathcal{S}$. If, on the other hand, $\psi \in \widehat\Phi_{0,n-1}$, then $\psi^{-1}\phi = \psi^{-1}\tilde\phi_{n|U_i} \in \mathcal{S}$ by the definition of U_i. The only case remaining is when $\phi \in \widehat\Phi_{0,n-1}$ and $\psi = \tilde\phi_{n|U_i}$ for some $i \in I$, which follows from the previous argument because $\mathcal{S}$ is symmetric. This completes the proof of Claim 1 by taking $\widehat\Phi_0 = \bigcup_n \widehat\Phi_{0,n}$.

We now turn to the task of defining Φ_0 and the generating pseudo∗group $\mathcal{T}$. Let $\{V_{\hat\phi} \mid \hat\phi \in \widehat\Phi_0\}$ be a shrinking of the covering $\{\operatorname{im}\hat\phi \mid \hat\phi \in \widehat\Phi_0\}$; that is, $\bigcup_{\hat\phi \in \Phi_0} V_{\hat\phi} = Y$ and $V_{\hat\phi}$ is an open set satisfying $\overline{V_{\hat\phi}} \subset \operatorname{im}\hat\phi$ for every $\hat\phi \in \widehat\Phi_0$.

Claim 2. *For every $x \in X$, there is a open neighborhood U_x of x such that*

$$U_x \cap V_{\hat\phi} \neq \emptyset \quad \Longrightarrow \quad U_x \subset \operatorname{im}\hat\phi$$

for every $\hat\phi \in \widehat\Phi_0$.

Since $\{\operatorname{im}\hat\phi \mid \hat\phi \in \widehat\Phi_0\}$ is locally finite by Claim 1 (c),

$$U_x = \Bigg(\bigcap_{\hat\phi \in \widehat\Phi_0,\ x \in \operatorname{im}\hat\phi} \operatorname{im}\hat\phi\Bigg) \setminus \Bigg(\bigcup_{\hat\phi \in \widehat\Phi_0,\ x \notin \overline{V_{\hat\phi}}} \overline{V_{\hat\phi}}\Bigg)$$

is an open set that satisfies the required properties, proving Claim 2.

For every $\hat\phi \in \widehat\Phi_0$, let $\{P_i \mid i \in I_{\hat\phi}\}$ be a locally finite open covering of the open set $V_{\hat\phi}$ such that every P_i satisfies

$$P_i \cap V_{\hat\psi} \neq \emptyset \implies P_i \subset \operatorname{im}\hat\psi \quad \forall \hat\psi \in \widehat\Phi_0, \tag{3.2}$$

$$\operatorname{diam}\hat\phi^{-1}(P_i) < d(\hat\phi^{-1}(P_i), \operatorname{dom}\hat\phi \setminus \hat\phi^{-1}(V_{\hat\phi})). \tag{3.3}$$

From now on, given $\hat\phi \in \widehat\Phi_0$ and $i \in I_{\hat\phi}$, let ϕ_i be shorthand for $\hat\phi|_{P_i}$. Let

$$\Phi_0 = \{\phi_i \mid \hat\phi \in \widehat\Phi_0,\ i \in I_{\hat\phi}\},$$
$$\mathcal{T} = \{\psi_j s \phi_i^{-1} \mid \phi_i, \psi_j \in \Phi_0,\ s \in \mathcal{S}\} \cup \{\mathrm{id}|_U : U \text{ open in Y}\}.$$

It is elementary to check that $\mathcal{T}$ is a pseudo$*$group, so let us prove that $\mathcal{T}$ generates $\mathcal{H}$. Let $h \in \mathcal{H}$ and let $x \in \operatorname{dom} h$. By the definitions of $\{V_{\hat\phi} \mid \phi \in \widehat\Phi_0\}$ and $\{P_i \mid i \in I_{\hat\phi}\}$, the collection

$$\{P_i \mid i \in I_{\hat\phi},\ \phi \in \widehat\Phi_0\} = \{\operatorname{im}\phi_i \mid i \in I_{\hat\phi},\ \hat\phi \in \widehat\Phi_0\}$$

is an open covering of Y. Therefore, there are $\phi_i, \psi_j \in \Phi_0$ so that $x \in \operatorname{im}\phi_i$, $f(x) \in \operatorname{im}\psi_j$. Thus $\psi_j^{-1} h \phi_i \in \mathcal{G}$ by Definition 2.6(v). Since $\mathcal{S}$ generates $\mathcal{G}$, there must be an open neighborhood U of $\phi_i^{-1}x$ so that the restriction of $\psi_j^{-1} h \phi_i$ to U belongs to $\mathcal{S}$ by Lemma 2.4. Then h coincides with $\psi_j s \phi_i^{-1} \in \mathcal{S}$ over $\phi_i(U)$. We have proved that, for every $h \in \mathcal{H}$ and $x \in \operatorname{dom} h$, the restriction of h to some open neighborhood of x belongs to $\mathcal{T}$, so $\mathcal{T}$ generates $\mathcal{H}$ by Lemma 2.4.

We now prove some preliminary results needed to define the metric d'. For each $\hat\phi \in \widehat\Phi_0$, let $D_{\hat\phi}\colon \operatorname{im}\hat\phi \times \operatorname{im}\hat\phi \to \mathbb{R}_{\geq 0}$ be the metric defined on the open set $\operatorname{im}\hat\phi$ by $D_{\hat\phi}(x,y) = d(\hat\phi^{-1}x, \hat\phi^{-1}y)$. If $u, v \in \operatorname{im}\hat\phi$ for some $\hat\phi \in \widehat\Phi_0$, let

$$\overline{D}(u,v) = \sup\{D_{\hat\phi}(u,v) \mid \hat\phi \in \widehat\Phi_0,\ u, v \in \operatorname{im}\hat\phi\}.$$

A pair $(u,v) \in Y \times Y$ is *admissible* if there is $\hat\phi \in \widehat\Phi_0$ such that $u, v \in V_{\hat\phi}$ and

$$\{u,v\} \cap V_{\hat\psi} \neq \emptyset \implies \{u,v\} \subset \operatorname{im}\hat\psi, \quad \forall \hat\psi \in \widehat\Phi_0.$$

Let $S_{u,v}$ be the set of sequences $(z_0, \dots, z_n)$ of arbitrary finite length with $z_0 = u$ and $z_n = v$, and such that (z_{i-1}, z_i) is an admissible pair for every $i = 1, \dots, n$. The following properties are elementary:

$$(u,u) \in S_{u,u}, \tag{3.4}$$

$$(z_0, \dots, z_n) \in S_{u,v} \implies (z_n, \dots, z_0) \in S_{v,u}, \tag{3.5}$$

$$\left.\begin{aligned} (z_0, \dots, z_m) &\in S_{u,v} \\ (z_m, \dots, z_{m+n}) &\in S_{v,w} \end{aligned}\right\} \implies (z_0, \dots, z_{m+n}) \in S_{u,w}. \tag{3.6}$$

Set

$$d'(u,v) = \begin{cases} \infty & \text{if } S_{u,v} = \emptyset, \\ \inf_{(z_0,\dots,z_n) \in S_{u,v}} \sum_{k=1}^n \overline{D}(z_{k-1}, z_k) & \text{if } S_{u,v} \neq \emptyset. \end{cases} \tag{3.7}$$

It follows from (3.4)–(3.6) that d' is a pseudometric in Y. To prove that it is actually a metric, we need the following result.

Claim 3. *Let* $\hat{\phi} \in \widehat{\Phi}_0$*, let* $u \in V_{\hat{\phi}}$*, and let* $v \in Y$ *be such that* $S_{u,v} \neq \emptyset$*. Then*

$$d'(u,v) \geq \begin{cases} \min\{D_{\hat{\phi}}(u,v), D_{\hat{\phi}}(u, \operatorname{im}\hat{\phi} \setminus V_{\hat{\phi}})\} & \text{if } v \in V_{\hat{\phi}}, \\ D_{\hat{\phi}}(u, \operatorname{im}\hat{\phi} \setminus V_{\hat{\phi}}) & \text{if } v \notin V_{\hat{\phi}}. \end{cases}$$

Let $(z_0, \dots, z_n) \in S_{u,v}$. Suppose first that $\{z_{i-1}, z_i\} \subset V_{\hat{\phi}}$ for every $i = 1, \dots, n$, then

$$\sum_{k=1}^{n} \overline{D}(z_{k-1}, z_k) \geq \sum_{k=1}^{n} D_{\hat{\phi}}(z_{k-1}, z_k) \geq D_{\hat{\phi}}(z_0, z_n) = D_{\hat{\phi}}(u,v)$$

by the triangle inequality. Assume now that m is the first index in $\{1, \dots, n\}$ satisfying $z_m \notin V_{\hat{\phi}}$. Since (z_{m-1}, z_m) is an admissible pair and $z_{m-1} \in \operatorname{im}\phi_i \subset V_{\hat{\phi}}$, we get $z_m \in \operatorname{im}\hat{\phi}$. Therefore

$$\sum_{k=1}^{n} \overline{D}(z_{k-1}, z_k) \geq \sum_{k=1}^{m} D_{\hat{\phi}}(z_{k-1}, z_k) \geq D_{\hat{\phi}}(z_0, z_m) \geq D_{\hat{\phi}}(u, \operatorname{im}\hat{\phi} \setminus V_{\hat{\phi}});$$

this completes the proof of Claim 3.

Claim 4. *d' is a compatible metric on Y.*

Let us prove that d' is a metric: Let $u, v \in Y$ be such that $d'(u,v) = 0$, so $S_{u,v} \neq \emptyset$. Take any $\hat{\phi} \in \widehat{\Phi}_0$ such that $u \in V_{\hat{\phi}}$. Since

$$D_{\hat{\phi}}(u, \operatorname{im}\hat{\phi} \setminus V_{\hat{\phi}}) > 0,$$

it follows from Claim 3 that $v \in V_{\hat{\phi}}$ and $D_{\hat{\phi}}(u,v) \leq d'(u,v) = 0$. But $D_{\hat{\phi}}$ is a metric on $\operatorname{im}\hat{\phi}$, so $u = v$ as desired.

Let us show that d' is a compatible metric. We start by showing that every neighborhood in X contains a d'-ball, so let U be a neighborhood of x. Since $\{\operatorname{im}\hat{\phi} \mid \hat{\phi} \in \widehat{\Phi}_0\}$ is a locally finite cover, we may assume

$$\{\hat{\phi} \in \widehat{\Phi}_0 \mid x \in V_{\hat{\phi}}\} = \{\hat{\phi}_1, \dots, \hat{\phi}_n\}$$

for some $n \in \mathbb{N}$. The metrics $D_{\hat{\phi}_i}$ are compatible over $\operatorname{im}\hat{\phi}_i$, so we can find some $r > 0$ satisfying

$$B_{D_{\hat{\phi}_i}}(x,r) \subset U \quad \text{and} \quad d(B_{D_{\hat{\phi}_i}}(x,r), \operatorname{im}\hat{\phi}_i \setminus V_{\hat{\phi}_i}) > r$$

for every $i = 1, \dots, n$; then, for every $y \in B_{d'}(x,r)$,

$$r > d'(x,y) \geq D_{\hat{\phi}_i}(x,y)$$

by Claim 3, so $y \in B_{D_{\hat{\phi}_i}}(x,r)$ and hence $y \in U$.

Consider now a ball $B_{d'}(x,r)$. Choose a neighborhood U of x small enough so that

$$U \subset V_{\hat{\phi}_i} \quad \text{and} \quad U \subset B_{D_{\hat{\phi}_i}}(x,r)$$

for $i = 1, \dots, n$. This means that $(x, u) \in S_{x,u}$ for every $u \in U$, so

$$d'(x,u) \le \overline{D}(x,u) = \sup_i D_{\hat{\phi}_i}(x,u) < r.$$

Hence $U \subset B_{d'}(x, r)$, proving Claim 4.

Let $\varepsilon, \delta > 0$, $x \in X$ and $y \in \Phi(x)$ be as in the statement of the theorem. By definition of $\mathcal{T}$, every map that is not the identity is of the form $\psi_j s \phi_i^{-1}$, where $s \in \mathcal{S}$ and $\psi_j, \phi_i \in \Phi_0$ for some $\hat{\psi}, \hat{\chi} \in \widehat{\Phi}_0$. Recall that the notation ψ_j means that ψ_j is of the form $\hat{\psi}|_{P_j}$, where $\{P_j \mid j \in I_{\hat{\psi}}\}$ is a locally finite covering of $V_{\hat{\psi}}$. Since the covering $\{V_{\hat{\chi}} \mid \hat{\chi} \in \widehat{\Phi}_0\}$ was also locally finite, there are only finitely many points in X that are sent to y by a map in Φ_0; denote them by $x_1, \dots, x_n$. Moreover, since Φ is an equivalence, all these points lie on the same orbit, so we can find a uniform δ_y satisfying

$$d(x_l, u) < \delta_y \implies d(sx_l, su) < \varepsilon \tag{3.8}$$

for every $l = 1, \dots, n$, $u \in X$ and $s \in \mathcal{S}$ with $x_l, u \in \operatorname{dom} s$.

Let $t \in T$ satisfy $y \in \operatorname{dom} t$, and let $v \in \operatorname{dom} t \cap B_{d'}(y, \delta_y)$. Let $t = \psi_j s \phi_i^{-1}$, then, in particular, (3.8) is also satisfied for $x_l = \phi_i^{-1}(y)$, which belongs to the set $x_1, \dots, x_n$.

Now $y, v \in \operatorname{dom} t$ implies $y, v \in \operatorname{im} \phi_i = P_i \subset V_{\hat{\phi}}$. Hence $\{y, v\} \in S_{y,v}$ by (3.2) and

$$D_{\hat{\phi}}(y, v) < D_{\hat{\phi}}(y, \operatorname{im} \hat{\phi} \setminus V_{\hat{\phi}})$$

by (3.3), so Claim 3 yields

$$D_{\hat{\phi}}(y, v) \le d'(y, v) < \delta_y,$$

and therefore

$$d(x_l, u) < \delta_y$$

by the definition of $D_{\hat{\phi}}$, where $u = \phi_i^{-1} v$.

Let $\chi_k \in \Phi_0$ be such that $ty, tv \in \operatorname{im} \chi_k$. Then

$$d(\chi_k^{-1} ty, \chi_k^{-1} tv) = d((\chi_k^{-1} \psi_j) s x_l, (\chi_k^{-1} \psi_j) s u).$$

Since $\chi_k^{-1} \psi_j \in \mathcal{S}$ by Claim 1 (b) and $d(x_l, u) < \delta_y$, we have

$$D_{\hat{\chi}}(ty, tv) = d(\chi_k^{-1} ty, \chi_k^{-1} tv) < \varepsilon.$$

The covering $\{\operatorname{im} \hat{\eta} \mid \hat{\eta} \in \widehat{\Phi}\}$ is locally finite, so again there are only finitely many maps such that ty and tv are contained in their image. In particular, this implies that $\overline{D}(ty, tv) < \varepsilon$ since we are taking the maximum of a finite set of values smaller than ε. Both sx and su belong to $\operatorname{dom} \psi_j \subset V_{\hat{\psi}}$, so (3.7) yields

$$d'(ty, tv) \le \overline{D}(ty, tv) < \varepsilon. \qquad \blacksquare$$

Corollary 3.19. *Being sensitive to initial conditions is invariant by equivalences of pseudogroups acting on locally compact spaces.*

Proof. Suppose that the pseudogroup $\mathcal{G}$ is not sensitive; then there are a metric d and a generating pseudo$*$group $\mathcal{S}$ such that, for every $\varepsilon > 0$, there are x_ε and δ_ε with

$$d(x_\varepsilon, u) < \delta_\varepsilon \quad \Longrightarrow \quad d(s(x_\varepsilon), s(u)) < \varepsilon$$

for all $u \in X$ and $s \in \mathcal{S}$ with $x_\varepsilon, u \in \operatorname{dom} s$.

Let $(Y, \mathcal{H})$ be a pseudogroup, and let $\Phi\colon (X, \mathcal{G}) \to (Y, \mathcal{H})$ be an equivalence. Proposition 3.18 yields a generating pseudo$*$group $\mathcal{T}$ for $\mathcal{H}$ and a metric d' on Y. Letting $y_\varepsilon \in \Phi(x_\varepsilon)$, Proposition 3.18 yields

$$d'(y_\varepsilon, v) < \delta_{\varepsilon, y_\varepsilon} \quad \Longrightarrow \quad d'(t(y_\varepsilon), t(v)) < \varepsilon$$

for every $\varepsilon > 0$, $v \in Y$ and $t \in \mathcal{T}$ with $y_\varepsilon, v \in \operatorname{dom} t$; this shows that $\mathcal{H}$ is not sensitive to initial conditions.

We have shown that if $\mathcal{G}$ is not sensitive, then neither is $\mathcal{H}$; the result now follows from the symmetry of the equivalence relation for pseudogroups. ■

Corollary 3.20. *Let $\Phi\colon (X, \mathcal{G}) \to (Y, \mathcal{H})$ be an equivalence of pseudogroups acting on locally compact spaces. If $x \in X$ is a point of $(\mathcal{S}, d)$-equicontinuity for $\mathcal{G}$, then, with the notation of Proposition 3.18, every $y \in \Phi_0 x$ is a point of $(\mathcal{T}, d')$-equicontinuity for $\mathcal{H}$. In particular, $\mathcal{G}$ is almost equicontinuous if and only if $\mathcal{H}$ is.*

Proof. The first assertion follows immediately from Preposition 3.18 and the definition of equicontinuous point. Suppose now that the points of $(\mathcal{S}, d)$-equicontinuity are dense in X; then they are also dense in the open set $\bigcup_{\phi\in\Phi_0} \operatorname{dom}\phi$. Since Φ_0 sends points of $(\mathcal{S}, d)$-equicontinuity to points of $(\mathcal{T}, d')$-equicontinuity and $\{\operatorname{im}\phi \mid \phi \in \Phi_0\}$ is an open covering of Y, the points of $(\mathcal{T}, d')$-equicontinuity are dense in Y. ■

Corollaries 3.19 and 3.20 together with Lemma 3.5 yield Theorem 1.9.

We turn our attention to the Auslander–Yorke dichotomy for pseudogroups, which we will subsequently use to prove Theorem 1.11.

Proof of Theorem 1.10. Suppose that $\mathcal{G}$ is not sensitive; thus, there are a metric d and a generating pseudo$*$group $\mathcal{S}$ so that, for every $n \in \mathbb{N}$, there are $x_n \in X$ and $\delta_n > 0$ satisfying

$$d(x_n, y) < \delta_n \quad \Longrightarrow \quad d(s(x_n), s(y)) < 1/n \tag{3.9}$$

for every $y \in X$ and $s \in \mathcal{S}$ with $x_n, y \in \operatorname{dom} s$.

Using Lemma 2.13, choose a system of compact generation $(U, F, \widetilde{F})$ with $\widetilde{F} \subset \mathcal{S}$; note that any point in $\mathcal{G}x_n$ still satisfies (3.9), perhaps with a different δ_n, so we may assume without loss of generality that the sequence x_n is contained in U. We also have $1/n < \sigma(U, F, \widetilde{F})$ for n large enough, and now Proposition 3.14 yields the existence of a sequence $r_n > 0$ such that

$$\operatorname{dom} f \cap B(x_n, r_n) \neq \emptyset \quad \Longrightarrow \quad B(x_n, r_n) \subset \operatorname{dom} \tilde{f} \tag{3.10}$$

for every $f \in \langle F \rangle$. We will suppose, by passing to a subsequence if necessary, that every x_n satisfies (3.10); moreover, we will also assume by decreasing r_n that $B(x_n, r_n) \subset U$ and that $r_n < \delta_n$.

Note that

$$\text{diam}\, f(B(x_n, r_n)) < \frac{2}{n} \tag{3.11}$$

for every $f \in F$. Indeed, otherwise there would be some $f \in F$ with

$$B(x_n, r_n) \subset \text{dom}\, \tilde{f} \quad \text{and} \quad \text{diam}\, \tilde{f}(B(x_n, r_n)) \geq \frac{2}{n}$$

by (3.10). But then the triangle inequality would yield $d(\tilde{f}(x_n), \tilde{f}(y)) \geq 1/n$ for some $y \in B(x_n, r_n) \subset B(x_n, \delta_n)$, contradicting (3.9).

Let

$$V_n = \bigcup_{f \in \langle F \rangle} f(B(x_n, r_n)) \quad \text{for } n \geq 1,$$

which are clearly open subsets of U. Moreover, topological transitivity implies that every V_n is dense in U, so by the Baire category theorem, $\bigcap_n V_n$ is also a dense subset of U.

Let us show that every $x \in \bigcap_{n \geq 1} V_n$ is a point of (F^*, d)-equicontinuity. Assume for the sake of contradiction that there is $c > 0$ such that, for every $r > 0$, there are $f \in F^*$ and $y \in B(x, r)$ such that

$$\text{for } x, y \in \text{dom}\, f, \quad d(f(x), f(y)) \geq c.$$

Choose m large enough so that $2/m < c/2$. Since $x \in \bigcap_{n \geq 1} V_n$, there is some $g \in F^*$ such that $g(x) \in B(x_m, r_m)$. By assumption, there are also $y \in X$ and $f \in F^*$ satisfying

$$y \in \text{dom}\, g \cap \text{dom}\, f, \quad g(y) \in B(x_m, r_m) \quad \text{and} \quad d(f(x), f(y)) \geq c.$$

But then

$$d(fg^{-1}(y'), fg^{-1}(x')) = d(f(x), f(y)) \geq c$$

for $x' = g(x)$, $y' = g(y)$, and now (3.10) and the triangle inequality yield

$$\max\{d(\tilde{f}\tilde{g}^{-1}(x_m), \tilde{f}\tilde{g}^{-1}(x')), d(\tilde{f}\tilde{g}^{-1}(x_m), \tilde{f}\tilde{g}^{-1}(y'))\} \geq c/2 > 2/m,$$

contradicting (3.11).

We have proved that, if $\mathcal{G} \curvearrowright X$ is topologically transitive and not sensitive to initial conditions, then there is a metric d on X and a system of compact generation $(U, F, \widetilde{F})$ such that the set of points of (F^*, d)-equicontinuity is dense in U. Since $\mathcal{G}$ is equivalent to $\mathcal{G}|_U$ by Lemma 2.8, Corollary 3.20 yields that $\mathcal{G}$ is almost equicontinuous.

If $\mathcal{G}$ is minimal, then it is trivial to check that $\bigcap_{n \geq 1} V_n = U$, so by the previous argument there are d and $(U, F, \widetilde{F})$ so that every point in U is a point of (F^*, d)-equicontinuity. The result then follows by Lemma 3.15 and Proposition 2.20. ∎

Proof of Theorem 1.11. Let $\mathcal{S}$ be a generating pseudo∗group for $\mathcal{G}$ and let d be a compatible metric. By Theorem 1.10 (Auslander–Yorke dichotomy for pseudogroups), it is enough to show that, for every $(\mathcal{S}, d)$, the set of points of $(\mathcal{S}, d)$-equicontinuity is empty. Let $(U, F, \widetilde{F})$ be a system of compact generation for $\mathcal{G}$ satisfying $\widetilde{F} \subset \mathcal{S}$.

Suppose for the sake of contradiction that $x \in U$ is a point of $(\mathcal{S}, d)$-equicontinuity. Then it must satisfy Proposition 3.14(i) with some $\rho > 0$. By Definition 3.4 and Corollary 3.6, there are points

$$q_1 \in B(x, \rho/2) \quad \text{and} \quad q_2 \in B(x, \rho/2) \setminus \mathcal{G} q_1$$

with finite $\mathcal{G}|_U$-orbits. Letting

$$\varepsilon < \frac{1}{2} d(\mathcal{G}|_U q_1, \mathcal{G}|_U q_2)$$

and using the triangle inequality, we can choose a point $q \in \{q_1, q_2\}$ satisfying

$$d(x, \mathcal{G} q) > \varepsilon. \tag{3.12}$$

For every $n \geq 1$, let p_n be a point in $B(x, \rho/n)$ with finite $\mathcal{G}|_U$-orbit. Since $\mathcal{G} p_n \cap B(x, \rho)$ is finite, there is a finite set $K_n \subset \langle F \rangle$ satisfying that, for every $y \in \mathcal{G} p_n \cap B(x, \rho)$, there is $k \in K_n$ with $ky = p_n$; moreover, since $y \in B(x, \rho)$, each map $k \in K_n$ may be extended to a map $\tilde{k} \in \langle \tilde{F} \rangle$ with $B(x, \rho) \subset \operatorname{dom} \tilde{k}$. For each n, let $\tilde{K}_n$ denote the collection of all such extensions. Since $\tilde{K}_n$ is a finite set of maps, there is a neighborhood W_n of q so that $W_n \subset B(x, \rho/2)$ and $\tilde{k}(W_n) \subset B(\tilde{k}(q), \varepsilon/4)$ for every $\tilde{k} \in \tilde{K}_n$.

As $\mathcal{G}|_U$ is topologically transitive, there are maps f_n and points $v_n \in B(x, \rho/n)$ such that $f_n v_n \in W_n$; again, $B(x, \rho) \subset \operatorname{dom} \tilde{f}_n$ for all n. If $d(\tilde{f}_n v_n, \tilde{f}_n p_n) \geq \rho/2$ for infinitely many n, then the triangle inequality yields

$$\max\{d(\tilde{f}_n x, \tilde{f}_n p_n), d(\tilde{f}_n x, \tilde{f}_n v_n)\} \geq \frac{\rho}{4},$$

showing that x is not a point of $(\mathcal{S}, d)$-equicontinuity; this is a contradiction. Hence, we may assume $d(\tilde{f}_n v_n, \tilde{f}_n p_n) < \rho/2$ for n large enough. In particular, since $\tilde{f}_n v_n \in W_n \subset B(x, \rho/2)$, we have $\tilde{f}_n p_n \in B(x, \rho)$, so there are maps k_n in K_n satisfying

$$k_n f_n(p_n) = p_n \quad \text{and} \quad B(x, \rho) \subset \operatorname{dom} \tilde{k}_n \tilde{f}_n$$

for n large enough. Now we have

$$\max\{d(\tilde{k}_n \tilde{f}_n p_n, \tilde{k}_n \tilde{f}_n x), d(\tilde{k}_n \tilde{f}_n x, \tilde{k}_n \tilde{f}_n v_n)\} \geq \frac{\varepsilon}{4} \tag{3.13}$$

for n large enough because, otherwise, the triangle inequality and $\tilde{k}_n \tilde{f}_n p_n = p_n$ would yield

$$\begin{aligned} d(x, \tilde{k}_n q) &\leq d(x, p_n) + d(\tilde{k}_n \tilde{f}_n p_n, \tilde{k}_n \tilde{f}_n x) + d(\tilde{k}_n \tilde{f}_n x, \tilde{k}_n \tilde{f}_n v_n) + d(\tilde{k}_n \tilde{f}_n v_n, \tilde{k}_n q) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon, \end{aligned}$$

contradicting (3.12). The inequality in (3.13) and the fact that the sequences $\{v_n\}$ and $\{p_n\}$ converge to x are at odds with the assumption that x was a point of $(\mathcal{S}, d)$-equicontinuity. Since x was an arbitrary point, we infer that the set of points of $(\mathcal{S}, d)$-equicontinuity is empty, as desired. ∎

3.6. A non-compactly generated, countably generated and topologically transitive pseudogroup that is not sensitive

In this section, we construct a counterexample showing that compact generation cannot be dropped in the statement of Theorem 1.11.

Let $Y = \{0,1\}^{\mathbb{Z}}$, which is a Cantor set. We use the greek letters $\alpha, \beta, \dots$ to denote elements of Y and the notation $\alpha = (\alpha_i)_{i\in\mathbb{Z}}, \beta = (\beta_i)_{i\in\mathbb{Z}}$. Let $\sigma\colon Y \to Y$ be the shift function, defined by

$$(\sigma(\alpha))_i = (\alpha_{i+1}).$$

Let G denote the subgroup of $\mathrm{Homeo}(Y)$ generated by σ, endowed with the obvious action $G \curvearrowright Y$. It is well-known that the action $G \curvearrowright Y$ is topologically transitive and has density of periodic orbits.

The point $\mu := (\dots, 0, 0, 0, \dots)$ is a fixed point of G. For $n \geq 0$, let

$$U_n = \{\alpha \in Y \mid \alpha_i = 0 \text{ for } |i| < n\}.$$

Note that $U_0 = Y$ and that $\{U_n\}$ is a system of clopen neighborhoods for μ. We also have

$$\sigma(U_{n+1}),\ \sigma^{-1}(U_{n+1}) \subset U_n \quad \text{for all } n \geq 0. \tag{3.14}$$

Let $X = \{(n,\alpha) \in \mathbb{N} \times Y \mid \alpha \in U_n\}$; that is, X is the disjoint union $\bigsqcup_{i\geq 0} U_i$. Let $f, g \in \mathrm{Ph}(X)$ be defined by

$$f(n,\alpha) = (n, \sigma(\alpha)), \qquad \operatorname{dom} f = \{(n,\alpha) \in X \mid \sigma(\alpha) \in U_n \setminus U_{n+2}\}, \tag{3.15}$$

$$g(n,\alpha) = (n+1, \alpha), \qquad \operatorname{dom} g = \{(n,\alpha) \in X \mid \alpha \in U_{n+1}\}. \tag{3.16}$$

In other words, f is defined for pairs (n,α) with $\alpha_i = 0$ for $i = -n, \dots, n+1$ but there is an index in the segment $[-n-1, n+3]$ where α takes the value 1. Note that $\operatorname{dom} f$, $\operatorname{im} f$, $\operatorname{dom} g$, and $\operatorname{im} g$ are clopen subsets of X. Finally, let $\mathcal{G}$ be the pseudogroup generated by f and g.

Lemma 3.21. *For every $(n,\alpha) \in X$, $\mathcal{G}(n,\alpha) = \{(m,\beta) \in X \mid \beta \in G\alpha\}$.*

Proof. It follows trivially from the definitions of f and g that every point in $\mathcal{G}(n,a)$ is of the form $(m,\beta) \in X$ for some $\beta \in G\alpha$, so let us prove the reverse inclusion. First note that, since μ is a fixed point of G, the lemma is trivial for $\alpha = \mu$, so assume $\alpha \neq \mu$; clearly, it is enough to show that

$$\{(m, \sigma(\alpha)) \in X\} \subset \mathcal{G}(n,\alpha), \tag{3.17}$$

so let us prove it. Let $(m,\alpha) \in X$, we will show first that

$$\text{there is } k \in \mathbb{N} \text{ such that} \quad (k,\alpha), (k, \sigma(\alpha)) \in X. \tag{3.18}$$

Since $\alpha \neq \mu$, there is a largest $l \geq 0$ such that $\sigma(\alpha) \in U_l$. If $l = 0$, then $(0, \sigma(\alpha)) \in U_0 \setminus U_2$, so $(0,\alpha) \in \operatorname{dom} f$ by (3.15); if $l \geq 1$, then $\alpha \in U_{l-1}$ by (3.14). We chose l so that $\sigma(\alpha) \notin U_{l+1}$, $(l-1,\alpha) \in \operatorname{dom} f$ and $f(l-1,\alpha) = (l-1, \sigma(\alpha))$ by (3.15), proving (3.18) and yielding

$$(m, \sigma(\alpha)) = g^{m-k} f g^{k-n}(n,\alpha).$$

This shows (3.17). ∎

Corollary 3.22. *$\mathcal{G}$ is topologically transitive and has density of periodic orbits.*

Proof. Let us prove transitivity first. By Lemma 3.3, it is enough to show that there is a dense orbit; choose $\alpha \in Y$ with a dense G-orbit, then $\mathcal{G}(0,\alpha)$ is dense in X by Lemma 3.21.

In order to prove density of periodic orbits, let $(m,\beta) \in X$ and let $V \subset U_m$ be an open neighborhood of β in Y; we will prove that there is a periodic point $(0,\alpha)$ with $\mathcal{G}(0,\alpha) \cap \{m\} \times V \neq \emptyset$. Since the finite G-orbits are dense, there is some periodic $\alpha \in Y$ such that $\alpha \neq \mu$ and $G\alpha \cap V \neq \emptyset$, so $\mathcal{G}(0,\alpha) \cap \{m\} \times V \neq \emptyset$ by Lemma 3.21. Let us show that $(0,\alpha)$ has a finite $\mathcal{G}$-orbit. Indeed, there is some k such that $G\alpha \notin U_k$, so the set

$$\{(m,\sigma^n(\alpha)) \mid \sigma^n(\alpha) \in U_n\}$$

is finite, and therefore $(0,\alpha)$ is a periodic point by Lemma 3.21. ∎

Lemma 3.23. *$(0,\mu)$ is a point of equicontinuity for $\mathcal{G}$.*

Proof. Let S be the generating set $\{f, f^{-1}, g, g^{-1}\}$. Then $(0,\mu) \notin \operatorname{dom} h$ for $h \in S$, and the result follows. ∎

4. Foliated dynamics

4.1. A non-chaotic foliated space with chaotic holonomy pseudogroup.

Using a construction inspired by Example 2.11, we will show that density of periodic orbits in the holonomy pseudogroup does not imply density of compact leaves. Chaos for foliated spaces is, therefore, a stronger condition than chaos for pseudogroups, at least with our definitions.

Think of $\mathbb{T}^2$ as the quotient $\mathbb{R}^2/\mathbb{Z}^2$ and consider Arnold's cat map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$, which is obtained by factoring the linear map

$$\tilde{f}\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad \tilde{f}(x,y) = (2x+y, x+y)$$

through the quotient $\pi\colon \mathbb{R}^2 \to \mathbb{T}^2$.

It is well known that the cat map f is chaotic, so by Theorem 1.7, the pseudogroup $\mathcal{G} \curvearrowright \mathbb{T}^2$ generated by f is chaotic too. It is also easy to check that the suspension foliation[2] induced by the representation $\pi_1(\mathbb{S}^1) \to \operatorname{Homeo}(\mathbb{T}^2)$ sending a generator to f satisfies Definition 1.13 and is, therefore, chaotic.

Consider now the pseudogroup $\mathcal{H} \curvearrowright \mathbb{R}^2$ generated by $\tilde{f}$ and the integer translations. As in Example 2.11,

$$\Phi := \{\pi|_U \mid U \subset \mathbb{R}^2 \text{ open}, \ \pi|_U\colon U \to \phi(U) \text{ is a homeomorphism}\}$$

is an equivalence $\Phi\colon (\mathbb{R}^2,\mathcal{H}) \to (\mathbb{T}^2,\mathcal{G})$; $\mathcal{H}$ is then a chaotic pseudogroup by Theorem 1.9. Let S denote the closed surface of genus three, whose fundamental group has presentation

$$\pi_1(S) = \langle a_1, b_1, a_2, b_2, a_3, b_3 \mid [a_1,b_1][a_2,b_2][a_3,b_3]\rangle,$$

[2] See Section 3.1 in [10] for an introduction to suspension foliations.

and consider the suspension foliation X induced by the representation

$$\phi\colon \pi_1(S) \to \mathrm{Homeo}(R^2)$$

defined by

$$\phi(a_1) = \tilde{f}, \qquad \phi(a_2) = [(x,y) \mapsto (x+1,y)],$$
$$\phi(a_3) = [(x,y) \mapsto (x,y+1)], \qquad \phi(b_1) = \phi(b_2) = \phi(b_3) = \mathrm{id}\,.$$

The holonomy pseudogroup is obviously equivalent to $\mathcal{H}$, hence chaotic. The foliated space does not have any compact leaves, however, because all $\mathcal{H}$-orbits are infinite, so X is not a chaotic foliated space according to Definition 1.13.

4.2. A non-compact topologically transitive foliated space with density of compact leaves which is not sensitive

This section shows that compactness cannot be omitted in the statement of Theorem 1.14.

Let us use the pseudogroup $\mathcal{G} \curvearrowright X$ from Section 3.6 to obtain a topologically transitive foliated space with density of compact leaves, but with an equicontinuous leaf. We start by constructing a directed graph Z, which can be defined as a pair $Z = (V(Z), E(Z))$ where $V(Z)$ is the vertex set and $E(Z) \subset V(Z) \times V(Z)$ is the set of directed edges. We think of z as the origin and z' as the end vertex of the directed edge $(z, z') \in E(Z)$.

Let $V(Z) = X$, and let $E(Z)$ consist of all edges of the form $((m,a), f(m,a))$ and $((n,b), g(n,b))$ for $(m,a) \in \mathrm{dom}\, f$ and for $(n,b) \in \mathrm{dom}\, g$. Let Y denote the set $\{\mathrm{dom}\, f, \mathrm{im}\, f, \mathrm{dom}\, g, \mathrm{im}\, g\}$, and define the map

$$\nu\colon X \to 2^Y, \quad \nu(x)(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{else,} \end{cases}$$

where $A \in Y$. Under the usual identification of maps in 2^Y and subsets of Y, $\nu(x)$ is the subset of Y containing exactly the elements of Y that contain x. Since $\mathrm{dom}\, f$, $\mathrm{im}\, f$, $\mathrm{dom}\, g$, and $\mathrm{im}\, g$ are clopen, the sets $\nu^{-1}(C)$ for $C \subset Y$ form a partition of Y by clopen sets. Choose disjoint open balls in the two-sphere $\mathbb{S}^2$ indexed by the elements $A \in Y$, and denote them by B_A.

For $C \subset Y$, let $S_C := \mathbb{S}^2 \setminus \bigcup_{A \in C} B_A$, and denote the boundary circles in S_C by Δ_A $(A \in C)$. Let

$$\mathfrak{Y}_1 = \bigsqcup_{C \subset Y} \nu^{-1}(C) \times S_C \quad \text{and} \quad \mathfrak{Y}_2 = E(Z) \times \mathbb{S} \times [0,1].$$

Assume that we have fixed identifications of the boundary circles Δ_A with $\mathbb{S}$. Let $\mathfrak{X}$ be the following quotient of $\mathfrak{Y}_1 \sqcup \mathfrak{Y}_2$: for each $((m,a), h(m,a)) \in E(Z)$ with $h \in \{f, g\}$, identify

$$\{((m,a), h(m,a))\} \times \mathbb{S} \times \{0\} \sim \{((m,a), h(m,a))\} \times \Delta_{\mathrm{dom}\, h},$$
$$\{((m,a), h(m,a))\} \times \mathbb{S} \times \{1\} \sim \{((m,a), h(m,a))\} \times \Delta_{\mathrm{im}\, h}.$$

First, note that, locally, the space looks like the product of $\mathbb{R}^2$ times a Cantor set, and is therefore a matchbox manifold. Take any point $u \in \mathbb{S}^2$ that is not contained in any B_A, $A \in Y$; the image by the quotient map $\pi\colon \mathfrak{Y}_1 \sqcup \mathfrak{Y}_2 \to \mathfrak{X}$ of

$$\bigsqcup_{C \subset Y} \nu^{-1}(C) \times \{u\} \cong X \times \{u\} \subset \mathfrak{Y}_1$$

meets every connected component of $\mathfrak{X}$, and is therefore a transversal meeting every leaf. Every path in $\mathfrak{X}$ is homotopic to the concatenation of paths of the form $\pi(\tau)$, where τ is either a path contained in a plaque $\{(n,a)\} \times S_C \subset \mathfrak{Y}_1$ for $(n,a) \in \nu^{-1}(C)$, $C \subset Y$, or the path $t \in [0,1] \mapsto (((n,a), h(n,a)), s, \pm t)$ for $((n,a), h(n,a)) \in E(Z)$ and s a point in a boundary circle. For τ of the first type, $\pi(\tau)$ is contained in the same plaque, so it induces an identity transformation on the transversal; for τ of the second kind, the path begins in the plaque corresponding to the point $(u,(n,a))$ and ends at the point $(u, h^{\pm 1}(n,a))$, where h is either f or g. So these paths induce f, f^{-1}, g or g^{-1} on the transversal, and we see that the holonomy pseudogroup is equivalent to $\mathcal{G} \curvearrowright X$.

We conclude that $\mathfrak{X}$ is a topologically transitive matchbox manifold and the leaf corresponding to $(0, 0^\infty) \in X$ is a leaf of equicontinuity by Lemma 3.23. Moreover, the periodic points of the action on X correspond to compact leaves in $\mathfrak{X}$, so we also have density of compact leaves.

4.3. An affine pseudogroup

Before embarking on the proof of Theorem 1.16, we need to obtain a modified version of the pseudogroup in Section 3.4. The reason is that we will construct the foliated space counterexample with a particular representative of the holonomy pseudogroup in mind, but we cannot use the pseudogroup of Section 3.4 because all its orbits are infinite.

We start by fixing the following notation:

$$l_n^- = \frac{1}{3 \cdot 2^{1+n}}, \quad r_n^- = \frac{1}{3 \cdot 2^n}, \quad l_n^+ = 1 - r_n^- \quad \text{and} \quad r_n^+ = 1 - l_n^-, \tag{4.1}$$

and then we fix the following intervals in order to define a sequence of toral linked twists:

$$\begin{aligned} H &= \{(x,y) \in \mathbb{T}^2 \mid 1/6 \le y \le 5/6\}, \\ V_0 &= \{(x,y) \in \mathbb{T}^2 \mid 1/6 \le x \le 5/6\}, \\ V_n^- &= \{(x,y) \in \mathbb{T}^2 \mid l_n^- \le x \le r_n^-\} \quad \text{for } n \ge 1, \\ V_n^+ &= \{(x,y) \in \mathbb{T}^2 \mid l_n^+ \le x \le r_n^+\} \quad \text{for } n \ge 1. \end{aligned}$$

Let $T_h\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the horizontal twist defined by

$$T_h(x,y) = \begin{cases} (x + 6(y - \frac{1}{6}), y) & \text{if } (x,y) \in H, \\ (x,y) & \text{else;} \end{cases} \tag{4.2}$$

and, for $m \in \mathbb{N}$, let $T_{v,m}\colon \mathbb{T}^2 \to \mathbb{T}^2$ be the vertical twist:

$$T_{v,m}(x,y) = \begin{cases} (x, y + 6(x - 1/6)) & \text{if } (x,y) \in V_0, \\ (x, y + 3 \cdot 2^{1+n}(x - l_n^-)) & \text{if } (x,y) \in V_n^-,\ n \le m, \\ (x, y + 3 \cdot 2^{1+n}(x - l_n^+)) & \text{if } (x,y) \in V_n^+,\ n \le m, \\ (x,y) & \text{else.} \end{cases} \tag{4.3}$$

Finally, let

$$T_m = T_{v,m} \circ T_h$$

be the corresponding linked twist map.

We denote by Δ_n the set of points in $\mathbb{T}^2$ where T_n is not smooth; i.e.,

$$\Delta_n = \partial H \cup \bigcup_{m \le n} T_h^{-1}(\partial V_m^-) \cup \bigcup_{m \le n} T_h^{-1}(\partial V_m^+).$$

Finally, let

$$\Delta = \bigcup_n \Delta_n \quad \text{and} \quad M_n := H \cup \bigcup_{m \le n} V_m^+ \cup \bigcup_{m \le n} V_m^-,$$

and note that $\Delta_n \subset M_n$ and M_n is the set where T_m is topologically transitive and sensitive to initial conditions by Theorem 2.15.

The linear nature of these linked twists will allow us to find a common set of periodic orbits. Let

$$Q_n = M_n \cap \left\{ \left(\frac{l_1}{2^n}, \frac{l_2}{2^n} \right) \,\middle|\, l_1, l_2 \in \{0, \dots, 2^n - 1\} \right\}, \quad n \ge 1.$$

Lemma 4.1. *$T_m(Q_n) = Q_n$ for every n and m.*

Proof. We can rewrite (4.2) and, using (4.1), also rewrite (4.3) as

$$T_h(x,y) = \begin{cases} (x + 6y, y) & \text{if } (x,y) \in H, \\ (x,y) & \text{else;} \end{cases}$$

$$T_{v,m}(x,y) = \begin{cases} (x, y + 6x) & \text{if } (x,y) \in V_0, \\ (x, y + 3 \cdot 2^{1+n} x) & \text{if } (x,y) \in V_n^-,\ n \le m, \\ (x, y + 3 \cdot 2^{1+n} x) & \text{if } (x,y) \in V_n^+,\ n \le m, \\ (x,y) & \text{else.} \end{cases}$$

The result is now obvious. ∎

Note that $Q_n \cap \partial H = \emptyset$ because the points in ∂H have y-coordinate $1/6$ or $5/6$, which cannot be expressed as a fraction whose denominator is a power of 2. Similarly, the above expression for T_h and the definitions of V_m^+ and V_m^- show that $Q_n \cap \Delta_m = \emptyset$ for every n and m, or, equivalently,

$$Q_n \cap \Delta = \emptyset \quad \text{for every } n. \tag{4.4}$$

Also, by defining

$$\widetilde{Q}_0 = Q_0 = \emptyset, \quad \widetilde{Q}_n = Q_n \setminus Q_{n-1} \quad \text{for } n \geq 1,$$

we obtain by Lemma 4.1

$$T_m(\widetilde{Q}_n) = \widetilde{Q}_n \quad \text{for every } n, m. \tag{4.5}$$

Up to this point, we have proceeded almost in the same way as in Section 3.4. There, we used the pseudogroup on $Y := \mathbb{T}^2 \times \mathbb{N}$ defined by the maps

$$((x, y), n) \mapsto (T_n(x, y), n) \quad \text{and} \quad ((x, y), n) \mapsto ((x, y), n + 1).$$

In order to get affine maps, we will restrict the first map to an appropiate subspace; to obtain density of finite orbits, we will "cut" open balls with center points in Q_n out of the domain of the second map. Let us start by finding suitable radii.

Lemma 4.2. *There is a decreasing sequence $r_0, r_1, \dots$ of positive radii such that, for every $n \geq 0$ and $(x, y) \in \widetilde{Q}_n$,*

(i) $B((x, y), r_n) \subset M_n$,

(ii) $d(B((x, y), r_n), \Delta_n) > 0$,

(iii) *for every $0 \leq m < n$ and $(x', y') \in \widetilde{Q}_m$,*

$$d((x, y), (x', y')) > r_m - r_n \implies d((x, y), (x', y')) > r_m + r_n,$$

(iv) *and $\partial B((x, y), r_n)$ consists of points with at least one irrational coordinate.*

Proof. We proceed by induction on $n \geq 0$; we will choose all radii r_n to be trascendental numbers. First, we may set r_0 arbitrarily because $\widetilde{Q}_0$ is empty. Assume now that we have chosen $r_0, \dots, r_{n-1}$ satisfying the above conditions. Then (i)–(ii) hold for r_n small enough because of (4.4). To prove that (iii) holds for small enough radii, note that every $\widetilde{Q}_m$ is a finite set consisting of points with rational coordinates, so $d((x, y), \widetilde{Q}_m)$ is an algebraic number for every $(x, y) \in \widetilde{Q}_n$ and $0 \leq m < n$. Since the r_m are transcendental numbers, (iii) is satisfied for small r_n. Finally, we can choose r_n transcendental and satisfying (iv) because there are countably many points with both coordinates rational but uncountably many transcendental radii satisfying (i)–(iii). ∎

Let

$$\mathcal{U}_n := \bigcup_{(x,y) \in \widetilde{Q}_n} B((x, y), r_n) \tag{4.6}$$

and

$$\mathcal{V}_n := \bigcup_{0 \leq m \leq n} \mathcal{U}_n. \tag{4.7}$$

By Lemma 4.2(iii), we can express $\mathcal{V}_n$ as a disjoint union of open balls. It follows that $\mathcal{V}_n \cap M_l$ is not dense in M_l for any $n, l \geq 0$.

We are finally prepared to define what will be the holonomy pseudogroup of our counterexample foliated space. Let $\tilde f$ and $\tilde g$ be maps on Y defined by

$$\tilde f((x,y),n) = (T_n(x,y),n), \qquad \operatorname{dom} \tilde f = \{((x,y),n) \mid (x,y) \notin \Delta_n\}, \tag{4.8}$$

$$\tilde g((x,y),n) = ((x,y),n+1), \qquad \operatorname{dom} \tilde g = \{((x,y),n) \mid (x,y) \notin \mathcal{V}_n\}, \tag{4.9}$$

and let $\widetilde G \curvearrowright Y$ be the pseudogroup generated by $\tilde f$ and $\tilde g$.

Lemma 4.3. *The pseudogroup $\widetilde{\mathscr{G}} \curvearrowright Y$ is topologically transitive.*

Proof. Since T_n is topologically transitive on M_n for every $n \geq 0$, there are residual sets $R_n \subset M_n$ consisting of points whose T_n-orbits are dense in M_n; hence, the fact that Δ is meager yields that

$$R := \Big(\bigcap_{n\geq 0, z\in\mathbb{Z}} T_0^z(R_n) \Big) \setminus \Big(\bigcup_{n\geq 0, z\in\mathbb{Z}} T_n^z(\Delta) \Big)$$

is a residual subset of M_0 satisfying

$$\Delta \cap \bigcup_{n\geq 0, z\in\mathbb{Z}} T_n^z(R) = \emptyset, \tag{4.10}$$

$$(x,y) \in R_0 \implies T_0^z(x,y) \in R_n \quad \text{for every } n \geq 0, z \in \mathbb{Z}. \tag{4.11}$$

We will prove that any point $((x,y),0) \in Y$ with $(x,y) \in R$ has a dense $\widetilde G$-orbit. Consider an open set $V \times \{n\}$ with $V \subset \mathbb{T}^2$ and $n \geq 0$, then there is a least $m \geq n$ such that $V \cap M_m \neq \emptyset$. Since $\mathcal{V}_l \cap M_0$ is not dense in M_0 for any $l \in \mathbb{N}$ and $T_0(x,y)$ is dense in M_0, there is $z_1 \in \mathbb{Z}$ such that

$$T_0^{z_1}(x,y) \notin \mathcal{V}_l \quad \text{for } l = 0, \dots, m.$$

This means that $((x,y),0) \in \operatorname{dom} \tilde g^m$ by (4.9). Equations (4.10) and (4.11) now yield

$$T_0^{z_1}(x,y) \in R_m \setminus \bigcup_{z\in\mathbb{Z}} T_m^z(\Delta),$$

and therefore

$$\tilde g^m \tilde f^{z_1}(x,y) \in (R_m \setminus \bigcup_{z\in\mathbb{Z}} T_m^z(\Delta)) \times \{m\}.$$

By the definition of R_m, the orbit

$$\bigcup_{z\in\mathbb{Z}} T_m^z(T_0^{z_1}(x,y))$$

is dense in M_m and disjoint from Δ, so, by (4.8), there is $z_2 \in \mathbb{Z}$ such that

$$(x',y') := \tilde f^{z_2} \tilde g^m \tilde f^{z_1}(x,y) \in (V \cap M_m) \times \{m\}.$$

We defined m as the least integer $\geq n$ satisfying $V \cap M_m \neq \emptyset$, so we have $V \cap M_l = \emptyset$ for every $n \leq l < m$. Hence $(x', y') \in \operatorname{dom}(\tilde{g}^{n-m})$ by (4.9) and Lemma 4.2(i), yielding

$$\tilde{g}^{n-m} \tilde{f}^{z_2} \tilde{g}^m \tilde{f}^{z_1}(x, y) \in V \times \{n\}.$$

We have proved that, for $(x, y) \in R$, the $\widetilde{\mathcal{G}}$-orbit of $((x, y), 0)$ meets every open set, so $\widetilde{\mathcal{G}} \curvearrowright Y$ is topologically transitive by Lemma 3.3. ∎

Lemma 4.4. *Let* $((x, y), m) \in \widetilde{Q}_n \times \{m\}$*, with* $m \leq n$*.Then the orbit* $\widetilde{\mathcal{G}}((x, y), m)$ *is contained in* $\widetilde{Q}_n \times \{0, \dots, n\}$*.*

Proof. We have $T_l(\widetilde{Q}_n) = \widetilde{Q}_n$ for every $0 \leq l \leq n$ by (4.5), so

$$\tilde{f}\Big(\bigcup_{m \leq n} \widetilde{Q}_n \times \{m\}\Big) \subset \bigcup_{m \leq n} \widetilde{Q}_n \times \{m\},$$

and similarly for $\tilde{f}^{-1}$. It is obvious from the definition of $\tilde{g}$ that

$$\tilde{g}^{-1}\Big(\bigcup_{m \leq n} \widetilde{Q}_n \times \{m\}\Big) \subset \bigcup_{m \leq n} \widetilde{Q}_n \times \{m\}, \quad \tilde{g}\Big(\widetilde{Q}_n \times \{m\}\Big) \subset \bigcup_{m \leq n} \widetilde{Q}_n \times \{m+1\}.$$

Hence, to finish the proof it is enough to show that $(\widetilde{Q}_n \times \{n\}) \cap \operatorname{dom} \tilde{g} = \emptyset$, but this follows from (4.6), (4.7) and (4.9). ∎

Corollary 4.5. *The finite* $\widetilde{\mathcal{G}}$*-orbits are dense in* Y*.*

The proof of the following statement is identical to that of Proposition 3.17.

Proposition 4.6. *There is a metric* d *on* Y *and a generating pseudo∗group* $\mathcal{S}$ *such that every map in* $\mathcal{S}$ *whose domain contains* $((0,0),0)$ *is an isometry. Hence,* $\widetilde{\mathcal{G}}$ *is not sensitive to initial conditions.*

4.4. A non-compact, topologically transitive affine foliation with a dense set of compact leaves which is not sensitive

We are now in position to prove Theorem 1.16 by constructing a suitable foliated space using the pseudogroup we have just defined.

Let Σ be a smooth surface of genus two divided into three smooth manifolds with boundary, Σ_0, Σ_α, and Σ_β, that overlap only on their boundaries. Let Σ_0 be a two-sphere with four open disks removed, and denote the boundary circles by S_α^-, S_α^+, S_β^- and S_β^+, with Σ_α attaching at S_α^- and S_α^+ (see Figure 2).

The main idea of the construction is as follows. We have constructed a pseudogroup $\widetilde{\mathcal{G}} \curvearrowright Y$ generated by two maps $\tilde{f}$ and $\tilde{g}$ satisfying suitable properties. In order to get a foliated manifold realizing this dynamics, the obvious candidate is to mimic a suspension foliation by taking products $\Sigma_\alpha \times \operatorname{dom} \tilde{f}$, $\Sigma_\beta \times \operatorname{dom} \tilde{g}$ and $\Sigma_0 \times Y$, and attaching each plaque in $\Sigma_\alpha \times \operatorname{dom} \tilde{f}$ so that we identify $(x, y) \in \Sigma_\alpha \times \operatorname{dom} \tilde{f}$ and $(x, y) \in \Sigma_0 \times \operatorname{dom} \tilde{f}$ if $x \in S_\alpha^-$ and we identify $(x, y) \in \Sigma_\alpha \times \operatorname{dom} \tilde{f}$ and $(x, \tilde{f}(y)) \in \Sigma_0 \times \operatorname{im} \tilde{f}$ if $x \in S_\alpha^+$. In this way, the plaques of $\Sigma_\alpha \times \operatorname{dom} \tilde{f}$ realize $\tilde{f}$ in the holonomy pseudogroup. Proceeding

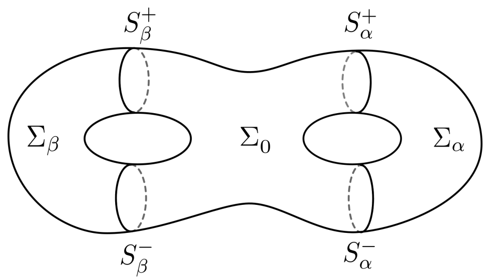

Figure 2. The surface Σ and its partition.

similarly, we realize $\tilde{g}$ using $\Sigma_\beta \times \operatorname{dom} \tilde{g}$. However, now there are singularities at the points

$$(x, y) \in (S_\alpha^- \times \partial \operatorname{dom} \tilde{f}) \sqcup (S_\alpha^+ \times \partial \operatorname{im} \tilde{f}),$$

and similarly for $\tilde{g}$, so the resulting space would not even be a manifold with boundary. Since we do not require a compact foliated manifold, we may get rid of the problematic points and consider $\mathfrak{Y}_0 := (\Sigma_0 \times Y) \setminus \mathfrak{C}$, where

$$\mathfrak{C} = \mathfrak{C}_\alpha^- \cup \mathfrak{C}_\alpha^+ \cup \mathfrak{C}_\beta^- \cup \mathfrak{C}_\beta^+,$$

with

$$\begin{aligned} \mathfrak{C}_\alpha^- &= S_\alpha^- \times \partial \operatorname{dom} \tilde{f}, \\ \mathfrak{C}_\alpha^+ &= S_\alpha^+ \times \partial \operatorname{im} \tilde{f}, \\ \mathfrak{C}_\beta^- &= S_\beta^- \times \partial \operatorname{dom} \tilde{g}, \\ \mathfrak{C}_\beta^+ &= S_\beta^+ \times \partial \operatorname{im} \tilde{g}. \end{aligned}$$

Recall that $\operatorname{dom} \tilde{f}$ is a dense open subset of Y and $\operatorname{dom} \tilde{g}$ is the complement of a disjoint union of open balls. Now $\mathfrak{Y}_0$ is a non-compact smooth manifold with boundary.

Now attach the boundaries of $\mathfrak{Y}_\alpha := \Sigma_\alpha \times \operatorname{dom} \tilde{f}$ and $\mathfrak{Y}_\beta := \Sigma_\beta \times \operatorname{dom} \tilde{g}$ to $\mathfrak{Y}_0$ using the identifications

$$\begin{aligned} (s, y) &\sim (s, y), & (s, y) &\in S_\alpha^- \times \operatorname{dom} \tilde{f}, \\ (s, y) &\sim (s, \tilde{f}(y)), & (s, y) &\in S_\alpha^+ \times \operatorname{dom} \tilde{f}, \\ (s, y) &\sim (s, y), & (s, y) &\in S_\beta^- \times \operatorname{dom} \tilde{g}, \\ (s, y) &\sim (s, \tilde{g}(y)), & (s, y) &\in S_\beta^+ \times \operatorname{dom} \tilde{g}. \end{aligned}$$

Denote by $\mathfrak{Y}$ the resulting space. The product foliated structures on $\mathfrak{Y}_0$, $\mathfrak{Y}_\alpha$, and $\mathfrak{Y}_\beta$ with leaves $\{y\} \times \Sigma_i$ (where $i = 0, \alpha$, or β, respectively) descend to $\mathfrak{Y}$. It is an elementary matter to check that, essentially by construction, $\mathfrak{Y}$ is C^∞ and its holonomy pseudogroup is equivalent to $\widetilde{\mathscr{G}} \curvearrowright Y$. Indeed, we can choose a point x in the interior of Σ_0, and then $\{x_0\} \times Y$ is a total transversal of $\mathfrak{Y}$. Since $\mathfrak{Y}_0$ has a product foliation structure, it has trivial dynamics; by gluing $\mathfrak{Y}_\alpha$ and $\mathfrak{Y}_\beta$, we realize $\tilde{f}$ and $\tilde{g}$, respectively, so the holonomy pseudogroup is equivalent to $\widetilde{\mathscr{G}} \curvearrowright Y$.

Corollary 4.7. *The foliated space $\mathfrak{Y}$ is C^∞, transversally affine and topologically transitive, but not sensitive to initial conditions.*

The space $\mathfrak{Y}$ is also a smooth manifold with boundary and, noticing that we removed $\mathfrak{C}$ from $(\Sigma_0 \times Y) \setminus \mathfrak{C}$ and keeping track of which points in the boundary of $\mathfrak{Y}_0$ we glued to $\mathfrak{Y}_\alpha$ and $\mathfrak{Y}_\beta$, we see that the boundary of $\mathfrak{Y}$ is

$$\partial \mathfrak{Y} = S_\beta^- \times (Y \setminus \overline{\operatorname{dom} \tilde{g}}) \sqcup S_\beta^+ \times (Y \setminus \overline{\operatorname{im} \tilde{g}}). \tag{4.12}$$

Recall that $\operatorname{dom} \tilde{f}$ is an open and dense subset so the terms containing $(Y \setminus \overline{\operatorname{dom} \tilde{g}})$ are empty; the same argument applies to $\operatorname{im} \tilde{f}$.

Lemma 4.8. *The foliated space $\mathfrak{Y}$ has a dense set of leaves that are compact manifolds, possibly with boundary.*

Proof. By Lemma 4.4, a leaf L corresponding to a point $((x, y), m)$ with $(x, y) \in Q_n$ corresponds to a finite orbit of $\tilde{\mathscr{G}}$. Consider the inverse image of L by the quotient map $\mathfrak{Y}_0 \sqcup (\operatorname{dom} \tilde{f} \times \Sigma_\alpha) \sqcup (\operatorname{dom} \tilde{g} \times \Sigma_\beta) \to \mathfrak{Y}$. Since it has a finite orbit, the inverse image of L only intersects finitely many plaques in $\mathfrak{Y}_0$, $\Sigma_\alpha \times \operatorname{dom} \tilde{f}$, and $\Sigma_\beta \times \operatorname{dom} \tilde{g}$. The plaques in $\Sigma_\alpha \times \operatorname{dom} \tilde{f}$, and $\Sigma_\beta \times \operatorname{dom} \tilde{g}$ are all compact, being copies of Σ_α and Σ_β, respectively. By Lemma 4.2 and (4.6)–(4.9),

$$Q_n \cap \bigcup_{l \geq 0} \partial \mathcal{V}_l = Q_n \cap \Delta = \emptyset,$$

so every $\tilde{\mathscr{G}}$-orbit of a point $((x, y), m)$ is disjoint from

$$\partial \operatorname{dom} \tilde{f} \cup \partial \operatorname{im} \tilde{f} \sqcup \partial \operatorname{dom} \tilde{g} \cup \partial \operatorname{im} \tilde{g}.$$

Looking back at the construction of $\mathfrak{Y}_0$ at the beginning of this section, this implies that the plaques in $\mathfrak{Y}_0$ that project to L are all compact, being copies of Σ_0; L is then the quotient of a finite union of compact plaques, hence compact. ∎

At this point, we have constructed a smooth and transitive foliated manifold with boundary $\mathfrak{Y}$ that has a dense set of compact leaves but is not sensitive to initial conditions. It only remains to modify it in order to get rid of the boundary: Take two copies $\mathfrak{Y}^-$ and $\mathfrak{Y}^+$, and let $\mathfrak{Y}^\pm \cong \mathfrak{Y}^- \cup \mathfrak{Y}^+ / \sim$ be the quotient space obtained by identifying their boundaries. This is sometimes called the "double" of a manifold with boundary, and it is known to admit a smooth structure making it a manifold without boundary. In our case, where we constructed our space $\mathfrak{Y}$ by gluing product manifolds endowed with product foliations, it is elementary to check that the foliated structure descends to the quotient and $\mathfrak{Y}^\pm$ is now a smooth foliated manifold without boundary.

Lemma 4.9. *The set of compact leaves is dense in $\mathfrak{Y}^\pm$.*

Proof. Denote by $\pi \colon \mathfrak{Y}^- \sqcup \mathfrak{Y}^+ \to \mathfrak{Y}^\pm$ the quotient map. Let U be a open subset of $\mathfrak{Y}^\pm$, which without loss of generality we may assume that is contained in $\pi(\mathfrak{Y}^- \setminus \partial \mathfrak{Y}^-)$. Now, by Lemma 4.8, there is a compact leaf L^- in $\mathfrak{Y}^-$ intersecting $\pi^{-1}(U)$. If L^- has empty boundary, then $L^- \cap \partial \mathfrak{Y}^- = \emptyset$ and $\pi(L^-)$ is a compact leaf in the quotient space $\mathfrak{Y}^{\pm}$ intersecting U. If L^- has non-empty boundary, then it is contained in the leaf $L = \pi(L^- \sqcup L^+)$, where L^+ is the leaf of $\mathfrak{Y}^+$ that corresponds to L^-. Then L intersects U and is a quotient of the compact space $L^- \sqcup L^+$, hence compact. ∎

Lemma 4.10. *$\mathfrak{Y}^{\pm}$ is topologically transitive but not sensitive to initial conditions.*

Proof. Let us inspect the holonomy pseudogroup of $\mathfrak{Y}^{\pm}$. Since $Y \cong \{x_0\} \times Y$ was a total transversal for $\mathfrak{Y}$, for $\mathfrak{Y}\pm$ we can take $Y^{\pm} \equiv Y^- \cup Y^+$, where Y^- and Y^+ denote the image by π of copies of $\{x_0\}$ in $\mathfrak{Y}^-$ and $\mathfrak{Y}^+$, respectively. On each of Y^- and Y^+ we have copies of $\widetilde{\mathscr{G}}$, denoted by $\widetilde{\mathscr{G}}^-$ and $\widetilde{\mathscr{G}}^+$. The maps between Y^- and Y^+ come from the gluing of the boundaries of $\mathfrak{Y}^-$ and $\mathfrak{Y}^+$. Recall that $Y = \mathbb{T}^2 \times \mathbb{N}$. Looking at (4.12), we see that, if we denote the points of Y^- and Y^+ by $(y,-)$ and $(y,+)$, respectively, we get a new map h defined by

$$
\begin{aligned}
\operatorname{dom} h &= \{(y,-) \mid y \in Y^- \setminus (\overline{\operatorname{dom} \tilde{g}^-} \cap \overline{\operatorname{im} \tilde{g}^-})\}, \\
\operatorname{im} h &= \{(y,+) \mid y \in Y^+ \setminus (\overline{\operatorname{dom} \tilde{g}^+} \cap \overline{\operatorname{im} \tilde{g}^+})\}, \\
h(y,-) &= (y,+),
\end{aligned}
$$

where $\tilde{g}^-$ and $\tilde{g}^+$ denote the respective copies of $\tilde{g}$ acting on Y^- and Y^+.

Denote by $\widetilde{G}^{\pm}$ the pseudogroup acting on $Y^{\pm}$ generated by $\widetilde{G}^-$, $\widetilde{G}^+$, and h, which is then the holonomy pseudogroup of $\mathfrak{Y}^{\pm}$. Let us prove that is topologically transitive. By Corollary 4.7, there is a dense orbit $\widetilde{\mathfrak{G}}(y)$ in Y. Then the orbit $\widetilde{\mathfrak{G}}^{\pm}(y,-)$ contains $\widetilde{\mathfrak{G}}^-(y,-)$, which is dense in Y^-, and similarly $\widetilde{\mathfrak{G}}^{\pm}(y,+)$ is dense in Y^+. Since $\operatorname{dom} \tilde{g}$ was a disjoint union of open balls in Y, $\widetilde{\mathfrak{G}}^-(y,-)$ meets $\operatorname{dom} h$, and we get

$$
\widetilde{\mathfrak{G}}^{\pm}(y,-) = \widetilde{\mathfrak{G}}^-(y,-) \cup \widetilde{\mathfrak{G}}^+(y,+),
$$

which is dense in $Y^{\pm}$.

Finally, let us prove that $\widetilde{\mathscr{G}}^{\pm}$ is not sensitive to initial conditions. By Proposition 4.6, there is a metric d on Y, a point $y \in Y$ and a generating pseudo$*$group S for $\widetilde{\mathscr{G}}$ such that every map of S whose domain contains y is an isometry. Let $d^{\pm}$ be metric on $Y^{\pm}$ such that points from Y^- and Y^+ are at infinite distance of each other, and the restrictions of $d^{\pm}$ to Y^- and Y^+ coincide with d. Finally, let $S^{\pm} = S^- \cup S^+ \cup \{h\}$, where S^- and S^+ are copies of S acting on Y^- and Y^+, respectively. It is immediate that $S^{\pm}$ generates $\widetilde{\mathscr{G}}^{\pm}$ and every map in $S^{\pm}$ whose domain contains $(y,-)$ is an isometry with respect to $d^{\pm}$, and the result follows. ∎

The pseudogroup $\mathscr{G} \curvearrowright Y$ was affine, and it is easily checked that so is $\widetilde{\mathscr{G}}^{\pm} \curvearrowright Y^{\pm}$, yielding that $\mathfrak{Y}^{\pm}$ is a transversally affine foliation. This completes the proof of Theorem 1.16.

Acknowledgments. The author would like to thank the anonymous referee for their thoughtful suggestions, which helped improved the paper.

Funding. This work was partially supported by grants PID2020-114474GB-I00 (AEI-FEDER, UE), ED431C 2019/10 (Xunta de Galicia, FEDER), and by the Research Promotion Program for Acquiring Grants in-Aid for Scientific Research (KAKENHI) by Ritsumeikan University.

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Received June 13, 2023; revised September 20, 2024.

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