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4-manifolds with boundary and fundamental group $\mathbb{Z}$

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Abstract. We classify topological 4-manifolds with boundary and fundamental group $\mathbb{Z}$, under some assumptions on the boundary. We apply this to classify surfaces in simply-connected 4-manifolds with S^3 boundary, where the fundamental group of the surface complement is $\mathbb{Z}$. We then compare these homeomorphism classifications with the smooth setting. For manifolds, we show that every Hermitian form over $\mathbb{Z}[t^{\pm 1}]$ arises as the equivariant intersection form of a pair of exotic smooth 4-manifolds with boundary and fundamental group $\mathbb{Z}$. For surfaces we have a similar result, and in particular we show that every 2-handlebody with S^3 boundary contains a pair of exotic discs.

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Introduction

In what follows a 4-manifold is understood to mean a compact, connected, oriented, topological 4-manifold. Freedman classified closed 4-manifolds with trivial fundamental group up to orientation-preserving homeomorphism. Other groups π for which

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classifications of closed 4-manifolds with fundamental group π are known, include $\pi \cong \mathbb{Z}$ (see [42, 85, 91]), π a finite cyclic group [52], and π a solvable Baumslag–Solitar group [53]. Complete classification results for manifolds with boundary essentially only include the simply-connected case [11, 12]; see also [83].

This paper classifies 4-manifolds with boundary and fundamental group $\mathbb{Z}$, under some extra assumptions on the boundary. We give an informal statement now. Fix a closed 3-manifold Y, an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$, a non-degenerate Hermitian form λ over $\mathbb{Z}[t^{\pm 1}]$, and an additional piece of data specifying how the Alexander module of Y interacts with λ. Then up to homeomorphism fixing Y, there exists a unique 4-manifold M filling Y inducing the specified data. Uniqueness is a consequence of [27, Theorem 1.10]. Existence is the main contribution of this paper, Theorem 2.4. We give a similar non-relative classification of such M in Theorem 2.8.

A feature of our classification, which we shall demonstrate in Section 8, is the existence of arbitrarily large sets of homeomorphism classes of such 4-manifolds, all of which have the same boundary Y and the same form λ. Recently, this was extended [20,23], using the results of this paper, to produce infinite sets of homeomorphism classes with this property. Thus this paper leads to the first classification of infinite families of orientable 4-manifolds, all with the same, non-trivial, equivariant intersection form. This can be compared with [14,61] and [70, Theorem 1.2], which produced infinite families of manifolds homotopy equivalent to $\mathbb{R}P^4 \# \mathbb{R}P^4$ and $L(p,q) \times S^1$ respectively; note that in both cases $\pi_2 = 0$ and so there is no intersection form.

We apply our results to study compact, oriented, locally flat, embedded surfaces in simply-connected 4-manifolds where the fundamental group of the exterior is infinite cyclic; we call these $\mathbb{Z}$-*surfaces*. The classification of closed surfaces in 4-manifolds whose exterior is simply-connected was carried out by Boyer [12]; see also [86]. Literature on the classification of discs in D^4 where the complement has fixed fundamental group includes [19,26,45]. For surfaces in more general 4-manifolds, [27] gave necessary and sufficient conditions for a pair of $\mathbb{Z}$-surfaces to be equivalent. In this work, for a 4-manifold N with boundary S^3 and a knot $K \subset S^3$, we classify $\mathbb{Z}$-surfaces in N with boundary K in terms of the equivariant intersection form of the surface exterior; see Theorem 1.7. An application to H-sliceness can be found in Corollary 1.9, while Theorem 1.11 classifies closed $\mathbb{Z}$-surfaces.

Finally, we compare these homeomorphism classifications with the smooth setting. We demonstrate that for every Hermitian form λ over $\mathbb{Z}[t^{\pm 1}]$ there are pairs of smooth 4-manifolds with boundary, $\pi_1 \cong \mathbb{Z}$, and equivariant intersection form λ which are homeomorphic rel. boundary but not diffeomorphic; see Theorem 1.15. We also show in Theorem 1.17 that for every Hermitian form λ satisfying conditions which are conjecturally necessary, there is a smooth 4-manifold N with S^3 boundary containing a pair of smoothly embedded $\mathbb{Z}$-surfaces whose exteriors have equivariant intersection form λ and which are topologically but not smoothly isotopic rel. boundary.

1. Statement of results

Before stating our main result, we introduce some terminology. Our 3-manifolds Y will always be oriented and will generally come equipped with an epimorphism, which we denote by $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$.

Definition 1.1. An oriented 4-manifold M together with an identification $\pi_1(M) \cong \mathbb{Z}$ is said to be a $\mathbb{Z}$*-manifold* if the inclusion induced map $\pi_1(\partial M) \to \pi_1(M)$ is surjective.

When we say that a $\mathbb{Z}$-manifold M has boundary (Y, φ), we mean that M comes equipped with a homeomorphism $\partial M \xrightarrow{\cong} Y$ such that the composition

$$\pi_1(Y) \twoheadrightarrow \pi_1(M) \xrightarrow{\cong} \mathbb{Z}$$

agrees with φ. We will always assume that the Alexander module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$-torsion; recall that the Alexander module is the first homology group of the infinite cyclic cover $Y^\infty \to Y$ corresponding to $\ker(\varphi)$. The action of the deck transformation group $\mathbb{Z} = \langle t \rangle$ makes the first homology into a $\mathbb{Z}[t^{\pm 1}]$-module.

1.1. The classification result

Our goal is to classify $\mathbb{Z}$-manifolds M whose boundary $\partial M \cong Y$ has $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ torsion, up to orientation-preserving homeomorphism. The isometry class of the *equivariant intersection form* λ_M on $H_2(M;\mathbb{Z}[t^{\pm 1}])$ is an invariant of such M (this definition is recalled in Section 3.1) and so, to classify such M, it is natural to first fix a non-degenerate Hermitian form λ over $\mathbb{Z}[t^{\pm 1}]$, and then to classify $\mathbb{Z}$-manifolds M with boundary $\partial M \cong Y$, and equivariant intersection form λ. The fact that λ is non-degenerate implies that the Alexander module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is torsion.

For such a 4-manifold M, the equivariant intersection form λ_M on $H_2(M;\mathbb{Z}[t^{\pm 1}])$ *presents* the *Blanchfield form* on $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ (see Section 3.2)

$$\mathrm{Bl}_Y\colon H_1\big(Y;\mathbb{Z}[t^{\pm 1}]\big) \times H_1\big(Y;\mathbb{Z}[t^{\pm 1}]\big) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}].$$

We make this algebraic notion precise next. If $\lambda\colon H \times H \to \mathbb{Z}[t^{\pm 1}]$ is a non-degenerate Hermitian form on a finitely generated free $\mathbb{Z}[t^{\pm 1}]$-module (for short, a *form*), then we write $\widehat{\lambda}\colon H \to H^*$ for the linear map $z \mapsto \lambda(-, z)$, and there is a short exact sequence

$$0 \to H \xrightarrow{\widehat{\lambda}} H^* \to \operatorname{coker}(\widehat{\lambda}) \to 0.$$

Such a presentation induces a *boundary linking form* $\partial\lambda$ on $\operatorname{coker}(\widehat{\lambda})$ in the following manner. For $[x] \in \operatorname{coker}(\widehat{\lambda})$ with $x \in H^*$, since $\operatorname{coker}(\widehat{\lambda})$ is $\mathbb{Z}[t^{\pm 1}]$-torsion there exist elements $z \in H$ and $p \in \mathbb{Z}[t^{\pm 1}] \setminus \{0\}$ such that $\lambda(-,z) = px \in H^*$. Then for $[x],[y] \in \operatorname{coker}(\widehat{\lambda})$ with $x, y \in H^*$, we define

$$\partial\lambda\big([x],[y]\big) := \frac{y(z)}{p} \in \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}].$$

One can check that $\partial\lambda$ is independent of the choices of p and z.

Definition 1.2. For T a torsion $\mathbb{Z}[t^{\pm 1}]$-module with a linking form $\ell\colon T \times T \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$, a non-degenerate Hermitian form (H,λ) *presents* (T,ℓ) if there is an isomorphism $h\colon \operatorname{coker}(\widehat{\lambda}) \to T$ such that $\ell(h(x),h(y)) = \partial\lambda(x,y)$. Such an isomorphism h is called an *isometry* of the forms, the set of isometries is denoted $\operatorname{Iso}(\partial\lambda,\ell)$. If (H,λ) presents $(H_1(Y;\mathbb{Z}[t^{\pm 1}]), -\operatorname{Bl}_Y)$ then we say (H,λ) *presents* Y.

This notion of a presentation is well known (see e.g. [28, 81]), and appeared in the classification of simply-connected 4-manifolds with boundary in [11, 12] and in [27] for 4-manifolds with $\pi_1 \cong \mathbb{Z}$. See also [10, 37]. Presentations capture the geometric relationship between the linking form of a 3-manifold and the intersection form of a 4-manifold filling. To see why the form $(H_2(M;\mathbb{Z}[t^{\pm 1}]), \lambda_M)$ presents ∂M, one first observes that the long exact sequence of the pair $(M, \partial M)$ with coefficients in $\mathbb{Z}[t^{\pm 1}]$ reduces to the short exact sequence

$$0 \to H_2\big(M;\mathbb{Z}[t^{\pm 1}]\big) \to H_2\big(M,\partial M;\mathbb{Z}[t^{\pm 1}]\big) \to H_1\big(\partial M;\mathbb{Z}[t^{\pm 1}]\big) \to 0,$$

where $H_2(M;\mathbb{Z}[t^{\pm 1}])$ and $H_2(M,\partial M;\mathbb{Z}[t^{\pm 1}])$ are finitely generated free $\mathbb{Z}[t^{\pm 1}]$-modules [27, Lemma 3.2]. The left term of the short exact sequence supports the equivariant intersection form λ_M and the right supports $\operatorname{Bl}_{\partial M}$. As explained in detail in [27, Remark 3.3], some algebraic topology gives the following commutative diagram of short exact sequences, where the isomorphism D_M is defined so that the right-most square commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2\big(M;\mathbb{Z}[t^{\pm 1}]\big) & \xrightarrow{\widehat{\lambda}_M} & H_2\big(M;\mathbb{Z}[t^{\pm 1}]\big)^* & \longrightarrow & \operatorname{coker}(\widehat{\lambda}_M) & \longrightarrow & 0 \\
 & & =\big\downarrow \mathrm{id} & & \cong\big\downarrow \mathrm{ev}^{-1}\circ \mathrm{PD} & & \cong\big\downarrow D_M & & \\
0 & \longrightarrow & H_2\big(M;\mathbb{Z}[t^{\pm 1}]\big) & \longrightarrow & H_2\big(M,\partial M;\mathbb{Z}[t^{\pm 1}]\big) & \longrightarrow & H_1\big(\partial M;\mathbb{Z}[t^{\pm 1}]\big) & \longrightarrow & 0.
\end{array}$$

It then follows that $(H_2(M;\mathbb{Z}[t^{\pm 1}]), \lambda_M)$ presents ∂M, where the isometry $\partial\lambda_M \cong -\operatorname{Bl}_{\partial M}$ is given by D_M. For details, see [27, Proposition 3.5].

Thus to classify the $\mathbb{Z}$-manifolds M with boundary $\partial M \cong Y$, it suffices to consider forms (H,λ) which present Y. In Section 2 we use D_M to define an additional *automorphism invariant*

$$b_M \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda).$$

Here, as we define precisely in equation (2.1) below, an isometry $F \in \operatorname{Aut}(\lambda)$ induces an isometry ∂F of $\partial\lambda$, and the action on $h \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)$ is then by

$$F \cdot h = h \circ \partial F^{-1}.$$

Additionally, recall that a Hermitian form (H, λ) is *even* if $\lambda(x,x) = q(x) + \overline{q(x)}$ for some $\mathbb{Z}[t^{\pm 1}]$-module homomorphism $q\colon H \to \mathbb{Z}[t^{\pm 1}]$ and is *odd* otherwise. Our first classification now reads as follows.

Theorem 1.3. *Fix the following data:*

(1) *a closed* 3*-manifold* Y*;*

(2) *an epimorphism* $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ *with respect to which the Alexander module of* Y *is torsion;*

(3) *a non-degenerate Hermitian form* $\lambda\colon H \times H \to \mathbb{Z}[t^{\pm 1}]$ *which presents* Y*;*

(4) *if* λ *is odd,* $k \in \mathbb{Z}_2$*;*

(5) *a class* $b \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$.

Up to homeomorphism rel. boundary, there exists a unique $\mathbb{Z}$*-manifold* M *with boundary* (Y, φ)*, equivariant intersection form* λ*, automorphism invariant* b *and, in the odd case, Kirby–Siebenmann invariant* k.

Here two 4-manifolds M_0 and M_1 with boundary Y are *homeomorphic rel. boundary* if there exists a homeomorphism $M_0 \xrightarrow{\cong} M_1$ such that the restriction composed with the given parametrisations of the boundary, $Y \cong \partial M_0 \xrightarrow{\cong} \partial M_1 \cong Y$ is the identity on Y. The uniqueness part of the theorem (which follows from [27]) can be thought of as answering whether or not a given pair of parametrisations $Y \cong \partial M_i$ extend to a homeomorphism $M_0 \cong M_1$. We refer to Remark 2.9 for a guide to applying the uniqueness statement of Theorem 1.3. We give the proof of Theorem 1.3 (modulo our main technical theorem) in Section 2.

Remark 1.4. We collect a couple of further remarks about this result.

(i) The *automorphism invariant* that distinguishes $\mathbb{Z}$-manifolds with the same equivariant form is non-trivial to calculate in practice, as its definition typically involves choosing identifications of the boundary 3-manifolds; see Section 2.

(ii) Theorem 1.3 should be thought of as an extension of the work of Boyer [11, 12] that classifies simply-connected 4-manifolds with boundary and fixed intersection form and an extension of the classification of closed 4-manifolds with $\pi_1 = \mathbb{Z}$ [42,85]. Boyer's main statements are formulated using presentations instead of isometries of linking forms, but both approaches can be shown to agree when the 3-manifold is a rational homology sphere [12, Corollary E]. By way of analogy, rational homology 3-spheres are to 1-connected 4-manifolds with boundary as pairs (Y, φ) with torsion Alexander module are to $\mathbb{Z}$-manifolds.

(iii) For (Y, φ) as above, it is implicit in Theorem 1.3 and in [27] that if M_0 and M_1 are spin 4-manifolds with $\pi_1(M_i) \cong \mathbb{Z}$, boundary homeomorphic to (Y, φ), isometric equivariant intersection form, and the same automorphism invariant, then their Kirby–Siebenmann invariants agree. The argument is given in Remark 2.5 below, whereas Section 5.7 shows that the assumption on the automorphism invariants cannot be dropped. We refer to [11, Proposition 4.1 (vi)] for the analogous fact in the simply-connected setting.

Example 1.5. We will show in Proposition 8.5 that there are examples of pairs (Y, φ) for which the set of 4-manifolds with fixed boundary Y and fixed (even) equivariant intersection form, up to homeomorphism rel. boundary, can have arbitrarily large cardinality (in the recent [20, 23] examples with infinite cardinality were obtained). Details are given in Section 8, but we note that the underlying algebra is similar to that which was used in [22] and [21] to construct closed manifolds of dimension $4k \geq 8$ with non-trivial homotopy stable classes. This arbitrarily large phenomenon also exists for simply-connected 4-manifolds bounding rational homology spheres, which can be deduced from Boyer's work [11, 12] with a similar proof. On the other hand in the simply-connected setting there can only ever be finite such families.

In Theorem 1.3, we fixed a parametrisation of the boundary. By changing the parametrisation by a homeomorphism of Y that intertwines φ, we can change the invariant $b \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$ by post-composition with the induced automorphism of $-\operatorname{Bl}_Y$. This leads to an absolute (i.e. non-rel. boundary) classification analogous to Theorem 1.3, which we will formalise in Theorem 2.8. For now we highlight the following example, which contrasts with Example 1.5.

Example 1.6. If $Y \cong \Sigma_g \times S^1$ and $\varphi\colon \pi_1(\Sigma_g \times S^1) \to \pi_1(S^1) \to \mathbb{Z}$ is induced by projection onto the second factor, then for a fixed non-degenerate Hermitian form λ that presents Y, if λ is even there is a unique homeomorphism class of 4-manifolds with $\pi_1 \cong \mathbb{Z}$, boundary Y, and equivariant intersection form λ, and if λ is odd there are exactly two such homeomorphism classes. Here we allow homeomorphisms to act non-trivially on the boundary. The key input is that every automorphism of Bl_Y can be realised by a homeomorphism of Y that intertwines φ ([27, Proposition 5.6]). Therefore, given two 4-manifolds for which the rest of the data coincide, by re-parametrising Y we can arrange for the automorphism invariants to agree.

In Section 2 we describe the automorphism invariant b from Theorem 1.3, give the statement of our main technical theorem on realisation of the invariants by $\mathbb{Z}$-manifolds, and explain how Theorem 1.3 implies a non-rel. boundary version of the result. But first, in Sections 1.2 and 1.3, we discuss some applications.

1.2. Classification of $\mathbb{Z}$-surfaces in simply-connected 4-manifolds with S^3 boundary

For a fixed simply-connected 4-manifold N with boundary S^3 and a fixed knot $K \subset \partial N = S^3$, we call two locally flat embedded compact surfaces $\Sigma, \Sigma' \subset N$ with boundary $K \subset S^3$ *equivalent rel. boundary* if there is an orientation-preserving homeomorphism $(N, \Sigma) \cong (N, \Sigma')$ that is pointwise the identity on $S^3 \cong \partial N$. We are interested in classifying the $\mathbb{Z}$-surfaces in N with boundary K up to equivalence rel. boundary.

As for manifolds, first we inventory some invariants of $\mathbb{Z}$-surfaces. The genus of Σ and the equivariant intersection form λ_{N_Σ} on $H_2(N_\Sigma; \mathbb{Z}[t^{\pm 1}])$ are invariants of such a surface Σ, where N_Σ denotes the exterior $N \setminus \nu(\Sigma)$. Write $E_K := S^3 \setminus \nu(K)$ for the exterior of K and recall that the boundary of N_Σ has a natural identification

$$\partial N_\Sigma \cong E_K \cup_\partial (\Sigma_{g,1} \times S^1) =: M_{K,g}.$$

As discussed previously in Section 1.1, there is a relationship between the equivariant intersection form λ_{N_Σ} on $H_2(N_\Sigma; \mathbb{Z}[t^{\pm 1}])$ and the Blanchfield form $\mathrm{Bl}_{M_{K,g}}$ on $H_1(M_{K,g}; \mathbb{Z}[t^{\pm 1}])$: the Hermitian form $(H_2(N_\Sigma; \mathbb{Z}[t^{\pm 1}]), \lambda_{N_\Sigma})$ presents $M_{K,g}$.

There is one additional necessary condition for a given form (H, λ) to be isometric to the intersection pairing $(H_2(N_\Sigma; \mathbb{Z}[t^{\pm 1}]), \lambda_{N_\Sigma})$ for some surface Σ. Observe that we can reglue the neighbourhood of Σ to N_Σ to recover N. This is reflected in the intersection form, as follows. We write $\lambda(1) := \lambda \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}_\varepsilon$, where $\mathbb{Z}_\varepsilon$ denotes $\mathbb{Z}$ with the trivial $\mathbb{Z}[t^{\pm 1}]$-module structure. If W is a $\mathbb{Z}$-manifold, then $\lambda_W(1) \cong Q_W$, where Q_W denotes the standard intersection form of W; see e.g. [27, Lemma 5.10]. Therefore, if $\lambda \cong \lambda_{N_\Sigma}$, then we have the isometries

$$\lambda(1) \cong \lambda_{N_\Sigma}(1) = Q_{N_\Sigma} \cong Q_N \oplus (0)^{\oplus 2g},$$

where the last isometry follows from a Mayer–Vietoris argument. The following theorem (which is stated slightly more generally in Theorem 6.2 below) shows that these invariants, with these two necessary conditions, are in fact also sufficient once an automorphism invariant is fixed.

Theorem 1.7. *Fix the following data:*

(1) *a simply-connected* 4*-manifold N with boundary S^3;*

(2) *an oriented knot* $K \subset S^3$*;*

(3) *an integer* $g \in \mathbb{Z}_{\geq 0}$*;*

(4) *a non-degenerate Hermitian form* (H, λ) *over* $\mathbb{Z}[t^{\pm 1}]$ *which presents* $M_{K,g}$ *and satisfies* $\lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}$*;*

(5) *a class* $b \in \operatorname{Aut}(\mathrm{Bl}_K)/\operatorname{Aut}(\lambda)$.

Up to equivalence rel. boundary, there exists a unique genus g $\mathbb{Z}$*-surface* $\Sigma \subset N$ *with boundary* K *whose exterior* N_Σ *has equivariant intersection form* λ *and automorphism invariant* b.

The action of the group $\operatorname{Aut}(\lambda)$ on the set $\operatorname{Aut}(\mathrm{Bl}_K)$ arises by restricting the action of $\operatorname{Aut}(\lambda)$ on $\operatorname{Aut}(\partial\lambda) \cong \operatorname{Aut}(\mathrm{Bl}_{M_{K,g}}) \cong \operatorname{Aut}(\mathrm{Bl}_K) \oplus \mathrm{Sp}_{2g}(\mathbb{Z})$ to the first summand. Here the (non-canonical) isomorphism $\operatorname{Aut}(\partial\lambda) \cong \operatorname{Aut}(\mathrm{Bl}_{M_{K,g}})$ holds because the form λ presents $M_{K,g}$, while the isomorphism $\operatorname{Aut}(\mathrm{Bl}_{M_{K,g}}) \cong \operatorname{Aut}(\mathrm{Bl}_K) \oplus \mathrm{Sp}_{2g}(\mathbb{Z})$ is a consequence of [27, Propositions 5.6 and 5.7].

Again, the construction is explicit. The idea is that the set of topological surfaces (up to equivalence rel. boundary) is in bijection with the set of surface complements (up to homeomorphism rel. boundary). So this theorem can be recovered from Theorem 1.3 by taking Y to be $M_{K,g}$. We detail this in Section 6 where we state the outcome as a bijection between $\operatorname{Aut}(\mathrm{Bl}_K)/\operatorname{Aut}(\lambda)$ and the set of rel. boundary isotopy classes of $\mathbb{Z}$-surfaces $\Sigma \subset N$ with boundary K and equivariant intersection form $\lambda_{N_\Sigma} \cong \lambda$. Finally, we note that when $N = D^4$, equivalence rel. boundary can be upgraded to isotopy rel. boundary via the Alexander trick. See also [77, Theorem F] for more cases when equivalence can be upgraded to isotopy.

Remark 1.8. Previous classification results of locally flat discs in 4-manifolds include $\mathbb{Z}$-discs in D^4 ([26, 42]), $BS(1,2)$-discs in D^4 ([26, 45]) and G-discs in D^4 (under some assumptions on the group G) ([19,45]). In the latter case it is not known whether there are groups satisfying the assumptions other than $\mathbb{Z}$ and $BS(1,2)$. Our result is the first classification of discs with non-simplyconnected exteriors in 4-manifolds other than D^4.

Before continuing with $\mathbb{Z}$-surfaces, we mention an application of Theorem 1.7 to H-sliceness. A knot K in ∂N is said to be (topologically) *H-slice* if K bounds a locally flat, embedded disc D in N that represents the trivial class in $H_2(N, \partial N)$. The study of H-slice knots has garnered some interest recently because of its potential applications towards producing small closed exotic 4-manifolds [24, 60, 69, 72–74]. Since $\mathbb{Z}$-slice knots are H-slice (see e.g. [27, Lemma 5.1]), Theorem 1.7 therefore gives a new criterion for topological H-sliceness. Our results also apply in higher genus. When $N = D^4$, this is reminiscent of the combination of [36, Theorems 2 and 3] and [9, Theorem 1.1] (and for $g = 0$ it is Freedman's theorem that Alexander polynomial one knots bound $\mathbb{Z}$-discs [41, 42]). In connected sums of copies of $\mathbb{C}P^2$, this is closely related to [69, Theorem 1.3]. Compare also [37, Theorem 1.10], which applies in connected sums of copies of $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ and $S^2 \times S^2$.

Corollary 1.9. *Let N be a simply-connected 4-manifold with boundary S^3 and let $K \subset S^3$ be a knot. If $\mathrm{Bl}_{M_{K,g}}$ is presented by a non-degenerate Hermitian matrix $A(t)$ such that $A(1)$ is congruent to $Q_N \oplus (0)^{\oplus 2g}$, then K bounds a genus g $\mathbb{Z}$-surface in N. In particular, when $g = 0$, K is H-slice in N.*

We study $\mathbb{Z}$-surfaces up to equivalence (instead of equivalence rel. boundary). Note that here an additional technical requirement is needed on the knot exterior $E_K := S^3 \setminus \nu(K)$.

Theorem 1.10. *Let K be a knot in S^3 such that every isometry of Bl_K is realised by an orientation-preserving homeomorphism $E_K \to E_K$. If a non-degenerate Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ presents $M_{K,g}$ and satisfies*

$$\lambda(1) \cong Q_N \oplus (0)^{\oplus 2g},$$

then up to equivalence, there exists a unique genus g surface $\Sigma \subset N$ with boundary K and whose exterior has equivariant intersection form λ.

The classification of closed $\mathbb{Z}$-surfaces then follows from Theorem 1.10. To state the result, given a closed simply-connected 4-manifold X, we use X_Σ to denote the exterior of a surface $\Sigma \subset X$ and $N := X \setminus \mathring{D}^4$ for the manifold obtained by puncturing X. The details are presented in Section 6.3. The idea behind the proof is that closed surfaces are in bijective correspondence, with surfaces with boundary U, so we can apply Theorem 1.10.

Theorem 1.11. *Let X be a closed simply-connected 4-manifold. If a non-degenerate Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ presents $\Sigma_g \times S^1$ and satisfies*

$$\lambda(1) \cong Q_X \oplus (0)^{\oplus 2g},$$

then there exists a unique (up to equivalence) genus g surface $\Sigma \subset X$ whose exterior has equivariant intersection form λ.

Note that the boundary 3-manifold in question here, $\Sigma_g \times S^1$, is the same one that appeared in Example 1.6. We conclude with a couple of remarks on Theorems 1.7, 1.10, and 1.11. Firstly, we note that for each theorem, the uniqueness statements follow from [27]. Our contributions in this work are the existence statements. Secondly, we note that similar results were obtained for closed surfaces with simply-connected complements by Boyer [12]. Some open questions concerning $\mathbb{Z}$-surfaces are discussed in Section 6.4.

1.3. Exotica for all equivariant intersection forms

So far, we have seen that the data in Theorems 1.3 and 1.7 determine the topological type of $\mathbb{Z}$-manifolds and $\mathbb{Z}$-surfaces respectively. In what follows, we investigate the smooth failure of these statements.

One of the driving questions in smooth 4-manifold topology is whether every smoothable simply-connected closed 4-manifold admits multiple smooth structures. This question has natural generalisations to 4-manifolds with boundary and with other fundamental groups; we set up these generalisations with the following definition.

Definition 1.12. For a 3-manifold Y, a (possibly degenerate) symmetric form Q over $\mathbb{Z}$ (resp. Hermitian form λ over $\mathbb{Z}[t^{\pm 1}]$) is *exotically realisable rel. Y* if there exists a pair of smooth simply-connected 4-manifolds M and M' with boundary Y (resp. $\mathbb{Z}$-manifolds with boundary Y) and intersection form Q (resp. equivariant intersection form λ) such that there is an orientation-preserving homeomorphism $F\colon M \to M'$ (for $\pi_1 \cong \mathbb{Z}$, we additionally require that F respects the identifications of $\pi_1(M)$ and $\pi_1(M')$ with $\mathbb{Z}$) but no diffeomorphism $G\colon M \to M'$.

In this language, the driving question above becomes (a subquestion of) the following: which symmetric bilinear forms over $\mathbb{Z}$ are exotically realisable rel. S^3? There is substantial literature demonstrating that some forms are exotically realisable rel. S^3 (we refer to [5,6] both for the state of the art and for a survey of results on the topic) but there remain many forms, such as definite forms or forms with $b_2 < 3$, for which determining exotic realisability rel. S^3 remains out of reach. For more general 3-manifolds, the situation is worse; in fact, it is an open question whether for every integer homology sphere Y there exists *some* symmetric form Q that is exotically realisable rel. Y [34].

Presently, there only seems to be traction on exotic realisability of intersection forms if one relinquishes control of the homeomorphism type of the boundary.

Definition 1.13. A symmetric form Q over $\mathbb{Z}$ (resp. a Hermitian form λ over $\mathbb{Z}[t^{\pm 1}]$) is *exotically realisable* if there exists pair of smooth simply-connected 4-manifolds M and M' with intersection form Q (resp. $\mathbb{Z}$-manifolds with equivariant intersection form λ) such that there is an orientation-preserving homeomorphism $F\colon M \to M'$ (for $\pi_1 \cong \mathbb{Z}$, we additionally require that F respects the identifications of $\pi_1(M)$ and $\pi_1(M')$ with $\mathbb{Z}$) but no diffeomorphism $G\colon M \to M'$.

The following theorem, which appears in [3] for $n = 0$ and [4] for $n > 1$, shows that contrarily to the closed setting, *every* symmetric bilinear form over $\mathbb{Z}$ is exotically realisable.

Theorem 1.14 (Akubulut–Yasui [4] and Akbulut–Ruberman [3]). *Every symmetric bilinear form $(\mathbb{Z}^n, Q)$ over $\mathbb{Z}$ is exotically realisable.*

Following our classification of $\mathbb{Z}$-manifolds with fixed boundary and fixed equivariant intersection form λ it is natural to ask which Hermitian forms λ are exotically realisable, with or without fixing a parametrisation of the boundary 3-manifold. We resolve the latter.

Theorem 1.15. *Every Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ is exotically realisable.*

Topologists are also interested in finding smooth surfaces which are topologically but not smoothly isotopic. While literature in the closed case includes [38, 39, 58, 65–67, 75] there has been a recent surge of interest in the relative setting on which we now focus [29, 54–56, 62]; see also [1]. Most relevant to us are the exotic ribbon discs from [54]. In order to prove that his discs in D^4 are topologically isotopic, Hayden showed that their exteriors have group $\mathbb{Z}$ and appealed to [26]. From the perspective of this paper and [27], any two $\mathbb{Z}$-ribbon discs are isotopic rel. boundary because their exteriors are aspherical and therefore have trivial equivariant intersection form. To generalise Hayden's result to other forms than the trivial one, we introduce some terminology.

Definition 1.16. For a fixed smooth simply-connected 4-manifold N, with boundary S^3, a form λ over $\mathbb{Z}[t^{\pm 1}]$ is *realised by exotic $\mathbb{Z}$-surfaces in N* if there exists a pair of smooth properly embedded $\mathbb{Z}$-surfaces Σ and Σ' in N, with the same boundary, whose exteriors have equivariant intersection forms isometric to λ, and which are topologically but not smoothly isotopic rel. boundary.

Using this terminology, Hayden's result states that the trivial form is realised by exotic $\mathbb{Z}$-discs (in D^4). The next result shows that in fact *every* form is realised by exotic $\mathbb{Z}$-discs.

Theorem 1.17. *Every Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$, such that $\lambda(1)$ is realised as the intersection form of a smooth simply-connected 4-dimensional 2-handlebody N with boundary S^3, is realised by exotic $\mathbb{Z}$-discs in N.*

Remark 1.18. We make a couple of remarks on Theorems 1.15 and 1.17.

(i) The $11/8$ conjecture predicts that every integer intersection form which is realisable by a smooth 4-manifold with S^3 boundary is realisable by a smooth 4-dimensional 2-handlebody with S^3 boundary, thus our hypothesis on the realisability of $\lambda(1)$ by 2-handlebodies is likely not an additional restriction (a nice exposition on why this follows from the $11/8$ conjecture is given in [59, page 24]).

(ii) The handlebody N is very explicit: it can be built from D^4 by attaching 2-handles according to $\lambda(1)$. In particular, when λ is the trivial form, then $N = D^4$ and so Theorem 1.17 demonstrates that there are exotic discs in D^4. This was originally proved in [54], and we note that our proof relies on techniques developed there.

(iii) The proof of Theorem 1.17 also shows that every smooth 2-handlebody with S^3 boundary contains a pair of exotic $\mathbb{Z}$-discs. We expand on this above the statement of Theorem 7.6.

We briefly mention the idea of the proof of Theorem 1.15. For a given Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$, we construct a Stein 4-manifold M with $\pi_1(M) \cong \mathbb{Z}$ and $\lambda_M \cong \lambda$ that contains a cork. Twisting along this cork produces the 4-manifold M' and the homeomorphism $F\colon M \cong M'$. We show that if $F|_\partial$ extended to a diffeomorphism $M \cong M'$, two auxiliary 4-manifolds W and W' (obtained from M and M' by adding a single 2-handle) would be diffeomorphic. We show this is not the case by proving that W is Stein whereas W' is not using work of Lisca–Matic [71]. This proves that M and M' are non-diffeomorphic rel. $F|_\partial$. We then use a result of [3] to show that there exists a pair of smooth manifolds V and V', which are homotopy equivalent to M and M' respectively, and which are homeomorphic but not diffeomorphic to each other. The proof of Theorem 1.17 uses similar ideas.

Organisation. In Section 2 we describe our main technical result and how it implies Theorem 1.3. In Section 3, we recall and further develop the theory of equivariant linking numbers. In Section 4 we review the facts we will need on Reidemeister torsion. Section 5, we prove our main technical result, Theorem 2.4. Section 6 is concerned with our applications to surfaces, and in particular we prove Theorems 1.7, 1.10 and 1.11. Our results in the smooth category, namely Theorems 1.15 and 1.17, are proved in Section 7. Finally, Section 8 exhibits the arbitrarily large collections promised in Example 1.5

Conventions. In Sections 2-6 and 8, we work in the topological category with locally flat embeddings unless otherwise stated. In Section 7, we work in the smooth category.

From now on, all manifolds are assumed to be compact, connected, based and oriented; if a manifold has a non-empty boundary, then the basepoint is assumed to be in the boundary.

If P is manifold and $Q \subseteq P$ is a submanifold with closed tubular neighbourhood $\overline{\nu}(Q) \subseteq P$, then $P_Q := P \setminus \nu(Q)$ will always denote the exterior of Q in P, that is the complement of the open tubular neighbourhood. The only exception to this use of notation is that the exterior of a knot K in S^3 will be denoted E_K instead of S^3_K.

We write $p \mapsto \overline{p}$ for the involution on $\mathbb{Z}[t^{\pm 1}]$ induced by $t \mapsto t^{-1}$. Given a $\mathbb{Z}[t^{\pm 1}]$-module H, we write $\overline{H}$ for the $\mathbb{Z}[t^{\pm 1}]$-module whose underlying abelian group is H but with module structure given by $p \cdot h = \overline{p}h$ for $h \in H$ and $p \in \mathbb{Z}[t^{\pm 1}]$. We write $H^* := \overline{\operatorname{Hom}_{\mathbb{Z}[t^{\pm 1}]}(H, \mathbb{Z}[t^{\pm 1}])}$.

If a pullback map F^* is invertible we shall abbreviate $(F^*)^{-1}$ to F^{-*}. Similarly, for an invertible square matrix A we write $A^{-T} := (A^T)^{-1}$.

2. The main technical realisation statement

The goal of this section is to formulate our main technical theorem, to explain how it implies Theorem 1.3 from the introduction, and to formulate its non-relative analogue. Along the way we also define the automorphism invariant in more detail. We begin by defining a set of $\mathbb{Z}$-manifolds $\mathcal{V}^0_\lambda(Y)$ with boundary Y and intersection form λ. Then we describe a map $b\colon \mathcal{V}^0_\lambda(Y) \to \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$. Theorem 1.3 (as formulated in Remark 2.2) then reduces to the statement that b is a bijection. As we will explain, the injectivity of b follows from [27, Theorem 1.10]. The main technical result of this paper is Theorem 2.4, which gives the surjectivity of b (and thus implies Theorem 1.3). We also prove in this section that Theorem 2.8, our absolute (i.e. non-rel. boundary) homeomorphism classification result, follows from Theorem 1.3. We finish the section with an outline of the proof of Theorem 2.4.

We start by describing the set $\mathcal{V}^0_\lambda(Y)$ from Theorem 1.3 more carefully.

Definition 2.1. Let Y be a 3-manifold with an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module is torsion, and let (H, λ) be a Hermitian form presenting Y. Consider the set $S_\lambda(Y)$ of pairs (M, g), where

- M is a $\mathbb{Z}$-manifold with a fixed identification $\pi_1(M) \xrightarrow{\cong} \mathbb{Z}$, equivariant intersection form isometric to λ, and boundary homeomorphic to Y;
- $g\colon \partial M \xrightarrow{\cong} Y$ is an orientation-preserving homeomorphism such that

$$Y \xrightarrow{g^{-1},\cong} \partial M \to M$$

  induces φ on fundamental groups.

Define $\mathcal{V}^0_\lambda(Y)$ as the quotient of $S_\lambda(Y)$ in which two pairs (M_1, g_1), (M_2, g_2) are deemed equal if and only if there is a homeomorphism $\Phi\colon M_1 \cong M_2$ such that

$$\Phi|_{\partial M_1} = g_2^{-1} \circ g_1.$$

Note that such a homeomorphism is necessarily orientation-preserving because g_1 and g_2 are. For conciseness, we will say that (M_1, g_1) and (M_2, g_2) are *homeomorphic rel. boundary* to indicate the existence of such a homeomorphism Φ.

Remark 2.2. Using Definition 2.1, Theorem 1.3 is equivalent to the following statement: If λ presents Y, then $\mathcal{V}^0_\lambda(Y)$ is non-empty and corresponds bijectively to

- $\operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$, if λ is an even form;
- $(\operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)) \times \mathbb{Z}_2$ if λ is an odd form. The map to $\mathbb{Z}_2$ is given by the Kirby–Siebenmann invariant.

The bijection is explicit and will be constructed in Construction 6.4.

Additionally, note that since (H, λ) is assumed to present Y, there is an isometry $\partial\lambda \cong -\mathrm{Bl}_Y$ and fixing a choice of one such isometry leads to a bijection

$$\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)/\operatorname{Aut}(\lambda) \approx \operatorname{Aut}(\partial\lambda)/\operatorname{Aut}(\lambda),$$

where $\operatorname{Aut}(\partial\lambda)$ denotes the group of self-isometries of $\partial\lambda$. Note however that this bijection is not canonical as it depends on the choice of the isometry $\partial\lambda \cong -\mathrm{Bl}_Y$.

Construction 2.3 (Constructing the map $b\colon \mathcal{V}^0_\lambda(Y) \to \mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)/\operatorname{Aut}(\lambda)$). Let Y be a 3-manifold with an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose corresponding Alexander module is torsion, and let (H, λ) be a form presenting Y. Let (M, g) be an element of $\mathcal{V}^0_\lambda(Y)$, i.e. M is a $\mathbb{Z}$-manifold with equivariant intersection form isometric to λ and $g\colon \partial M \cong Y$ is a homeomorphism as in Definition 2.1.

In the text preceding Theorem 1.3, we showed how M determines an isometry $D_M \in \mathrm{Iso}(\partial\lambda_M, -\mathrm{Bl}_{\partial M})$. Morally, one should think that this isometry D_M is the invariant we associate to M. For this to be meaningful however, we instead need an isometry that takes value in a set defined in terms of just the 3-manifold Y and the form (H, λ), without referring to M itself. We resolve this by composing D_M with other isometries, so that our invariant is ultimately an element of $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)$. Once we have built the invariant, we will show it is well defined up to an action by $\operatorname{Aut}(\lambda)$.

We first use g to describe an isometry $\mathrm{Bl}_{\partial M} \cong \mathrm{Bl}_Y$. Since on the level of fundamental groups g intertwines the maps to $\mathbb{Z}$, [27, Proposition 3.7] implies that g induces an isometry

$$g_*\colon \mathrm{Bl}_{\partial M} \cong \mathrm{Bl}_Y .$$

Next we describe an isometry $\partial\lambda \cong \partial\lambda_M$. The assumption that M has equivariant intersection form λ means by definition that there is an isometry $F\colon \lambda \cong \lambda_M$, i.e. an isomorphism $F\colon H \to H_2(M; \mathbb{Z}[t^{\pm 1}])$ that intertwines the forms λ and λ_M. Note that there is no preferred choice of F. Any such F induces an isometry $\partial F \in \operatorname{Aut}(\partial\lambda, \partial\lambda_M)$ as follows: $F\colon H \to H_2(M; \mathbb{Z}[t^{\pm 1}])$ gives an isomorphism

$$(F^*)^{-1}\colon H^* \to H_2\big(M; \mathbb{Z}[t^{\pm 1}]\big)^*$$

that descends to an isomorphism $\operatorname{coker}(\widehat{\lambda}) \cong \operatorname{coker}(\widehat{\lambda}_M)$ and is in fact an isometry; this is by definition

$$\partial F := (F^*)^{-1}\colon \partial\lambda \cong \partial\lambda_M.$$

This construction is described in greater generality in [27, Section 2.2]. We shall henceforth abbreviate $(F^*)^{-1}$ to F^{-*}.

We are now prepared to associate an isometry in $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)$ to $(M, g) \in \mathcal{V}^0_\lambda(Y)$ as follows: choose an isometry $F\colon \lambda_M \cong \lambda$ and consider the isometry

$$b_{(M,g,F)} := g_* \circ D_M \circ \partial F \in \mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y).$$

We are not quite done, because we need to ensure that our invariant is independent of the choice of F and that b defines a map on $\mathcal{V}^0_\lambda(Y)$.

First, we will make our invariant independent of the choice of F. We require the following observation. Given a Hermitian form (H,λ) and linking form (T,ℓ), there is a natural left action $\operatorname{Aut}(\lambda) \curvearrowright \operatorname{Iso}(\partial\lambda,\ell)$ defined via

$$G \cdot h := h \circ \partial G^{-1} \quad \text{for } G \in \operatorname{Aut}(\lambda) \text{ and } h \in \operatorname{Iso}(\partial\lambda,\ell). \tag{2.1}$$

In particular, we can consider

$$b_{(M,g)} := g_* \circ D_M \circ \partial F \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda).$$

It is now not difficult to check that $b_{(M,g)}$ is independent of the choice of F.

The fact that if (M_0,g_0) and (M_1,g_1) are homeomorphic rel. boundary (recall Definition 2.1), then $b_{(M_0,g_0)} = b_{(M_1,g_1)}$ follows fairly quickly. From now on we omit the boundary identification $g\colon \partial M \cong Y$ from the notation, writing b_M instead of $b_{(M,g)}$. This concludes the construction of our automorphism invariant.

We are now ready to state our main technical theorem.

Theorem 2.4. *Let Y be a 3-manifold with an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module is torsion, and let (H,λ) be a non-degenerate Hermitian form presenting Y. If $b \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$ is an isometry, then there is a $\mathbb{Z}$-manifold M with equivariant intersection form $\lambda_M \cong \lambda$, boundary Y and $b_M = b$. If the form is odd, then M can be chosen to have either* $\operatorname{ks}(M) = 0$ *or* $\operatorname{ks}(M) = 1$.

We now describe how to obtain Theorem 1.3 (as formulated in Remark 2.2) by combining this result with [27].

Proof of Theorem 1.3 assuming Theorem 2.4. First, notice that Theorem 2.4 implies the surjectivity portion of the statement in Theorem 1.3. It therefore suffices to prove that the assignment $\mathcal{V}^0_\lambda(Y) \to \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$ which sends M to b_M is injective for λ even, and that the assignment $\mathcal{V}^0_\lambda(Y) \to (\operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)) \times \mathbb{Z}_2$ which sends M to $(b_M, \operatorname{ks}(M))$ is injective for λ odd.

Let (M_0,g_0) and (M_1,g_1) be two pairs representing elements in $\mathcal{V}^0_\lambda(Y)$. Each 4-manifold M_i comes with an isometry $F_i\colon (H,\lambda) \to (H_2(M_i;\mathbb{Z}[t^{\pm1}]), \lambda_{M_i})$ and for $i = 0,1$, the homeomorphisms $g_i\colon \partial M_i \to Y$ are as in Definition 2.1. We then get epimorphisms

$$(g_i)_* \circ D_{M_i} \circ \partial F_i \circ \pi\colon H^* \twoheadrightarrow H_1\big(Y;\mathbb{Z}[t^{\pm1}]\big).$$

Here $\pi\colon H^* \to \operatorname{coker}(\widehat{\lambda})$ denotes the canonical projection. We assume that $b_{M_0} = b_{M_1}$ and, if λ is odd, then we additionally assume that $\operatorname{ks}(M_0) = \operatorname{ks}(M_1)$. The fact that

$b_{M_0} = b_{M_1}$ implies that there is an isometry $F\colon (H,\lambda) \cong (H,\lambda)$ that makes the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{\widehat{\lambda}} & H^* & \xrightarrow{(g_0)_*\circ D_{M_0}\circ\partial F_0\circ\pi} & H_1\big(Y;\mathbb{Z}[t^{\pm1}]\big) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F^{-*}} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & H & \xrightarrow{\widehat{\lambda}} & H^* & \xrightarrow{(g_1)_*\circ D_{M_1}\circ\partial F_1\circ\pi} & H_1\big(Y;\mathbb{Z}[t^{\pm1}]\big) & \longrightarrow & 0. \end{array}$$

But now, by considering the isometry $G\colon \lambda_{M_0} \cong \lambda_{M_1}$ defined by $G := F_1 \circ F \circ F_0^{-1}$, a quick verification shows that (G, id_Y) is a compatible pair in the sense of [27]. Consequently, [27, Theorem 1.10] shows that there is a homeomorphism $M_0 \cong M_1$ extending id_Y and inducing G; in particular, M_0 and M_1 are homeomorphic rel. boundary. ∎

Remark 2.5. For (Y,φ) as in Theorem 2.4, we explain the fact (already mentioned in Remark 1.4) that if M_0 and M_1 are spin 4-manifolds with $\pi_1(M_i) \cong \mathbb{Z}$, boundary homeomorphic to (Y,φ), isometric equivariant intersection form, and the same automorphism invariant, then their Kirby–Siebenmann invariants agree. As explained during the proof of Theorem 1.3, these assumptions ensure the existence of a compatible pair (G, id_Y). This in turn implies that $M := M_0 \cup_{g_0\circ g_1^{-1}} M_1$ is spin and has fundamental group $\mathbb{Z}$ [27, Theorem 3.12]. The assertion now follows from additivity of ks and Novikov additivity of the signature:

$$\mathrm{ks}(M_0) + \mathrm{ks}(M_1) = \mathrm{ks}(M) \equiv \frac{\sigma(M)}{8} = \frac{\sigma(M_0) - \sigma(M_1)}{8} = 0 \qquad (\mathrm{mod}\ 2).$$

We also use that the signatures of M, M_0, and M_1 can be obtained from the respective equivariant intersection forms by specialising to $t = 1$ and taking the signature.

In Section 5.7, we exhibit examples of spin 4-manifolds with boundary homeomorphic to $-L(8,1)\#(S^1 \times S^2)$ and isometric equivariant intersection form that have different Kirby–Siebenmann invariants, demonstrating that the automorphism invariant was needed in the argument of this remark.

Next we outline the strategy of the proof of Theorem 2.4.

Outline of the proof of Theorem 2.4. The idea is to perform surgeries on Y along a set of generators of $H_1(Y;\mathbb{Z}[t^{\pm1}])$ to obtain a 3-manifold Y' with $H_1(Y';\mathbb{Z}[t^{\pm1}]) = 0$. The verification that $H_1(Y';\mathbb{Z}[t^{\pm1}]) = 0$ uses Reidemeister torsion. We then use surgery theory to show that this Y' bounds a 4-manifold B with $B \simeq S^1$; this step relies on Freedman's work in the topological category [7,40,42]. The 4-manifold M is then obtained as the union of the trace of these surgeries with B. To show that in the odd case both values of the Kirby–Siebenmann invariant are realised, we use the star construction [42,83]. The main difficulty of the proof is to describe the correct surgeries on Y to obtain Y'; this is where the fact that λ presents Bl_Y comes into play: we show that generators of $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ can be represented by a framed link $\widetilde{L}$ with equivariant linking matrix equal to minus the transposed inverse of a matrix representing λ. ■

This is a strategy similar to the one employed in Boyer's classification of simply-connected 4-manifolds with a given boundary [12]. The argument is also reminiscent of [10, Theorem 2.9], where Borodzik and Friedl obtain bounds (in terms of a presentation matrix for Bl_K) on the number of crossing changes required to turn K into an Alexander polynomial one knot: they perform surgeries on the zero-framed surgery $Y = M_K$ to obtain $Y' = M_{K'}$, where K' is an Alexander polynomial one knot.

Remark 2.6. As we mentioned in Construction 2.3, if M_0 and M_1 are homeomorphic rel. boundary, then $b_{M_0} = b_{M_1}$ in $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)/\operatorname{Aut}(\lambda)$. In fact, the same proof shows more. If two 4-manifolds M_0 and M_1 that represent elements of $\mathcal{V}^0_\lambda(Y)$ are *homotopy equivalent* rel. boundary, then $b_{M_0} = b_{M_1}$ in $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)/\operatorname{Aut}(\lambda)$.

Next, we describe how the classification in the case where the homeomorphisms need not fix the boundary pointwise follows from Theorem 1.3. To this effect, we use $\mathrm{Homeo}^+_\varphi(Y)$ to denote the orientation-preserving homeomorphisms of Y such that the induced map on π_1 commutes with $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ and we describe the set of homeomorphism classes of $\mathbb{Z}$-manifolds that we will be working with.

Definition 2.7. For Y and (H, λ) as in Definition 2.1, define $\mathcal{V}_\lambda(Y)$ as the quotient of $S_\lambda(Y)$ in which two pairs $(M_1, g_1), (M_2, g_2)$ are deemed equal if and only if there is a homeomorphism $\Phi\colon M_1 \cong M_2$ such that

$$\Phi|_{\partial M_1} = g_2^{-1} \circ f \circ g_1$$

for some $f \in \mathrm{Homeo}^+_\varphi(Y)$; note that such a homeomorphism Φ is necessarily orientation-preserving.

We continue to set up notation to describe how the non-relative classification follows from Theorem 1.3. Observe that the group $\mathrm{Homeo}^+_\varphi(Y)$ acts on $\mathcal{V}^0_\lambda(Y)$ by setting $f \cdot (M, g) := (M, f \circ g)$ for $f \in \mathrm{Homeo}^+_\varphi(Y)$. Further, observe that

$$\mathcal{V}_\lambda(Y) = \mathcal{V}^0_\lambda(Y)/\,\mathrm{Homeo}^+_\varphi(Y). \tag{2.2}$$

Recall that any $f \in \mathrm{Homeo}^+_\varphi(Y)$ induces an isometry f_* of the Blanchfield form Bl_Y. Thus the group $\mathrm{Homeo}^+_\varphi(Y)$ acts on $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)$ by $f \cdot h := f_* \circ h$. Finally, there is a natural left action $\operatorname{Aut}(\lambda) \times \mathrm{Homeo}^+_\varphi(Y)$ on $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)$ defined via

$$(F, f) \cdot h := f_* \circ h \circ \partial F^{-1}.$$

The non-relative classification statement reads as follows.

Theorem 2.8. *Let Y be a 3-manifold with an epimorphism $\pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module is torsion, let (H,λ) be a non-degenerate Hermitian form over $\mathbb{Z}[t^{\pm 1}]$. Consider the set $\mathcal{V}_\lambda(Y)$ of $\mathbb{Z}$-manifolds with boundary $\partial M \cong Y$, and $\lambda_M \cong \lambda$, considered up to orientation-preserving homeomorphism.*

If the form (H,λ) presents Y, then $\mathcal{V}_\lambda(Y)$ is non-empty and corresponds bijectively to

(1) $\operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/(\operatorname{Aut}(\lambda) \times \operatorname{Homeo}^+_\varphi(Y))$ *if λ is an even form;*

(2) $(\operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/(\operatorname{Aut}(\lambda) \times \operatorname{Homeo}^+_\varphi(Y))) \times \mathbb{Z}_2$ *if λ is an odd form. The map to $\mathbb{Z}_2$ is given by the Kirby–Siebenmann invariant.*

Proof. Thanks to Theorem 1.3 (as formulated in Remark 2.2) and (2.2), it suffices to prove that the map b respects the $\operatorname{Homeo}^+_\varphi(Y)$ actions, i.e. that $b_{f\cdot(M,g)} = f \cdot b_{(M,g)}$, where $g\colon \partial M \cong Y$ is a homeomorphism as in Definition 2.1 and $f \in \operatorname{Homeo}^+_\varphi(Y)$. This now follows from the following formal calculation:

$$b_{f\cdot(M,g)} = b_{(M,f\circ g)} = f_* \circ g_* \circ D_M \circ \partial F = f \cdot b_{(M,g)},$$

where $F\colon \lambda_M \cong \lambda$ is an isometry and we used the definitions of the $\operatorname{Homeo}^+_\varphi(Y)$ actions and of the map b. ■

Remark 2.9. To make the results as user friendly as possible, we spell out how to apply them in practice. Fix an oriented 3-manifold Y with torsion Alexander module. Two orientable $\mathbb{Z}$-manifolds M_0 and M_1 with boundary Y are homeomorphic if and only if they have the same Kirby–Siebenmann invariants, and the following hold:

(1) there are identifications $\psi_i\colon \pi_1(M_i) \xrightarrow{\cong} \mathbb{Z}$ for $i = 0,1$, and

(2) there are a homeomorphism $g_i\colon Y \xrightarrow{\cong} \partial M_i$ for $i = 0,1$, and a surjection $\pi_1(Y) \to \mathbb{Z}$ such that

$$\psi_i \circ \operatorname{incl}_i \circ g_i = \varphi \quad \text{for } i = 0,1,$$

and such that

(3) using the coefficient system induced by the ψ_i, and the orientations induced by the g_i to define the intersection forms, there is an isometry

$$F\colon \big(H_2\big(M_0;\mathbb{Z}[t^{\pm 1}]\big),\lambda_{M_0}\big) \cong \big(H_2\big(M_1;\mathbb{Z}[t^{\pm 1}]\big),\lambda_{M_1}\big),$$

and

(4) with respect to this isometry we have that

$$b_{M_0} = b_{M_1} \in \operatorname{Iso}(\partial\lambda_{M_0}, -\operatorname{Bl}_Y)/\big(\operatorname{Aut}(\lambda_{M_0}) \times \operatorname{Homeo}^+_\varphi(Y)\big)$$

or, equivalently, there exists an isometry $F\colon \lambda_{M_0} \cong \lambda_{M_1}$ whose algebraic boundary $\partial F\colon \partial\lambda_{M_0} \cong \partial\lambda_{M_1}$ is induced by some orientation-preserving homeomorphism $f\colon Y \to Y$ that intertwines φ. In [27] such a pair (f, F) is called *compatible*.

The next few sections are devoted to proving Theorem 2.4.

3. Equivariant linking and longitudes

We collect some preliminary notions that we will need later on. In Section 3.1 we fix our notation for twisted homology and equivariant intersections. In Section 3.2, we collect some facts about linking numbers in infinite cyclic covers, while in Section 3.3, we define an analogue of integer framings of a knot in S^3 for knots in infinite cyclic covers.

3.1. Covering spaces and twisted homology

We fix our conventions on twisted homology and recall some facts about equivariant intersection numbers. We refer the reader interested in the intricacies of transversality in the topological category to [43, Section 10].

We first introduce some notation for infinite cyclic covers. Given a space X that has the homotopy type of a finite CW complex, together with an epimorphism $\varphi\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$, we write $p\colon X^\infty \to X$ for the infinite cyclic cover corresponding to $\ker(\varphi)$. If $A \subset X$ is a subspace, then we set $A^\infty := p^{-1}(A)$ and we will often write $H_*(X, A; \mathbb{Z}[t^{\pm 1}])$ instead of $H_*(X^\infty, A^\infty)$. Similarly, since $\mathbb{Q}(t)$ is flat over $\mathbb{Z}[t^{\pm 1}]$, we often write $H_*(X, A; \mathbb{Q}(t))$ or $H_*(X, A; \mathbb{Z}[t^{\pm 1}]) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}(t)$ instead of $H_*(X^\infty, A^\infty) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}(t)$.

Remark 3.1. The *Alexander polynomial* of X, denoted Δ_X is the order of the *Alexander module* $H_1(X; \mathbb{Z}[t^{\pm 1}])$. While we refer to Remark 4.3 below for some recollections on orders of modules, here we simply note that Δ_X is a Laurent polynomial that is well defined up to multiplication by $\pm t^k$ with $k \in \mathbb{Z}$ and that if $X = M_K$ is the 0-framed surgery along a knot K, then Δ_X is the Alexander polynomial of K.

Next, we move on to equivariant intersections in covering spaces.

Definition 3.2. Let M be an n-manifold (with possibly non-empty boundary) with an epimorphism $\pi_1(M) \twoheadrightarrow \mathbb{Z}$. For a k-dimensional closed submanifold $A \subset M^\infty$ and an $(n-k)$-dimensional closed submanifold $A' \subset M^\infty$ such that A and $t^j A'$ intersect transversely for all $j \in \mathbb{Z}$, the *equivariant intersection* $A \cdot_{\infty,M} A' \in \mathbb{Z}[t^{\pm 1}]$

is defined as

$$A \cdot_{\infty,M} A' = \sum_{j \in \mathbb{Z}} \big(A \cdot_{M^\infty} (t^j A')\big) t^{-j},$$

where $\cdot_{M^\infty}$ denotes the usual (algebraic) signed count of points of intersection. If the boundary of M is non-empty and $A' \subset M$ is properly embedded, then we can make the same definition and also write $A \cdot_{\infty,M} A' \in \mathbb{Z}[t^{\pm 1}]$.

Remark 3.3. We collect a couple of observations about equivariant intersections.

(1) Equivariant intersections are well defined on homology and in fact $A \cdot_{\infty,M} A' = \lambda([A'],[A])$, where λ denotes the equivariant intersection form

$$\lambda \colon H_k\big(M;\mathbb{Z}[t^{\pm 1}]\big) \times H_{n-k}\big(M;\mathbb{Z}[t^{\pm 1}]\big) \to \mathbb{Z}[t^{\pm 1}].$$

The reason $A \cdot_{\infty,M} A'$ is equal to $\lambda([A'],[A]) = \overline{\lambda([A],[A'])}$ instead of $\lambda([A],[A'])$ is due to the fact that we are following the conventions from [27, Section 2] in which the adjoint of a Hermitian form $\lambda \colon H \times H \to \mathbb{Z}[t^{\pm 1}]$ is defined by the equation $\widehat{\lambda}(y)(x) = \lambda(x,y)$. With these conventions λ is linear in the first variable and anti-linear in the second, whereas $\cdot_{\infty,M}$ is linear in the second variable and anti-linear in the first.

(2) When $\partial M \neq \emptyset$ and $A \subset M$ is a properly embedded submanifold with boundary, then again $A \cdot_{\infty,M} A' = \lambda^\partial([A'],[A])$ where this time λ^∂ denotes the pairing

$$\lambda^\partial \colon H_k\big(M;\mathbb{Z}[t^{\pm 1}]\big) \times H_{n-k}\big(M,\partial M;\mathbb{Z}[t^{\pm 1}]\big) \to \mathbb{Z}[t^{\pm 1}].$$

As previously, λ^∂ is linear in the first variable and anti-linear in the second.

(3) The definition of the pairings λ and λ^∂ can be made with arbitrary twisted coefficients. In order to avoid extraneous generality, we simply mention that there are $\mathbb{Q}(t)$-valued pairings $\lambda_{\mathbb{Q}(t)}$ and $\lambda^\partial_{\mathbb{Q}(t)}$ defined on homology with $\mathbb{Q}(t)$-coefficients and that if $A, B \subset M^\infty$ are closed submanifolds of complementary dimension, then

$$\lambda_{\mathbb{Q}(t)}\big([A],[B]\big) = \lambda\big([A],[B]\big),$$

and similarly for properly embedded submanifolds with boundary.

3.2. Equivariant linking

We recall definitions and properties of equivariant linking numbers. Other papers that feature discussions of the topic include [9, 68, 80].

We assume for the rest of the section that Y is a 3-manifold and $\varphi \colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ is an epimorphism such that the corresponding Alexander module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is torsion, i.e. $H_*(Y;\mathbb{Q}(t)) = 0$. We also write $p \colon Y^\infty \to Y$ for the infinite cyclic cover corresponding to $\ker(\varphi)$ so that $H_1(Y;\mathbb{Z}[t^{\pm 1}]) = H_1(Y^\infty)$. Given a simple closed curve $\widetilde{a} \subset Y^\infty$, we write $a^\infty := \bigcup_{k\in\mathbb{Z}} t^k \widetilde{a}$ for the union of all the translates of $\widetilde{a}$ and $a := p(\widetilde{a}) \subset Y$ for the projection of $\widetilde{a}$ down to Y. This way, the covering map $p\colon Y^\infty \to Y$ restricts to a covering map

$$Y^\infty \setminus \nu(a^\infty) \to Y \setminus \nu(a) =: Y_a.$$

Since the Alexander module of Y is torsion, a short Mayer–Vietoris argument shows that the vector space $H_*(Y_a;\mathbb{Q}(t)) = \mathbb{Q}(t)$ is generated by $[\widetilde{\mu}_a]$, the class of a meridian of $\widetilde{a} \subset Y^\infty$.

Definition 3.4. The *equivariant linking number* of two disjoint simple closed curves $\widetilde{a}, \widetilde{b} \subset Y^\infty$ is the unique rational function $\ell k_{\mathbb{Q}(t)}(\widetilde{a},\widetilde{b}) \in \mathbb{Q}(t)$ such that

$$[\widetilde{b}] = \ell k_{\mathbb{Q}(t)}(\widetilde{a},\widetilde{b})[\widetilde{\mu}_a] \in H_1\big(Y \setminus \nu(a);\mathbb{Q}(t)\big).$$

Observe that this linking number is only defined for *disjoint* pairs of simple closed curves. We give a second, more geometric, description of the equivariant linking number.

Remark 3.5. Since $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is torsion, for any simple closed curve $\widetilde{a}$ in Y^∞, there is some polynomial $p(t) = \sum_i c_i t^i$ such that $p(t)[\widetilde{a}] = 0$. Thus there is a surface $F \subset Y^\infty \setminus \nu(a^\infty)$ with boundary consisting of the disjoint union of c_i parallel copies of $t^i \cdot \widetilde{a}'$ and d_j meridians of $t^j \cdot \widetilde{a}'$ where $\widetilde{a}'$ is some pushoff of $\widetilde{a}$ in $\partial\overline{\nu}(\widetilde{a})$ and $j \neq i$; we abusively write $\partial F = p(t)\widetilde{a}$.

Proposition 3.6. *Let Y be a 3-manifold, let $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ be an epimorphism such that the Alexander module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is torsion, and let $\widetilde{a},\widetilde{b} \subset Y^\infty$ be disjoint simple closed curves.*

Let F and $p(t)$ be respectively a surface and a polynomial associated to $\widetilde{a}$ as in Remark 3.5. The equivariant linking of $\widetilde{a}$ and $\widetilde{b}$ can be written as

$$\ell k_{\mathbb{Q}(t)}(\widetilde{a},\widetilde{b}) = \frac{1}{p(t^{-1})}\sum_{k\in\mathbb{Z}}(F\cdot t^k\widetilde{b})t^{-k} = \frac{1}{p(t^{-1})}(F \cdot_{\infty,Y_a} \widetilde{b}). \tag{3.1}$$

In particular, this expression is independent of the choices of F and $p(t)$.

Proof. As in Section 3.1, write λ^∂ for the (homological) intersection pairing

$$H_1\big(Y_a;\mathbb{Z}[t^{\pm 1}]\big) \times H_2\big(Y_a,\partial Y_a;\mathbb{Z}[t^{\pm 1}]\big) \to \mathbb{Z}[t^{\pm 1}]$$

and $\lambda^\partial_{\mathbb{Q}(t)}$ for the pairing involving $\mathbb{Q}(t)$-homology.

Write $\ell := \ell k(\widetilde{a},\widetilde{b})$ so that $[\widetilde{b}] = \ell[\widetilde{\mu}_a] \in H_1(Y_a;\mathbb{Q}(t))$. From this and Remark 3.3, for a surface F as in the statement, we obtain

$$\begin{aligned} F \cdot_{\infty,Y_a} \widetilde{b} = \lambda^\partial\big([\widetilde{b}],[F]\big) &= \lambda^\partial_{\mathbb{Q}(t)}\big([\ell\widetilde{\mu}_a],[F]\big) \\ &= \ell\lambda^\partial_{\mathbb{Q}(t)}\big([\widetilde{\mu}_a],[F]\big) = \ell(F \cdot_{\infty,Y_a} \widetilde{\mu}_a) = \ell p(t^{-1}). \end{aligned}$$

The last equality here follows from inspection; since $F \hookrightarrow Y^\infty \setminus \nu(a^\infty)$ has boundary along c_i copies of $t^i \cdot \widetilde{a}'$ and d_j copies of $t^j \widetilde{\mu}_a$, each meridian $t^i \cdot \mu_{\widetilde{a}}$ intersects F in c_i points. The result now follows after dividing out by $p(t^{-1})$. ■

Just as for linking numbers in rational homology spheres, the equivariant linking number is not well defined on homology, unless the target is replaced by $\mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$. To describe the resulting statement, we briefly recall the definition of the Blanchfield form.

Remark 3.7. Using the same notation and assumptions as in Proposition 3.6, the Blanchfield form is a non-singular sesquilinear, Hermitian pairing that can be defined as follows:

$$\begin{aligned} \mathrm{Bl}_Y \colon H_1\big(Y;\mathbb{Z}[t^{\pm 1}]\big) \times H_1\big(Y;\mathbb{Z}[t^{\pm 1}]\big) &\to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}], \\ \big([\widetilde{b}],[\widetilde{a}]\big) &\mapsto \left[\frac{1}{p(t)}(F \cdot_{\infty, Y_a} \widetilde{b})\right]. \end{aligned} \tag{3.2}$$

We refer to [44,79] for further background and homological definitions of this pairing.

We summarise this discussion and collect another property of equivariant linking in the next proposition.

Proposition 3.8. *Let Y be a 3-manifold and let $\varphi \colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ be an epimorphism such that the Alexander module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is torsion. For disjoint simple closed curves $\widetilde{a}, \widetilde{b} \subset Y^\infty$, the equivariant linking number satisfies the following properties:*

1. sesquilinearity: $\ell k_{\mathbb{Q}(t)}(p\widetilde{a}, q\widetilde{b}) = \overline{p}q\,\ell k_{\mathbb{Q}(t)}(\widetilde{a},\widetilde{b})$ *for all* $p, q \in \mathbb{Z}[t^{\pm 1}]$*;*
2. symmetry: $\ell k_{\mathbb{Q}(t)}(\widetilde{a},\widetilde{b}) = \overline{\ell k_{\mathbb{Q}(t)}(\widetilde{b},\widetilde{a})}$*;*
3. relation to the Blanchfield form: $[\ell k_{\mathbb{Q}(t)}(\widetilde{a},\widetilde{b})] = \mathrm{Bl}_Y([\widetilde{b}],[\widetilde{a}]) \in \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$.

Proof. The first property follows from (3.1). Before proving the second and third properties,we note that in (3.1) and (3.2), we can assume that $p(t) = p(t^{-1})$. Indeed, both formulas are independent of the choice of $p(t)$ and if $q(t)$ satisfies $q(t)[\widetilde{a}] = 0$, then so does $p(t) := q(t)q(t^{-1})$. The proof of the second assertion now follows as in [9, Lemma 3.3], whereas the third follows by inspecting (3.1) and (3.2). ■

The reader will have observed that the formulas in Proposition 3.6 and 3.8 depend heavily on conventions chosen for adjoints, module structures, equivariant intersections and twisted homology. It is for this reason that the formulas presented here might differ (typically up to switching variables) from others in the literature.

3.3. Parallels, framings, and longitudes

Continuing with the notation and assumptions from the previous section, we fix some terminology regarding parallels and framings in infinite cyclic covers. The goal is to be able to describe a notion of integer surgery for appropriately nullhomologous knots in the setting of infinite cyclic covers. Our approach is inspired by [12, 13].

Definition 3.9. Let $\widetilde{K} \subset Y^\infty$ be a knot, let $p\colon Y^\infty \to Y$ be the covering map, and denote $K := p(\widetilde{K}) \subset Y$ the projection of $\widetilde{K}$.

(1) A *parallel* to $\widetilde{K}$ is a simple closed curve $\pi \subset \partial\overline{\nu}(\widetilde{K})$ that is isotopic to $\widetilde{K}$ in $\overline{\nu}(\widetilde{K})$.

(2) Given any parallel π of $\widetilde{K}$, we use $\overline{\nu}_\pi(\widetilde{K})$ to denote the parametrisation

$$S^1 \times D^2 \xrightarrow{\cong} \overline{\nu}(\widetilde{K})$$

which sends $S^1 \times \{x\}$ to π for some $x \in \partial D^2$.

(3) A *framed link* is a link $\widetilde{L} \subset Y^\infty$ together with a choice of a parallel for each of its components.

(4) We say that the knot $\widetilde{K}$ *admits framing coefficient* $r(t) \in \mathbb{Q}(t)$ if there is a parallel π with $\ell k_{\mathbb{Q}(t)}(\widetilde{K}, \pi) = r(t)$. We remark that, unlike in the setting of homology with integer coefficients where every knot K admits any integer r as a framing coefficient, when we work with $\mathbb{Z}[t^{\pm 1}]$-homology, a fixed knot $\widetilde{K}$ will have many $r(t) \in \mathbb{Q}(t)$ (in fact, even in $\mathbb{Z}[t^{\pm 1}]$) which it does not admit as a framing coefficient. We will refer to π as a *framing curve* of $\widetilde{K}$ with framing $r(t)$.

(5) A framed n-component link $\widetilde{L}$ which admits framing coefficients $\mathbf{r}(t) := (r_i(t))_{i=1}^n$, together with a choice of parallels realising those framing coefficients, is called an $\mathbf{r}(t)$-framed link.

(6) The *equivariant linking matrix* of an $\mathbf{r}(t)$-framed link $\widetilde{L}$ is the matrix $A_{\widetilde{L}}$ with diagonal term $(A_{\widetilde{L}})_{ii} = r_i(t)$ and off-diagonal terms $(A_{\widetilde{L}})_{ij} = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j)$ for $i \neq j$.

(7) For a link $\widetilde{L}$ in Y^∞, we define L^∞ to be the set of all the translates of $\widetilde{L}$. We also set $L := p(\widetilde{L})$. We say that $\widetilde{L}$ is in *covering general position* if the map $p\colon L^\infty \to L$ is a trivial $\mathbb{Z}$-covering isomorphic to the pullback cover

$$\begin{array}{ccc} L^\infty & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ L & \xrightarrow{c} & S^1, \end{array}$$

where c is a constant map. In particular, each component of L^∞ is mapped by p, via a homeomorphism, to some component of L. From now on we will always assume that our links $\widetilde{L}$ are in covering general position. This assumption is to avoid pathologies, and holds generically.

(8) For an n-component link $\widetilde{L}$ which admits the framing coefficients $\mathbf{r}(t) := (r_i(t))_{i=1}^n$, the $\mathbf{r}(t)$-*surgery* along $\widetilde{L}$ is the covering space $Y^\infty_{\mathbf{r}(t)}(\widetilde{L}) \to Y_{\mathbf{r}}(L)$ defined by Dehn filling $Y^\infty \setminus \nu(L^\infty)$ along all the translates of all the parallels $\pi_1^\infty, \dots, \pi_n^\infty$ as follows:

$$Y^\infty_{\mathbf{r}(t)}(\widetilde{L}) = Y^\infty \setminus \left(\bigcup_{k \in \mathbb{Z}} \bigcup_{i=1}^n \big(t^k \overline{\nu}_{\pi_i}(\widetilde{K}_i)\big) \right) \cup \left(\bigcup_{k \in \mathbb{Z}} \bigcup_{i=1}^n (D^2 \times S^1) \right).$$

Since $\widetilde{L}$ is in covering general position, for all $\widetilde{K}_i$ the covering map $p|_{\widetilde{K}_i} \colon \widetilde{K}_i \to K_i$ is a homeomorphism, so $p|_{\overline{\nu}(\widetilde{K}_i)} \colon \overline{\nu}(\widetilde{K}_i) \to \nu(K_i)$ is a homeomorphism. Thus any parallel π_i of $\widetilde{K}_i$ projects to a parallel of K, so we may also define $\mathbf{r}$-surgery along L downstairs:

$$Y_{\mathbf{r}}(L) = Y \setminus \left(\bigcup_{i=1}^n \overline{\nu}_{p(\pi_i)}\big(p(\widetilde{K}_i)\big) \right) \cup \left(\bigcup_{i=1}^n (D^2 \times S^1) \right).$$

Observe that there is a naturally induced cover $Y^\infty_{\mathbf{r}(t)}(\widetilde{L}) \to Y_{\mathbf{r}}(L)$ obtained by restricting $p \colon Y^\infty \to Y$ to the link exterior and then extending it to the trivial disconnected $\mathbb{Z}$-cover over each of the surgery solid tori.

(9) The *dual framed link* $\widetilde{L}' \subset Y^\infty_{\mathbf{r}(t)}(\widetilde{L})$ associated to a framed link $\widetilde{L} \subset Y^\infty$ is defined as follows:

- the i-th component $\widetilde{K}_i'$ of the underlying link $\widetilde{L}' \subset Y^\infty_{\mathbf{r}(t)}(\widetilde{L})$ is obtained by considering the core of the i-th surgery solid torus $D^2 \times S^1$;
- the framing of $\widetilde{K}_i'$ is given by the S^1-factor $S^1 \times \{\mathrm{pt}\}$ of the parametrised solid torus used to define $\widetilde{K}_i'$.

(10) We also define analogues of these notions (except (6) and (7)) for a link L in the 3-manifold Y, without reference to the cover.

The next lemma provides a sort of analogue for the Seifert longitude of a knot in S^3; it is inspired by [13, Lemma 1.2]. The key difference with the Seifert longitude is that in our setting this class, which we denote by $\lambda_{\widetilde{K}}$, is just a homology class in $H_1(\partial \overline{\nu}(\widetilde{K}); \mathbb{Q}(t))$; it will frequently not be represented by a simple closed curve.

Lemma 3.10. *For every knot $\widetilde{K} \subset Y^\infty$, there is a unique homology class $\lambda_{\widetilde{K}} \in H_1(\partial \overline{\nu}(\widetilde{K}); \mathbb{Q}(t))$ called the* longitude *of $\widetilde{K}$ such that the following two conditions hold:*

(1) *the algebraic equivariant intersection number of $[\mu_{\widetilde{K}}]$ and $\lambda_{\widetilde{K}}$ is one:*

$$\lambda_{\partial \overline{\nu}(K), \mathbb{Q}(t)}\big([\mu_{\widetilde{K}}], \lambda_{\widetilde{K}}\big) = 1;$$

(2) *the class* $\lambda_{\widetilde{K}}$ *maps to zero in* $H_1(Y_K;\mathbb{Q}(t))$.

For any parallel π *of* $\widetilde{K}$, *this class satisfies*

$$\lambda_{\widetilde{K}} = [\pi] - \ell k_{\mathbb{Q}(t)}(\widetilde{K},\pi)[\mu_{\widetilde{K}}].$$

Proof. We first prove existence and then uniqueness. For existence, pick any parallel π to $\widetilde{K}$, i.e. any curve in $\partial\overline{\nu}(\widetilde{K})$ that is isotopic to $\widetilde{K}$ in $\overline{\nu}(\widetilde{K})$ and define

$$\lambda_{\widetilde{K}} := [\pi] - \ell k_{\mathbb{Q}(t)}(\widetilde{K},\pi)[\mu_{\widetilde{K}}].$$

Here recall that the equivariant linking $r := \ell k_{\mathbb{Q}(t)}(\widetilde{K},\pi)$ is the unique element of $\mathbb{Q}(t)$ such that $[\pi] = r[\mu_{\widetilde{K}}]$ in $H_1(Y_K;\mathbb{Q}(t))$. The two axioms now follow readily.

For uniqueness, we suppose that $\lambda_{\widetilde{K}}$ and $\lambda'_{\widetilde{K}}$ are two homology classes as in the statement of the lemma. Choose a parallel π of $\widetilde{K}$ and base $H_1(\partial\overline{\nu}(K);\mathbb{Q}(t))$ by the pair $(\mu_{\widetilde{K}},\pi)$. This way, we can write $\lambda_{\widetilde{K}} = r_1[\mu_{\widetilde{K}}] + r_2[\pi]$ and $\lambda'_{\widetilde{K}} = r'_1[\mu_{\widetilde{K}}] + r'_2[\pi]$. The first condition on $\lambda_{\widetilde{K}}$ now promptly implies that $r_2 = r'_2 = 1$; formally

$$1 = \lambda_{\partial\overline{\nu}(K),\mathbb{Q}(t)}\big([\mu_{\widetilde{K}}],\lambda_{\widetilde{K}}\big) = r_2\lambda_{\partial\overline{\nu}(K),\mathbb{Q}(t)}\big([\mu_{\widetilde{K}}],[\pi]\big) = r_2$$

and similarly for r'_2. To see that $r_1 = r'_1$, observe that since $r_2 = r'_2$, we have that

$$\lambda_{\widetilde{K}} = \lambda'_{\widetilde{K}} + (r'_1 - r_1)[\mu_{\widetilde{K}}].$$

Recall that $[\mu_{\widetilde{K}}]$ is a generator of the vector space $H_1(Y_K;\mathbb{Q}(t)) = \mathbb{Q}(t)$ and that $\lambda'_{\widetilde{K}}$, $\lambda'_{\widetilde{K}}$ are zero in $H_1(Y_K;\mathbb{Q}(t))$. We conclude that $(r'_1 - r_1) = 0$, as required. ■

As motivation, we observe that for a link $L = K_1 \cup \cdots \cup K_n \subset S^3$, the group $H_1(E_L;\mathbb{Z})$ is freely generated by the meridians μ_{K_i} and, if L is framed with integral linking matrix A, then the framing curves π_i can be written in this basis as

$$[\pi_i] = \sum_{j=1}^{n} A_{ij}[\mu_{K_j}] \in H_1(E_L;\mathbb{Z}).$$

The situation is similar in our setting.

Proposition 3.11. *Let* $\widetilde{L} \subset Y^\infty$ *be an* n*-component framed link in covering general position whose components have framing curves* $\pi_1,\dots,\pi_n$. *Recall that*

$$H_1\big(Y_L;\mathbb{Q}(t)\big) = \mathbb{Q}(t)^n$$

is generated by the homology classes of the meridians $\mu_{\widetilde{K}_1},\dots,\mu_{\widetilde{K}_n}$. *The homology classes of the* π_i *in* $H_1(Y_L;\mathbb{Q}(t)) \cong \mathbb{Q}(t)^n$ *are related to the meridians by the formula*

$$[\pi_i] = \sum_{j=1}^{n}(A_{\widetilde{L}})_{ij}[\mu_{\widetilde{K}_j}] \in H_1\big(Y_L;\mathbb{Q}(t)\big).$$

Proof. By definition of the equivariant linking matrix $A_{\widetilde{L}}$, we must prove that

$$[\pi_i] = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi_i)[\mu_{\widetilde{K}_i}] + \sum_{j \neq i} \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j)[\mu_{\widetilde{K}_j}] \in H_1\big(Y_L; \mathbb{Q}(t)\big) \qquad (3.3)$$

for each i. Since the sum of the inclusion induced maps give rise to an isomorphism

$$H_1\big(Y_L; \mathbb{Q}(t)\big) \cong \bigoplus_{j=1}^{n} H_1\big(Y_{K_j}; \mathbb{Q}(t)\big)$$

it suffices to prove the equality after applying the inclusion map $H_1(Y_L; \mathbb{Q}(t)) \to H_1(Y_{K_j}; \mathbb{Q}(t))$, for each j. Since π_i is a parallel of $\widetilde{K}_i$, applying Lemma 3.10, we have

$$[\pi_i] = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi_i)[\mu_{\widetilde{K}_i}] + \lambda_{\widetilde{K}_i} \in H_1\big(\partial Y_{K_i}; \mathbb{Q}(t)\big).$$

We consider the image of this homology class in $H_1(Y_{K_j}; \mathbb{Q}(t))$ for $j = 1, \ldots, n$. In the vector space $H_1(Y_{K_i}; \mathbb{Q}(t)) = \mathbb{Q}(t)[\mu_{\widetilde{K}_i}]$, the longitude class $\lambda_{\widetilde{K}_i}$ vanishes (again by Lemma 3.10). For $j \neq i$, the class $[\mu_{\widetilde{K}_i}]$ vanishes in $H_1(Y_{K_j}; \mathbb{Q}(t))$; thus the image of $[\pi_i]$ in $H_1(Y_{K_j}; \mathbb{Q}(t))$ is

$$\ell k_{\mathbb{Q}(t)}(\pi_i, \widetilde{K}_j)[\mu_{\widetilde{K}_j}] = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j)[\mu_{\widetilde{K}_j}].$$

This concludes the proof of (3.3). ∎

From now on, we will be working with $\mathbb{Z}[t^{\pm 1}]$-coefficient homology both for Y and for the result $Y' := Y_{\mathbf{r}(t)}(L)$ of surgery on a framed link $L \subset Y$. Let W denote the trace of the surgery from Y to Y'. We therefore record a fact about the underlying coefficient systems for later reference.

Lemma 3.12. *The epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ can be extended to an epimorphism $\pi_1(W) \twoheadrightarrow \mathbb{Z}$, which by precomposition with the inclusion map induces an epimorphism $\varphi'\colon \pi_1(Y') \twoheadrightarrow \mathbb{Z}$.*

Proof. Note that $\pi_1(W)$ is obtained from $\pi_1(Y)$ by adding relators that kill each of the $[K_i] \in \pi_1(Y)$ (indeed W is obtained by adding 2-handles to $Y \times [0,1]$ along the K_i). Since φ is trivial on the $K_i \subset Y$ (because they lift to Y^∞), we deduce that φ descends to an epimorphism on $\pi_1(W)$.

The composition $\pi_1(Y') \to \pi_1(W) \twoheadrightarrow \mathbb{Z}$ is also surjective because $\pi_1(W)$ is obtained from $\pi_1(Y')$ by adding relators that kill each of the $[K_i'] \in \pi_1(Y')$; indeed W is obtained by adding 2-handles to $Y' \times [0,1]$ along the dual knots K_i'. ∎

Remark 3.13. In particular, note from the proof of Lemma 3.12 that the homomorphism $\varphi'\colon \pi_1(Y') \twoheadrightarrow \mathbb{Z}$ vanishes on the knots $K_i' \subset Y$ dual to the original $K_i \subset Y$.

The next lemma proves an infinite cyclic cover analogue of the following familiar statement: performing surgery on a framed link $L \subset S^3$ whose linking matrix is invertible over $\mathbb{Q}$ results in a rational homology sphere.

Lemma 3.14. *Let Y be a 3-manifold and let $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ be an epimorphism such that the Alexander module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is torsion. If $\widetilde{L} \subset Y^\infty$ is an n-component framed link in covering general position, whose equivariant linking matrix $A_{\widetilde{L}}$ is invertible over $\mathbb{Q}(t)$, then the result Y' of surgery on L satisfies $H_1(Y';\mathbb{Q}(t)) = 0$.*

Proof. The result will follow by studying the portion

$$\cdots \to H_2\big(Y, Y_L;\mathbb{Q}(t)\big) \xrightarrow{\partial} H_1\big(Y_L;\mathbb{Q}(t)\big) \to H_1\big(Y';\mathbb{Q}(t)\big) \to H_1\big(Y', Y_L;\mathbb{Q}(t)\big)$$

of the long exact sequence of the pair (Y, Y_L) with $\mathbb{Q}(t)$-coefficients, and arguing that $H_1(Y', Y_L;\mathbb{Q}(t)) = 0$ and that ∂ is an isomorphism.

The fact that $H_1(Y', Y_L;\mathbb{Q}(t)) = 0$ can be deduced from excision, replacing (Y', Y_L) with the pair $(\sqcup^n S^1 \times D^2, \sqcup^n S^1 \times S^1)$. For the same reason, the vector space $H_2(Y, Y_L;\mathbb{Q}(t)) = \mathbb{Q}(t)^n$ is based by the classes of the discs

$$\big(D^2 \times \{\mathrm{pt}\}\big)_i \subset (D^2 \times S^1)_i$$

whose boundaries are the framing curves π_i. To conclude that ∂ is indeed an isomorphism, note that $H_1(Y_L;\mathbb{Q}(t)) = \mathbb{Q}(t)^n$ is generated by the $[\mu_{\widetilde{K}_i}]$ (because the Alexander module of Y is torsion) and use Proposition 3.11 to deduce that with respect to these bases, ∂ is represented by the equivariant linking matrix $A_{\widetilde{L}}$. Since this matrix is by assumption invertible over $\mathbb{Q}(t)$, we deduce that ∂ is an isomorphism. It follows that $H_1(Y';\mathbb{Q}(t)) = 0$, as desired. ∎

The next lemma describes the framing on the dual of a framed link. The statement resembles [13, Lemma 1.5] and [80, Theorem 1.1].

Lemma 3.15. *Let Y be a 3-manifold and let $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ be an epimorphism such that the Alexander module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is torsion. If $\widetilde{L} \subset Y^\infty$ is a framed link in covering general position whose equivariant linking matrix $A_{\widetilde{L}}$ is invertible over $\mathbb{Q}(t)$, then the equivariant linking matrix of the dual framed link $\widetilde{L}'$ is*

$$A_{\widetilde{L}'} = -A_{\widetilde{L}}^{-1}.$$

Proof. Consider the exterior $Y_L = Y'_{L'}$ and recall that

$$H_1\big(Y_L;\mathbb{Q}(t)\big) = \mathbb{Q}(t)^n$$

is generated by the meridians $\mu_{\widetilde{K}_1}, \dots, \mu_{\widetilde{K}_n}$ of the link $\widetilde{L}$ because we assumed that $H_1(Y;\mathbb{Q}(t)) = 0$. Since we assumed that $H_1(Y;\mathbb{Q}(t)) = 0$ and $\det(A_{\widetilde{L}}) \neq 0$, we can apply Lemma 3.14 to deduce that $H_1(Y';\mathbb{Q}(t)) = 0$, and hence

$$H_1\big(Y_L;\mathbb{Q}(t)\big) = H_1\big(Y'_{L'};\mathbb{Q}(t)\big)$$

is also generated by the meridians $\mu_{\widetilde{K}'_1},\dots,\mu_{\widetilde{K}'_n}$ of the link $\widetilde{L}'$.

Thus the vector space $H_1(Y_L;\mathbb{Q}(t)) = \mathbb{Q}(t)^n$ has bases both

$$\boldsymbol{\mu} = \big([\mu_{\widetilde{K}_1}],\dots,[\mu_{\widetilde{K}_n}]\big) \quad \text{and} \quad \boldsymbol{\mu}' = \big([\mu_{\widetilde{K}'_1}],\dots,[\mu_{\widetilde{K}'_n}]\big),$$

and we let B be the change of basis matrix between these two bases so that $B\boldsymbol{\mu} = \boldsymbol{\mu}'$. Here and in the remainder of this proof, we adopt the following convention: if C is a matrix over $\mathbb{Q}(t)^n$ and if $\boldsymbol{x} = (x_1,\dots,x_n)$ is a collection of n vectors in $\mathbb{Q}(t)^n$, then we write $C\boldsymbol{x}$ for the collection of n vectors $Cx_1,\dots,Cx_n$.

Recall that for $i = 1,\dots,n$, the framing curves of the $\widetilde{K}_i$ and $\widetilde{K}'_i$ are respectively denoted by $\pi_i \subset Y^\infty$ and $\pi'_i \subset Y'^\infty$. Slightly abusing notation, we also write $[\pi_i]$ for the class of π_i in $H_1(Y_{K_i};\mathbb{Q}(t))$. We set $\boldsymbol{\pi} = ([\pi_1],\dots,[\pi_n])$ and $\boldsymbol{\pi}' = ([\pi'_1],\dots,[\pi'_n])$ and use Proposition 3.11 to deduce that

$$\boldsymbol{\pi} = A_{\widetilde{L}}\boldsymbol{\mu}, \quad \boldsymbol{\pi}' = A_{\widetilde{L}'}\boldsymbol{\mu}'$$

Inspecting the surgery instructions, we also have the relations

$$\boldsymbol{\mu}' = -\boldsymbol{\pi}, \quad \boldsymbol{\mu} = \boldsymbol{\pi}'.$$

We address the sign in Remark 3.16 below. Combining these equalities, we obtain

$$\boldsymbol{\mu} = \boldsymbol{\pi}' = A_{\widetilde{L}'}\boldsymbol{\mu}' = A_{\widetilde{L}'}B\boldsymbol{\mu}, \quad \boldsymbol{\mu}' = -\boldsymbol{\pi} = -A_{\widetilde{L}}\boldsymbol{\mu} = -A_{\widetilde{L}}B^{-1}\boldsymbol{\mu}'.$$

Unpacking the equality $A_{\widetilde{L}'}B\boldsymbol{\mu} = \boldsymbol{\mu}$, we deduce that $A_{\widetilde{L}'}B[\mu_{\widetilde{K}_i}] = [\mu_{\widetilde{K}_i}]$ for $i = 1,\dots,n$. But since the $[\mu_{\widetilde{K}_1}],\dots,[\mu_{\widetilde{K}_n}]$ form a basis for $\mathbb{Q}(t)^n$, this implies that

$$A_{\widetilde{L}'}B = I_n.$$

The same argument shows that $-A_{\widetilde{L}}B^{-1} = I_n$, and therefore both matrices $A_{\widetilde{L}}$, $A_{\widetilde{L}'}$ are invertible, with $-A_{\widetilde{L}} = B = A_{\widetilde{L}'}^{-1}$. ■

Remark 3.16. In the above proposition, we were concerned with the relationship between the curves $(\boldsymbol{\mu},\boldsymbol{\pi})$ and $(\boldsymbol{\mu}',\boldsymbol{\pi}')$, all of which represent classes in $H_1(\partial Y_L,\mathbb{Q}(t))$. We know from the surgery instructions that $g(\boldsymbol{\mu}) = \boldsymbol{\pi}'$. We are free to choose the collection of curves $g(\boldsymbol{\pi})$ so long as we choose each $g(\pi_i)$ to intersect π'_i geometrically once (as unoriented curves). We choose the unoriented curves $\pm\boldsymbol{\mu}'$. Since we know that the surgery was done to produce an oriented manifold, it must be the case that the gluing transformation $g\colon \partial Y_L \to \partial Y_L$ is orientation-preserving. The fact that g is orientation-preserving implies that it preserves intersections numbers, we deduce that $\delta_{ij} = \mu_i \cdot \pi_j = g(\mu_i)\cdot g(\pi_j) = \pi'_j \cdot (\pm\mu'_i)$. This forces $g(\boldsymbol{\pi}) = -\boldsymbol{\mu}'$.

4. Reidemeister torsion

We recall the definition of the Reidemeister torsion of a based chain complex as well as the corresponding definition for CW complexes. This will be primarily used in Section 5.3. References on Reidemeister torsion include [17, 89, 90].

Let $\mathbb{F}$ be a field. Given two bases u, v of a r-dimensional $\mathbb{F}$-vector space, we write $\det(u/v)$ for the determinant of the matrix taking v to u, i.e. the determinant of the matrix $A = (A_{ij})$ that satisfies $v^i = \sum_{j=1}^{r} A_{ij} u^j$. A *based chain complex* is a finite chain complex

$$C = \big(0 \to C_m \xrightarrow{\partial_{m-1}} C_{m-1} \xrightarrow{\partial_{m-2}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_0} C_0 \to 0\big)$$

of $\mathbb{F}$-vector spaces together with a basis c_i for each C_{i+1}. Given a based chain complex, fix a basis b_i for $B_i = \operatorname{im}(\partial_{i+1})$ and pick a lift $\widetilde{b}_i$ of b_i to C_i. Additionally, fix a basis h_i for each homology group $H_i(C)$ and let $\widetilde{h}_i$ be a lift of h_i to C_i. One checks that $(b_i, \widetilde{h}_i, \widetilde{b}_{i-1})$ forms a basis of C_i.

Definition 4.1. Let C be a based chain complex over $\mathbb{F}$ and let $\mathcal{B} = \{h_i\}$ be a basis for $H_*(C)$. The *Reidemeister torsion* of $(C, \mathcal{B})$ is defined as

$$\tau(C, \mathcal{B}) = \frac{\prod_i \det((b_{2i+1}, \widetilde{h}_{2i+1}, \widetilde{b}_{2i})|c_{2i+1})}{\prod_i \det((b_{2i}, \widetilde{h}_{2i}, \widetilde{b}_{2i-1})|c_{2i})} \in \mathbb{F} \setminus \{0\}.$$

Implicit in this definition is the fact that $\tau(C, \mathcal{B})$ depends neither on the choice of the basis b_i, nor on the choice of the lifts $\widetilde{b}_i$, nor on the choice of the lifts $\widetilde{h}_i$ of the h_i. It does depend on $\mathcal{B} = \{h_i\}$.

When C is acyclic, we drop $\mathcal{B}$ from the notation and simply write $\tau(C)$.

Note that we are following Turaev's sign convention [89, 90]; Milnor's convention [76] yields the multiplicative inverse of $\tau(C, \mathcal{B})$ (see [90, Remark 1.4 (5)]). The next result collects two properties of the torsion that will be used later on.

Proposition 4.2. *The following statements hold.*

(1) *Suppose that* $0 \to C' \to C \to C'' \to 0$ *is a short exact sequence of based chain complexes and that* $\mathcal{B}', \mathcal{B}$, *and* $\mathcal{B}''$ *are bases for* $H_*(C'), H_*(C)$ *and* $H_*(C'')$ *respectively. If we view the associated homology long exact sequence as an acyclic complex* $\mathcal{H}$, *based by* $\mathcal{B}, \mathcal{B}'$, *and* $\mathcal{B}''$ *respectively, then*

$$\tau(C, \mathcal{B}) = \tau(C', \mathcal{B}')\tau(C'', \mathcal{B}'')\tau(\mathcal{H}).$$

(2) *If* $C = (0 \to C_1 \xrightarrow{\partial_0} C_0 \to 0)$ *is an isomorphism between* n*-dimensional vector spaces, so that* C *is an acyclic based chain complex, then*

$$\tau(C) = \det(A)^{-1},$$

where A denotes the $n \times n$-matrix which represents ∂_0 with respect to the given bases.

Proof. The multiplicativity statement is proved in [76], The second statement follows from Definition 4.1; details are in [90, Remark 1.4 (3)]. ■

We now recall the definition of the torsion of a pair of CW complexes. We focus on the case where the spaces come with a map of their fundamental group to $\mathbb{Z}$. This is a special case of an analogous general theory for the case of an arbitrary group [90], and for more general twisted coefficients [46].

Let (X, A) be a finite CW pair, let $\varphi\colon \pi_1(X) \to \mathbb{Z}$ be a homomorphism, and let $\mathcal{B}$ be a basis for the $\mathbb{Q}(t)$-vector space $H_*(X, A; \mathbb{Q}(t))$. Write $p\colon X^\infty \to X$ for the cover corresponding to $\ker(\varphi)$ and set $A^\infty := p^{-1}(A)$. The chain complex $C_*(X^\infty, A^\infty)$ can be based over $\mathbb{Z}[t^{\pm 1}]$ by choosing a lift of each cell of (X, A) and orienting it; this also gives a basis of

$$C_*\big(X, A; \mathbb{Q}(t)\big) = C_*(X^\infty, A^\infty) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}(t).$$

Let $\mathcal{E}$ denote the resulting choice of basis for $C_*(X, A; \mathbb{Q}(t))$. We then define the torsion of (X, A, φ) as

$$\tau(X, A, \mathcal{B}, \mathcal{E}) := \tau\big(C_*(X, A; \mathbb{Q}(t)), \mathcal{B}, \mathcal{E}\big) \in \mathbb{Q}(t) \setminus \{0\}.$$

Given $p(t), q(t) \in \mathbb{Q}(t)$, we write $p(t) \doteq q(t)$ to indicate that $p(t)$ and $q(t)$ agree up to multiplication by $\pm t^k$, for some $k \in \mathbb{Z}$. This will enable us to obtain an invariant that does not depend on the choice of $\mathcal{E}$. We write

$$\tau(X, A, \mathcal{B}) := [\tau(X, A, \mathcal{B}, \mathcal{E})] \in \big(\mathbb{Q}(t) \setminus \{0\}\big)/\doteq,$$

for some choice of $\mathcal{E}$. It is known that $\tau(X, A, \mathcal{B})$ is well defined and is invariant under simple homotopy equivalence preserving $\mathcal{B}$ (see [90, Theorem 9.1]). We drop the $\mathcal{B}$ from the notation if $H_*(X, A; \mathbb{Q}(t)) = 0$.

Additionally, Chapman proved that $\tau(X, A, \mathcal{B})$ only depends on the underlying homeomorphism type of (X, A) (see [18]), and not on the particular CW structure. In particular, when (M, N) is a manifold pair, we can define $\tau(M, N, \mathcal{B})$ for any finite CW-structure on (M, N), We will only consider the Reidemeister torsion of 3-manifolds, and so every pair (M, N) we consider will admit a CW structure. It will not be relevant in this paper, but we note that it is possible to define Reidemeister torsion for topological 4-manifolds not known to admit a CW structure; see [43, Section 14] for a discussion.

Remark 4.3. The reason we consider Reidemeister torsion is its relation with Alexander polynomials; see Section 5.3 below. To this effect, we recall some relevant algebra. Let P be a $\mathbb{Z}[t^{\pm 1}]$-module with presentation

$$\mathbb{Z}[t^{\pm 1}]^m \xrightarrow{f} \mathbb{Z}[t^{\pm 1}]^n \to P \to 0.$$

Consider elements of the free modules $\mathbb{Z}[t^{\pm 1}]^m$ and $\mathbb{Z}[t^{\pm 1}]^n$ as row vectors and represent f by an $m \times n$ matrix A, acting on the right of the row vectors. By adding rows of zeros, corresponding to trivial relations, we may assume that $m \geq n$. The 0*-th elementary ideal* $E_0(P)$ of a finitely presented $\mathbb{Z}[t^{\pm 1}]$-module P is the ideal of $\mathbb{Z}[t^{\pm 1}]$ generated by all $n \times n$ minors of A. This definition is independent of the choice of the presentation matrix A. The *order* of P, denoted Δ_P, is then by definition a generator of the smallest principal ideal containing $E_0(P)$, i.e. the greatest common divisor of the minors. The order of P is well defined up to multiplication by units of $\mathbb{Z}[t^{\pm 1}]$ and if P admits a square presentation matrix, then $\Delta_P \doteq \det(A)$, where A is some square presentation matrix for P. It follows that for a $\mathbb{Z}[t^{\pm 1}]$-module P which admits a square presentation matrix, one has $P = 0$ if and only if $\Delta_P \doteq 1$. For more background on these topics,, we refer the reader to [90, Section 1.4].

5. Proof of Theorem 2.4

Now we prove Theorem 2.4 from the introduction. For the reader's convenience, we recall the statement of this result.

Theorem 5.1. *Let Y be a 3-manifold with an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module is torsion, and let (H, λ) be a non-degenerate Hermitian form over $\mathbb{Z}[t^{\pm 1}]$ presenting Y. If $b \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$ is an isometry, then there is a $\mathbb{Z}$-manifold M with equivariant intersection form $\lambda_M \cong \lambda$, boundary Y and with $b_M = b$. If the form is odd, then M can be chosen to have either $\operatorname{ks}(M) = 0$ or $\operatorname{ks}(M) = 1$.*

For the remainder of the section, we let Y be a 3-manifold, let $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ be an epimorphism, and let $p\colon Y^\infty \to Y$ be the infinite cyclic cover associated to (Y, φ). We assume that $H_1(Y; \mathbb{Z}[t^{\pm 1}]) := H_1(Y^\infty)$ is $\mathbb{Z}[t^{\pm 1}]$-torsion. We first describe the strategy of the proof and then carry out each of the steps successively.

5.1. Plan

Let $b\colon (\operatorname{coker}(\widehat{\lambda}), \partial\lambda) \to (H_1(Y; \mathbb{Z}[t^{\pm 1}]), -\operatorname{Bl}_Y)$ be an isometry. Precompose b with the projection $H^* \twoheadrightarrow \operatorname{coker}(\widehat{\lambda})$ to get an epimorphism $\pi\colon H^* \twoheadrightarrow H_1(Y; \mathbb{Z}[t^{\pm 1}])$.

In particular,

$$0 \to H \xrightarrow{\widehat{\lambda}} H^* \xrightarrow{\varpi} H_1\big(Y; \mathbb{Z}[t^{\pm 1}]\big) \to 0$$

is a presentation of Y. Pick generators $x_1, \dots, x_n$ for H and endow H^* with the dual basis $x_1^*, \dots, x_n^*$. Write Q for the matrix of λ in this basis. Note that $Q = \overline{Q}^T$ since λ is Hermitian. The strategy to prove Theorem 5.1 is as follows.

Step 1. Prove that one can represent the classes $\pi(x_1^*), \dots, \pi(x_n^*)$ by an n-component framed link $\widetilde{L} = \widetilde{K}_1 \cup \dots \cup \widetilde{K}_n$ with equivariant linking matrix $A_{\widetilde{L}} = -Q^{-T}$.

Step 2. Argue that the result Y' of surgery on $L = p(\widetilde{L})$ satisfies $H_1(Y'; \mathbb{Z}[t^{\pm 1}]) = 0$.

Step 3. There is a topological 4-manifold $B \simeq S^1$ with boundary Y' following [42, Section 11.6].

Step 4. Argue that the equivariant intersection form of the 4-manifold M defined below with boundary Y is represented by Q and prove that $b_M = b$. Here, the 4-manifold M and its infinite cyclic cover M^∞ are defined via

$$-M^\infty := \left((Y^\infty \times [0,1]) \cup \bigcup_{i=1}^{n} \bigcup_{j_i \in \mathbb{Z}} t^{j_i} h_i^{(2)} \right) \cup_{Y'^\infty} -B^\infty,$$

$$-M := \left((Y \times [0,1]) \cup \bigcup_{i=1}^{n} h_i^{(2)} \right) \cup_{Y'} -B,$$

where upstairs the 2-handles $h_i^{(2)}$ are attached along the link L^∞; downstairs, one attaches the 2-handles along the projection $L = p(L^\infty)$ of this link.

Step 5. If λ is odd, then we use the star construction [42, 84] to show that both values of the Kirby–Siebenmann invariant can occur.

5.2. Step 1: Constructing a link with the appropriate equivariant linking matrix

We continue with the notation from the previous section. In particular, we have a presentation

$$0 \to H \xrightarrow{\widehat{\lambda}} H^* \xrightarrow{\varpi} H_1\big(Y; \mathbb{Z}[t^{\pm 1}]\big) \to 0$$

and a basis $x_1, \dots, x_n$ for H with dual basis $x_1^*, \dots, x_n^*$ for H^*. The aim of this section is to prove that it is possible to represent the generators $\pi(x_1^*), \dots, \pi(x_n^*)$ of $H_1(Y; \mathbb{Z}[t^{\pm 1}])$ by a framed link $\widetilde{L} = \widetilde{K}_1 \cup \dots \cup \widetilde{K}_n \subset Y^\infty$ whose transposed equivariant linking matrix agrees with $-Q^{-1}$; see Proposition 5.4. In other words, we must have

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_j, \widetilde{K}_i) = -(Q^{-1})_{ij} \quad \text{and} \quad \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi_i) = -(Q^{-1})_{ii},$$

where π_i is the framing curve of $\widetilde{K}_i$. Since the Blanchfield form Bl_Y is represented by the $\mathbb{Q}(t)$-coefficient matrix $-Q^{-1}$ (see [27, Section 3]), we know from Proposition 3.8 that any link representing the $\pi(x_i^*)$ must satisfy these relations up to adding a polynomial in $\mathbb{Z}[t^{\pm 1}]$. Most of this section therefore concentrates on showing that the equivariant linking (resp. framing) of an arbitrary framed link in Y^∞ can be changed by any polynomial (resp. symmetric polynomial) in $\mathbb{Z}[t^{\pm 1}]$, without changing the homology classes defined by the components of this link.

We start by showing how to modify the equivariant linking between distinct components of a link, without changing the homology class of the link.

Lemma 5.2. *Let $\widetilde{L} = \widetilde{K}_1 \cup \cdots \cup \widetilde{K}_n \subset Y^\infty$ be an n-component framed link in covering general position, with parallels $\pi_1, \dots, \pi_n$. For every distinct i, j and every polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$, there is a framed link*

$$\widetilde{L}' := \widetilde{K}_1 \cup \cdots \cup \widetilde{K}_{i-1} \cup \widetilde{K}_i' \cup \widetilde{K}_{i+1} \cup \cdots \cup \widetilde{K}_n,$$

also in covering general position, such that:

(1) *the knot $\widetilde{K}_i'$ is isotopic to $\widetilde{K}_i$ in Y^∞. In particular,*

$$[\widetilde{K}_i'] = [\widetilde{K}_i]$$

in $H_1(Y; \mathbb{Z}[t^{\pm 1}])$;

(2) *the equivariant linking between $\widetilde{K}_i$ and $\widetilde{K}_j$ is changed by $p(t)$, i.e.*

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \widetilde{K}_j) = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j) + p(t);$$

(3) *the equivariant linking between $\widetilde{K}_i$ and $\widetilde{K}_\ell$ is unchanged for $\ell \neq i, j$;*

(4) *the framing coefficients are unchanged; that is, there is a parallel γ_i for $\widetilde{K}_i'$ such that*

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \gamma_i) = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi_i).$$

Proof. Without loss of generality we can assume that $p(t) = mt^k$ for $m, k \in \mathbb{Z}$. The new knot $\widetilde{K}_i'$ is then obtained by band summing $\widetilde{K}_i$ with m meridians of $t^{-k}\widetilde{K}_j$, framed using the bounding framing induced by meridional discs. The first, third, and fourth properties of $\widetilde{K}_i'$ are immediate: clearly the linking of $\widetilde{K}_i$ with $\widetilde{K}_\ell$ is unchanged for $\ell \neq i, j$ and since the aforementioned meridians bound discs in Y^∞ over which the framing extends, we see that $\widetilde{K}_i'$ is framed isotopic (and in particular homologous) to $\widetilde{K}_i$ in Y^∞. It follows that the framing coefficient is unchanged.

The second property is obtained from a direct calculation using the sesquilinearity of equivariant linking numbers:

$$\begin{aligned}\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \widetilde{K}_j) &= \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j) + m\ell k_{\mathbb{Q}(t)}(t^{-k}\mu_{\widetilde{K}_j}, \widetilde{K}_j) \\ &= \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j) + mt^k.\end{aligned}$$

■

Next, we show how to modify the framing of a framed link component by a symmetric polynomial $p = \overline{p}$, without changing the homology class of the link.

Lemma 5.3. *Let $\widetilde{L} = \widetilde{K}_1 \cup \cdots \cup \widetilde{K}_n \subset Y^\infty$ be an n-component framed link in covering general position. Fix a parallel π_i for $\widetilde{K}_i$. For each $i = 1, \ldots, n$ and every symmetric polynomial $p(t) = p(t^{-1})$, there exists a knot $\widetilde{K}_i' \subset Y^\infty$ and a parallel γ_i of $\widetilde{K}_i'$ such that*

1. *the knot $\widetilde{K}_i'$ is isotopic to $\widetilde{K}_i$ in $Y^\infty \setminus \cup_{j \neq i} \widetilde{K}_j$, and in particular $[\widetilde{K}_i'] = [\widetilde{K}_i]$ in $H_1(Y; \mathbb{Z}[t^{\pm 1}])$;*
2. *the framing coefficient of $\widetilde{K}_i$ is changed by $p(t)$, i.e.*
$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \gamma_i) = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi_i) + p(t);$$
3. *the other linking numbers are unchanged: $\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \widetilde{K}_j) = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{K}_j)$ for all $j \neq i$.*

Proof. We first prove the lemma when $p(t)$ has no constant term. In this case, it suffices to show how to change the self-linking number by $m(t^k + t^{-k})$ for $k \neq 0$. To achieve this, band sum $\widetilde{K}_i$ with m meridians of $t^k \widetilde{K}_i$. As in the proof of Lemma 5.2, the first and third properties of $\widetilde{K}_i$ are clear. To define γ_i and prove the second property, define $\mu'_{\widetilde{K}_i}$ to be a parallel of $\mu_{\widetilde{K}_i}$ with $\ell k_{\mathbb{Q}(t)}(\mu_{\widetilde{K}_i}, \mu'_{\widetilde{K}_i}) = 0$ in Y^∞. Define γ_i to be the parallel of $\widetilde{K}_i'$ obtained by banding π_i to m copies of $t^k \mu'_{\widetilde{K}_i}$, using bands which are push-offs of the bands used to define $\widetilde{K}_i'$, and parallel copies of the meridian chosen with the zero-framing with respect to the framing induced by the associated meridional disc. Using the sesquilinearity of equivariant linking numbers, we obtain

$$\begin{aligned}\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \gamma_i) &= \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi_i) + m\, \ell k_{\mathbb{Q}(t)}(t^k \mu_{\widetilde{K}_i}, \pi_i) \\ &\qquad + m\, \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, t^k \mu'_{\widetilde{K}_i}) + \ell k_{\mathbb{Q}(t)}(\mu_{\widetilde{K}_i}, \mu'_{\widetilde{K}_i}) \\ &= \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \pi) + m(t^k + t^{-k}).\end{aligned}$$

We have therefore shown how to modify the self-linking within a fixed homology class by a symmetric polynomial with no constant term.

The general case follows: thanks to the previous paragraph, it suffices to describe how to change the self-linking by a constant, and this can be arranged by varying the choice of the parallel γ_i i.e. by additionally winding an initial choice of γ_i around the appropriate number of meridians of $\widetilde{K}_i'$. ∎

By combining the previous two lemmas, we can now prove the main result of this section.

Proposition 5.4. *Let $0 \to H \xrightarrow{\widehat{\lambda}} H^* \xrightarrow{\varpi} H_1(Y; \mathbb{Z}[t^{\pm 1}]) \to 0$ be a presentation of Y. Pick generators $x_1, \ldots, x_n$ for H and endow H^* with the dual basis $x_1^*, \ldots, x_n^*$. Let Q*

be the matrix of λ with respect to these bases. The classes $\pi(x_1^*), \dots, \pi(x_n^*)$ can be represented by simple closed curves $\widetilde{K}_1, \dots, \widetilde{K}_n \subset Y^\infty$ such that $\widetilde{L} = \widetilde{K}_1 \cup \dots \cup \widetilde{K}_n$ is in covering general position and satisfies the following properties:

(1) *the equivariant linking of the $\widetilde{K}_i$ satisfy*

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_j, \widetilde{K}_i) = -(Q^{-1})_{ij} \quad \text{for } i \neq j;$$

(2) *there exist parallels $\gamma_1, \dots, \gamma_n$ of $\widetilde{K}_1, \dots, \widetilde{K}_n$ such that*

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \gamma_i) = -(Q^{-1})_{ii}.$$

In particular, the parallel γ_i represents the homology class $-(Q^{-1})_{ii}[\mu_{\widetilde{K}_i}] + \lambda_{\widetilde{K}_i} \in H_1(\partial\overline{\nu}(K_i); \mathbb{Q}(t))$ and the transpose of the equivariant linking matrix of $\widetilde{L}$ is equal to $-Q^{-1}$.

Proof. Represent the classes $\pi(x_1^*), \dots, \pi(x_n^*)$ by an n-component link in Y^∞ that can be assumed to be in covering general position. Use $\widetilde{J}_1, \dots, \widetilde{J}_n$ to denote the components of this link. Thanks to Lemma 5.2, we can assume that the equivariant linking numbers of these knots coincide with the off-diagonal terms of Q^{-1}; we can apply this lemma because for $i \neq j$ the rational functions $\ell k_{\mathbb{Q}(t)}(\widetilde{J}_j, \widetilde{J}_i)$ and the corresponding $-(Q^{-1})_{ij}$ both reduce mod $\mathbb{Z}[t^{\pm 1}]$ to $\mathrm{Bl}_Y(\pi(x_i^*), \pi(x_j^*))$ and thus differ by a Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$.

We arrange the framings and last assertion simultaneously. For brevity, from now on we write $r_i := -(Q^{-1})_{ii}$. By Lemma 3.10, for each i, the class $r_i[\mu_{\widetilde{J}_i}] + \lambda_{\widetilde{J}_i}$ can be rewritten as

$$\big(r_i - \ell k_{\mathbb{Q}(t)}(\widetilde{J}_i, \pi_i)\big)[\mu_{\widetilde{J}_i}] + [\pi_i]$$

for any choice of parallel π_i for $\widetilde{J}_i$. Note that $r_i - \ell k_{\mathbb{Q}(t)}(\widetilde{J}_i, \pi_i)$ is a Laurent polynomial: indeed both r_i and $\ell k_{\mathbb{Q}(t)}(\widetilde{J}_i, \pi_i)$ reduce mod $\mathbb{Z}[t^{\pm 1}]$ to $\mathrm{Bl}_Y(\pi([x_i^*]), \pi([x_i^*]))$.

Claim. *The polynomial $r_i - \ell k_{\mathbb{Q}(t)}(\widetilde{J}_i, \pi_i)$ is symmetric.*

Proof. We first assert that if σ is a parallel of $\widetilde{J}_i$, then $\ell k_{\mathbb{Q}(t)}(\sigma, \widetilde{J}_i)$ is symmetric. The rational function $\ell k_{\mathbb{Q}(t)}(\sigma, \widetilde{J}_i)$ is symmetric if and only if

$$\ell k_{\mathbb{Q}(t)}(\sigma, \widetilde{J}_i) = \overline{\ell k_{\mathbb{Q}(t)}(\sigma, \widetilde{J}_i)}.$$

By the symmetry property of the equivariant linking form mentioned in Proposition 3.8, this is equivalent to the equality $\ell k_{\mathbb{Q}(t)}(\sigma, \widetilde{J}_i) = \ell k_{\mathbb{Q}(t)}(\widetilde{J}_i, \sigma)$ and in turn this equality holds because the ordered link $(\sigma, \widetilde{J}_i)$ is isotopic to the ordered link $(\widetilde{J}_i, \sigma)$ in Y^∞. This concludes the proof of the assertion that $\ell k_{\mathbb{Q}(t)}(\sigma, \widetilde{J}_i)$ is symmetric.

We conclude the proof of the claim. Thanks to the assertion, it now suffices to prove that r_i is symmetric. To see this, note that since the matrix Q^{-1} is Hermitian (because Q is) we have

$$r_i(t^{-1}) = -(\overline{Q^{-1}})_{ii} = -(\overline{Q^{-T}})_{ii} = -(Q^{-1})_{ii} = r_i(t),$$

as required. ∎

We can now apply Lemma 5.3 to $p(t) := r_i - \ell k_{\mathbb{Q}(t)}(\widetilde{J}_i, \pi_i)$ (which is symmetric by the claim) to isotope the $\widetilde{J}_i$ to knots $\widetilde{K}_i$ (without changing the equivariant linking) and to find parallels $\gamma_1, \dots, \gamma_n$ of $\widetilde{K}_1, \dots, \widetilde{K}_n$ that satisfy the equalities $-(Q^{-1})_{ii} = r_i = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \gamma_i)$. This proves the second item of the proposition and the assertions in the last sentence follow because $r_i[\mu_{\widetilde{K}_i}] + \lambda_{\widetilde{K}_i} = [\gamma_i]$ (by Lemma 3.10) and from the definition of the equivariant linking matrix. ∎

5.3. Step 2: The result of surgery is a $\mathbb{Z}[t^{\pm 1}]$-homology $S^1 \times S^2$

Let $\widetilde{L} \subset Y^\infty$ be a framed link in covering general position. Let Y' be the effect of surgery on the framed link $L = p(\widetilde{L})$ with equivariant linking matrix $A_{\widetilde{L}}$ over $\mathbb{Q}(t)$. We assume throughout this subsection that $\det(A_{\widetilde{L}}) \neq 0$. Our goal is to calculate the Alexander polynomial $\Delta_{Y'}$ in terms of Δ_Y and of the equivariant linking matrix of $\widetilde{L} \subset Y^\infty$. In Theorem 5.8 we will show that

$$\Delta_{Y'} \doteq \Delta_Y \det(A_{\widetilde{L}}). \tag{5.1}$$

We then apply this to the framed link $\widetilde{L} \subset Y^\infty$ that we built in Proposition 5.4; this framed link satisfies $\det(A_{\widetilde{L}}) = \det(Q^{-T}) \neq 0$. Continuing with the notation from that proposition, we have

$$\det(A_{\widetilde{L}}) = \det(-Q^{-T}) \doteq \frac{1}{\Delta_Y}$$

(because Q presents $H_1(Y; \mathbb{Z}[t^{\pm 1}])$) so in this case (5.1) implies that $\Delta_{Y'} \doteq 1$, which in turn implies that Y' is a $\mathbb{Z}[t^{\pm 1}]$-homology $S^1 \times S^2$; see Remark 4.3 and Proposition 5.9.

We start by outlining the proof of (5.1), which will later be recorded as Theorem 5.8.

Outline of proof of Theorem 5.8. We plan to compute the Reidemeister torsion $\tau(Y')$ in terms of the Reidemeister torsion $\tau(Y)$, and then, for $Z = Y, Y'$ to use the relation

$$\Delta_Z = \tau(Z)(t-1)^2 \tag{5.2}$$

from [89, Theorem 1.1.2] to derive (5.1). We note that in our setting we are allowed to write $\tau(Y)$ and $\tau(Y')$ for the Reidemeister torsions without having to choose bases $\mathcal{B}$; this is because both $H_*(Y;\mathbb{Q}(t)) = 0$ and $H_*(Y';\mathbb{Q}(t)) = 0$, recall Lemma 3.14 and Section 4; here note that we can apply Lemma 3.14 because we are assuming that $\det(A_{\tilde{L}}) \neq 0$.

We will calculate $\tau(Y')$ from $\tau(Y)$ by studying the long exact sequence of the pairs (Y, Y_L) and (Y', Y_L) with $\mathbb{Q}(t)$ coefficients. More concretely, in Construction 5.5, we endow the $\mathbb{Q}(t)$-vector spaces $H_*(Y, Y_L;\mathbb{Q}(t))$, $H_*(Y', Y_L;\mathbb{Q}(t))$, and $H_*(Y_L;\mathbb{Q}(t))$ with bases that we denote by $\mathcal{B}_{Y,Y_L}$, $\mathcal{B}_{Y',Y_L}$, and $\mathcal{B}_{Y_L}$ respectively. In Lemma 5.6, we then show that

$$\tau(Y)\tau(\mathcal{H}_L)^{-1} \doteq \tau(Y_L, \mathcal{B}_{Y_L}) \doteq \tau(Y')\tau(\mathcal{H}_{L'})^{-1},$$

where $\mathcal{H}_L$ and $\mathcal{H}_{L'}$ respectively denote the long exact sequences in $\mathbb{Q}(t)$-homology of the pairs (Y, Y_L) and (Y', Y_L). Finally, we prove that $\tau(\mathcal{H}_L) \doteq 1$ and $\tau(\mathcal{H}_{L'}) \doteq \det(A_{\tilde{L}})$. From (5.2) and the previous equation we then deduce

$$\frac{\Delta_Y}{(t-1)^2 \cdot 1} \doteq \tau(Y)\tau(\mathcal{H}_L)^{-1} \doteq \tau(Y')\tau(\mathcal{H}_{L'})^{-1} \doteq \frac{\Delta_{Y'}}{(t-1)^2 \cdot \det(A_{\tilde{L}})}.$$

The equality $\Delta_{Y'} \doteq \Delta_Y \det(A_{\tilde{L}})$ follows promptly. ■

We start filling in the details with our choice of bases for the previously mentioned $\mathbb{Q}(t)$-homology vector spaces.

Construction 5.5. We fix the bases for $H_*(Y, Y_L;\mathbb{Q}(t))$, $H_*(Y', Y_L;\mathbb{Q}(t))$, and $H_*(Y_L;\mathbb{Q}(t))$, that we will respectively denote by $\mathcal{B}_{Y,Y_L}$, $\mathcal{B}_{Y',Y_L}$, and $\mathcal{B}_{Y_L}$.

(1) We base the $\mathbb{Q}(t)$-vector spaces $H_*(Y, Y_L;\mathbb{Q}(t))$ and $H_*(Y', Y_L;\mathbb{Q}(t))$. Excising $\mathring{Y}_L$, we obtain

$$H_i\big(Y, Y_L;\mathbb{Q}(t)\big) = \bigoplus_{i=1}^{n} H_i\big(D^2 \times S^1, S^1 \times S^1;\mathbb{Q}(t)\big),$$

where n is the number of components of L. Similarly, by excising $\mathring{Y}_L \cong \mathring{Y}_{L'}$, we have

$$H_i\big(Y', Y_L;\mathbb{Q}(t)\big) = \bigoplus_{i=1}^{n} H_i\big(S^1 \times D^2, S^1 \times S^1;\mathbb{Q}(t)\big).$$

Since the map $\pi_1(S^1) \to \mathbb{Z}$ determining the coefficients is trivial,

$$\begin{aligned}\bigoplus_{i=1}^{n} H_i\big(S^1 \times D^2, S^1 \times S^1;\mathbb{Q}(t)\big) &\cong \bigoplus_{i=1}^{n} H^{3-i}\big(S^1;\mathbb{Q}(t)\big)\\ &\cong \bigoplus_{i=1}^{n} H^{3-i}(S^1;\mathbb{Z}) \otimes \mathbb{Q}(t).\end{aligned}$$

These homology vector spaces are only non-zero when $i = 2, 3$. in which case they are isomorphic to $\mathbb{Q}(t)^n$.

We now pick explicit generators for these vector spaces. Endow $S^1 \times S^1$ with its usual cell structure, with one 0-cell, two 1-cells and one 2-cell $e^2_{S^1 \times S^1}$. Note that $D^2 \times S^1$ is obtained from $S^1 \times S^1 \times I$ by additionally attaching a 3-dimensional 2-cell $e^2_{D^2 \times S^1}$ and 3-cell, $e^3_{D^2 \times S^1}$, where on the chain level

$$\partial e^3_{D^2\times S^1} = e^2_{D^2\times S^1} + e^2_{S^1\times S^1} - e^2_{D^2\times S^1} = e^2_{S^1\times S^1}.$$

We now fix once and for all lifts of these cells to the covers. It follows that for $k = 2, 3$:

$$\begin{aligned} H_k\big(Y, Y_L; \mathbb{Q}(t)\big) &= C_k\big(Y, Y_L; \mathbb{Q}(t)\big) \\ &= C_k\big(D^2 \times S^1, S^1 \times S^1; \mathbb{Q}(t)\big) = \bigoplus_{i=1}^n \mathbb{Q}(t)\big(\widetilde{e}^k_{D^2\times S^1}\big)_i, \\ H_k\big(Y', Y_L; \mathbb{Q}(t)\big) &= C_k\big(Y', Y_L; \mathbb{Q}(t)\big) \\ &= C_k\big(S^1 \times D^2, S^1 \times S^1; \mathbb{Q}(t)\big) = \bigoplus_{i=1}^n \mathbb{Q}(t)\big(\widetilde{e}^k_{S^1\times D^2}\big)_i. \end{aligned}$$

(2) We now base $H_*(Y_L; \mathbb{Q}(t))$. Since $H_*(Y; \mathbb{Q}(t)) = 0$, a Mayer–Vietoris argument shows that

$$H_1\big(Y_L; \mathbb{Q}(t)\big) \cong \mathbb{Q}(t)^n,$$

generated by the meridians $\mu_{\widetilde{K}_i}$ of $\widetilde{L}$. Mayer–Vietoris also shows that the inclusion of the boundary induces an isomorphism $\mathbb{Q}(t)^n = H_2(\partial Y_L; \mathbb{Q}(t)) \cong H_2(Y_L; \mathbb{Q}(t))$. We can then base $H_2(Y_L; \mathbb{Q}(t))$ using fixed lifts of the aforementioned 2-cells $(e^2_{S^1\times S^1})_i$ generating each of the torus factors of ∂Y_L. Summarising, we have

$$H_1\big(Y_L; \mathbb{Q}(t)\big) = \bigoplus_{i=1}^n \mathbb{Q}(t)\mu_{\widetilde{K}_i}, \quad H_2\big(Y_L; \mathbb{Q}(t)\big) = \bigoplus_{i=1}^n \mathbb{Q}(t)\big(\widetilde{e}^2_{S^1\times S^1}\big)_i.$$

The next lemma reduces the calculation of $\Delta_{Y'}$ to the calculation of $\tau(\mathcal{H}_L)$ and $\tau(\mathcal{H}_{L'})$. Here, recall that $\tau(\mathcal{H}_L)$ and $\tau(\mathcal{H}_{L'})$ denote the torsion of the long exact sequences $\mathcal{H}_L$ and $\mathcal{H}_{L'}$ of the pairs (Y, Y_L) and (Y', Y_L), viewed as based acyclic complexes with bases $\mathcal{B}_{Y_L}$, $\mathcal{B}_{Y,Y_L}$, and $\mathcal{B}_{Y',Y_L}$.

Lemma 5.6. *If* $H_1(Y; \mathbb{Q}(t)) = 0$ *and* $\det(A_{\widetilde{L}}) \neq 0$, *then we have*

$$\tau(Y) \doteq \tau(Y_L, \mathcal{B}_{Y_L}) \cdot \tau(\mathcal{H}_L), \quad \tau(Y') \doteq \tau(Y_L, \mathcal{B}_{Y_L}) \cdot \tau(\mathcal{H}_{L'}).$$

In particular, we have

$$\Delta_{Y'} \cdot \tau(\mathcal{H}_L) \doteq \Delta_Y \cdot \tau(\mathcal{H}_{L'}).$$

Proof. We start by proving that the last statement follows from the first. First note that since the vector spaces $H_1(Y;\mathbb{Q}(t))$ and $H_1(Y';\mathbb{Q}(t))$ vanish (for the latter we use Lemma 3.14 which applies since $\det(A_{\widetilde{L}}) \neq 0$), the Alexander polynomials of Y and Y' are non-zero. Next, [89, Theorem 1.1.2] implies that $\tau(Y)(t-1)^2 = \Delta_Y$ and similarly for Y'. Therefore, $\Delta_{Y'}/\Delta_Y = \tau(Y')/\tau(Y)$. The first part of the lemma implies that $\tau(Y')/\tau(Y) = \tau(\mathcal{H}_{L'})/\tau(\mathcal{H}_L)$. Combining these equalities,

$$\frac{\Delta_{Y'}}{\Delta_Y} = \frac{\tau(Y')}{\tau(Y)} = \frac{\tau(\mathcal{H}_{L'})}{\tau(\mathcal{H}_L)},$$

from which the required statement follows immediately.

To prove the first statement of the lemma, it suffices to prove that

$$\tau(Y, Y_L, \mathcal{B}_{Y,Y_L}) = 1,$$

as well as $\tau(Y', Y_L, \mathcal{B}_{Y',Y_L}) = 1$: indeed, the required equalities then follow by applying the multiplicativity of Reidemeister torsion (the first item of Proposition 4.2) to the short exact sequences

$$0 \to C_*\big(Y_L;\mathbb{Q}(t)\big) \to C_*\big(Y;\mathbb{Q}(t)\big) \to C_*\big(Y, Y_L;\mathbb{Q}(t)\big) \to 0,$$

leading to

$$\tau(Y) = \tau(Y_L)\cdot\tau(Y, Y_L, \mathcal{B}_{Y,Y_L})\cdot\tau(\mathcal{H}_L) = \tau(Y_L)\cdot 1\cdot\tau(\mathcal{H}_L),$$

as desired. And similarly for the pair (Y', Y_L).

We use Definition 4.1 to prove that $\tau(Y, Y_L, \mathcal{B}_{Y,Y_L}) = 1$; again the proof for L' is analogous. We endow Y and Y_L with cell structures for which Y_L and ∂Y_L are subcomplexes of Y, and Y is obtained from Y_L by attaching n solid tori to ∂Y_L. By definition of the relative chain complex, we have

$$C_*\big(Y, Y_L;\mathbb{Q}(t)\big) = C_*\big(Y;\mathbb{Q}(t)\big)/C_*\big(Y_L;\mathbb{Q}(t)\big).$$

Since we are working with cellular chain complexes we deduce that

$$\begin{aligned} C_*\big(Y, Y_L;\mathbb{Q}(t)\big) &= C_*\big(Y;\mathbb{Q}(t)\big)/C_*\big(Y_L;\mathbb{Q}(t)\big) \\ &= \bigoplus_{i=1}^{n} C_*\big(D^2\times S^1;\mathbb{Q}(t)\big)/C_*\big(S^1\times S^1;\mathbb{Q}(t)\big). \end{aligned}$$

Using the cell structures described in Construction 5.5, $D^2\times S^1$ is obtained from $S^1\times S^1$ by attaching a 2-cell and a 3-cell. By the above sequence of isomorphisms, this shows that $C_i(Y, Y_L;\mathbb{Q}(t)) = 0$ for $i \neq 2, 3$ and gives a basis for $C_2(Y, Y_L;\mathbb{Q}(t))$ and $C_3(Y, Y_L;\mathbb{Q}(t))$. In fact, this also implies that

$$C_i\big(Y, Y_L;\mathbb{Q}(t)\big) = H_i\big(Y, Y_L;\mathbb{Q}(t)\big)$$

and that the differentials in the chain complex are zero, as was mentioned in Construction 5.5. Thus, the basis of $C_*(Y, Y_L; \mathbb{Q}(t))$ corresponds exactly to the way we based $H_*(Y, Y_L; \mathbb{Q}(t))$ in Construction 5.5. Therefore the change of basis matrix is the identity and so the torsion is equal to 1. This concludes the proof of the lemma. ■

Our goal is now to show that $\tau(\mathcal{H}_L) \doteq 1$ and $\tau(\mathcal{H}_{L'}) \doteq \det(A_{\widetilde{L}})$. We start by describing the long exact sequences $\mathcal{H}_L$ and $\mathcal{H}_{L'}$.

Lemma 5.7. *Assume that* $H_1(Y_L; \mathbb{Q}(t)) = 0$ *and* $\det(A_{\widetilde{L}}) \neq 0$. *The only non-trivial portions of the long exact sequence of the pairs* (Y, Y_L) *and* $(Y, Y_{L'})$ *with* $\mathbb{Q}(t)$-*coefficients are of the following form:*

$$\begin{aligned}\mathcal{H}_L = \Big(0 \to H_3\big(Y, Y_L; \mathbb{Q}(t)\big) \\ \xrightarrow{\partial_3^L} H_2\big(Y_L; \mathbb{Q}(t)\big) \to 0 \to H_2\big(Y, Y_L; \mathbb{Q}(t)\big) \xrightarrow{\partial_2^L} H_1\big(Y_L; \mathbb{Q}(t)\big) \to 0\Big),\\ \mathcal{H}_{L'} = \Big(0 \to H_3\big(Y', Y_L; \mathbb{Q}(t)\big) \\ \xrightarrow{\partial_3^{L'}} H_2\big(Y_L; \mathbb{Q}(t)\big) \to 0 \to H_2\big(Y', Y_L; \mathbb{Q}(t)\big) \xrightarrow{\partial_2^{L'}} H_1\big(Y_L; \mathbb{Q}(t)\big) \to 0\Big).\end{aligned}$$

Additionally, with respect to the bases of Construction 5.5,

- *the homomorphism* $\partial_2^{L'}$ *is represented by* $-A_{\widetilde{L}}^{-1}$, *i.e. minus the inverse of the equivariant linking matrix for* $\widetilde{L}$;
- *the homomorphisms* ∂_2^L, ∂_3^L, *and* $\partial_3^{L'}$ *are represented by identity matrices.*

Proof. Since Y^∞ and Y'^∞ are connected, we have

$$H_0\big(Y; \mathbb{Z}[t^{\pm 1}]\big) = \mathbb{Z} \quad \text{and} \quad H_0\big(Y'; \mathbb{Z}[t^{\pm 1}]\big) = \mathbb{Z},$$

so $H_0(Y; \mathbb{Q}(t)) = 0$ and $H_0(Y'; \mathbb{Q}(t)) = 0$. Since we are working with field coefficients, Poincaré duality and the universal coefficient theorem imply that

$$H_3\big(Y; \mathbb{Q}(t)\big) = 0 \quad \text{and} \quad H_3\big(Y'; \mathbb{Q}(t)\big) = 0.$$

As observed in Construction 5.5 above, by excision, the only non-zero relative homology groups of (Y, Y_L) and (Y', Y_L) are

$$H_i\big(Y, Y_L; \mathbb{Q}(t)\big) = \mathbb{Q}(t)^n \quad \text{and} \quad H_i\big(Y', Y_L; \mathbb{Q}(t)\big) = \mathbb{Q}(t)^n n$$

for $i = 2, 3$. Next, since by assumption $H_1(Y; \mathbb{Q}(t)) = 0$, duality and the universal coefficient theorem imply that $H_2(Y; \mathbb{Q}(t)) = 0$. Since we proved in Lemma 3.14 that

$$H_1\big(Y'; \mathbb{Q}(t)\big) = 0,$$

(here we used $\det(A_{\widetilde{L}}) \neq 0$) the same argument shows that $H_2(Y';\mathbb{Q}(t)) = 0$. This establishes the first part of the lemma.

We now prove the statement concerning ∂_2^L and $\partial_2^{L'}$. Recall from Construction 5.5 that we based the vector spaces $H_2(Y, Y_L;\mathbb{Q}(t))$ and $H_2(Y', Y_L;\mathbb{Q}(t))$ by meridional discs to the $\widetilde{K}_i$ and $\widetilde{K}_i'$ respectively. The map ∂_2^L takes each disc to its boundary, the meridian $\mu_{\widetilde{K}_i}$; since these meridians form our chosen basis for $H_1(Y_L;\mathbb{Q}(t))$, we deduce that ∂_2^L is represented by the identity matrix. The map $\partial_2^{L'}$ also takes each meridional disc to its boundary, the meridian $\widetilde{\mu}_{K_i'}$ to the dual knot. It follows that $\partial_2^{L'}$ is represented by the change of basis matrix B such that $\boldsymbol{\mu}' = B\boldsymbol{\mu}$. But during the proof of Lemma 3.15 we saw that $B = -A_{\widetilde{L}}^{-1}$.

Finally, we prove that ∂_3^L and $\partial_3^{L'}$ are represented by identity matrices. In Construction 5.5, we based $H_3(Y, Y_L;\mathbb{Q}(t))$ and $H_3(Y', Y_L;\mathbb{Q}(t))$ using respectively (lifts of) the 3-cells of the $(D^2 \times S^1)_i$ and $(S^1 \times D^2)_i$. Now both ∂_3^L and $\partial_3^{L'}$ take these 3-cells to their boundaries. But as we noted in Construction 5.5, these boundaries are (algebraically) the 2-cells $(e^2_{S^1\times S^1})_i$. In other words both ∂_3^L and $\partial_3^{L'}$ map our choice of ordered bases to our other choice of ordered bases, and are therefore represented in these bases by identity matrices, as required. This concludes the proof of Lemma 5.6. ∎

As we now understand the exact sequences $\mathcal{H}_L$ and $\mathcal{H}_{L'}$ we can calculate their torsions, leading to the proof of the main result of this subsection.

Theorem 5.8. *If $H_1(Y_L;\mathbb{Q}(t)) = 0$ and $\det(A_{\widetilde{L}}) \neq 0$, then we have*

$$\Delta_{Y'} \doteq \det(A_{\widetilde{L}})\Delta_Y.$$

Proof. Use the bases from Construction 5.5. Combine the second item of Proposition 4.2 with Lemma 5.7 to obtain

$$\tau(\mathcal{H}_L) \doteq \frac{\det(\partial_3^L)}{\det(\partial_2^L)} \doteq 1 \quad \text{and} \quad \tau(\mathcal{H}_{L'}) \doteq \frac{\det(\partial_3^{L'})}{\det(\partial_2^{L'})} \doteq \det(A_{\widetilde{L}}).$$

We deduce that $\tau(\mathcal{H}_{L'})/\tau(\mathcal{H}_L) \doteq \det(A_{\widetilde{L}})$. Apply Lemma 5.6 to obtain

$$\frac{\Delta_{Y'}}{\Delta_Y} \doteq \frac{\tau(\mathcal{H}_{L'})}{\tau(\mathcal{H}_L)} \doteq \det(A_{\widetilde{L}}).$$

Rearranging yields the desired equality. ∎

As a consequence, we complete the second step of the plan from Section 5.1.

Proposition 5.9. *Let $0 \to H \xrightarrow{\widehat{\lambda}} H^* \xrightarrow{\varpi} H_1(Y;\mathbb{Z}[t^{\pm 1}]) \to 0$ be a presentation of Y. Pick generators $x_1, \dots, x_n$ for H and endow H^* with the dual basis $x_1^*, \dots, x_n^*$. Let Q be the matrix of λ with respect to these bases. The classes $\pi(x_1^*), \dots, \pi(x_n^*)$*

can be represented by a framed link $\widetilde{L}$ in covering general position with equivariant linking matrix $A_{\widetilde{L}} = -Q^{-T}$. In addition, the 3-manifold Y' obtained by surgery on Y along L satisfies $H_1(Y';\mathbb{Z}[t^{\pm 1}]) = 0$.

Proof. The existence of $\widetilde{L}$ representing the given generators and with equivariant linking matrix $A_{\widetilde{L}} = -Q^{-T}$ is proved in Proposition 5.4. As Q^T presents $H_1(Y;\mathbb{Z}[t^{\pm 1}])$, we have $\det(Q) \doteq \Delta_Y$, and therefore $\det(A_{\widetilde{L}}) \doteq 1/\Delta_Y$. Theorem 5.8 now implies that $\Delta_{Y'} \doteq 1$.

A short argument is now needed to use Remark 4.3 in order to conclude that $H_1(Y';\mathbb{Z}[t^{\pm 1}]) = 0$: we require that this torsion module admits a square presentation matrix, i.e. has projective dimension at most 1, denoted $\mathrm{pd}(H_1(Y';\mathbb{Z}[t^{\pm 1}])) \leq 1$. Here recall that a $\mathbb{Z}[t^{\pm 1}]$-module P *has projective dimension at most k* if

$$\mathrm{Ext}^i_{\mathbb{Z}[t^{\pm 1}]}(P;V) = 0$$

for every $\mathbb{Z}[t^{\pm 1}]$-module V and every $i \geq k+1$, and that for a short exact sequence $0 \to A \to B \to C \to 0$ of $\mathbb{Z}[t^{\pm 1}]$-modules, the associated long exact sequence in $\mathrm{Ext}(-;V)$ groups implies that:

(a) if $\mathrm{pd}(C) \leq 1$ and A is free, then $\mathrm{pd}(B) \leq 1$;

(b) if $\mathrm{pd}(B) \leq 1$ and A is free, then $\mathrm{pd}(C) \leq 1$.

The following paragraph proves that $\mathrm{pd}(H_1(Y';\mathbb{Z}[t^{\pm 1}])) \leq 1$. As $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ and $H_1(Y';\mathbb{Z}[t^{\pm 1}])$ are torsion (for the latter recall Lemma 3.14), a duality argument implies that $H_2(Y;\mathbb{Z}[t^{\pm 1}]) = \mathbb{Z}$ and $H_2(Y';\mathbb{Z}[t^{\pm 1}]) = \mathbb{Z}$ (see e.g. the first item of [27, Lemma 3.2]). Since these modules are torsion and since excision implies that

$$\begin{aligned} H_2\big(Y, Y_L;\mathbb{Z}[t^{\pm 1}]\big) &= \mathbb{Z}[t^{\pm 1}]^n \quad \text{and} \quad H_2\big(Y', Y_L;\mathbb{Z}[t^{\pm 1}]\big) = \mathbb{Z}[t^{\pm 1}]^n, \\ H_1\big(Y, Y_L;\mathbb{Z}[t^{\pm 1}]\big) &= 0 \qquad\quad\;\; \text{and} \quad H_1\big(Y', Y_L;\mathbb{Z}[t^{\pm 1}]\big) = 0, \end{aligned}$$

we deduce that the maps $H_2(Y;\mathbb{Z}[t^{\pm 1}]) \to H_2(Y, Y_L;\mathbb{Z}[t^{\pm 1}])$ and $H_2(Y';\mathbb{Z}[t^{\pm 1}]) \to H_2(Y', Y_L;\mathbb{Z}[t^{\pm 1}])$ are both trivial leading to the short exact sequences

$$\begin{aligned} &0 \to H_2\big(Y, Y_L;\mathbb{Z}[t^{\pm 1}]\big) \to H_1\big(Y_L;\mathbb{Z}[t^{\pm 1}]\big) \to H_1\big(Y;\mathbb{Z}[t^{\pm 1}]\big) \to 0, \\ &0 \to H_2\big(Y', Y_L;\mathbb{Z}[t^{\pm 1}]\big) \to H_1\big(Y_L;\mathbb{Z}[t^{\pm 1}]\big) \to H_1\big(Y';\mathbb{Z}[t^{\pm 1}]\big) \to 0. \end{aligned}$$

Next we apply the facts (a) and (b) on projective dimension given above. Since the torsion module $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ is presented by (H, λ), it has projective dimension at most 1 and since $H_2(Y, Y_L;\mathbb{Z}[t^{\pm 1}])$ is free, the first short exact sequence implies that $H_1(Y_L;\mathbb{Z}[t^{\pm 1}])$ has projective dimension at most 1. Since $H_2(Y', Y_L;\mathbb{Z}[t^{\pm 1}])$ is free, the second short exact sequence now implies that

$$\mathrm{pd}\big(H_1\big(Y';\mathbb{Z}[t^{\pm 1}]\big)\big) \leq 1,$$

as required.

As explained above, since $\operatorname{pd}(H_1(Y';\mathbb{Z}[t^{\pm 1}])) \leq 1$ and $\Delta_{Y'} \doteq 1$, Remark 4.3 now allow us to conclude that

$$H_1\big(Y';\mathbb{Z}[t^{\pm 1}]\big) = 0,$$

as required. ■

5.4. Step 3: Every $\mathbb{Z}[t^{\pm 1}]$-homology $S^1 \times S^2$ bounds a homotopy circle

The goal of this subsection is to prove the following theorem, which is a generalisation of a key step in the proof that Alexander polynomial one knots are topologically slice.

Theorem 5.10. *Let Y be a 3-manifold with an epimorphism $\pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module vanishes, i.e. $H_1(Y;\mathbb{Z}[t^{\pm 1}]) = 0$. Then there exists a 4-manifold B with a homotopy equivalence $g\colon B \xrightarrow{\simeq} S^1$ so that $\partial B \cong Y$ and $\pi_1(Y) \twoheadrightarrow \pi_1(B) \xrightarrow{g_*} \pi_1(S^1) \cong \mathbb{Z}$ agrees with φ.*

Proof. This proof can be deduced by combining various arguments from [42, Section 11.6], so we only outline the main steps. First we use framed bordism to find some 4-manifold whose boundary is Y, with a map to S^1 realising φ, as in [42, Lemma 11.6B]. This map might not be a homotopy equivalence, but we then we will use surgery theory to show that W is bordant rel. boundary to a homotopy circle.

To start the first step, recall that every oriented 3-manifold admits a framing of its tangent bundle. Using the axioms of a generalised homology theory, we have

$$\Omega_3^{\mathrm{fr}}(B\mathbb{Z}) \cong \Omega_3^{\mathrm{fr}} \oplus \Omega_2^{\mathrm{fr}} \cong \mathbb{Z}/24 \oplus \mathbb{Z}/2.$$

We consider the image of (Y,φ) in $\Omega_3^{\mathrm{fr}}(B\mathbb{Z})$. The first summand can be killed by changing the choice of framing of the tangent bundle of Y; see the proof of [42, Lemma 11.6B] for details. The second summand is detected by an Arf invariant, which vanishes thanks to the assumption that $H_1(Y;\mathbb{Z}[t^{\pm 1}]) = 0$; details are again in the proof of [42, Lemma 11.6B]. Therefore, there exists a framed 4-manifold W with framed boundary Y, such that the map $Y \to S^1$ associated with φ extends over W.

Now we use surgery theory to show that W is bordant rel. boundary to a homotopy circle. Consider the mapping cylinder

$$X := \mathcal{M}(Y \xrightarrow{\varphi} S^1). \tag{5.3}$$

We claim that (X,Y) is a Poincaré pair. The argument is similar to [42, Proposition 11.C]. As $X \simeq S^1$, the connecting homomorphism from the exact sequence of the pair (X,Y) gives an isomorphism $\partial\colon H_4(X,Y) \cong H_3(Y) \cong \mathbb{Z}$. We then define the required fundamental class as

$$[X,Y] := \partial^{-1}\big([Y]\big) \in H_4(X,Y).$$

Using $H_1(Y;\mathbb{Z}[t^{\pm 1}]) = 0$, one can now use the same argument as in [45, Lemma 3.2] to show that the following cap product is an isomorphism:

$$- \cap [X,Y]\colon H^i\big(X,Y;\mathbb{Z}[t^{\pm 1}]\big) \to H_{4-i}\big(X;\mathbb{Z}[t^{\pm 1}]\big).$$

This concludes the proof of the fact that (X,Y) is a Poincaré pair.

The end of the argument follows from the exactness of the surgery sequence for (X,Y) as in [42, Proposition 11.6A] but we outline some details for the reader unfamiliar with surgery theory. Since (X,Y) is a Poincaré pair, we can consider its set $\mathcal{N}(X,Y)$ of normal invariants. The set $\mathcal{N}(X,Y)$ consists of normal bordism classes of degree one normal maps to X that restrict to a homeomorphism on the boundary, where a bordism restricts to a product cobordism homeomorphic to $Y \times I$ between the boundaries. The next paragraph uses the map $W \to S^1$ to define an element of $\mathcal{N}(X,Y)$.

Via the homotopy equivalence $X \simeq S^1$, the map $Y \to S^1 \simeq X$ extends to

$$F\colon W \to S^1 \simeq X.$$

It then follows from the naturality of the long exact sequence of the pairs (W,Y) and (X,Y) that F has degree one. We therefore obtain a degree one map

$$(F, \mathrm{id}_Y)\colon (W,Y) \to (X,Y).$$

To upgrade (F,id_Y) to a degree one normal map, we take a trivial (stable) bundle $\xi \to X$ over the codomain. Normal data is determined by a (stable) trivialisation of $TW \oplus F^*\xi$. The framing of W provides a trivialisation for the first summand, while any choice of trivialisation for $F^*\xi$ can be used for the second summand. We therefore have a degree one normal map

$$\big((F,\mathrm{id}_Y)\colon (W,Y) \to (X,Y)\big) \in \mathcal{N}(X,Y).$$

Our goal is to change W to $W\#^{\ell} Z$, where $Z = E_8$, and then to do surgery on the interior of the domain $(W\#^{\ell} Z, Y)$ to convert F into a homotopy equivalence $(F', \mathrm{id}_Y)\colon (B,Y) \to (X,Y)$. Since the fundamental group $\mathbb{Z}$ is a good group, surgery theory says that this is possible if and only if $\ker(\sigma)$ is non-empty [42, Section 11.3]. Here $\sigma\colon \mathcal{N}(X,Y) \to L_4(\mathbb{Z}[t^{\pm 1}])$ is the surgery obstruction map. Essentially, it takes the intersection pairing on $H_2(W;\mathbb{Z}[t^{\pm 1}])$ and considers it in the Witt group of non-singular, Hermitian, even forms over $\mathbb{Z}[t^{\pm 1}]$ up to stable isometry, where stabilisation is by hyperbolic forms

$$\left(\mathbb{Z}[t^{\pm 1}] \oplus \mathbb{Z}[t^{\pm 1}], \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

Shaneson splitting [82] implies that

$$L_4\big(\mathbb{Z}[t^{\pm 1}]\big) \cong L_4(\mathbb{Z}) \oplus L_3(\mathbb{Z}) \cong L_4(\mathbb{Z}) \cong 8\mathbb{Z}.$$

The last isomorphism is given by taking the signature. We take the connected sum of $W \to X$ with copies of $(E_8 \to S^4)$ or $(-E_8 \to S^4)$, to arrange that the signature becomes zero. Then the resulting normal map $W\#^\ell Z \to X$ has trivial surgery obstruction in $L_4(\mathbb{Z}[t^{\pm 1}])$ (i.e. lies in $\ker(\sigma)$) and therefore is normally bordant to a homotopy equivalence $(F', \mathrm{id}_Y)\colon (B, Y) \to (X, Y)$, as desired. Since the mapping cylinder X from (5.3) is a homotopy circle, so is B. This concludes the proof of the theorem. ∎

5.5. Step 4: Constructing a 4-manifold that induces the given boundary isomorphism

We begin by recalling the notation and outcome of Proposition 5.9 as follows. We let $b \in \mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)$ be an isometry of linking forms. Pulling this back to H, we obtain a presentation

$$0 \to H \xrightarrow{\widehat{\lambda}} H^* \xrightarrow{\varpi} H_1\big(Y; \mathbb{Z}[t^{\pm 1}]\big) \to 0$$

of Y. Pick generators $x_1, \dots, x_n$ for H and endow H^* with the dual basis $x_1^*, \dots, x_n^*$. Let Q be the matrix of λ with respect to these bases. By Propositions 5.4 and 5.9, the classes $\pi(x_1^*), \dots, \pi(x_n^*)$ can be represented by a framed link $\widetilde{L} \subset Y^\infty$ in covering general position with transposed equivariant linking matrix $-Q^{-1}$ and the 3-manifold Y' obtained by surgery on $L = p(\widetilde{Y})$ satisfies $H_1(Y'; \mathbb{Z}[t^{\pm 1}]) = 0$. Applying Theorem 5.10, there is a topological 4-manifold B with boundary Y' and such that $B \simeq S^1$.

We now define a 4-manifold M with boundary Y as follows: begin with $Y \times I$ and attach 2-handles to $Y \times \{1\}$ along the framed link $L := p(\widetilde{L})$ (here recall that $p\colon Y^\infty \to Y$ denotes the covering map), so that the resulting boundary is Y'. Call this 2-handle cobordism W, and observe that $\partial^- W = -Y$. We can now cap $\partial^+ W \cong Y'$ with $-B$. Since $W \cup -B$ has boundary $-Y$, we define M to be $-W \cup B$. We can then consider the corresponding $\mathbb{Z}$-cover:

$$-M^\infty := \left((Y^\infty \times [0,1]) \cup \bigcup_{i=1}^{n} \bigcup_{j_i \in \mathbb{Z}} t^{j_i} h_i^{(2)} \right) \cup_{Y'^\infty} -B^\infty = W^\infty \cup_{Y'^\infty} -B^\infty,$$

$$-M := \left((Y \times [0,1]) \cup \bigcup_{i=1}^{n} h_i^{(2)} \right) \cup_{Y'} -B =: W \cup_{Y'} -B,$$

in which the 2-handles are attached along the framed link $\widetilde{L}$ upstairs and its framed projection L downstairs.

We begin by verifying some properties of M.

Lemma 5.11. *The $\mathbb{Z}$-manifold M has boundary Y.*

Proof. We first prove that $\pi_1(M) \cong \mathbb{Z}$. A van Kampen argument shows that $\pi_1(M)$ is obtained from $\pi_1(B)$ by modding out the $[\iota(\widetilde{K}_i')]$ where $\widetilde{K}_1', \dots, \widetilde{K}_n'$ denote the components of the framed link dual to $\widetilde{L}$ and where $\iota\colon \pi_1(Y') \to \pi_1(B)$ is the inclusion induced map. Recall from Lemma 3.12 and Remark 3.13 that the epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ induces an epimorphism $\varphi'\colon \pi_1(Y') \twoheadrightarrow \mathbb{Z}$ and that $\varphi'([K_i']) = 0$ for $i = 1, \dots, n$. Since Theorem 5.10 ensures that ι agrees with φ', we deduce that the classes $[\iota(\widetilde{K}_i')]$ are trivial and therefore $\pi_1(M) \cong \pi_1(B) \cong \mathbb{Z}$.

Next we argue that as a $\mathbb{Z}$-manifold M has boundary Y. Since the inclusion induced map $\pi_1(Y) \to \pi_1(W)$ is surjective, it suffices to prove that the inclusion induced map $\pi_1(W) \to \pi_1(M)$ is surjective. This follows from van Kampen's theorem: as $\pi_1(Y') \to \pi_1(B)$ is surjective, so is $\pi_1(W) \to \pi_1(M)$. ■

It is not too difficult to compute, as we will do in Proposition 5.13 below, that $H_2(M;\mathbb{Z}[t^{\pm 1}])$ is f.g. free of rank n. To complete Step 4, we must prove the following two claims.

(1) The equivariant intersection form λ_M of M is represented by Q; i.e. λ_M is isometric to λ.

(2) The 4-manifold M satisfies $b_M = b \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$.

The proof of the first claim follows a standard outline; for the hasty reader we will give the outline here, and for the record we provide a detailed proof at the end of the subsection.

Proof outline of claim (1). Since by setup the transposed equivariant linking matrix of the framed link $\widetilde{L}$ is $-Q^{-1}$, Proposition 3.15 shows that the transposed equivariant linking matrix of the dual link $\widetilde{L}'$ is Q. Thus, it suffices to show that λ_M is presented by the transposed equivariant linking matrix of $\widetilde{L}'$.

While it was natural initially to build W^∞ by attaching 2-handles to $Y^\infty \times I$, in what follows it will be more helpful to view $-W^\infty$ as being obtained from $Y' \times I$ by attaching 2-handles to the framed link $\widetilde{L}'$ dual to $\widetilde{L}$. In particular, the components of $\widetilde{L}'$ bound the cores of the 2-handles.

Recall that $H_1(Y';\mathbb{Z}[t^{\pm 1}]) = 0$ by Proposition 5.9 and that $H_2(B;\mathbb{Z}[t^{\pm 1}]) = 0$ by Proposition 5.10. Let Σ_i denote a surface in Y'^∞ with boundary $\widetilde{K}_i'$, and let F_i be the surface in M formed by Σ_i capped with the core of the 2-handle attached along $\widetilde{K}_i'$. The proof that $H_2(M;\mathbb{Z}[t^{\pm 1}])$ is freely generated by the $[F_i]$ and that the equivariant intersection form λ_M is represented by the transposed equivariant linking matrix of $\widetilde{L}'$ (which we showed above is Q), is now routine; the details are expanded in Propositions 5.13 and 5.14 below. ■

As promised, the section now concludes with a detailed proof of the claims. Firstly in Construction 5.12, we give the detailed construction of the surfaces F_i that were mentioned in the proof outline. Secondly, in Proposition 5.13 we show that these surfaces lead to a basis of $H_2(M; \mathbb{Z}[t^{\pm 1}])$. Thirdly, in Proposition 5.14 we conclude the proof of the first claim by showing that with respect to this basis, λ_M is represented by the transposed equivariant linking matrix of $\widetilde{L}'$. Finally, in Proposition 5.15, we prove the second claim.

Construction 5.12. For $i = 1, \dots, n$, we define the closed surfaces $F_i \subset -W^\infty \subset M^\infty$ that were mentioned in the outline. As $H_1(Y'; \mathbb{Z}[t^{\pm 1}]) = 0$ (by Step 2), each component $\widetilde{K}_i'$ of $\widetilde{L}'$ bounds a surface $\Sigma_i \subset Y'^\infty$. Additionally, each $\widetilde{K}_i'$ (considered in $Y' \times \{1\}$) bounds the core of one of the (lifted) 2-handles in the dual handle decomposition of $-W$. Define the surface $F_i \subset -W^\infty \subset M^\infty$ by taking the union of Σ_i with this core.

The next proposition shows that the surfaces F_i' give a basis for $H_2(M; \mathbb{Z}[t^{\pm 1}])$. It is with respect to this basis that we will calculate λ_M in Proposition 5.14 below.

Proposition 5.13. *The following isomorphisms hold:*

$$H_2\big(-W; \mathbb{Z}[t^{\pm 1}]\big) = \mathbb{Z} \oplus \bigoplus_{i=1}^{n} \mathbb{Z}[t^{\pm 1}][F_i],$$

$$H_2\big(M; \mathbb{Z}[t^{\pm 1}]\big) = \bigoplus_{i=1}^{n} \mathbb{Z}[t^{\pm 1}][F_i].$$

Proof. These follow by standard arguments using Mayer–Vietoris, which we outline now.

The first equality follows from the observation that $-W^\infty$ is obtained from $Y'^\infty \times [0,1]$ by attaching the dual 2-handles to the $h_i^{(2)}$. Morally, since $H_1(Y'; \mathbb{Z}[t^{\pm 1}]) = 0$ (Step 2), each dual 2-handle contributes a free generator. The additional $\mathbb{Z}$ summand comes from

$$H_2\big(Y' \times [0,1]; \mathbb{Z}[t^{\pm 1}]\big) \cong \mathbb{Z}.$$

More formally, one applies Mayer–Vietoris with $\mathbb{Z}[t^{\pm 1}]$-coefficients to the decomposition of W as the union of $Y' \times [0,1]$ with the dual 2-handles, which since the dual 2-handles are contractible and $H_1(Y'; \mathbb{Z}[t^{\pm 1}]) = 0$ yields the short exact sequence:

$$0 \to H_2\big(Y' \times [0,1]; \mathbb{Z}[t^{\pm 1}]\big) \to H_2\big(-W; \mathbb{Z}[t^{\pm 1}]\big) \xrightarrow{\partial} H_1\big(\overline{\nu}(L'); \mathbb{Z}[t^{\pm 1}]\big) \to 0.$$

Since $\varphi'([L']) = 0$, then $H_1(\overline{\nu}(L'); \mathbb{Z}[t^{\pm 1}]) \cong \bigoplus_{i=1}^n \mathbb{Z}[t^{\pm 1}]$, generated by the $[K_i']$. Mapping each $[K_i']$ to $[F_i]$ determines a splitting.

For the second equality, note that since B is a homotopy circle and $g_*\colon \pi_1(B) \to \mathbb{Z}$ is an isomorphism, B has no (reduced) $\mathbb{Z}[t^{\pm 1}]$-homology. The Mayer–Vietoris exact sequence associated to the decomposition $M = -W \cup_{Y' \times \{1\}} B$ therefore yields the short exact sequence

$$0 \to H_2\big(Y'; \mathbb{Z}[t^{\pm 1}]\big) \to H_2\big(-W; \mathbb{Z}[t^{\pm 1}]\big) \to H_2\big(M; \mathbb{Z}[t^{\pm 1}]\big) \to 0.$$

Appealing to our computation of $H_2(-W; \mathbb{Z}[t^{\pm 1}])$, we deduce that $H_2(M; \mathbb{Z}[t^{\pm 1}])$ is freely generated by the $[F_i]$. ■

Now we prove the first claim of the previously mentioned outline.

Proposition 5.14. *With respect to the basis of $H_2(M; \mathbb{Z}[t^{\pm 1}])$ given by $[F_1], \dots, [F_n]$, the equivariant intersection form λ_M of M is given by the transposed equivariant linking matrix of the framed link $\widetilde{L}'$ dual to $\widetilde{L}$.*

Proof. Recall from Construction 5.12 that for $i = 1, \dots, n$, the surface $F_i \subset -W^\infty \subset M^\infty$ was obtained as the union of a surface $\Sigma_i \subset Y'^\infty$ whose boundary is $\widetilde{K}_i'$ with the core of a (lifted) 2-handle in the dual handle decomposition of W. For $i = 1, \dots, n$, define F_i' to be a surface isotopic to F_i obtained by pushing the interior of Σ_i into B^∞. Let Σ_i' be such a push-in. Since F_i and F_i' are isotopic for every $i = 1, \dots, n$, we can use the F_i' to calculate λ_M. Fix real numbers $0 < s_1 < \cdots < s_n < 1$. We model Σ_i' in the coordinates of a collar neighbourhood $\partial B \times [0, 1]$ as

$$\Sigma_i' := \big(\partial \Sigma_i \times [0, s_i]\big) \cup \big(\Sigma_i \times \{s_i\}\big).$$

We start by calculating the equivariant intersection form $\lambda_M([F_i'], [F_j'])$ for $i \neq j$. Since the aforementioned cores of the dual 2-handles are pairwise disjoint, we obtain

$$\overline{\lambda_M\big([F_i'], [F_j']\big)} = F_i' \cdot_{\infty, M} F_j' = \Sigma_i' \cdot_{\infty, B} \Sigma_j'.$$

Recall that we use $A_{\widetilde{L}'}$ to be the linking matrix of the framed link L'. It therefore remains to show that $\Sigma_i' \cdot_{\infty, B} \Sigma_j' = (A_{\widetilde{L}'})_{ij}$. Assume without loss of generality that $i > j$, and so $s_i > s_j$. Also note that $\partial \Sigma_i \cap \partial \Sigma_j = \emptyset$. By inspecting the locations of the intersections, it follows that

$$\begin{aligned} \Sigma_i' \cdot_{\infty, B} \Sigma_j' &= \big(\partial \Sigma_i \times [0, s_i]\big) \cdot_{\infty, B} \big(\Sigma_j \times \{s_j\}\big) \\ &= \partial \Sigma_i \cdot_{\infty, \partial B} \Sigma_j = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \widetilde{K}_j'), \end{aligned}$$

where the last equality makes use of the definition of the equivariant linking number in $\partial B = Y'$. For $i \neq j$, we have therefore proved that

$$\lambda_M\big([F_j'], [F_i']\big) = \Sigma_i' \cdot_{\infty, B} \Sigma_j' = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_i', \widetilde{K}_j').$$

It remains to prove that $\lambda_M([F_i'], [F_i']) = (A_{\widetilde{L}'})_{ii}$.

By definition of the dual framed knot $\widetilde{K}'_i$, we have $(A_{\widetilde{L}'})_{ii} = \ell k_{\mathbb{Q}(t)}(\widetilde{K}'_i, \pi'_i)$, where π'_i denotes the framing curve of $\widetilde{K}'_i$.

Perform a small push-off of the surface $\Sigma'_i \subset B^\infty$ to obtain a surface $\Sigma''_i \subset B^\infty$ isotopic to $\Sigma'_i \subset B^\infty$ with boundary $\partial \Sigma''_i = \pi'_i$. Cap off Σ''_i with a parallel disjoint copy of the cocore of the 2-handle, yielding a closed surface F''_i that is isotopic to F'_i, and such that all the intersections between the two occur between Σ'_i and Σ''_i. As in the $i \neq j$ case, we then have

$$\lambda_M\big([F'_i], [F'_i]\big) = \Sigma'_i \cdot_{\infty,B} \Sigma''_i = \ell k_{\mathbb{Q}(t)}(\widetilde{K}'_i, \pi'_i).$$

We have therefore shown that the equivariant intersection form of M is represented by the transposed linking matrix $A^T_{\widetilde{L}'}$ and this concludes the proof of the proposition. ■

Finally, we prove the second claim of our outline, thus completing Step 4.

Proposition 5.15. *Let Y be a 3-manifold with an epimorphism $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module is torsion, and let (H, λ) be a non-degenerate Hermitian form presenting Y. If $b \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$ is an isometry, then there is a $\mathbb{Z}$-manifold M with equivariant intersection form $\lambda_M \cong \lambda$, boundary Y and with $b_M = b$.*

Proof. Let M be the 4-manifold with boundary Y constructed as described above. The manifold $M = -W \cup_{Y'} B$ comes with a homeomorphism $g\colon \partial M \cong Y$, since $-W$ is obtained from $Y \times [0, 1]$ by adding 2-handles. We already explained why M has intersection form isometric to λ but we now make the isometry more explicit. Define an isomorphism $F\colon H \to H_2(M; \mathbb{Z}[t^{\pm 1}])$ by mapping x_i to $[F_i]$, where the $F_i \subset M^\infty$ are the surfaces built in Construction 5.12. This is an isometry because, by combining Proposition 5.14 with Lemma 3.15, we get

$$\lambda_M\big([F_i], [F_j]\big) = (A_{\widetilde{L}'})_{ji} = -(A^{-1}_{\widetilde{L}})_{ji} = Q_{ij} = \lambda(x_i, x_j).$$

We now check that $b_M = b$ by proving that $b = g_* \circ D_M \circ \partial F$. This amounts to proving that the bottom square of the following diagram commutes (we refer to Construction 2.3 if a refresher on the notation is needed):

$$\begin{array}{ccccc}
H^* & \xrightarrow{F^{-*},\cong} & H_2\big(M; \mathbb{Z}[t^{\pm 1}]\big)^* & \xrightarrow{\mathrm{PD}\circ \mathrm{ev}^{-1},\cong} & H_2\big(M, \partial M; \mathbb{Z}[t^{\pm 1}]\big) \\
\Big\downarrow{\scriptstyle \mathrm{proj}} & & \Big\downarrow{\scriptstyle \mathrm{proj}} & & \Big\downarrow{\scriptstyle \delta_M} \\
\operatorname{coker}(\widehat{\lambda}) & \xrightarrow{\partial F,\cong} & \operatorname{coker}(\widehat{\lambda}_M) & \xrightarrow{D_M,\cong} & H_1\big(\partial M; \mathbb{Z}[t^{\pm 1}]\big) \\
\Big\downarrow{\scriptstyle =} & & & & \Big\downarrow{\scriptstyle g_*,\cong} \\
\operatorname{coker}(\widehat{\lambda}) & & \xrightarrow{\qquad b,\cong \qquad} & & H_1\big(Y; \mathbb{Z}[t^{\pm 1}]\big).
\end{array}$$

The top squares of this diagram commute by definition of ∂F and D_M. Since the top vertical maps are surjective, the commutativity of the bottom square is now equivalent to the commutativity of the outer square. It therefore remains to prove that

$$g_* \circ \delta_M \circ (\mathrm{PD} \circ \mathrm{ev}^{-1}) \circ F^{-*} = \pi;$$

(recall that by definition $\pi = b \circ \mathrm{proj}$). In fact, it suffices to prove this on the x_i^* as they form a basis of H^*. Writing c_i for the core of the 2-handles attached to $Y \times [0,1]$, union a product of their attaching circles with $[0,1]$ in $Y \times [0,1]$, note that the c_i intersects F_j in δ_{ij} points, since F_j is built from a surface in Y'^{∞} union the cocore of the j-th 2-handle. We have

$$\begin{aligned} g_* \circ \delta_M \circ (\mathrm{PD} \circ \mathrm{ev}^{-1}) \circ F^{-*}(x_i^*) &= g_* \circ \delta_M \circ (\mathrm{PD} \circ \mathrm{ev}^{-1})\big([F_i]^*\big) \\ &= g_* \circ \delta_M\big([\widetilde{c}_i]\big) = [\widetilde{K}_i] = \pi(x_i^*). \end{aligned}$$

Here we use successively the definition of F, as well as the geometric interpretation of $\mathrm{PD} \circ \mathrm{ev}^{-1}$, the fact that $\widetilde{g}(\partial \widetilde{c}_i) = \widetilde{K}_i$ and the definition of the $\widetilde{K}_i$. Therefore, the outer square commutes as asserted. This concludes the proof that $b = g_* \circ D_M \circ \partial F$ and therefore $b_M = b$, as required. ■

5.6. Step 5: Fixing the Kirby–Siebenmann invariant and concluding

The conclusion of Theorem 5.1 will follow promptly from Proposition 5.15 once we recall how, in the odd case, it is possible to modify the Kirby–Siebenmann invariant of a given 4-manifold with fundamental group $\mathbb{Z}$. This is achieved using the star construction, a construction which we now recall following [42] and [83]. In what follows, $*\mathbb{C}P^2$ denotes the Chern manifold, i.e. the unique simply-connected topological 4-manifold homotopy equivalent to $\mathbb{C}P^2$ but with $\mathrm{ks}(*\mathbb{C}P^2) = 1$.

Let M be a topological 4-manifold with (potentially empty) boundary, good fundamental group π and such that the second Stiefel–Whitney class of the universal cover $w_2(\widetilde{M})$ is non-trivial. There is a 4-manifold $*M$, called the *star partner of M* that is rel. boundary homotopy equivalent to M but has the opposite Kirby–Siebenmann invariant from that of M ([42, Theorem 10.3 (1)]). See [87] or [64, Proposition 5.8] for a more general condition under which a star partner exists.

Remark 5.16. For fundamental group $\mathbb{Z}$, every non-spin 4-manifold has $w_2(\widetilde{M}) \neq 0$. To see this, we use the exact sequence

$$0 \to H^2(B\pi; \mathbb{Z}/2) \to H^2(M; \mathbb{Z}/2) \xrightarrow{p^*} H^2(\widetilde{M}; \mathbb{Z}/2)^{\pi},$$

where $\pi := \pi_1(M)$. This can be deduced from the Leray–Serre spectral sequence for the fibration $\widetilde{M} \to M \to B\pi$; see e.g. [63, Lemma 3.17]. For $\pi = \mathbb{Z}$, the first term vanishes, so p^* is injective. By naturality, $p^*(w_2(M)) = w_2(\widetilde{M})$, so $w_2(M) \neq 0$ implies $w_2(\widetilde{M}) \neq 0$, as desired. It follows that for a non-spin 4-manifold M with fundamental group $\mathbb{Z}$ ([42, Theorem 10.3]) applies and there is a star partner.

To describe $*M$, consider the 4-manifold $W := M\#(*\mathbb{C}P^2)$ and note that the inclusions $M \hookrightarrow W$ and $*\mathbb{C}P^2 \hookrightarrow W$ induce a splitting

$$\pi_2(M) \oplus \big(\pi_2(*\mathbb{C}P^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi]\big) \xrightarrow{\cong} \pi_2(W). \tag{5.4}$$

By [42, Theorem 10.3 (1)] (cf. [64, Proposition 5.8]) there exists a 4-manifold $*M$ and an orientation-preserving homeomorphism

$$h\colon W \xrightarrow{\cong} *M\#\mathbb{C}P^2$$

that respects the splitting on π_2 displayed in (5.4). The star partner $*M$ is also unique up to homeomorphism, by [84, Corollary 1.2].

To be more precise about the condition on h, let

$$\iota\colon \pi_2(*\mathbb{C}P^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi] \to \pi_2\big(M\#(*\mathbb{C}P^2)\big) = \pi_2(W)$$

denote the split isometric injection induced by the zigzag $*\mathbb{C}P^2 \leftarrow *\mathbb{C}P^2 \setminus \mathring{D}^4 \rightarrow W$, and let $\mathrm{incl}_*\colon \pi_2(\mathbb{C}P^2) \to \pi_2(*M\#\mathbb{C}P^2)$ be defined similarly. Then we say that h *respects the splitting on* π_2 if for some isomorphism $f\colon \pi_2(*\mathbb{C}P^2) \xrightarrow{\cong} \pi_2(\mathbb{C}P^2)$, the following diagram commutes:

$$\begin{array}{ccc} \pi_2(*\mathbb{C}P^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi] & \xhookrightarrow{\iota} & \pi_2(W) \\ {\scriptstyle f\otimes \mathrm{id}}\downarrow \cong & & \cong \downarrow {\scriptstyle h_*} \\ \pi_2(\mathbb{C}P^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi] & \xhookrightarrow{\mathrm{incl}_*} & \pi_2(*M\#\mathbb{C}P^2). \end{array}$$

Since both horizontal maps in this diagram are split, this implies that h_* induces an isomorphism $g\colon \pi_2(M) \xrightarrow{\cong} \pi_2(*M)$, and so h_* splits as follows:

$$\begin{aligned} h_* = (g_*, f_* \otimes \mathrm{id})\colon \pi_2(M) &\oplus \big(\pi_2(*\mathbb{C}P^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi]\big) \\ &\xrightarrow{\cong} \pi_2(*M) \oplus \big(\pi_2(\mathbb{C}P^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\pi]\big). \end{aligned}$$

We recall that M and $*M$ are orientation-preserving homotopy equivalent rel. boundary. This will ensure that their automorphism invariants agree. The argument is due to Stong [84, Section 2], and a proof can also be found in [64, Lemma 5.7].

Proposition 5.17. *If M is a topological 4-manifold with boundary, good fundamental group π and whose universal cover has non-trivial second Stiefel–Whitney class, then M is an orientation-preserving homotopy equivalent rel. boundary to its star partner $*M$.*

We are ready to prove Theorem 5.1, whose statement we recall for the reader's convenience. Let Y be a 3-manifold with an epimorphism $\pi_1(Y) \twoheadrightarrow \mathbb{Z}$ whose Alexander module is torsion, and let (H, λ) be a form presenting Y. If $b \in \operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)$ is an isometry, then there is a $\mathbb{Z}$-manifold M with equivariant intersection form λ_M, boundary Y and with $b_M = b$. If the form is odd, then M can be chosen to have either $\operatorname{ks}(M) = 0$ or $\operatorname{ks}(M) = 1$. We now conclude the proof of this theorem.

Proof of Theorem 5.1. In Proposition 5.15, we proved the existence of a $\mathbb{Z}$ manifold M with equivariant intersection form λ_M, boundary Y and with $b_M = b$. It remains to show that if λ is odd, then M can be chosen to have either $\operatorname{ks}(M) = 0$ or $\operatorname{ks}(M) = 1$. This is possible by using the star partner $*M$ of M. Indeed Proposition 5.17 implies that M and $*M$ are homotopy equivalent rel. boundary, and therefore Remark 2.6 ensures that $b_{*M} = b_M$ is unchanged. ■

5.7. An example

Remark 2.5 shows that if M_0 and M_1 are spin 4-manifolds with $\pi_1(M_i) \cong \mathbb{Z}$, boundary homeomorphic to (Y, φ), isometric equivariant intersection form, and the same automorphism invariant, then their Kirby–Siebenmann invariants agree. The next proposition shows that the condition on the automorphism invariant is necessary. After the proof, we offer an extended example to illustrate the proof of Theorem 2.4 and to show that it is possible to work with the automorphism invariants and the $\mathbb{Q}(t)$-valued linking numbers explicitly.

Proposition 5.18. *There are two spin 4-manifolds M_0 and M_1 with $\pi_1 \cong \mathbb{Z}$, equivariant intersection form isometric to $\lambda := (-8)$ and boundary homeomorphic to $Y := -L(8,1)\#(S^1 \times S^2)$ that are distinguished both by their Kirby–Siebenmann invariants and their automorphism invariants.*

Proof. The manifolds M_0, M_1 are obtained by boundary connect summing $S^1 \times D^3$ to simply-connected 4-manifolds V_0 and V_1 that we now describe. Up to homeomorphism, there are two simply-connected 4-manifolds V_0 and V_1 with intersection form $\lambda' = (-8)\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, and boundary homeomorphic to the lens space $Y' := -L(8,1)$. They are distinguished by Boyer's simply-connected version of the automorphism invariant [12, Corollary E]. We construct them explicitly and show that $\operatorname{ks}(V_0) \neq \operatorname{ks}(V_1)$.

The (-8)-trace on the unknot, $V_0 := X_{-8}(U)$, gives the first of these 4-manifolds. Towards describing V_1, first note that from $-L(8,1)$ one can obtain the integer homology sphere $S^3_{+1}(T_{2,3})$ by a Dehn surgery along the framed knot K_1 illustrated in Figure 1. Note also that $S^3_{+1}(T_{2,3})$ bounds a contractible topological 4-manifold C.

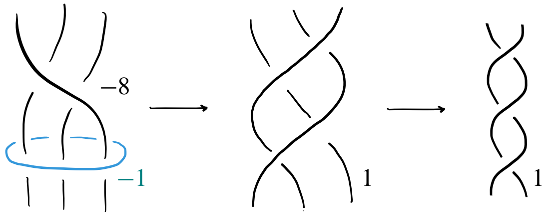

Figure 1. Performing -1 surgery on the blue knot K_1 in the lens space $L(-8, 1)$ yields the 3-manifold obtained by $+1$ surgery on the right handed trefoil in S^3. Each frame of the figure should be imagined to be vertically braid closed. The first homeomorphism indicated is a Rolfsen twist, the second is an isotopy in S^3.

We can now build $-V_1$ by beginning with $-L(8, 1) \times I$, attaching a $+1$ framed 2-handle along K_1, and capping off with $-C$. The resulting manifold $-V_1$ has $\partial(-V_1) = L(8, 1)$, so $\partial V_1 = -L(8, 1)$, as desired.

The manifolds V_0 and V_1 are simply-connected, spin, have boundary homeomorphic to $-L(8, 1)$, and intersection form isometric to (-8). We have $\mathrm{ks}(V_0) = 0$ (because V_0 is smooth), whereas $\mathrm{ks}(V_1) = \mathrm{ks}(C) = \mu(S^3_{+1}(T_{2,3})) = \mathrm{Arf}(T_{2,3}) = 1$. Here μ denotes the Rochlin invariant and the relation between ks and μ is due to Gonzáles-Acuña [50].

The manifolds M_0 and M_1 are now obtained by setting

$$M_0 := V_0 \natural (S^1 \times D^3) \quad \text{and} \quad M_1 := V_1 \natural (S^1 \times D^3).$$

The manifolds M_0 and M_1 have $\pi_1(M_i) \cong \mathbb{Z}$, boundary homeomorphic to $Y = -L(8, 1)\#(S^1 \times S^2)$, and equivariant intersection form isometric to

$$(-8) \colon \mathbb{Z}[t^{\pm 1}] \times \mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}[t^{\pm 1}].$$

The additivity of the Kirby–Siebenmann invariant implies that $\mathrm{ks}(M_0) = \mathrm{ks}(V_0) = 0$, whereas $\mathrm{ks}(M_1) = \mathrm{ks}(V_1) = 1$. The manifolds must have distinct automorphism invariants, since otherwise by the classification (Theorem 2.8) they would be homeomorphic and hence would have the same Kirby–Siebenmann invariants. ∎

Example 5.19. To provide an explicit example of our realisation procedure from the proof of Theorem 2.4, we describe how the manifolds M_0 and M_1 realise two distinct, explicit automorphism invariants.

Fix a model of $Y := -L(8, 1)\#(S^1 \times S^2)$ as surgery on a 2-component unlink $L_1 \cup L_2$ with framings $(-8, 0)$. Consider the epimorphism

$$\varphi \colon \pi_1(Y) \cong \mathbb{Z}_8 * \mathbb{Z} \to \mathbb{Z}$$

given by sending the meridian μ_{L_1} of L_1 to 0 and the meridian μ_{L_2} to 1. Fix a lift $\widetilde{\mu}_{L_1}$ of μ_{L_1} to the infinite cyclic cover and note that it generates $H_1(Y;\mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z}[t^{\pm 1}]/(8)$ and satisfies $\mathrm{Bl}_Y(\widetilde{\mu}_{L_1}, \widetilde{\mu}_{L_1}) = 1/8$. One way to see this latter equality is to use the calculation of the linking form of lens spaces.

A verification shows that Y is presented by the Hermitian form

$$\begin{aligned} \lambda\colon \mathbb{Z}[t^{\pm 1}] \times \mathbb{Z}[t^{\pm 1}] &\to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}], \\ (x, y) &\mapsto 8x\overline{y}. \end{aligned}$$

Note also that multiplication by 3 induces an isometry of $\mathrm{Bl}_Y \cong -\partial\lambda$. Using the notation from the proof of Step 1 in Section 5.1, we let x_1 be a generator of $\mathbb{Z}[t^{\pm 1}]$, and we let $x_1^* \in \mathbb{Z}[t^{\pm 1}]^*$ be the dual generator. In these bases, the matrix of λ is $Q = (-8)$. We therefore obtain two elements of $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)$ by considering

$$\begin{aligned} b_0\colon \mathbb{Z}[t^{\pm 1}]/(8) &\xrightarrow{\cong} H_1\big(Y;\mathbb{Z}[t^{\pm 1}]\big), \quad [x_1^*] \mapsto [\widetilde{\mu}_{L_1}], \\ b_1\colon \mathbb{Z}[t^{\pm 1}]/(8) &\xrightarrow{\cong} H_1\big(Y;\mathbb{Z}[t^{\pm 1}]\big), \quad [x_1^*] \mapsto 3[\widetilde{\mu}_{L_1}]. \end{aligned}$$

Since $\mathrm{Aut}(\lambda) = \{\pm t^k\}_{k\in\mathbb{Z}}$, it follows that b_0 and b_1 remain distinct in

$$\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_Y)/\,\mathrm{Aut}(\lambda).$$

That they remain in distinct orbits of the action of $\mathrm{Homeo}^+_\varphi(Y)$ requires the following claim.

Claim. *The group* $\mathrm{Homeo}^+_\varphi(Y)$ *acts on* $H_1(Y;\mathbb{Z}[t^{\pm 1}])$ *as follows: for each* $\psi \in \mathrm{Homeo}^+_\varphi(Y)$*, we have that* $\psi \cdot x = \pm t^k x$ *for some* $k \in \mathbb{Z}$.

Proof. In [16, Theorem 3] we find the statement that every automorphism of a connected sum of 3-manifolds is a composition of slides, permutations, and automorphisms of the factors. That article was an announcement, and the theorem is actually due to Hendriks–Laudenbach [57, §5, Théorème]. For our purposes the statement in [16] is easier to apply, which is why we mention it. Permutations are irrelevant here since there is a unique irreducible factor. Sliding the $-L(8,1)$ factor around the generator of $S^1 \times S^2$ exactly corresponds to an action by t^n. Sliding the handle sends a generator $t \in \pi_1(S^1 \times S^2)$ to $g \cdot t$ where $g \in \mathbb{Z}_8$. However it acts trivially on a generator of $\pi_1(L(8,1))$ and hence acts trivially on $H_1(Y;\mathbb{Z}[t^{\pm 1}])$. It remains to consider automorphisms of the irreducible factor, i.e. of $L(8,1)$. Bonahon [8] proved that every element of $\mathrm{Homeo}^+(L(8,1))$ acts by ± 1 on $H_1(L(8,1))$, and hence such an element acts by ± 1 on $H_1(Y;\mathbb{Z}[t^{\pm 1}])$. Combining these conclusions, we see that every homeomorphism $f \in \mathrm{Homeo}^+_\varphi(Y)$ acts by $\pm t^n$ on $H_1(Y;\mathbb{Z}[t^{\pm 1}])$, for some sign and some $n \in \mathbb{Z}$, as asserted. ∎

The claim implies that the isometries b_0 and b_1 determine distinct elements in the orbit set $\operatorname{Iso}(\partial\lambda, -\operatorname{Bl}_Y)/\operatorname{Aut}(\lambda)\times\operatorname{Homeo}^+_\varphi(Y)$. We will show that applying the realisation process of Theorem 5.1 to these elements results in M_0 and M_1 respectively. Following the notation of Section 5.1, for $i=0,1$, precompose b_i with the canonical projection $\mathbb{Z}[t^{\pm1}]^* \to \mathbb{Z}[t^{\pm1}]/(8)$ to get the epimorphism

$$\varpi_i\colon \mathbb{Z}[t^{\pm1}]^* \to \mathbb{Z}[t^{\pm1}]/(8) \xrightarrow{b_i} H_1\big(Y;\mathbb{Z}[t^{\pm1}]\big).$$

For $i=0,1$, let $\widetilde{K}_i \subset Y^\infty$ be a framed knot representing $\varpi_i(x_1^*)$ and let $K_i \subset Y$ be its projection down to Y. We can assume that

$$\widetilde{K}_i \subset -L(8,1) \subset (S^2\times\mathbb{R})\#_{k\in\mathbb{Z}} t^k\big(-L(8,1)\big) = Y^\infty.$$

Thinking of Y as the $(-8,0)$-framed surgery on the unlink $L_1\cup L_2$, one can arrange also for K_i to be disjoint from $L_1\cup L_2$. Consider the 3-component link

$$K_i \cup L_1 \cup L_2 \subset S^3.$$

Note that $K_i\cup L_2$ is split from L_1, $\ell k(K_0,L_1)=1$ and $\ell k(K_1,L_1)=3$. When we refer to a framing of K_i, it will be as a knot in S^3. Let π_{K_1} (resp. π_{K_0}) be the (-1)-parallel of K_1 (resp. 0-parallel of K_0), and let $\widetilde{\pi}_{\widetilde{K}_i}$ be a lift of π_{K_i} to Y^∞, which is a parallel of $\widetilde{K}_i$ for $i=0,1$.

The next claim carries out by hand the first step of the plan described in Section 5.1.

Claim. *For $i=0,1$, the knot $\widetilde{K}_i\subset Y^\infty$ represents the homology class $\varpi_i(x_1^*)$, and the parallel $\widetilde{\pi}_{\widetilde{K}_i}$ satisfies*

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_i, \widetilde{\pi}_{\widetilde{K}_i}) = \frac{1}{8}.$$

In particular, $\widetilde{K}_i$ has equivariant linking matrix $A_{\widetilde{K}_i} = (1/8) = -(1/-8) = -Q^{-T}$ for $i=0,1$.

Proof. The assertion concerning the homology class holds by construction and so we focus on the equivariant linking number calculation. The proofs are similar for M_0 and M_1, so we give the most details for M_1, since that is the more complicated case, and then we sketch the easier case of M_0. We will use the equation

$$[\widetilde{\pi}_{\widetilde{K}_1}] = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_1, \widetilde{\pi}_{\widetilde{K}_1})[\mu_{\widetilde{K}_1}] \in H_1\big(Y\setminus\nu(K_1);\mathbb{Q}(t)\big) \tag{5.5}$$

from Definition 3.4. The $\mathbb{Z}$-cover Y^∞ of Y is $(S^2\times\mathbb{R})\#_{k\in\mathbb{Z}}t^k(-L(8,1))$, and there is no linking between curves in different $L(8,1)$ summands. Thus it suffices to investigate the $\mathbb{Q}$-valued linking number of K_1 and π_{K_1} in $Y' := -L(8,1)$, and consider the result as an element of $\mathbb{Q}(t)$. Formally speaking, we use an isomorphism

$$H_1\big(Y^\infty \setminus \cup_{i\in\mathbb{Z}} t^i \cdot \nu(\widetilde{K}_1)\big) \cong H_1\big(Y' \setminus \nu(K_1)\big) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}],$$

and then tensor both sides further by $- \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}(t)$. We compute in the right-hand side and translate to a conclusion about the left-hand side.

Since $Y' := -L(8,1) = S^3_{(-8)}(L_1)$, the manifold $Y' \setminus \nu(K_1)$ is obtained from the exterior of the 2-component link $L_1 \cup K_1 \subset S^3$ by Dehn filling L_1 with surgery coefficient -8. Since $\ell k(L_1, K_1) = 3$, the homology is therefore

$$H_1(Y' \setminus \nu(K_1)) \cong \frac{\mathbb{Z}\langle \mu_{L_1}\rangle \oplus \mathbb{Z}\langle \mu_{K_1}\rangle}{\langle -8\mu_{L_1} + 3\mu_{K_1}\rangle} \cong \mathbb{Z}.$$

We now express $[\pi_{K_1}]$ as a multiple of $[\mu_{K_1}]$, as required to calculate the framing of K_1. Since π_{K_1} is a (-1)-parallel of K_1 we have $[\pi_{K_1}] = 3[\mu_{L_1}] - [\mu_{K_1}]$. One checks that $\left(\begin{smallmatrix} 1 & 3 \\ -3 & -8 \end{smallmatrix}\right)\left(\begin{smallmatrix} -8 \\ 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, so one can use the invertible matrix $\left(\begin{smallmatrix} 1 & 3 \\ -3 & -8 \end{smallmatrix}\right)$ to change coordinates to the presentation

$$\mathbb{Z} \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} \mathbb{Z} \oplus \mathbb{Z} \to H_1\big(Y' \setminus \nu(K_1)\big) \to 0.$$

In this presentation, we compute that

$$\begin{aligned} [\mu_{K_1}] &= \mathrm{proj}_2 \circ \left(\begin{smallmatrix} 1 & 3 \\ -3 & -8 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = -8 \in \mathbb{Z} \cong H_1\big(Y' \setminus \nu(K_1)\big), \\ [\pi_{K_1}] &= \mathrm{proj}_2 \circ \left(\begin{smallmatrix} 1 & 3 \\ -3 & -8 \end{smallmatrix}\right)\left(\begin{smallmatrix} 3 \\ -1 \end{smallmatrix}\right) = -1 \in \mathbb{Z} \cong H_1\big(Y' \setminus \nu(K_1)\big). \end{aligned}$$

Hence, passing to the $\mathbb{Z}$-cover, tensoring up to $\mathbb{Q}(t)$ coefficients, and applying (5.5), we see that $-1 = \ell k_{\mathbb{Q}(t)}(\widetilde{K}_1, \widetilde{\pi}_{\widetilde{K}_1}) \cdot (-8)$ so, as asserted

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_1, \widetilde{\pi}_{\widetilde{K}_1}) = \frac{1}{8} \in \mathbb{Q}(t).$$

As indicated above, a similar computation shows the same result for M_0. Here are some details. The space $Y' \setminus \nu(K_0)$ is obtained from the exterior of the link $K_0 \cup L_1 \subset S^3$ by Dehn filling L_1 with framing -8. Since $\ell k(L_1, K_0) = 1$, it follows that

$$H_1\big(Y' \setminus \nu(K_0)\big) \cong \frac{\mathbb{Z}\langle \mu_{L_1}\rangle \oplus \mathbb{Z}\langle \mu_{K_0}\rangle}{\langle -8\mu_{L_1} + \mu_{K_0}\rangle} \cong \mathbb{Z}.$$

We now express $[\pi_{K_0}]$ as a multiple of $[\mu_{K_0}]$, as required to calculate the framing of K_0. Since π_{K_0} is a 0-parallel of K_0 we have $[\pi_{K_0}] = [\mu_{L_1}]$. Use the invertible matrix $\left(\begin{smallmatrix} 1 & 8 \\ 0 & 1 \end{smallmatrix}\right)$ to change coordinates to the presentation

$$\mathbb{Z} \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} \mathbb{Z} \oplus \mathbb{Z} \to H_1\big(Y' \setminus \nu(K_1)\big) \to 0.$$

In this presentation, we compute that

$$[\mu_{K_0}] = \mathrm{proj}_2 \circ \left(\begin{smallmatrix} 1 & 8 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = 8 \in \mathbb{Z} \cong H_1\big(Y' \setminus \nu(K_0)\big),$$
$$[\pi_{K_0}] = \mathrm{proj}_2 \circ \left(\begin{smallmatrix} 1 & 8 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = 1 \in \mathbb{Z} \cong H_1\big(Y' \setminus \nu(K_0)\big).$$

Hence, passing to the $\mathbb{Z}$ cover, tensoring up to $\mathbb{Q}(t)$ coefficients, one obtains

$$\ell k_{\mathbb{Q}(t)}(\widetilde{K}_0, \widetilde{\pi}_{\widetilde{K}_0}) = \frac{1}{8} \in \mathbb{Q}(t).$$

This concludes the proof of the claim. ∎

The combination of the claim with Step 2 of the plan from Section 5.1 implies that surgery along K_i yields a $\mathbb{Z}[t^{\pm 1}]$-homology 3-sphere for $i = 0, 1$. In order to recover the construction described during the proof of Proposition 5.18 however, we take $\widetilde{K}_i$ (and therefore $K_i \subset -L(8,1)$) to be the unknot for $i = 0, 1$: as described in the proposition, surgery on Y along K_0 and K_1 then yields $S^1 \times S^2$ and $(S^1 \times S^2)\# S^3_{+1}(T_{2,3})$ respectively. The infinite cyclic covers of these manifolds have vanishing Alexander modules yielding a "by hand" version of Step 2.

Step 3 is carried out by capping off with $S^1 \times D^3$ and $(S^1 \times D^3)\natural C$ respectively; both of these are homotopy $S^1 \times D^3$s. Thus M_0 and M_1 are obtained by the realisation process of our main theorem. It follows that $b_{M_0} = b_0 \neq b_1 = b_{M_1}$, as asserted.

In summary, the Kirby–Siebenmann invariant of spin 4-manifolds is not always controlled by the boundary and the intersection form. Rather, the automorphism invariant must be taken into account as well.

An explanation for this is that the automorphism invariant can act non-trivially on the spin structures. Using b_0 to fix an isometry $\partial\lambda \cong -\mathrm{Bl}_Y$, b_1 determines an automorphism of Bl_Y. If this automorphism preserved the quadratic enhancement of Bl_Y determined by a spin structure (or by the presentation of $\partial\lambda \cong \mathrm{Bl}_Y$ as the boundary of an even Hermitian form [81, p. 243], [20, Definition 2.5]) then the induced spin structures on Y would agree. Then M_0 and M_1 would be stably homeomorphic and hence their Kirby–Siebenmann invariants would be the same; see [20, Proposition 4.2]. But when we consider an automorphism of the linking form that does not preserve the quadratic enhancement, as is the case for b_1 above, then the Kirby–Siebenmann invariants can be different, as with the example just given.

Finally, we note that the example just given, without adding the copies of $S^1 \times D^3$, is also compelling in the simply-connected case. We gave it for infinite cyclic fundamental group since that is the topic of the present paper.

6. Application to $\mathbb{Z}$-surfaces in 4-manifolds

Recall that a $\mathbb{Z}$-*surface* refers to a locally flat, embedded surface in a 4-manifold whose complement has infinite cyclic fundamental group. In this section we apply our classification of 4-manifolds with fundamental group $\mathbb{Z}$ to the study of $\mathbb{Z}$-surfaces in simply-connected 4-manifolds and prove Theorems 1.7, 1.10, and 1.11 from the introduction. In Section 6.1, we focus on $\mathbb{Z}$-surfaces with boundary up to equivalence rel. boundary. In the shorter Sections 6.2 and 6.3, we respectively study surfaces with boundary up to equivalence (not necessarily rel. boundary) and closed surfaces. Section 6.4 lists some open problems.

6.1. Surfaces with boundary up to equivalence rel. boundary

Let N be a simply-connected 4-manifold with boundary homeomorphic to S^3. We fix once and for all a particular homeomorphism $h\colon \partial N \cong S^3$. Let $K \subset S^3$ be a knot. Thus K and h determine a knot in ∂N, which we also denote by K. The goal of this subsection is to give an algebraic description of the set of $\mathbb{Z}$-surfaces in N with boundary K up to equivalence rel. boundary.

We begin with some conventions. Given a properly embedded $\mathbb{Z}$-surface $\Sigma \subset N$ in a simply-connected 4-manifold, denote its exterior by $N_\Sigma := N \setminus \nu(\Sigma)$. Throughout this section, we will refer to embedded surfaces simply as Σ, and abstract surfaces as $\Sigma_{g,b}$, where g is the genus and b is the number of boundary components; we may sometimes write Σ_g when $b = 0$. Recall that throughout, $\Sigma_{g,b}$ and N will be oriented. This data determines orientations on S^3, K, and every meridian of an embedding of $\Sigma_{g,b}$. Observe that the $\pi_1(N_\Sigma) \cong \mathbb{Z}$ hypothesis implies that $[\Sigma, \partial\Sigma] = 0 \in H_2(N, \partial N)$ by [27, Lemma 5.1], so the relative Euler number of the normal bundle of Σ, with respect to the zero-framing of $\nu(\partial N)$, vanishes [27, Lemma 5.2]. From now on, we choose a framing $\nu(\Sigma) \cong \Sigma \times \mathring{D}^2 \cong \Sigma \times \mathbb{R}^2$ compatible with the orientation and with the property that for each simple closed curve $\gamma_k \subset \Sigma$, we have $\gamma_k \times \{e_1\} \subset N \setminus \Sigma$ is nullhomologous in $N \setminus \Sigma$. We will refer to such a framing as a *good framing*. As such, when $\partial\Sigma = K \subset \partial N$, we can identify the boundary of N_Σ as

$$\partial N_\Sigma \cong E_K \cup_\partial (\Sigma_{g,1} \times S^1) =: M_{K,g},$$

where the gluing ∂ takes λ_K to $\partial\Sigma \times \{\mathrm{pt}\}$.

We call two locally flat surfaces $\Sigma, \Sigma' \subset N$ with boundary $K \subset \partial N \cong S^3$ *equivalent rel. boundary* if there is an orientation-preserving homeomorphism of pairs $(N, \Sigma) \cong (N, \Sigma')$ that is pointwise the identity on $\partial N \cong S^3$. Note that if $\Sigma \subset N$ is a $\mathbb{Z}$-surface with boundary K, then N_Σ is a $\mathbb{Z}$-manifold with boundary $\partial N_\Sigma \cong M_{K,g}$ (see [27, Lemma 5.4]) and $H_1(M_{K,g}; \mathbb{Z}[t^{\pm 1}]) \cong H_1(E_K; \mathbb{Z}[t^{\pm 1}]) \oplus \mathbb{Z}^{2g}$ is torsion because the Alexander module $H_1(E_K;\mathbb{Z}[t^{\pm 1}])$ of K is torsion [27, Lemma 5.5]. Additionally, note that the equivariant intersection form λ_{N_Σ} of a surface exterior N_Σ must present $M_{K,g}$.

Consequently, as we did for manifolds, it is natural to fix a form (H,λ) that presents $M_{K,g}$ and to consider the set $\operatorname{Surf}(g)^0_\lambda(N,K)$ of genus g $\mathbb{Z}$-surfaces in N with boundary K and $\lambda_{N_\Sigma}\cong\lambda$.

Definition 6.1. For a non-degenerate Hermitian form (H,λ) over $\mathbb{Z}[t^{\pm 1}]$ that presents $M_{K,g}$, we set

$$\operatorname{Surf}(g)^0_\lambda(N,K) := \{\mathbb{Z}\text{-surfaces } \Sigma\subset N \text{ for } K \text{ with } \lambda_{N_\Sigma}\cong\lambda\}/\text{equivalence rel. } \partial.$$

There is an additional necessary condition for this set to be non-empty. For conciseness, we write $\lambda(1) := \lambda\otimes_{\mathbb{Z}[t^{\pm 1}]}\mathbb{Z}_\varepsilon$, where $\mathbb{Z}_\varepsilon$ denotes $\mathbb{Z}$ with the trivial $\mathbb{Z}[t^{\pm 1}]$-module structure. This way, if $A(t)$ is a matrix that represents λ, then $A(1)$ represents $\lambda(1)$. Additionally, recall that if W is a $\mathbb{Z}$-manifold, then $\lambda_W(1)\cong Q_W$, where Q_W denotes the standard intersection form of W; see e.g. [27, Lemma 5.10]. Thus, if we take $W=N_\Sigma$ and assume that $\lambda\cong\lambda_{N_\Sigma}$, then

$$\lambda(1)\cong\lambda_{N_\Sigma}(1)\cong Q_{N_\Sigma}\cong Q_N\oplus(0)^{\oplus 2g},$$

where the last isometry follows from a Mayer–Vietoris argument. Thus, for the set $\operatorname{Surf}(g)^0_\lambda(N,K)$ to be non-empty, it is also necessary that $\lambda(1)\cong Q_N\oplus(0)^{\oplus 2g}$.

For the final piece of setup for the statement of the main result of the section, we describe an action of $\operatorname{Homeo}^+(\Sigma_{g,1},\partial)$ on the set $\operatorname{Iso}(\partial\lambda,-\operatorname{Bl}_{M_{K,g}})$ as follows. First, a rel. boundary homeomorphism $x\colon\Sigma_{g,1}\to\Sigma_{g,1}$ induces an isometry

$$x''_*\colon\operatorname{Bl}_{M_{K,g}}\cong\operatorname{Bl}_{M_{K,g}}$$

as follows. Extend x to a self homeomorphism x' of $\Sigma_{g,1}\times S^1$ by defining $x'(s,\theta)=(x(s),\theta)$. Then extend x' by the identity over E_K; in total one obtains a self homeomorphism x'' of $M_{K,g}$. Now lift this homeomorphism to the covers and take the induced map on H_1 to get

$$x''_*\colon\operatorname{Bl}_{M_{K,g}}\cong\operatorname{Bl}_{M_{K,g}}.$$

The required action is now by postcomposition; for $f\in\operatorname{Iso}(\partial\lambda,-\operatorname{Bl}_{M_{K,g}})$, define $x\cdot f := x''_*\circ f$. The main result of this section proves Theorem 1.7 from the introduction. The formulation of the result is different than in the introduction, but clearly equivalent.

Theorem 6.2. *Let N be a simply-connected 4-manifold with boundary $\partial N\cong S^3$ and let $K\subset S^3$ be a knot. Given a non-degenerate Hermitian form (H,λ) over $\mathbb{Z}[t^{\pm 1}]$, the following assertions are equivalent:*

(1) *the Hermitian form* (H,λ) *presents* $M_{K,g}$ *and satisfies* $\lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}$*;*

(2) *the set* $\mathrm{Surf}(g)^0_\lambda(N,K)$ *is non-empty and there is a bijection*

$$\mathrm{Surf}(g)^0_\lambda(N,K) \approx \mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_{M_{K,g}})/\big(\mathrm{Aut}(\lambda)\times \mathrm{Homeo}^+(\Sigma_{g,1},\partial)\big).$$

Remark 6.3. We collect some remarks concerning Theorem 6.2.

(i) If (H,λ) presents $M_{K,g}$, then there is a non-canonical bijection

$$\frac{\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_{M_{K,g}})}{(\mathrm{Aut}(\lambda)\times \mathrm{Homeo}^+(\Sigma_{g,1},\partial))} \approx \frac{\mathrm{Aut}(\partial\lambda)}{(\mathrm{Aut}(\lambda)\times \mathrm{Homeo}^+(\Sigma_{g,1},\partial))}.$$

In addition, we have the isomorphism

$$\mathrm{Aut}(\partial\lambda) \cong \mathrm{Aut}(\mathrm{Bl}_{M_{K,g}}) \cong \mathrm{Aut}(\mathrm{Bl}_K)\oplus \mathrm{Sp}_{2g}(\mathbb{Z}),$$

where the latter is the group of automorphisms of the symplectic intersection pairing of $\Sigma_{g,1}$ (see [27, Propositions 5.6 and 5.7]). The group $\mathrm{Homeo}^+(\Sigma_{g,1},\partial)$ acts trivially on the first summand and transitively on the second. Therefore one can express the quotients above as

$$\mathrm{Aut}(\mathrm{Bl}_K)/\,\mathrm{Aut}(\lambda),$$

where the action of $\mathrm{Aut}(\lambda)$ on $\mathrm{Aut}(\mathrm{Bl}_K)$ arises by restricting the action of $\mathrm{Aut}(\lambda)$ on

$$\mathrm{Aut}(\partial\lambda) \cong \mathrm{Aut}(\mathrm{Bl}_{M_{K,g}}) \cong \mathrm{Aut}(\mathrm{Bl}_K)\oplus \mathrm{Sp}_{2g}(\mathbb{Z})$$

to the first summand. We stress again that the isomorphism $\mathrm{Aut}(\partial\lambda)\cong \mathrm{Aut}(\mathrm{Bl}_{M_{K,g}})$ is not canonical. The set $\mathrm{Aut}(\mathrm{Bl}_K)/\,\mathrm{Aut}(\lambda)$ was mentioned in Theorem 1.7 from the introduction.

(ii) The action of $\mathrm{Homeo}^+(\Sigma_{g,1},\partial)$ on $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_{M_{K,g}})$ factors through the corresponding mapping class group

$$\mathrm{Mod}^+(\Sigma_{g,1},\partial) := \pi_0\big(\mathrm{Homeo}^+(\Sigma_{g,1},\partial)\big).$$

In particular, Theorem 6.2 could have equally well been stated using $\mathrm{Mod}^+(\Sigma_{g,1},\partial)$ instead of $\mathrm{Homeo}^+(\Sigma_{g,1},\partial)$.

(iii) Our surface set $\mathrm{Surf}(g)^0_\lambda(N,K)$ is defined up to equivalence, hence Theorem 6.2 only gives a classification of surfaces up to equivalence (instead of ambient isotopy). This is because we prove Theorem 6.2 as a consequence of Theorem 1.3 and the equivalence on $\mathcal{V}^0_\lambda(M_{K,g})$ is up to *any* homeomorphism rel. boundary, not just homeomorphisms in a prescribed isotopy class. As a consequence, when N admits homeomorphisms which are not isotopic to the identity rel. boundary, there can be $\mathbb{Z}$-surfaces that are equivalent rel. boundary but not ambient isotopic. Here is an example.

Let $K \subset S^3$ be a knot with non-trivial Alexander polynomial Δ_K, that bounds a $\mathbb{Z}$-disc in a punctured $\mathbb{C}P^2$ with intersection form represented by the 1×1 matrix (Δ_K). Let N be given by the boundary connected sum with another punctured $\mathbb{C}P^2$ (so that N is a punctured $\mathbb{C}P^2 \# \mathbb{C}P^2$), and denote the same $\mathbb{Z}$-disc considered in N by D. There is a self-homeomorphism $\tau: N \to N$ that induces $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on $H_2(N) \cong \mathbb{Z}^2$. Isotope τ to be the identity on $\partial N \cong S^3$. The discs D and $\tau(D)$ are equivalent rel. boundary. But a short computation shows that the equivariant intersection forms of the exteriors are $\left(\begin{smallmatrix} \Delta_K & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \Delta_K \end{smallmatrix}\right)$ respectively. A straightforward computation shows that every $\mathbb{Z}[t^{\pm 1}]$-isometry between these two forms augments over $\mathbb{Z}$ to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. It follows that there is no ambient isotopy between D and $\tau(D)$.

Theorem 6.2 will be proved in three steps.

(1) We define a map Θ from a set of equivalence classes of embeddings

$$\Sigma_{g,1} \hookrightarrow N,$$

which we denote $\mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K)$ and which we will define momentarily, to the set of manifolds $\mathcal{V}^0_\lambda(M_{K,g})$ from Definition 2.1. By Theorem 1.3, $\mathcal{V}^0_\lambda(M_{K,g})$ corresponds bijectively to the set of isometries $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_{M_{K,g}})/\operatorname{Aut}(\lambda)$.

(2) We prove that the map Θ is a bijection, by defining a map Ψ in the other direction, from the set of manifolds to the set of embeddings, and showing that both $\Theta \circ \Psi$ and $\Psi \circ \Theta$ are the identity maps.

(3) In the final step, we describe the set of surfaces $\mathrm{Surf}(g)^0_\lambda(N, K)$ as a quotient of $\mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K)$ by $\mathrm{Homeo}^+(\Sigma_{g,1}, \partial)$. We show that this action and the actions of $\mathrm{Homeo}^+(\Sigma_{g,1}, \partial)$ on $\mathcal{V}^0_\lambda(M_{K,g})$ and $\mathrm{Iso}(\partial\lambda, -\mathrm{Bl}_{M_{K,g}})/\operatorname{Aut}(\lambda)$ are all compatible. Passing to orbits leads to the bijection in Theorem 6.2. This step is largely formal.

Step (1): From embeddings to manifolds. For the first step, we give some definitions and construct the map which will be the bijection in Theorem 6.2.

Consider the following set:

$$\mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K) = \frac{\{e: \Sigma_{g,1} \hookrightarrow N \mid e(\Sigma_{g,1}) \text{ is a } \mathbb{Z}\text{-surface for } K \text{ with } \lambda_{N_{e(\Sigma_{g,1})}} \cong \lambda\}}{\text{equivalence rel. } \partial}.$$

Two embeddings e_1, e_2 are *equivalent rel. boundary* if there exists a homeomorphism $\Phi: N \to N$ that is the identity on $\partial N \cong S^3$ and satisfies $\Phi \circ e_1 = e_2$.

In what follows, we let $\varphi: \pi_1(M_{K,g}) \twoheadrightarrow \mathbb{Z}$ be the epimorphism such that the induced map $\varphi': H_1(M_{K,g}) \twoheadrightarrow \mathbb{Z}$ is the unique epimorphism that maps the meridian of K to 1 and the other generators to zero. When we write $\mathcal{V}^0_\lambda(M_{K,g})$, it is with respect to this epimorphism φ. Recall also that we have a fixed homeomorphism $h: \partial N \to S^3$; whenever we say $\partial N \cong S^3$, it is with this fixed h.

In addition to our homeomorphism $h\colon \partial N \to S^3$, we fix once and for all the following data.

(i) A closed tubular neighbourhood $\overline{\nu}(K) \subset \partial N$. Since we have already fixed h, and since we are abusively using K for both the knot K in ∂N and for the image $h(K)$ in S^3, this choice of $\overline{\nu}(K) \subset \partial N$ also determines a particular neighbourhood $\overline{\nu}(K) \subset S^3$. We will use E_K exclusively to denote the complement of $\nu(K)$ in S^3.

(ii) A homeomorphism $D\colon \partial\Sigma_{g,1} \times S^1 \to \partial\overline{\nu}(K)$ that takes $\partial\Sigma_{g,1} \times \{1\}$ to the 0-framed longitude of K and $\{\mathrm{pt}\} \times S^1$ to the meridian of K such that

$$M_{K,g} = E_K \cup_D \Sigma_{g,1} \times S^1.$$

These choices can change the bijection, however we are interested only in the existence of a bijection, so this is not an issue.

Next we define the map which will be the bijection in Theorem 6.2.

Construction 6.4. We construct a map $\Theta\colon \mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K) \to \mathcal{V}^0_\lambda(M_{K,g})$. Let $e\colon \Sigma_{g,1} \hookrightarrow N$ be an embedding that belongs to $\mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K)$. We will assign to e a pair $(N_{e(\Sigma_{g,1})}, f)$, where $f\colon \partial N_{e(\Sigma_{g,1})} \to M_{K,g}$ is a homeomorphism. The pair we construct will depend on several choices, but we will show that the outcome is independent of these choices up to equivalence in $\mathcal{V}^0_\lambda(M_{K,g})$.

To cut down on notation we set $\Sigma := e(\Sigma_{g,1})$ and describe the choices on which our pair (N_Σ, f) will a priori depend.

(1) An embedding $\iota\colon \overline{\nu}(\Sigma) \hookrightarrow N$ of the normal bundle of Σ such that $\iota(\overline{\nu}(\Sigma)) \cap \partial N$ agrees with our fixed tubular neighbourhood of K.

(2) A good framing $\gamma\colon \overline{\nu}(\Sigma) \cong \Sigma_{g,1} \times D^2$ such that $h| \circ \iota \circ \gamma^{-1} = D$:

$$\begin{array}{ccc} \partial\Sigma_{g,1} \times S^1 & \xrightarrow{\quad D \quad} & \partial\overline{\nu}(K) \subset E_K \\ \Big\downarrow{\scriptstyle \gamma^{-1}} & & \Big\uparrow{\scriptstyle h|} \\ \gamma^{-1}(\partial\Sigma_{g,1} \times S^1) & \xrightarrow{\ \iota|\ } & \iota\big(\gamma^{-1}(\partial\Sigma_{g,1} \times S^1)\big) \subset \partial N \setminus \nu(K). \end{array} \tag{6.1}$$

In this diagram, $h|$ denotes the restriction of our fixed identification $h\colon \partial N \cong S^3$ and $D\colon \partial\Sigma_{g,1} \times S^1 \to \partial\overline{\nu}(K)$ is the homeomorphism that we fixed above.

We also record some of the notation that stems from these choices.

(i) The boundary of the surface exterior N_Σ decomposes as

$$\partial N_\Sigma \cong \big(\partial N \setminus \nu(K)\big) \cup \big(\partial\iota(\overline{\nu}(\Sigma)) \setminus \big(\iota(\nu(\Sigma)) \cap \partial N\big)\big). \tag{6.2}$$

Here the first part of this union is homeomorphic to a knot exterior, while the second is homeomorphic to $\Sigma_{g,1} \times S^1$.

(ii) Restricting our fixed homeomorphism $h\colon \partial N \cong S^3$ to the knot exterior part in (6.2), we obtain the homeomorphism

$$h|\colon \partial N \setminus \nu(K) \to E_K \subset M_{K,g}.$$

(iii) On the circle bundle part of (6.2), we consider the homeomorphism

$$\gamma| \circ \iota^{-1}\colon \big(\partial\iota(\overline{\nu}(\Sigma)) \setminus \big(\iota(\nu(\Sigma)) \cap \partial N\big)\big) \to \Sigma_{g,1} \times S^1 \subset M_{K,g}.$$

Here by the slightly abusive notation ι^{-1}, we mean that since $\iota\colon \overline{\nu}(\Sigma) \hookrightarrow N$ is an embedding, it is a homeomorphism onto its image, whence the inverse.

The diagram in (6.1) ensures that $h|$ and $\gamma| \circ \iota^{-1}$ can be glued together to give rise to the homeomorphism we have been building towards:

$$f_\gamma\colon \partial N_\Sigma \to M_{K,g}, \quad f_\gamma := \big(h|\big) \cup \big(\gamma| \circ \iota^{-1}\big). \tag{6.3}$$

Set $\Theta(e) := (N_\Sigma, f_\gamma)$. We need to verify that Θ gives rise to a map

$$\mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K) \to \mathcal{V}^0_\lambda(M_{K,g}).$$

In other words, we need to check that modulo homeomorphisms rel. boundary, $\Theta(e)$ does not depend on the embedding $\iota\colon \overline{\nu}(\Sigma) \hookrightarrow N$ nor on the particular choice of the good framing γ subject to the condition in (6.1). We also have to verify that equivalent embeddings produce equivalent manifolds.

(a) First we show that the construction is independent of γ and ι. Pick another embedding

$$\iota'\colon \overline{\nu}\big(e(\Sigma_{g,1})\big) \hookrightarrow N$$

of the normal bundle and another good framing

$$\gamma'\colon \overline{\nu}\big(e(\Sigma_{g,1})\big) \cong \Sigma_{g,1} \times D^2$$

with the same hypothesis about compatibility with D. This leads to boundary homeomorphisms $f_\gamma := (h|) \cup (\gamma| \circ \iota^{-1})$ and $f_{\gamma'} := (h|) \cup (\gamma'| \circ \iota'^{-1})$ and we must show that the following pairs are equivalent rel. boundary:

$$\big(N_{e_\iota(\Sigma_{g,1})}, f_\gamma\big) \quad \text{and} \quad \big(N_{e_{\iota'}(\Sigma_{g,1})}, f_{\gamma'}\big). \tag{6.4}$$

For a moment we are keeping track of the embeddings ι and ι' in our notation for exteriors. More explicitly, we set $N_{e_\iota(\Sigma_{g,1})} := N \setminus \iota(\nu(e(\Sigma_{g,1})))$ and similarly for ι'.

By uniqueness of tubular neighbourhoods [42, Theorem 9.3D], there is an isotopy of embeddings $\Gamma_t\colon \Sigma_{g,1} \times D^2 \hookrightarrow N$ such that $\Gamma_0 = \iota \circ \gamma^{-1}$ and $\Gamma_1 = \iota' \circ \gamma'^{-1}$ that fixes a neighbourhood of $\partial\Sigma_{g,1} \times D^2$. Then by the Edwards–Kirby isotopy extension theorem [31], there is an isotopy of homeomorphisms $F_t\colon N \to N$ with $F_1 \circ \iota \circ \gamma^{-1} = \iota' \circ \gamma'^{-1}$ and $F_0 = \mathrm{id}_N$ and such that F_t is the identity on a neighbourhood of the boundary ∂N for every $t \in [0,1]$. We will argue that this F_1 restricted to the exteriors $N_{e_\iota(\Sigma_{g,1})}$ and $N_{e_{\iota'}(\Sigma_{g,1})}$ gives a rel. boundary homeomorphism between the pairs in (6.4).

We wish to argue that the restriction of F_1 to the surface exteriors identifies $(N_{e_\iota(\Sigma_{g,1})}, f_\gamma)$ with $(N_{e_{\iota'}(\Sigma_{g,1})}, f_{\gamma'})$ as elements of $\mathcal{V}^0_\lambda(M_{K,g})$. Consider the following diagram:

$$\begin{array}{ccccccc}
M_{K,g} & \xleftarrow{f_\gamma=(h|)\cup(\gamma|\circ\iota^{-1})} & \partial N_{e_\iota(\Sigma_{g,1})} & \xrightarrow{\quad\subset\quad} & N_{e_\iota(\Sigma_{g,1})} & \xrightarrow{\quad\subset\quad} & N \\
\downarrow{=} & & \downarrow{F_1} & & \downarrow{F_1} & & \downarrow{F_1} \\
M_{K,g} & \xleftarrow{f_{\gamma'}=(h|)\cup(\gamma'|\circ\iota'^{-1})} & \partial N_{e_{\iota'}(\Sigma_{g,1})} & \xrightarrow{\quad\subset\quad} & N_{e_{\iota'}(\Sigma_{g,1})} & \xrightarrow{\quad\subset\quad} & N.
\end{array}$$

The right two squares certainly commute, while the left square commutes because the homeomorphism $F_1\colon N \to N$ is rel. boundary and because, by construction,

$$\gamma| \circ \iota^{-1} = F_1 \circ \gamma'| \circ \iota'^{-1}.$$

In total, we have

$$\begin{aligned}
f_{\gamma'} \circ F_1 &= \big((h|) \cup (\gamma'| \circ \iota'^{-1})\big) \circ F_1 \\
&= (h| \circ F_1) \cup (\gamma'| \circ \iota'^{-1} \circ F_1) = h| \cup (\gamma \circ \iota^{-1}) = f_\gamma. \qquad (6.5)
\end{aligned}$$

(b) We now show that the map Θ from Construction 6.4 is well defined up to rel. boundary homeomorphisms of N; recall that this is the equivalence relation on the domain $\mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K)$. Assume that $e, e'\colon \Sigma_{g,1} \hookrightarrow N$ are embeddings that are homeomorphic rel. boundary via a homeomorphism $F\colon N \to N$. Pick good framings γ and γ' for $\overline{\nu}(e(\Sigma_{g,1}))$ and $\overline{\nu}(e'(\Sigma_{g,1}))$ as well as an embedding $\iota'\colon \overline{\nu}(e'(\Sigma_{g,1})) \hookrightarrow N$. We now consider the embedding $\iota := F^{-1} \circ \iota' \circ (\gamma')^{-1} \circ \gamma$. The following diagram commutes:

$$\begin{array}{ccccccc}
\Sigma_{g,1} \times D^2 & \xrightarrow{\gamma^{-1},\cong} & \overline{\nu}\big(e(\Sigma_{g,1})\big) & \xrightarrow{\iota,\cong} & \iota\big(\overline{\nu}\big(e'(\Sigma_{g,1})\big)\big) & \xrightarrow{\quad\subset\quad} & N \\
\downarrow{=} & & & & \downarrow{F|} & & \downarrow{F} \\
\Sigma_{g,1} \times D^2 & \xrightarrow{\gamma'^{-1},\cong} & \overline{\nu}\big(e'(\Sigma_{g,1})\big) & \xrightarrow{\iota',\cong} & \iota'\big(\overline{\nu}\big(e(\Sigma_{g,1})\big)\big) & \xrightarrow{\quad\subset\quad} & N.
\end{array} \qquad (6.6)$$

As in Construction 6.4, the choice of framings leads to boundary homeomorphisms

$$\begin{aligned}
f &= (h|) \cup (\gamma| \circ \iota^{-1})\colon \partial N_{e_\iota(\Sigma_{g,1})} \xrightarrow{\cong} M_{K,g}, \\
f' &= (h|) \cup (\gamma'| \circ \iota'^{-1})\colon \partial N_{e'_{\iota'}(\Sigma_{g,1})} \xrightarrow{\cong} M_{K,g}.
\end{aligned}$$

As in (6.5), using the diagram from (6.6) and the fact that F is a rel. boundary homeomorphism, we deduce that $F| = f'^{-1} \circ f$ and that F restricts to a rel. boundary homeomorphism

$$F|\colon N_{e_\iota(\Sigma_{g,1})} \to N_{e'_{\iota'}(\Sigma_{g,1})}.$$

We conclude that $(N_{e(\Sigma_{g,1})}, f)$ is equivalent to $(N_{e'(\Sigma_{g,1})}, f')$ in $\mathcal{V}^0_\lambda(M_{K,g})$.

This concludes the verification that the map Θ from Construction 6.4 is well defined.

Remark 6.5. From now on, we continue to use the notation $\Sigma := e(\Sigma_{g,1})$ and we omit the choice of an embedding $\iota\colon \overline{\nu}(\Sigma_{g,1}) \hookrightarrow N$ from the notation since we have shown that $\Theta(e)$ is independent of the choice of embedding ι up to equivalence in $\mathcal{V}^0_\lambda(M_{K,g})$. In practice this means that we will simply write $\overline{\nu}(\Sigma) \subset N$. Since we omit ι from the notation, we also allow ourselves to think of (the inverse of) a good framing γ as giving an embedding

$$\gamma^{-1}\colon \Sigma_{g,1} \times D^2 \hookrightarrow \overline{\nu}(\Sigma) \subset N.$$

Similarly, given a choice of such a good framing, we now write the homeomorphism from (6.3) as

$$f_\gamma\colon \partial N_\Sigma \to M_{K,g}, \quad f_\gamma := (h|) \cup (\gamma|),$$

once again omitting ι from the notation. We sometimes also omit the choice of the framing γ from the notation, writing instead $\Theta(e) = (N_\Sigma, f)$.

Step (2): From manifolds to embeddings. We set up some notation aimed towards proving that Θ is a bijection when the form λ is even, and that Θ is a bijection when λ is odd and the Kirby–Siebenmann is fixed. Set $\varepsilon := \mathrm{ks}(N)$ and write $\mathcal{V}^{0,\varepsilon}_\lambda(M_{K,g})$ for the subset of those manifolds in $\mathcal{V}^0_\lambda(M_{K,g})$ whose Kirby–Siebenmann invariant equals ε. Observe that by additivity of the Kirby–Siebenmann invariant (see e.g. [43, Theorem 8.2]), if λ is odd and $\Sigma \subset N$ is a $\mathbb{Z}$-surface, then $\mathrm{ks}(N_\Sigma) = \mathrm{ks}(N) = \varepsilon$, so the image of Θ lies in $\mathcal{V}^{0,\varepsilon}_\lambda(M_{K,g})$. The next proposition is the next step in the proof of Theorem 6.2.

Proposition 6.6. *Let N be a simply-connected 4-manifold with boundary $\partial N \cong S^3$, let $K \subset S^3$ be a knot and let (H, λ) be a non-degenerate Hermitian form with $\lambda(1) \cong Q_N \oplus (0)^{2g}$.*

(1) *If λ is even, then the map Θ from Construction 6.4 determines a bijection*

$$\mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K) \to \mathcal{V}^0_\lambda(M_{K,g}).$$

(2) *If λ is odd, then the map Θ from Construction 6.4 determines a bijection*

$$\mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K) \to \mathcal{V}^{0,\varepsilon}_\lambda(M_{K,g}),$$

where $\varepsilon = \mathrm{ks}(N)$.

Proof. We construct an inverse Ψ to the assignment

$$\Theta\colon e \mapsto \big(N_{e(\Sigma_{g,1})}, f\big)$$

from Construction 6.4; this will in fact take up most of the proof. Let (W, f) be a pair, where W is a 4-manifold with fundamental group $\pi_1(W) \cong \mathbb{Z}$, equivariant intersection form $\lambda_W \cong \lambda$ and, in the odd case, Kirby–Siebenmann invariant $\operatorname{ks}(W) = \varepsilon$, and $f\colon \partial W \cong M_{K,g}$ is a homeomorphism.

The inverse $\Psi(W, f)$ is an embedding $\Sigma_{g,1} \hookrightarrow N$ which we define as follows. Glue $\Sigma_{g,1} \times D^2$ to W via the homeomorphism $f^{-1}|_{\Sigma_{g,1}\times S^1}$. This produces a 4-manifold $\widehat{W}$ with boundary

$$\partial\widehat{W} = \big(\partial W \setminus f^{-1}(\Sigma_{g,1} \times S^1)\big) \cup (\partial\Sigma_{g,1} \times D^2),$$

together with an embedding

$$\times\{0\}\colon \Sigma_{g,1} \hookrightarrow \widehat{W}, \quad x \mapsto (x, 0) \in \Sigma_{g,1} \times \{0\} \subset \Sigma_{g,1} \times D^2.$$

Note for now that $\partial\Sigma_{g,1} \times \{0\} \subset \partial\widehat{W}$ bounds a genus g $\mathbb{Z}$-surface in $\widehat{W}$ (with exterior W).

We will use the homeomorphism $f\colon \partial W \to M_{K,g}$ to define a homeomorphism $f'\colon \partial\widehat{W} \to \partial N$ and then use Freedman's classification of compact simply-connected 4-manifolds with S^3 boundary, to deduce that this homeomorphism extends to a homeomorphism $F\colon \widehat{W} \to N$. We will then take our embedding to be

$$\Psi(W, f) := F \circ \big(\times\{0\}\big)\colon \Sigma_{g,1} \hookrightarrow N.$$

The next paragraphs flesh out the details of this construction. Namely, firstly we build $f'\colon \partial\widehat{W} \to \partial N$ and secondly we argue it extends to a homeomorphism $F\colon \widehat{W} \to N$.

(i) Towards building this f', first observe that we get a natural homeomorphism $\partial\widehat{W} \to S^3$ as follows. Restricting f gives a homeomorphism

$$f|\colon \partial W \setminus f^{-1}(\Sigma_{g,1} \times S^1) \cong S^3 \setminus \nu(K).$$

Recall that the homeomorphism $D\colon \partial\Sigma_{g,1} \times S^1 \to \partial\overline{\nu}(K)$ sends $\partial\Sigma_{g,1} \times \{\mathrm{pt}\}$ to λ_K and $\{\cdot\} \times \partial D^2$ to μ_K, where λ_K and μ_K respectively denote the Seifert longitude and meridian of $K \subset S^3$. Since μ_K bounds a disc in $\overline{\nu}(K)$, this homeomorphism extends to a homeomorphism

$$\vartheta\colon \partial\Sigma_{g,1} \times D^2 \to \overline{\nu}(K). \tag{6.7}$$

Note that ϑ is well defined up to isotopy. Consider the following diagram:

$$\begin{array}{ccc} \partial W \setminus f^{-1}(\Sigma^\circ_{g,1} \times S^1) & \xrightarrow{f|,\cong} & S^3 \setminus \nu(K) \\ \uparrow {\scriptstyle f^{-1}|_{\partial \Sigma_{g,1} \times S^1}} & & \uparrow {\scriptstyle \subset} \\ \partial \Sigma_{g,1} \times S^1 & \xrightarrow{D,\cong} & \partial \overline{\nu}(K) \\ \downarrow {\scriptstyle \subset} & & \downarrow {\scriptstyle \subset} \\ \partial \Sigma_{g,1} \times D^2 & \xrightarrow{\vartheta,\cong} & \overline{\nu}(K). \end{array}$$

The bottom square commutes by definition of ϑ, whereas the top square commutes because $f|$ is obtained by restricting $f\colon \partial W \to M_{K,g} = (S^3 \setminus \nu(K)) \cup_D \Sigma_{g,1} \times S^1$. The commutativity of this diagram implies that f and ϑ combine to a homeomorphism

$$f| \cup \vartheta \colon= \partial \widehat{W} \to S^3.$$

Then $h^{-1} \circ (f| \cup \vartheta)$ gives the required homeomorphism

$$f' := h|^{-1} \circ (f| \cup \vartheta)\colon \partial \widehat{W} \to \partial N.$$

Further, we observe that $f'(\partial \Sigma_{g,1}) = K$.

(ii) To prove that this homeomorphism extends to a homeomorphism $\widehat{W} \cong N$, we will appeal to Freedman's theorem that for every pair of simply-connected topological 4-manifolds with boundary homeomorphic to S^3, the same intersection form, and the same Kirby–Siebenmann invariant, every homeomorphism between the boundaries extends to a homeomorphism between the 4-manifolds [40]. We check now that the hypotheses are satisfied.

We first argue that $\widehat{W}$ is simply-connected as follows. The hypothesis that W lies in $\mathcal{V}^0_\lambda(M_{K,g})$ implies that there is an isomorphism

$$\widehat{\varphi}\colon \pi_1(W) \xrightarrow{\cong} \mathbb{Z}$$

such that $\varphi = \widehat{\varphi} \circ \kappa$, where κ is the inclusion induced map $\pi_1(M_{K,g}) \to \pi_1(W)$ (see Definition 2.1). Since we required that $\varphi(\mu_K)$ generates $\mathbb{Z}$, we must have that $\kappa(\mu_K)$ generates $\pi_1(W) \cong \mathbb{Z}$. Since gluing $\Sigma_{g,1} \times D^2$ along $\Sigma_{g,1} \times S^1$ has the effect of killing $\kappa(\mu_K)$, we conclude that $\widehat{W}$ is simply-connected as claimed.

Next we must show that $Q_{\widehat{W}}$ is isometric to Q_N. A Mayer–Vietoris argument establishes the isometry $Q_{\widehat{W}} \oplus (0)^{\oplus 2g} \cong Q_W$. It then follows from our assumption on the Hermitian form (H, λ) that we have the isometries

$$Q_{\widehat{W}} \oplus (0)^{\oplus 2g} \cong Q_W \cong \lambda_W(1) \cong \lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}.$$

This implies $Q_{\widehat{W}} \cong Q_N$ since both forms are non-singular (indeed $\partial \widehat{W} \cong \partial N \cong S^3$).

In the even case, we deduce that both $\widehat{W}$ and N are spin. In the odd case, using the additivity of the Kirby–Siebenmann invariant (see e.g. [43, Theorem 8.2]), we have $\mathrm{ks}(\widehat{W}) = \mathrm{ks}(W) = \varepsilon = \mathrm{ks}(N)$.

Therefore $\widehat{W}$ and N are simply-connected topological 4-manifolds with boundary S^3, with the same intersection form and the same Kirby–Siebenmann invariant. Freedman's classification of simply-connected 4-manifolds with boundary S^3 now ensures that the homeomorphism $f' \colon \partial\widehat{W} \to \partial N$ extends to a homeomorphism $F \colon \widehat{W} \to N$ that induces the isometry $Q_{\widehat{W}} \cong Q_N$ and fits into the following commutative diagram

$$\begin{array}{ccccc} \big(\partial W \setminus f^{-1}(\Sigma_{g,1} \times S^1)\big) \cup (\partial\Sigma_{g,1} \times D^2) & \xrightarrow{=} & \partial\widehat{W} & \xrightarrow{\subset} & \widehat{W} \\ \downarrow{\scriptstyle h|^{-1}\circ(f|\cup\vartheta|)} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle F} \\ \big(\partial N \setminus \nu(K)\big) \cup \overline{\nu}(K) & \xrightarrow{=} & \partial N & \xrightarrow{\subset} & N. \end{array} \tag{6.8}$$

As mentioned above, we obtain an embedding as

$$\Psi(W, f) := \big(e \colon \Sigma_{g,1} \xrightarrow{\times\{0\}} \widehat{W} \xrightarrow{F,\cong} N\big). \tag{6.9}$$

This concludes the construction of our embedding $\Psi(W, f)$.

We must check that this construction gives rise to a map

$$\Psi \colon \mathcal{V}^0_\lambda(M_{K,g}) \to \mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K).$$

In other words, we verify that, up to homeomorphisms of N rel. boundary, the embedding e from (6.9) depends neither on the choice of isometry $Q_{\widehat{W}} \cong Q_N$ nor the choice of ϑ from (6.7) nor the homeomorphism $\widehat{W} \cong N$ extending our boundary homeomorphism nor on the homeomorphism rel. boundary type of (W, f).

(i) The precise embedding e depends on the homeomorphism $\widehat{W} \cong N$ chosen to extend a given f'. This homeomorphism in turn depends on the choice of isometry $Q_{\widehat{W}} \cong Q_N$. However for any two choices F_1 and F_2 of homeomorphisms $\widehat{W} \cong N$ extending f', the resulting embeddings are equivalent rel. boundary, as can be seen by composing one choice of homeomorphism with the inverse of the other:

$$\begin{array}{ccccc} \Sigma & \xrightarrow{[\times 0]} & \widehat{V} & \xrightarrow{F_1} & W \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle F_2\circ F_1^{-1}} \\ \Sigma & \xrightarrow{[\times 0]} & \widehat{V} & \xrightarrow{F_2} & W. \end{array}$$

So the equivalence class of the surface $\Psi(W, f)$ does not depend on the choice of isometry $Q_{\widehat{W}} \cong Q_N$ nor on the choice of homeomorphism $\widehat{W} \cong N$ realising this isometry and extending f'.

(ii) Next, we show that the definition is independent of the choice of

$$\vartheta\colon \partial\Sigma_{g,1} \times D^2 \to \overline{\nu}(K)$$

within its isotopy class. If $\vartheta_0, \vartheta_1\colon \partial\Sigma_{g,1} \times D^2 \to \overline{\nu}(K)$ are isotopic, then so are the resulting homeomorphisms $f'_0 := (f| \cup \vartheta_0|)$, $f'_1 := (f| \cup \vartheta_1|)\colon \partial\widehat{W} \to \partial N$ via an isotopy f'_s.

Claim. *There is an isotopy $F_s\colon \widehat{W} \to N$ extending f'_s.*

Proof. Pick a homeomorphism $F_0\colon \widehat{W} \to N$ extending f'_0; note that when we constructed $\Psi(W, f)$, we argued that such an F_0 exists. There are collars $\partial\widehat{W} \times [0,1]$ and $\partial N \times [0,1]$ such that $F_0|_{\partial\widehat{W}\times[0,1]} = f'_0 \times [0,1]$. Here it is understood that the boundaries of $\widehat{W}$ and N are respectively given by $\partial\widehat{W} \times \{0\}$ and $\partial N \times \{0\}$.

The idea is to implant the isotopy f'_s between f'_0, f'_1 in these collars in order to obtain an isotopy between F_0 and a homeomorphism F_1 that restricts to f'_1 on the boundary. To carry out this idea, consider the restriction

$$F_0|\colon \widehat{W} \setminus \big(\partial\widehat{W} \times [0,1]\big) \to N \setminus \big(\partial N \times [0,1]\big).$$

Define an isotopy of homeomorphisms between the collars via the formula

$$\begin{aligned} G_s\colon \partial\widehat{W} \times [0,1] &\to \partial N \times [0,1], \\ (x,t) &\mapsto \big(f'_{(1-t)s}(x), t\big). \end{aligned}$$

Since we have that $G_s(x,1) = (f'_0(x), 1)$ for every s, we obtain the required isotopy as $F_s := G_s \cup F_0$. By construction F_i restricts to f'_i on the boundary for $i = 0, 1$, thus concluding the proof of the claim. ∎

Thanks to the claim, we can use F_0 and F_1 to define the embeddings

$$e_0 := F_0 \circ \big(\times\{0\}\big) \quad \text{and} \quad e_1 := F_1 \circ \big(\times\{0\}\big).$$

This way, $F_1 \circ F_0^{-1}\colon N \to N$ is an equivalence rel. boundary between e_0 and e_1 so that the definition of Ψ is independent of the choice of ϑ within its isotopy class.

(iii) Next we check the independence of the rel. boundary homeomorphism type of (W, f). If we have (W_1, f_1) and (W_2, f_2) that are equivalent rel. boundary, then there is a homeomorphism $\Phi\colon W_1 \to W_2$ that satisfies $f_2 \circ \Phi| = f_1$. This homeomorphism extends to

$$\widehat{\Phi} := \Phi \cup \mathrm{id}_{\Sigma_{g,1}\times D^2}\colon \widehat{W}_1 \to \widehat{W}_2,$$

and therefore to a homeomorphism $N \to N$ that is, by construction rel. boundary. A formal verification using this latter homeomorphism then shows that the embeddings $\Psi(W_1, f_1)$ and $\Psi(W_2, f_2)$ are equivalent rel. boundary.

Now we prove that the maps Θ and Ψ are mutually inverse.

(i) First we prove that $\Psi \circ \Theta = \mathrm{id}$. Start with an embedding $e\colon \Sigma_{g,1} \hookrightarrow N$ and write $\Theta(e) = (N_{e(\Sigma_{g,1})}, f)$ with $f = (h|) \cup (\gamma|)\colon \partial N_{e(\Sigma_{g,1})} \to M_{K,g}$ the homeomorphism described in Construction 6.4. Then $\Psi(\Theta(e))$ is an embedding

$$\Sigma_{g,1} \xrightarrow{\times\{0\}} N_{e(\Sigma_{g,1})} \cup_f (\Sigma_{g,1} \times D^2) \xrightarrow{F,\cong} N.$$

We showed that the equivalence class of this embedding is independent of the homeomorphism F that extends f. It suffices to show that we can make choices so that $\Psi(\Theta(e))$ recovers e. This can be done explicitly as follows. Choose

$$\vartheta := h \circ \gamma^{-1}\colon \partial\Sigma_{g,1} \times D^2 \to \overline{\nu}(K).$$

Then we have

$$f' = \mathrm{id}_{\partial N \setminus \nu(K)} \cup \big(h^{-1} \circ (h \circ \gamma^{-1})\big) = \mathrm{id}_{\partial N \setminus \nu(K)} \cup \gamma|^{-1},$$

where the notation is as in (6.8) (with $W = N_{e(\Sigma_{g,1})}$). We already know an extension of f', namely $\mathrm{id}_{N_{e(\Sigma_{g,1})}} \cup \gamma^{-1}$, which we take to be F. Thus,

$$\Psi\big(\Theta(e)\big) = \gamma^{-1}|_{\Sigma_{g,1}\times\{0\}}\colon \Sigma_{g,1} \hookrightarrow N$$

which, by definition of a normal bundle, agrees with the initial embedding e.

(ii) Next we prove that $\Theta \circ \Psi = \mathrm{id}$. This time we start with a pair (W, f) consisting of a 4-manifold W and a homeomorphism $f\colon \partial W \to M_{K,g}$. Then $\Psi(W, f)$ is represented by an embedding

$$e\colon \Sigma_{g,1} \xrightarrow{\times\{0\}} \widehat{W} \xrightarrow{F,\cong} N.$$

Recall that we write $h\colon \partial N \to S^3$ for our preferred homeomorphism and that by construction, on the boundaries, F restricts to

$$h|^{-1} \circ (f| \cup \vartheta)\colon \partial\widehat{W} \to \partial N$$

where (the isotopy class of) $\vartheta\colon \partial\Sigma_{g,1} \times D^2 \to \overline{\nu}(K)$ satisfies the properties listed below equation (6.7).

We frame $\Sigma_{g,1} \times \{0\} \subset \widehat{W}$ via the unique homeomorphism $\mathrm{fr}\colon \overline{\nu}(\Sigma_{g,1} \times \{0\}) \to \Sigma_{g,1} \times D^2$ that makes the following diagram commute:

$$\begin{array}{ccc} \overline{\nu}\big(\Sigma_{g,1} \times \{0\}\big) & \xrightarrow{\mathrm{fr}} & \Sigma_{g,1} \times D^2 \\ & {\scriptstyle\mathrm{incl}}\searrow \quad \swarrow{\scriptstyle\mathrm{incl}} & \\ & \widehat{W} = W \cup (\Sigma_{g,1} \times D^2). & \end{array}$$

We then frame $e(\Sigma_{g,1}) \subset N$ via

$$\gamma := \mathrm{fr} \circ F^{-1}|\colon \overline{\nu}\big(e(\Sigma_{g,1})\big) \cong \Sigma_{g,1} \times D^2.$$

This framing is good thanks to the definition of $\varphi\colon \pi_1(M_{K,g}) \to \mathbb{Z}$ as the unique epimorphism that maps the meridian of K to 1 and the other generators to zero: indeed this implies that the curves on $\Sigma_{g,1} \times \{0\}$ are nullhomologous in W and therefore the same thing holds for $e(\Sigma_{g,1}) \subset N$. It can be verified that this framing satisfies the condition from (6.1).

We then obtain

$$\Theta\big(\Psi(W, f)\big) = \big(N_\Sigma := N \setminus \nu\big(e(\Sigma_{g,1})\big), h| \cup \gamma|\big),$$

where, as dictated by Construction 6.4, the boundary homeomorphism is

$$h| \cup \gamma|\colon \partial N_\Sigma \to M_{K,g}.$$

Here we are making use of the fact that up to equivalence, we can choose any framing in the definition of Θ.

We have to prove that $(N_\Sigma, h| \cup \gamma|)$ is homeomorphic rel. boundary to (W, f). We claim that the restriction of $F\colon \widehat{W} \to N$ gives the required homeomorphism. To see this, consider the following diagram

$$\begin{array}{ccccccccc}
M_{K,g} & \xleftarrow{f,\cong} & \big(\partial W \setminus f^{-1}(\Sigma_{g,1} \times S^1)\big) \cup \big(f^{-1}(\Sigma_{g,1} \times S^1)\big) & \xrightarrow{=} & \partial W & \xrightarrow{\subset} & W & \xrightarrow{\subset} & \widehat{W} \\
\Big\downarrow{=} & & \Big\downarrow{f' := (h|^{-1} \circ f|) \cup F|} & & \Big\downarrow{F|} & & \Big\downarrow{F|} & & \Big\downarrow{F} \\
M_{K,g} & \xleftarrow{h| \cup \gamma|} & \big(\partial N \setminus \nu(K)\big) \cup \big(\partial \overline{\nu}(\Sigma) \setminus \big(\nu(\Sigma) \cap \partial N\big)\big) & \xrightarrow{=} & \partial N_\Sigma & \xrightarrow{\subset} & N_\Sigma & \xrightarrow{\subset} & N.
\end{array}$$

The right two squares certainly commute. In the second-from-left square, we have just expanded out ∂W and ∂N_Σ, as well as written $F|$ explicitly on the regions where we have an explicit description from the construction of Ψ. So this square commutes.

It remains to argue that the left square commutes. By construction, $F|_{\partial \widehat{W}} = f' = h^{-1} \circ (f| \cup \vartheta)$. Thus on the knot exteriors, we have that $F| = h^{-1} \circ f|$ and so the left portion of the square commutes on the knot exteriors.

Now it remains to prove that $\gamma| \circ F| = f$. By definition of $\gamma = \mathrm{fr} \circ F^{-1}$, we must show that $\mathrm{fr}| = f|$ on $f^{-1}(\Sigma_{g,1} \times S^1)$. First note that fr has domain

$$\overline{\nu}\big(\Sigma_{g,1} \times \{0\}\big) \subset \widehat{W} = W \cup (\Sigma_{g,1} \times D^2),$$

so it appears we are attempting to compare maps which have different domains. However, the definition of $\widehat{W}$ identifies the portion of the boundary of $\overline{\nu}(\Sigma_{g,1})$ that we are interested in with $f^{-1}(\Sigma_{g,1} \times S^1) \subset \partial W$ via $f^{-1}| \circ \mathrm{fr}|$, so it makes sense to compare f on $f^{-1}(\Sigma_{g,1} \times S^1)$ with $\mathrm{fr}|$ on $\mathrm{fr}|^{-1} \circ f|_{f^{-1}(\Sigma_{g,1}\times S^1)}$. These maps are tautologically equal. Therefore, the left-hand side of the diagram commutes and this concludes the proof that $\Theta \circ \Psi = \mathrm{id}$.

We have shown that Θ and Ψ are mutually inverse, and so both are bijections. This completes the proof of Proposition 6.6. ∎

Step (3): From embeddings to submanifolds. Now we deduce a description of $\mathrm{Surf}(g)^0_\lambda(N, K)$ from Proposition 6.6. Note that $\mathrm{Surf}(g)^0_\lambda(N, K)$ arises as the orbit set

$$\mathrm{Surf}(g)^0_\lambda(N, K) = \mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K)/\operatorname{Homeo}^+(\Sigma_{g,1}, \partial),$$

where the left action of $x \in \operatorname{Homeo}^+(\Sigma_{g,1}, \partial)$ on $e \in \mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K)$ is defined by $x \cdot e = e \circ x^{-1}$. There is a surjective map

$$\mathrm{Emb}^0_\lambda(\Sigma_{g,1}, N; K) \to \mathrm{Surf}(g)^0_\lambda(N, K)$$

that maps an embedding $e\colon \Sigma_{g,1} \hookrightarrow N$ onto its image. One then verifies that this map descends to a bijection on the orbit set. Next, we note that $\operatorname{Homeo}^+(\Sigma_g, \partial)$ acts on the sets $\mathcal{V}^0_\lambda(M_{K,g})$ and $\mathcal{V}^{0,\varepsilon}_\lambda(M_{K,g})$ as follows. A rel. boundary homeomorphism $x\colon \Sigma_{g,1} \to \Sigma_{g,1}$ extends to a self homeomorphism x' of $\Sigma_{g,1} \times S^1$ by defining

$$x'(s, \theta) = \big(x(s), \theta\big).$$

Then extend x' by the identity over E_K; in total one obtains a self homeomorphism x'' of $M_{K,g}$. The required action is now by postcomposition: for (W, f) representing an element of $\mathcal{V}^0_\lambda(M_{K,g})$ or $\mathcal{V}^{0,\varepsilon}_\lambda(M_{K,g})$, define

$$x \cdot (W, f) := (W, x'' \circ f).$$

The following proposition is now a relatively straightforward consequence of Proposition 6.6.

Proposition 6.7. *Let N be a simply-connected 4-manifold with boundary $\partial N \cong S^3$, let $K \subset S^3$ be a knot and let (H, λ) be a non-degenerate Hermitian form with $\lambda(1) \cong Q_N \oplus (0)^{2g}$.*

(1) *If λ is even, then the map Θ from Construction 6.4 descends to a bijection*

$$\mathrm{Surf}(g)^0_\lambda(N, K) \to \mathcal{V}^0_\lambda(M_{K,g})/\operatorname{Homeo}^+(\Sigma_{g,1}, \partial).$$

(2) *If λ is odd, then the map Θ from Construction 6.4 descends to a bijection*

$$\mathrm{Surf}(g)^0_\lambda(N, K) \to \mathcal{V}^{0,\varepsilon}_\lambda(M_{K,g})/\operatorname{Homeo}^+(\Sigma_{g,1}, \partial),$$

where $\varepsilon = \mathrm{ks}(N)$.

Proof. Thanks to Proposition 6.6, it is enough to check that $\Theta(x \cdot e) = x \cdot \Theta(e)$ for $x \in \operatorname{Homeo}(\Sigma_{g,1}, \partial)$ and $e\colon \Sigma_{g,1} \hookrightarrow N$ an embedding representing an element of $\operatorname{Emb}^0_\lambda(\Sigma_{g,1}, N; K)$. By definition of Θ, we know $\Theta(x \cdot e)$ is $(N_{e(x^{-1}(\Sigma_{g,1}))}, f_{e\circ x^{-1}})$ and $x \cdot \Theta(e) = (N_{e(\Sigma_{g,1})}, x'' \circ f_e)$, where the $f_e, f_{e\circ x^{-1}}$ are homeomorphisms from the boundaries of these surface exteriors to $M_{K,g}$ that can be constructed, up to equivalence rel. boundary, using any choice of good framing; recall Construction 6.4. In what follows, we will make choices of framings so that the pairs

$$\Theta(x \cdot e) = \big(N_{e(x^{-1}(\Sigma_{g,1}))}, f_{e\circ x^{-1}}\big) \quad \text{and} \quad x \cdot \Theta(e) = \big(N_{e(\Sigma_{g,1})}, x'' \circ f_e\big)$$

are equivalent rel. boundary.

Pick a good framing $\gamma\colon \overline{\nu}(e(\Sigma_{g,1})) \cong \Sigma_{g,1} \times D^2$, so that

$$\Theta(e) = \big(N_{e(\Sigma_{g,1})}, f_e\big) = \big(N_{e(\Sigma_{g,1})}, h| \cup \gamma|\big).$$

Since $\gamma^{-1}\colon \Sigma_{g,1} \times D^2 \hookrightarrow N$ satisfies $\gamma^{-1}|_{\Sigma_{g,1}\times\{0\}} = e$, we can deduce that $\gamma^{-1} \circ (x^{-1} \times \mathrm{id}_{D^2})$ gives an embedding of the normal bundle of $e \circ x^{-1}$. We can therefore choose the inverse $\gamma_{e\circ x} := (x \times \mathrm{id}_{D^2}) \circ \gamma$ as a good framing for the embedding $e \circ x^{-1}$. Using this choice of good framing to construct $f_{e\circ x^{-1}}$, we have

$$\Theta(e \circ x^{-1}) = \big(N_{e\circ x^{-1}(\Sigma_{g,1})}, h| \cup \big((x \times \mathrm{id}_{D^2}) \circ \gamma|\big)\big).$$

Using these observations and the fact that x is rel. boundary, we obtain

$$\begin{aligned}
\Theta(x \cdot e) &= \Theta(e \circ x^{-1}) \\
&= \big(N_{e\circ x^{-1}(\Sigma_{g,1})}, h| \cup \big((x \times \mathrm{id}_{D^2}) \circ \gamma|\big)\big) \\
&= \big(N_{e\circ x^{-1}(\Sigma_{g,1})}, x'' \circ (h| \cup \gamma|)\big) = x \cdot \big(N_{e(\Sigma_{g,1})}, f_e\big) = x \cdot \Theta(e).
\end{aligned}$$

This proves that $\Theta(x \cdot e) = (N_{e(x^{-1}(\Sigma_{g,1}))}, f_{e\circ x^{-1}})$ and $x \cdot \Theta(e) = (N_{e(\Sigma_{g,1})}, f_e)$ are equivalent rel. boundary and thus concludes the proof of the proposition. ∎

We now deduce our description of the surface set, thus proving the main result of this section.

Proof of Theorem 6.2. Proposition 6.7 shows that if λ is even then the map Θ from Construction 6.4 induces a bijection

$$\operatorname{Surf}(g)^0_\lambda(N, K) \to \mathcal{V}^0_\lambda(M_{K,g}) / \operatorname{Homeo}^+(\Sigma_{g,1}, \partial),$$

while if λ is odd, for $\varepsilon := \operatorname{ks}(N)$, the map Θ induces a bijection

$$\operatorname{Surf}(g)^0_\lambda(N, K) \to \mathcal{V}^{0,\varepsilon}_\lambda(M_{K,g}) / \operatorname{Homeo}^+(\Sigma_{g,1}, \partial).$$

Thus the theorem will follows once we show that the map

$$b\colon V^0_\lambda(M_{K,g}) \to \operatorname{Iso}(\partial\lambda, -\mathrm{Bl}_{M_{K,g}})/\operatorname{Aut}(\lambda)$$

from Construction 2.3 intertwines the $\operatorname{Homeo}^+(\Sigma_{g,1}, \partial)$-actions, that is, it satisfies $b_{x\cdot(W,f)} = x \cdot b_{(W,f)}$ for every $x \in \operatorname{Homeo}^+(\Sigma_{g,1}, \partial)$ and for every pair (W, f) representing an element of $V^0_\lambda(M_{K,g})$.

This follows formally from the definitions of the actions: on the one hand, for some isometry $F\colon \lambda \cong \lambda_W$, we have $b_{x\cdot(W,f)} = b_{(W,x''\circ f)} = x''_* \circ f_* \circ D_W \circ \partial F$; on the other hand, we have $x \cdot b_{(W,f)}$ is $x \cdot (f_* \circ D_W \circ \partial F)$ and this gives the same result. This concludes the proof of Theorem 6.2. ∎

6.2. Surfaces with boundary up to equivalence

The study of surfaces up to equivalence (instead of equivalence rel.boundary) presents additional challenges: while there is still a map $\Theta\colon \operatorname{Emb}_\lambda(\Sigma_{g,1}, N; K) \to \mathcal{V}_\lambda(M_{K,g})$, the proof of Proposition 6.6 (in which we constructed an inverse Ψ of Θ) breaks down because if W and W' are homeomorphic $\mathbb{Z}$-fillings of $M_{K,g}$, it is unclear whether we can always find a homeomorphism $W \cup (\Sigma_{g,1} \times D^2) \cong W' \cup (\Sigma_{g,1} \times D^2)$. We nevertheless obtain the following result.

Theorem 6.8. *Let N be a simply-connected 4-manifold with boundary $\partial N \cong S^3$, let K be a knot such that every isometry of Bl_K is realised by an orientation-preserving homeomorphism $E_K \to E_K$ and let (H, λ) be a non-degenerate Hermitian form over $\mathbb{Z}[t^{\pm 1}]$. The following assertions are equivalent:*

1. *the Hermitian form λ presents $M_{K,g}$ and $\lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}$;*
2. *up to equivalence, there exists a unique genus g surface $\Sigma \subset N$ with boundary K and whose exterior has equivariant intersection form λ, i.e.*

$$|\operatorname{Surf}(g)_\lambda(N, K)| = 1.$$

Proof. We already proved the fact that the second statement implies the first, so we focus on the converse. We can apply Theorem 6.2 to deduce that $\operatorname{Surf}(g)^0_\lambda(N, K)$ is non-empty, this implies in particular that $\operatorname{Surf}(g)_\lambda(N, K)$ is non-empty. Since this set is non-empty, we assert that the hypothesis on K ensures we can apply [27, Theorem 1.3] to deduce that $|\operatorname{Surf}(g)_\lambda(N, K)| = 1$.

In contrast to Theorem 6.8, the statement of [27, Theorem 1.3] contains the additional condition that the orientation-preserving homeomorphism $f\colon E_K \to E_K$ be the identity on ∂E_K. We show that this assumption is superfluous, so that we can apply [27, Theorem 1.3] without assuming that $f|_{\partial E_K} = \mathrm{id}_{\partial E_K}$.

First, note that since f realises an isometry of Bl_K, it is understood that f preserves a basepoint x_0 and satisfies $f([\mu_K]) = [\mu_K]$, where $[\mu_K] \in \pi_1(E_K, x_0)$ is the based homotopy class of a meridian of K. An application of the Gordon–Luecke theorem [51] now implies that $f|_{\partial E_K}$ is isotopic to $\mathrm{id}_{\partial E_K}$; this isotopy can be assumed to be basepoint preserving by [35, p. 57]. Implanting this basepoint preserving isotopy in a collar neighbourhood of ∂E_K implies that f itself is basepoint preserving isotopic to a homeomorphism $E_K \to E_K$ that restricts to the identity on ∂E_K. This completes the proof that the extra assumption in the statement of [27, Theorem 1.3] can be assumed to hold without loss of generality. ∎

6.3. Closed surfaces

We now turn our attention to closed $\mathbb{Z}$-surfaces. Let X be a closed simply-connected 4-manifold and let $\Sigma \subset X$ be a closed $\mathbb{Z}$-surface with genus g, whose normal bundle we frame as in the case with boundary. With this framing, we can now identify the boundary of $X_\Sigma := X \setminus \nu(\Sigma)$ as

$$\partial X_\Sigma \cong \Sigma_g \times S^1.$$

Two such surfaces Σ and Σ' are *equivalent* if there exists an orientation-preserving homeomorphism $(X, \Sigma) \cong (X, \Sigma')$. Again as in the case of surfaces with boundary, X_Σ is a $\mathbb{Z}$ manifold and $H_1(\Sigma_g \times S^1; \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z}^{2g}$ is torsion. Additionally, note that the equivariant intersection form λ_{X_Σ} of a surface exterior X_Σ must present $\Sigma_g \times S^1$.

Definition 6.9. For a non-degenerate Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ presenting $\Sigma_g \times S^1$, set

$$\mathrm{Surf}(g)_\lambda(X) := \{\mathbb{Z}\text{-surface } \Sigma \subset X \text{ with } \lambda_{X_\Sigma} \cong \lambda\}/\text{ equivalence}.$$

As for $\mathbb{Z}$-surfaces with non-empty boundary, in order for $\mathrm{Surf}(g)_\lambda(X)$ to be non-empty it is additionally necessary that $\lambda(1) \cong Q_X \oplus (0)^{\oplus 2g}$. It was proved in [27, Theorem 1.4] that whenever $\mathrm{Surf}(g)_\lambda(X)$ is non-empty, it contains a single element. We improve this statement to include an existence clause.

Theorem 6.10. *Let X be a closed simply-connected 4-manifold. Given a non-degenerate Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$, the following assertions are equivalent:*

(1) *the Hermitian form λ presents $\Sigma_g \times S^1$ and $\lambda(1) \cong Q_X \oplus (0)^{\oplus 2g}$;*

(2) *there exists a unique (up to equivalence) genus g $\mathbb{Z}$-surface $\Sigma \subset X$ whose exterior has equivariant intersection form λ; i.e. $|\mathrm{Surf}(g)_\lambda(X)| = 1$.*

Proof. We have already argued that (2) $\Rightarrow$ (1) and so we focus on the converse. Use $U \subset S^3$ to denote the unknot and use N to denote the simply-connected 4-manifold with boundary S^3 obtained from X by removing a small open 4-ball. Note that $M_{U,g} = \Sigma_g \times S^1$ and that $Q_N = Q_X$. Since the Blanchfield form of U is trivial, Theorem 6.8 applies; this shows us that item (1) in Theorem 6.10 is equivalent to the existence of a unique (up to equivalence) genus g surface $\Sigma \subset N$ with boundary U and equivariant intersection form λ, in other words:

$$|\operatorname{Surf}(g)_\lambda(N, U)| = 1.$$

Since [27, Theorem 1.4] shows that $|\operatorname{Surf}(g)_\lambda(X)| \in \{0, 1\}$, it suffices to show that $\operatorname{Surf}(g)_\lambda(X)$ surjects onto $\operatorname{Surf}(g)_\lambda(N, U)$: this will imply $|\operatorname{Surf}(g)_\lambda(X)| = 1$.

Given a closed genus g $\mathbb{Z}$-surface $\Sigma \subset X$, a $\mathbb{Z}$-surface $\mathring{\Sigma} \subset N$ with boundary U can be obtained by removing a $(\mathring{D}^4, \mathring{D}^2)$-pair from (X, Σ). Because $\lambda_{X_\Sigma} \cong \lambda_{N_{\mathring{\Sigma}}}$ and because an equivalence from Σ to Σ' in X, restricts to an equivalence from $\mathring{\Sigma}$ to $\mathring{\Sigma}'$ in N, this puncturing operation gives rise to a map

$$\operatorname{Surf}(g)_\lambda(X) \to \operatorname{Surf}(g)_\lambda(N, U).$$

The surjectivity of this map is straightforward: a pair (N, Σ) where Σ has boundary U can be capped off by a pair (D^4, D^2) to get a closed surface in X. As explained above, (1) is equivalent to $|\operatorname{Surf}(g)_\lambda(N, U)| = 1$, which implies $|\operatorname{Surf}(g)_\lambda(X)| = 1$, as required. ∎

6.4. Problems and open questions

We conclude with some problems in the theory of $\mathbb{Z}$-surfaces, both in the closed case and in the case with boundary. In what follows, we set

$$\mathcal{H}_2 := \begin{pmatrix} 0 & t-1 \\ t^{-1}-1 & 0 \end{pmatrix}.$$

We start with closed surfaces in closed manifolds where the statements are a little cleaner.

Problem 1. *Fix a closed, simply-connected 4-manifold X. Characterise the non-degenerate Hermitian forms (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ that arise as λ_{X_Σ} where $\Sigma \subset X$ is a closed $\mathbb{Z}$-surface of genus g.*

It is known that if λ is as in Problem 1, then it must present $\Sigma_g \times S^1$, that $\lambda(1) \cong Q_X \oplus (0)^{\oplus 2g}$ and that $\lambda \oplus \mathcal{H}_2^{\oplus n} \cong Q_X \oplus \mathcal{H}_2^{\oplus(g+n)}$ for some $n \geq 0$. The necessity of the first two conditions was mentioned in Section 6.3 while the necessity of third was proved in [27, Proposition 1.6].

Here is what is known about Problem 1:

- if $X = S^4$ and $g \neq 1, 2$, then $\lambda \cong \mathcal{H}_2^{\oplus g}$ [27, Section 7];

- for $X = \mathbb{C}P^2$ and $g = 0$, the equivariant intersection form is necessarily the form $(x, y) \mapsto x\overline{y}$ and it follows that $\mathbb{Z}$-spheres in X are unique up to isotopy [25, Proposition A.1];
- if $b_2(X) \geq |\sigma(X)| + 6$, then [86, Theorem 7.2] implies that $\lambda \cong Q_X \oplus \mathcal{H}_2^{\oplus g}$.

This leads to the following question, a positive answer to which would solve Problem 1.

Question 1. *Let X be a closed simply-connected 4-manifold and let (H, λ) be a non-degenerate Hermitian form over $\mathbb{Z}[t^{\pm 1}]$. Is it the case that if λ presents $\Sigma_g \times S^1$, $\lambda(1) \cong Q_X \oplus (0)^{\oplus 2g}$ and $\lambda \oplus \mathcal{H}_2^{\oplus n} \cong Q_X \oplus \mathcal{H}_2^{\oplus (g+n)}$ for some $n \geq 0$, then $\lambda \cong Q_X \oplus \mathcal{H}_2^{\oplus g}$?*

If the answer to Question 1 were positive, then using Theorem 6.10 one could completely classify closed $\mathbb{Z}$-surfaces in closed simply-connected 4-manifolds: for every $g \geq 0$, in a closed simply-connected 4-manifold X, there would exist a unique $\mathbb{Z}$-surface of genus g in X up to equivalence.

Next, we discuss the analogous (but more challenging) problem for surfaces with boundary.

Problem 2. *Fix a simply-connected 4-manifold N with boundary S^3. Characterise the non-degenerate Hermitian forms (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ that arise as λ_{N_Σ}, where $\Sigma \subset N$ is a $\mathbb{Z}$-surface of genus g with boundary a fixed knot K. For brevity, we call such forms (N, K, g)*-realisable.

It is known that if λ is (N, K, g)-realisable, then it must present $M_{K,g}$, satisfy $\lambda(1) \cong Q_N \oplus (0)^{\oplus 2g}$ as well as $\lambda \oplus \mathcal{H}_2^{\oplus n} \cong Q_N \oplus \mathcal{H}_2^{\oplus (g+n)}$ for some $n \geq 0$. The necessity of the first two conditions was mentioned in Section 6.1 while the necessity of third was proved in [27, Proposition 1.6].

Here is what is known about Problem 2:

- if $N = D^4$, $g \neq 1, 2$ and K has Alexander polynomial one, then $\lambda \cong \mathcal{H}_2^{\oplus g}$ [27, Section 7];
- for $N = \mathbb{C}P^2 \setminus \mathring{D}^4$ and $g = 0$, the equivariant intersection form λ is necessarily the form $(x, y) \mapsto x\Delta_K\overline{y}$. After this article appeared, the classification $\mathbb{Z}$-discs in $\mathbb{C}P^2 \setminus \mathring{D}^4$ was studied in [23].

We conclude by listing consequences of further solutions to Problem 2.

(1) Using Theorem 6.2, a solution to Problem 2 would make it possible to fully determine the classification of properly embedded $\mathbb{Z}$-surfaces in a simply-connected 4-manifold N with boundary S^3 up to equivalence rel. boundary: for every $g \geq 0$, there would be precisely one $\mathbb{Z}$-surface of genus g in N with boundary K for every element of $\operatorname{Aut}(\operatorname{Bl}_K)/\operatorname{Aut}(\lambda)$, where λ ranges across all (N, K, g)-realisable forms.

(2) If one dropped the rel. boundary condition, then one might conjecture that for every $g \geq 0$, in a simply-connected 4-manifold N with boundary S^3, there is precisely one $\mathbb{Z}$-surface of genus g with boundary K for every element of $\mathrm{Aut}(\partial\lambda)/(\mathrm{Aut}(\lambda) \times \mathrm{Homeo}^+(E_K, \partial))$, where λ ranges across (N, K, g)-realisable forms. If the conjecture were true, then a solution to Problem 2 would provide a complete description of the set of properly embedded $\mathbb{Z}$-surfaces in a simply-connected 4-manifold N with boundary S^3, up to equivalence.

7. Ubiquitous exotica

In this section we demonstrate the failure of our topological classification to hold in the smooth setting. In Section 7.1 we set up some preliminaries we will require about Stein 4-manifolds and corks. In Section 7.2 we give the proofs of Theorems 1.15 and 1.17 from the introduction. In this section, all manifolds and embeddings are understood to be smooth.

7.1. Background on Stein structures and corks

We will be concerned with arranging that certain compact 4-manifolds with boundary admit a Stein structure. The unfamiliar reader can think of this as a particularly nice symplectic structure. Abusively, we will say that any smooth 4-manifold which admits a Stein structure is Stein. The reason for this sudden foray into geometry is to take advantage of restrictions on the genera of smoothly embedded surfaces representing certain homology classes in Stein manifolds. These restrictions will aid us in demonstrating that two 4-manifolds are not diffeomorphic. In this section, we will recall both a combinatorial condition for ensuring that a 4-manifold is Stein and the restrictions on smooth representatives of certain homology classes in Stein manifolds. We use the conventions and setup of [48] throughout.

We begin by recalling a criterion to ensure that a handle diagram with a unique 0-handle and no 3 or 4-handles describes a Stein 4-manifold. Recall that we can describe $\natural_{i=1}^r S^1 \times B^3$ using the dotted circle notation for 1-handles as in the left frame of Figure 2. It is not hard to show that any link in $\#_{i=1}^r S^1 \times S^2$ can be isotoped into the position shown in the right frame of Figure 2, where inside the tangle marked T we require that the diagram meet the conventions of a front diagram for the standard contact structure on S^3. For details on front diagrams, see [33]; stated briefly this amounts to isotoping the diagram so that all vertical tangencies are replaced by cusps and so that at each crossing the more negatively sloped strand goes over. We note that front diagrams require oriented links; we can choose orientations on our 2-handle attaching spheres arbitrarily, since orienting the link does not affect the 4-manifold.

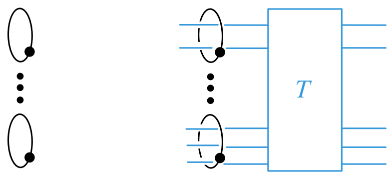

Figure 2. The left-hand side shows a handle diagram for a boundary connected sum of $S^1 \times D^3$. On the right-hand side, the tangle diagram T satisfies the conventions of a front diagram.

Thus any handle diagram with a unique 0-handle and no 3 or 4-handles can be isotoped into the form of the right frame of Figure 2; we say that such a diagram is in *Gompf standard form*.

For a diagram in Gompf standard form, let L_i^T denote the tangle diagram obtained by restricting the i-th component L_i of the diagram of L to T. For a diagram in Gompf standard form, the *Thurston–Bennequin number* $TB(L_i)$ of L_i is defined as

$$TB(L_i^D) = w(L_i^T) - c(L_i^T),$$

where $w(L_i^T)$ denotes the writhe of the tangle and $c(L_i^T)$ denotes the number of left cusps.

In this setup, the following criterion is helpful to prove that handlebodies are Stein.

Theorem 7.1 ([32, 48], see also [49, Theorem 11.2.2]). *A smooth* 4*-manifold X with boundary is Stein if and only if it admits a handle diagram in Gompf standard form such that the framing f_i on each* 2*-handle attaching curve L_i has $f_i = TB(L_i) - 1$.*

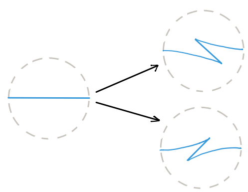

Figure 3. Stabilising a front diagram.

Remark 7.2. The 'if' direction of the Theorem 7.1 holds under the weaker hypothesis that each 2-handle attaching curve L_i has $f_i \leq TB(L_i) - 1$. To see this, observe that any 2-handle L_i can be locally isotoped via the *stabilisations* demonstrated in Figure 3 and observe that stabilisation preserves the condition on T and lowers the Thurston–Bennequin number of L_i by one. The claim now follows since we can stabilise any 2-handle in a diagram in Gompf standard form to lower its Thurston–Bennequin number without changing the smooth 4-manifold described.

We will also make use of the following special case of the adjunction inequality for Stein manifolds.

Theorem 7.3 ([71]). *In a Stein manifold X, any homology class $\alpha \in H_2(X)$ with $\alpha \cdot \alpha = -1$ cannot be represented by a smoothly embedded sphere.*

Proof. The proof can be deduced by combining [71, Theorem 3.2] with [15, 47]; further exposition can be found in [2, Theorems 1.2 and 1.3]. ■

In order to handily construct pairs of homeomorphic 4-manifolds, we will make use of *cork twisting*. Define C to be the contractible 4-manifold in the left frame of Figure 4, which is commonly refereed to as the *Akbulut cork*. Observe that ∂C admits another contractible filing C' given by the right frame of Figure 4, and that there is a natural homeomorphism $\tau := (\delta')^{-1} \circ \delta \colon \partial C \to \partial C'$ demonstrated in the figure. Using the work of Freedman [40], the homeomorphism τ extends to a homeomorphism $T \colon C \to C'$. As a result, for any 4-manifold W with $\iota \colon C \hookrightarrow W$, one can construct a new 4-manifold $W' := W \setminus \iota(C) \cup_{(\iota|_\partial)\circ\tau^{-1}} C'$ and, combining the identity homeomorphism $\mathrm{id}_{W\setminus\iota(C)}$ with T, one sees that W and W' are homeomorphic.

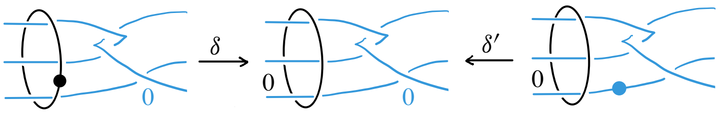

Figure 4. Two fillings of the boundary of the Akbulut cork, with boundary homeomorphism $\delta'^{-1} \circ \delta$. Here and throughout the rest of the paper, all handle diagrams drawn in this horizontal format should be braid closed.

Historically, the literature has been concerned with two types of exotic phenomena. If smooth 4-manifolds X, X' with boundary admit a homeomorphism $F \colon X \to X'$ but no diffeomorphism $G \colon X \to X'$ such that $G|_\partial$ is isotopic to $F|_\partial$, we call X and X' *relatively exotic*. If smooth 4-manifolds X, X' admit a homeomorphism $F \colon X \to X'$ but no diffeomorphism $G \colon X \to X'$ we call X and X' *absolutely exotic*. It is easier to build relatively exotic pairs in practice. Fortunately, work of Akbulut and Ruberman shows that all relative exotica contains absolute exotica.

Theorem 7.4 ([3, Theorem A]). *Let M, M' be smooth 4-manifolds. Let $F \colon M \to M'$ be a homeomorphism whose restriction to the boundary is a diffeomorphism that does not extend to a diffeomorphism $M \to M'$. Then M (resp. M') contains a smooth codimension 0 submanifold V (resp. V') which is orientation-preserving homotopy equivalent to M (resp. M') such that V is homeomorphic but not diffeomorphic to V'.*

If ∂M and $\partial M'$ are non-empty, then V and V' necessarily also have non-empty boundaries since they are codimension zero submanifolds of manifolds with boundary. We remark that Akbulut–Ruberman's theorem is only stated when M is diffeomorphic to M' (hence by applying a reference identification, they can in fact just call both manifolds M). However their proof works verbatim when M and M' are just homeomorphic smooth manifolds, which is the hypothesis we take above.

7.2. Proof of Theorems 1.15 and 1.17

We prove Theorem 1.15 from the introduction, which for convenience we state again here in more detail.

Theorem 7.5. *For every Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ there exists a pair of smooth $\mathbb{Z}$-manifolds M and M' with boundary and fundamental group $\mathbb{Z}$, such that:*

(1) *there is a homeomorphism $F: M \to M'$;*

(2) *F induces an isometry $\lambda_M \cong \lambda_{M'}$, and both forms are isometric to λ;*

(3) *there is no diffeomorphism from M to M'.*

In other words, every Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ is exotically realisable.

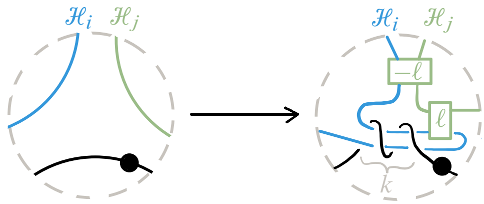

Figure 5. Arbitrary Hermitian forms can be realised as equivariant intersection forms by repeatedly performing the following local move, which we illustrate for $k = 2$.

Proof. Let $A(t)$ be a matrix representing the given form λ, so that $A(1)$ is an integer valued matrix. Choose any framed link $L = \cup L_i \subset S^3$ with linking matrix $A(1)$ and let M_1 be the 4-manifold obtained from D^4 by attaching $A(1)_{ii}$-framed 2-handles to D^4 along L_i. Let M_2 be the 4-manifold obtained from M_1 by attaching a 1-handle (which we will think of as removing the tubular neighbourhood of a trivial disc for an unknot split from L). Thus $\pi_1(M_2) \cong \mathbb{Z}$ and both the integer valued intersection form Q_{M_2} and the equivariant intersection form λ_{M_2} are represented by a matrix for $\lambda(1)$.

Now we will modify the handle diagram of M_2 in a way which will preserve the fundamental group and integer valued intersection form, but will result in an M_3 with equivariant intersection form $\lambda_{M_3} \cong \lambda$. For pairs i, j with $i < j$, for each monomial ℓt^k in the polynomial $A(t)_{ij}$, perform the local modification exhibited in Figure 5. Observe (for later use) that this move does not change the framed link type of the link of attaching spheres of 2-handles. Furthermore, the modification does not change the fundamental group or the integer valued intersection form of M_2. We exhibit in Figure 6 what the cover looks like locally after the modification.

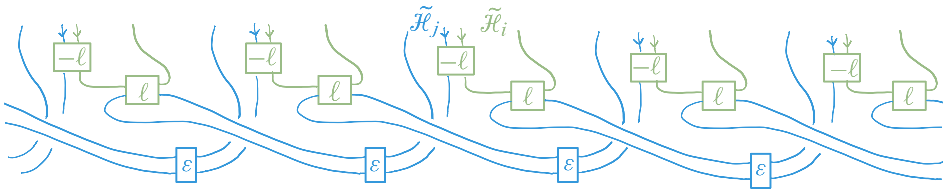

Figure 6. A local picture of the cover after our local modification with $k = 2$. When $k > 0$ the twist parameter ε is $1 - k$, when $k < 0$ it is $-k - 1$.

Recall from Remark 3.3 that for elements $[\widetilde{a}], [\widetilde{b}] \in H_2(M_2, \mathbb{Z}[t^{\pm 1}])$ the equivariant intersection form satisfies

$$\lambda_{M_2}\big([\widetilde{b}], [\widetilde{a}]\big) = \sum_k (\widetilde{a} \cdot_{M_2^\infty} t^k \widetilde{b}) t^{-k}.$$

Thus we see that after each iteration of the local move we have that

$$\lambda_{M_2'}(t)_{ij} = \lambda_{M_2}(t)_{ij} - \ell + \ell t^k \quad \text{and} \quad \lambda_{M_2'}(t)_{ji} = \lambda_{M_2}(t)_{ji} - \ell + \ell t^{-k}.$$

For pairs $i = j$, for each monomial ℓt^k with $k > 0$ in the polynomial $A(t)_{ii}$, again perform the local modification in Figure 5. In this case, one finds that

$$\lambda_{M_2'}(t)_{ii} = \lambda_{M_2}(t)_{ii} - 2\ell + \ell t^k + \ell t^{-k}. \tag{7.1}$$

The non-constant terms of (7.1) are straightforward to deduce. The constant term is computed by considering a parallel of $\mathcal{H}_i$ downstairs which is 0-framed in the modification region, lifting the framing curve into the cover, and then computing the linking of the lift of the framing with $\widetilde{\mathcal{H}}_i$.

Once these modifications are complete, we obtain a 4-manifold M_3 with λ_{M_3} agreeing with λ everywhere except *a priori* on the constant terms of each $A(t)_{ij}$. Observe however that since these local modifications do not change the integer valued intersection form $\lambda(1)$, we have that λ_{M_3} must also agree with λ on the constant terms of each $A(t)_{ij}$. Thus, when we are finished, we have a smooth 4-manifold M_3 with no 3-handles, $\pi_1(M_3) \cong \mathbb{Z}$ and $\lambda_{M_3} \cong \lambda$.

Next we will modify the 2-handles of our handle diagram $\mathcal{H}$ of M_3 to get a Stein 4-manifold M_4 with the same fundamental group and equivariant intersection form

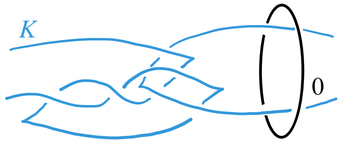

Figure 7. The knot K in $S^1 \times S^2$. A handle diagram for the 4-manifold X is obtained from this diagram by dotting the black unknot and attaching a 0-framed 2-handle to K.

as M_3. We will do this by getting the handle diagram into a form where we can apply Eliashberg's theorem 7.1, which requires arranging that each 2-handle has a suitably large Thurston–Bennequin number. To begin, isotope $\mathcal{H}$ into Gompf standard form, so that we think of the 2-handles of $\mathcal{H}$ as a Legendrian link in the standard tight contact structure on $S^1 \times S^2$. If any of the 2-handle attaching curves do not have any cusps, stabilise once so that they do. Let $A_3(t)$ be the equivariant linking matrix of $\mathcal{H}$; note that $A_3(t) = A(t)$ is a matrix representing the equivariant intersection form λ. Let K be the knot in $S^1 \times S^2$ exhibited in Figure 7. Observe that if we use K to describe a 4-manifold X via attaching a 0-framed 2-handle to $S^1 \times B^3$ along K, then $\pi_1(X) \cong \mathbb{Z}$ and the equivariant intersection form λ_X is represented by the size one matrix (0). Observe further that K has a Legendrian representative $\mathcal{K}$ (illustrated in Figure 7) in the standard tight contact structure on $S^1 \times S^2$ with $\operatorname{TB}(\mathcal{K}) = 1$. In our handle diagram $\mathcal{H}$ of M_3, let $\mathring{K}$ be a copy of K in $S^1 \times S^2$ which is split from all of the 2-handles of $\mathcal{H}$, as depicted in the left frame of Figure 8.

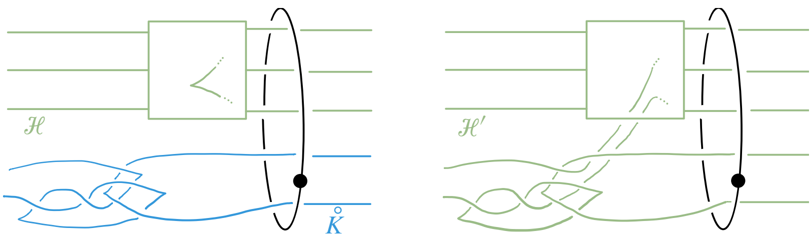

Figure 8. The connect sum band can be taken with a sufficiently positive slope that choosing it to pass under any strands in the tangle T causes the diagram to remain in Gompf standard form.

Now for any handle $\mathcal{H}_i$ of $\mathcal{H}$ with $A_3(1)_{ii} > \operatorname{TB}(\mathcal{H}_i) - 2$ form $\mathcal{H}_i'$ by taking the connected sum of $\mathcal{H}_i$ with a split copy of $\mathring{K}$ in the manner depicted in Figure 8. Frame $\mathcal{H}_i'$ using the same diagrammatic framing instruction that was used to frame $\mathcal{H}_i$. One computes readily from the right frame of Figure 8 that $\operatorname{TB}(\mathcal{H}_i') = \operatorname{TB}(\mathcal{H}_i) + 1$. Repeat this process until $A_3(1)_{ii} \leq \operatorname{TB}(\mathcal{H}_i) - 2$ for all 2-handles. Let M_4 be the resulting 4-manifold. Then M_4 is Stein by Theorem 7.1 and Remark 7.2. Further, since X contributes neither to the equivariant intersection form nor to π_1, we

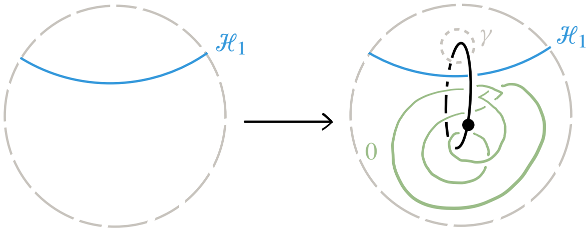

Figure 9. The local modification performed on the handle $\mathcal{H}_1$ of the manifold M_3.

have that M_4 has the same equivariant intersection form and π_1 as M_3. We record (for later use) the observation that the link in S^3 consisting of the attaching spheres of the 2-handles is unchanged by these modifications; one can see this by ignoring the 1-handle in Figure 8 and doing a bit of isotopy.

Now we will make a final modification to M_4 to get a 4-manifold $M_5 =: M$ which we can cork twist to get M'. Choose any 2-handle, without loss of generality we choose $\mathcal{H}_1$, and perform the local modification described in Figure 9; the resulting 4-manifold is our M.

One can readily check that this local modification does not impact π_1 or the equivariant intersection form. Further, this local diagram can be readily converted to Gompf standard form, (see the blue and green handles of Figure 10) where we have $A_3(1)_{ii} \leq \mathrm{TB}(\mathcal{H}_i) - 1$ for all 2-handles, hence M is Stein. By construction, M contains a copy of the Akbulut cork C. Because M has no 3-handles, $\pi_1(\partial M)$ surjects $\pi_1(M)$.

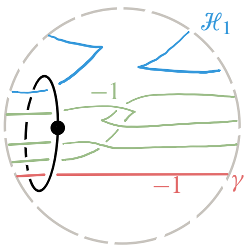

Figure 10. A handle diagram for the manifold W in Gompf standard form.

Now define M' to be the 4-manifold obtained from M by twisting C. Since there is a homeomorphism $T\colon C \to C$ extending the twist homeomorphism $\tau\colon \partial C \to \partial C$, then there is a natural homeomorphism $F\colon M \to M'$; we let f denote the restriction $f\colon \partial M \to \partial M'$.

It remains to show that M and M' are not diffeomorphic. We will begin by showing the relative statement, i.e. there is no diffeomorphism $G\colon M \to M'$ such that $G|_\partial = f$. It would be convenient if at this point we could distinguish M and M' directly by showing that one is Stein and one is not. Unfortunately, both are Stein. So instead we will consider auxiliary manifolds W and W' constructed as follows. Suppose for a contradiction that there were such a diffeomorphism G. Construct a 4-manifold W by attaching a (-1)-framed 2-handle to M along γ (where γ is the curve in ∂M marked in Figure 9) and a second 4-manifold W' from M' by attaching a 2-handle to M' with attaching sphere and framing given by $(f(\gamma), -1)$.[1] Notice that the image under f of a (-1)-framing curve for γ is in fact a (-1)-framing curve for $f(\gamma)$. The diffeomorphism G extends to give a diffeomorphism $\widehat{G}\colon W \to W'$. In Figure 10, we have exhibited the natural handle diagram for W in Gompf standard form, from which Theorem 7.1 implies that W admits a Stein structure.

We will finish showing that f does not extend by demonstrating that W' does not admit any Stein structure, thus W cannot be diffeomorphic to W'. Since W' is obtained from W by reversing the dot and the zero on the handles of C, $f(\gamma)$ is just a meridian of a 2-handle of M'. Thus the final 2-handle of W' is attached along a curve which bounds a disc in M', implying that there is a (-1)-framed sphere embedded in W'. But the adjunction inequality for Stein manifolds (recall Theorem 7.3) indicates that no 4-manifold which admits a Stein structure can contain an embedded sphere with self-intersection -1. Hence, W is not diffeomorphic to W', thus there cannot be a diffeomorphism $G\colon M \to M'$ extending f.

Now we would like to extend this to a statement about absolute exotica. To do so, we apply Theorem 7.4 to our M, M', and f to produce a pair of smooth 4-manifolds V and V' (both of which have non-empty boundary) which are homeomorphic but not diffeomorphic. Since V and V' are orientation-preserving homotopy equivalent to M and M' respectively, the equivariant intersection forms λ_V and $\lambda_{V'}$ are also isometric to λ, and both V and V' have fundamental group $\mathbb{Z}$. Since V and V' are homeomorphic, so are ∂V and $\partial V'$. ■

Next, we prove Theorem 1.17 from the introduction, again stated here in more detail. If one wants to show that any 2-handlebody N with boundary S^3 contains a pair of exotic $\mathbb{Z}$-discs one can run the same proof, where in the first line $\mathcal{H}'$ is chosen to be a handle diagram for N; this was mentioned in Remark 1.18.

Theorem 7.6. *For every Hermitian form (H, λ) over $\mathbb{Z}[t^{\pm 1}]$ such that $\lambda(1)$ is realised as the intersection form of a smooth simply-connected 4-dimensional 2-handlebody N with $\partial N \cong S^3$, there exists a pair of smooth $\mathbb{Z}$-discs D and D' in N with the same boundary and the following properties:*

[1] The (-1)-framing instruction for $f(\gamma)$ requires a diagram of $f(\gamma)$ in $\partial M'$. Because f is a dot-zero homeomorphism, we can use the exact same diagram as we used for γ in ∂M.

(1) *the equivariant intersection forms* λ_{N_D} *and* $\lambda_{N_{D'}}$ *are isometric to* λ*;*

(2) D *is topologically isotopic to* D' *rel. boundary;*

(3) D *is not smoothly equivalent to* D' *rel. boundary.*

Proof. Let $\mathcal{H}'$ be a handle diagram for a 2-handlebody with S^3 boundary and such that Q_N isometric to $\lambda(1)$. Let D be the standard disc for a local unknot in ∂N, and as usual let N_D be its exterior, which has handle diagram $\mathcal{H} := \mathcal{H}' \cup$ 1-handle.

Akin to the proof of Theorem 1.15, we will now modify the linking of the handles of $\mathcal{H}$ to get a Stein manifold with equivariant intersection form λ. However, we also want to do so in such a way that the manifold presented by $\mathcal{H}$ is still $N_{D'}$ for some smooth disc D' properly embedded in N.

We claim that if we modify only the linking of the 2-handles with the 1-handle, and not the linking of the 2-handles with each other nor the knot type or framing of the 2-handles, we will have that $\mathcal{H}$ presents such an $N_{D'}$. To prove the claim, first observe that X is the exterior of a disc in N if and only if N can be obtained from X by adding on a single 2-handle. Observe that adding a 0-framed 2-handle to the meridian of a 1-handle in dotted circle notation allows us to erase both the new 2-handle and the 1-handle. Thus, if our modifications only change the way the 2-handles of N link the new one-handle, we will still have the property that after a single 2-handle addition we obtain N, thus our manifold is the exterior of a disc embedded in N. This concludes the proof of the claim.

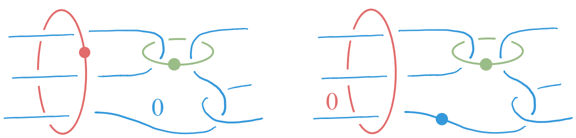

Figure 11. In both frames the red and blue handles give a non-standard handle diagram for D^4, and in both frames the green knot $K \subset S^3$ bounds a disc disjoint from the 1-handle; these are our two discs Σ and Σ' for K in D^4. The handle diagrams here present D^4_Σ and $D^4_{\Sigma'}$.

Now observe that all of the modifications we performed in the proof of Theorem 1.15 to get from M_2 to M_4 modified only the linking of the 2-handles with the 1-handle, and not the linking of the 2-handles with each other nor the knot type or framing of the 2-handles. Thus we can again perform those same modifications to our $\mathcal{H}$ to obtain a smooth $\mathbb{Z}$-disc D' properly embedded in N such that the resulting $\mathcal{H}$ is a handle diagram for $N_{D'}$ in Gompf standard form satisfying Eliashberg's criteria and such that the equivariant intersection form of the exterior is $\lambda_{N_{D'}} \cong \lambda$. Notice in particular that $N_{D'}$ is Stein.

Now let Σ, Σ' be the pair of slice discs for K in D^4 exhibited in Figure 11. These discs were constructed following the techniques of [54]. It is elementary to check

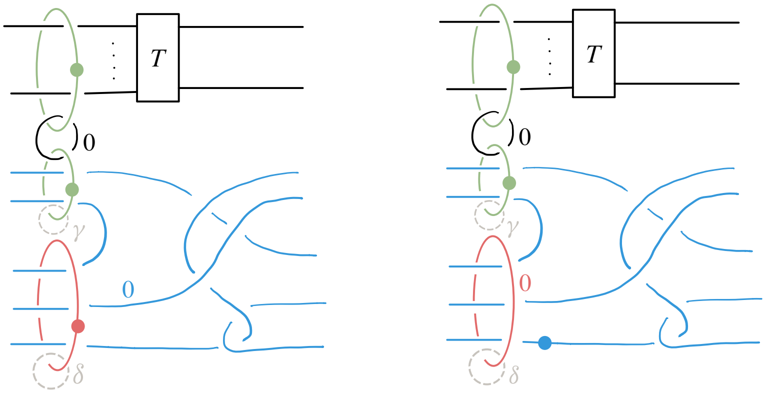

Figure 12. The left frame gives a handle diagram for N_R, and the right for $N_{R'}$. The top black 2-handles and tangle T represent the handle diagram of $N_{D'}$ in Gompf standard form which we already constructed.

from the exhibited handle diagrams that both discs have $\pi_1(D^4_\Sigma) = \pi_1(D^4_{\Sigma'}) = \mathbb{Z}$ and are ribbon. It is then a consequence of [26, Theorem 1.2] that Σ is topologically isotopic to Σ' rel. boundary.

We will construct discs R and R' in N by taking the boundary connect sum of pairs $(N, R) := (N, D')\natural(D^4, \Sigma)$, $(N, R') := (N, D')\natural(D^4, \Sigma')$. We demonstrate natural handle decompositions for N_R and $N_{R'}$ in Figure 12. It is straightforward to confirm that

$$\pi_1(N_R) \cong \pi_1(N_{R'}) \cong \mathbb{Z}.$$

Further, since Σ is topologically isotopic to Σ' in D^4 rel. boundary, R is topologically isotopic in N to R' rel. boundary. Since Σ and Σ' are $\mathbb{Z}$-discs in D^4, their exteriors are aspherical [27, Lemma 2.1] and so both λ_{N_Σ} and $\lambda_{N_{\Sigma'}}$ are trivial. It is then not hard to show that band summing D' with Σ or Σ' does not change the equivariant intersection form, so

$$\lambda_{N_R} \cong \lambda_{N_{R'}} \cong \lambda_{N_{D'}}.$$

It remains to show that R is not smoothly equivalent to R' rel. boundary. If R were equivalent to R' rel. boundary then there would be a diffeomorphism $F\colon N_R \to N_{R'}$ which is the identity on the boundary. Let γ and δ be the curves in $\partial N_R = \partial N_{R'}$ demonstrated in Figure 12, and let W (similarly W') be formed from N_R by attaching (-1)-framed 2-handles along γ and δ.

If a diffeomorphism $F\colon N_R \to N_{R'}$ extending the identity exists, then W is diffeomorphic to W'. Observe that W' does not admit a Stein structure, because the 2-handle along δ naturally introduces a (-1)-framed 2-sphere embedded in W', which violates the Stein adjunction inequality in Theorem 7.3. However, W admits the handle

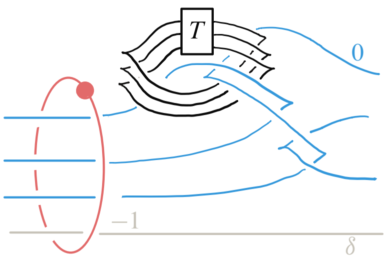

Figure 13. The black 2-handles here have both framing and TB one less than they had in Figure 12; since we had already arranged that the tangle T in Figure 12 satisfied the framing criteria of Theorem 7.1, this handle diagram also satisfies the criteria.

decomposition given in Figure 13, which is in Gompf standard form, so Theorem 7.1 ensures that W admits a Stein structure. Therefore W is not diffeomorphic to W', so there can be no such F, so R is not smoothly equivalent to R' rel. boundary. ∎

Remark 7.7. In the above proof, R is smoothly isotopic to R' *not* rel. boundary, because Σ is smoothly isotopic to Σ' not rel. boundary. If we wanted to produce R and R' which are not smoothly isotopic (without a boundary condition), we could have instead used a Σ and Σ' which are not isotopic rel. boundary and run a similar argument. Such Σ and Σ' are produced in [54]; we have not pursued this here because the diagrams are somewhat more complicated.

8. Non-trivial boundary automorphism set

We prove that there are examples of pairs (Y, φ) for which the set of 4-manifolds with fixed boundary Y and equivariant intersection form, up to homeomorphism, can have arbitrarily large cardinality. This was alluded to in Example 1.5. The main step in this process is to find a sequence of Hermitian forms (H_i, λ_i) for which

$$\{|\operatorname{Aut}(\partial\lambda_i)/\operatorname{Aut}(\lambda_i)|\}$$

is unbounded. The most direct way to achieve this is when H has rank 1. Indeed, in this case, $\operatorname{Aut}(\partial\lambda)/\operatorname{Aut}(\lambda)$ can be described in terms of certain units of $\mathbb{Z}[t^{\pm 1}]/\lambda$, as we now make precise.

Given a ring R with involution $x \mapsto \overline{x}$, the group of *unitary units* $U(R)$ refers to those $u \in R$ such that $u\overline{u} = 1$. For example, when $R = \mathbb{Z}[t^{\pm 1}]$, all units are unitary and are of the form $\pm t^k$ with $k \in \mathbb{Z}$.

In what follows, we make no distinction between rank one Hermitian forms and symmetric Laurent polynomials. The next lemma follows by unwinding the definition of $\operatorname{Aut}(\partial\lambda)$; see also [27, Remark 1.16].

Lemma 8.1. *If $\lambda \in \mathbb{Z}[t^{\pm 1}]$ is a symmetric Laurent polynomial, then*

$$\operatorname{Aut}(\partial\lambda)/\operatorname{Aut}(\lambda) = U\big(\mathbb{Z}[t^{\pm 1}]/\lambda\big)/U\big(\mathbb{Z}[t^{\pm 1}]\big).$$

Given a symmetric Laurent polynomial $P \in \mathbb{Z}[t^{\pm 1}]$, use n_P to denote the number of ways P can be written as an unordered product ab of symmetric polynomials $a, b \in \mathbb{Z}[t^{\pm 1}]$ such that there exists $x, y \in \mathbb{Z}[t^{\pm 1}]$ with $ax + by = 1$, where the factorisations ab and $(-a)(-b)$ are deemed equal.

Lemma 8.2. *If $P \in \mathbb{Z}[t^{\pm 1}]$ is a symmetric Laurent polynomial, then*

$$U\big(\mathbb{Z}[t^{\pm 1}]/2P\big)/U\big(\mathbb{Z}[t^{\pm 1}]\big)$$

contains at least n_P elements.

Proof. A first verification shows that if P factorises as $P = ab$ where $a, b \in \mathbb{Z}[t^{\pm 1}]$ are symmetric polynomials and satisfy $ax + by = 1$, then

$$\Phi(a, b) := -ax + by$$

is a unitary unit in $\mathbb{Z}[t^{\pm 1}]/2P$, i.e. belongs to $U(\mathbb{Z}[t^{\pm 1}]/2P)$:

$$\begin{aligned}(-ax + by)\overline{(-ax + by)} &= a\bar{a}x\bar{x} + b\bar{b}y\bar{y} - ax\bar{b}\bar{y} - \bar{a}\bar{x}by \\ &= a\bar{a}x\bar{x} + b\bar{b}y\bar{y} - ab(x\bar{y} + \bar{x}y) \\ &\equiv a\bar{a}x\bar{x} + b\bar{b}y\bar{y} + ab(x\bar{y} + \bar{x}y) \\ &= (ax + by)\overline{(ax + by)} = 1.\end{aligned}$$

It can also be verified that $\Phi(a, b)$ depends neither on the ordering of a, b nor on the choice of x, y. The former check is immediate from the definition of Φ because $-1 \in U(\mathbb{Z}[t^{\pm 1}])$. We verify that the assignment does not depend on the choice of x, y. Assume that $ax + by = 1 = ax' + by'$ for $x, x', y, y' \in \mathbb{Z}[t^{\pm 1}]$. We deduce that

$$ax' = 1 = ax \bmod b \quad \text{and} \quad by' = 1 = by \bmod a.$$

But now $x' \equiv (ax)x' = x(ax') = x \bmod b$, and similarly $y' = y \bmod a$ so that $x' = x + kb$ and $y' = y + \ell a$ for $k, l \in \mathbb{Z}[t^{\pm 1}]$. Expanding $ax' + by' = 1$, it follows that $k = -l$. Therefore,

$$-ax' + by' = -a(x + kb) + b(y - ka) \equiv -ax + by.$$

We will prove that if $\Phi(a,b) = v \cdot \Phi(a',b')$ for some unit $v \in U(\mathbb{Z}[t^{\pm 1}])$, then $(a,b) = \pm(a',b')$ or $(a,b) = \pm(b',a')$. It then follows that for any two ways (a,b) and (a',b') of factorising P, distinct up to sign and up to reordering, the resulting elements $\Phi(a,b)$ and $\Phi(a',b')$ are distinct in $U(\mathbb{Z}[t^{\pm 1}]/2P)/U(\mathbb{Z}[t^{\pm 1}])$, from which the proposition follows.

Assume that $x, x', y, y' \in \mathbb{Z}[t^{\pm 1}]$ are such that

$$ax + by = 1 = a'x' + b'y' \quad \text{and} \quad -ax + by = -a'x' + b'y' \bmod 2P.$$

Next add $2ax + 2a'x'v$ to both sides of the congruence $-ax + by = v(-a'x' + b'y')$ mod $2P$. Using that $ax + by = 1$ and $a'x' + b'y' = 1$, we obtain the congruence

$$2ax + v = 2a'x'v + 1 \bmod 2P. \tag{8.1}$$

Similarly, we add $-2by + 2a'x'v$ to both sides of $-ax + by = v(-a'x' + b'y')$ mod $2P$. Using that $ax + by = 1$ and $a'x' + b'y' = 1$, we obtain the equation

$$-2by + v = 2a'x'v - 1 \bmod 2P. \tag{8.2}$$

We deduce from the previous two equations that $v + 1$ and $v - 1$ are divisible by 2. Since $v = \pm t^k$, we deduce that $\pm t^k \pm 1$ is divisible by 2 and so $v = \pm 1$.

First, we treat the case where the unit is $v = 1$.

Claim 1. *We have* (i) *a divides a', and* (ii) *a' divides a.*

Proof. As $v = 1$, (8.1) implies that $2ax = 2a'x'$ mod $2P$. Writing $2P = 2ab$, and simplifying the 2s, we deduce that a divides $a'x'$. Similarly, writing $2P = 2a'b'$, and simplifying the 2s, we deduce that a' divides ax. Next, multiply the equations $1 = ax + by$ (resp. $1 = a'x' + b'y'$) by a (resp. a') to obtain

$$a = a^2x + aby,$$
$$a' = a'^2x' + a'b'y'.$$

Since a' divides ax and $ab = P = a'b'$, it follows that a' divides a. The same reasoning with the second equation shows that a divides a'. This concludes the proof of the claim. ∎

Using the claim we have $a = ua'$ for some unit u; this unit is necessarily symmetric since both a and a' are symmetric. It follows that $a'b' = ab = ua'b$ with $u = \pm 1$. We deduce $b' = ub$ and therefore $b = b'/u$. Thus $(a,b) = u \cdot (a',b')$ as required, in the case $v = 1$.

Next, we treat the case where the unit is $v = -1$.

Claim 2. *We have* (i) *b divides a', and* (ii) *a' divides b.*

Proof. As $v = -1$, (8.2) implies that $-2by = 2a'x' \bmod 2P$. Writing $2P = 2ab$, and simplifying the 2s, we deduce that b divides $a'x'$. Similarly, writing $2P = 2a'b'$, and simplifying the 2s, we deduce that a' divides by. Next, multiply the equations $1 = ax + by$ (resp. $1 = a'x' + b'y'$) by b (resp. a') to obtain

$$b = abx + b^2 y,$$
$$a' = a'^2 x' + a'b'y'.$$

Since a' divides by and $ab = P = a'b'$, it follows that a' divides b. The same reasoning with the second equation shows that b divides a'. This concludes the proof of the claim. ∎

Using the claim we have $b = ua'$ for some unit u; this unit is necessarily symmetric since both b and a' are symmetric. It follows that $a'b' = ab = uaa'$ with $u = \pm 1$. We deduce $b' = ua$, and therefore $a = b'/u$. Thus $(a, b) = u \cdot (b', a')$, as required, in the case that $v = -1$. This completes the proof that $\Phi(a, b) = v \cdot \Phi(a', b')$ implies $(a, b) = \pm(a', b')$ or $(a, b) = \pm(b', a')$, which completes the proof of the proposition. ∎

Over $\mathbb{Z}$, it is not difficult to show that if N is an integer that can be factored as a product of n distinct primes, then $U(\mathbb{Z}/N)/U(\mathbb{Z})$ contains precisely 2^{n-1} elements. Using Lemma 8.2, the next example shows that a similar lower bound (which is not in general sharp) holds over $\mathbb{Z}[t^{\pm 1}]$.

Example 8.3. The reader can check that if P is an integer than can be factored as a product $p_1 \cdots p_n$ of n distinct primes, then $n_P = 2^{n-1}$. Lemma 8.2 implies that $U(\mathbb{Z}[t^{\pm 1}]/2P)/U(\mathbb{Z}[t^{\pm 1}])$ contains at least 2^{n-1} elements.

Remark 8.4. In order to produce examples, there is no need to restrict P an integer. Take $P = q_1 \cdots q_n$, where the q_i are symmetric Laurent polynomials such that for every i, j, there exists $x, y \in \mathbb{Z}[t^{\pm 1}]$ with $q_i x + q_j y = 1$. The latter condition implies, via a straightforward induction on n, that there exists such x, y for any pair of polynomials $q_{i_1} \cdots q_{i_k}$ and $q_{i_{k+1}} \cdots q_{i_n}$ with $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$ obtained from factoring P. Then by applying Φ we can obtain examples of P such that $U(\mathbb{Z}[t^{\pm 1}]/2P)/U(\mathbb{Z}[t^{\pm 1}])$ has cardinality at least 2^{n-1}. However, this level of generality is not strictly necessary, as Example 8.3, in which P is an integer, suffices to prove Proposition 8.5 below.

We now prove the main result of this section that was mentioned in Example 1.5 from the introduction: there are examples of pairs (Y, φ) for which the set of 4-manifolds with fixed boundary Y and equivariant intersection form, up to homeomorphism, can have arbitrarily large cardinality. Recall that $\mathcal{V}^0_\lambda(Y)$ and $\mathcal{V}_\lambda(Y)$ were defined in Definitions 2.1 and 2.7 respectively.

Proposition 8.5. *For every $m \geq 0$, there is a pair (Y, φ) and a Hermitian form (H, λ) so that $\mathcal{V}^0_\lambda(Y)$ and $\mathcal{V}_\lambda(Y)$ have at least m elements.*

Proof. Since the cardinality of $\mathcal{V}^0_\lambda(Y)$ is greater than that of $\mathcal{V}_\lambda(Y)$, it suffices to prove that the latter set can be made arbitrarily large. However since proof involving $\mathcal{V}^0_\lambda(Y)$ is substantially less demanding, we include it as a quick warm up.

Set $\lambda := 2P$ where P is an integer than can be factored as a product $p_1 \cdots p_k$ of k distinct primes with $2^{k-1} \geq m$. Example 8.3 and Proposition 8.2 imply that $U(\mathbb{Z}[t^{\pm 1}]/\lambda)/U(\mathbb{Z}[t^{\pm 1}])$ has at least 2^{k-1} elements. By Proposition 8.1, this means that $\operatorname{Aut}(\partial\lambda)/\operatorname{Aut}(\lambda)$ has at least 2^{k-1} elements. As in the proof of Theorem 7.5, construct a smooth $\mathbb{Z}$-manifold W with equivariant intersection form λ. In our setting, where $\lambda := 2P$, the manifold produced will be $X_\lambda(U)\natural(S^1 \times D^3)$, where $X_\lambda(U)$ is the manifold obtained by attaching a λ-framed 2-handle to D^4 along the unknot U. Let Y' be the boundary of this 4-manifold and let $\varphi\colon \pi_1(Y') \to \pi_1(W) \cong \mathbb{Z}$ be the inclusion induced map. Since λ presents Y', Theorem 1.3 implies that $\mathcal{V}^0_\lambda(Y')$ has at least $2^{k-1} \geq m$ elements, as required.

We now turn to the statement involving $\mathcal{V}_\lambda(Y)$.

Claim. *There is an integer $N > 0$ so that for any $n > N$, there exists a smooth $\mathbb{Z}$-manifold W_n with equivariant intersection form (n) and such that ∂W_n has trivial mapping class group.*

Proof. Let L be the 3-component link in the left frame of Figure 14 and let Z be the 3-manifold obtained from L by 0-surgering both the red and blue components, and removing a tubular neighbourhood of the green component γ. Using verified computations in Snappy inside of Sage, we find that Z is hyperbolic and has trivial mapping class group.[2] By Thurston's hyperbolic Dehn surgery theorem [88, Theorem 5.8.2], there exists $N > 0$ such that for $n > N$, the manifold Z_n obtained by $-1/n$ filling γ is hyperbolic and has trivial symmetry group; for the mapping class group part of this statement, see for example [30, Lemma 2.2].

Let W_n be the 4-manifold described in the right frame of Figure 14 and observe that $\partial W_n \cong Z_n$. It is not difficult to verify that W_n is a $\mathbb{Z}$-manifold with equivariant intersection form (n). This concludes the proof of the claim. ■

We conclude the proof of the proposition. Fix $m \geq 0$ and choose an integer P such that

- P can be factored as a product $p_1 \cdots p_k$ of k distinct primes with $2^{k-1} \geq m$;
- $2P > N$, where N is as in the claim.

[2] Transcripts of the computation are available at [78].

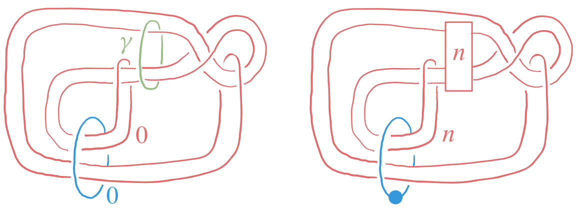

Figure 14. Left frame: the complement of γ is a hyperbolic 3-manifold Z with trivial mapping class group. Right frame: This $\mathbb{Z}$-manifold W_n has equivariant intersection form (n) and, for n sufficiently large, boundary ∂W_n with trivial mapping class group.

Since $2P > N$, the claim implies that $Y := \partial W_{2P}$ has trivial mapping class group. The proof is now concluded as in the warm up, but we spell out the details. As we already mentioned, W_{2P} has equivariant intersection form $\lambda := 2P$. Example 8.3 and Proposition 8.2 imply that $U(\mathbb{Z}[t^{\pm 1}]/\lambda)/U(\mathbb{Z}[t^{\pm 1}])$ has at least 2^{k-1} elements. By Proposition 8.1, this means that $\mathrm{Aut}(\partial\lambda)/\mathrm{Aut}(\lambda)$ has at least 2^{k-1} elements. Since Y has trivial mapping class group, either of Theorem 1.3 or Theorem 2.8 implies that $\mathcal{V}_\lambda(Y) = \mathcal{V}^0_\lambda(Y)$ has at least $2^{k-1} \geq m$ elements. ∎

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