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First isola of modulational instability of Stokes waves in deep water

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Abstract. We prove high-frequency modulational instability of small-amplitude Stokes waves in deep water under longitudinal perturbations, providing the first isola of unstable eigenvalues branching off from $\mathrm{i}\frac{3}{4}$. Unlike the finite depth case this is a degenerate problem and the real part of the unstable eigenvalues has a much smaller size than in finite depth. By a symplectic version of Kato theory, we reduce to search the eigenvalues of a 2×2 Hamiltonian and reversible matrix which has eigenvalues with nonzero real part if and only if a certain analytic function is not identically zero. In deep water, we prove that the Taylor coefficients up to order three of this function vanish, but not the fourth-order one.

Dedicated to Thomas Kappeler, who taught us beautiful mathematics and to be always grateful for the good things happening in life

1. Introduction and main result

Stokes waves are periodic solutions of the pure gravity water waves equations, traveling at constant speed. Since their discovery by Stokes [31] in 1847 and their rigorous mathematical existence proof in [27,29,32], they have been the object of intense studies, regarded as a key first step toward better understanding of the complicated flow evolution of the water waves equations. Pioneering experimental and formal works by Benjamin and Feir [3, 4], Lighthill [28], Zakharov [35], and Whitham [33] highlighted, more than fifty years ago, that Stokes waves are unstable under long wave perturbations, a phenomenon that nowadays goes by the name of Benjamin–Feir or *modulational instability*. Rigorous mathematical proofs were later given by Bridges–Mielke [9] in finite depth and by Nguyen–Strauss [30] in deep water. These works prove existence of unstable spectrum near the origin of the complex plane of the linearized water waves operator $\mathscr{L}_\varepsilon$ (see (1.8)) at a Stokes wave of small amplitude ε, when it is regarded as an unbounded operator on $L^2(\mathbb{R})$. We also mention the recent nonlinear modulational instability result in Chen–Su [10].

In a recent series of works, we gave the complete description of the portion of the spectrum $\sigma_{L^2(\mathbb{R})}(\mathscr{L}_\varepsilon)$ near the origin of the complex plane both in deep water [6] and finite

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depth [7, 8], proving the existence of an unstable spectral branch of eigenvalues outside the imaginary axis forming a figure "8", as first observed numerically by Deconinck–Oliveras [19] (see also the formal computations in [16]).

The behavior of the spectrum $\sigma_{L^2(\mathbb{R})}(\mathscr{L}_\varepsilon)$ away from zero is still not sufficiently investigated, although highly relevant for the stability/instability of the Stokes waves. Numerical results by Deconinck–Oliveras [19] shed light on the presence of "*isolas* of instability"—isolated elliptic shaped islands of unstable eigenvalues away from the origin—which suggests the appearance of unstable spectrum along the whole imaginary axis, with real part decreasing exponentially at infinity. Because of the Hamiltonian character of the operator $\mathscr{L}_\varepsilon$, unstable eigenvalues can occur only as perturbations of multiple purely imaginary eigenvalues of $\mathscr{L}_0$. The nonzero multiple eigenvalues of $\mathscr{L}_0$ are enumerated in literature by an integer $p \geq 2$—it turns out they are all double—and we will follow this convention. Formal expansions describing the first two isolas ($p = 2, 3$) of unstable eigenvalues were obtained by Creedon–Deconinck–Tritchenko [17] in both finite and infinite depth, see also [14, 15, 20] for other asymptotic models. A rigorous analytical result about the first isola of instability $p = 2$ was given recently by Hur and Yang [24] in finite depth for pure gravity waves and in [23] for gravity-capillary waves. Unfortunately, the approach in [24] does not apply in infinite depth, as it relies on spatial dynamics which fails in deep water (similarly to [9]). We finally mention the very recent paper [18] which, relying on the spectral approach in [6], proved the instability of the Stokes waves under transversal perturbations.

The goal of this paper is to rigorously prove the existence of the first isola of instability ($p = 2$) in the deep-water case, under longitudinal perturbations. As we explain below, this problem is considerably more difficult than in finite depth, because it is *degenerate*. In Theorem 1, we show that the spectrum $\sigma_{L^2(\mathbb{R})}(\mathscr{L}_\varepsilon)$ has a branch of eigenvalues near $\mathrm{i}\,\frac{3}{4}$ with non zero real part, forming a *very narrow* elliptic shaped curve in the complex plane, see Figures 1 and 2. Formula (1.4) provides the ellipse which asymptotically approximates the $p = 2$-isola of unstable eigenvalues.

Let us now state precisely our main result. Since the operator $\mathscr{L}_\varepsilon$ has 2π-periodic coefficients, arising by the linearization of the water waves equations at a 2π-periodic Stokes wave, its spectrum is conveniently described via Bloch–Floquet theory, which ensures that

$$\sigma_{L^2(\mathbb{R})}(\mathscr{L}_\varepsilon) = \bigcup_{\mu \in [0, \frac{1}{2})} \sigma_{L^2(\mathbb{T})}(\mathscr{L}_{\mu,\varepsilon}),$$

where $\mathscr{L}_{\mu,\varepsilon} := e^{-\mathrm{i}\mu x}\mathscr{L}_\varepsilon e^{\mathrm{i}\mu x}$. Our main result describes the spectrum near $\mathrm{i}\,\frac{3}{4}$ of the operator $\mathscr{L}_{\mu,\varepsilon}$, for any (μ, ε) sufficiently close to $(\frac{1}{4}, 0)$, value at which $\mathscr{L}_{\frac{1}{4},0}$ has a double eigenvalue (the closest one to 0).

Notation. Along the paper we denote by $r(y^{m_1}\varepsilon^{n_1}, \dots, y^{m_p}\varepsilon^{n_p})$ (the variable y is called μ, δ, ν depending on the context) a real analytic function satisfying, for some $C > 0$ and for any small value of (y, ε), $|r(y^{m_1}\varepsilon^{n_1}, \dots, y^{m_p}\varepsilon^{n_p})| \leq C \sum_{j=1}^{p} |y|^{m_j}|\varepsilon|^{n_j}$.

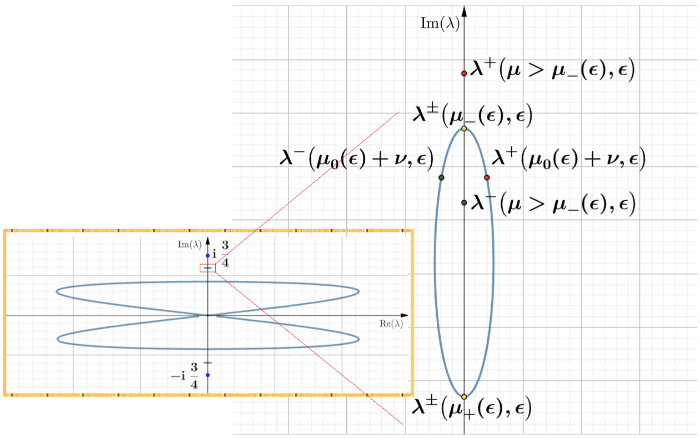


Figure 1. Part of the spectrum of $\mathcal{L}_\varepsilon$ in the deep-water case. The figure “8” was proved in [6]. The two small symmetric isolas (detailed in Figure 2) are the subject of this article.

Theorem 1. *There exist* $\varepsilon_1, \delta_0 > 0$ *and real analytic functions* $\mu_0, \mu_\pm \colon [0, \varepsilon_1) \to B_{\delta_0}(\frac{1}{4})$ *with* $\mu_+(\varepsilon) \le \mu_0(\varepsilon) \le \mu_-(\varepsilon)$ *of the form*

$$\mu_0(\varepsilon) = \frac{1}{4} - \frac{57}{64}\varepsilon^2 + r(\varepsilon^3), \quad \mu_\pm(\varepsilon) = \mu_0(\varepsilon) \mp \frac{111\sqrt{3}}{1024}\varepsilon^4 + r(\varepsilon^5) \tag{1.1}$$

such that for any $(\mu, \varepsilon) \in B_{\delta_0}(\frac{1}{4}) \times [0, \varepsilon_1)$ *the following holds true. Defining* $\nu_\pm(\varepsilon) := \mu_\pm(\varepsilon) - \mu_0(\varepsilon)$*, the operator* $\mathcal{L}_{\mu,\varepsilon}$ *at* $(\mu, \varepsilon) = (\mu_0(\varepsilon) + \nu, \varepsilon)$ *possesses two eigenvalues of the form*

$$\lambda^\pm\big(\mu_0(\varepsilon) + \nu, \varepsilon\big) = \begin{cases} \mathrm{i}\,\frac{3}{4} + \mathrm{i}\,\frac{4}{3}\nu - \mathrm{i}\,\frac{55}{32}\varepsilon^2 + \mathrm{i}\,r(\varepsilon^3, \nu\varepsilon^2, \nu^2) \pm \frac{1}{2}\sqrt{D(\mu_0(\varepsilon) + \nu, \varepsilon)} \\ \quad \text{if } \nu \in \big(\nu_+(\varepsilon), \nu_-(\varepsilon)\big), \\ \mathrm{i}\,\frac{3}{4} + \mathrm{i}\,\frac{4}{3}\nu - \mathrm{i}\,\frac{55}{32}\varepsilon^2 + \mathrm{i}\,r(\varepsilon^3, \nu\varepsilon^2, \nu^2) \pm \frac{\mathrm{i}}{2}\sqrt{|D(\mu_0(\varepsilon) + \nu, \varepsilon)|} \\ \quad \text{if } \nu \notin \big(\nu_+(\varepsilon), \nu_-(\varepsilon)\big), \end{cases} \tag{1.2}$$

where

$$D(\mu_0(\varepsilon) + \nu, \varepsilon) = \frac{4107}{65536}\varepsilon^8 - \frac{16}{9}\nu^2 - \frac{37}{128}\nu\varepsilon^6 + r(\varepsilon^9, \nu\varepsilon^7, \nu^2\varepsilon^2, \nu^3). \tag{1.3}$$

For any fixed $\varepsilon \in (0, \varepsilon_1)$*, the pair of unstable eigenvalues* $\lambda^\pm(\mu, \varepsilon)$ *depicts, as* μ *varies in* $(\mu_+(\varepsilon), \mu_-(\varepsilon))$*, an ellipse-like curve in the complex plane, i.e., a closed analytic curve that intersects orthogonally the imaginary axis and encircles a convex region. See Figure 2.*

Let us make some comment on the result.

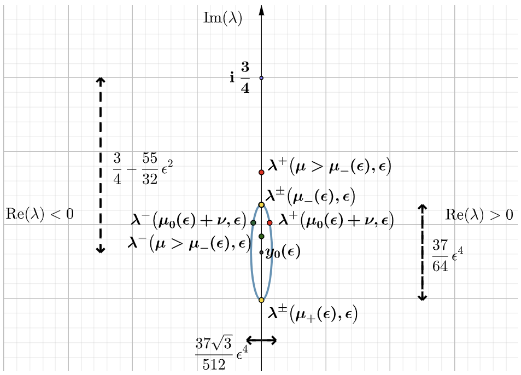


Figure 2. First isola depicted by the two symmetric non-purely imaginary eigenvalues of $\mathcal{L}_{\mu,\varepsilon}$ close to $\mathrm{i}\,\frac{3}{4}$. The isola does not encircle $\mathrm{i}\,\frac{3}{4}$.

(1) According to (1.2), for any Floquet parameter $\mu \in (\mu_+(\varepsilon), \mu_-(\varepsilon))$ the eigenvalues $\lambda^\pm(\mu,\varepsilon)$ have opposite real part. As $\mu \to \mu_\pm(\varepsilon)$, the eigenvalues collide on the imaginary axis. For $\mu < \mu_+(\varepsilon)$ or $\mu > \mu_-(\varepsilon)$, the two eigenvalues are purely imaginary. Being $\mathcal{L}_\varepsilon$ a real operator, also the mirror spectral branch of eigenvalues $\overline{\lambda^\pm(-\mu,\varepsilon)}$ is present.

(2) By fixing $\varepsilon > 0$ and letting μ vary in the interval $(\mu_+(\varepsilon), \mu_-(\varepsilon))$, the two eigenvalues $\lambda^\pm(\mu,\varepsilon)$ depict an ellipse-like curve. By discarding the remainders $r(\nu^\alpha)$, $\alpha = 2, 3$, in the expressions (1.2) and (1.3), the real and imaginary parts of the approximate eigenvalues parameterize the ellipse

$$x^2 + \frac{1}{4}\big(y - y_0(\varepsilon)\big)^2\big(1 + r(\varepsilon^2)\big) = \frac{4107}{262144}\varepsilon^8(1 + r(\varepsilon)), \quad y_0(\varepsilon) = \frac{3}{4} - \frac{55}{32}\varepsilon^2 + r(\varepsilon^4). \tag{1.4}$$

(3) Theorem 1 actually provides the expansion of the two eigenvalues of $\mathcal{L}_{\mu,\varepsilon}$ close to $\mathrm{i}\,\frac{3}{4}$ for all values of (μ,ε) in $(\frac{1}{4} - \delta_0, \frac{1}{4} + \delta_0) \times [0, \varepsilon_1)$. The analytic curves $\mu_\pm(\varepsilon)$ in (1.1) divide such rectangle in two separated regions: one where $\mathcal{L}_{\mu,\varepsilon}$ has two eigenvalues with nonzero real part, and another one where the eigenvalues are purely imaginary. See Figure 3.

(4) The work [24] describes the first isola in finite depth. Such a case is non-degenerate (see (2.24)) for almost any depth. On the contrary, the infinite depth case is *degenerate*, and its analysis requires the fourth-order expansion of the Stokes waves, as we now explain.

We briefly describe the proof and its difficulties. Exploiting that $\mathcal{L}_{\mu,\varepsilon}$ is a Hamiltonian and reversible operator, we use a symplectic version of Kato's similarity transformation theory to compute a symplectic basis of the two-dimensional invariant subspace $\mathcal{V}_{\mu,\varepsilon}$ associated with the two eigenvalues of $\mathcal{L}_{\mu,\varepsilon}$ close to $\mathrm{i}\,\frac{3}{4}$. The action of $\mathcal{L}_{\mu,\varepsilon}|_{\mathcal{V}_{\mu,\varepsilon}}$ is then represented by a 2×2 Hamiltonian and reversible matrix of the form

$$\mathtt{L}(\mu,\varepsilon) = \begin{pmatrix} -\mathrm{i}\,\alpha(\mu,\varepsilon) & \beta(\mu,\varepsilon) \\ \beta(\mu,\varepsilon) & \mathrm{i}\,\gamma(\mu,\varepsilon) \end{pmatrix},$$

where $\alpha(\mu,\varepsilon)$, $\beta(\mu,\varepsilon)$, and $\gamma(\mu,\varepsilon)$ are real analytic functions.

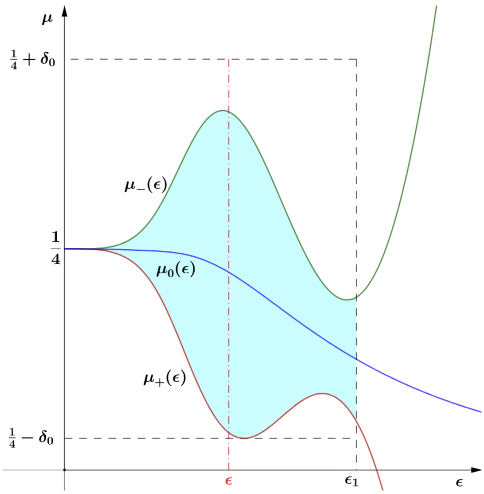


Figure 3. The degenerate instability region. We boxed with black dashed lines the validity box of Theorem 1. At any fixed ε, for (μ, ε) in the colored cusp-shape region, one has the formation of the isola of instability in Figure 2.

We like to remember that this method, that we initiated in [6, 7], was inspired to us by the approach that Thomas Kappeler developed in [25], in the context of selfadjoint operators, to study the spectral gaps of the Lax-operator of the KdV equation (this line of research was further developed in [2, 26]).

After our symplectic reduction one needs only to investigate the eigenvalues of the matrix $\mathtt{L}(\mu, \varepsilon)$. In Section 2, we prove a necessary and sufficient condition for a 2×2 matrix of this form to have non-purely imaginary eigenvalues. Theorem 9 proves $\mathtt{L}(\mu, \varepsilon)$ has two unstable eigenvalues if and only if the real analytic function $\varepsilon \mapsto \beta(\mu_0(\varepsilon), \varepsilon)$ is not identically zero, where $\mu_0(\varepsilon)$ is the analytic function such that $\alpha(\mu_0(\varepsilon), \varepsilon) + \gamma(\mu_0(\varepsilon), \varepsilon) \equiv 0$. This instability criterion amounts to prove that $\beta(\mu_0(\varepsilon), \varepsilon)$ has a nonzero Taylor coefficient at $\varepsilon = 0$. The first Taylor coefficients of $\beta(\mu_0(\varepsilon), \varepsilon)$ are computed in (2.23) in terms of the Taylor coefficients of $\alpha(\mu, \varepsilon)$, $\beta(\mu, \varepsilon)$ and $\gamma(\mu, \varepsilon)$. In deep water, it turns out that the coefficient β_1 in (2.23) vanishes, cf., (5.2b); this is the degeneracy we mentioned above (whereas β_1 is not zero for almost any finite depth). We are then led to compute the coefficient of ε^4 in (2.23), which depends on a higher-order Taylor expansion of $\alpha(\mu, \varepsilon)$, $\beta(\mu, \varepsilon)$ and $\gamma(\mu, \varepsilon)$ given in Theorem 4 and proved throughout Sections 3 and 5. Via a careful analysis to isolate only the relevant terms, we finally prove that the fourth-order coefficient of (2.23) is $\frac{37\sqrt{3}}{512} \neq 0$. We thus conclude that the eigenvalues of $\mathtt{L}(\mu, \varepsilon)$, i.e., the ones of $\mathscr{L}_{\mu,\varepsilon}$ close to $\mathrm{i}\,\frac{3}{4}$, are unstable in a certain region of (μ, ε) as stated in Theorem 1.

Theorem 1 is a direct consequence of Theorem 4 that we now rigorously present.

The water waves equations. We consider the Euler equations for a 2-dimensional incompressible and irrotational fluid under the action of gravity filling the region $\mathcal{D}_\eta := \{(x, y) \in \mathbb{T} \times \mathbb{R} : y < \eta(t, x)\}$, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, with space periodic boundary conditions. The irrotational velocity field is the gradient of a harmonic scalar potential $\Phi = \Phi(t, x, y)$ determined by its trace $\psi(t, x) = \Phi(t, x, \eta(t, x))$ at the free surface $y = \eta(t, x)$. Actually, Φ is the unique solution of $\Delta\Phi = 0$ in $\mathcal{D}_\eta$ satisfying the Dirichlet condition $\Phi(t, x, \eta(t, x)) = \psi(t, x)$ and $\nabla\Phi(t, x, y) \to 0$ as $y \to -\infty$. The time evolution of the fluid is determined by two boundary conditions at the free surface. The first is that the fluid particles remain, along the evolution, on the free surface (kinematic boundary condition), and the second one is that the pressure of the fluid is equal, at the free surface, to the constant atmospheric pressure (dynamic boundary condition). Then, as shown in [12, 35], the evolution of the fluid is determined by the following equations for the unknowns $(\eta(t, x), \psi(t, x))$:

$$\eta_t = G(\eta)\psi, \quad \psi_t = -g\eta - \frac{\psi_x^2}{2} + \frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2, \tag{1.5}$$

where $g > 0$ is the gravity constant and $G(\eta)$ denotes the Dirichlet–Neumann operator $[G(\eta)\psi](x) := \Phi_y(x, \eta(x)) - \Phi_x(x, \eta(x))\eta_x(x)$. Without loss of generality, we set the gravity constant $g = 1$. The equations (1.5) are the Hamiltonian system

$$\partial_t \begin{bmatrix} \eta \\ \psi \end{bmatrix} = \mathcal{J} \begin{bmatrix} \nabla_\eta \mathcal{H} \\ \nabla_\psi \mathcal{H} \end{bmatrix}, \quad \mathcal{J} := \begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}, \tag{1.6}$$

where ∇ denote the L^2-gradient, and the Hamiltonian $\mathcal{H}(\eta, \psi) := \frac{1}{2}\int_{\mathbb{T}}(\psi\, G(\eta)\psi + \eta^2)\mathrm{d}x$ is the sum of the kinetic and potential energy of the fluid. In addition to being Hamiltonian, the water waves system (1.5) is time reversible with respect to the involution

$$\rho \begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} := \begin{bmatrix} \eta(-x) \\ -\psi(-x) \end{bmatrix}, \quad \text{i.e., } \mathcal{H} \circ \rho = \mathcal{H}. \tag{1.7}$$

Stokes waves, linearization, and Bloch–Floquet expansion. Equations (1.5) admit an analytic family of traveling periodic Stokes waves $(\eta_\varepsilon(t, x), \psi_\varepsilon(t, x)) = (\breve{\eta}_\varepsilon(x - c_\varepsilon t), \breve{\psi}_\varepsilon(x - c_\varepsilon t))$, where $(\breve{\eta}_\varepsilon(x), \breve{\psi}_\varepsilon(x))$ are 2π-periodic functions of the form

$$\breve{\eta}_\varepsilon(x) = \varepsilon\cos(x) + \mathcal{O}(\varepsilon^2), \quad \breve{\psi}_\varepsilon(x) = \varepsilon\sin(x) + \mathcal{O}(\varepsilon^2), \quad c_\varepsilon = 1 + \mathcal{O}(\varepsilon^2).$$

In a reference frame moving with the speed c_ε, the linearized water waves equations at the Stokes waves turn out to be[1] the linear system $h_t = \mathcal{L}_\varepsilon h$, where $\mathcal{L}_\varepsilon : H^1(\mathbb{T}, \mathbb{R}^2) \to L^2(\mathbb{T}, \mathbb{R}^2)$ is the Hamiltonian and reversible real operator

$$\mathcal{L}_\varepsilon = \begin{bmatrix} \partial_x \circ (1 + p_\varepsilon(x)) & |D| \\ -(1 + a_\varepsilon(x)) & (1 + p_\varepsilon(x))\partial_x \end{bmatrix} = \mathcal{J} \begin{bmatrix} 1 + a_\varepsilon(x) & -(1 + p_\varepsilon(x))\partial_x \\ \partial_x \circ (1 + p_\varepsilon(x)) & |D| \end{bmatrix} \tag{1.8}$$

[1] After conjugating with the “good unknown of Alinhac” and the “Levi-Civita” transformations, we refer to [6–8] for details.

and $p_\varepsilon(x)$, $a_\varepsilon(x)$ are real even analytic functions. We will need their fourth order Taylor expansion which is

$$\begin{aligned} p_\varepsilon(x) &= \sum_{n\geq 1} \varepsilon^n p_n(x) = -2\varepsilon\cos(x) + \varepsilon^2\Big(\frac{3}{2} - 2\cos(2x)\Big) \\ &\quad + 3\varepsilon^3\big(\cos(x) - \cos(3x)\big) \\ &\quad + \varepsilon^4\Big(\frac{1}{8} + 4\cos(2x) - \frac{16}{3}\cos(4x)\Big) + \mathcal{O}(\varepsilon^5), \qquad (1.9a) \\ a_\varepsilon(x) &= \sum_{n\geq 1} \varepsilon^n a_n(x) = -2\varepsilon\cos(x) + 2\varepsilon^2\big(1 - \cos(2x)\big) \\ &\quad + \varepsilon^3\big(4\cos(x) - 3\cos(3x)\big) \\ &\quad + \varepsilon^4\Big(-1 + 4\cos(2x) - \frac{16}{3}\cos(4x)\Big) + \mathcal{O}(\varepsilon^5), \qquad (1.9b) \end{aligned}$$

as shown taking the infinite depth limit in [8, (A.59)–(A.60)] (it follows by the fourth-order expansion of the Stokes waves in [8, (A.1)] or [18, Proposition 2.2]).

By Bloch–Floquet theory, by introducing the Floquet exponent μ, the spectrum

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_\varepsilon) = \bigcup_{\mu\in[-\frac12,\frac12)} \sigma_{L^2(\mathbb{T})}(\mathcal{L}_{\mu,\varepsilon}), \quad \text{where } \mathcal{L}_{\mu,\varepsilon} := e^{-\mathrm{i}\mu x}\,\mathcal{L}_\varepsilon\, e^{\mathrm{i}\mu x},$$

and if λ is an eigenvalue of $\mathcal{L}_{\mu,\varepsilon}$ with a 2π-periodic eigenvector $v(x)$, then $h(t,x) = e^{\lambda t}e^{\mathrm{i}\mu x}v(x)$ is a solution of $h_t = \mathcal{L}_\varepsilon h$ whose growth in time is determined by $\operatorname{Re}\lambda$.

Remark 2. Being $\sigma(\mathcal{L}_{-\mu,\varepsilon}) = \overline{\sigma(\mathcal{L}_{\mu,\varepsilon})}$ and $\sigma(\mathcal{L}_{\mu,\varepsilon})$ a 1-periodic set with respect to μ, we study the $L^2(\mathbb{T})$-spectrum of $\mathcal{L}_{\mu,\varepsilon}$ for μ in the "first zone of Brillouin" $0 \leq \mu < \frac{1}{2}$.

The operator $\mathcal{L}_{\mu,\varepsilon}$ is the complex *Hamiltonian* and *reversible* operator

$$\begin{aligned} \mathcal{L}_{\mu,\varepsilon} &:= \begin{bmatrix} (\partial_x + \mathrm{i}\mu)\circ(1+p_\varepsilon(x)) & |D+\mu| \\ -(1+a_\varepsilon(x)) & (1+p_\varepsilon(x))(\partial_x+\mathrm{i}\mu) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}}_{=\mathcal{J}} \underbrace{\begin{bmatrix} 1+a_\varepsilon(x) & -(1+p_\varepsilon(x))(\partial_x+\mathrm{i}\mu) \\ (\partial_x+\mathrm{i}\mu)\circ(1+p_\varepsilon(x)) & |D+\mu| \end{bmatrix}}_{=:\mathcal{B}(\mu,\varepsilon)=\mathcal{B}^*(\mu,\varepsilon)}, \qquad (1.10) \end{aligned}$$

which we regard as an operator with domain $H^1(\mathbb{T}) := H^1(\mathbb{T},\mathbb{C}^2)$ and range $L^2(\mathbb{T}) := L^2(\mathbb{T},\mathbb{C}^2)$, equipped with the complex scalar product[2]

$$(f,g) := \frac{1}{2\pi}\int_{\mathbb{T}} \big(f_1\overline{g_1} + f_2\overline{g_2}\big)\,\mathrm{d}x, \quad \forall f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},\ g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in L^2(\mathbb{T},\mathbb{C}^2). \qquad (1.11)$$

[2] The operator $\mathcal{B}^*(\mu,\varepsilon)$ in (1.10) is the adjoint with respect to the complex scalar product (1.11).

We also denote $\|f\|^2 = (f, f)$. The complex Hilbert space $L^2(\mathbb{T}, \mathbb{C}^2)$ is also equipped with the sesquilinear, skew-Hermitian and non-degenerate complex symplectic form

$$\mathcal{W}_c \colon L^2(\mathbb{T}, \mathbb{C}^2) \times L^2(\mathbb{T}, \mathbb{C}^2) \to \mathbb{C}, \quad \mathcal{W}_c(f, g) := (\mathcal{J} f, g), \tag{1.12}$$

where $\mathcal{J}$ is defined in (1.6). The complex operator $\mathcal{L}_{\mu,\varepsilon}$ in (1.10) is also reversible, namely,

$$\mathcal{L}_{\mu,\varepsilon} \circ \overline{\rho} = -\overline{\rho} \circ \mathcal{L}_{\mu,\varepsilon}, \quad \text{equivalently } \mathcal{B}(\mu,\varepsilon) \circ \overline{\rho} = \overline{\rho} \circ \mathcal{B}(\mu,\varepsilon), \tag{1.13}$$

where $\overline{\rho}$ is the complex involution (cf., (1.7))

$$\overline{\rho} \begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} := \begin{bmatrix} \overline{\eta}(-x) \\ -\overline{\psi}(-x) \end{bmatrix}. \tag{1.14}$$

In addition $(\mu, \varepsilon) \to \mathcal{L}_{\mu,\varepsilon} \in \mathcal{L}(H^1(\mathbb{T}), L^2(\mathbb{T}))$ is analytic, and $\mathcal{L}_{\mu,\varepsilon}$ is linear in μ, being

$$|D + \mu| = |D| + \mu \operatorname{sgn}^+(D) \quad \forall \mu > 0, \tag{1.15}$$

where $\operatorname{sgn}^+(j) := 1$ if $j \geq 0$ and $\operatorname{sgn}^+(j) := -1$ for any $j < 0$.

We aim to describe a *far-from-the-origin* spectral branching of eigenvalues of $\mathcal{L}_{\mu,\varepsilon}$ out of the imaginary axis. The Hamiltonian structure of $\mathcal{L}_{\mu,\varepsilon}$ allows such a branching to form only as perturbation of a *multiple* purely imaginary eigenvalue of $\mathcal{L}_{\mu,0}$.

The spectrum of $\mathcal{L}_{\mu,0}$. The spectrum of the Fourier multiplier matrix operator

$$\mathcal{L}_{\mu,0} = \begin{bmatrix} \partial_x + \mathrm{i}\,\mu & |D+\mu| \\ -1 & \partial_x + \mathrm{i}\,\mu \end{bmatrix} \tag{1.16}$$

on $L^2(\mathbb{T}, \mathbb{C}^2)$ is given by

$$\begin{aligned} &\lambda_j^\sigma(\mu) = \mathrm{i}\,\omega_j^\sigma(\mu), \quad \omega_j^\sigma(\mu) := j + \mu - \sigma\sqrt{|j+\mu|}, \quad j \in \mathbb{Z},\ \sigma = \pm, \\ &\omega_j^\sigma(\mu) = \omega^\sigma(j+\mu) := j + \mu - \sigma\Omega_j(\mu), \quad \Omega_j(\mu) := \Omega(j+\mu) := \sqrt{|j+\mu|}. \end{aligned} \tag{1.17}$$

For any $j + \mu \neq 0$, $\sigma = \pm$, we associate to the eigenvalue $\mathrm{i}\,\omega_j^\sigma(\mu)$ the eigenvector

$$f_j^\sigma(\mu) := \frac{1}{\sqrt{2\Omega_j(\mu)}} \begin{bmatrix} -\sqrt{\sigma}\,\Omega_j(\mu) \\ \sqrt{-\sigma} \end{bmatrix} e^{\mathrm{i}\,jx}, \quad \mathcal{L}_{\mu,0} f_j^\sigma(\mu) = \mathrm{i}\,\omega_j^\sigma(\mu) f_j^\sigma(\mu), \tag{1.18}$$

which satisfies, recalling (1.14), the reversibility property

$$\overline{\rho} f_j^+(\mu) = f_j^+(\mu), \quad \overline{\rho} f_j^-(\mu) = -f_j^-(\mu). \tag{1.19}$$

For any $\mu \notin \mathbb{Z}$, the family of eigenvectors (1.18) forms a *complex symplectic basis* of $L^2(\mathbb{T}, \mathbb{C}^2)$ with respect to the complex symplectic form $\mathcal{W}_c$ in (1.12), namely, its elements are linearly independent, span densely $L^2(\mathbb{T}, \mathbb{C}^2)$ and satisfy, for any $j \in \mathbb{Z}$,

$$\mathcal{W}_c\big(f_j^\sigma(\mu), f_{j'}^{\sigma'}(\mu)\big) = \begin{cases} -\mathrm{i} & \text{if } j = j' \text{ and } \sigma = \sigma' = +, \\ \mathrm{i} & \text{if } j = j' \text{ and } \sigma = \sigma' = -, \\ 0 & \text{otherwise}. \end{cases} \tag{1.20}$$

The choice of the normalization constant in (1.18) implies (1.20).

All the multiple nonzero eigenvalues of $\mathcal{L}_{\mu,0}$ are given by the following lemma, cf. [1], where, in view of Remark 2, we consider $\mu \in [0, \frac{1}{2})$.

Lemma 3 (Multiple eigenvalues of $\mathcal{L}_{\mu,0}$ away from 0). *For any $\mu \in [0, \frac{1}{2})$, the spectrum of $\mathcal{L}_{\mu,0}$ away from 0 contains only simple or double eigenvalues:*

(1) *for any μ in $(0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{1}{2})$, the eigenvalues of $\mathcal{L}_{\mu,0}$ are all simple;*

(2) *for $\mu = \frac{1}{4}$, the double eigenvalues of $\mathcal{L}_{\frac{1}{4},0}$ form the set $\{\mathrm{i}\,\omega_*^{(p)}\}_{\substack{p\geq 2\\ p\ \text{even}}}$, where*

$$\omega_*^{(p)} := \frac{p^2-1}{4}, \qquad p \in \mathbb{N}; \tag{1.21}$$

(3) *for $\mu = 0$, the double eigenvalues of $\mathcal{L}_{0,0}$ form the set $\{\pm\mathrm{i}\,\omega_*^{(p)}\}_{\substack{p\geq 3\\ p\ \text{odd}}}$.*

Let $p \in \mathbb{N}$, $p \geq 2$. The eigenspace associated with the double eigenvalue $\mathrm{i}\,\omega_*^{(p)}$ is spanned by the eigenvectors $f_k^-(\underline{\mu})$, $f_{k'}^+(\underline{\mu})$ in (1.18), where

$$\begin{cases} \underline{\mu} = \frac{1}{4}, & k = n^2 - n, \quad k' = k + p = n^2 + n, \quad \text{if } p = 2n \text{ is even,} \\ \underline{\mu} = 0, & k = n^2, \quad k' = k + p = (n+1)^2, \quad \text{if } p = 2n+1 \text{ is odd.} \end{cases} \tag{1.22}$$

In view of Remark 2, the splitting of the double eigenvalue $-\mathrm{i}\,\omega_*^{(p)}$ for $\underline{\mu} = 0$ can be obtained by complex conjugation. This paper aims to study the splitting of the closest-to-zero double eigenvalue of $\mathcal{L}_{\frac{1}{4},0}$. *Hence, in Section 5, in view of Lemma 3, we will fix*

$$p = 2, \quad k = 0, \quad k' = 2, \quad \underline{\mu} := \frac{1}{4} \quad \text{and denote } \omega_* := \omega_*^{(2)} = \frac{3}{4}. \tag{1.23}$$

We will consider Floquet parameters μ close to $\underline{\mu}$, so that $\mu \notin \mathbb{Z}$ and the family of eigenvectors (1.18) forms a complex symplectic basis according to (1.20).

The spectrum $\sigma(\mathcal{L}_{\underline{\mu},0})$ decomposes into two disjoint parts:

$$\sigma(\mathcal{L}_{\underline{\mu},0}) = \sigma'(\mathcal{L}_{\underline{\mu},0}) \cup \sigma''(\mathcal{L}_{\underline{\mu},0}), \quad \text{where } \sigma'(\mathcal{L}_{\underline{\mu},0}) := \left\{\mathrm{i}\,\omega_*^{(p)}\right\} \tag{1.24}$$

is the double eigenvalue in (1.23) and $\sigma''(\mathcal{L}_{\underline{\mu},0}) := \{\lambda_j^\sigma(\underline{\mu}),\ (j,\sigma) \notin \Sigma\}$, where $\Sigma := \{(k',+),(k,-)\}$, collects the other eigenvalues $\lambda_j^\sigma(\underline{\mu})$ in (1.17).

By Kato's perturbation theory (Lemma 5) for any (μ,ε) sufficiently close to $(\underline{\mu},0)$, the perturbed spectrum $\sigma(\mathcal{L}_{\mu,\varepsilon})$ admits a disjoint decomposition $\sigma(\mathcal{L}_{\mu,\varepsilon}) = \sigma'(\mathcal{L}_{\mu,\varepsilon}) \cup \sigma''(\mathcal{L}_{\mu,\varepsilon})$, where $\sigma'(\mathcal{L}_{\mu,\varepsilon})$ consists of 2 eigenvalues close to the double eigenvalue $\mathrm{i}\,\omega_*^{(p)}$ of $\mathcal{L}_{\underline{\mu},0}$. We denote by $\mathcal{V}_{\mu,\varepsilon}$ the spectral subspace associated to $\sigma'(\mathcal{L}_{\mu,\varepsilon})$, which has dimension 2 and it is invariant under $\mathcal{L}_{\mu,\varepsilon}$. The next result provides the expansion of the matrix representing the action of the operator $\mathcal{L}_{\mu,\varepsilon}\colon \mathcal{V}_{\mu,\varepsilon} \to \mathcal{V}_{\mu,\varepsilon}$. We denote by $B(r) := \{y \in \mathbb{R}\colon |y| < r\}$ the real interval of length $2r$ centered in 0.

Theorem 4. *Assume (1.23). There exist ε_0, $\delta_0 > 0$ such that the operator $\mathscr{L}_{\mu,\varepsilon}\colon \mathcal{V}_{\mu,\varepsilon} \to \mathcal{V}_{\mu,\varepsilon}$ for any $(\mu,\varepsilon) = (\frac14 + \delta, \varepsilon) \in B_{\delta_0}(\frac14) \times B_{\varepsilon_0}(0)$ is represented by a 2×2 matrix with identical real off-diagonal entries and purely imaginary diagonal entries of the form*

$$\begin{pmatrix} \mathrm{i}\,\frac34 + \mathrm{i}\,\frac23\delta - \mathrm{i}\,\frac98\varepsilon^2 + \mathrm{i}\,r_1(\varepsilon^3, \delta\varepsilon^2, \delta^2) & -\frac{\sqrt3}{6}\delta\varepsilon^2 - \frac{39\sqrt3}{512}\varepsilon^4 + r_2(\varepsilon^5, \delta\varepsilon^4, \delta^2\varepsilon^2, \delta^4\varepsilon) \\ -\frac{\sqrt3}{6}\delta\varepsilon^2 - \frac{39\sqrt3}{512}\varepsilon^4 + r_2(\varepsilon^5, \delta\varepsilon^4, \delta^2\varepsilon^2, \delta^4\varepsilon) & \mathrm{i}\,\frac34 + \mathrm{i}\,2\delta + \frac{\mathrm{i}}{16}\varepsilon^2 + \mathrm{i}\,r_3(\varepsilon^3, \delta\varepsilon^2, \delta^2) \end{pmatrix}. \tag{1.25}$$

Theorem 1 follows directly from Theorem 4 as shown in the beginning of Section 5.

2. Perturbation of separated eigenvalues and instability criteria

We briefly recall Kato's similarity transformation theory as developed in [6, Section 3]. The following result is proved as in [6, Lemmata 3.1, 3.2] with the only difference that concerns perturbations of a non zero eigenvalue $\mathrm{i}\,\omega_*^{(p)}$ of $\mathscr{L}_{\underline{\mu},0}$. We remind that the operators $\mathscr{L}_{\mu,\varepsilon} : Y \subset X \to$ have domain $Y := H^1(\mathbb{T}) := H^1(\mathbb{T}, \mathbb{C}^2)$ and range $X := L^2(\mathbb{T}) := L^2(\mathbb{T}, \mathbb{C}^2)$.

Lemma 5. *Fix $p \in \mathbb{N}$, $p \geq 2$. Let Γ be a closed counterclockwise oriented curve winding around the double eigenvalue $\mathrm{i}\,\omega_*^{(p)}$ of $\mathscr{L}_{\underline{\mu},0}$ given by Lemma 3 in the complex plane, separating $\sigma'(\mathscr{L}_{\underline{\mu},0})$ and the other part of the spectrum $\sigma''(\mathscr{L}_{\underline{\mu},0})$ in (1.24). Then, there exist ε_0, $\delta_0 > 0$ such that for any $(\mu,\varepsilon) \in B_{\delta_0}(\underline{\mu}) \times B_{\varepsilon_0}(0)$ the following hold.*

(1) *The curve Γ belongs to the resolvent set of the operator $\mathscr{L}_{\mu,\varepsilon} : Y \subset X \to X$ defined in (1.10). The operators*

$$P(\mu,\varepsilon) := -\frac{1}{2\pi\mathrm{i}} \oint_\Gamma (\mathscr{L}_{\mu,\varepsilon} - \lambda)^{-1} \mathrm{d}\lambda : X \to Y \tag{2.1}$$

are projectors commuting with $\mathscr{L}_{\mu,\varepsilon}$, i.e., $P(\mu,\varepsilon)^2 = P(\mu,\varepsilon)$ and $P(\mu,\varepsilon)\mathscr{L}_{\mu,\varepsilon} = \mathscr{L}_{\mu,\varepsilon}P(\mu,\varepsilon)$. The map $(\mu,\varepsilon) \mapsto P(\mu,\varepsilon)$ is analytic from $B_{\delta_0}(\underline{\mu}) \times B_{\varepsilon_0}(0)$ to $\mathcal{L}(X,Y)$. The projectors $P(\mu,\varepsilon)$ are skew-Hamiltonian, namely, $\mathcal{J}P(\mu,\varepsilon) = P(\mu,\varepsilon)^*\mathcal{J}$, and reversibility preserving, i.e., $\overline{\rho}P(\mu,\varepsilon) = P(\mu,\varepsilon)\overline{\rho}$.

(2) *The domain Y of $\mathscr{L}_{\mu,\varepsilon}$ decomposes as the direct sum $Y = \mathcal{V}_{\mu,\varepsilon} \oplus \mathrm{Ker}(P(\mu,\varepsilon))$ of the closed subspaces $\mathcal{V}_{\mu,\varepsilon} := \mathrm{Rg}(P(\mu,\varepsilon))$, $\mathrm{Ker}(P(\mu,\varepsilon))$, which are invariant under $\mathscr{L}_{\mu,\varepsilon}$, and*

$$\sigma(\mathscr{L}_{\mu,\varepsilon}) \cap \{z \in \mathbb{C} \text{ inside } \Gamma\} = \sigma(\mathscr{L}_{\mu,\varepsilon}|_{\mathcal{V}_{\mu,\varepsilon}}) = \sigma'(\mathscr{L}_{\mu,\varepsilon}).$$

(3) *The projector $P(\mu,\varepsilon)$ is conjugated to $P(\underline{\mu},0)$ through an operator $U(\mu,\varepsilon)$, bounded and invertible in Y and in X, and*

$$\begin{aligned} &U(\mu,\varepsilon)P(\underline{\mu},0) \\ &\quad = P(\mu,\varepsilon)U(\mu,\varepsilon) = \big(\mathrm{Id} - (P(\mu,\varepsilon) - P(\underline{\mu},0))^2\big)^{-1/2} P(\mu,\varepsilon)P(\underline{\mu},0), \end{aligned} \tag{2.2}$$

The map $(\mu,\varepsilon)\mapsto U(\mu,\varepsilon)$ is analytic from $B_{\delta_0}(\underline{\mu})\times B_{\varepsilon_0}(0)$ to $\mathcal{L}(Y)$. The transformation operator $U(\mu,\varepsilon)$ is symplectic, i.e., $U(\mu,\varepsilon)^*\mathcal{J}U(\mu,\varepsilon)=\mathcal{J}$, and reversibility preserving. One has $\mathcal{V}_{\mu,\varepsilon}=U(\mu,\varepsilon)\mathcal{V}_{\underline{\mu},0}$ and $\dim\mathcal{V}_{\mu,\varepsilon}=\dim\mathcal{V}_{\underline{\mu},0}=2$.

Remark 6. The proof that $P_{\mu,\varepsilon}$ is skew-Hamiltonian and reversibility preserving holds as in [6, Lemma 3.1] choosing $\gamma(t)=\mathrm{i}\,\omega_*^{(p)}+re^{\mathrm{i}t}$ so that $-\overline{\gamma}(t)$ winds around $\mathrm{i}\,\omega_*^{(p)}$ clockwise.

We consider the basis

$$\mathcal{F}:=\{f^+(\mu,\varepsilon),f^-(\mu,\varepsilon)\},\quad f^+(\mu,\varepsilon):=U(\mu,\varepsilon)f_{k'}^+,\quad f^-(\mu,\varepsilon):=U(\mu,\varepsilon)f_k^-, \tag{2.3}$$

of the subspace $\mathcal{V}_{\mu,\varepsilon}$, obtained applying the transformation operators $U(\mu,\varepsilon)$ of Lemma 5 to the eigenvectors $f_{k'}^+:=f_{k'}^+(\underline{\mu})$, $f_k^-:=f_k^-(\underline{\mu})$ in (1.18) of $\mathcal{L}_{\underline{\mu},0}$, which, by Lemma 3, form a basis of the eigenspace $\mathcal{V}_{\underline{\mu},0}$ associated with $\mathrm{i}\,\omega_*^{(p)}$, for any fixed integer $p\geq 2$.

Lemma 7 (Matrix representation of $\mathcal{L}_{\mu,\varepsilon}$ on $\mathcal{V}_{\mu,\varepsilon}$). *Fix $p\in\mathbb{N}$, $p\geq 2$. The operator $\mathcal{L}_{\mu,\varepsilon}:\mathcal{V}_{\mu,\varepsilon}\to\mathcal{V}_{\mu,\varepsilon}$ in (1.10) is represented on the basis $\mathcal{F}$ in (2.3) by the 2×2 Hamiltonian and reversible matrix*

$$\mathtt{L}(\mu,\varepsilon)=\mathtt{J}\mathtt{B}(\mu,\varepsilon),\quad \mathtt{J}:=\begin{pmatrix}-\mathrm{i}&0\\0&\mathrm{i}\end{pmatrix}\quad \forall(\mu,\varepsilon)\in B_{\delta_0}(\underline{\mu})\times B_{\varepsilon_0}(0),$$

where

$$\mathtt{B}(\mu,\varepsilon):=\begin{pmatrix}(\mathfrak{B}(\mu,\varepsilon)f_{k'}^+,f_{k'}^+)&(\mathfrak{B}(\mu,\varepsilon)f_k^-,f_{k'}^+)\\(\mathfrak{B}(\mu,\varepsilon)f_{k'}^+,f_k^-)&(\mathfrak{B}(\mu,\varepsilon)f_k^-,f_k^-)\end{pmatrix}:=\begin{pmatrix}\alpha(\mu,\varepsilon)&\mathrm{i}\,\beta(\mu,\varepsilon)\\-\mathrm{i}\,\beta(\mu,\varepsilon)&\gamma(\mu,\varepsilon)\end{pmatrix}, \tag{2.4}$$

the functions $\alpha(\mu,\varepsilon)$, $\beta(\mu,\varepsilon)$, $\gamma(\mu,\varepsilon)$ are real analytic, and

$$\mathfrak{B}(\mu,\varepsilon):=P(\underline{\mu},0)^*\,U(\mu,\varepsilon)^*\,\mathcal{B}(\mu,\varepsilon)\,U(\mu,\varepsilon)\,P(\underline{\mu},0). \tag{2.5}$$

Furthermore,

$$\mathtt{B}(\mu,0)=\begin{pmatrix}\alpha(\mu,0)&\mathrm{i}\,\beta(\mu,0)\\-\mathrm{i}\,\beta(\mu,0)&\gamma(\mu,0)\end{pmatrix}=\begin{pmatrix}-\omega_{k'}^+(\mu)&0\\0&\omega_k^-(\mu)\end{pmatrix} \tag{2.6}$$

with $\omega_j^\sigma(\mu)$ in (1.17), in particular, $\mathtt{B}(\underline{\mu},0)=\begin{pmatrix}-\omega_*^{(p)}&0\\0&\omega_*^{(p)}\end{pmatrix}$.

Proof. In view of (1.20) and (1.19), the basis $\{f_{k'}^+,f_k^-\}$ of $\mathcal{V}_{\underline{\mu},0}$ is complex symplectic and reversible and, since $U(\mu,\varepsilon)$ is symplectic and reversibility preserving, the basis $\mathcal{F}$ in (2.3) is a complex symplectic and reversible basis of $\mathcal{V}_{\mu,\varepsilon}$. Given a complex symplectic basis $\{\mathtt{f}^+,\mathtt{f}^-\}$, any vector $\mathtt{f}$ in $\langle\mathtt{f}^+,\mathtt{f}^-\rangle$ verifies $\mathtt{f}=\mathrm{i}(\mathcal{J}\mathtt{f},\mathtt{f}^+)\mathtt{f}^+-\mathrm{i}(\mathcal{J}\mathtt{f},\mathtt{f}^-)\mathtt{f}^-$ whence we obtain (2.4)-(2.5). The function β is real by the reversibility property (1.13) and (1.19).

Let us prove (2.6). The operator $\mathfrak{B}(\mu,0)$ in (2.5) is a Fourier multiplier, since $\mathcal{L}_{\mu,0}$ in (1.10) is a Fourier multiplier, and so is $\mathcal{B}(\mu,0)$ in (1.10), $P(\mu,0)$ in (2.1), and finally

$U(\mu,0)$ in (2.2). As a consequence, $\beta(\mu,0) = (\mathfrak{B}(\mu,0) f_k^-, f_{k'}^+) = 0$. Then, we exploit that $\mathcal{L}_{\mu,0}$ has eigenvectors

$$\mathcal{L}_{\mu,0} f_{k'}^{\pm}(\mu) = \mathrm{i}\,\omega_{k'}^{\pm}(\mu) f_2^{\pm}(\mu), \quad \mathcal{L}_{\mu,0} f_k^{\pm}(\mu) = \mathrm{i}\,\omega_k^{\pm}(\mu) f_k^{\pm}(\mu),$$

in (1.18) and $U(\mu,0) f_{k'}^+$ and $U(\mu,0) f_k^-$ are also eigenvectors of $\mathcal{L}_{\mu,0}$ which, by continuity, are multiples, respectively, of $f_{k'}^+(\mu)$ and $f_k^-(\mu)$. ∎

The eigenvalues of the matrix $\mathtt{L}(\mu,\varepsilon)$ in (2.4) are

$$\lambda^{\pm}(\mu,\varepsilon) = \frac{\mathrm{i}}{2} S(\mu,\varepsilon) \pm \frac{1}{2}\sqrt{D(\mu,\varepsilon)}, \tag{2.7}$$

where

$$S(\mu,\varepsilon) := \gamma(\mu,\varepsilon) - \alpha(\mu,\varepsilon), \tag{2.8}$$

$$D(\mu,\varepsilon) := 4\beta^2(\mu,\varepsilon) - T^2(\mu,\varepsilon), \quad T(\mu,\varepsilon) := \alpha(\mu,\varepsilon) + \gamma(\mu,\varepsilon). \tag{2.9}$$

The goal of the next sections is to prove, for (μ,ε) close to $(\underline{\mu},0)$, the existence of eigenvalues of the matrix $\mathtt{L}(\mu,\varepsilon)$ with non zero real part. We now formulate abstract instability criteria to completely describe the spectrum of a 2×2 matrix of the form (2.4).

Abstract instability criteria. We consider a 2×2 Hamiltonian and reversible matrix

$$\mathtt{L}(\mu,\varepsilon) = \mathtt{J}\mathtt{B}(\mu,\varepsilon), \quad \mathtt{B}(\mu,\varepsilon) = \begin{pmatrix} \alpha(\mu,\varepsilon) & \mathrm{i}\,\beta(\mu,\varepsilon) \\ -\mathrm{i}\,\beta(\mu,\varepsilon) & \gamma(\mu,\varepsilon) \end{pmatrix}, \tag{2.10}$$

where $\alpha(\mu,\varepsilon)$, $\beta(\mu,\varepsilon)$, $\gamma(\mu,\varepsilon)$ are real analytic functions defined in a neighborhood $B_{\delta_0}(\underline{\mu}) \times B_{\varepsilon_0}(0)$ of $(\underline{\mu},0)$, $\underline{\mu} \in \mathbb{R}$. We make the following.

Assumption 8. *The real analytic entries $\alpha(\mu,\varepsilon)$, $\beta(\mu,\varepsilon)$, $\gamma(\mu,\varepsilon)$ of the matrix (2.10) admit for $(\mu,\varepsilon) = (\underline{\mu}+\delta,\varepsilon) \in B_{\delta_0}(\underline{\mu}) \times B_{\varepsilon_0}(0)$ an expansion of the form*

$$\alpha(\underline{\mu}+\delta,\varepsilon) = -\alpha_0(\delta) + \alpha_2\varepsilon^2 + r(\varepsilon^3, \delta\varepsilon^2, \delta^2\varepsilon), \tag{2.11a}$$

$$\beta(\underline{\mu}+\delta,\varepsilon) = \beta_1\varepsilon^2 + \beta_2\delta\varepsilon^2 + \beta_3\varepsilon^4 + r(\varepsilon^5, \delta\varepsilon^3, \delta^2\varepsilon^2, \delta^3\varepsilon), \tag{2.11b}$$

$$\gamma(\underline{\mu}+\delta,\varepsilon) = \gamma_0(\delta) + \gamma_2\varepsilon^2 + r(\varepsilon^3, \delta\varepsilon^2, \delta^2\varepsilon), \tag{2.11c}$$

where

$$\alpha_0(\delta) = \omega_*^{(p)} - \alpha_1\delta + r(\delta^2), \quad \gamma_0(\delta) = \omega_*^{(p)} + \gamma_1\delta + r(\delta^2) \tag{2.12}$$

and

$$\alpha_1 + \gamma_1 > 0. \tag{2.13}$$

In view of Assumption 8, the trace $T(\mu,\varepsilon) := \mathrm{Tr}\,\mathtt{B}(\mu,\varepsilon) = \alpha(\mu,\varepsilon) + \gamma(\mu,\varepsilon)$ expands as

$$\begin{aligned} &T(\underline{\mu}+\delta,\varepsilon) = T(\underline{\mu}+\delta,0) + T_2\varepsilon^2 + r(\varepsilon^3, \delta\varepsilon^2, \delta^2\varepsilon), \quad T_2 := \alpha_2 + \gamma_2, \\ &T(\underline{\mu}+\delta,0) = T_1\delta + r(\delta^2), \quad T_1 := \alpha_1 + \gamma_1 > 0. \end{aligned} \tag{2.14}$$

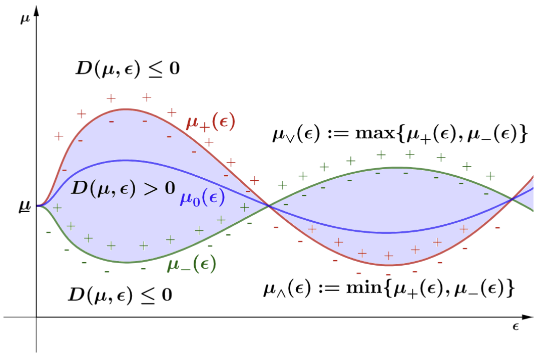


Figure 4. The instability region around the curve $\mu_0(\varepsilon)$ delimited by the curves $\mu_\vee(\varepsilon)$ and $\mu_\wedge(\varepsilon)$.

By (2.14) and the analytic implicit function theorem, there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that the zero sets of the trace $T(\mu, \varepsilon)$ and of the functions

$$d_\pm(\mu,\varepsilon) := T(\mu,\varepsilon) \pm 2\beta(\mu,\varepsilon) \tag{2.15}$$

are, in the set $B_{\delta_0}(\underline{\mu}) \times B_{\varepsilon_1}(0)$, graphs of analytic functions

$$\begin{aligned} &\mu_0, \mu_\pm : (-\varepsilon_1, \varepsilon_1) \mapsto \mu_0(\varepsilon), \quad \mu_\pm(\varepsilon), \\ &\text{i.e.,} \quad T(\mu_0(\varepsilon),\varepsilon) \equiv 0 \quad \text{and} \quad d_\pm(\mu_\pm(\varepsilon),\varepsilon) \equiv 0, \end{aligned} \tag{2.16}$$

satisfying, by (2.14), the Taylor expansions

$$\mu_0(\varepsilon) = \underline{\mu} - \frac{T_2}{T_1}\varepsilon^2 + r(\varepsilon^3), \quad \mu_\pm(\varepsilon) = \underline{\mu} - \frac{T_2 \pm 2\beta_1}{T_1}\varepsilon^2 + r_\pm(\varepsilon^3). \tag{2.17}$$

Since $T_1 > 0$, the functions $d_\pm(\mu,\varepsilon)$ are strictly positive (resp., negative) for $\mu > \mu_\pm(\varepsilon)$ (resp., $\mu < \mu_\pm(\varepsilon)$). In addition, since $d_\pm(\mu_0(\varepsilon),\varepsilon) = \pm 2\beta(\mu_0(\varepsilon),\varepsilon)$, we deduce that for any $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$

$$\begin{cases} \text{if } \beta(\mu_0(\varepsilon),\varepsilon) > 0 & \text{then } \mu_+(\varepsilon) < \mu_0(\varepsilon) < \mu_-(\varepsilon), \\ \text{if } \beta(\mu_0(\varepsilon),\varepsilon) < 0 & \text{then } \mu_-(\varepsilon) < \mu_0(\varepsilon) < \mu_+(\varepsilon), \\ \text{if } \beta(\mu_0(\varepsilon),\varepsilon) = 0 & \text{then } \mu_0(\varepsilon) = \mu_+(\varepsilon) = \mu_-(\varepsilon). \end{cases} \tag{2.18}$$

The graphs of these functions look like in Figure 4. Thus, $\mu_\wedge(\varepsilon) \le \mu_0(\varepsilon) \le \mu_\vee(\varepsilon)$, where

$$\mu_\wedge(\varepsilon) := \min\{\mu_+(\varepsilon), \mu_-(\varepsilon)\} \quad \text{and} \quad \mu_\vee(\varepsilon) := \max\{\mu_+(\varepsilon), \mu_-(\varepsilon)\}, \tag{2.19}$$

and the inequalities are strict if and only if $\beta(\mu_0(\varepsilon),\varepsilon) \neq 0$.

The following result provides a *necessary and sufficient* criterion for the existence of eigenvalues of the matrix $\mathtt{L}(\mu,\varepsilon)$ with nonzero real part.

Theorem 9 (Criterion of instability). *Assume that the* 2×2 *Hamiltonian and reversible matrix* $\mathtt{L}(\mu,\varepsilon) = \mathtt{J}\mathtt{B}(\mu,\varepsilon)$ *in* (2.10) *satisfies Assumption* 8. *Then, the following hold.*

(i) *The matrix* $\mathtt{L}(\mu,\varepsilon)$ *has eigenvalues with nonzero real part, for* (μ,ε) *close to* $(\underline{\mu},0)$, *if and only if the analytic function*

$$(-\varepsilon_1,\varepsilon_1) \ni \varepsilon \mapsto \beta(\mu_0(\varepsilon),\varepsilon) \not\equiv 0 \tag{2.20}$$

is not identically zero, where $\mu_0(\varepsilon)$ *is the analytic function defined in* (2.16)*; equivalently*

$$\exists\, n \in \mathbb{N} \quad \textit{such that} \ \frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\beta(\mu_0(\varepsilon),\varepsilon)_{|\varepsilon=0} \neq 0. \tag{2.21}$$

(ii) *The spectrum of* $\mathtt{L}(\mu,\varepsilon)$, *for* (μ,ε) *close to* $(\underline{\mu},0)$, *consists of two eigenvalues* $\lambda^{\pm}(\mu,\varepsilon)$ *with opposite nonzero real part if and only if* (μ,ε) *lies inside the region (see Figure* 4*)*

$$\begin{aligned} R :&= \big\{(\mu,\varepsilon) \in B_{\delta_0}(\underline{\mu}) \times B_{\varepsilon_1}(0) : D(\mu,\varepsilon) = -d_+(\mu,\varepsilon)\,d_-(\mu,\varepsilon) > 0\big\} \\ &= \big\{(\mu,\varepsilon) \in B_{\delta_0}(\underline{\mu}) \times B_{\varepsilon_1}(0) : \mu_\wedge(\varepsilon) < \mu < \mu_\vee(\varepsilon)\big\}, \end{aligned} \tag{2.22}$$

whereas, for $(\mu,\varepsilon) \notin R$, *the eigenvalues* $\lambda^{\pm}(\mu,\varepsilon)$ *are purely imaginary.*

Proof. The eigenvalues of $\mathtt{L}(\mu,\varepsilon)$ have the form (2.7) and have nonzero real part if and only if the discriminant $D(\mu,\varepsilon) = 4\beta^2(\mu,\varepsilon) - T^2(\mu,\varepsilon) = -d_+(\mu,\varepsilon)d_-(\mu,\varepsilon)$ is positive. In view of (2.18) and (2.19), we deduce item (ii). The region R is not empty if and only if condition (2.20) holds. This proves item (i). ∎

Let us now show sufficient conditions to verify (2.20). In view of the expansions (2.11b) and (2.17), we have

$$\beta(\mu_0(\varepsilon),\varepsilon) = \beta_1\varepsilon^2 + \left(\beta_3 - \beta_2\frac{T_2}{T_1}\right)\varepsilon^4 + r(\varepsilon^5). \tag{2.23}$$

Hence, the following conditions imply the instability criterion (2.21):

$$\begin{aligned} &(1)\ \text{(Non-degenerate case)} \quad \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\beta(\mu_0(\varepsilon),\varepsilon)_{|\varepsilon=0} = \beta_1 \neq 0; \\ &(2)\ \text{(First degenerate case)} \quad \beta_1 = 0 \text{ and } \frac{1}{4!}\frac{\mathrm{d}^4}{\mathrm{d}\varepsilon^4}\beta(\mu_0(\varepsilon),\varepsilon)_{|\varepsilon=0} = \beta_3 - \frac{T_2}{T_1}\beta_2 \neq 0. \end{aligned} \tag{2.24}$$

We will prove in Section 5 that, for $p = 2$ and considering the deep water case, the coefficient β_1 in the expansion (2.11) of the matrix (2.4) vanishes, but $\beta_3 - \frac{T_2}{T_1}\beta_2 \neq 0$, hence we are in the setup of the first degenerate case.

In order to carefully describe the unstable eigenvalues, it is convenient to translate the Floquet exponent μ around the value $\mu_0(\varepsilon)$, where the $T(\mu_0(\varepsilon),\varepsilon) = 0$ vanishes, cf., (2.16), namely, we introduce the new parameter ν such that

$$\mu = \mu_0(\varepsilon) + \nu, \quad \text{i.e., } \nu := \delta + \underline{\mu} - \mu_0(\varepsilon). \tag{2.25}$$

Accordingly, we write the functions $\mu_\pm(\varepsilon)$ in (2.17) as $\mu_\pm(\varepsilon) = \mu_0(\varepsilon) + \nu_\pm(\varepsilon)$, where

$$\nu_\pm(\varepsilon) := \mu_\pm(\varepsilon) - \mu_0(\varepsilon) \overset{(2.17)}{=} \mp \frac{2\beta_1}{T_1}\varepsilon^2 + r(\varepsilon^3). \tag{2.26}$$

Along the proof of Theorem 11, we need the following expansion.

Lemma 10. *If $\beta_1 = 0$, the functions $\nu_\pm(\varepsilon)$ in (2.26) admit the expansion*

$$\nu_\pm(\varepsilon) = \mp \frac{2}{T_1}\Big(\beta_3 - \beta_2 \frac{T_2}{T_1}\Big)\varepsilon^4 + r(\varepsilon^5). \tag{2.27}$$

Proof. The function $\nu_+(\varepsilon)$ solves

$$d_+(\mu_0(\varepsilon) + \nu_+(\varepsilon), \varepsilon) = T\big(\mu_0(\varepsilon) + \nu_+(\varepsilon), \varepsilon\big) + 2\beta\big(\mu_0(\varepsilon) + \nu_+(\varepsilon), \varepsilon\big) = 0.$$

Expanding this identity at $\mu = \mu_0(\varepsilon)$, we have

$$\underbrace{T\big(\mu_0(\varepsilon), \varepsilon\big)}_{=0 \text{ by } (2.16)} + \underbrace{(\partial_\mu T)\big(\mu_0(\varepsilon), \varepsilon\big)}_{=T_1 + r(\varepsilon^2) \text{ by } (2.14) \text{ and } (2.17)} \nu_+(\varepsilon) + \underbrace{r\big(\nu_+^2(\varepsilon)\big)}_{=r(\varepsilon^6) \text{ by } (2.26) \text{ with } \beta_1 = 0}$$
$$+ \underbrace{2\beta\big(\mu_0(\varepsilon), \varepsilon\big)}_{=2(\beta_3 - \beta_2 \frac{T_2}{T_1})\varepsilon^4 + r(\varepsilon^5) \text{ by } (2.23)} + 2\underbrace{(\partial_\mu \beta)\big(\mu_0(\varepsilon), \varepsilon\big)}_{=r(\varepsilon^2) \text{ by } (2.11b)} \nu_+(\varepsilon) = 0,$$

which gives (2.27). Analogously, one obtains the expansion of $\nu_-(\varepsilon)$. ■

We define

$$\nu_\wedge(\varepsilon) := \min\{\nu_+(\varepsilon),\ \nu_-(\varepsilon)\} \le 0, \quad \nu_\vee(\varepsilon) := \max\{\nu_+(\varepsilon),\ \nu_-(\varepsilon)\} \ge 0. \tag{2.28}$$

Note that $\nu_\wedge(\varepsilon)$ and $\nu_\vee(\varepsilon)$ are the points where the discriminant $D(\mu_0(\varepsilon) + \nu, \varepsilon)$ in (2.9) of the matrix $\mathtt{L}(\mu_0(\varepsilon) + \nu, \varepsilon)$ vanishes and for $\nu_\wedge(\varepsilon) < \nu < \nu_\vee(\varepsilon)$ the discriminant $D(\mu_0(\varepsilon) + \nu, \varepsilon)$ is positive. We now describe the first degenerate case.

Theorem 11 (First degenerate case). *Assume* (2.11a)–(2.11c) *and*

$$\beta_1 = 0, \quad \beta_3 - \beta_2 \frac{T_2}{T_1} \neq 0, \tag{2.29}$$

where T_1 and T_2 are defined in (2.14). Then, the matrix $\mathtt{L}(\mu, \varepsilon)$ in (2.10) possesses two unstable eigenvalues $\lambda^\pm(\mu_0(\varepsilon) + \nu, \varepsilon)$ for any $\nu_\wedge(\varepsilon) < \nu < \nu_\vee(\varepsilon)$, where $\nu_\wedge(\varepsilon)$, $\nu_\vee(\varepsilon)$ are defined in (2.28). The eigenvalues are

$$\lambda^\pm(\mu_0(\varepsilon) + \nu, \varepsilon)$$
$$= \begin{cases} \frac{\mathrm{i}}{2} S(\mu_0(\varepsilon) + \nu, \varepsilon) \pm \frac{\mathrm{i}}{2}\sqrt{|D(\mu_0(\varepsilon) + \nu, \varepsilon)|} & \text{if } \nu \le \nu_\wedge(\varepsilon) \text{ or } \nu \ge \nu_\vee(\varepsilon), \\ \frac{\mathrm{i}}{2} S(\mu_0(\varepsilon) + \nu, \varepsilon) \pm \frac{1}{2}\sqrt{D(\mu_0(\varepsilon) + \nu, \varepsilon)} & \text{if } \nu_\wedge(\varepsilon) < \nu < \nu_\vee(\varepsilon), \end{cases} \tag{2.30}$$

where $\mu_0(\varepsilon)$ is defined in (2.16) and it has the form (2.17),

$$D(\mu_0(\varepsilon)+\nu,\varepsilon) = 4\Big(\beta_3-\beta_2\frac{T_2}{T_1}\Big)^2\varepsilon^8 - T_1^2\nu^2 + 8\beta_2\Big(\beta_3-\beta_2\frac{T_2}{T_1}\Big)\nu\varepsilon^6 + r(\varepsilon^9,\nu\varepsilon^7,\nu^2\varepsilon^2,\nu^3), \tag{2.31}$$

and

$$S(\mu_0(\varepsilon)+\nu,\varepsilon) = 2\omega_*^{(p)} + (\gamma_1-\alpha_1)\nu + \Big(\gamma_2-\alpha_2-(\gamma_1-\alpha_1)\frac{T_2}{T_1}\Big)\varepsilon^2 + r(\varepsilon^3,\nu\varepsilon^2,\nu^2). \tag{2.32}$$

The maximum absolute value of the real part of the unstable eigenvalues in (2.30) is

$$\max \operatorname{Re}\lambda^{\pm}(\mu_0(\varepsilon)+\nu,\varepsilon) = \Big|\beta_3-\beta_2\frac{T_2}{T_1}\Big|\varepsilon^4(1+r(\varepsilon)). \tag{2.33}$$

(Isola). Assume in addition that the coefficients in (2.11a)–(2.11c) satisfy $\alpha_1\neq\gamma_1$. Then, for any ε small enough, the pair of unstable eigenvalues $\lambda^{\pm}(\mu_0(\varepsilon)+\nu,\varepsilon)$ depicts in the complex λ-plane, as ν varies in the interval $(\nu_\wedge(\varepsilon),\nu_\vee(\varepsilon))$ a closed analytic curve which intersects orthogonally the imaginary axis and encircles a convex region.

Proof. (Unstable eigenvalues) The criterion of instability in Theorem 9 is satisfied in view of (2.23) and (2.29). By (2.7), (2.11), and (2.25), the eigenvalues of $\mathtt{L}(\mu,\varepsilon)$ have the form (2.30). We now prove the expansion (2.31) of the discriminant

$$D(\mu_0(\varepsilon)+\nu,\varepsilon) = 4\beta^2(\mu_0(\varepsilon)+\nu,\varepsilon) - T^2(\mu_0(\varepsilon)+\nu,\varepsilon). \tag{2.34}$$

By (2.16) and (2.14), we get that

$$T(\mu_0(\varepsilon)+\nu,\varepsilon) = \partial_\mu T(\mu_0(\varepsilon),\varepsilon)\nu + r(\nu^2) = T_1\nu + r(\nu\varepsilon^2,\nu^2). \tag{2.35}$$

By (2.11b) with $\beta_1=0$, (2.17), (2.25), and (2.29),

$$\beta(\mu_0(\varepsilon)+\nu,\varepsilon) = \Big(\beta_3-\beta_2\frac{T_2}{T_1}\Big)\varepsilon^4 + \beta_2\nu\varepsilon^2 + r(\varepsilon^5,\nu\varepsilon^3,\nu^2\varepsilon^2,\nu^3\varepsilon). \tag{2.36}$$

Then, the expansion (2.31) follows by (2.34) and taking the square of (2.35) and (2.36). The expansion of $S(\mu_0(\varepsilon)+\nu,\varepsilon)$ in (2.32) follows from (2.8), (2.11a)–(2.11c), (2.14) and (2.17). The absolute-value maximum of the real parts of the eigenvalues is attained at $\nu=\nu_{\mathrm{Re}}$, with ν_{Re} such that $(\partial_\mu D)(\mu_0(\varepsilon)+\nu_{\mathrm{Re}},\varepsilon)=0$. By (2.31) we have the expansion

$$\nu_{\mathrm{Re}}(\varepsilon) = 4\frac{\beta_2}{T_1^2}\Big(\beta_3-\beta_2\frac{T_2}{T_1}\Big)\varepsilon^6 + r(\varepsilon^7). \tag{2.37}$$

By plugging (2.37) into (2.30)-(2.31) one obtains expansion (2.33).

(Isola). In view of (2.30), for any fixed ε small enough the unstable eigenvalues branch off from the imaginary axis at $\nu = \nu_\wedge(\varepsilon)$, evolve specularly as ν increases and rejoin at $\nu = \nu_\vee(\varepsilon)$ thus forming a *closed* curve. With the hypothesis $\alpha_1 \neq \gamma_1$, the imaginary part of the eigenvalues $I(\nu,\varepsilon) := \operatorname{Im}\lambda^\pm(\mu_0(\varepsilon)+\nu,\varepsilon) = \frac{1}{2}S(\mu_0(\varepsilon)+\nu,\varepsilon)$ is monotone w.r.t. $\nu \in (\nu_\wedge(\varepsilon), \nu_\vee(\varepsilon))$ because its derivative fulfills

$$\partial_\nu I(\nu,\varepsilon) \overset{(2.32)}{=} \frac{\gamma_1-\alpha_1}{2} + r(\varepsilon^2,\nu) \neq 0, \quad \nu_\wedge(\varepsilon) \leq \nu \leq \nu_\vee(\varepsilon). \tag{2.38}$$

Thus, the map $\nu \mapsto I(\nu,\varepsilon)$ is a diffeomorphism between $(\nu_\wedge(\varepsilon), \nu_\vee(\varepsilon))$ and its image $(y_\wedge(\varepsilon), y_\vee(\varepsilon))$. Let us denote by $\nu(y,\varepsilon)$ the inverse of $y = I(\nu,\varepsilon)$, with y varying in $y_\wedge(\varepsilon) < y < y_\vee(\varepsilon)$. The curves covered by the two unstable eigenvalues in (2.30) in the complex plane are the two specular graphs on the imaginary axis

$$\begin{aligned} \Gamma_r &:= \big\{\big(X(y,\varepsilon), y\big) : y_\wedge(\varepsilon) < y < y_\vee(\varepsilon)\big\}, \\ \Gamma_l &:= \big\{\big(-X(y,\varepsilon), y\big) : y_\wedge(\varepsilon) < y < y_\vee(\varepsilon)\big\}, \end{aligned} \tag{2.39}$$

where

$$X(y,\varepsilon) := \frac{1}{2}\sqrt{D\big(\mu(y,\varepsilon),\varepsilon\big)}, \quad \mu(y,\varepsilon) := \mu_0(\varepsilon) + \nu(y,\varepsilon). \tag{2.40}$$

At the bottom and top of the isola, i.e., at $y = y_\wedge(\varepsilon)$ and $y = y_\vee(\varepsilon)$, the real parts $\pm X(y,\varepsilon)$ of the unstable eigenvalues vanish with derivative that tends to infinity. Indeed,

$$\partial_y X(y,\varepsilon) \overset{(2.40)}{=} \partial_y \frac{1}{2}\sqrt{D(\mu(y,\varepsilon),\varepsilon)} = \frac{(\partial_\mu D)(\mu(y,\varepsilon),\varepsilon)}{4\sqrt{D(\mu(y,\varepsilon),\varepsilon)}}(\partial_y \mu)(y,\varepsilon), \tag{2.41}$$

and, by (2.31), (2.27) and (2.40), we have

$$\begin{aligned} \lim_{y\to y_\wedge, y_\vee} (\partial_\mu D)(\mu(y,\varepsilon),\varepsilon) &= (\partial_\mu D)(\mu_0(\varepsilon)+\nu_\pm(\varepsilon),\varepsilon) \\ &= \pm 4\left(\beta_3 - \beta_2\frac{T_2}{T_1}\right)T_1\varepsilon^4 + r(\varepsilon^5) \neq 0, \\ \lim_{y\to y_\wedge, y_\vee} (\partial_y \mu)(y,\varepsilon) &= \frac{1}{(\partial_\nu I)(\mu_\pm(\varepsilon),\varepsilon)} \overset{(2.38)}{\neq} 0, \end{aligned}$$

and, since $D(\mu(y,\varepsilon),\varepsilon)$ tends to 0 as $y \to y_\wedge(\varepsilon), y_\vee(\varepsilon)$, we deduce that $|\partial_y X(y,\varepsilon)|$ in (2.41) tends to $+\infty$.

Finally, we claim that the region encircled by the two graphs (2.39) is convex. It is sufficient to prove that $\partial_{yy}X(y,\varepsilon)$ is negative for any $y_\wedge(\varepsilon) < y < y_\vee(\varepsilon)$. Indeed, by (2.40),

$$\partial_{yy}X(y,\varepsilon) = \frac{2[(\partial_\mu^2 D)(\mu(y,\varepsilon),\varepsilon)(\partial_y\mu(y,\varepsilon))^2 + (\partial_\mu D)(\mu(y,\varepsilon),\varepsilon)\partial_y^2\mu(y,\varepsilon)]D(\mu(y,\varepsilon),\varepsilon) - (\partial_\mu D(\mu(y,\varepsilon),\varepsilon)\partial_y\mu(y,\varepsilon))^2}{8(D(\mu(y,\varepsilon),\varepsilon))^{\frac{3}{2}}} \tag{2.42}$$

In view of (2.31) and (2.27) we have, for any $y_\wedge(\varepsilon) < y < y_\vee(\varepsilon)$,

$$\partial_\mu^2 D(\mu(y,\varepsilon),\varepsilon) \leq -T_1^2, \quad |\partial_\mu D(\mu(y,\varepsilon),\varepsilon)| \leq 8\left|T_1\left(\beta_3 - \beta_2\frac{T_2}{T_1}\right)\right|\varepsilon^4. \tag{2.43}$$

Moreover, by (2.38), (2.30), and (2.32) there is $c > 0$ such that, for any $\nu_\wedge(\varepsilon) < \nu < \nu_\vee(\varepsilon)$,

$$|\partial_\nu I(\nu,\varepsilon)| \geq \frac{1}{4}|\gamma_1 - \alpha_1|, \quad |\partial_{\nu\nu} I(\nu,\varepsilon)| \leq c,$$

and therefore, for some $C_1 > 0$ and for any $y_\wedge(\varepsilon) < y < y_\vee(\varepsilon)$,

$$|\partial_y \mu(y,\varepsilon)| = |\partial_y \nu(y,\varepsilon)| \leq \frac{C_1}{|\gamma_1-\alpha_1|}, \quad |\partial_{yy}\mu(y,\varepsilon)| = |\partial_{yy}\nu(y,\varepsilon)| \leq \frac{C_1}{|\gamma_1-\alpha_1|}. \tag{2.44}$$

By (2.44) and (2.43), we have, for some $\widetilde{C} > 0$,

$$\begin{aligned} &\partial_\mu^2 D(\mu(y,\varepsilon),\varepsilon)(\partial_y\mu(y,\varepsilon))^2 + \partial_\mu D(\mu(y,\varepsilon),\varepsilon)\partial_y^2\mu(y,\varepsilon) \\ &\leq -\frac{\widetilde{C}T_1^2}{(\gamma_1-\alpha_1)^2} + \frac{\widetilde{C}}{|\gamma_1-\alpha_1|}\varepsilon^4 < 0 \end{aligned} \tag{2.45}$$

for ε small. By (2.42) and (2.45), the function $y \mapsto X(y,\varepsilon)$ is concave. ∎

A first approximation $\widetilde{\lambda}^\pm(\nu,\varepsilon)$ of the eigenvalues $\lambda^\pm(\mu_0(\varepsilon)+\nu,\varepsilon)$ of Lemma 11, which neglects the remainders $r(\nu^3)$ of $D(\mu_0(\varepsilon)+\nu,\varepsilon)$ in (2.31) and $r(\nu^2)$ of $S(\mu_0(\varepsilon)+\nu,\varepsilon)$ in (2.32), is

$$\begin{cases} x := \operatorname{Re}\widetilde{\lambda}^\pm(\nu,\varepsilon) \\ \quad := \pm\frac{1}{2}\sqrt{4(\beta_3-\beta_2\frac{T_2}{T_1})^2\varepsilon^8(1+r(\varepsilon)) - T_1^2\nu^2(1+r(\varepsilon^2)) + 8\beta_2(\beta_3-\beta_2\frac{T_2}{T_1})\nu\varepsilon^6(1+r(\varepsilon))}, \\ y := \operatorname{Im}\widetilde{\lambda}^\pm(\nu,\varepsilon) := \omega_*^{(p)} + (\frac{\gamma_2-\alpha_2}{2} - \frac{T_2(\gamma_1-\alpha_1)}{2T_1})\varepsilon^2(1+r(\varepsilon^2)) + \frac{\gamma_1-\alpha_1}{2}\nu(1+r(\varepsilon^2)). \end{cases} \tag{2.46}$$

The functions $\widetilde{\lambda}^\pm(\nu,\varepsilon)$ are defined for ν in the interval $\widetilde{\nu}_\wedge(\varepsilon) \leq \nu \leq \widetilde{\nu}_\vee(\varepsilon)$, where the argument of the square root in (2.46) is non-negative. These approximating eigenvalues describe an ellipse in the (x,y)-plane.

Lemma 12 (Approximating ellipse). *Suppose the coefficients α_1 and γ_1 in (2.46) are different, i.e., $\gamma_1 - \alpha_1 \neq 0$. As ν varies between $\widetilde{\nu}_\wedge(\varepsilon)$ and $\widetilde{\nu}_\vee(\varepsilon)$ the approximating eigenvalues $\widetilde{\lambda}^\pm(\nu,\varepsilon)$ in (2.46) form an ellipse of equation*

$$x^2 + \frac{T_1^2(1+r(\varepsilon^2))}{(\gamma_1-\alpha_1)^2}(y-y_0(\varepsilon))^2 = \left(\beta_3 - \beta_2\frac{T_2}{T_1}\right)^2\varepsilon^8(1+r(\varepsilon)), \tag{2.47}$$

centered at $(0, y_0(\varepsilon))$, where $y_0(\varepsilon)$ is an analytic function of the form

$$y_0(\varepsilon) = \omega_*^{(p)} + \left(\frac{\gamma_2-\alpha_2}{2} - \frac{T_2(\gamma_1-\alpha_1)}{2T_1}\right)\varepsilon^2 + r(\varepsilon^4). \tag{2.48}$$

Proof. We invert the second equation in (2.46) and obtain the function

$$\nu(y,\varepsilon) = \frac{2}{\gamma_1-\alpha_1}(y-\widetilde{y}(\varepsilon))(1+r(\varepsilon^2)), \tag{2.49}$$

where

$$\tilde{y}(\varepsilon) := \operatorname{Im} \widetilde{\lambda}^{\pm}(0,\varepsilon) = \omega_*^{(p)} + \Big(\frac{\gamma_2-\alpha_2}{2} - \frac{T_2(\gamma_1-\alpha_1)}{2T_1}\Big)\varepsilon^2(1+r(\varepsilon^2)).$$

By plugging the expansion (2.49) for $\nu = \nu(y,\varepsilon)$ in the equation for x^2, obtained by squaring the first line in (2.46), we get the equation of a conic $0 = e(x,y) := x^2 - [\operatorname{Re} \widetilde{\lambda}^{\pm}(\nu(y,\varepsilon),\varepsilon)]^2$, with

$$\begin{aligned} e(x,y) &:= x^2 + \frac{T_1^2}{(\gamma_1-\alpha_1)^2}(y-\tilde{y}(\varepsilon))^2(1+r(\varepsilon^2)) \\ &\quad - 4\frac{\beta_2(\beta_3-\beta_2\frac{T_2}{T_1})}{\gamma_1-\alpha_1}(y-\tilde{y}(\varepsilon))\varepsilon^6(1+r(\varepsilon)) - \Big(\beta_3-\beta_2\frac{T_2}{T_1}\Big)^2\varepsilon^8(1+r(\varepsilon)). \end{aligned}$$

Then, one puts the conic into its canonical form (2.47) with $y_0(\varepsilon) - \tilde{y}(\varepsilon) = r(\varepsilon^6)$. ∎

3. Taylor expansion of $\mathcal{B}(\mu,\varepsilon)$, $P(\mu,\varepsilon)$, and $\mathfrak{B}(\mu,\varepsilon)$

In this section, we provide the Taylor expansion of the operators $\mathcal{B}(\mu,\varepsilon)$ in (1.10), the projectors $P(\mu,\varepsilon)$ and the operators $\mathfrak{B}(\mu,\varepsilon)$ defined in (2.5) around $(\underline{\mu},0)$.

Notation. For an operator $A = A(\mu,\varepsilon;x)$, we denote

$$A_{i,j} := A_{i,j}(\underline{\mu}+\delta,\varepsilon;x) := \frac{1}{i!j!}\big(\partial_\mu^i\partial_\varepsilon^j A\big)(\underline{\mu},0;x)\,\delta^i\varepsilon^j, \quad A_k := \sum_{\substack{i+j=k\\ i,j\geq 0}} A_{i,j}. \quad (3.1a)$$

We also denote by $A_{i,j}^{[\kappa]}$ the part of the operator $A_{i,j}$ with Fourier harmonic $e^{\mathrm{i}\kappa x}$, i.e.,

$$A_{i,j}^{[\kappa]} := \frac{e^{\mathrm{i}\kappa x}}{2\pi}\int_0^{2\pi} A_{i,j}(\underline{\mu}+\delta,\varepsilon;y)e^{-\mathrm{i}\kappa y}\,\mathrm{d}y, \quad A_\ell^{[\kappa]} := \sum_{\substack{i+j=\ell\\ i,j\geq 0}} A_{i,j}^{[\kappa]}. \quad (3.1b)$$

It results

$$\big[A_\ell^{[\kappa]}\big]^* = (A^*)_\ell^{[-\kappa]}. \quad (3.2)$$

We will occasionally split $A_{i,j} = A_{i,j}^{[\mathrm{ev}]} + A_{i,j}^{[\mathrm{odd}]}$, where $A_{i,j}^{[\mathrm{ev}]}$ is the part of the operator $A_{i,j}$ having only even harmonics, whereas $A_{i,j}^{[\mathrm{odd}]}$ is the part with only odd ones.

We denote by $\mathcal{O}(\delta^{m_1}\varepsilon^{n_1},\ldots,\delta^{m_p}\varepsilon^{n_p})$, $m_j, n_j \in \mathbb{N}$, analytic functions of (δ,ε) with values in a Banach space X which satisfy the estimate $\|\mathcal{O}(\delta^{m_j}\varepsilon^{n_j})\|_X \leq C\sum_{j=1}^p |\delta|^{m_j}|\varepsilon|^{n_j}$ for some $C>0$ and for small values of (δ,ε). For any $k\in\mathbb{N}$ we denote by $\mathcal{O}_k$ an operator mapping $H^1(\mathbb{T},\mathbb{C}^2)$ into $L^2(\mathbb{T},\mathbb{C}^2)$-functions with size $\varepsilon^k, \delta\varepsilon^{k-1},\ldots,\delta^{k-1}\varepsilon$ or δ^k.

We directly have the following expansion recalling (1.9a)–(1.9b) and (1.15).

Lemma 13. *The operator $\mathcal{B}(\mu,\varepsilon)$ in (1.10) expands as*

$$\mathcal{B}(\underline{\mu}+\delta,\varepsilon)=\mathcal{B}_0+\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_4+\mathcal{O}_5,$$

where

$$\mathcal{B}_0=\mathcal{B}_{0,0}=\begin{bmatrix}1 & -\partial_x-\mathrm{i}\,\underline{\mu}\\ \partial_x+\mathrm{i}\,\underline{\mu} & |D+\underline{\mu}|\end{bmatrix}, \tag{3.3a}$$

$$\begin{aligned}\mathcal{B}_1=\mathcal{B}_{0,1}+\mathcal{B}_{1,0}&=\varepsilon\begin{bmatrix}a_1(x) & -p_1(x)(\partial_x+\mathrm{i}\,\underline{\mu})\\ (\partial_x+\mathrm{i}\,\underline{\mu})\circ p_1(x) & 0\end{bmatrix}\\ &+\delta\begin{bmatrix}0 & -\mathrm{i}\\ \mathrm{i} & \mathrm{sgn}^+(D)\end{bmatrix},\end{aligned} \tag{3.3b}$$

$$\begin{aligned}\mathcal{B}_2=\mathcal{B}_{0,2}+\mathcal{B}_{1,1}&=\varepsilon^2\begin{bmatrix}a_2(x) & -p_2(x)(\partial_x+\mathrm{i}\,\underline{\mu})\\ (\partial_x+\mathrm{i}\,\underline{\mu})\circ p_2(x) & 0\end{bmatrix}\\ &+\delta\varepsilon\begin{bmatrix}0 & -\mathrm{i}\,p_1(x)\\ \mathrm{i}\,p_1(x) & 0\end{bmatrix},\end{aligned} \tag{3.3c}$$

$$\begin{aligned}\mathcal{B}_3=\mathcal{B}_{0,3}+\mathcal{B}_{1,2}&=\varepsilon^3\begin{bmatrix}a_3(x) & -p_3(x)(\partial_x+\mathrm{i}\,\underline{\mu})\\ (\partial_x+\mathrm{i}\,\underline{\mu})\circ p_3(x) & 0\end{bmatrix}\\ &+\delta\varepsilon^2\begin{bmatrix}0 & -\mathrm{i}\,p_2(x)\\ \mathrm{i}\,p_2(x) & 0\end{bmatrix},\end{aligned} \tag{3.3d}$$

$$\begin{aligned}\mathcal{B}_4=\mathcal{B}_{0,4}+\mathcal{B}_{1,3}&=\varepsilon^4\begin{bmatrix}a_4(x) & -p_4(x)(\partial_x+\mathrm{i}\,\underline{\mu})\\ (\partial_x+\mathrm{i}\,\underline{\mu})\circ p_4(x) & 0\end{bmatrix}\\ &+\delta\varepsilon^3\begin{bmatrix}0 & -\mathrm{i}\,p_3(x)\\ \mathrm{i}\,p_3(x) & 0\end{bmatrix},\end{aligned} \tag{3.3e}$$

with $p_k(x)$ and $a_k(x)$, $k=1,\dots,4$, in (1.9a)–(1.9b).

Note that the functions $p_k(x)$ and $a_k(x)$ in (1.9) have only even (resp., odd) harmonics when k is even (resp., odd). Consequently, with the notation introduced below (3.2), we have

$$\mathcal{B}_{i,j}^{[\mathrm{ev}]}=\begin{cases}\mathcal{B}_{i,j} & \text{if } j \text{ is even},\\ 0 & \text{if } j \text{ is odd},\end{cases}\qquad \mathcal{B}_{i,j}^{[\mathrm{odd}]}=\begin{cases}0 & \text{if } j \text{ is even},\\ \mathcal{B}_{i,j} & \text{if } j \text{ is odd}.\end{cases} \tag{3.4}$$

We remark that sum and composition of operators satisfying (3.4) still satisfy (3.4).

Analogously, we expand the projectors $P(\mu,\varepsilon)$ in (2.1) as

$$P(\underline{\mu}+\delta,\varepsilon)=P_0+P_1+P_2+P_3+\mathcal{O}_4,$$

where

$$\begin{aligned}&P_0:=P(\underline{\mu},0),\quad P_1:=\mathcal{P}[\mathcal{B}_1],\quad P_2:=\mathcal{P}[\mathcal{B}_2]+\mathcal{P}[\mathcal{B}_1,\mathcal{B}_1],\\ &P_3:=\mathcal{P}[\mathcal{B}_3]+\mathcal{P}[\mathcal{B}_2,\mathcal{B}_1]+\mathcal{P}[\mathcal{B}_1,\mathcal{B}_2]+\mathcal{P}[\mathcal{B}_1,\mathcal{B}_1,\mathcal{B}_1],\end{aligned} \tag{3.5}$$

and, for any $k \in \mathbb{N}$,

$$\mathcal{P}[A_1, \ldots, A_k] := \frac{(-1)^{k+1}}{2\pi \mathrm{i}} \oint_\Gamma \big(\mathcal{L}_{\underline{\mu},0} - \lambda\big)^{-1} \mathcal{J} A_1 \big(\mathcal{L}_{\underline{\mu},0} - \lambda\big)^{-1} \cdots \mathcal{J} A_k \big(\mathcal{L}_{\underline{\mu},0} - \lambda\big)^{-1} \mathrm{d}\lambda, \tag{3.6}$$

with Γ the same circuit of Lemma 5. In virtue of (3.5)–(3.6) and (3.4), we obtain

$$P_{i,j}^{[\mathrm{ev}]} = \begin{cases} P_{i,j} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd,} \end{cases} \qquad P_{i,j}^{[\mathrm{odd}]} = \begin{cases} 0 & \text{if } j \text{ is even,} \\ P_{i,j} & \text{if } j \text{ is odd.} \end{cases} \tag{3.7}$$

Now, we provide the expansion of the operators $\mathfrak{B}(\mu, \varepsilon)$. Let $\mathbf{Sym}[A] := \frac{1}{2}A + \frac{1}{2}A^*$.

Lemma 14 (Expansion of $\mathfrak{B}(\mu, \varepsilon)$). *The operator $\mathfrak{B}(\mu, \varepsilon)$ in (2.5) has the Taylor expansion*

$$\mathfrak{B}(\underline{\mu} + \delta, \varepsilon) = \mathfrak{B}_0 + \mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3 + \mathfrak{B}_4 + \mathcal{O}_5,$$

where

$$\mathfrak{B}_0 := P_0^* \mathcal{B}_0 P_0, \quad \mathfrak{B}_1 := P_0^* \mathcal{B}_1 P_0, \quad \mathfrak{B}_2 := P_0^* \,\mathbf{Sym}[\mathcal{B}_2 + \mathcal{B}_1 P_1] P_0, \tag{3.8a}$$

$$\mathfrak{B}_3 := P_0^* \,\mathbf{Sym}[\mathcal{B}_3 + \mathcal{B}_2 P_1 + \mathcal{B}_1 (\mathrm{Id} - P_0) P_2] P_0, \tag{3.8b}$$

$$\mathfrak{B}_4 = P_0^* \,\mathbf{Sym}[\mathcal{B}_4 + \mathcal{B}_3 P_1 + \mathcal{B}_2 (\mathrm{Id} - P_0) P_2 + \mathcal{B}_1 (\mathrm{Id} - P_0) P_3 - \mathcal{B}_1 P_1 P_0 P_2] P_0, \tag{3.8c}$$

with $\mathcal{B}_j$, $j = 0, \ldots, 4$, in (3.3) and P_j, $j = 0, \ldots, 3$, in (3.5).

Proof. The proof follows as Lemma 3.6 of [8], obtaining the same expansions (3.24a)–(3.24c) of [8]. In the present case, the last operator in formula (3.24c) of [8], namely, $P_0^* \,\mathbf{Sym}[\mathfrak{N} P_0 P_2] P_0$, where $\mathfrak{N} := \frac{1}{4}(P_2^* \mathcal{B}_0 - \mathcal{B}_0 P_2)$, actually vanishes, in view of the identity $P_0^* \mathcal{B}_0 P_2 P_0 = P_0^* P_2^* \mathcal{B}_0 P_0$ that we now prove. First, we have[3] $\mathcal{B}_0 P_0 = P_0^* \mathcal{B}_0 = -\mathrm{i}\, \omega_*^{(p)} \mathcal{J} P_0$, that follows from

$$(\mathcal{B}_0 P_0 f, g) = (\mathcal{L}_{\underline{\mu},0} P_0 f, \mathcal{J} g) = \mathrm{i}\, \omega_*^{(p)} (P_0 f, \mathcal{J} g) = -\mathrm{i}\, \omega_*^{(p)} (\mathcal{J} P_0 f, g).$$

This identity, together with $\mathcal{J} P_j = P_j^* \mathcal{J}$ (a consequence of $P(\mu, \varepsilon)$ be skew-Hamiltonian), gives $P_0^* \mathcal{B}_0 P_j P_0 = P_0^* P_j^* \mathcal{B}_0 P_0$ for any $j \in \mathbb{N}$. ∎

In virtue of (3.1), (3.4), (3.7), and (3.8), we obtain

$$\mathfrak{B}_{i,j}^{[\mathrm{ev}]} = \begin{cases} \mathfrak{B}_{i,j} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd,} \end{cases} \qquad \mathfrak{B}_{i,j}^{[\mathrm{odd}]} = \begin{cases} 0 & \text{if } j \text{ is even,} \\ \mathfrak{B}_{i,j} & \text{if } j \text{ is odd.} \end{cases} \tag{3.9}$$

[3] This identity does not hold in [8] because of the presence of a generalized eigenvector.

4. Entanglement coefficients

In this section, we introduce the *entanglement coefficients* that represent how the jets of the operator $\mathfrak{B}(\mu,\varepsilon)$ act on the unperturbed eigenvector basis (1.18).

4.1. Abstract representation formulas

Take $\underline{\mu} \in \mathbb{R} \setminus \mathbb{Z}$, so that the eigenvectors $\{f_j^\sigma\}_{j\in\mathbb{Z},\sigma=\pm}$, $f_j^\sigma := f_j^\sigma(\underline{\mu})$ in (1.18), form a complex symplectic basis of $L^2(\mathbb{T},\mathbb{C}^2)$. Our goal is to describe the action of the operators $\mathcal{J}\mathcal{B}_\ell$ and $\mathcal{P}[\mathcal{B}_{\ell_1},\dots,\mathcal{B}_{\ell_s}]$ (recall (3.6)) on a vector f_j^σ of the basis. To do so, we introduce the *entanglement coefficients*

$$\mathrm{B}_{\ell\ \,j',j}^{[\kappa]\,\sigma',\sigma} := \big(\mathcal{B}_\ell^{[\kappa]} f_j^\sigma, f_{j'}^{\sigma'}\big), \quad \ell \in \mathbb{N}_0^2,\ j',j \in \mathbb{Z},\ \sigma,\sigma' = \pm, \tag{4.1}$$

where $\mathcal{B}_\ell^{[\kappa]}$ is the κth Fourier coefficient of the operator $\mathcal{B}_\ell$ in (3.3) (according to (3.1)). We stress that in this section ℓ is always a pair $\ell = (i,j) \in \mathbb{N}_0^2$.

The entanglement coefficients fulfill

$$\mathrm{B}_{\ell\ \,j',j}^{[\kappa]\,\sigma',\sigma} = 0 \quad \text{if } j' \neq j+\kappa \quad \text{and} \quad \overline{\mathrm{B}_{\ell\ \,j',j}^{[\kappa]\,\sigma',\sigma}} = \mathrm{B}_{\ell\ \ \,j,j'}^{[-\kappa]\,\sigma,\sigma'}. \tag{4.2}$$

The next lemma provides effective formulas to compute the action of the operators $\mathcal{J}\mathcal{B}_\ell$ and $\mathcal{P}[\mathcal{B}_{\ell_1},\dots,\mathcal{B}_{\ell_s}]$ on the vector basis.

Lemma 15. *Let* $\mathrm{B}_{\ell\ \,j',j}^{[\kappa]\,\sigma',\sigma}$ *denote the entanglement coefficients in* (4.1) *and* $f_j^\sigma := f_j^\sigma(\underline{\mu})$ *in* (1.18) *with* $\underline{\mu} \in \mathbb{R}\setminus\mathbb{Z}$. *Then, the following statements hold:*

(i) *for any* $\ell \in \mathbb{N}_0^2$ *and* $j,\kappa \in \mathbb{Z}$ *and* $\sigma = \pm$ *one has*

$$\mathcal{J}\mathcal{B}_\ell^{[\kappa]} f_j^\sigma = \sum_{\sigma_1=\pm} -\mathrm{i}\,\sigma_1\, \mathrm{B}_{\ell\ \,j+\kappa,j}^{[\kappa]\,\sigma_1,\sigma}\, f_{j+\kappa}^{\sigma_1} = \mathrm{i}\,\mathrm{B}_{\ell\ \,j+\kappa,j}^{[\kappa]\,-,\sigma}\, f_{j+\kappa}^{-} - \mathrm{i}\,\mathrm{B}_{\ell\ \,j+\kappa,j}^{[\kappa]\,+,\sigma}\, f_{j+\kappa}^{+}; \tag{4.3}$$

(ii) *for any* $q \in \mathbb{N}$, $\ell_1,\dots,\ell_q \in \mathbb{N}_0^2$, $j,\kappa_1,\dots,\kappa_q \in \mathbb{Z}$ *and* $\sigma = \pm$, *the operator* $\mathcal{P}[\mathcal{B}_{\ell_q}^{[\kappa_q]},\dots,\mathcal{B}_{\ell_1}^{[\kappa_1]}]$ *defined via* (3.6), *satisfies*

$$\begin{aligned}&\mathcal{P}\big[\mathcal{B}_{\ell_q}^{[\kappa_q]},\dots,\mathcal{B}_{\ell_1}^{[\kappa_1]}\big] f_j^\sigma \\ &\quad= \sum_{\sigma_1,\dots,\sigma_q=\pm} \sigma_1\cdots\sigma_q\, \mathrm{B}_{\ell_q\ \,j_q,j_{q-1}}^{[\kappa_q]\,\sigma_q,\sigma_{q-1}} \cdots \mathrm{B}_{\ell_2\ \,j_2,j_1}^{[\kappa_2]\,\sigma_2,\sigma_1}\, \mathrm{B}_{\ell_1\ \,j_1,j}^{[\kappa_1]\,\sigma_1,\sigma}\, \mathrm{R}_{j,j_1,\dots,j_q}^{\sigma,\sigma_1,\dots,\sigma_q}\, f_{j_q}^{\sigma_q},\end{aligned} \tag{4.4}$$

where $j_1 := j+\kappa_1$, $j_2 := j_1+\kappa_2,\dots, j_q = j_{q-1}+\kappa_q$, *and*

$$\mathrm{R}_{j,j_1,\dots,j_q}^{\sigma,\sigma_1,\dots,\sigma_q} := \frac{(-\mathrm{i})^q}{2\pi\mathrm{i}} \oint_\Gamma \frac{\mathrm{d}\lambda}{(\lambda - \mathrm{i}\,\omega_j^\sigma)(\lambda-\mathrm{i}\,\omega_{j_1}^{\sigma_1})\cdots(\lambda-\mathrm{i}\,\omega_{j_q}^{\sigma_q})} \tag{4.5}$$

with Γ *is a circuit winding once around* $\mathrm{i}\,\omega_*^{(p)}$ *counterclockwise and* $\omega_j^{\pm} := \omega_j^{\pm}(\underline{\mu})$ *in* (1.17).

The coefficients $\mathtt{R}^{\sigma_0,\sigma_1,\ldots,\sigma_q}_{j_0,j_1,\ldots,j_q}$ *are real and invariant by permutations of the indexes, namely,*

$$\mathtt{R}^{\sigma_0,\sigma_1,\ldots,\sigma_q}_{j_0,j_1,\ldots,j_q} = \overline{\mathtt{R}^{\sigma_0,\sigma_1,\ldots,\sigma_q}_{j_0,j_1,\ldots,j_q}},$$

$$\mathtt{R}^{\sigma_0,\sigma_1,\ldots,\sigma_q}_{j_0,j_1,\ldots,j_q} = \mathtt{R}^{\sigma_{\tau(0)},\sigma_{\tau(1)},\ldots,\sigma_{\tau(q)}}_{j_{\tau(0)},j_{\tau(1)},\ldots,j_{\tau(q)}} \quad \text{for any permutation } \tau \text{ of } \{0,1,\ldots,q\}; \tag{4.6}$$

(iii) *for any* $q \in \mathbb{N}$, $\ell_1,\ldots,\ell_{q+1} \in \mathbb{N}_0^2$, $j, j', \kappa_1,\ldots,\kappa_{q+1} \in \mathbb{Z}$ *and* $\sigma,\sigma' = \pm$, *one has*

$$\Big(\mathcal{B}^{[\kappa_{q+1}]}_{\ell_{q+1}} \mathcal{P}\big[\mathcal{B}^{[\kappa_q]}_{\ell_q},\ldots,\mathcal{B}^{[\kappa_1]}_{\ell_1}\big] f^{\sigma}_j, f^{\sigma'}_{j'}\Big)$$
$$= \sum_{\sigma_1,\ldots,\sigma_q=\pm} \sigma_1\cdots\sigma_q \mathtt{B}^{[\kappa_{q+1}]\sigma',\sigma_q}_{\ell_{q+1}\ j',j_q} \mathtt{B}^{[\kappa_q]\sigma_q,\sigma_{q-1}}_{\ell_q\ j_q,j_{q-1}} \cdots \mathtt{B}^{[\kappa_2]\sigma_2,\sigma_1}_{\ell_2\ j_2,j_1} \mathtt{B}^{[\kappa_1]\sigma_1,\sigma}_{\ell_1\ j_1,j} \mathtt{R}^{\sigma,\sigma_1,\ldots,\sigma_q}_{j,j_1,\ldots,j_q} \tag{4.7a}$$

with $j_1 := j + \kappa_1$, $j_2 := j_1 + \kappa_2, \ldots, j_q = j_{q-1} + \kappa_q$, *and*

$$\Big(\mathcal{B}^{[\kappa_1]}_{\ell_1} f^{\sigma}_j, \mathcal{P}\big[\mathcal{B}^{[-\kappa_2]}_{\ell_2},\ldots,\mathcal{B}^{[-\kappa_{q+1}]}_{\ell_{q+1}}\big] f^{\sigma'}_{j'}\Big)$$
$$= \sum_{\sigma_1,\ldots,\sigma_q=\pm} \sigma_1\cdots\sigma_q\, \mathtt{B}^{[\kappa_{q+1}]\sigma',\sigma_q}_{\ell_{q+1}\ j',\xi_q} \mathtt{B}^{[\kappa_q]\sigma_q,\sigma_{q-1}}_{\ell_q\ \xi_q,\xi_{q-1}} \cdots \mathtt{B}^{[\kappa_2]\sigma_2,\sigma_1}_{\ell_2\ \xi_2,\xi_1} \mathtt{B}^{[\kappa_1]\sigma_1,\sigma}_{\ell_1\ \xi_1,j} \mathtt{R}^{\sigma',\sigma_q,\ldots,\sigma_1}_{j',\xi_q,\ldots,\xi_1} \tag{4.7b}$$

with $\xi_q := j' - \kappa_{q+1}$, $\xi_{q-1} := \xi_q - \kappa_q, \ldots, \xi_1 := \xi_2 - \kappa_2$.

Proof. (i) Since the operator $\mathcal{B}^{[\kappa]}_{\ell}$ shifts the harmonic j to $j+\kappa$ and $\mathcal{J}$ leaves it invariant, there are scalars $\alpha^{\pm} \in \mathbb{C}$ such that

$$\mathcal{J}\mathcal{B}^{[\kappa]}_{\ell} f^{\sigma}_j = \alpha^- f^-_{j+\kappa} + \alpha^+ f^+_{j+\kappa}. \tag{4.8}$$

Then,

$$\big(\mathcal{J}\mathcal{B}^{[\kappa]}_{\ell} f^{\sigma}_j, \mathcal{J} f^-_{j+\kappa}\big) = \alpha^-\big(f^-_{j+\kappa}, \mathcal{J} f^-_{j+\kappa}\big) + \alpha^+\big(f^+_{j+\kappa}, \mathcal{J} f^-_{j+\kappa}\big) \overset{(1.20)}{=} -\mathrm{i}\,\alpha^-. \tag{4.9}$$

On the other hand,

$$\big(\mathcal{J}\mathcal{B}^{[\kappa]}_{\ell} f^{\sigma}_j, \mathcal{J} f^-_{j+\kappa}\big) = \big(\mathcal{B}^{[\kappa]}_{\ell} f^{\sigma}_j, f^-_{j+\kappa}\big) \overset{(4.1)}{=} \mathtt{B}^{[\kappa]-,\sigma}_{\ell\ j+\kappa,j}. \tag{4.10}$$

Similarly, $\mathtt{B}^{[\kappa]+,\sigma}_{\ell\ j+\kappa,j} = (\mathcal{J}\mathcal{B}^{[\kappa]}_{\ell} f^{\sigma}_j, \mathcal{J} f^+_{j+\kappa}) = \mathrm{i}\alpha^+$. By (4.8), (4.9), and (4.10), we get (4.3).

(ii) By (1.18), we have $(\mathcal{L}_{\underline{\mu},0}-\lambda)^{-1} f_j^\sigma = \frac{1}{\mathrm{i}\omega_j^\sigma - \lambda} f_j^\sigma$. Thus, in view of (3.6),

$$\mathcal{P}[\mathcal{B}_{\ell_q}^{[\kappa_q]},\dots,\mathcal{B}_{\ell_1}^{[\kappa_1]}] f_j^\sigma = \frac{(-1)^{q+1}}{2\pi\mathrm{i}} \oint_\Gamma \frac{(\mathcal{L}_{\underline{\mu},0}-\lambda)^{-1}}{\mathrm{i}\,\omega_j^\sigma-\lambda} \mathcal{J}\mathcal{B}_{\ell_q}^{[\kappa_q]} \cdots (\mathcal{L}_{\underline{\mu},0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{\ell_1}^{[\kappa_1]} f_j^\sigma \,\mathrm{d}\lambda\,.$$

Now, it is enough to use repeatedly the formula above, setting $j_0 := j$ and $\sigma_0 := \sigma$,

$$(\mathcal{L}_{\underline{\mu},0}-\lambda)^{-1}\mathcal{J}\mathcal{B}_{\ell_\iota}^{[\kappa_\iota]} f_{j_{\iota-1}}^{\sigma_{\iota-1}} = \sum_{\sigma_\iota=\pm} -\mathrm{i}\,\sigma_\iota \frac{\mathtt{B}_{\ell_\iota\ \, j_\iota,j_{\iota-1}}^{[\kappa_\iota]\,\sigma_\iota,\sigma_{\iota-1}}}{\mathrm{i}\,\omega_{j_\iota}^{\sigma_\iota}-\lambda} f_{j_\iota}^{\sigma_\iota}, \quad j_\iota := j_{\iota-1}+\kappa_\iota,\ \iota=1,\dots,q,$$

(which follows from (4.3)) to obtain (4.4).

Next, we prove properties (4.6). The second one follows trivially from the definition (4.5). For the first one, we have, by (4.5),

$$\overline{\mathtt{R}^{\sigma_0,\sigma_1,\dots,\sigma_q}_{j_0,j_1,\dots,j_q}} = \frac{\mathrm{i}^q}{2\pi\mathrm{i}} \int_0^{2\pi} \frac{-\overline{\gamma}'(t)\mathrm{d}t}{(\overline{\gamma}(t)+\mathrm{i}\,\omega_{j_0}^{\sigma_0})\cdots(\overline{\gamma}(t)+\mathrm{i}\,\omega_{j_q}^{\sigma_q})},$$

for a closed path $\gamma(t)$ winding around $\mathrm{i}\,\omega_*^{(p)}$ counterclockwise. Thus, being $-\overline{\gamma}(t)$ reverse oriented,

$$\overline{\mathtt{R}^{\sigma_0,\sigma_1,\dots,\sigma_q}_{j_0,j_1,\dots,j_q}} = -\frac{\mathrm{i}^q}{2\pi\mathrm{i}} \oint_\Gamma \frac{\mathrm{d}\lambda}{(\mathrm{i}\,\omega_{j_0}^{\sigma_0}-\lambda)\cdots(\mathrm{i}\,\omega_{j_q}^{\sigma_q}-\lambda)} = \mathtt{R}^{\sigma_0,\sigma_1,\dots,\sigma_q}_{j_0,j_1,\dots,j_q},$$

whence the first identity in (4.6) follows.

(iii) Identity (4.7a) is a consequence of (4.4) and (4.1). By a similar argument, one proves (4.7b), using also (4.2) and (4.6). ∎

The following identities will be particularly useful to identify expressions of the form (4.7a)–(4.7b) that coincide.

Lemma 16. *Let (j,σ), (j',σ') be such that $\omega_j^\sigma = \omega_{j'}^{\sigma'}$. Then,*

$$\Big(\mathcal{B}_{\ell_{q+1}}^{[\kappa_{q+1}]}\mathcal{P}[\mathcal{B}_{\ell_q}^{[\kappa_q]},\dots,\mathcal{B}_{\ell_1}^{[\kappa_1]}] f_j^\sigma, f_{j'}^{\sigma'}\Big) = \Big(\mathcal{B}_{\ell_1}^{[\kappa_1]} f_j^\sigma, \mathcal{P}[\mathcal{B}_{\ell_2}^{[-\kappa_2]},\dots,\mathcal{B}_{\ell_{q+1}}^{[-\kappa_{q+1}]}] f_{j'}^{\sigma'}\Big). \tag{4.11}$$

Proof. If $j' \neq j+\kappa_1+\cdots+\kappa_{q+1}$, then both sides of identity (4.11) vanish. Otherwise, $j' = j+\kappa_1+\cdots+\kappa_{q+1}$ and, recalling the definitions of j_s, ξ_s in (4.7a), (4.7b), one has

$$\xi_s = j'-\kappa_{s+1}-\cdots-\kappa_{q+1} = j+\kappa_1+\cdots+\kappa_s = j_s \quad \forall s=1,\dots,q.$$

Using (4.5), the second property of (4.6) and $\omega_j^\sigma = \omega_{j'}^{\sigma'}$ then $\mathtt{R}^{\sigma,\sigma_1,\dots,\sigma_q}_{j,j_1,\dots,j_q} = \mathtt{R}^{\sigma',\sigma_q,\dots,\sigma_1}_{j',\xi_q,\dots,\xi_1}$. Thus, by (4.7a), (4.7b), the two sides of (4.11) are equal. ∎

4.2. Entanglement coefficients for $p = 2$

We conclude this section by giving explicit formulas for the entanglement coefficients in (4.1) and the residue term in (4.5) for the particular case $p = 2$, where we assume (1.23), namely, we fix $\underline{\mu} = \frac{1}{4}$, $k = 0$, $k' = 2$ and $\omega_* = \omega_*^{(2)}$.

We first consider the Fourier series expansions of the even functions $p_n(x)$, $a_n(x)$, $n \in \mathbb{N}$, defined in (1.9),

$$
\begin{aligned}
p_n(x) &= \frac{1}{2}p_n^{[0]} + \sum_{\kappa\geq 1} p_n^{[\kappa]}\cos(\kappa x) = \frac{1}{2}p_n^{[0]} + \sum_{\kappa\geq 1} \frac{p_n^{[\kappa]}}{2}e^{\mathrm{i}\kappa x} + \frac{p_n^{[-\kappa]}}{2}e^{-\mathrm{i}\kappa x},\\
a_n(x) &= \frac{1}{2}a_n^{[0]} + \sum_{\kappa\geq 1} a_n^{[\kappa]}\cos(\kappa x) = \frac{1}{2}a_n^{[0]} + \sum_{\kappa\geq 1} \frac{a_n^{[\kappa]}}{2}e^{\mathrm{i}\kappa x} + \frac{a_n^{[-\kappa]}}{2}e^{-\mathrm{i}\kappa x},
\end{aligned}
\tag{4.12}
$$

with $p_n^{[-\kappa]} := p_n^{[\kappa]}$ and $a_n^{[-\kappa]} := a_n^{[\kappa]}$ for any $\kappa \in \mathbb{N}$. In view of (1.9) for any $n = 1, \dots, 4$ the non zero Fourier coefficients $p_n^{[\kappa]}, a_n^{[\kappa]}$ are

$$
\begin{aligned}
&p_1^{[\pm 1]} = a_1^{[\pm 1]} := -2, \quad p_2^{[0]} = 3, \quad a_2^{[0]} = 4, \quad p_2^{[\pm 2]} = a_2^{[\pm 2]} = -2,\\
&p_3^{[\pm 1]} = 3, \quad a_3^{[\pm 1]} = 4, \quad p_3^{[\pm 3]} = a_3^{[\pm 3]} = -3,\\
&p_4^{[0]} = \frac{1}{4}, \quad a_4^{[0]} = -2, \quad p_4^{[\pm 2]} = a_4^{[\pm 2]} = 4, \quad p_4^{[\pm 4]} = a_4^{[\pm 4]} = -\frac{16}{3}.
\end{aligned}
\tag{4.13}
$$

In view of (3.1), the operators in (3.3) have jets

$$
\mathcal{B}_{1,0}^{[0]} = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & \mathrm{sgn}^+(D) \end{bmatrix}\delta, \tag{4.14}
$$

and, for any $n = 1, \dots, 4$ (recall also (4.12), (4.13))

$$
\mathcal{B}_{i,n}^{[\kappa]} = \begin{cases}
\frac{e^{\mathrm{i}\kappa x}}{2}\begin{bmatrix} a_n^{[\kappa]} & -p_n^{[\kappa]}(\partial_x + \mathrm{i}\frac{1}{4}) \\ p_n^{[\kappa]}(\partial_x + \mathrm{i}\frac{1}{4} + \mathrm{i}\kappa) & 0 \end{bmatrix}\varepsilon^n, \\
\qquad \text{if } i = 0,\ \kappa \equiv n \pmod 2,\ |\kappa| \leq n,\\
\frac{e^{\mathrm{i}\kappa x}}{2}\begin{bmatrix} 0 & -\mathrm{i}\, p_{n-1}^{[\kappa]} \\ \mathrm{i}\, p_{n-1}^{[\kappa]} & 0 \end{bmatrix}\delta\varepsilon^{n-1}, \\
\qquad \text{if } i = 1,\ \kappa \not\equiv n \pmod 2,\ |\kappa| \leq n-1,\\
0, \quad \text{otherwise.}
\end{cases}
\tag{4.15}
$$

Lemma 17. *For any $j \in \mathbb{Z}$ and $\sigma, \sigma' = \pm 1$, the nonzero entanglement coefficients in (4.1) are as follows.*

- *If $i = 1$, $n = 0$, then*

$$
\mathrm{B}_{1,0\,j,j}^{[0]\,\sigma',\sigma} = \frac{\sqrt{\sigma}\,\overline{\sqrt{\sigma'}}}{2\Omega_j}\big(\sigma\sigma'\mathrm{sgn}^+(j) - (\sigma + \sigma')\Omega_j\big)\delta, \tag{4.16a}
$$

where $\Omega_j := \sqrt{|j + \frac{1}{4}|}$;

- *If* $i = 0$, $n = 1, 2, 3, 4$, $\kappa \in \mathbb{Z}$, *then*

$$\begin{aligned}&\mathrm{B}^{[\kappa]\ \sigma',\sigma}_{0,nj+\kappa,j}\\ &\quad = \frac{1}{4}\sqrt{\sigma}\overline{\sqrt{\sigma'}}\sqrt{\Omega_{j+\kappa}\Omega_j}\big(a_n^{[\kappa]} - \sigma p_n^{[\kappa]}\Omega_j \mathrm{sgn}^+(j) - \sigma' p_n^{[\kappa]}\Omega_{j+\kappa}\mathrm{sgn}^+(j+\kappa)\big)\varepsilon^n\end{aligned} \tag{4.16b}$$

  if $\kappa \equiv n$ *(mod 2) and* $|\kappa| \le n$, *with constants* $p_n^{[\kappa]}$, $a_n^{[\kappa]}$ *given in* (4.13), *and vanish otherwise.*

- *If* $i = 1$, $n = 1, 2, 3$, $\kappa \in \mathbb{Z}$, *then*

$$\mathrm{B}^{[\kappa]\ \sigma',\sigma}_{1,nj+\kappa,j} = -\frac{1}{4}\sqrt{\sigma}\overline{\sqrt{\sigma'}}\frac{1}{\sqrt{\Omega_j\Omega_{j+\kappa}}}\big(\sigma\Omega_{j+\kappa} + \sigma'\Omega_j\big)p_{n-1}^{[\kappa]}\delta\varepsilon^{n-1} \tag{4.16c}$$

  if $\kappa \not\equiv n$ *(mod 2) and* $|\kappa| \le n-1$, *with constants* $p_n^{[\kappa]}$ *given in* (4.13), *and vanish otherwise.*

Proof. Recall that $f_j^\sigma = f_j^\sigma(\frac{1}{4})$ are given in (1.18). In view of (4.14), we have

$$\mathcal{B}^{[0]}_{0,1} f_j^\sigma = \frac{e^{\mathrm{i}jx}}{\sqrt{2\Omega_j}}\begin{bmatrix} -\mathrm{i}\sqrt{-\sigma} \\ -\mathrm{i}\sqrt{\sigma}\Omega_j + \sqrt{-\sigma}\mathrm{sgn}^+(j)\end{bmatrix}\delta,$$

and by (4.1),

$$\mathrm{B}^{[0]\ \sigma',\sigma}_{0,1j,j} = \frac{1}{2\Omega_j}\Big(\sqrt{-\sigma}\,\overline{\sqrt{-\sigma'}}\mathrm{sgn}^+(j) + \mathrm{i}\sqrt{-\sigma}\,\overline{\sqrt{\sigma'}}\,\Omega_j - \mathrm{i}\sqrt{\sigma}\overline{\sqrt{-\sigma'}}\Omega_j\Big)\delta.$$

Then, (4.16a) follows because, for any $\sigma, \sigma' = \pm$, we have

$$\begin{aligned}\sqrt{-\sigma}\overline{\sqrt{-\sigma'}} &= \sigma\sigma'\sqrt{\sigma}\overline{\sqrt{\sigma'}},\\ \mathrm{i}\sqrt{-\sigma}\overline{\sqrt{\sigma'}} &= -\sigma\sqrt{\sigma}\overline{\sqrt{\sigma'}},\\ -\mathrm{i}\sqrt{\sigma}\overline{\sqrt{-\sigma'}} &= -\sigma'\sqrt{\sigma}\overline{\sqrt{\sigma'}}.\end{aligned} \tag{4.17}$$

We now prove (4.16b). In this case, in view of (4.15) and (1.18), we obtain

$$\mathcal{B}^{[\kappa]}_{0,n} f_j^\sigma = \frac{e^{\mathrm{i}(j+\kappa)x}}{2\sqrt{2\Omega_j}}\begin{bmatrix} -a_n^{[\kappa]}\sqrt{\sigma}\Omega_j - \mathrm{i}\sqrt{-\sigma}p_n^{[\kappa]}(j+\frac{1}{4}) \\ -\mathrm{i}\sqrt{\sigma}p_n^{[\kappa]}\Omega_j(j+\kappa+\frac{1}{4})\end{bmatrix}\varepsilon^n;$$

thus, by (4.1), we have

$$\begin{aligned}\mathrm{B}^{[\kappa]\ \sigma',\sigma}_{0,nj+\kappa,j} = \frac{1}{4\sqrt{\Omega_j\Omega_{j+\kappa}}}\Big(&\sqrt{\sigma}\overline{\sqrt{\sigma'}}a_n^{[\kappa]}\Omega_j\Omega_{j+\kappa} + \mathrm{i}\sqrt{-\sigma}\overline{\sqrt{\sigma'}}p_n^{[\kappa]}\Omega_{j+\kappa}\Big(j+\frac{1}{4}\Big)\\ &- \mathrm{i}\sqrt{\sigma}\overline{\sqrt{-\sigma'}}p_n^{[k]}\Omega_j\Big(j+\kappa+\frac{1}{4}\Big)\Big)\varepsilon^n.\end{aligned}$$

Formula (4.16b) follows by (4.17) and

$$j+\frac{1}{4}=\Omega_j^2\mathrm{sgn}^+(j),\quad j+\kappa+\frac{1}{4}=\Omega_{j+\kappa}^2\mathrm{sgn}^+(j+\kappa).$$

Similarly, we prove (4.16c). By (4.15) and (1.18),

$$\mathcal{B}_{1,n}^{[\kappa]}f_j^\sigma=-\mathrm{i}\,p_{n-1}^{[\kappa]}\frac{e^{\mathrm{i}(j+\kappa)x}}{2\sqrt{2\Omega_j}}\begin{bmatrix}\sqrt{-\sigma}\\ \sqrt{\sigma}\Omega_j\end{bmatrix}\delta\varepsilon^{n-1};$$

thus, by (4.1), we have

$$\mathtt{B}_{1,n\,j+\kappa,j}^{[\kappa]\,\sigma',\sigma}=\frac{p_{n-1}^{[\kappa]}}{4\sqrt{\Omega_j\Omega_{j+\kappa}}}\Big(\mathrm{i}\sqrt{-\sigma}\,\overline{\sqrt{\sigma'}}\Omega_{j+\kappa}-\mathrm{i}\sqrt{\sigma}\,\overline{\sqrt{-\sigma'}}\Omega_j\Big)\delta\varepsilon^{n-1},$$

and by (4.17), we conclude (4.16c). ∎

We now give some effective formulas to compute the residue term in (4.5).

Lemma 18. *Let $j_0,j_1,\dots,j_q\in\mathbb{N}_0$, $\sigma_0,\dots,\sigma_q=\pm$. Then, the coefficient $\mathtt{R}_{j_0,j_1,\dots,j_q}^{\sigma_0,\sigma_1,\dots,\sigma_q}$ in (4.5) fulfills the following.*

(I) *If for any $\iota=0,\dots,q$ one has $\omega_{j_\iota}^{\sigma_\iota}\neq\omega_*$ (no pole), then*

$$\mathtt{R}_{j_0,j_1,\dots,j_q}^{\sigma_0,\sigma_1,\dots,\sigma_q}=0. \tag{4.18}$$

(II) *If there is one and only one index $\iota\in\{0,\dots,q\}$ such that $\omega_{j_\iota}^{\sigma_\iota}=\omega_*$ (single pole), then*

$$\mathtt{R}_{j_0,j_1,\dots,j_q}^{\sigma_0,\sigma_1,\dots,\sigma_q}=\frac{1}{(\omega_{j_0}^{\sigma_0}-\omega_*)\cdots(\omega_{j_{\iota-1}}^{\sigma_{\iota-1}}-\omega_*)(\omega_{j_{\iota+1}}^{\sigma_{\iota+1}}-\omega_*)\cdots(\omega_{j_q}^{\sigma_q}-\omega_*)}. \tag{4.19}$$

(III) *If there are two and only two indices $\iota_1,\iota_2\in\{0,\dots,q\}$ such that $\omega_{\iota_1}^{\sigma_{\iota_1}}=\omega_{\iota_2}^{\sigma_{\iota_2}}=\omega_*$ (double pole), then*

$$\mathtt{R}_{j_0,j_1,\dots,j_q}^{\sigma_0,\sigma_1,\dots,\sigma_q}=-\Bigg(\sum_{\substack{m=0\\ m\neq\iota_1,\iota_2}}^{q}\frac{1}{\omega_{j_m}^{\sigma_m}-\omega_*}\Bigg)\Bigg(\prod_{\substack{k=0\\ k\neq\iota_1,\iota_2}}^{q}\frac{1}{\omega_{j_k}^{\sigma_k}-\omega_*}\Bigg). \tag{4.20}$$

(IV) *If $q\geq1$, $\omega_{j_0}^{\sigma_0}=\dots=\omega_{j_q}^{\sigma_q}=\omega_*$ (i.e., pole of order $q+1$), then $\mathtt{R}_{j_0,j_1,\dots,j_q}^{\sigma_0,\sigma_1,\dots,\sigma_q}=0$.*

Proof. Apply the residue theorem to formula (4.5). ∎

Remark 19. From (4.4), one checks that

$$
(\mathrm{Id}-P_0)\mathcal{P}\big[\mathcal{B}^{[\kappa_q]}_{\ell_q},\dots,\mathcal{B}^{[\kappa_1]}_{\ell_1}\big]f^\sigma_j
= \begin{cases}
\sum\limits_{\sigma_1,\dots,\sigma_{q-1}=\pm} \sigma_1\cdots\sigma_{q-1}\mathtt{B}^{[\kappa_1]\,\sigma_1,\sigma}_{\ell_1\ \ j_1,j}\mathtt{B}^{[\kappa_2]\,\sigma_2,\sigma_1}_{\ell_2\ \ j_2,j_1}\cdots\mathtt{B}^{[\kappa_q]\,+,\sigma_{q-1}}_{\ell_q\ \ 0,j_{q-1}}\mathtt{R}^{\sigma,\sigma_1,\dots,+}_{j,j_1,\dots,0} f^+_0, \\
\quad \text{if } j+\sum\limits_{i=1}^q \kappa_i=0, \\
-\sum\limits_{\sigma_1,\dots,\sigma_{q-1}=\pm} \sigma_1\cdots\sigma_{q-1}\mathtt{B}^{[\kappa_1]\,\sigma_1,\sigma}_{\ell_1\ \ j_1,j}\mathtt{B}^{[\kappa_2]\,\sigma_2,\sigma_1}_{\ell_2\ \ j_2,j_1}\cdots\mathtt{B}^{[\kappa_q]\,-,\sigma_{q-1}}_{\ell_q\ \ 2,j_{q-1}}\mathtt{R}^{\sigma,\sigma_1,\dots,-}_{j,j_1,\dots,2} f^-_2, \\
\quad \text{if } j+\sum\limits_{i=1}^q \kappa_i=2, \\
\mathcal{P}\big[\mathcal{B}^{[\kappa_q]}_{\ell_q},\dots,\mathcal{B}^{[\kappa_1]}_{\ell_1}\big]f^\sigma_j, \quad \text{otherwise},
\end{cases}
\tag{4.21a}
$$

and

$$
P_0\mathcal{P}\big[\mathcal{B}^{[\kappa_q]}_{\ell_q},\dots,\mathcal{B}^{[\kappa_1]}_{\ell_1}\big]f^\sigma_j
= \begin{cases}
-\sum\limits_{\sigma_1,\dots,\sigma_{q-1}=\pm} \sigma_1\cdots\sigma_{q-1}\mathtt{B}^{[\kappa_1]\,\sigma_1,\sigma}_{\ell_1\ \ j_1,j}\mathtt{B}^{[\kappa_2]\,\sigma_2,\sigma_1}_{\ell_2\ \ j_2,j_1}\cdots\mathtt{B}^{[\kappa_q]\,-,\sigma_{q-1}}_{\ell_q\ \ 0,j_{q-1}}\mathtt{R}^{\sigma,\sigma_1,\dots,-}_{j,j_1,\dots,0} f^-_0, \\
\quad \text{if } j+\sum\limits_{i=1}^q \kappa_i=0, \\
\sum\limits_{\sigma_1,\dots,\sigma_{q-1}=\pm} \sigma_1\cdots\sigma_{q-1}\mathtt{B}^{[\kappa_1]\,\sigma_1,\sigma}_{\ell_1\ \ j_1,j}\mathtt{B}^{[\kappa_2]\,\sigma_2,\sigma_1}_{\ell_2\ \ j_2,j_1}\cdots\mathtt{B}^{[\kappa_q]\,+,\sigma_{q-1}}_{\ell_q\ \ 2,j_{q-1}}\mathtt{R}^{\sigma,\sigma_1,\dots,+}_{j,j_1,\dots,2} f^+_2, \\
\quad \text{if } j+\sum\limits_{i=1}^q \kappa_i=2, \\
0, \quad \text{otherwise}.
\end{cases}
\tag{4.21b}
$$

5. Taylor expansion of $\mathtt{B}(\mu,\varepsilon)$

Let us assume from now on (1.23). The main result of this section is the following expansion of the matrix $\mathtt{B}(\mu,\varepsilon)$ in (2.4) which directly implies Theorems 4 and 1.

Proposition 20 (Expansion of $\mathtt{B}(\mu,\varepsilon)$). *The coefficients $\alpha(\mu,\varepsilon)$, $\beta(\mu,\varepsilon)$ and $\gamma(\mu,\varepsilon)$ of the 2×2 Hermitian matrix $\mathtt{B}(\mu,\varepsilon)$ in (2.4) admit the expansions at $(\mu,\varepsilon)=(\frac14+\delta,\varepsilon)$*

$$
\alpha\Big(\frac14+\delta,\varepsilon\Big)=\alpha\Big(\frac14+\delta,0\Big)+\widetilde{\alpha_1}\varepsilon+\alpha_2\varepsilon^2+\widetilde{\alpha_2}\delta\varepsilon+r(\varepsilon^3,\delta\varepsilon^2,\delta^2\varepsilon),
$$

$$\beta\left(\frac{1}{4}+\delta,\varepsilon\right)=\beta\left(\frac{1}{4}+\delta,0\right)+\widetilde{\beta_1}\varepsilon+\beta_1\varepsilon^2+\widetilde{\beta_2}\delta\varepsilon+\beta_2\delta\varepsilon^2+\widetilde{\beta}_3\varepsilon^3+\beta_3\varepsilon^4$$
$$+\widetilde{\beta_4}\delta^2\varepsilon+r(\varepsilon^5,\delta\varepsilon^3,\delta^2\varepsilon^2,\delta^3\varepsilon),$$
$$\gamma\left(\frac{1}{4}+\delta,\varepsilon\right)=\gamma\left(\frac{1}{4}+\delta,0\right)+\widetilde{\gamma_1}\varepsilon+\gamma_2\varepsilon^2+\widetilde{\gamma_2}\delta\varepsilon+r(\varepsilon^3,\delta\varepsilon^2,\delta^2\varepsilon), \tag{5.1}$$

where

$$\begin{aligned}
\widetilde{\alpha_1}\varepsilon &:= \left(\mathfrak{B}_{0,1}f_2^+,f_2^+\right)=0, & \widetilde{\alpha_2}\delta\varepsilon &:= \left(\mathfrak{B}_{1,1}f_2^+,f_2^+\right)=0,\\
\mathrm{i}\,\widetilde{\beta_1}\varepsilon &:= \left(\mathfrak{B}_{0,1}f_0^-,f_2^+\right)=0, & \mathrm{i}\,\widetilde{\beta}_3\varepsilon^3 &:= \left(\mathfrak{B}_{0,3}f_0^-,f_2^+\right)=0,\\
\widetilde{\gamma_1}\varepsilon &:= \left(\mathfrak{B}_{0,1}f_0^-,f_0^-\right)=0, & \widetilde{\gamma_2}\delta\varepsilon &:= \left(\mathfrak{B}_{1,1}f_0^-,f_0^-\right)=0,\\
\widetilde{\beta_2}\delta\varepsilon &:= \left(\mathfrak{B}_{1,1}f_0^-,f_2^+\right)=0, & \widetilde{\beta_4}\delta^2\varepsilon &:= \left(\mathfrak{B}_{2,1}f_0^-,f_2^+\right)=0
\end{aligned} \tag{5.2a}$$

and

$$\begin{aligned}
\alpha_2\varepsilon^2 &:= \left(\mathfrak{B}_{0,2}f_2^+,f_2^+\right)=\frac{9}{8}\varepsilon^2,\\
\gamma_2\varepsilon^2 &:= \left(\mathfrak{B}_{0,2}f_0^-,f_0^-\right)=\frac{1}{16}\varepsilon^2,\\
\mathrm{i}\,\beta_1\varepsilon^2 &:= \left(\mathfrak{B}_{0,2}f_0^-,f_2^+\right)=0,\\
\mathrm{i}\,\beta_2\delta\varepsilon^2 &:= \left(\mathfrak{B}_{1,2}f_0^-,f_2^+\right)=-\frac{1}{2\sqrt{3}}\delta\varepsilon^2,\\
\mathrm{i}\,\beta_3\varepsilon^4 &:= \left(\mathfrak{B}_{0,4}f_0^-,f_2^+\right)=-\frac{39\sqrt{3}}{512}\varepsilon^4.
\end{aligned} \tag{5.2b}$$

For any μ, we have $\beta(\mu,0)=0$ and $\alpha(\mu,0)=-\omega_2^+(\mu)$, $\gamma(\mu,0)=\omega_0^-(\mu)$ (cf. (2.6)).

Proof of Theorems 1 and 4. Theorem 4 follows by (5.1)–(5.2) and using that, by (1.17),

$$\begin{aligned}
\alpha\left(\frac{1}{4}+\delta,0\right) &= -\left(\frac{9}{4}+\delta-\Omega\left(\frac{9}{4}+\delta\right)\right)=-\omega_*+\alpha_1\delta+r(\delta^2),\\
\alpha_1 &:= \Omega'\left(\frac{9}{4}\right)-1\overset{(1.17)}{=}-\frac{2}{3},\\
\gamma\left(\frac{1}{4}+\delta,0\right) &= \frac{1}{4}+\delta+\Omega\left(\frac{1}{4}+\delta\right)=\omega_*+\gamma_1\delta+r(\delta^2),\\
\gamma_1 &:= \Omega'\left(\frac{1}{4}\right)+1\overset{(1.17)}{=}2.
\end{aligned}$$

Furthermore, Assumption 8 is fulfilled because $T_1:=\alpha_1+\gamma_1=\frac{4}{3}$. Then, we are in the first degenerate case of (2.24) since $\beta_1=0$, $T_2=\frac{19}{16}$ and $\beta_3-\beta_2\frac{T_2}{T_1}=\frac{37\sqrt{3}}{512}$. Finally, $\gamma_1-\alpha_1=\frac{8}{3}\neq 0$, then the abstract Theorem 11 applies and proves Theorem 1 together with (1.4). The expansions in (1.1) descend from (2.17) and (2.27). ■

The rest of the section is devoted to the proof of Proposition 20. We first prove (5.2a).

Lemma 21. *The coefficients* $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, $\widetilde{\beta}_1$, $\widetilde{\beta}_2$, $\widetilde{\beta}_3$, $\widetilde{\beta}_4$, $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$ *in* (5.2a) *vanish.*

Proof. Each scalar product in (5.2a) splits into two terms accordingly to the splitting of the operators in even and odd harmonics defined below (3.2). The term with odd harmonics vanishes because the difference between the harmonics of f_0^- and f_2^+ is 2. Thus, the coefficients are, respectively, given by $(\mathfrak{B}_{0,1}^{[\mathrm{ev}]} f_2^+, f_2^+)$, $(\mathfrak{B}_{1,1}^{[\mathrm{ev}]} f_2^+, f_2^+)$, $(\mathfrak{B}_{0,1}^{[\mathrm{ev}]} f_0^-, f_2^+)$, $(\mathfrak{B}_{1,1}^{[\mathrm{ev}]} f_0^-, f_2^+)$, $(\mathfrak{B}_{2,1}^{[\mathrm{ev}]} f_0^-, f_2^+)$, $(\mathfrak{B}_{0,3}^{[\mathrm{ev}]} f_0^-, f_2^+)$, $(\mathfrak{B}_{0,1}^{[\mathrm{ev}]} f_0^-, f_0^-)$, $(\mathfrak{B}_{1,1}^{[\mathrm{ev}]} f_0^-, f_0^-)$, and vanish because, by (3.9), the operators $\mathfrak{B}_{0,1}^{[\mathrm{ev}]}$, $\mathfrak{B}_{1,1}^{[\mathrm{ev}]}$, $\mathfrak{B}_{0,1}^{[\mathrm{ev}]}$, $\mathfrak{B}_{0,3}^{[\mathrm{ev}]}$, $\mathfrak{B}_{0,1}^{[\mathrm{ev}]}$, $\mathfrak{B}_{1,1}^{[\mathrm{ev}]}$ and $\mathfrak{B}_{2,1}^{[\mathrm{ev}]}$ are zero. ∎

We now compute the remaining coefficients in (5.2b). We give their algebraic expression in terms of the entanglement coefficients, and their numerical values exploiting Mathematica. The code can be found at the link in footnote[4]. We start with the quadratic terms.

Computation of α_2. In view of (5.2b), (3.8a), since $P_1^{[\pm 1]} = \mathcal{P}[\mathcal{B}_1^{[\pm 1]}]$, where by (4.15) $\mathcal{B}_{0,1}^{[\kappa]}$ (and similarly $P_{0,1}^{[\kappa]}$) is nonzero only for $\kappa = \pm 1$, one has

$$
\begin{aligned}
\alpha_2 \varepsilon^2 &= \big(\mathfrak{B}_{0,2} f_2^+, f_2^+\big) \\
&= \big(\mathcal{B}_{0,2} f_2^+, f_2^+\big) + \frac{1}{2}\big(\mathcal{B}_{0,1} P_{0,1} f_2^+, f_2^+\big) + \frac{1}{2}\big(\mathcal{B}_{0,1} f_2^+, P_{0,1} f_2^+\big) \\
&= \underbrace{\big(\mathcal{B}_{0,2}^{[0]} f_2^+, f_2^+\big)}_{=\mathrm{Aa}} + \frac{1}{2}\underbrace{\big(\mathcal{B}_{0,1}^{[+1]} P_{0,1}^{[-1]} f_2^+, f_2^+\big)}_{=:\mathrm{Ab}} + \frac{1}{2}\underbrace{\big(\mathcal{B}_{0,1}^{[-1]} P_{0,1}^{[+1]} f_2^+, f_2^+\big)}_{=:\mathrm{Ac}} \\
&\quad + \frac{1}{2}\underbrace{\big(\mathcal{B}_{0,1}^{[+1]} f_2^+, P_{0,1}^{[+1]} f_2^+\big)}_{\overset{(4.11)}{=}\mathrm{Ac}} + \frac{1}{2}\underbrace{\big(\mathcal{B}_{0,1}^{[-1]} f_2^+, P_{0,1}^{[-1]} f_2^+\big)}_{\overset{(4.11)}{=}\mathrm{Ab}} = \mathrm{Aa} + \mathrm{Ab} + \mathrm{Ac}
\end{aligned}
$$

(for the last two terms we also used (3.2)). One then has, also by using (4.2),

$$
\begin{aligned}
\mathrm{Aa} &\overset{(4.1)}{=} \mathrm{B}_{0,2\,2,2}^{[0]\ +,+} = -\frac{15}{8}\varepsilon^2; \\
\mathrm{Ab} &= \big(\mathcal{B}_{0,1}^{[+1]} P_{0,1}^{[-1]} f_2^+, f_2^+\big) \overset{(4.11)}{=} \big(\mathcal{B}_{0,1}^{[-1]} f_2^+, P_{0,1}^{[-1]} f_2^+\big) \\
&\overset{(4.7),(4.19)}{=} \sum_{\sigma=\pm} \frac{\sigma |\mathrm{B}_{0,1\ 1,2}^{[-1]\sigma,+}|^2}{\omega_1^\sigma - \omega_*} = -\frac{15}{8}\varepsilon^2; \\
\mathrm{Ac} &= \big(\mathcal{B}_{0,1}^{[-1]} P_{0,1}^{[+1]} f_2^+, f_2^+\big) \overset{(4.11)}{=} \big(\mathcal{B}_{0,1}^{[+1]} f_2^+, P_{0,1}^{[+1]} f_2^+\big) \\
&\overset{(4.7),(4.19)}{=} \sum_{\sigma=\pm} \frac{\sigma |\mathrm{B}_{0,1\ 3,2}^{[+1]\sigma,+}|^2}{\omega_3^\sigma - \omega_*} = \frac{39}{8}\varepsilon^2.
\end{aligned}
\tag{5.3}
$$

We conclude that $\alpha_2 = \frac{9}{8}$ as claimed in (5.2b).

[4] https://git-scm.sissa.it/amaspero/first-isola-of-modulational-instability-of-stokes-waves-\protect\@normalcr\relaxin-deep-water

Computation of β_1. In view of (5.2b), (3.8a), since $P_1^{[\pm 1]} = \mathcal{P}[\mathcal{B}_1^{[\pm 1]}]$, where by (4.15) $\mathcal{B}_{0,1}^{[\kappa]}$ (and similarly $P_{0,1}^{[\kappa]}$) is nonzero only for $\kappa = \pm 1$, one has

$$\begin{aligned} \mathrm{i}\,\beta_1 \varepsilon^2 &= \big(\mathfrak{B}_{0,2} f_0^-,\, f_2^+\big) \\ &= \underbrace{\big(\mathcal{B}_{0,2}^{[+2]} f_0^-,\, f_2^+\big)}_{=:\mathrm{Ba}} + \frac{1}{2}\underbrace{\big(\mathcal{B}_{0,1}^{[+1]} P_{0,1}^{[+1]} f_0^-,\, f_2^+\big)}_{=:\mathrm{Bb}} + \frac{1}{2}\underbrace{\big(\mathcal{B}_{0,1}^{[+1]} f_0^-,\, P_{0,1}^{[-1]} f_2^+\big)}_{\overset{(4.11)}{=}\mathrm{Bb}}, \end{aligned}$$

(for the last term we also used (3.2)). We have

$$\begin{aligned} \mathrm{Ba} &= \big(\mathcal{B}_{0,2}^{[+2]} f_0^-,\, f_2^+\big) \overset{(4.1)}{=} \mathrm{B}_{0,2\ \ 2,0}^{[+2]\,+,-} \overset{(4.16)}{=} 0 \\ \mathrm{Bb} &= \big(\mathcal{B}_{0,1}^{[+1]} P_{0,1}^{[+1]} f_0^-,\, f_2^+\big) \overset{(4.11)}{=} \big(\mathcal{B}_{0,1}^{[+1]} f_0^-,\, P_{0,1}^{[-1]} f_2^+\big) \\ &\overset{(4.7)}{=} -\frac{\mathrm{B}_{0,1\ \ 1,0}^{[+1]\,-,-}\mathrm{B}_{0,1\ \ 2,1}^{[+1]\,+,-}}{\omega_1^- - \omega_*} + \frac{\mathrm{B}_{0,1\ \ 1,0}^{[+1]\,+,-}\mathrm{B}_{0,1\ \ 2,1}^{[+1]\,+,+}}{\omega_1^+ - \omega_*} \overset{(4.16)}{=} 0. \end{aligned} \tag{5.4}$$

We conclude that $\beta_1 = 0$ as claimed in (5.2b).

Remark 22. In the finite depth case the coefficient $\beta_1 \neq 0$. In infinite depth the degeneracy $\beta_1 = 0$ descends from peculiar identities of the expansion of the Stokes wave. These are coherent with the structure of the completely integrable quartic Birkhoff normal form of the pure gravity water waves in deep water proved in [5, 11, 13, 22], see also [21, 34].

Computation of γ_2. Proceeding in a similar way as for the computation of α_2, we get

$$\begin{aligned} \gamma_2 \varepsilon^2 &= \big(\mathfrak{B}_{0,2} f_0^-, f_0^-\big) = \underbrace{\big(\mathcal{B}_{0,2}^{[0]} f_0^-, f_0^-\big)}_{=:\mathrm{Ca}} \\ &+ \frac{1}{2}\underbrace{\big(\mathcal{B}_{0,1}^{[+1]} P_{0,1}^{[-1]} f_0^-, f_0^-\big)}_{=:\mathrm{Cb}} + \frac{1}{2}\underbrace{\big(\mathcal{B}_{0,1}^{[-1]} P_{0,1}^{[+1]} f_0^-, f_0^-\big)}_{=:\mathrm{Cc}} \\ &+ \frac{1}{2}\underbrace{\big(\mathcal{B}_{0,1}^{[+1]} f_0^-, P_{0,1}^{[+1]} f_0^-\big)}_{\overset{(4.11)}{=}\mathrm{Cc}} + \frac{1}{2}\underbrace{\big(\mathcal{B}_{0,1}^{[-1]} f_0^-, P_{0,1}^{[-1]} f_0^-\big)}_{\overset{(4.11)}{=}\mathrm{Cb}} = \mathrm{Ca} + \mathrm{Cb} + \mathrm{Cc}, \end{aligned}$$

where

$$\begin{aligned} \mathrm{Ca} &= \big(\mathcal{B}_{0,2}^{[0]} f_0^-, f_0^-\big) = \mathrm{B}_{0,2\,0,0}^{[0]\,-,-} = \frac{7}{8}\varepsilon^2 \\ \mathrm{Cb} &= \big(\mathcal{B}_{0,1}^{[+1]} P_{0,1}^{[-1]} f_0^-, f_0^-\big) = \big(\mathcal{B}_{0,1}^{[-1]} f_0^-, P_{0,1}^{[-1]} f_0^-\big) \\ &= \frac{|\mathrm{B}_{0,1\ -1,0}^{[-1]\,+,-}|^2}{\omega_{-1}^+ - \omega_*} - \frac{|\mathrm{B}_{0,1\ -1,0}^{[-1]\,-,-}|^2}{\omega_{-1}^- - \omega_*} = -\frac{3}{16}\varepsilon^2 \\ \mathrm{Cc} &= \big(\mathcal{B}_{0,1}^{[-1]} P_{0,1}^{[+1]} f_0^-, f_0^-\big) = \big(\mathcal{B}_{0,1}^{[+1]} f_0^-, P_{0,1}^{[+1]} f_0^-\big) \\ &= \frac{|\mathrm{B}_{0,1\ \ 1,0}^{[+1]\,+,-}|^2}{\omega_1^+ - \omega_*} - \frac{|\mathrm{B}_{0,1\ \ 1,0}^{[+1]\,-,-}|^2}{\omega_1^- - \omega_*} = -\frac{5}{8}\varepsilon^2. \end{aligned} \tag{5.5}$$

We conclude that $\gamma_2 = \frac{1}{16}$ as claimed in (5.2b).

We now proceed with the cubic term γ_2.

Computation of β_2. We will use the following identities.

Lemma 23. *For any* $(j,\sigma) \in \{(0,-),(2,+)\}$ *we have*

$$P_0\mathcal{P}\big[\mathcal{B}_{0,2}^{[-2]}\big]f_2^+ = P_0\mathcal{P}\big[\mathcal{B}_{0,2}^{[0]}\big]f_2^+ = P_0\mathcal{P}\big[\mathcal{B}_{0,2}^{[+2]}\big]f_0^- = P_0\mathcal{P}\big[\mathcal{B}_{0,2}^{[0]}\big]f_0^- = 0, \quad (5.6)$$

$$P_0\mathcal{P}\big[\mathcal{B}_{1,1}^{[\pm1]}\big]f_j^\sigma = P_0\mathcal{P}\big[\mathcal{B}_{0,1}^{[\pm1]},\mathcal{B}_{1,0}^{[0]}\big]f_j^\sigma = P_0\mathcal{P}\big[\mathcal{B}_{1,0}^{[0]},\mathcal{B}_{0,1}^{[\pm1]}\big]f_j^\sigma = 0. \quad (5.7)$$

Proof. Identities (5.6) follow from formula (4.21b) with $q=1$ and $(j,\sigma)=(2,+)$ or $(0,-)$, since in both cases, by (4.5), the residue coefficients $\mathtt{R}_{j,0}^{\sigma,-}$ and $\mathtt{R}_{j,2}^{\sigma,+}$ vanish. Identities (5.7) follow from (4.21b) too, with $j+\kappa_0+\cdots+k_q \neq 0,2$. ■

We claim that the coefficient β_2 is a linear combination of the following terms:

$$\begin{aligned}
\Theta &:= \big(\mathcal{B}_{1,2}^{[+2]}f_0^-,\ f_2^+\big), \\
\mathrm{I} &:= \big(\mathcal{B}_{1,1}^{[+1]}\mathcal{P}[\mathcal{B}_{0,1}^{[+1]}]f_0^-,\ f_2^+\big) = \big(\mathcal{B}_{0,1}^{[+1]}f_0^-,\ \mathcal{P}[\mathcal{B}_{1,1}^{[-1]}]f_2^+\big), \\
\mathrm{II} &:= \big(\mathcal{B}_{0,2}^{[+2]}\mathcal{P}[\mathcal{B}_{1,0}^{[0]}]f_0^-,\ f_2^+\big) = \big(\mathcal{B}_{1,0}^{[0]}f_0^-,\ (\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_{0,2}^{[-2]}]f_2^+\big), \\
\mathrm{III\,a} &:= \big(\mathcal{B}_{0,1}^{[+1]}\mathcal{P}[\mathcal{B}_{1,1}^{[+1]}]f_0^-,\ f_2^+\big) = \big(\mathcal{B}_{1,1}^{[+1]}f_0^-,\ \mathcal{P}[\mathcal{B}_{0,1}^{[-1]}]f_2^+\big), \\
\mathrm{III\,b} &:= \big(\mathcal{B}_{0,1}^{[+1]}\mathcal{P}[\mathcal{B}_{1,0}^{[0]},\mathcal{B}_{0,1}^{[+1]}]f_0^-,\ f_2^+\big) = \big(\mathcal{B}_{0,1}^{[+1]}f_0^-,\ \mathcal{P}[\mathcal{B}_{1,0}^{[0]},\mathcal{B}_{0,1}^{[-1]}]f_2^+\big), \\
\mathrm{III\,c} &:= \big(\mathcal{B}_{1,0}^{[0]}f_0^-,\ (\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_{0,1}^{[-1]},\mathcal{B}_{0,1}^{[-1]}]f_2^+\big), \\
\mathrm{III\,d} &:= \big(\mathcal{B}_{0,1}^{[+1]}\mathcal{P}[\mathcal{B}_{0,1}^{[+1]},\mathcal{B}_{1,0}^{[0]}]f_0^-,\ f_2^+\big) = \mathrm{III\,c} + \big(\mathcal{B}_{1,0}^{[0]}f_0^-,\ P_0\mathcal{P}[\mathcal{B}_{0,1}^{[-1]},\mathcal{B}_{0,1}^{[-1]}]f_2^+\big), \\
\mathrm{IV\,a} &:= \big(\mathcal{B}_{1,0}^{[0]}(\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_{0,2}^{[+2]}]f_0^-,\ f_2^+\big) = \big(\mathcal{B}_{0,2}^{[+2]}f_0^-,\ \mathcal{P}[\mathcal{B}_{1,0}^{[0]}]f_2^+\big), \\
\mathrm{IV\,b} &:= \big(\mathcal{B}_{1,0}^{[0]}(\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_{0,1}^{[+1]},\mathcal{B}_{0,1}^{[+1]}]f_0^-,\ f_2^+\big), \\
\mathrm{IV\,c} &:= \big(\mathcal{B}_{0,1}^{[+1]}f_0^-,\ \mathcal{P}[\mathcal{B}_{0,1}^{[-1]},\mathcal{B}_{1,0}^{[0]}]f_2^+\big) = \mathrm{IV\,b} + \big(\mathcal{B}_{1,0}^{[0]}P_0\mathcal{P}[\mathcal{B}_{0,1}^{[+1]},\mathcal{B}_{0,1}^{[+1]}]f_0^-,\ f_2^+\big),
\end{aligned} \quad (5.8)$$

where the identities come from (4.11) and (5.6). The claim follows by (3.8b), (3.9), (3.4), (3.7) together with

$$\begin{aligned}
&P_{0,1}^{[+1]} = \mathcal{P}\big[\mathcal{B}_{0,1}^{[+1]}\big], \quad P_{0,2}^{[\pm2]} = \mathcal{P}\big[\mathcal{B}_{0,2}^{[\pm2]}\big] + \mathcal{P}\big[\mathcal{B}_{0,1}^{[\pm1]},\mathcal{B}_{0,1}^{[\pm1]}\big], \\
&P_{1,1}^{[\pm1]} = \mathcal{P}\big[\mathcal{B}_{1,1}^{[\pm1]}\big] + \mathcal{P}\big[\mathcal{B}_{0,1}^{[\pm1]},\mathcal{B}_{1,0}^{[0]}\big] + \mathcal{P}\big[\mathcal{B}_{1,0}^{[0]},\mathcal{B}_{0,1}^{[\pm1]}\big],
\end{aligned} \quad (5.9)$$

whence, setting $\widehat{P}_2 := (\mathrm{Id}-P_0)P_2P_0$, one has

$$\begin{aligned}
\mathrm{i}\,\beta_2\delta\varepsilon^2 &= \big(\mathfrak{B}_{1,2}^{[2]}f_0^-,\ f_2^+\big) \\
&= \underbrace{\big(\mathcal{B}_{1,2}^{[+2]}f_0^-,\ f_2^+\big)}_{=\Theta} + \frac12\underbrace{\big(\mathcal{B}_{1,1}^{[+1]}P_{0,1}^{[+1]}f_0^-,\ f_2^+\big)}_{=\mathrm{I}} + \frac12\underbrace{\big(\mathcal{B}_{1,1}^{[+1]}f_0^-,\ P_{0,1}^{[-1]}f_2^+\big)}_{=\mathrm{III\,a}} \\
&\quad + \frac12\underbrace{\big(\mathcal{B}_{0,2}^{[+2]}P_{1,0}^{[0]}f_0^-,\ f_2^+\big)}_{=\mathrm{II}} + \frac12\underbrace{\big(\mathcal{B}_{0,2}^{[+2]}f_0^-,\ P_{1,0}^{[0]}f_2^+\big)}_{=\mathrm{IV\,a}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\underbrace{\big(\mathcal{B}_{0,1}^{[+1]}\widehat{P}_{1,1}^{[+1]}f_0^-,\ f_2^+\big)}_{=\text{III a}+\text{III b}+\text{III d by (5.9), (5.7)}} \\
&+\frac{1}{2}\underbrace{\big(\mathcal{B}_{0,1}^{[+1]}f_0^-,\ \widehat{P}_{1,1}^{[-1]}f_2^+\big)}_{=\text{I}+\text{III b}+\text{IV c by (5.9), (5.7)}}+\frac{1}{2}\underbrace{\big(\mathcal{B}_{1,0}^{[0]}\widehat{P}_{0,2}^{[+2]}f_0^-,\ f_2^+\big)}_{=\text{IV a}+\text{IV b}} \\
&+\frac{1}{2}\underbrace{\big(\mathcal{B}_{1,0}^{[0]}f_0^-,\ \widehat{P}_{0,2}^{[-2]}f_2^+\big)}_{=\text{II}+\text{III c by (5.9)}} \\
&=\Theta+\text{I}+\text{II}+\text{III a}+\text{III b}+\frac{1}{2}(\text{III c}+\text{III d})+\text{IV a}+\frac{1}{2}(\text{IV b}+\text{IV c}). \qquad (5.10)
\end{aligned}
$$

By (5.8), (4.7), Lemma 18 and finally (4.16), we obtain

$$
\begin{aligned}
\Theta &= \mathtt{B}_{1,2\ 2,0}^{[+2]+,-} = -\frac{\mathrm{i}}{\sqrt{3}}\delta\varepsilon^2, \\
\text{I} &= \frac{\mathtt{B}_{0,1\ 1,0}^{[+1]+,-}\mathtt{B}_{1,1\ 2,1}^{[+1]+,+}}{\omega_1^+-\omega_*}-\frac{\mathtt{B}_{0,1\ 1,0}^{[+1]-,-}\mathtt{B}_{1,1\ 2,1}^{[+1]+,-}}{\omega_1^--\omega_*}=\frac{\mathrm{i}}{4\sqrt{3}}\delta\varepsilon^2, \\
\text{II} &= \frac{\mathtt{B}_{1,0\,0,0}^{[0]\ +,-}\mathtt{B}_{0,2\ 2,0}^{[+2]+,+}}{\omega_0^+-\omega_*}=\mathrm{i}\,\frac{\sqrt{3}}{4}\delta\varepsilon^2, \\
\text{III a} &= \frac{\mathtt{B}_{1,1\ 1,0}^{[+1]+,-}\mathtt{B}_{0,1\ 2,1}^{[+1]+,+}}{\omega_1^+-\omega_*}-\frac{\mathtt{B}_{1,1\ 1,0}^{[+1]-,-}\mathtt{B}_{0,1\ 2,1}^{[+1]+,-}}{\omega_1^--\omega_*}=\mathrm{i}\,\frac{\sqrt{3}}{4}\delta\varepsilon^2, \\
\text{III b} &= \frac{\mathtt{B}_{0,1\ 1,0}^{[+1]-,-}\mathtt{B}_{1,0\,1,1}^{[0]\ -,-}\mathtt{B}_{0,1\ 2,1}^{[+1]+,-}}{(\omega_1^--\omega_*)^2}-\frac{\mathtt{B}_{0,1\ 1,0}^{[+1]-,-}\mathtt{B}_{1,0\,1,1}^{[0]\ +,-}\mathtt{B}_{0,1\ 2,1}^{[+1]+,+}}{(\omega_1^+-\omega_*)(\omega_1^--\omega_*)} \\
&\quad+\frac{\mathtt{B}_{0,1\ 1,0}^{[+1]+,-}\mathtt{B}_{1,0\,1,1}^{[0]\ +,+}\mathtt{B}_{0,1\ 2,1}^{[+1]+,+}}{(\omega_1^+-\omega_*)^2}-\frac{\mathtt{B}_{0,1\ 1,0}^{[+1]+,-}\mathtt{B}_{1,0\,1,1}^{[0]\ -,+}\mathtt{B}_{0,1\ 2,1}^{[+1]+,-}}{(\omega_1^+-\omega_*)(\omega_1^--\omega_*)}=\mathrm{i}\,\frac{\sqrt{3}}{2}\delta\varepsilon^2, \\
\text{III c} &= \frac{\mathtt{B}_{1,0\,0,0}^{[0]\ +,-}\mathtt{B}_{0,1\ 1,0}^{[+1]+,+}\mathtt{B}_{0,1\ 2,1}^{[+1]+,+}}{(\omega_0^+-\omega_*)(\omega_1^+-\omega_*)}-\frac{\mathtt{B}_{1,0\,0,0}^{[0]\ +,-}\mathtt{B}_{0,1\ 1,0}^{[+1]-,+}\mathtt{B}_{0,1\ 2,1}^{[+1]+,-}}{(\omega_0^+-\omega_*)(\omega_1^--\omega_*)}=-\mathrm{i}\,\frac{5\sqrt{3}}{16}\delta\varepsilon^2, \\
\text{III d} &= \text{III c}+\frac{\mathtt{B}_{1,0\,0,0}^{[0]\ -,-}\mathtt{B}_{0,1\ 1,0}^{[+1]+,-}\mathtt{B}_{0,1\ 2,1}^{[+1]+,+}}{(\omega_1^+-\omega_*)^2}-\frac{\mathtt{B}_{1,0\,0,0}^{[0]\ -,-}\mathtt{B}_{0,1\ 1,0}^{[+1]-,-}\mathtt{B}_{0,1\ 2,1}^{[+1]+,-}}{(\omega_1^--\omega_*)^2}=-\mathrm{i}\,\frac{15\sqrt{3}}{16}\delta\varepsilon^2, \\
\text{IV a} &= -\frac{\mathtt{B}_{0,2\ 2,0}^{[+2]-,-}\mathtt{B}_{1,0\,2,2}^{[0]\ +,-}}{\omega_2^--\omega_*}=-\mathrm{i}\,\frac{1}{4\sqrt{3}}\delta\varepsilon^2, \\
\text{IV b} &= \frac{\mathtt{B}_{0,1\ 1,0}^{[+1]-,-}\mathtt{B}_{1\ \ \ 2,1}^{[+1]-,-}\mathtt{B}_{1,0\,2,2}^{[0]\ +,-}}{(\omega_1^--\omega_*)(\omega_2^--\omega_*)}-\frac{\mathtt{B}_{0,1\ 1,0}^{[+1]+,-}\mathtt{B}_{0,1\ 2,1}^{[+1]-,+}\mathtt{B}_{1,0\,2,2}^{[0]\ +,-}}{(\omega_1^+-\omega_*)(\omega_2^--\omega_*)}=-\mathrm{i}\,\frac{5}{16\sqrt{3}}\delta\varepsilon^2, \\
\text{IV c} &= \text{IV b}+\frac{\mathtt{B}_{0,1\ 1,0}^{[+1]-,-}\mathtt{B}_{0,1\ 2,1}^{[+1]+,-}\mathtt{B}_{1,0\,2,2}^{[0]\ +,+}}{(\omega_1^--\omega_*)^2}-\frac{\mathtt{B}_{0,1\ 1,0}^{[+1]+,-}\mathtt{B}_{0,1\ 2,1}^{[+1]+,+}\mathtt{B}_{1,0\,2,2}^{[0]\ +,+}}{(\omega_1^+-\omega_*)^2}=-\mathrm{i}\,\frac{5\sqrt{3}}{16}\delta\varepsilon^2.
\end{aligned}
$$

By inserting the resulting values in (5.10) we obtain that $\beta_2=-\frac{1}{2\sqrt{3}}$ as claimed in (5.2b).

We now conclude with the quartic term β_3.

Computation of $\boldsymbol{\beta_3}$. In this part, to simplify notation, we will simply denote

$$\mathcal{B}_{0,n} \equiv \mathcal{B}_n, \quad P_{0,n} \equiv P_n, \quad \mathsf{B}^{[\kappa]\,\sigma,\sigma'}_{0,n\,j,j'} \equiv \mathsf{B}^{[\kappa]\,\sigma,\sigma'}_{n\;\;j,j'}, \tag{5.11}$$

since all the operators that enter in the computation of β_3 have this form. By (3.8c), we have

$$\begin{aligned} \mathrm{i}\,\beta_3\varepsilon^4 = \big(\mathfrak{B}_{0,4} f_0^-, f_2^+\big) &= \big(\mathcal{B}_4 f_0^-, f_2^+\big) + \frac{1}{2}\big(\mathcal{B}_3 P_1 f_0^-, f_2^+\big) + \frac{1}{2}\big(\mathcal{B}_3 f_0^-, P_1 f_2^+\big) \\ &+ \frac{1}{2}\big(\mathcal{B}_2 \widehat{P}_2 f_0^-, f_2^+\big) + \frac{1}{2}\big(\mathcal{B}_2 f_0^-, \widehat{P}_2 f_2^+\big) + \frac{1}{2}\big(\mathcal{B}_1 \widehat{P}_3 f_0^-, f_2^+\big) \\ &+ \frac{1}{2}\big(\mathcal{B}_1 f_0^-, \widehat{P}_3 f_2^+\big) - \frac{1}{2}\big(\mathcal{B}_1 P_1 P_0 P_2 f_0^-, f_2^+\big) - \frac{1}{2}\big(\mathcal{B}_1 f_0^-, P_1 P_0 P_2 f_2^+\big), \end{aligned} \tag{5.12}$$

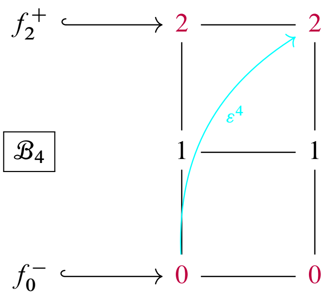
where, for brevity, we denote

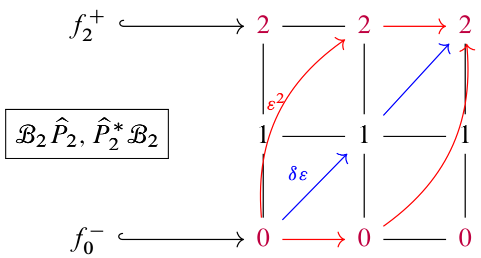
$$\widehat{P}_2 := (\mathrm{Id} - P_0) P_2 P_0, \quad \text{and} \quad \widehat{P}_3 := (\mathrm{Id} - P_0) P_3 P_0. \tag{5.13}$$

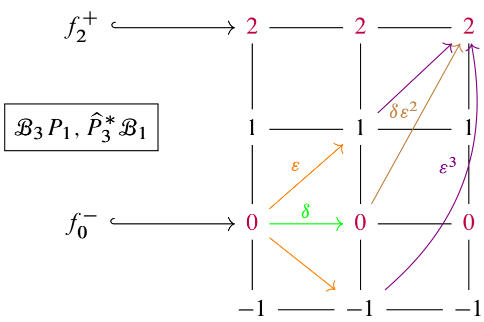
Each scalar product in (5.12) is a sum of terms of the form (recall (3.1))

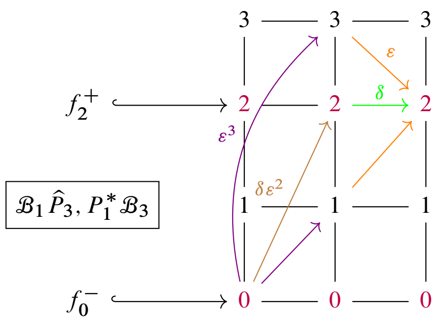
$$\Big(\mathcal{B}^{[\kappa_0]}_{\ell_0}\, \mathcal{P}\big[\mathcal{B}^{[\kappa_1]}_{\ell_1}, \ldots, \mathcal{B}^{[\kappa_p]}_{\ell_p}\big] f_0^-, \mathcal{P}\big[\mathcal{B}^{[\kappa_{p+1}]}_{\ell_1}, \ldots, \mathcal{B}^{[\kappa_{p+q}]}_{\ell_{p+q}}\big] f_2^+\Big),$$

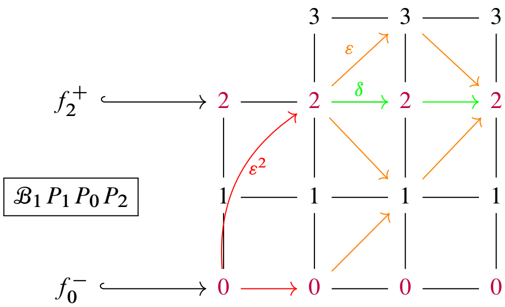
which vanishes unless $\kappa_0 + \kappa_1 + \cdots + \kappa_p - \kappa_{p+1} - \kappa_{p+q} = 2$. To compute the non zero scalar products, we represent the action of the nine operators in (5.12)—i.e., $\mathcal{B}_4$, $\mathcal{B}_3 P_1$, $P_1^* \mathcal{B}_3$, $\mathcal{B}_2 \widehat{P}_2$, $\widehat{P}_2^* \mathcal{B}_2$, $\mathcal{B}_2 \widehat{P}_2$, $\widehat{P}_2^* \mathcal{B}_2$, $\mathcal{B}_1 P_1 P_0 P_2$, $P_2^* P_0^* P_1^* \mathcal{B}_1$—on the eigenfunctions f_2^+, f_0^- in (1.18) through the following graphs, where we only highlight the paths connecting f_0^- to f_2^+, with the corresponding order in $\delta^i \varepsilon^j$ (recall formulas (3.3), (3.5), and also (3.4), (3.7), (3.2)):

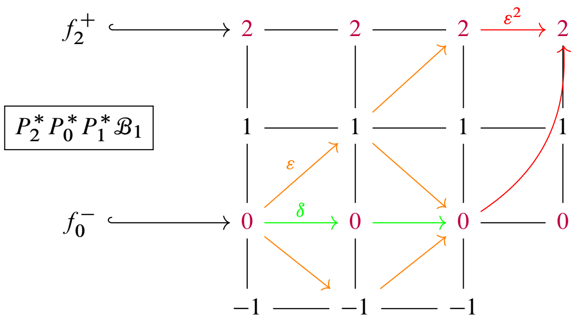












In particular, by (3.5), the only nontrivial terms of order ε^4 involve, besides (5.9), the terms

$$
\begin{aligned}
P_2^{[0]} &= \mathcal{P}\big[\mathcal{B}_2^{[0]}\big] + \mathcal{P}\big[\mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[-1]}\big] + \mathcal{P}\big[\mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[+1]}\big], \\
P_3^{[\pm 1]} &= \mathcal{P}\big[\mathcal{B}_1^{[\mp 1]}, \mathcal{B}_1^{[\pm 1]}, \mathcal{B}_1^{[\pm 1]}\big] + \mathcal{P}\big[\mathcal{B}_1^{[\pm 1]}, \mathcal{B}_1^{[\mp 1]}, \mathcal{B}_1^{[\pm 1]}\big] \\
&\quad + \mathcal{P}\big[\mathcal{B}_1^{[\pm 1]}, \mathcal{B}_1^{[\pm 1]}, \mathcal{B}_1^{[\mp 1]}\big] \\
&\quad + \mathcal{P}\big[\mathcal{B}_2^{[0]}, \mathcal{B}_1^{[\pm 1]}\big] + \mathcal{P}\big[\mathcal{B}_1^{[\pm 1]}, \mathcal{B}_2^{[0]}\big] + \mathcal{P}\big[\mathcal{B}_2^{[\pm 2]}, \mathcal{B}_1^{[\mp 1]}\big] \qquad (5.14) \\
&\quad + \mathcal{P}\big[\mathcal{B}_1^{[\mp 1]}, \mathcal{B}_2^{[\pm 2]}\big] + \mathcal{P}\big[\mathcal{B}_3^{[\pm 1]}\big], \\
P_3^{[\pm 3]} &= \mathcal{P}\big[\mathcal{B}_3^{[\pm 3]}\big] + \mathcal{P}\big[\mathcal{B}_2^{[\pm 2]}, \mathcal{B}_1^{[\pm 1]}\big] + \mathcal{P}\big[\mathcal{B}_1^{[\pm 1]}, \mathcal{B}_2^{[\pm 2]}\big] \\
&\quad + \mathcal{P}\big[\mathcal{B}_1^{[\pm 1]}, \mathcal{B}_1^{[\pm 1]}, \mathcal{B}_1^{[\pm 1]}\big].
\end{aligned}
$$

Indeed, using in (5.15)–(5.16) that $P_0 P_3^{[\pm 1]} P_0 = P_0 P_3^{[\pm 3]} P_0 = 0$,

$$
\begin{aligned}
\mathrm{i}\,\varepsilon^4 \beta_3 := {} & \underbrace{\big(\mathcal{B}_4^{[+2]} f_0^-, f_2^+\big)}_{=\Theta} + \frac{1}{2} \underbrace{\big(\mathcal{B}_3^{[+1]} P_1^{[+1]} f_0^-, f_2^+\big)}_{=\mathrm{I}} \\
& + \frac{1}{2} \underbrace{\big(\mathcal{B}_3^{[+3]} P_1^{[-1]} f_0^-, f_2^+\big)}_{=\mathrm{II}} \\
& + \frac{1}{2} \underbrace{\big(\mathcal{B}_3^{[+1]} f_0^-,\ P_1^{[-1]} f_2^+\big)}_{=\mathrm{V\,a}} + \frac{1}{2} \underbrace{\big(\mathcal{B}_3^{[+3]} f_0^-,\ P_1^{[+1]} f_2^+\big)}_{=\mathrm{VI\,a}} \\
& + \frac{1}{2} \underbrace{\big(\mathcal{B}_2^{[0]} (\mathrm{Id} - P_0) P_2^{[+2]} f_0^-,\ f_2^+\big)}_{=\mathrm{III\,a + III\,b}\ (\text{by (5.9)})} + \frac{1}{2} \underbrace{\big(\mathcal{B}_2^{[+2]} (\mathrm{Id} - P_0) P_2^{[0]} f_0^-,\ f_2^+\big)}_{=\mathrm{IV\,a + IV\,b + IV\,c}\ (\text{by (5.14)})} \\
& + \frac{1}{2} \underbrace{\big(\mathcal{B}_2^{[0]} f_0^-,\ (\mathrm{Id} - P_0) P_2^{[-2]} f_2^+\big)}_{=\mathrm{IV\,a + V\,c}\ (\text{by (5.9)})} + \frac{1}{2} \underbrace{\big(\mathcal{B}_2^{[+2]} f_0^-,\ (\mathrm{Id} - P_0) P_2^{[0]} f_2^+\big)}_{=\mathrm{III\,a + V\,e + VI\,c}\ (\text{by (5.14)})} \\
& + \frac{1}{2} \underbrace{\big(\mathcal{B}_1^{[+1]} P_3^{[+1]} f_0^-, f_2^+\big)}_{=\mathrm{V\,a + V\,b} + \widehat{\mathrm{V}}\mathrm{c} + \mathrm{V\,d} + \widehat{\mathrm{V}}\mathrm{e} + \mathrm{V\,f + V\,g + V\,h}\ (\text{by (5.14)})} + \frac{1}{2} \underbrace{\big(\mathcal{B}_1^{[-1]} P_3^{[+3]} f_0^-, f_2^+\big)}_{=\mathrm{VI\,a + VI\,b} + \widehat{\mathrm{VI\,c}} + \mathrm{VI\,d}\ (\text{by (5.14)})}
\end{aligned} \tag{5.15}
$$

$$
+ \frac{1}{2} \underbrace{\big(\mathcal{B}_1^{[+1]} f_0^-,\ P_3^{[-1]} f_2^+\big)}_{=\mathrm{I + V\,b} + \widehat{\mathrm{III\,b}} + \mathrm{VI\,b} + \widehat{\mathrm{IV\,c}} + \mathrm{V\,g + V\,f + VI\,d}\ (\text{by (5.14)})} + \frac{1}{2} \underbrace{\big(\mathcal{B}_1^{[-1]} f_0^-,\ P_3^{[-3]} f_2^+\big)}_{=\mathrm{II + V\,d} + \widehat{\mathrm{IV\,b}} + \mathrm{V\,h}\ (\text{by (5.14)})} \tag{5.16}
$$

$$
\left.
\begin{aligned}
& - \frac{1}{2} \big(\mathcal{B}_1^{[+1]} P_1^{[+1]} P_0 P_2^{[0]} f_0^-, f_2^+\big) - \frac{1}{2} \big(\mathcal{B}_1^{[-1]} P_1^{[+1]} P_0 P_2^{[+2]} f_0^-, f_2^+\big) \\
& - \frac{1}{2} \big(\mathcal{B}_1^{[+1]} P_1^{[-1]} P_0 P_2^{[+2]} f_0^-, f_2^+\big) - \frac{1}{2} \big(\mathcal{B}_1^{[+1]} f_0^-, P_1^{[-1]} P_0 P_2^{[0]} f_2^+\big) \\
& - \frac{1}{2} \big(\mathcal{B}_1^{[-1]} f_0^-, P_1^{[-1]} P_0 P_2^{[-2]} f_2^+\big) - \frac{1}{2} \big(\mathcal{B}_1^{[+1]} f_0^-, P_1^{[+1]} P_0 P_2^{[-2]} f_2^+\big)
\end{aligned}
\right\} =: \mathbf{L}, \tag{5.17}
$$

namely,

$$
\begin{aligned}
\mathrm{i}\,\varepsilon^4 \beta_3 = {} & \Theta + \mathrm{I} + \mathrm{II} + \mathrm{III\,a} + \frac{1}{2} \mathrm{III\,b} + \frac{1}{2} \widehat{\mathrm{III\,b}} + \mathrm{IV\,a} + \frac{1}{2} \mathrm{IV\,b} + \frac{1}{2} \widehat{\mathrm{IV\,b}} + \frac{1}{2} \mathrm{IV\,c} + \frac{1}{2} \widehat{\mathrm{IV\,c}} \\
& + \mathrm{V\,a} + \mathrm{V\,b} + \frac{1}{2} \mathrm{V\,c} + \frac{1}{2} \widehat{\mathrm{V}}\mathrm{c} + \mathrm{V\,d} + \frac{1}{2} \mathrm{V\,e} + \frac{1}{2} \widehat{\mathrm{V}}\mathrm{e} + \mathrm{V\,f} + \mathrm{V\,g} + \mathrm{V\,h} \\
& + \mathrm{VI\,a} + \mathrm{VI\,b} + \frac{1}{2} \mathrm{VI\,c} + \frac{1}{2} \widehat{\mathrm{VI\,c}} + \mathrm{VI\,d} + \mathbf{L}
\end{aligned} \tag{5.18}
$$

with, by (4.11),

$$
\begin{aligned}
\Theta &:= \big(\mathcal{B}_4^{[+2]} f_0^-,\ f_2^+\big),\\
\mathrm{I} &:= \big(\mathcal{B}_3^{[+1]} P_1^{[+1]} f_0^-,\ f_2^+\big) \overset{(5.9),(4.11)}{=} \big(\mathcal{B}_1^{[+1]} f_0^-,\ \mathcal{P}[\mathcal{B}_3^{[-1]}] f_2^+\big),\\
\mathrm{II} &:= \big(\mathcal{B}_3^{[+3]} P_1^{[-1]} f_0^-,\ f_2^+\big) \overset{(5.9)}{=} \big(\mathcal{B}_1^{[-1]} f_0^-,\ \mathcal{P}[\mathcal{B}_3^{[-3]}] f_2^+\big),\\
\mathrm{III\,a} &:= \big(\mathcal{B}_2^{[0]}(\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_2^{[+2]}] f_0^-,\ f_2^+\big) \overset{(5.6)}{=} \big(\mathcal{B}_2^{[+2]} f_0^-, (\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_2^{[0]}]\, f_2^+\big),\\
\mathrm{III\,b} &:= \big(\mathcal{B}_2^{[0]}(\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[+1]}] f_0^-,\ f_2^+\big),\\
\widehat{\mathrm{III}\,\mathrm{b}} &:= \big(\mathcal{B}_1^{[+1]} f_0^-, \mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_2^{[0]}]\, f_2^+\big) = \mathrm{III\,b} + \big(\mathcal{B}_2^{[0]} P_0 \mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[+1]}] f_0^-,\ f_2^+\big),\\
\mathrm{IV\,a} &:= \big(\mathcal{B}_2^{[+2]}(\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_2^{[0]}] f_0^-,\ f_2^+\big) \overset{(5.6)}{=} \big(\mathcal{B}_2^{[0]} f_0^-,\ (\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_2^{[-2]}] f_2^+\big),\\
\mathrm{IV\,b} &:= \big(\mathcal{B}_2^{[+2]}(\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[-1]}] f_0^-,\ f_2^+\big),\\
\mathrm{IV\,c} &:= \big(\mathcal{B}_2^{[+2]}(\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[+1]}] f_0^-,\ f_2^+\big),\\
\widehat{\mathrm{IV}\,\mathrm{b}} &:= \big(\mathcal{B}_1^{[-1]} f_0^-, \mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_2^{[-2]}] f_2^+\big) = \mathrm{IV\,b} + \big(\mathcal{B}_2^{[+2]} P_0 \mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[-1]}] f_0^-, f_2^+\big),\\
\widehat{\mathrm{IV}\,\mathrm{c}} &:= \big(\mathcal{B}_1^{[+1]} f_0^-, \mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_2^{[-2]}] f_2^+\big) = \mathrm{IV\,c} + \big(\mathcal{B}_2^{[+2]} P_0 \mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[+1]}] f_0^-, f_2^+\big),\\
\mathrm{V\,a} &:= \big(\mathcal{B}_1^{[+1]} \mathcal{P}[\mathcal{B}_3^{[+1]}] f_0^-, f_2^+\big) \overset{(4.11)}{=} \big(\mathcal{B}_3^{[+1]} f_0^-, \mathcal{P}[\mathcal{B}_1^{[-1]}] f_2^+\big),\\
\mathrm{V\,b} &:= \big(\mathcal{B}_1^{[+1]} \mathcal{P}[\mathcal{B}_2^{[0]}, \mathcal{B}_1^{[+1]}] f_0^-,\ f_2^+\big) = \big(\mathcal{B}_1^{[+1]} f_0^-,\ \mathcal{P}[\mathcal{B}_2^{[0]}, \mathcal{B}_1^{[-1]}] f_2^+\big),\\
\mathrm{V\,c} &:= \big(\mathcal{B}_2^{[0]} f_0^-,\ (\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[-1]}] f_2^+\big),\\
\widehat{\mathrm{V}\,\mathrm{c}} &:= \big(\mathcal{B}_1^{[+1]} \mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_2^{[0]}] f_0^-,\ f_2^+\big) = \mathrm{V\,c} + \big(\mathcal{B}_2^{[0]} f_0^-,\ P_0 \mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[-1]}] f_2^+\big),\\
\mathrm{V\,d} &:= \big(\mathcal{B}_1^{[+1]} \mathcal{P}[\mathcal{B}_2^{[+2]}, \mathcal{B}_1^{[-1]}] f_0^-,\ f_2^+\big) = \big(\mathcal{B}_1^{[-1]} f_0^-,\ \mathcal{P}[\mathcal{B}_2^{[-2]}, \mathcal{B}_1^{[-1]}] f_2^+\big),\\
\mathrm{V\,e} &:= \big(\mathcal{B}_2^{[+2]} f_0^-,\ (\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[-1]}] f_2^+\big),\\
\widehat{\mathrm{V}\,\mathrm{e}} &:= \big(\mathcal{B}_1^{[+1]} \mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_2^{[+2]}] f_0^-,\ f_2^+\big)\\
&\overset{(4.11)}{=} \mathrm{V\,e} + \big(\mathcal{B}_2^{[+2]} f_0^-,\ P_0 \mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[-1]}] f_2^+\big),\\
\mathrm{V\,f} &:= \big(\mathcal{B}_1^{[+1]} \mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[+1]}] f_0^-,\ f_2^+\big)\\
&= \big(\mathcal{B}_1^{[+1]} f_0^-,\ \mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[-1]}] f_2^+\big),\\
\mathrm{V\,g} &:= \big(\mathcal{B}_1^{[+1]} \mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[+1]}] f_0^-,\ f_2^+\big)\\
&= \big(\mathcal{B}_1^{[+1]} f_0^-,\ \mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[-1]}] f_2^+\big),\\
\mathrm{V\,h} &:= \big(\mathcal{B}_1^{[+1]} \mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[-1]}] f_0^-,\ f_2^+\big)\\
&= \big(\mathcal{B}_1^{[-1]} f_0^-,\ \mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[-1]}] f_2^+\big),\\
\mathrm{VI\,a} &:= \big(\mathcal{B}_1^{[-1]} \mathcal{P}[\mathcal{B}_3^{[+3]}] f_0^-,\ f_2^+\big) \overset{(4.11)}{=} \big(\mathcal{B}_3^{[+3]} f_0^-,\ \mathcal{P}[\mathcal{B}_1^{[+1]}] f_2^+\big),\\
\mathrm{VI\,b} &:= \big(\mathcal{B}_1^{[-1]} \mathcal{P}[\mathcal{B}_2^{[+2]}, \mathcal{B}_1^{[+1]}] f_0^-,\ f_2^+\big) = \big(\mathcal{B}_1^{[+1]} f_0^-,\ \mathcal{P}[\mathcal{B}_2^{[-2]}, \mathcal{B}_1^{[+1]}] f_2^+\big),\\
\mathrm{VI\,c} &:= \big(\mathcal{B}_2^{[+2]} f_0^-,\ (\mathrm{Id}-P_0)\mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[+1]}] f_2^+\big),
\end{aligned}
$$

$$\widehat{\mathrm{VI}}\,\mathrm{c} := \big(\mathcal{B}_1^{[-1]}\mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_2^{[+2]}]f_0^-, f_2^+\big) = \mathrm{VI}\,\mathrm{c} + \big(\mathcal{B}_2^{[+2]}f_0^-, P_0\mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[+1]}]f_2^+\big),$$
$$\begin{aligned}\mathrm{VI}\,\mathrm{d} &:= \big(\mathcal{B}_1^{[-1]}\mathcal{P}[\mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[+1]}]f_0^-, f_2^+\big)\\ &= \big(\mathcal{B}_1^{[+1]}f_0^-, \mathcal{P}[\mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[+1]}]f_2^+\big).\end{aligned}$$

By Lemma 15, Remark 19, Lemma 18 and (4.2) we represent these expressions in terms of the entanglement coefficients; we then compute them by means of Lemma 17. We have, recalling the notation $\mathrm{B}_{0,n\,j,j'}^{[\kappa]\;\sigma,\sigma'} \equiv \mathrm{B}_{n\;\;j,j'}^{[\kappa]\sigma,\sigma'}$,

$$\Theta = \mathrm{B}_{4\;\;2,0}^{[+2]+,-} = 0, \tag{5.19}$$

$$\mathrm{I} = \frac{\mathrm{B}_{1\;\;1,0}^{[+1]+,-}\mathrm{B}_{3\;\;2,1}^{[+1]+,+}}{\omega_1^+ - \omega_*} - \frac{\mathrm{B}_{1\;\;1,0}^{[+1]-,-}\mathrm{B}_{3\;\;2,1}^{[+1]+,-}}{\omega_1^- - \omega_*} = \mathrm{i}\,\frac{5\sqrt{3}}{32}\varepsilon^4,$$

$$\mathrm{II} = \frac{\mathrm{B}_{1\;\;-1,0}^{[-1]+,-}\mathrm{B}_{3\;\;2,-1}^{[+3]+,+}}{\omega_{-1}^+ - \omega_*} - \frac{\mathrm{B}_{1\;\;-1,0}^{[-1]-,-}\mathrm{B}_{3\;\;2,-1}^{[+3]+,-}}{\omega_{-1}^- - \omega_*} = -\frac{9\mathrm{i}\sqrt{3}}{32}\varepsilon^4,$$

$$\mathrm{III}\,\mathrm{a} \overset{(5.6)}{=} -\frac{\mathrm{B}_{2\;\;2,0}^{[+2]-,-}\mathrm{B}_{2\;\;2,2}^{[0]+,-}}{\omega_2^- - \omega_*} = \mathrm{i}\,\frac{3\sqrt{3}}{8}\varepsilon^4,$$

$$\mathrm{III}\,\mathrm{b} = \frac{\mathrm{B}_{1\;\;1,0}^{[+1]-,-}\mathrm{B}_{1\;\;2,1}^{[+1]-,-}\mathrm{B}_{2\;\;2,2}^{[0]+,-}}{(\omega_1^- - \omega_*)(\omega_2^- - \omega_*)} - \frac{\mathrm{B}_{1\;\;1,0}^{[+1]+,-}\mathrm{B}_{1\;\;2,1}^{[+1]-,+}\mathrm{B}_{2\;\;2,2}^{[0]+,-}}{(\omega_1^+ - \omega_*)(\omega_2^- - \omega_*)} = \mathrm{i}\,\frac{15\sqrt{3}}{32}\varepsilon^4,$$

$$\widehat{\mathrm{III}}\,\mathrm{b} - \mathrm{III}\,\mathrm{b} = \frac{\mathrm{B}_{1\;\;1,0}^{[+1]-,-}\mathrm{B}_{1\;\;2,1}^{[+1]+,-}\mathrm{B}_{2\;\;2,2}^{[0]+,+}}{(\omega_1^- - \omega_*)^2} - \frac{\mathrm{B}_{1\;\;1,0}^{[+1]+,-}\mathrm{B}_{1\;\;2,1}^{[+1]+,+}\mathrm{B}_{2\;\;2,2}^{[0]+,+}}{(\omega_1^+ - \omega_*)^2} = -\mathrm{i}\,\frac{75\sqrt{3}}{128}\varepsilon^4,$$

$$\mathrm{IV}\,\mathrm{a} \overset{(5.6)}{=} \frac{\mathrm{B}_{2\;\;0,0}^{[0]+,-}\mathrm{B}_{2\;\;2,0}^{[+2]+,+}}{\omega_0^+ - \omega_*} = -\mathrm{i}\,\frac{\sqrt{3}}{8}\varepsilon^4,$$

$$\mathrm{IV}\,\mathrm{b} = \frac{\mathrm{B}_{1\;\;-1,0}^{[-1]+,-}\mathrm{B}_{1\;\;0,-1}^{[+1]+,+}\mathrm{B}_{2\;\;2,0}^{[+2]+,+}}{(\omega_{-1}^+ - \omega_*)(\omega_0^+ - \omega_*)} - \frac{\mathrm{B}_{1\;\;-1,0}^{[-1]-,-}\mathrm{B}_{1\;\;0,-1}^{[+1]+,-}\mathrm{B}_{2\;\;2,0}^{[+2]+,+}}{(\omega_{-1}^- - \omega_*)(\omega_0^+ - \omega_*)} = \mathrm{i}\,\frac{3\sqrt{3}}{64}\varepsilon^4,$$

$$\widehat{\mathrm{IV}}\,\mathrm{b} - \mathrm{IV}\,\mathrm{b} := \frac{\mathrm{B}_{1\;\;-1,0}^{[-1]+,-}\mathrm{B}_{1\;\;0,-1}^{[+1]-,+}\mathrm{B}_{2\;\;2,0}^{[+2]+,-}}{(\omega_{-1}^+ - \omega_*)^2} - \frac{\mathrm{B}_{1\;\;-1,0}^{[-1]-,-}\mathrm{B}_{1\;\;0,-1}^{[+1]-,-}\mathrm{B}_{2\;\;2,0}^{[+2]+,-}}{(\omega_{-1}^- - \omega_*)^2} = 0,$$

$$\mathrm{IV}\,\mathrm{c} = \frac{\mathrm{B}_{1\;\;1,0}^{[+1]+,-}\mathrm{B}_{1\;\;0,1}^{[-1]+,+}\mathrm{B}_{2\;\;2,0}^{[+2]+,+}}{(\omega_1^+ - \omega_*)(\omega_0^+ - \omega_*)} - \frac{\mathrm{B}_{1\;\;1,0}^{[+1]-,-}\mathrm{B}_{1\;\;0,1}^{[-1]+,-}\mathrm{B}_{2\;\;2,0}^{[+2]+,+}}{(\omega_1^- - \omega_*)(\omega_0^+ - \omega_*)} = \mathrm{i}\,\frac{5\sqrt{3}}{64}\varepsilon^4,$$

$$\widehat{\mathrm{IV}}\,\mathrm{c} - \mathrm{IV}\,\mathrm{c} = \frac{\mathrm{B}_{1\;\;1,0}^{[+1]+,-}\mathrm{B}_{1\;\;0,1}^{[-1]-,+}\mathrm{B}_{2\;\;2,0}^{[+2]+,-}}{(\omega_1^+ - \omega_*)^2} - \frac{\mathrm{B}_{1\;\;1,0}^{[+1]-,-}\mathrm{B}_{1\;\;0,1}^{[-1]-,-}\mathrm{B}_{2\;\;2,0}^{[+2]+,-}}{(\omega_1^- - \omega_*)^2} = 0,$$

$$\mathrm{V}\,\mathrm{a} = \frac{\mathrm{B}_{3\;\;1,0}^{[+1]+,-}\mathrm{B}_{1\;\;2,1}^{[+1]+,+}}{\omega_1^+ - \omega_*} - \frac{\mathrm{B}_{3\;\;1,0}^{[+1]-,-}\mathrm{B}_{1\;\;2,1}^{[+1]+,-}}{\omega_1^- - \omega_*} = -\mathrm{i}\,\frac{5\sqrt{3}}{32}\varepsilon^4,$$

$$
\begin{aligned}
\mathrm{V\,b} = {} & \frac{\mathrm{B}^{[+1]\,+,-}_{1\;\;1,0}\,\mathrm{B}^{[0]\,+,+}_{2\;\;1,1}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}}{(\omega_1^+ - \omega_*)^2} - \frac{\mathrm{B}^{[+1]\,-,-}_{1\;\;1,0}\,\mathrm{B}^{[0]\,+,-}_{2\;\;1,1}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}}{(\omega_1^- - \omega_*)(\omega_1^+ - \omega_*)} \\
& + \frac{\mathrm{B}^{[+1]\,-,-}_{1\;\;1,0}\,\mathrm{B}^{[0]\,-,-}_{2\;\;1,1}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}}{(\omega_1^- - \omega_*)^2} - \frac{\mathrm{B}^{[+1]\,+,-}_{1\;\;1,0}\,\mathrm{B}^{[0]\,-,+}_{2\;\;1,1}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}}{(\omega_1^- - \omega_*)(\omega_1^+ - \omega_*)} = -\mathrm{i}\,\frac{25\sqrt{3}}{128}\varepsilon^4, \\
\mathrm{V\,c} = {} & \frac{\mathrm{B}^{[0]\,+,-}_{2\;\;0,0}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;1,0}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}}{(\omega_0^+ - \omega_*)(\omega_1^+ - \omega_*)} - \frac{\mathrm{B}^{[0]\,+,-}_{2\;\;0,0}\,\mathrm{B}^{[+1]\,-,+}_{1\;\;1,0}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}}{(\omega_0^+ - \omega_*)(\omega_1^- - \omega_*)} = \mathrm{i}\,\frac{5\sqrt{3}}{32}\varepsilon^4, \\
\widehat{\mathrm{V\,c}} - \mathrm{V\,c} := {} & \frac{\mathrm{B}^{[0]\,-,-}_{2\;\;0,0}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;1,0}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}}{(\omega_1^+ - \omega_*)^2} - \frac{\mathrm{B}^{[0]\,-,-}_{2\;\;0,0}\,\mathrm{B}^{[+1]\,-,-}_{1\;\;1,0}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}}{(\omega_1^- - \omega_*)^2} = -\mathrm{i}\,\frac{35\sqrt{3}}{128}\varepsilon^4, \\
\mathrm{V\,d} = {} & \frac{\mathrm{B}^{[-1]\,+,-}_{1\;\;-1,0}\,\mathrm{B}^{[+2]\,+,+}_{2\;\;1,-1}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}}{(\omega_{-1}^+ - \omega_*)(\omega_1^+ - \omega_*)} - \frac{\mathrm{B}^{[-1]\,-,-}_{1\;\;-1,0}\,\mathrm{B}^{[+2]\,+,-}_{2\;\;1,-1}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}}{(\omega_{-1}^- - \omega_*)(\omega_1^+ - \omega_*)} \\
& + \frac{\mathrm{B}^{[-1]\,-,-}_{1\;\;-1,0}\,\mathrm{B}^{[+2]\,-,-}_{2\;\;1,-1}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}}{(\omega_{-1}^- - \omega_*)(\omega_1^- - \omega_*)} - \frac{\mathrm{B}^{[-1]\,+,-}_{1\;\;-1,0}\,\mathrm{B}^{[+2]\,-,+}_{2\;\;1,-1}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}}{(\omega_{-1}^+ - \omega_*)(\omega_1^- - \omega_*)} = \mathrm{i}\,\frac{15\sqrt{3}}{64}\varepsilon^4, \\
\mathrm{V\,e} = {} & \frac{\mathrm{B}^{[+2]\,-,-}_{2\;\;2,0}\,\mathrm{B}^{[-1]\,-,-}_{1\;\;1,2}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}}{(\omega_2^- - \omega_*)(\omega_1^- - \omega_*)} - \frac{\mathrm{B}^{[+2]\,-,-}_{2\;\;2,0}\,\mathrm{B}^{[-1]\,+,-}_{1\;\;1,2}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}}{(\omega_2^- - \omega_*)(\omega_1^+ - \omega_*)} = \mathrm{i}\,\frac{15\sqrt{3}}{64}\varepsilon^4, \\
\widehat{\mathrm{V\,e}} - \mathrm{V\,e} = {} & \frac{\mathrm{B}^{[+2]\,+,-}_{2\;\;2,0}\,\mathrm{B}^{[-1]\,-,+}_{1\;\;1,2}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}}{(\omega_1^- - \omega_*)^2} - \frac{\mathrm{B}^{[+2]\,+,-}_{2\;\;2,0}\,\mathrm{B}^{[-1]\,+,+}_{1\;\;1,2}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}}{(\omega_1^+ - \omega_*)^2} = 0, \\
\mathrm{V\,f} = {} & -2\frac{\mathrm{B}^{[+1]\,+,-}_{1\;\;1,0}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}\,\mathrm{B}^{[-1]\,+,+}_{1\;\;1,2}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}}{(\omega_1^+ - \omega_*)^3} \\
& - 2\frac{\mathrm{B}^{[+1]\,-,-}_{1\;\;1,0}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}\,\mathrm{B}^{[-1]\,-,+}_{1\;\;1,2}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}}{(\omega_1^- - \omega_*)^3} \\
& + \frac{(\omega_1^+ + \omega_1^- - 2\omega_*)\mathrm{B}^{[+1]\,-,-}_{1\;\;1,0}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}\,\mathrm{B}^{[-1]\,+,+}_{1\;\;1,2}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}}{(\omega_1^+ - \omega_*)^2(\omega_1^- - \omega_*)^2} \\
& + \frac{(\omega_1^+ + \omega_1^- - 2\omega_*)\mathrm{B}^{[+1]\,+,-}_{1\;\;1,0}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}\,\mathrm{B}^{[-1]\,-,+}_{1\;\;1,2}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}}{(\omega_1^+ - \omega_*)^2(\omega_1^- - \omega_*)^2} \\
& + \frac{\mathrm{B}^{[+1]\,+,-}_{1\;\;1,0}\,\mathrm{B}^{[+1]\,-,+}_{1\;\;2,1}\,\mathrm{B}^{[-1]\,-,-}_{1\;\;1,2}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}}{(\omega_1^+ - \omega_*)(\omega_2^- - \omega_*)(\omega_1^- - \omega_*)} \\
& - \frac{\mathrm{B}^{[+1]\,+,-}_{1\;\;1,0}\,\mathrm{B}^{[+1]\,-,+}_{1\;\;2,1}\,\mathrm{B}^{[-1]\,+,-}_{1\;\;1,2}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}}{(\omega_1^+ - \omega_*)^2(\omega_2^- - \omega_*)} \\
& + \frac{\mathrm{B}^{[+1]\,-,-}_{1\;\;1,0}\,\mathrm{B}^{[+1]\,-,-}_{1\;\;2,1}\,\mathrm{B}^{[-1]\,+,-}_{1\;\;1,2}\,\mathrm{B}^{[+1]\,+,+}_{1\;\;2,1}}{(\omega_1^- - \omega_*)(\omega_2^- - \omega_*)(\omega_1^+ - \omega_*)} \\
& - \frac{\mathrm{B}^{[+1]\,-,-}_{1\;\;1,0}\,\mathrm{B}^{[+1]\,-,-}_{1\;\;2,1}\,\mathrm{B}^{[-1]\,-,-}_{1\;\;1,2}\,\mathrm{B}^{[+1]\,+,-}_{1\;\;2,1}}{(\omega_1^- - \omega_*)^2(\omega_2^- - \omega_*)} = -\mathrm{i}\,\frac{75\sqrt{3}}{256}\varepsilon^4,
\end{aligned}
$$

$$
\begin{aligned}
V g = {} & 2\frac{B_1^{[+1]}{}^{+,-}_{1,0}\, B_1^{[-1]}{}^{\neg,+}_{0,1}\, B_1^{[+1]}{}^{+,-}_{1,0}\, B_1^{[+1]}{}^{+,+}_{2,1}}{(\omega_1^+ - \omega_*)^3} \\
& + 2\frac{B_1^{[+1]}{}^{\neg,-}_{1,0}\, B_1^{[-1]}{}^{\neg,-}_{0,1}\, B_1^{[+1]}{}^{\neg,-}_{1,0}\, B_1^{[+1]}{}^{+,-}_{2,1}}{(\omega_1^- - \omega_*)^3} \\
& - \frac{(\omega_1^+ + \omega_1^- - 2\omega_*) B_1^{[+1]}{}^{+,-}_{1,0}\, B_1^{[-1]}{}^{\neg,+}_{0,1}\, B_1^{[+1]}{}^{\neg,-}_{1,0}\, B_1^{[+1]}{}^{+,-}_{2,1}}{(\omega_1^+ - \omega_*)^2(\omega_1^- - \omega_*)^2} \\
& - \frac{(\omega_1^+ + \omega_1^- - 2\omega_*) B_1^{[+1]}{}^{\neg,-}_{1,0}\, B_1^{[-1]}{}^{\neg,-}_{0,1}\, B_1^{[+1]}{}^{+,-}_{1,0}\, B_1^{[+1]}{}^{+,+}_{2,1}}{(\omega_1^+ - \omega_*)^2(\omega_1^- - \omega_*)^2} \\
& + \frac{B_1^{[+1]}{}^{+,-}_{1,0}\, B_1^{[-1]}{}^{+,+}_{0,1}\, B_1^{[+1]}{}^{+,+}_{1,0}\, B_1^{[+1]}{}^{+,+}_{2,1}}{(\omega_1^+ - \omega_*)^2(\omega_0^+ - \omega_*)} \\
& - \frac{B_1^{[+1]}{}^{\neg,-}_{1,0}\, B_1^{[-1]}{}^{+,-}_{0,1}\, B_1^{[+1]}{}^{+,+}_{1,0}\, B_1^{[+1]}{}^{+,+}_{2,1}}{(\omega_1^+ - \omega_*)(\omega_0^+ - \omega_*)(\omega_1^- - \omega_*)} \\
& + \frac{B_1^{[+1]}{}^{\neg,-}_{1,0}\, B_1^{[-1]}{}^{+,-}_{0,1}\, B_1^{[+1]}{}^{\neg,+}_{1,0}\, B_1^{[+1]}{}^{+,-}_{2,1}}{(\omega_1^- - \omega_*)^2(\omega_0^+ - \omega_*)} \\
& - \frac{B_1^{[+1]}{}^{+,-}_{1,0}\, B_1^{[-1]}{}^{+,+}_{0,1}\, B_1^{[+1]}{}^{\neg,+}_{1,0}\, B_1^{[+1]}{}^{+,-}_{2,1}}{(\omega_1^+ - \omega_*)(\omega_0^+ - \omega_*)(\omega_1^- - \omega_*)} = \mathrm{i}\,\frac{25\sqrt{3}}{256}\varepsilon^4, \\
V h = {} & \frac{(\omega_{-1}^+ + \omega_1^+ - 2\omega_*) B_1^{[-1]}{}^{+,-}_{-1,0} B_1^{[+1]}{}^{\neg,+}_{0,-1} B_1^{[+1]}{}^{+,-}_{1,0}\, B_1^{[+1]}{}^{+,+}_{2,1}}{(\omega_{-1}^+ - \omega_*)^2(\omega_1^+ - \omega_*)^2} \\
& + \frac{(\omega_{-1}^- + \omega_1^- - 2\omega_*) B_1^{[-1]}{}^{\neg,-}_{-1,0} B_1^{[+1]}{}^{\neg,-}_{0,-1} B_1^{[+1]}{}^{\neg,-}_{1,0}\, B_1^{[+1]}{}^{+,-}_{2,1}}{(\omega_{-1}^- - \omega_*)^2(\omega_1^- - \omega_*)^2} \\
& - \frac{(\omega_{-1}^- + \omega_1^+ - 2\omega_*) B_1^{[-1]}{}^{\neg,-}_{-1,0} B_1^{[+1]}{}^{\neg,-}_{0,-1} B_1^{[+1]}{}^{+,-}_{1,0}\, B_1^{[+1]}{}^{+,+}_{2,1}}{(\omega_{-1}^- - \omega_*)^2(\omega_1^+ - \omega_*)^2} \\
& - \frac{(\omega_{-1}^+ + \omega_1^- - 2\omega_*) B_1^{[-1]}{}^{+,-}_{-1,0} B_1^{[+1]}{}^{\neg,+}_{0,-1} B_1^{[+1]}{}^{\neg,-}_{1,0}\, B_1^{[+1]}{}^{+,-}_{2,1}}{(\omega_{-1}^+ - \omega_*)^2(\omega_1^- - \omega_*)^2} \\
& + \frac{B_1^{[-1]}{}^{+,-}_{-1,0} B_1^{[+1]}{}^{+,+}_{0,-1} B_1^{[+1]}{}^{+,+}_{1,0}\, B_1^{[+1]}{}^{+,+}_{2,1}}{(\omega_{-1}^+ - \omega_*)(\omega_0^+ - \omega_*)(\omega_1^+ - \omega_*)} \\
& - \frac{B_1^{[-1]}{}^{\neg,-}_{-1,0} B_1^{[+1]}{}^{+,-}_{0,-1} B_1^{[+1]}{}^{+,+}_{1,0}\, B_1^{[+1]}{}^{+,+}_{2,1}}{(\omega_{-1}^- - \omega_*)(\omega_0^+ - \omega_*)(\omega_1^+ - \omega_*)} \\
& + \frac{B_1^{[-1]}{}^{\neg,-}_{-1,0} B_1^{[+1]}{}^{+,-}_{0,-1} B_1^{[+1]}{}^{\neg,+}_{1,0}\, B_1^{[+1]}{}^{+,-}_{2,1}}{(\omega_{-1}^- - \omega_*)(\omega_0^+ - \omega_*)(\omega_1^- - \omega_*)} \\
& - \frac{B_1^{[-1]}{}^{+,-}_{-1,0} B_1^{[+1]}{}^{+,+}_{0,-1} B_1^{[+1]}{}^{\neg,+}_{1,0}\, B_1^{[+1]}{}^{+,-}_{2,1}}{(\omega_{-1}^+ - \omega_*)(\omega_0^+ - \omega_*)(\omega_1^- - \omega_*)} = 0,
\end{aligned}
$$

$$\mathrm{VI\,a} := \frac{\mathtt{B}_{3\ \ 3,0}^{[+3]+,-}\mathtt{B}_{1\ \ 2,3}^{[-1]+,+}}{\omega_3^+ - \omega_*} - \frac{\mathtt{B}_{3\ \ 3,0}^{[+3]-,-}\mathtt{B}_{1\ \ 2,3}^{[-1]+,-}}{\omega_3^- - \omega_*} = 0,$$

$$\begin{aligned}
\mathrm{VI\,b} =& \frac{\mathtt{B}_{1\ \ 1,0}^{[+1]+,-}\mathtt{B}_{2\ \ 3,1}^{[+2]+,+}\mathtt{B}_{1\ \ 2,3}^{[-1]+,+}}{(\omega_1^+ - \omega_*)(\omega_3^+ - \omega_*)} - \frac{\mathtt{B}_{1\ \ 1,0}^{[+1]-,-}\mathtt{B}_{2\ \ 3,1}^{[+2]+,-}\mathtt{B}_{1\ \ 2,3}^{[-1]+,+}}{(\omega_1^- - \omega_*)(\omega_3^+ - \omega_*)} \\
&+ \frac{\mathtt{B}_{1\ \ 1,0}^{[+1]-,-}\mathtt{B}_{2\ \ 3,1}^{[+2]-,-}\mathtt{B}_{1\ \ 2,3}^{[-1]+,-}}{(\omega_1^- - \omega_*)(\omega_3^- - \omega_*)} - \frac{\mathtt{B}_{1\ \ 1,0}^{[+1]+,-}\mathtt{B}_{2\ \ 3,1}^{[+2]-,+}\mathtt{B}_{1\ \ 2,3}^{[-1]+,-}}{(\omega_1^+ - \omega_*)(\omega_3^- - \omega_*)} = 0,
\end{aligned}$$

$$\mathrm{VI\,c} = \frac{\mathtt{B}_{2\ \ 2,0}^{[+2]-,-}\mathtt{B}_{1\ \ 3,2}^{[+1]-,-}\mathtt{B}_{1\ \ 2,3}^{[-1]+,-}}{(\omega_2^- - \omega_*)(\omega_3^- - \omega_*)} - \frac{\mathtt{B}_{2\ \ 2,0}^{[+2]-,-}\mathtt{B}_{1\ \ 3,2}^{[+1]+,-}\mathtt{B}_{1\ \ 2,3}^{[-1]+,+}}{(\omega_2^- - \omega_*)(\omega_3^+ - \omega_*)} = -\mathrm{i}\,\frac{39\sqrt{3}}{64}\varepsilon^4,$$

$$\widehat{\mathrm{VI\,c}} - \mathrm{VI\,c} := \frac{\mathtt{B}_{2\ \ 2,0}^{[+2]+,-}\mathtt{B}_{1\ \ 3,2}^{[+1]-,+}\mathtt{B}_{1\ \ 2,3}^{[-1]+,-}}{(\omega_3^- - \omega_*)^2} - \frac{\mathtt{B}_{2\ \ 2,0}^{[+2]+,-}\mathtt{B}_{1\ \ 3,2}^{[+1]+,+}\mathtt{B}_{1\ \ 2,3}^{[-1]+,+}}{(\omega_3^+ - \omega_*)^2} = 0,$$

$$\begin{aligned}
\mathrm{VI\,d} =& \frac{\mathtt{B}_{1\ \ 1,0}^{[+1]-,-}\mathtt{B}_{1\ \ 2,1}^{[+1]-,-}\mathtt{B}_{1\ \ 3,2}^{[+1]+,-}\mathtt{B}_{1\ \ 2,3}^{[-1]+,+}}{(\omega_1^- - \omega_*)(\omega_2^- - \omega_*)(\omega_3^+ - \omega_*)} - \frac{\mathtt{B}_{1\ \ 1,0}^{[+1]+,-}\mathtt{B}_{1\ \ 2,1}^{[+1]-,+}\mathtt{B}_{1\ \ 3,2}^{[+1]+,-}\mathtt{B}_{1\ \ 2,3}^{[-1]+,+}}{(\omega_1^+ - \omega_*)(\omega_2^- - \omega_*)(\omega_3^+ - \omega_*)} \\
&+ \frac{\mathtt{B}_{1\ \ 1,0}^{[+1]+,-}\mathtt{B}_{1\ \ 2,1}^{[+1]-,+}\mathtt{B}_{1\ \ 3,2}^{[+1]-,-}\mathtt{B}_{1\ \ 2,3}^{[-1]+,-}}{(\omega_1^+ - \omega_*)(\omega_2^- - \omega_*)(\omega_3^- - \omega_*)} \\
&- \frac{\mathtt{B}_{1\ \ 1,0}^{[+1]-,-}\mathtt{B}_{1\ \ 2,1}^{[+1]-,-}\mathtt{B}_{1\ \ 3,2}^{[+1]-,-}\mathtt{B}_{1\ \ 2,3}^{[-1]+,-}}{(\omega_1^- - \omega_*)(\omega_2^- - \omega_*)(\omega_3^- - \omega_*)} \\
&+ \frac{(\omega_1^- + \omega_3^+ - 2\omega_*)\mathtt{B}_{1\ \ 1,0}^{[+1]-,-}\mathtt{B}_{1\ \ 2,1}^{[+1]+,-}\mathtt{B}_{1\ \ 3,2}^{[+1]+,+}\mathtt{B}_{1\ \ 2,3}^{[-1]+,+}}{(\omega_1^- - \omega_*)^2(\omega_3^+ - \omega_*)^2} \\
&- \frac{(\omega_1^+ + \omega_3^+ - 2\omega_*)\mathtt{B}_{1\ \ 1,0}^{[+1]+,-}\mathtt{B}_{1\ \ 2,1}^{[+1]+,+}\mathtt{B}_{1\ \ 3,2}^{[+1]+,+}\mathtt{B}_{1\ \ 2,3}^{[-1]+,+}}{(\omega_1^+ - \omega_*)^2(\omega_3^+ - \omega_*)^2} \\
&+ \frac{(\omega_1^+ + \omega_3^- - 2\omega_*)\mathtt{B}_{1\ \ 1,0}^{[+1]+,-}\mathtt{B}_{1\ \ 2,1}^{[+1]+,+}\mathtt{B}_{1\ \ 3,2}^{[+1]-,+}\mathtt{B}_{1\ \ 2,3}^{[-1]+,-}}{(\omega_1^+ - \omega_*)^2(\omega_3^- - \omega_*)^2} \\
&- \frac{(\omega_1^- + \omega_3^- - 2\omega_*)\mathtt{B}_{1\ \ 1,0}^{[+1]-,-}\mathtt{B}_{1\ \ 2,1}^{[+1]+,-}\mathtt{B}_{1\ \ 3,2}^{[+1]-,+}\mathtt{B}_{1\ \ 2,3}^{[-1]+,-}}{(\omega_1^- - \omega_*)^2(\omega_3^- - \omega_*)^2} = \mathrm{i}\,\frac{195\sqrt{3}}{256}\varepsilon^4.
\end{aligned} \tag{5.20}$$

Finally, we compute the last term $\mathbf{L}$ in (5.17). By (3.5), (5.14) and (5.6),

$$\begin{aligned}
P_0 P_2^{[+2]} f_0^- &= P_0 \mathcal{P}\big[\mathcal{B}_1^{[+1]}, \mathcal{B}_1^{[+1]}\big] f_0^- = \zeta_1 f_2^+ \varepsilon^2, \\
P_0 P_2^{[-2]} f_2^+ &= P_0 \mathcal{P}\big[\mathcal{B}_1^{[-1]}, \mathcal{B}_1^{[-1]}\big] f_2^+ = \zeta_2 f_0^- \varepsilon^2, \\
P_0 P_2^{[0]} f_0^- &= \zeta_3 f_0^- \varepsilon^2, \quad P_0 P_2^{[0]} f_2^+ = \zeta_4 f_2^+ \varepsilon^2,
\end{aligned} \tag{5.21}$$

with $\zeta_3, \zeta_4 \in \mathbb{C}$, and, again by (4.21b) and (4.20),

$$\begin{aligned}
\zeta_1 &:= \frac{\mathrm{B}_{1\ \ 1,0}^{[+1]-,-}\,\mathrm{B}_{1\ \ 2,1}^{[+1]+,-}}{(\omega_1^- - \omega_*)^2} - \frac{\mathrm{B}_{1\ \ 1,0}^{[+1]+,-}\,\mathrm{B}_{1\ \ 2,1}^{[+1]+,+}}{(\omega_1^+ - \omega_*)^2} = \mathrm{i}\,\frac{5\sqrt{3}}{16} \\
\zeta_2 &:= \frac{\mathrm{B}_{1\ \ 1,2}^{[-1]+,+}\,\mathrm{B}_{1\ \ 0,1}^{[-1]-,+}}{(\omega_1^+ - \omega_*)^2} - \frac{\mathrm{B}_{1\ \ 1,2}^{[-1]-,+}\,\mathrm{B}_{1\ \ 0,1}^{[-1]-,-}}{(\omega_1^- - \omega_*)^2} = \mathrm{i}\,\frac{5\sqrt{3}}{16}.
\end{aligned} \tag{5.22}$$

Note that $\zeta_1 = -\overline{\zeta}_2$ by (4.2). By (5.21) the term $\mathbf{L}$ in (5.17) is given by

$$\begin{aligned}
-\frac{2\mathbf{L}}{\varepsilon^2} = {} & \zeta_3 \underbrace{\left(\mathcal{B}_1^{[+1]} P_1^{[+1]} f_0^-,\ f_2^+\right)}_{=\mathrm{Bb}=0} + \zeta_1 \underbrace{\left(\mathcal{B}_1^{[-1]} P_1^{[+1]} f_2^+,\ f_2^+\right)}_{=\mathrm{Ac}} \\
& + \zeta_1 \underbrace{\left(\mathcal{B}_1^{[+1]} P_1^{[-1]} f_2^+,\ f_2^+\right)}_{=\mathrm{Ab}} \\
& + \overline{\zeta}_4 \underbrace{\left(\mathcal{B}_1^{[+1]} f_0^-,\ P_1^{[-1]} f_2^+\right)}_{=\mathrm{Bb}=0} + \overline{\zeta}_2 \underbrace{\left(\mathcal{B}_1^{[-1]} f_0^-,\ P_1^{[-1]} f_0^-\right)}_{=\mathrm{Cb}} \\
& + \overline{\zeta}_2 \underbrace{\left(\mathcal{B}_1^{[+1]} f_0^-,\ P_1^{[+1]} f_0^-\right)}_{=\mathrm{Cc}},
\end{aligned} \tag{5.23}$$

with Ab, Ac, Bb, Cb and Cc defined and computed in (5.3), (5.4), (5.5) (recall also (5.11)). We then have

$$\mathbf{L} = -\mathrm{i}\,\frac{305}{512}\sqrt{3}\varepsilon^4. \tag{5.24}$$

By (5.18), (5.20) and (5.24) we conclude that $\beta_3 = -\frac{39\sqrt{3}}{512}$ as stated in (5.2b). This completes the proof of Proposition 20.

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References

[1] B. Akers and D. P. Nicholls, Spectral stability of deep two-dimensional gravity water waves: repeated eigenvalues. *SIAM J. Appl. Math.* **72** (2012), no. 2, 689–711 Zbl 1322.76029 MR 2914345

[2] D. Bambusi and A. Maspero, Birkhoff coordinates for the Toda lattice in the limit of infinitely many particles with an application to FPU. *J. Funct. Anal.* **270** (2016), no. 5, 1818–1887 Zbl 1335.37047 MR 3452718

[3] T. Benjamin, Instability of periodic wavetrains in nonlinear dispersive systems. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **299** (1967), no. 1456, article no. 0123

[4] T. Benjamin and J. Feir, The disintegration of wave trains on deep water, Part 1. Theory. *J. Fluid Mech.* **27** (1967), no. 3, 417–430 Zbl 0144.47101

[5] M. Berti, R. Feola, and F. Pusateri, Birkhoff normal form and long time existence for periodic gravity water waves. *Comm. Pure Appl. Math.* **76** (2023), no. 7, 1416–1494 Zbl 1531.76016 MR 4588319

[6] M. Berti, A. Maspero, and P. Ventura, Full description of Benjamin–Feir instability of Stokes waves in deep water. *Invent. Math.* **230** (2022), no. 2, 651–711 Zbl 1498.76037 MR 4493325

[7] M. Berti, A. Maspero, and P. Ventura, Benjamin–Feir instability of Stokes waves in finite depth. *Arch. Ration. Mech. Anal.* **247** (2023), no. 5, article no. 91 Zbl 1525.76049 MR 4632837

[8] M. Berti, A. Maspero, and P. Ventura, Stokes waves at the critical depth are modulationally unstable. *Comm. Math. Phys.* **405** (2024), no. 3, article no. 56 Zbl 1542.76008 MR 4709095

[9] T. J. Bridges and A. Mielke, A proof of the Benjamin–Feir instability. *Arch. Rational Mech. Anal.* **133** (1995), no. 2, 145–198 Zbl 0845.76029 MR 1367360

[10] G. Chen and Q. Su, Nonlinear modulational instabililty of the Stokes waves in 2D full water waves. *Comm. Math. Phys.* **402** (2023), no. 2, 1345–1452 Zbl 1522.76012 MR 4627323

[11] W. Craig and C. Sulem, Numerical simulation of gravity waves. *J. Comput. Phys.* **108** (1993), no. 1, 73–83 Zbl 0778.76072 MR 1239970

[12] W. Craig and C. Sulem, Mapping properties of normal forms transformations for water waves. *Boll. Unione Mat. Ital.* **9** (2016), no. 2, 289–318 Zbl 1382.37066 MR 3502161

[13] W. Craig and P. A. Worfolk, An integrable normal form for water waves in infinite depth. *Phys. D* **84** (1995), no. 3-4, 513–531 Zbl 0883.35092 MR 1336546

[14] R. Creedon, B. Deconinck, and O. Trichtchenko, High-frequency instabilities of the Kawahara equation: a perturbative approach. *SIAM J. Appl. Dyn. Syst.* **20** (2021), no. 3, 1571–1595 Zbl 1478.35072 MR 4309857

[15] R. Creedon, B. Deconinck, and O. Trichtchenko, High-frequency instabilities of the Kawahara equation: a perturbative approach. *SIAM J. Appl. Dyn. Syst.* **20** (2021), no. 3, 1571–1595 Zbl 1478.35072 MR 4309857

[16] R. P. Creedon and B. Deconinck, A high-order asymptotic analysis of the Benjamin–Feir instability spectrum in arbitrary depth. *J. Fluid Mech.* **956** (2023), article no. A29 Zbl 1516.76032 MR 4546129

[17] R. P. Creedon, B. Deconinck, and O. Trichtchenko, High-frequency instabilities of Stokes waves. *J. Fluid Mech.* **937** (2022), article no. A24 Zbl 07482844 MR 4386807

[18] R. P. Creedon, H. Q. Nguyen, and W. A. Strauss, Proof of the transverse instability of Stokes waves. *Ann. PDE* **11** (2025), no. 1, article no. 4 MR 4850595

[19] B. Deconinck and K. Oliveras, The instability of periodic surface gravity waves. *J. Fluid Mech.* **675** (2011), 141–167 Zbl 1241.76212 MR 2801039

[20] B. Deconinck and O. Trichtchenko, High-frequency instabilities of small-amplitude solutions of Hamiltonian PDEs. *Discrete Contin. Dyn. Syst.* **37** (2017), no. 3, 1323–1358 Zbl 1365.37057 MR 3640555

[21] Y. Deng, A. D. Ionescu, and F. Pusateri, On the wave turbulence theory of 2D gravity waves, I: Deterministic energy estimates. *Comm. Pure Appl. Math.* **78** (2025), no. 2, 211–322 Zbl 07960943 MR 4837431

[22] A. I. D'yachenko and V. E. Zakharov, Is free-surface hydrodynamics an integrable system? *Phys. Lett. A* **190** (1994), no. 2, 144–148 Zbl 0961.76511 MR 1283779

[23] V. M. Hur and Z. Yang, Unstable capillary-gravity waves. 2023, arXiv:2311.01368v2

[24] V. M. Hur and Z. Yang, Unstable Stokes waves. *Arch. Ration. Mech. Anal.* **247** (2023), no. 4, article no. 62 Zbl 1521.35139 MR 4600217

[25] T. Kappeler, Fibration of the phase space for the Korteweg–de Vries equation. *Ann. Inst. Fourier (Grenoble)* **41** (1991), no. 3, 539–575 Zbl 0731.58033 MR 1136595

[26] S. Kuksin and G. Perelman, Vey theorem in infinite dimensions and its application to KdV. *Discrete Contin. Dyn. Syst.* **27** (2010), no. 1, 1–24 Zbl 1193.37076 MR 2600759

[27] T. Levi-Civita, Détermination rigoureuse des ondes permanentes d'ampleur finie. *Math. Ann.* **93** (1925), no. 1, 264–314 Zbl 51.0671.06 MR 1512238

[28] M. Lighthill, Contribution to the theory of waves in nonlinear dispersive systems. *IMA J. Appl. Math.* **1** (1965), no. 3, 269–306 Zbl 0218.76015

[29] A. Nekrasov, On steady waves. *Izv. Ivanovo-Voznesenk. Politekhn. In-ta* **3** (1921), 52–65

[30] H. Q. Nguyen and W. A. Strauss, Proof of modulational instability of Stokes waves in deep water. *Comm. Pure Appl. Math.* **76** (2023), no. 5, 1035–1084 Zbl 1526.76011 MR 4569610

[31] G. Stokes, On the theory of oscillatory waves. *Trans. Cambridge Phil. Soc.* **8** (1847), 441–455

[32] D. J. Struik, Détermination rigoureuse des ondes irrotationelles périodiques dans un canal à profondeur finie. *Math. Ann.* **95** (1926), no. 1, 595–634 Zbl 52.0876.04 MR 1512296

[33] G. B. Whitham, *Linear and nonlinear waves*. Pure Appl. Math., Wiley-Interscience John Wiley, New York-London-Sydney, 1974 Zbl 0373.76001 MR 0483954

[34] S. Wu, The quartic integrability and long time existence of steep water waves in 2D. [v1] 2020, [v2] 2021, arXiv:2010.09117v2

[35] V. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Zhurnal Prikladnoi Mekhaniki i Teckhnicheskoi Fiziki* **9** (1968), no. 2, 86–94

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