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Stokes waves are unstable, even very small ones

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Abstract. WS: it is difficult to write about Thomas Kappeler in the past tense. He was a brilliant mathematician, but more importantly he was a wonderfully open, generous, and friendly person. I was fortunate that we had many opportunities to spend time together and discuss mathematics. I greatly miss him.

A Stokes wave is a traveling free-surface periodic water wave that is constant in the direction transverse to the direction of propagation. Even Stokes waves of very small amplitude are unstable when subjected to various perturbations. We present a brief survey of this phenomenon with emphasis on transverse perturbations.

Dedicated to the memory of Thomas Kappeler

We consider classical water waves that are irrotational, inviscid, and incompressible with constant density. The water is modeled by the Euler equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla P = -g\mathbf{e}_z, \quad \nabla \cdot \mathbf{u} = 0,$$

in three dimensions, where $\mathbf{u}$ is the velocity, P is the pressure and ρ is the density. The term $-g\mathbf{e}_z$ represents the force of gravity, pointing downwards. In addition, the water lies below an unknown free surface S. On S there are the two boundary conditions. (i) The fluid velocity $\mathbf{u}$ is tangential to S (due to lack of spouts, etc.). (ii) The pressure P is a constant on S (due to air pressure and assumed lack of surface tension).

Such waves have been studied for over two centuries, notably by Stokes [27]. A *Stokes wave* is a two-dimensional steady wave traveling in a fixed horizontal direction at a fixed speed c. It has been known for a century that a curve of small-amplitude Stokes waves exists [17, 21, 28]. Several decades ago, it was proven that the curve extends to large amplitudes as well [16].

Of course, water waves can become turbulent, especially large ones. Before the 1960s, it was widely believed that the very small Stokes waves should be stable because the dispersion relation gives no hint of temporal growth. However, in 1967, Benjamin and Feir [1] discovered, to the general surprise of the fluids community, that a *small* long-wave perturbation of a *small* Stokes wave in the same direction of propagation will lead to

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exponential instability. This is known as the *modulational* (or Benjamin–Feir or sideband) *instability*, a phenomenon whereby deviations from a periodic wave are reinforced by the nonlinearity, leading to the eventual breakup of the wave into a train of pulses. Rigorous proofs of the modulational instability were discovered in 1995 by Bridges and Mielke [6] in the case of finite depth, provided the depth is larger than a critical depth d_0, and in 2020 by two of the current authors [22] for infinite depth. A more detailed description of the instability, including the figure-8 pattern of the unstable eigenvalues, was found numerically in [12] and asymptotically by another of the current authors [9]. This detailed description was proven rigorously by Berti et al., first in the deep water case [2] and then in the finite-depth case [3] if the depth is larger than d_0. Recently, the much more subtle critical depth case was treated in [4].

A different type of instability due to perturbations in the same direction of propagation (that is, the longitudinal direction) was detected in 1981 in the numerical work of McLean [18, 19]. It is called *high-frequency instabilities* because they develop away from the origin of the complex plane, appearing as small isolas (bubbles) centered on the imaginary axis. In contrast to modulational instabilities, high-frequency instabilities occur at all values of the depth. The first plot of the high-frequency instabilities was due to Deconinck and Oliveras [12], thirty years after McLean's work. Among the challenges in plotting these instabilities was to find the longitudinal wave numbers of the perturbation that correspond to each high-frequency isola, which exist in narrow intervals that drift as the amplitude of the Stokes wave increases. In [10], a perturbation method was developed to obtain an asymptotic expansion of these intervals in addition to an asymptotic expansion of the maximum growth rates of the high-frequency instabilities. This revealed for the first time analytically that such instabilities can grow faster than the modulational instability at certain finite depths. These high-frequency results have since been made rigorous in the recent work [14].

Both modulational and high-frequency instabilities are created by longitudinal perturbations that have *different periods* compared to those of the Stokes waves. On the other hand, what was unanswered was whether a small Stokes wave could be unstable *when perturbed in both horizontal directions while keeping the longitudinal period unperturbed*. This *transverse instability* problem was studied numerically first by Bryant [7] and was followed by much more detailed work of McLean et al. [18–20]. While these remarkable papers did detect transverse instabilities, a mathematical proof has been missing ever since. This problem is truly three dimensional.

Here, we announce the first rigorous proof of transverse instability of small Stokes waves [11].

Before describing the proof, it is important to note that there are several other models of water waves for which the transverse instability has been studied rigorously. One such model includes the presence of surface tension, that is, gravity-capillary waves. However, it should be kept in mind that the presence of surface tension drastically changes the mathematical problem. The transverse instability for solitary (non-periodic) waves in such a model was rigorously discussed by a number of authors, including Bridges [5], Pego and

Sun [25], and Rousset and Tzvetkov [26]. The transverse instability for periodic waves in this model was recently studied by Haragus, Truong, and Wahlen [13].

We now specify the parameters of our problem. Let x and y denote the horizontal variables and z the vertical one. For simplicity, we assume here that the depth is infinite. We are confident that our proof generalizes to the finite-depth case. Consider the curve of Stokes waves traveling in the x-direction and with a given period, say, 2π without loss of generality. This curve is parametrized by a small parameter ε which represents the wave amplitude of the Stokes waves. Such a steady wave can be described in the moving (x, z) plane (where $x - ct$ is replaced by x) by its free surface $S = \{(x, y, z) \mid z = \eta^*(x; \varepsilon)\}$ and by its velocity potential $\psi^*(x; \varepsilon)$ restricted to S.

Our perturbation of η^* takes the form $\overline{\eta}(x)e^{\lambda t + i\alpha y}$, where $\overline{\eta}$ has the *same period* 2π as the Stokes wave, $\lambda \in \mathbb{C}$ is the growth rate of the perturbation, and $\alpha \in \mathbb{R}$ is the transverse wave number of the perturbation. The goal is to find at least one value of α that leads to instability, that is, $\operatorname{Re}\lambda > 0$. After linearizing the nonlinear water wave system about a Stokes wave and performing a conformal mapping change of variables, we find that the exponents λ are eigenvalues of a linear operator $\mathcal{L}_{\varepsilon,\beta}$, where $\beta = \alpha^2$. Motivated by [20], we first determine a *resonant transverse wave number* α_* so that the unperturbed operator $\mathcal{L}_{0,\beta_*}$ with $\beta_* = \alpha_*^2$ has an imaginary double eigenvalue $\lambda_0 = i\sigma$. This eigenvalue corresponds to the lowest-possible resonance that generates a Type II transverse instability according to McLean [20], of which there are infinitely many higher-order resonances that have the potential to generate higher-order transverse instabilities. (We expect however that higher-order transverse instabilities have slower growth rates for small Stokes waves.) In order to capture the transverse instabilities, we introduce a small parameter δ for the perturbation of β about β_*. Our main result is that the perturbed operator $\mathcal{L}_{\varepsilon,\beta_*+\delta}$ has eigenvalues $\lambda_\pm$ with non-zero real parts that bifurcate from λ_0, as stated in the following theorem.

Theorem 1. *There exist* $\varepsilon_0 > 0$ *and* $\delta_0 > 0$ *such that, for all* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ *and* $\delta \in (-\delta_0, \delta_0)$*, the operator* $\mathcal{L}_{\varepsilon,\beta_*+\delta}$ *has a pair of eigenvalues*

$$\lambda_\pm = i\left(\sigma + \frac{1}{2}T(\varepsilon,\delta)\right) \pm \frac{1}{2}\sqrt{\Delta(\varepsilon,\delta)}, \tag{1}$$

where T *and* Δ *are real-valued, real-analytic functions such that* $T(\varepsilon,\delta) = O(\delta)$ *and* $\Delta(\varepsilon,\delta) = O(\delta^2)$ *as* $(\varepsilon,\delta) \to (0,0)$*. Furthermore, there exist* $\kappa_0 \in \mathbb{R}$ *and* $\kappa_1 > 0$ *such that, for*

$$\delta = \delta(\varepsilon,\theta) = \kappa_0\varepsilon^2 + \theta\varepsilon^3 \quad \text{with } |\theta| < \kappa_1,$$

we have $\Delta(\varepsilon,\delta(\varepsilon,\theta)) > 0$ *for sufficiently small* ε*. Thus, the eigenvalue* λ_+ *has positive real part for such* δ *and for* ε *is sufficiently small. Moreover,* $\operatorname{Re}\lambda_+ = O(\varepsilon^3)$ *as* $\varepsilon \to 0$ *for each* θ*. This means that there exist transverse perturbations of the given Stokes wave whose amplitudes grow temporally like* $e^{t\operatorname{Re}\lambda_+}$*.*

Substituting $\delta = \delta(\varepsilon,\theta)$ into (1), we obtain an asymptotic expansion of the unstable eigenvalues with an $O(\varepsilon^4)$ remainder. By eliminating θ from this expansion in favor of its

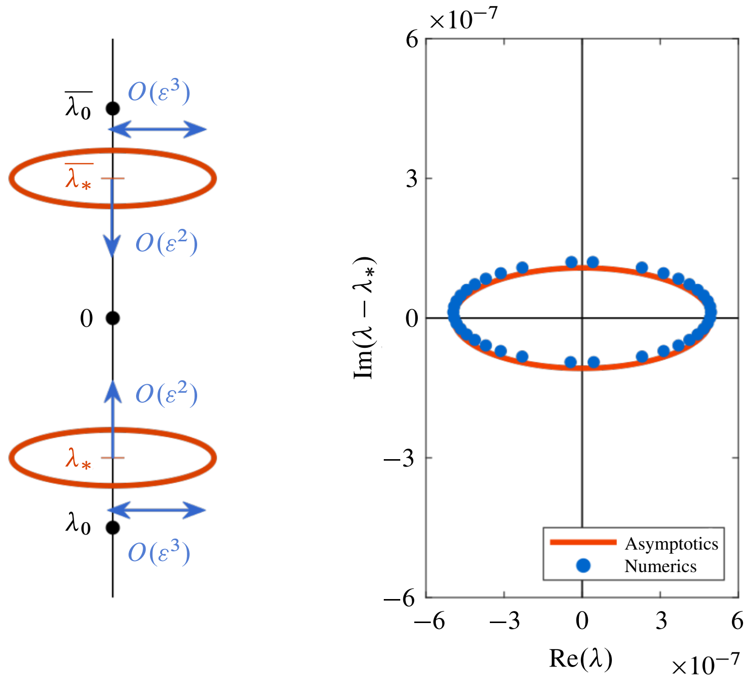

Figure 1. (Left) A schematic of the transverse instability isolas (orange curves) of width $O(\varepsilon^3)$ drifting from λ_0 like $O(\varepsilon^2)$. Here, λ_* represents the center of the isola. (Right) A comparison of the asymptotic approximation of the transverse instability isola (2) (orange curve) and numerical computations of the unstable eigenvalues (blue dots) when $\varepsilon = 0.01$. The center of the isola is subtracted from the imaginary part to show a sense of scale. The difference between the numerical and asymptotic results is $O(\varepsilon^4)$.

real and imaginary parts, denoted by λ_r and λ_i, respectively, we find that the eigenvalues lie on an isola which is approximately the ellipse

$$\frac{4.085\lambda_r^2}{\varepsilon^6} + \frac{86.059(\lambda_i + 0.389 - 0.467\varepsilon^2)^2}{\varepsilon^6} = 1. \tag{2}$$

We have numerically evaluated the coefficients for the sake of readability. The center of the ellipse drifts away from the double eigenvalue $i\sigma \approx -0.389i$ along the imaginary axis like $O(\varepsilon^2)$, while its semi-major and semi-minor axes scale like $O(\varepsilon^3)$. See the left panel of Figure 1.

We can compare (2) to the *numerical* computations [24] of the unstable eigenvalues obtained by the Floquet–Fourier–Hill method applied to the Ablowitz–Fokas–Musslimani formulation of the transverse spectral problem. The right panel of Figure 1 shows the results of these numerical computations on a Stokes wave with amplitude parameter $\varepsilon = 0.01$. Also plotted is the corresponding asymptotic ellipse (2). The difference between the asymptotic and numerical results is $O(\varepsilon^4)$, demonstrating excellent agreement between the theoretical results and the numerical computations to $O(\varepsilon^3)$. Even better agreement can be found by retaining higher-order corrections of the unstable eigenvalues in a manner similar to [10].

The isola of unstable eigenvalues found above is reminiscent of the high-frequency isolas that appear in the *longitudinal* stability spectrum. It is therefore natural to compare the growth rates of the transverse instability obtained in Theorem 1 to the known growth rates of the longitudinal instabilities of Stokes waves, including both the high-frequency and the Benjamin–Feir instabilities. In the infinite-depth longitudinal case, the largest high-frequency isola has semi-major and semi-minor axes that scale like $O(\varepsilon^4)$ [10]. Thus, our transverse instability grows at the *faster* rate $O(\varepsilon^3)$ for sufficiently small amplitude waves. Moreover, our instability grows *slower* than the largest high-frequency instability in finite depth, which grows like $O(\varepsilon^2)$ [10, 14]. On the other hand, our instability grows *slower* than the Benjamin–Feir instability rate, which is $O(\varepsilon^2)$ in both finite and infinite depth [2, 3, 6, 9, 14, 23].

Now, we briefly turn to the main ideas in the proof of Theorem 1. First, we introduce the Stokes waves and proceed with the linearization and then the flattening of the fluid domain by means of a conformal mapping. We study the behavior of the three-dimensional Dirichlet–Neumann operator [8] under a two-dimensional conformal mapping and prove its analyticity in ε and δ. The problem is reduced to studying the eigenvalues of the linearized operator $\mathcal{L}_{\varepsilon,\beta}$, which has a Hamiltonian form and is reversible. We find an expression for $\mathcal{L}_{\varepsilon,\beta}$ by a method analogous to that in [23]. However, for the present three-dimensional instability problem, $\mathcal{L}_{\varepsilon,\beta}$ involves a genuine pseudo-differential operator as opposed to the simpler Fourier multiplier in the two-dimensional problem considered in [2, 23]. In fact, the linear operator $\mathcal{L}_{\varepsilon,\beta}$ is

$$\mathcal{L}_{\varepsilon,\beta} = J\mathcal{H}_{\varepsilon,\beta}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{H}_{\varepsilon,\beta} = \begin{bmatrix} \frac{1+q(x,\varepsilon)}{\zeta'(x,\varepsilon)} & -p(x,\varepsilon)\partial_x \\ \partial_x(p(x,\varepsilon)\cdot) & \mathcal{G}_{\varepsilon,\beta} \end{bmatrix},$$

where $\mathcal{H}_{\varepsilon,\beta}$ is self-adjoint. The coefficients $p(x,\varepsilon), q(x,\varepsilon), \zeta'(x,\varepsilon)$ depend analytically on the Stokes wave. $\mathcal{G}_{\varepsilon,\delta}$ is the Dirichlet–Neumann operator modified by the conformal map and the introduction of $e^{i\alpha y}$. All four entries of $\mathcal{H}_{\varepsilon,\beta}$ depend on ε, but only the lower right corner depends also on β.

In case $\varepsilon = 0$, the linear operator around the laminar (flat) flow reduces to

$$\mathcal{L}_{0,\beta} = \begin{bmatrix} \partial_x & (|D|^2+\beta)^{\frac{1}{2}} \\ -1 & \partial_x \end{bmatrix},$$

where $\beta = \alpha^2$. The spectrum of $\mathcal{L}_{0,\beta}$ consists of the purely imaginary eigenvalues

$$\lambda^0_\pm(k,\beta) = i\big[k \pm (k^2+\beta)^{\frac{1}{4}}\big], \quad k \in \mathbb{Z}.$$

There is a *double eigenvalue* (a resonance) whenever

$$\lambda^0_+(-(m+1),\beta) = \lambda^0_-(m,\beta).$$

For our purposes, we choose the one with $m = 1$, which is closest to the origin. Thus, we define $\beta_* \approx 2.7275211479$ to be the unique positive solution of $-2 + (\beta_* + 4)^{\frac{1}{4}} = 1 - (\beta_* + 1)^{\frac{1}{4}}$.

The proof continues by following the method of [2] that uses a Kato similarity transformation [15] to reduce the relevant spectral data of $\mathcal{L}_{\varepsilon,\beta}$ to a 2×2 matrix $\mathrm{L}_{\varepsilon,\delta}$ with the property that $i\mathrm{L}_{\varepsilon,\delta}$ is real and skew-adjoint. We show that the entries of this matrix are real-analytic functions of ε and δ, and we obtain convenient functional expressions for its eigenvalues, resulting in (1). We perform lengthy expansions of $\mathcal{L}_{\varepsilon,\beta_*+\delta}$ out to third order in both ε and δ. This is a major new difficulty compared to the two-dimensional instabilities studied in [2,23]. We use the expansions of $\mathcal{L}_{\varepsilon,\beta_*+\delta}$ to compute the expansions of the Kato basis vectors and of the matrix $\mathrm{L}_{\varepsilon,\delta}$. Next, we analyze the leading terms in the characteristic discriminant $\Delta(\varepsilon,\delta)$ of $\mathrm{L}_{\varepsilon,\delta}$. At third order, the instability eventually becomes apparent. In order to conclude that $\Delta(\varepsilon,\beta) > 0$ for $\delta = \delta(\varepsilon,\theta)$ and sufficiently small ε, we must expand the entries of the matrix in a power series up to third order in the pair (ε,δ). If the expansions were terminated before third order, one would find $\Delta(\varepsilon,\delta) \le 0$ for any choice of δ, which would be insufficient for eigenvalues with positive real part. With the third-order expansions, however, we are able to show that $\Delta(\varepsilon,\delta) > 0$ if $\delta = O(\varepsilon^2)$ is chosen appropriately.

Our theorem turned out to be considerably more difficult than we had anticipated. Originally, we began by attempting to take the transverse perturbation at a fixed period, that is, $\delta = 0$. For the reasons stated above, that approach did not yield an instability. We used residue calculations related to the eigenfunction expansions of the unstable eigenvalues which led to many non-zero terms. This, coupled with the introduction of the small parameter δ, led to extremely arduous calculations, so we took advantage of Mathematica to check our calculations and carry out the longest ones.

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