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Criticality transition for positive powers of the discrete Laplacian on the half line

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Abstract. We study the criticality and subcriticality of powers $(-\Delta)^\alpha$ with $\alpha > 0$ of the discrete Laplacian $-\Delta$ acting on $\ell^2(\mathbb{N})$. We prove that these positive powers of the Laplacian are critical if and only if $\alpha \geq 3/2$. We complement our analysis with Hardy-type inequalities for $(-\Delta)^\alpha$ in the subcritical regimes $\alpha \in (0, 3/2)$. As an illustration of the critical case $\alpha \geq 3/2$, we analyze asymptotic properties of discrete eigenvalues emerging by coupling $(-\Delta)^\alpha$ with a localized potential.

1. Introduction

1.1. Physical motivation

The uniqueness of the world we live in consists in that $\mathbb{R}^3$ is the lowest dimensional Euclidean space for which the Brownian motion is *transient*. Indeed, it is well known that the Brownian particle in $\mathbb{R}^d$ will escape from any bounded set after some time forever if $d \geq 3$, while the opposite holds true in low dimensions, i.e., the Brownian motion is *recurrent* in $\mathbb{R}^1$ and $\mathbb{R}^2$. This is a well-known criticality transition in dimensions.

Since the Brownian motion is mathematically introduced via the heat equation, it is not surprising that the transiency is closely related to spectral properties of the Laplacian. Indeed, the self-adjoint realization $-\Delta$ in $L^2(\mathbb{R}^d)$ is *subcritical* if and only if $d \geq 3$, meaning that there exists a non-trivial non-negative function V such that the Hardy-type inequality $-\Delta \geq V$ holds in the sense of quadratic forms. On the other hand, $-\Delta$ is critical if $d = 1, 2$ in the sense that $\inf \sigma(-\Delta + V) < 0$ for every non-trivial non-positive function V. The Hardy inequality has other physical consequences, namely in quantum mechanics, where it can be interpreted in terms of the uncertainty principle and leads to the stability of matter in $\mathbb{R}^3$.

The case of Brownian particles dying on massive subsets of $\mathbb{R}^d$ is less interesting in the sense that the Dirichlet Laplacian $-\Delta$ in $\mathbb{R}^d \setminus \overline{\Omega}$, with any Ω non-empty and open, is always subcritical. In particular, the Brownian motion in the half-space $\mathbb{R}^{d-1} \times (0, \infty)$ is transient for every $d \geq 1$, so no criticality transition in dimensions occurs.

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There is a probabilistic interpretation of powers of the Laplacian in terms of an anomalous diffusion. From this perspective, the case of the half-line is equally uninteresting because all the powers $(-\Delta)^k$ in $L^2(0,\infty)$, with $k \in \mathbb{N}$, are subcritical. There is no criticality transition in powers.

The objective of this paper is to disclose the surprising fact that the situation is very different in the discrete setting. Indeed, we demonstrate that the integer powers of the discrete Laplacian $(-\Delta)^k$ on $\ell^2(\mathbb{N})$ are subcritical if and only if $k = 1$. What is more curious in fine properties of this transition, we consider possibly non-integer powers and reveal the following precise threshold in all positive powers:

$$(-\Delta)^\alpha \text{ on } \ell^2(\mathbb{N}) \text{ is subcritical if and only if } \alpha < 3/2.$$

1.2. Mathematical formulation

The discrete Laplacian on the (discrete) half line $\mathbb{N} = \{1, 2, 3, \dots\}$ is defined as the second-order difference operator $-\Delta$ given by the formula

$$(-\Delta u)_n := -u_{n-1} + 2u_n - u_{n+1}, \quad n \in \mathbb{N},$$

where $u = \{u_n\}_{n=1}^\infty$ is a complex sequence, together with the convention $u_0 := 0$. When regarded as an operator on the Hilbert space $\ell^2(\mathbb{N})$, the discrete Laplacian is bounded and self-adjoint with spectrum $\sigma(-\Delta) = [0, 4]$. The matrix representation of $-\Delta$ with respect to the standard basis of $\ell^2(\mathbb{N})$ coincides with the tridiagonal Toeplitz matrix

$$-\Delta = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

It is well known that $-\Delta$ is *subcritical*, meaning that there exists a non-trivial diagonal operator $V \geq 0$ such that $-\Delta \geq V$ in the sense of quadratic forms. Indeed, one has the classical Hardy inequality $-\Delta \geq V^{\mathrm{H}}$, where

$$V_n^{\mathrm{H}} := \frac{1}{4n^2}, \quad n \in \mathbb{N}.$$

Interestingly, even though the constant $1/4$ in the Hardy weight is optimal, the shifted operator $-\Delta - V^{\mathrm{H}}$ is still subcritical. An improved Hardy-type inequality $-\Delta \geq V^{\mathrm{KPP}}$, with

$$V_n^{\mathrm{KPP}} := 2 - \sqrt{\frac{n-1}{n}} - \sqrt{\frac{n+1}{n}}, \quad n \in \mathbb{N},$$

was found only recently by Keller, Pinchover, and Pogorzelski in [23]; see also [25] for a simple proof. Moreover, it is proved in these references that $-\Delta - V^{\mathrm{KPP}}$ is *critical* in the sense of the spectral instability $\inf \sigma(-\Delta - V^{\mathrm{KPP}} + V) < 0$ for any non-trivial diagonal $V \leq 0$. In fact, not only the criticality, but an optimality of the weight V^{KPP} in a stronger sense was proven in [23], together with more general results on discrete Laplacians on graphs; see also [22, 24], and moreover [5, 6], for related works in the continuous framework. We emphasize the contrast with the (continuous) Dirichlet Laplacian on the half line, where $-\Delta - V^{\mathrm{H}}$ with the classical Hardy weight $V^{\mathrm{H}}(x) := 1/(4x^2)$ is critical in $L^2(0,\infty)$.

While the (sub)criticality and the related Hardy-type inequalities for $-\Delta$ on $\ell^2(\mathbb{N})$ are well understood, much less is known about their generalization to the discrete polyharmonic operator or any positive power of $-\Delta$. The primary goal of this paper is to investigate the criticality or subcriticality of $(-\Delta)^\alpha$ depending on $\alpha > 0$, and to make the first attempt towards Hardy-type inequalities in the subcritical regimes. The operator $(-\Delta)^\alpha$ is defined by the usual functional calculus (see Section 2). Except for a partial result in [19] for $\alpha = 2$, the topic seems not to be studied for any non-trivial exponent $\alpha \neq 1$ so far.

The paper is organized as follows. In Subsection 1.3, our main results are formulated as Theorems 1.1, 1.3, and 1.4. Subsection 1.4 summarizes some relevant results on positive powers of Laplacians on the half line or the line in both the discrete and continuous settings. After Section 2, where preliminary results on powers of the Laplacian and their Green kernel are presented, the three main theorems are proven in Section 3. Moreover, weak and strong coupling regimes with localized potentials are studied in Section 4. The paper is concluded by an appendix with auxiliary integral identities and asymptotics.

1.3. Main results

In this paper, we adopt the following definition of subcriticality/criticality in terms of spectral stability/instability against small perturbations. Given any bounded self-adjoint operator H on $\ell^2(\mathbb{N})$, we say that H is *critical* if $\inf \sigma(H+V) < 0$ whenever $V \leq 0$ is non-trivial. Generically throughout this paper, V denotes a potential (or a weight), i.e., a diagonal operator acting on $\ell^2(\mathbb{N})$. As usual, our notation does not distinguish between a diagonal operator V on $\ell^2(\mathbb{N})$ and the sequence $V = \{V_n\}_{n=1}^\infty$ of its diagonal entries. We say H is *subcritical* if it is not critical, which is equivalent to the existence of a non-trivial weight $V \geq 0$ such that $H \geq V$ in the sense of forms (i.e., $\langle \psi, (H-V)\psi \rangle \geq 0$ for all $\psi \in \ell^2(\mathbb{N})$).

The question of the criticality of $(-\Delta)^\alpha$ for $\alpha > 0$ is answered by our first main result.

Theorem 1.1. *Suppose $\alpha > 0$. Then $(-\Delta)^\alpha$ is critical if and only if $\alpha \geq 3/2$.*

The proof of Theorem 1.1 is given in Subsection 3.1.

Remark 1.2. Notice that $(-\Delta)^\alpha$ is bounded from above by 4^α, since $\sigma((-\Delta)^\alpha) = [0, 4^\alpha]$. Therefore, one could also study the stability of the upper bound 4^α when adding a perturbation $V \geq 0$, i.e., the criticality of the operator $4^\alpha - (-\Delta)^\alpha$ in our setting. We briefly comment on this in Subsection 3.2, where we prove $4^\alpha - (-\Delta)^\alpha$ to be subcritical for all $\alpha > 0$. This result has no continuous analogue, since the Dirichlet Laplacian on $L^2(0,\infty)$ is not bounded from above.

For the particular case of the discrete bilaplacian Δ^2 on $\ell^2(\mathbb{N})$, it has already been observed in [19] that no direct analogue of the classical Rellich inequality holds, i.e., that there exists *no* $c > 0$ such that $\Delta^2 \geq V^{\mathrm{R}}$ with

$$V_n^{\mathrm{R}} := \frac{c}{n^4}, \quad n \in \mathbb{N};$$

this answers a question from [10]. Our Theorem 1.1 shows that there exists no discrete Rellich inequality on $\ell^2(\mathbb{N})$, neither any Hardy-type inequality for the discrete polyharmonic operator $(-\Delta)^k$ on $\ell^2(\mathbb{N})$ with $k = 3, 4, \ldots$ This can be surprising when compared to the continuous setting, where the polyharmonic operator $(-\Delta)^k$ is subcritical in $L^2(0,\infty)$ for all $k \in \mathbb{N}$ (while one should also take into account other possible definitions of powers of Laplacians, cf. Subsection 1.4 below). From this perspective, it would be an interesting topic for a separate study to investigate transitions between discrete and continuum operators via continuum limits with the focus on criticality and Hardy inequalities; see [8, 20, 28] for related recent works.

Theorem 1.1 implies the existence of Hardy-type inequalities for $(-\Delta)^\alpha$ with $\alpha \in (0, 3/2)$. A sufficient condition for admissible Hardy weights is given in the next theorem. Our analysis is based on a Birman–Schwinger argument, which in turn relies on upper bounds for the Green kernel of the free resolvent of $(-\Delta)^\alpha$. The estimate in Lemma 2.5 leads to a sufficient condition in terms of a sequence of functions naturally appearing in this resolvent bound, namely

$$g_n(\alpha) := \left(1 - \frac{(\alpha)_{2n}}{(1-\alpha)_{2n}}\right)\tan(\pi\alpha) \tag{1.1}$$

for $\alpha \in (0, 3/2)$ and $n \in \mathbb{N}$, where $(\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1)$ is the Pochhammer symbol. For $\alpha = 1/2$ and $\alpha = 1$, the values of $g_n(\alpha)$ are given by the respective limits

$$g_n\left(\frac{1}{2}\right) = \frac{2}{\pi}\sum_{j=1}^{2n}\frac{1}{2j-1} \quad \text{and} \quad g_n(1) = 2\pi n. \tag{1.2}$$

Notice that $g_n(\alpha) > 0$ for all $\alpha \in (0, 3/2)$ and $n \in \mathbb{N}$. In the sequel, Γ denotes the Gamma function.

Theorem 1.3. *Let $\alpha \in (0, 3/2)$. If a potential $V \geq 0$ satisfies the condition*

$$\sum_{n=1}^{\infty} g_n(\alpha)\, V_n \leq 2\pi\,\frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)}, \tag{1.3}$$

where g_n is as in (1.1) and (1.2), then

$$(-\Delta)^\alpha \geq V.$$

The proof of Theorem 1.3 is worked out in Subsection 3.3.

To obtain more concrete Hardy-type inequalities in the subcritical regimes $\alpha \in (0, 3/2)$, we consider weights which are of power form. It turns out, however, that an application of Theorem 1.3 does not yield the expected optimal decay rate of such weights. This is due to the underlying resolvent bound being symmetric in m and n with an asymptotic behavior

$$g_n(\alpha) = \begin{cases} O(1) & \text{if } \alpha \in (0, 1/2), \\ O(\ln n) & \text{if } \alpha = 1/2, \\ O(n^{2\alpha-1}) & \text{if } \alpha \in (1/2, 3/2), \end{cases} \tag{1.4}$$

as $n \to \infty$. Nevertheless, a slightly different approach leads to a refined bound on the Green kernel respecting its structure and thus eventually to the desired powers in the weights below. We emphasize that while our definition of subcriticality merely involves the existence of non-negative weights, the Hardy weights obtained in the following theorem are in fact strictly positive.

Theorem 1.4. *For every $\alpha \in (0, 3/2)$, there exists a positive constant $\gamma = \gamma(\alpha)$ such that*

$$(-\Delta)^\alpha \geq V,$$

where

$$V_n = V_n(\alpha) := \begin{cases} \gamma/n^{2\alpha} & \text{if } \alpha \neq 1/2, \\ \gamma/(n \ln(n+1)) & \text{if } \alpha = 1/2. \end{cases}$$

Theorem 1.4 is proven in Subsection 3.4.

Remark 1.5. Even though there is room for optimization of $\gamma(\alpha)$ in Theorem 1.4, we doubt that our method provides the optimal result. For this reason, we do not state the explicit constants nor attempt to optimize them, and leave the quest for the largest constant $\gamma(\alpha)$ as an open problem. An even more difficult question, which currently seems to be out of reach, is whether one can find explicit $V = V(\alpha) \geq 0$ such that $(-\Delta)^\alpha - V$ is critical for every value $\alpha \in (0, 3/2)$.

1.4. Relevant literature

We briefly discuss several closely related results and summarize the state of the art concerning mainly the criticality of positive powers of Laplacians on the half or the full line in both the discrete and the continuous settings.

(1) **Discrete polyharmonic operators on $H_0^k(\mathbb{N})$.**

Let $k \in \mathbb{N}$ and let $\{e_n \mid n \in \mathbb{N}\}$ denote the standard basis of $\ell^2(\mathbb{N})$. When the discrete polyharmonic operator $(-\Delta)^k$ is restricted to the subspace $H_0^k(\mathbb{N}) := \{e_1, \dots, e_{k-1}\}^\perp$ of $\ell^2(\mathbb{N})$, there exist discrete analogues of the continuous Birman inequality [2, 12] for the polyharmonic operator $(-\Delta)^k$ in $L^2(0, \infty)$, i.e., the inequality $(-\Delta)^k \geq V^{\mathrm{B},k}$ with the weight

$$V^{\mathrm{B},k}(x) := \frac{((2k)!)^2}{16^k (k!)^2} \frac{1}{x^{2k}}, \tag{1.5}$$

see also [11] for a recent proof. The discrete version of the Birman inequality $(-\Delta)^k \geq V^{\mathrm{B},k}$ on $H_0^k(\mathbb{N})$, where $V_n^{\mathrm{B},k}$ is as in (1.5) with x replaced by n, was proven in [19] (while it was deduced with a smaller constant in the PhD thesis [14], see also [16]). An improved discrete Birman inequality on $H_0^k(\mathbb{N})$ with a weight strictly larger than $V^{\mathrm{B},k}$ has been only conjectured in [10] for $k \geq 3$.

(2) **The discrete bilaplacian on $H_0^2(\mathbb{N})$.**

More is known about improved inequalities for the discrete bilaplacian Δ^2 on $H_0^2(\mathbb{N})$. In [10], the discrete Rellich inequality $\Delta^2 \geq V^{\mathrm{GKS}}$ was derived on $H_0^2(\mathbb{N})$ with

$$V_n^{\mathrm{GKS}} := 6 - 4\left(1 + \frac{1}{n}\right)^{3/2} - 4\left(1 - \frac{1}{n}\right)^{3/2} + \left(1 + \frac{2}{n}\right)^{3/2} + \left(1 - \frac{2}{n}\right)^{3/2}, \quad n \in \mathbb{N}.$$

This weight is asymptotically equal but strictly larger than the discrete analogue of $V^{\mathrm{B},2}$. Further improvements upon V^{GKS} were obtained only recently in [19]. Nevertheless, a critical (or even optimal) discrete Rellich weight on $H_0^2(\mathbb{N})$ remains unknown at the moment.

(3) **Discrete fractional Laplacians on $\ell^2(\mathbb{Z})$.**

When the discrete Laplacian $-\Delta$ is considered on the full line $\mathbb{Z}$, the picture is more complete. For fractional powers $\alpha \in (0, 1/2)$, it was shown in [4] that $(-\Delta)^\alpha \geq V^{\mathrm{CR},\alpha}$ on $\ell^2(\mathbb{Z})$ with the weight

$$V_n^{\mathrm{CR},\alpha} := 4^\alpha \frac{\Gamma^2\left(\frac{1+2\alpha}{4}\right)}{\Gamma^2\left(\frac{1-2\alpha}{4}\right)} \frac{\Gamma\left(|n| + \frac{1-2\alpha}{4}\right)\Gamma\left(|n| + \frac{3-2\alpha}{4}\right)}{\Gamma\left(|n| + \frac{1+2\alpha}{4}\right)\Gamma\left(|n| + \frac{3+2\alpha}{4}\right)}, \quad n \in \mathbb{Z}. \tag{1.6}$$

Interestingly, the weight $V^{\mathrm{CR},\alpha}$ turns out to be optimal; this was proven only recently in [21]. It particularly follows that $(-\Delta)^\alpha - V^{\mathrm{CR},\alpha}$ is critical for all $\alpha \in (0, 1/2)$. Although this seems not to be mentioned in the papers [4, 21], we remark without a proof that, for $\alpha \geq 1/2$, the operator $(-\Delta)^\alpha$ is critical. This can be verified by the same method which we use in the proof of Theorem 1.1. It means that there are no Hardy-type inequalities for $(-\Delta)^\alpha$ on $\ell^2(\mathbb{Z})$ when $\alpha \geq 1/2$, completing the picture of positive powers of the discrete Laplacian on $\mathbb{Z}$.

(4) **Fractional Laplacians in $L^2(0, \infty)$.**

The subcriticality of the polyharmonic operators in $L^2(0, \infty)$, due to the Birman inequalities [2, 11, 12] mentioned in point (2), are further complemented by inequalities $(-\Delta)^\alpha \geq V^{\mathrm{BD},\alpha}$ proved for fractional powers $\alpha \in (0, 1)$ in [3], where

$$V^{\mathrm{BD},\alpha}(x) := \frac{c_\alpha}{x^{2\alpha}}$$

with a constant $c_\alpha \geq 0$; see [3] for an explicit formula. The constant c_α is positive if $\alpha \neq 1/2$, implying $(-\Delta)^\alpha$ to be subcritical for $\alpha \in (0, 1) \setminus \{1/2\}$.

(5) **Fractional Laplacians in $L^2(\mathbb{R})$.**

The continuous setting on the full line resembles its discrete analogue on $\mathbb{Z}$. Indeed, for fractional powers $\alpha \in (0, 1/2)$, an inequality $(-\Delta)^\alpha \geq V^{\mathrm{He},\alpha}$ holds in $L^2(\mathbb{R})$ with the weight

$$V^{\mathrm{He},\alpha}(x) := 4^\alpha \frac{\Gamma^2\left(\frac{1+2\alpha}{4}\right)}{\Gamma^2\left(\frac{1-2\alpha}{4}\right)} \frac{1}{x^{2\alpha}},$$

see [18] and also [1,29]. This is in line with the discrete setting on $\ell^2(\mathbb{Z})$, where the leading term in the asymptotic expansion of (1.6) as $n \to \infty$ is equal to the discrete analogue of $V^{\mathrm{He},\alpha}$.

(6) **Powers of Laplacians in the higher dimensional setting.**

An asymptotic behavior of optimal constants in the discrete Hardy and Rellich inequalities on $\mathbb{Z}^d$ is studied in [15] for $d \to \infty$. For further numerous research works related to fractional Laplacians defined on various subspaces of $L^2(\Omega)$ on open domains $\Omega \subset \mathbb{R}^d$ and general dimension $d \geq 1$, we refer to the recent review [9] and references therein.

2. Preliminaries

2.1. Powers of the discrete Laplacian on $\ell^2(\mathbb{N})$

We give more details on the general properties of the discrete Laplacian and its positive powers on $\ell^2(\mathbb{N})$, mainly their diagonalization and matrix representation.

First, one can employ basic properties of Chebyshev polynomials of the second kind U_n to diagonalize the discrete Laplacian $-\Delta$. Recall that the sequence of Chebyshev polynomials U_n is determined by the recurrence

$$U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0, \quad n \in \mathbb{N}, \tag{2.1}$$

with the initial setting $U_0(x) := 1$ and $U_1(x) := 2x$; we refer the reader to Section 10.11 of [7] for general properties of Chebyshev polynomials. Further, the set of functions

$$\left\{\sqrt{2/\pi}\, U_n \mid n \in \mathbb{N}_0\right\}$$

forms an orthonormal basis in the Hilbert space $L^2((-1,1), \sqrt{1-x^2}\, \mathrm{d}x)$. Therefore, the mapping $\mathcal{U}$ defined as

$$\mathcal{U}e_n := \sqrt{\frac{2}{\pi}}\, U_{n-1}, \quad n \in \mathbb{N},$$

where e_n is the n-th vector of the standard basis of $\ell^2(\mathbb{N})$, extends to a unitary operator $\mathcal{U}\colon \ell^2(\mathbb{N}) \to L^2((-1,1), \sqrt{1-x^2}\, \mathrm{d}x)$. With the aid of (2.1), it is straightforward to verify that

$$\mathcal{U}(-\Delta)\,\mathcal{U}^{-1} = M_{2(1-x)}, \tag{2.2}$$

where $M_{f(x)}$ denotes the multiplication operator by a measurable function f in

$$L^2\big((-1,1), \sqrt{1-x^2}\, \mathrm{d}x\big).$$

From this observation, the spectral representation of $(-\Delta)^\alpha$ readily follows. Moreover, one can also compute the matrix representation of $(-\Delta)^\alpha$ with respect to the standard basis of $\ell^2(\mathbb{N})$ which turns out to be a particular Hankel plus Toeplitz matrix. The formula for the matrix elements of $(-\Delta)^\alpha$ is of no explicit use is this paper but can be of independent interest.

Proposition 2.1. *Let $\alpha > 0$. Then*

$$(-\Delta)^\alpha = \mathcal{U}^{-1} M_{2^\alpha(1-x)^\alpha}\, \mathcal{U}.$$

Further, for $m, n \in \mathbb{N}$, the matrix entries of $(-\Delta)^\alpha$ read

$$(-\Delta)^\alpha_{m,n} = (-1)^{m+n}\left[\binom{2\alpha}{\alpha+m-n} - \binom{2\alpha}{\alpha+m+n}\right], \tag{2.3}$$

where the generalized binomial number is defined by the formula

$$\binom{a}{b} := \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}. \tag{2.4}$$

(*Recall that the reciprocal Gamma function is an entire function vanishing at the points* $0, -1, -2, \dots$)

Proof. The first claim follows readily from (2.2) and the functional calculus for self-adjoint operators.

Furthermore, the first claim implies that for the matrix entries

$$(-\Delta)^{\alpha}_{m,n} = \langle e_m, (-\Delta)^{\alpha} e_n \rangle,$$

we have the integral representation

$$(-\Delta)^{\alpha}_{m,n} = \frac{2^{\alpha+1}}{\pi} \int_{-1}^{1} (1-x)^{\alpha}\, U_{m-1}(x)\, U_{n-1}(x) \sqrt{1-x^2}\, \mathrm{d}x, \quad m,n \in \mathbb{N}.$$

Formula (2.3) follows from an explicit calculation of the above integral which is postponed to the Appendix, see Lemma A.1. ■

As an immediate corollary of the last proposition, we state an integral representation for the Green kernel of $(-\Delta)^{\alpha}$.

Corollary 2.2. *Let $\alpha > 0$. Then we have*

$$((-\Delta)^{\alpha} - \lambda)^{-1}_{m,n} = \frac{2}{\pi} \int_{-1}^{1} \frac{U_{m-1}(x) U_{n-1}(x)}{2^{\alpha}(1-x)^{\alpha} - \lambda} \sqrt{1-x^2}\, dx \tag{2.5}$$

for all $m,n \in \mathbb{N}$ and $\lambda \notin [0, 4^{\alpha}]$.

Remark 2.3. Although negative powers α are not in the scope of the current paper, we remark on a possible extension of (2.3) to $\alpha < 0$, in which case $(-\Delta)^{\alpha}$ is an unbounded operator. Investigating the convergence of the resulting integrals, one sees that the Chebyshev polynomials U_n belong to the domain or form domain of $M_{2^{\alpha}(1-x)^{\alpha}}$, respectively, if and only if $\alpha > -3/4$ or $\alpha > -3/2$. The same conditions thus hold for the standard basis vectors in $\ell^2(\mathbb{N})$ to lie in the domain or form domain of $(-\Delta)^{\alpha}$. Formula (2.3) remains valid even for $\alpha > -3/2$, with the left-hand side interpreted as the corresponding quadratic form. For the apparent singularities $\alpha = -1/2$ and $\alpha = -1$, the right-hand side of (2.3) is to be understood as the respective limit

$$\begin{aligned}(-\Delta)^{-1/2}_{m,n} &= \frac{(-1)^{m+n}}{2} \left[\frac{\psi(1/2+m+n) + \psi(1/2-m-n)}{\Gamma(1/2+m+n)\Gamma(1/2-m-n)} \right. \\ &\qquad\qquad \left. - \frac{\psi(1/2+m-n) + \psi(1/2-m+n)}{\Gamma(1/2+m-n)\Gamma(1/2-m+n)} \right], \\ (-\Delta)^{-1}_{m,n} &= \min(m,n),\end{aligned}$$

where $\psi := \Gamma'/\Gamma$ is the digamma function.

2.2. Uniform bounds on the Green kernel

An important ingredient to our proof of Theorem 1.1 is a bound on the modulus of the Green kernel of $(-\Delta)^{\alpha}$ which is uniform in the spectral parameter. We prove three such bounds. First, a rather rough but sufficient bound for the proof of the subcriticality of $(-\Delta)^{\alpha}$ for $\alpha \in (0, 3/2)$. Second, a refined estimate which will be used in the proof of Theorem 1.3 and third, a qualitatively more precise bound which eventually leads to the Hardy weights with expected optimal decay in Theorem 1.4.

Lemma 2.4. *Let $\alpha \in (0, 3/2)$. Then there exists a constant $C_\alpha > 0$ such that, for all $m, n \in \mathbb{N}$ and $\lambda < 0$, we have*

$$|((-\Delta)^\alpha - \lambda)^{-1}_{m,n}| \leq C_\alpha mn.$$

Proof. Observe that the modulus of the integrand in the integral representation (2.5) is an increasing function of $\lambda < 0$. Therefore we may estimate it from above by taking $\lambda = 0$. The resulting integral remains convergent due to the assumption $\alpha \in (0, 3/2)$. This reasoning yields the upper estimate

$$|((-\Delta)^\alpha - \lambda)^{-1}_{m,n}| \leq \frac{2}{\pi} \int_{-1}^{1} \frac{|U_{m-1}(x)\, U_{n-1}(x)|}{2^\alpha (1-x)^\alpha - \lambda} \sqrt{1-x^2}\, \mathrm{d}x \leq C_\alpha \|U_{m-1}\|_\infty \|U_{n-1}\|_\infty$$

with the positive constant

$$C_\alpha := \frac{1}{2^{\alpha-1}\pi} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{(1-x)^\alpha}\, \mathrm{d}x < \infty,$$

and where

$$\|U_n\|_\infty := \max_{x \in [-1,1]} |U_n(x)|.$$

We conclude the proof by showing that

$$\|U_n\|_\infty \leq n + 1$$

for all $n \in \mathbb{N}_0$, where the above is actually an equality since $U_n(1) = n + 1$. Using the identity

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \tag{2.6}$$

(see equation (2) in Section 10.11 of [7]), we obtain the expression

$$U_n(\cos\theta) = \frac{e^{\mathrm{i}(n+1)\theta} - e^{-\mathrm{i}(n+1)\theta}}{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}} = e^{-\mathrm{i}n\theta} \sum_{k=0}^{n} e^{2\mathrm{i}k\theta},$$

from which we immediately deduce that

$$|U_n(\cos\theta)| \leq \sum_{k=0}^{n} 1 = n + 1$$

for all $\theta \in (0, \pi)$ and $n \in \mathbb{N}_0$. The proof is complete. ∎

Lemma 2.5. *Let $\alpha \in (0, 3/2)$. Then for all $m, n \in \mathbb{N}$ and $\lambda < 0$, we have*

$$|((-\Delta)^\alpha - \lambda)^{-1}_{m,n}| \leq \frac{1}{2\pi} \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} \sqrt{g_m(\alpha)}\, \sqrt{g_n(\alpha)},$$

where g_n is as in (1.1) and (1.2).

Proof. Since $\alpha \in (0, 3/2)$, the integral in formula (2.5) is convergent for $\lambda = 0$. Therefore, with fixed $m, n \in \mathbb{N}$ and $\lambda < 0$, we can estimate the Green kernel as follows:

$$\begin{aligned} |((-\Delta)^\alpha - \lambda)^{-1}_{m,n}| &\leq \frac{1}{2^{\alpha-1}\pi} \int_{-1}^{1} \frac{|U_{m-1}(x)\, U_{n-1}(x)|}{(1-x)^\alpha} \sqrt{1-x^2}\, \mathrm{d}x \\ &\leq \frac{1}{2^{\alpha-1}\pi} \sqrt{I_m(\alpha)}\, \sqrt{I_n(\alpha)}, \end{aligned}$$

where we used the Cauchy–Schwarz inequality and defined

$$I_n(\alpha) := \int_{-1}^{1} \frac{U_{n-1}^2(x)}{(1-x)^\alpha} \sqrt{1-x^2}\, \mathrm{d}x. \tag{2.7}$$

The rest of the proof follows from a formula for $I_n(\alpha)$ in the Appendix, see Lemma A.2, where it is proven that

$$I_n(\alpha) = 2^{\alpha-2} \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\, g_n(\alpha)$$

for any $\alpha \in (0, 3/2)$ and $n \in \mathbb{N}$ (see therein also the limiting formulas for the cases $\alpha = 1/2$ and $\alpha = 1$). ■

Lemma 2.6. *Let $\alpha \in (0, 3/2)$. Then there exists a constant $C_\alpha > 0$ such that, for all $m, n \in \mathbb{N}$ and $\lambda < 0$, we have*

$$|((-\Delta)^\alpha - \lambda)^{-1}_{m,n}| \leq C_\alpha \begin{cases} \frac{1}{\max(|n-m|^{1-2\alpha}, 1)} & \text{if } \alpha \in (0, 1/2), \\ \min\left(\frac{n}{m} \ln(m+1), \frac{m}{n} \ln(n+1)\right) & \text{if } \alpha = 1/2, \\ \min(m^{2\alpha-2} n, n^{2\alpha-2} m) & \text{if } \alpha \in (1/2, 3/2). \end{cases} \tag{2.8}$$

Proof. We define $t := -\lambda/2^{2\alpha} > 0$ and derive a formula for the resolvent kernel which exhibits a convenient structure for our analysis. Recalling identity (2.6), the substitution $x := \cos\theta$ in formula (2.5) and further elementary manipulations yield

$$\begin{aligned} ((-\Delta)^\alpha - \lambda)^{-1}_{m,n} &= \frac{2^{2-\alpha}}{\pi} \int_0^\pi \frac{\sin(m\theta)\sin(n\theta)}{(1-\cos\theta)^\alpha - 2^{-\alpha}\lambda} \frac{\mathrm{d}\theta}{2} \\ &= \frac{2^{2-2\alpha}}{\pi} \int_0^{\pi/2} \frac{\sin(2m\varphi)\sin(2n\varphi)}{\sin^{2\alpha}(\varphi) + t}\, \mathrm{d}\varphi \\ &= \frac{2^{1-2\alpha}}{\pi} \int_0^{\pi/2} \frac{\cos(2(m-n)\varphi) - \cos(2(m+n)\varphi)}{\sin^{2\alpha}(\varphi) + t}\, \mathrm{d}\varphi. \end{aligned} \tag{2.9}$$

Case $\alpha \in (0, 1/2)$.

Since $\alpha < 1/2$, one can split the last integral above and study its summands separately. Let first $m \neq m$. Setting either $k := |m-n|$ or $k := n+m$ and considering that $|m-n| \leq m+n$, it is sufficient to prove an estimate of the form

$$\left| \int_0^{\pi/2} \frac{\cos(2k\varphi)}{\sin^{2\alpha}(\varphi) + t}\, \mathrm{d}\varphi \right| \leq \frac{C}{k^{1-2\alpha}}, \quad k \in \mathbb{N},$$

with a constant $C = C(\alpha) > 0$ which does not depend on $t \in \mathbb{R}_+$. To this end, we first estimate the integral over a small neighborhood around the singularity by

$$\Big|\int_0^{\pi/(4k)} \frac{\cos(2k\varphi)}{\sin^{2\alpha}(\varphi)+t}\,\mathrm{d}\varphi\Big| \le \int_0^{\pi/(4k)} \sin^{-2\alpha}\varphi\,\mathrm{d}\varphi \le \Big(\frac{\pi}{2\sqrt{2}}\Big)^{2\alpha}\int_0^{\pi/(4k)} \varphi^{-2\alpha}\,\mathrm{d}\varphi = \frac{\pi}{2^{2-\alpha}(1-2\alpha)k^{1-2\alpha}}.$$

Here we have employed the elementary fact that

$$\frac{2\sqrt{2}}{\pi}\varphi \le \sin\varphi \le \varphi, \quad \text{for } 0 \le \varphi \le \frac{\pi}{4}. \tag{2.10}$$

For the bulk of the integral, by the second mean value theorem for definite integrals, one calculates

$$\int_{\pi/(4k)}^{\pi/2} \frac{\cos(2k\varphi)}{\sin^{2\alpha}(\varphi)+t}\,\mathrm{d}\varphi = \frac{1}{\sin^{2\alpha}\left(\frac{\pi}{4k}\right)+t}\int_{\pi/(4k)}^{\eta}\cos(2k\varphi)\,\mathrm{d}\varphi = \frac{1}{\sin^{2\alpha}\left(\frac{\pi}{4k}\right)+t}\,\frac{\sin(2k\eta)-1}{2k},$$

with some $\eta \in (\pi/(4k), \pi/2]$. Using again (2.10), it then follows for the modulus that

$$\Big|\int_{\pi/(4k)}^{\pi/2} \frac{\cos(2k\varphi)}{\sin^{2\alpha}(\varphi)+t}\,\mathrm{d}\varphi\Big| \le \frac{1}{\sin^{2\alpha}\left(\frac{\pi}{4k}\right)k} \le \frac{2^{\alpha}}{k^{1-2\alpha}},$$

and the claim follows for $m \ne n$. For $m = n$ on the other hand, the claimed bound (which then reduces to a constant) is a consequence of Lemma 2.5 and the asymptotic behavior (1.4) of $g_n(\alpha)$ as $n \to \infty$.

Case $\alpha \in [1/2, 3/2)$.

Since $\alpha \ge 1/2$, one cannot split the integrals in the last line of (2.9) and the second line therein is of more convenient form. We derive the upper bound by the respective first entry in the minimum in (2.8), the bounds by the second entry (and thus the resulting bounds by the minima) then follow from the underlying symmetry in m and n. Splitting the area of integration as before, we see that

$$\Big|\int_0^{\pi/(4m)} \frac{\sin(2m\varphi)\sin(2n\varphi)}{\sin^{2\alpha}(\varphi)+t}\,\mathrm{d}\varphi\Big| \le 4mn\int_0^{\pi/(4m)} \frac{\varphi^2}{\sin^{2\alpha}\varphi}\,\mathrm{d}\varphi$$
$$\le 4\Big(\frac{\pi}{2\sqrt{2}}\Big)^{2\alpha} mn\int_0^{\pi/(4m)} \varphi^{2-2\alpha}\,\mathrm{d}\varphi = \frac{2^{\alpha-4}\pi^3}{3-2\alpha}m^{2\alpha-2}n. \tag{2.11}$$

In the remaining part, we integrate by parts and obtain

$$\int_{\pi/(4m)}^{\pi/2} \frac{\sin(2m\varphi)\sin(2n\varphi)}{\sin^{2\alpha}(\varphi)+t}\,\mathrm{d}\varphi$$
$$= -\Big[\frac{\cos(2m\varphi)}{2m}\,\frac{\sin(2n\varphi)}{\sin^{2\alpha}(\varphi)+t}\Big]_{\pi/(4m)}^{\pi/2} + \int_{\pi/(4m)}^{\pi/2}\frac{\cos(2m\varphi)}{2m}\,\frac{\mathrm{d}}{\mathrm{d}\varphi}\Big(\frac{\sin(2n\varphi)}{\sin^{2\alpha}(\varphi)+t}\Big)\,\mathrm{d}\varphi$$
$$= \int_{\pi/(4m)}^{\pi/2}\frac{\cos(2m\varphi)}{2m}\Big(\frac{2n\cos(2n\varphi)}{\sin^{2\alpha}(\varphi)+t} - \frac{\sin(2n\varphi)2\alpha\sin^{2\alpha-1}\varphi\cos\varphi}{(\sin^{2\alpha}(\varphi)+t)^2}\Big)\,\mathrm{d}\varphi.$$

Taking absolute values and using (2.10), this leads to the estimate

$$\begin{aligned}(2.12)\qquad &\left|\int_{\pi/(4m)}^{\pi/2} \frac{\sin(2m\varphi)\sin(2n\varphi)}{\sin^{2\alpha}(\varphi)+t}\,\mathrm{d}\varphi\right| \\ &\qquad \leq \frac{n}{m}\left(\frac{\pi}{2\sqrt{2}}\right)^{2\alpha}\left(1+2\alpha\left(\frac{\pi}{2\sqrt{2}}\right)^{2\alpha}\right)\int_{\pi/(4m)}^{\pi/2}\varphi^{-2\alpha}\,\mathrm{d}\varphi.\end{aligned}$$

The claim then finally follows from

$$(2.13)\qquad \int_{\pi/(4m)}^{\pi/2}\varphi^{-2\alpha}\,\mathrm{d}\varphi = \begin{cases} \ln 2 + \ln m & \text{if } \alpha = 1/2, \\ \frac{1}{1-2\alpha}\left(\frac{\pi}{2}\right)^{1-2\alpha}\left(1-(2m)^{2\alpha-1}\right) & \text{if } \alpha \in (1/2, 3/2) \end{cases}$$

(and since the bound obtained from (2.12) and (2.13) is larger at infinity than (2.11) with $\alpha = 1/2$). Notice that in the second case, the term $m^{2\alpha-1}$ diverges due to $\alpha > 1/2$. ■

3. Proofs of Theorems 1.1, 1.3, and 1.4

Our method relies on the Birman–Schwinger principle, see Theorem 1 and Corollary 6 in [17], which allows to relate both criticality and subcriticality to the behavior of the Green kernel (2.5) as the spectral parameter λ approaches the spectrum. The uniform bounds from Lemmas 2.4, 2.5 and 2.6 guarantee a finite limit of the Green kernel as $\lambda \to 0^-$. This leads to Hardy-type inequalities and thus to the subcriticality of $(-\Delta)^\alpha$ for $\alpha \in (0, 3/2)$. On the other hand, if $\alpha \geq 3/2$, the diagonal entries of the Green kernel have a singularity as $\lambda \to 0^-$, which results in the criticality of $(-\Delta)^\alpha$.

3.1. Proof of Theorem 1.1

Recall that we always assume $\alpha > 0$. The statement of Theorem 1.1 is proven in two steps:

(i) if $\alpha < 3/2$, then $(-\Delta)^\alpha$ is subcritical;

(ii) if $\alpha \geq 3/2$, then $(-\Delta)^\alpha$ is critical.

Step (i): *Proof of the subcriticality in Theorem* 1.1. Suppose that $\alpha \in (0, 3/2)$. Consider the potential

$$V_n := \frac{\gamma}{n^4}, \quad n \in \mathbb{N},$$

where $\gamma > 0$. Since V is compact (even trace class), the spectrum of $(-\Delta)^\alpha - V$ below 0 can contain only eigenvalues. With this particular choice of V, we show that there exists a sufficiently small $\gamma > 0$ such that the operator norm of the Birman–Schwinger operator

$$(3.1)\qquad K(\lambda) := -V^{1/2}((-\Delta)^\alpha - \lambda)^{-1}V^{1/2}$$

fulfills

$$(3.2)\qquad \sup_{\lambda<0}\|K(\lambda)\| < 1.$$

By the Birman–Schwinger principle (Theorem 1 in [17]), it follows that there exists no negative eigenvalue of $(-\Delta)^\alpha - V$. Consequently, $(-\Delta)^\alpha \geq V$ and $(-\Delta)^\alpha$ is therefore subcritical.

By means of Lemma 2.4, we derive the following bound on the Hilbert–Schmidt (and thus operator) norm of $K(\lambda)$:

$$\|K(\lambda)\| \leq \|K(\lambda)\|_{\mathrm{HS}} = \Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|K_{m,n}(\lambda)|^2\Big)^{1/2} \leq C_\alpha \sum_{n=1}^{\infty} n^2 V_n = C_\alpha \sum_{n=1}^{\infty}\frac{\gamma}{n^2} \tag{3.3}$$

for all $\lambda < 0$. Thus, for any $0 < \gamma < 6/(\pi^2 C_\alpha)$, inequality (3.2) holds true. ■

For the proof of Step (ii), we will need two auxiliary results.

Lemma 3.1. *Let $\alpha \geq 3/2$. Then, for all $n \in \mathbb{N}$, we have*

$$\lim_{\lambda\to 0^-}((-\Delta)^\alpha - \lambda)^{-1}_{n,n} = +\infty.$$

Proof. Taking the formal limit $\lambda \to 0^-$ under the integral in (2.5) produces a singularity in the integrand at $x = 1$. The strategy is to split off an interval touching this singularity where the integrand is strictly positive (and one can thus easily show this portion of the integral to diverge). The remaining term can be easily bounded uniformly in λ. For this partition, the sign of the Chebyshev polynomials U_n is of interest. As we know from the proof of Lemma 2.4,

$$\max_{x\in[-1,1]}|U_{n-1}(x)| = U_{n-1}(1) = n, \quad n \in \mathbb{N}. \tag{3.4}$$

Therefore, we can pick a point larger than the largest zero of U_{n-1}, which are explicitly $\cos(k\pi/n)$, $k = 1,\dots,n-1$, see (2.6), e.g.,

$$\max\{x \in [-1,1] \mid U_{n-1}(x) = 0\} = \cos\Big(\frac{\pi}{n}\Big) < a_n := \cos\Big(\frac{\pi}{n+1}\Big) \in [0,1), \quad n \in \mathbb{N},$$

and we have

$$\min_{x\in[a_n,1]} U_{n-1}(x) > 0, \quad n \in \mathbb{N}.$$

For arbitrary fixed $n \in \mathbb{N}$ and a spectral parameter $\lambda < 0$, we start estimating (2.5) with $m = n$ from below. Since the integrand is non-negative, we have

$$((-\Delta)^\alpha - \lambda)^{-1}_{n,n} \geq \frac{2}{\pi}\int_{a_n}^1 \frac{U_{n-1}^2(x)}{2^\alpha(1-x)^\alpha - \lambda}\sqrt{1-x^2}\,\mathrm{d}x. \tag{3.5}$$

We show that the integral in (3.5) tends to infinity as $\lambda \to 0^-$. Using that $\alpha \geq 3/2$ and $a_n \geq 0$, we have

$$(1-x)^\alpha \leq (1-x)^{3/2}$$

for all $x \in [a_n, 1] \subset [0,1]$. We therefrom conclude a lower estimate on the integral in (3.5) as follows:

$$\begin{aligned}\int_{a_n}^1 &\frac{U_{n-1}^2(x)}{2^\alpha(1-x)^\alpha - \lambda}\sqrt{1-x^2}\,\mathrm{d}x \\ &\geq \sqrt{1+a_n}\min_{x\in[a_n,1]}(U_{n-1}^2(x))\int_{a_n}^1 \frac{\sqrt{1-x}}{2^\alpha(1-x)^{3/2}-\lambda}\,\mathrm{d}x.\end{aligned}$$

Hence, it suffices to check that the integral on the right-hand side tends to infinity as $\lambda \to 0^-$. This is the case indeed as, by the monotone convergence, one has

$$\lim_{\lambda\to 0^-} \int_{a_n}^{1} \frac{\sqrt{1-x}}{2^\alpha(1-x)^{3/2}-\lambda}\,\mathrm{d}x = \frac{1}{2^\alpha}\int_{a_n}^{1}\frac{\mathrm{d}x}{1-x} = \infty. \qquad \blacksquare$$

We next use the above lemma to show that, when $\alpha \geq 3/2$, for any arbitrarily small localized perturbation of $(-\Delta)^\alpha$ a unique eigenvalue emerges from the bottom of the spectrum. We denote by

$$\delta_n := \langle e_n, \cdot\rangle e_n$$

the delta potential localized at $n \in \mathbb{N}$.

Lemma 3.2. *Let $\alpha \geq 3/2$. Then for any $n \in \mathbb{N}$ and $c > 0$, the operator $(-\Delta)^\alpha - c\delta_n$ has a unique negative eigenvalue.*

Proof. Fix $c > 0$ and $n \in \mathbb{N}$. Since δ_n is a projection, one has $\delta_n = \delta_n^2$, and the corresponding Birman–Schwinger operator reads

$$K(\lambda) := -c\delta_n((-\Delta)^\alpha - \lambda)^{-1}\delta_n, \quad \lambda \in \mathbb{C}\setminus[0,4^\alpha].$$

For all $\lambda \in \mathbb{C}\setminus[0,4^\alpha]$, the Birman–Schwinger principle (Theorem 1 in [17]) gives the equivalence

$$\lambda \in \sigma_{\mathrm{p}}((-\Delta)^\alpha - c\delta_n) \quad \Longleftrightarrow \quad -1 \in \sigma_{\mathrm{p}}(K(\lambda)), \tag{3.6}$$

where σ_{p} denotes the point spectrum. Since $K(\lambda)$ is a rank one operator with the single non-zero eigenvalue

$$\mu(\lambda) := -c((-\Delta)^\alpha - \lambda)^{-1}_{n,n}, \tag{3.7}$$

equivalence (3.6) can be rewritten as

$$\lambda \in \sigma_{\mathrm{p}}((-\Delta)^\alpha - c\delta_n) \quad \Longleftrightarrow \quad \mu(\lambda) = -1. \tag{3.8}$$

The strategy is to show that there exists a unique $\lambda = \lambda_n(c) < 0$ such that $\mu(\lambda) = -1$. For this, it is sufficient to verify that μ, as a function on $(-\infty, 0)$, has the following properties:

(a) μ is strictly decreasing on $(-\infty, 0)$,

(b) μ is continuous on $(-\infty, 0)$,

(c) $\mu(\lambda) \to 0$ as $\lambda \to -\infty$,

(d) $\mu(\lambda) \to -\infty$ as $\lambda \to 0^-$.

Property (a) is immediate from the integral representation

$$\mu(\lambda) = -\frac{2c}{\pi}\int_{-1}^{1} \frac{U_{n-1}^2(x)}{2^\alpha(1-x)^\alpha - \lambda}\sqrt{1-x^2}\,\mathrm{d}x, \tag{3.9}$$

see (2.5). Since the resolvent $((-\Delta)^\alpha - \lambda)^{-1}$ is an analytic (operator-valued) function of λ on $\mathbb{C}\setminus[0,4^\alpha]$, (b) follows. By monotone convergence applied to the integral in (3.9), one easily verifies property (c). Finally, (d) is a consequence of Lemma 3.1. $\blacksquare$

We are now ready to prove the remaining part of Theorem 1.1.

Step (ii): *Proof of the criticality in Theorem* 1.1. Let $\alpha \geq 3/2$. From Lemma 3.2, it follows that if

$$(-\Delta)^\alpha \geq c\delta_n$$

with any $n \in \mathbb{N}$ and $c \geq 0$, then necessarily $c = 0$. Suppose that $V \geq 0$ is a given bounded potential such that $(-\Delta)^\alpha \geq V$. Then, for all $n \in \mathbb{N}$, one has $V \geq V_n\delta_n$ and thus $(-\Delta)^\alpha \geq V_n\delta_n$. By the observation above, it follows that $V_n = 0$. Since $n \in \mathbb{N}$ is arbitrary, we conclude $V = 0$. ■

3.2. A comment on the subcriticality of $(-\Delta)^\alpha$ from above

Recall that $\sigma((-\Delta)^\alpha) = [0, 4^\alpha]$ for any $\alpha > 0$. Analogously to the usual notion of criticality of $(-\Delta)^\alpha$, one may ask about the stability of the upper bound 4^α when $(-\Delta)^\alpha$ is perturbed by a bounded potential $V \geq 0$. In other words, we may investigate the criticality or subcriticality of the operator $4^\alpha - (-\Delta)^\alpha$. It turns out that this operator is subcritical for all $\alpha > 0$, which can be seen from essentially the same arguments as in the proof of Theorem 1.1.

Proposition 3.3. *The operator* $4^\alpha - (-\Delta)^\alpha$ *is subcritical for all* $\alpha > 0$.

Proof. Fix any $\alpha > 0$. First, from (2.5) we deduce the integral representation of the Green function

$$(4^\alpha - (-\Delta)^\alpha - \lambda)^{-1}_{m,n} = \frac{2}{\pi}\int_{-1}^{1} \frac{U_{m-1}(x)\,U_{n-1}(x)}{4^\alpha - 2^\alpha(1-x)^\alpha - \lambda}\sqrt{1-x^2}\,\mathrm{d}x$$

for any $\lambda \notin [0, 4^\alpha]$ and $m, n \in \mathbb{N}$. Next, similarly as in Lemma 2.4, we deduce the uniform bound

$$|\,(4^\alpha - (-\Delta)^\alpha - \lambda)^{-1}_{m,n}\,| \leq C_\alpha mn$$

for all $\lambda < 0$, with the positive constant

$$C_\alpha := \frac{2}{\pi}\int_{-1}^{1} \frac{\sqrt{1-x^2}}{4^\alpha - 2^\alpha(1-x)^\alpha}\,\mathrm{d}x.$$

The only difference to the proof of the subcriticality of $(-\Delta)^\alpha$ is that the above integral is always finite regardless the value of $\alpha > 0$. Indeed, its integrand is a continuous function on $(-1, 1]$ which equals $O(\sqrt{x+1})$ as $x \to -1^+$. The rest of the proof is analogous to Step (i) in the proof of Theorem 1.1. ■

3.3. Proof of Theorem 1.3

As a byproduct of our method, in the proof of the subcriticality in Theorem 1.1 we already obtain a (rather rough) Hardy inequality for $(-\Delta)^\alpha$. This is done using the bound from Lemma 2.4 when estimating the Hilbert–Schmidt norm of the Birman–Schwinger operator, see (3.3). We proceed similarly with the refined bound in Lemma 2.5 to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\alpha \in (0, 3/2)$. Suppose first that a potential $V \geq 0$ satisfies condition (1.3) with the strict inequality, i.e., that

$$\sum_{n=1}^{\infty} g_n(\alpha) V_n < 2\pi \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)}. \tag{3.10}$$

Using Lemma 2.5, we estimate the norm of the corresponding Birman–Schwinger operator, cf. (3.1) and (3.3), as follows:

$$\begin{aligned} \|K(\lambda)\| &\leq \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |K_{m,n}(\lambda)|^2 \Big)^{1/2} \\ &= \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_m \big|((-\Delta)^\alpha - \lambda)^{-1}_{m,n}\big|^2 V_n \Big)^{1/2} \leq \frac{1}{2\pi} \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} \sum_{n=1}^{\infty} V_n \, g_n(\alpha) \end{aligned}$$

for all $\lambda < 0$. Hence, assumption (3.10) implies $\|K(\lambda)\| < 1$ for all $\lambda < 0$, and thus further

$$(-\Delta)^\alpha \geq V$$

by the Birman–Schwinger principle (Corollary 6 in [17]).

Suppose now that $V \geq 0$ satisfies (1.3). We introduce the auxiliary potentials

$$V(q) := qV \quad \text{for all } q \in (0, 1).$$

Then (3.10) holds for $V(q)$, and therefore

$$(-\Delta)^\alpha \geq V(q)$$

for all $q \in (0, 1)$ by the first part of this proof. Since $V(q) \to V$ converges strongly as $q \to 1^-$, the above inequality in sense of forms remains valid in the limit, and we conclude that $(-\Delta)^\alpha \geq V$. ∎

3.4. Proof of Theorem 1.4

Similarly to Theorem 1.3, the Hardy-type inequality is achieved by using the bound in Lemma 2.6 on the resolvent kernel to derive a uniform estimate on the norm of the Birman–Schwinger operator. The structure of the resolvent estimate thereby allows for a weighted Schur test along the lines of [26, 27] instead of having to bound the Hilbert–Schmidt norm.

Proof of Theorem 1.4. As before, the claimed inequality follows from the uniform bound $\|K(\lambda)\| < 1$ on the norm of the Birman–Schwinger operator for all $\lambda < 0$, cf. the proof of Theorem 1.3 and Theorem 1 and Corollary 6 in [17]. We suitably employ a weighted Schur test for the matrix operator

$$K_{m,n}(\lambda) = -\sqrt{V_m}\, ((-\Delta)^\alpha - \lambda)^{-1}_{m,n} \sqrt{V_n}, \quad m, n \in \mathbb{N}.$$

Case $\alpha \in (0, 1/2)$.

In view of Lemma 2.6, it suffices to apply the weighted Schur test to $K(\lambda)$ with the resolvent kernel replaced by the upper bound in (2.8). Choosing both weights in the Schur test as $1/n^{\alpha+\varepsilon}$ with suitable $\varepsilon > 0$, and since the problem is symmetric in m and n, it thus suffices to show that there exists $M_\alpha > 0$ such that for all $n \in \mathbb{N}$ one has

$$\sum_{m\in\mathbb{N}, m\neq n} \frac{1}{m^\alpha} \frac{1}{|m-n|^{1-2\alpha}} \frac{1}{n^\alpha} \frac{1}{m^{\alpha+\varepsilon}} + \frac{1}{n^{3\alpha+\varepsilon}} \leq \frac{M_\alpha}{n^{\alpha+\varepsilon}}.$$

The second summand (which is the contribution from $m = n$) can be trivially bounded by the right-hand side and can hence be omitted. We multiply the rest of the claimed inequality by n^α, split the sum into three parts and estimate them separately. For the first part of the sum, we observe that if $m \leq n/2$, then $|m-n| \geq n/2$, and thus

$$\sum_{m=1}^{\lfloor n/2\rfloor} \frac{1}{|m-n|^{1-2\alpha}} \frac{1}{m^{2\alpha+\varepsilon}} \leq \left(\frac{2}{n}\right)^{1-2\alpha} \sum_{m=1}^{\lfloor n/2\rfloor} \frac{1}{m^{2\alpha+\varepsilon}} \leq \left(\frac{2}{n}\right)^{1-2\alpha} \left(1 + \int_1^{n/2} \frac{\mathrm{d}m}{m^{2\alpha+\varepsilon}}\right).$$

The needed bound for this sum follows from

$$\int_1^{n/2} \frac{\mathrm{d}m}{m^{2\alpha+\varepsilon}} = \frac{1}{1-2\alpha-\varepsilon}\left(\left(\frac{n}{2}\right)^{1-2\alpha-\varepsilon} - 1\right)$$

and choosing $\varepsilon < 1-2\alpha$ such that the integral diverges like $n^{1-2\alpha-\varepsilon}$ (which is possible since $\alpha < 1/2$). In the next part of the sum, we use $n/2 \leq m \leq 2n$ to estimate

$$\begin{aligned}
&\sum_{m=\lfloor n/2\rfloor+1, m\neq n}^{2n} \frac{1}{|m-n|^{1-2\alpha}} \frac{1}{m^{2\alpha+\varepsilon}} \\
&\qquad \leq \left(\frac{2}{n}\right)^{2\alpha+\varepsilon} \left(\sum_{m=\lfloor n/2\rfloor+1}^{n-2} \frac{1}{(n-m)^{1-2\alpha}} + 2 + \sum_{m=n+2}^{2n} \frac{1}{(m-n)^{1-2\alpha}} \right) \\
&\qquad \leq \left(\frac{2}{n}\right)^{2\alpha+\varepsilon} \left(\int_{n/2}^{n-1} (n-m)^{2\alpha-1}\,\mathrm{d}m + 2 + \int_{n+1}^{2n} (m-n)^{2\alpha-1}\,\mathrm{d}m \right).
\end{aligned}$$

As before, the bound straightforwardly follows by calculating the above integrals. Finally, observing that if $m \geq 2n$ then $|m-n| \geq m/2$, the tail of the sum is estimated as

$$\begin{aligned}
\sum_{m=2n+1}^{\infty} \frac{1}{|m-n|^{1-2\alpha}} \frac{1}{m^{2\alpha+\varepsilon}} &\leq 2^{1-2\alpha} \sum_{m=2n+1}^{\infty} \frac{1}{m^{1+\varepsilon}} \\
&\leq 2^{1-2\alpha} \int_{2n}^{\infty} m^{-1-\varepsilon}\,\mathrm{d}m = \frac{2^{1-2\alpha-\varepsilon}}{\varepsilon n^\varepsilon}.
\end{aligned}$$

Case $\alpha = 1/2$.

In this case, no weight is needed and by the symmetry we again only need to estimate the sum over $m \in \mathbb{N}$. It again suffices to employ the Schur test with the bound on the

Green kernel of the resolvent from Lemma 2.6, while we split the summation at $m = n$ and use the respective more convenient member of the minimum in (2.8) as follows:

$$\sum_{m\in\mathbb{N}} |K_{m,n}(\lambda)| \leq C_{1/2} \sum_{m=1}^{n} \frac{1}{m^{1/2}\ln^{1/2}(m+1)} \frac{m}{n} \ln(n+1) \frac{1}{n^{1/2}\ln^{1/2}(n+1)}$$
$$+ C_{1/2} \sum_{m=n+1}^{\infty} \frac{1}{m^{1/2}\ln^{1/2}(m+1)} \frac{n}{m} \ln(m+1) \frac{1}{n^{1/2}\ln^{1/2}(n+1)}.$$

We need to prove that the right-hand side is bounded in $n \in \mathbb{N}$. As the function $m \mapsto m/\log(m+1)$ is increasing on $[1,\infty)$, we may estimate the first sum by

$$\frac{\ln^{1/2}(n+1)}{n^{3/2}} \sum_{m=1}^{n} \frac{m^{1/2}}{\ln^{1/2}(m+1)} \leq \frac{\ln^{1/2}(n+1)}{n^{3/2}} \frac{n^{3/2}}{\ln^{1/2}(n+1)} = 1.$$

For the tail, we proceed by an integral estimate:

$$\frac{n^{1/2}}{\ln^{1/2}(n+1)} \sum_{m=n+1}^{\infty} \frac{\ln^{1/2}(m+1)}{m^{3/2}} \leq \frac{n^{1/2}}{\ln^{1/2}(n+1)} \int_n^\infty \frac{\ln^{1/2}(m+1)}{m^{3/2}}\, \mathrm{d}m.$$

Integrating by parts gives

$$\int_n^\infty \frac{\ln^{1/2}(m+1)}{m^{3/2}}\, \mathrm{d}m = -2\left[\frac{\ln^{1/2}(m+1)}{m^{1/2}}\right]_n^\infty + \int_n^\infty \frac{\mathrm{d}m}{m^{1/2}(m+1)\ln^{1/2}(m+1)}.$$

Further estimations show that

$$\int_n^\infty \frac{\ln^{1/2}(m+1)}{m^{3/2}}\, \mathrm{d}m \leq 2\,\frac{\ln^{1/2}(n+1)}{n^{1/2}} + \frac{1}{\sqrt{\ln 2}} \int_n^\infty \frac{\mathrm{d}m}{m^{3/2}}.$$

Since the last integral decays as $n^{-1/2}$, we see that there exists a constant $M > 0$ such that

$$\left|\int_n^\infty \frac{\ln^{1/2}(m+1)}{m^{3/2}}\, \mathrm{d}m\right| \leq \frac{M\ln^{1/2}(n+1)}{n^{1/2}}, \quad n \in \mathbb{N}.$$

It follows that the tail of the sum is also uniformly bounded, which completes the proof in the case $\alpha = 1/2$.

Case $\alpha \in (1/2, 3/2)$.

Here we choose both weights in the Schur test to be $1/n^{\alpha-1+\varepsilon}$ with suitable $\varepsilon > 0$, and again estimate only the sum over $m \in \mathbb{N}$ due to the symmetry. Splitting the summation along the diagonal and using the respective suitable bound from the minimum in (2.8), it suffices to bound

$$\sum_{m\in\mathbb{N}} |K_{m,n}(\lambda)| \frac{1}{m^{\alpha-1+\varepsilon}} \leq C_\alpha \sum_{m=1}^{n} \frac{1}{n^\alpha} n^{2\alpha-2} m \frac{1}{m^\alpha} \frac{1}{m^{\alpha-1+\varepsilon}}$$
$$+ C_\alpha \sum_{m=n+1}^{\infty} \frac{1}{n^\alpha} m^{2\alpha-2} n \frac{1}{m^\alpha} \frac{1}{m^{\alpha-1+\varepsilon}}$$

by a constant multiple of $1/n^{\alpha-1+\varepsilon}$. For the first sum, we estimate

$$\frac{1}{n^{2-\alpha}} \sum_{m=1}^{n} \frac{1}{m^{2\alpha-2+\varepsilon}} \leq \frac{1}{n^{2-\alpha}} \Big(1 + \int_1^n \frac{\mathrm{d}m}{m^{2\alpha-2+\varepsilon}} + \frac{1}{n^{2\alpha-2+\varepsilon}}\Big),$$

and calculating the integral

$$\int_1^n \frac{\mathrm{d}m}{m^{2\alpha-2+\varepsilon}} = \frac{1}{3-2\alpha-\varepsilon}\,(1-n^{3-2\alpha-\varepsilon}).$$

The needed bound then readily follows by requiring $\varepsilon < 3-2\alpha$ such that the above integral diverges like $n^{3-2\alpha-\varepsilon}$ (which is possible since $\alpha < 3/2$). For the tail of the sum, the claimed bound is immediate from

$$\frac{1}{n^{\alpha-1}} \sum_{m=n+1}^{\infty} \frac{1}{m^{1+\varepsilon}} \leq \frac{1}{n^{\alpha-1}} \int_n^\infty m^{-1-\varepsilon}\,\mathrm{d}m = \frac{1}{\varepsilon n^{\alpha-1+\varepsilon}}. \qquad \blacksquare$$

4. Coupling with localized potentials

We study the asymptotic behavior of the unique negative eigenvalue of the perturbation $(-\Delta)^\alpha - c\delta_n$ from Lemma 3.2 in the regime of small and large coupling constants. Recall that U_n denotes the nth Chebyshev polynomial of the second kind.

Proposition 4.1. *For $\alpha \geq 3/2$ and for $n \in \mathbb{N}$, the unique negative eigenvalue $\lambda_n(c)$ of $(-\Delta)^\alpha - c\delta_n$ with $c > 0$ satisfies the asymptotic formula*

$$\lambda_n(c) = \begin{cases} -e^{-\frac{3\pi}{2n^2c}(1+\mathcal{O}(c))} & \text{if } \alpha = 3/2, \\ -\Big(\frac{n^2 c}{\alpha \sin\left(\frac{3\pi}{2\alpha}\right)}\Big)^{\frac{2\alpha}{2\alpha-3}} (1+r(c)) & \text{if } \alpha > 3/2, \end{cases} \qquad \text{as } c \to 0^+,$$

with the decaying remainder

$$r(c) = \begin{cases} \mathcal{O}(c) & \text{if } \alpha \in (3/2, 5/2), \\ \mathcal{O}(c \ln \frac{1}{c}) & \text{if } \alpha = 5/2, \\ \mathcal{O}\big(c^{\frac{2}{2\alpha-3}}\big) & \text{if } \alpha > 5/2. \end{cases}$$

For large coupling constants, one has

$$\lambda_n(c) = -c\Big(1 + \mathcal{O}\Big(\frac{1}{c}\Big)\Big), \quad \text{as } c \to +\infty.$$

Proof. According to (3.8) and (3.9), the defining equation for $\lambda = \lambda_n(c)$ reads

$$\int_{-1}^{1} \frac{U_{n-1}^2(x)}{2^\alpha(1-x)^\alpha - \lambda} \sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{2c}.$$

From this formula, it is clear that λ tends to zero as $c \to 0^+$, and to negative infinity as $c \to +\infty$. We first consider the regime $c \to 0^+$. Using Lemma A.3, it follows that one can write

$$\frac{\pi}{2c} = \begin{cases} \frac{n^2}{3} \ln \frac{1}{|\lambda|} + \mathcal{O}(1) & \text{if } \alpha = 3/2, \\ \frac{n^2\pi}{2\alpha \sin\left(\frac{3\pi}{2\alpha}\right)} |\lambda|^{\frac{3-2\alpha}{2\alpha}} + s(|\lambda|) & \text{if } \alpha > 3/2, \end{cases} \tag{4.1}$$

with

$$s(|\lambda|) = \begin{cases} \mathcal{O}(1) & \text{if } \alpha \in (3/2, 5/2), \\ \mathcal{O}\left(\ln \frac{1}{|\lambda|}\right) & \text{if } \alpha = 5/2, \\ \mathcal{O}\left(|\lambda|^{\frac{5}{2\alpha}-1}\right) & \text{if } \alpha > 5/2. \end{cases}$$

Reducing $|\lambda| \equiv -\lambda$ in equation (4.1) for $\alpha = 3/2$, one immediately obtains the respective claim. Further, from (4.1) for $\alpha > 3/2$, one deduces the relation

$$|\lambda| = \mathcal{O}\left(c^{\frac{2\alpha}{2\alpha-3}}\right)$$

and the equation can be written as

$$\frac{\pi}{2c}\left(1 + \tilde{r}(c)\right) = \frac{n^2\pi}{2\alpha \sin\left(\frac{3\pi}{2\alpha}\right)} |\lambda|^{\frac{3-2\alpha}{2\alpha}},$$

where the remainder

$$\tilde{r}(c) := -cs(|\lambda|)$$

decays as a function of c as specified in the claim. Solving the last equation for $|\lambda|$ yields the asymptotic formula for $c \to 0^+$ when $\alpha > 3/2$.

In the regime $c \to +\infty$, the claim follows in an analogous way from Lemma A.4. ■

For the discrete bilaplacian, the emerging negative eigenvalue can be characterized by an implicit equation. If $n = 1$, it can even be expressed fully explicitly as a function of the coupling constant.

Proposition 4.2. *For $n \in \mathbb{N}$ and $c > 0$, the unique negative eigenvalue of $\Delta^2 - c\delta_n$ is given as*

$$\lambda_n(c) = -\frac{(1-r^2)^4}{r^2(1+r^2)^2},$$

where r is the unique solution of the implicit equation

$$\frac{r^2}{1-r^2} \sum_{j=0}^{n-1} r^{2j}\, U_{2j}\left(\frac{2r}{1+r^2}\right) = \frac{1}{c}$$

in $(0,1)$. In particular, for $n = 1$ it is

$$\lambda_1(c) = -\frac{c^4}{(c+1)(c+2)^2}.$$

The proof relies on an explicit formula for the Green kernel of the resolvent. The latter might be of independent interests and is therefore stated in a lemma. It uses a convenient transformation of the spectral parameter. Recall that the Joukowski transform is the bijection

$$\phi : \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus [-2, 2], \quad \phi(z) := z + z^{-1},$$

where $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. For every $\lambda \notin \sigma(\Delta^2) = [0, 16]$, there exist unique $0 \neq \xi, \eta \in \mathbb{D}$ such that

$$2 + \sqrt{\lambda} = \xi + \xi^{-1} \quad \text{and} \quad 2 - \sqrt{\lambda} = \eta + \eta^{-1}. \tag{4.2}$$

Notice that the resolvent formula below does not depend on the particular definition of the complex square root. Choosing another branch only exchanges the roles of ξ and η, in which the formula indeed commutes.

Lemma 4.3. *For $\lambda \in \mathbb{C} \setminus [0, 16]$ and $m, n \in \mathbb{N}$, the Green kernel of the bilaplacian is given by*

$$(\Delta^2 - \lambda)^{-1}_{m,n} = \frac{\xi\eta}{(1 - \xi\eta)(\xi - \eta)}\Big(\frac{\xi^{m+n} - \xi^{|m-n|}}{\xi - \xi^{-1}} - \frac{\eta^{m+n} - \eta^{|m-n|}}{\eta - \eta^{-1}}\Big).$$

Here $0 \neq \xi, \eta \in \mathbb{D}$ are uniquely determined by (4.2).

Proof. Follows immediately from (2.5) and Lemma A.5. ∎

Proof of Proposition 4.2. Notice first that if $\lambda < 0$, then $\sqrt{\lambda} \in \mathrm{i}\mathbb{R}$, such that $\xi = \overline{\eta}$ follows easily from (4.2). Hence, using Lemma 4.3 with $m = n$, the defining relations (3.7) and (3.8) for $\lambda = \lambda_n(c)$ become

$$\frac{1}{c} = \frac{|\xi|^2}{1 - |\xi|^2}\frac{1}{\operatorname{Im}\xi}\operatorname{Im}\Big(\frac{\xi^{2n} - 1}{\xi - \xi^{-1}}\Big) = \frac{|\xi|^2}{1 - |\xi|^2}\sum_{j=0}^{n-1}\frac{\operatorname{Im}(\xi^{2j+1})}{\operatorname{Im}\xi}.$$

Next, from (4.2) it is elementary to derive

$$\operatorname{Re}\xi = \frac{2|\xi|^2}{1 + |\xi|^2} \quad \text{and} \quad \operatorname{Im}^2\xi = -\frac{\lambda|\xi|^4}{(1 - |\xi|^2)^2}, \tag{4.3}$$

and thus for the cosine of the argument,

$$\cos(\operatorname{Arg}\xi) = \frac{\operatorname{Re}\xi}{|\xi|} = \frac{2|\xi|}{1 + |\xi|^2}.$$

Using (2.6), we can further compute

$$\frac{\operatorname{Im}(\xi^{2j+1})}{\operatorname{Im}\xi} = |\xi|^{2j}\frac{\sin((2j + 1)\operatorname{Arg}\xi)}{\sin(\operatorname{Arg}\xi)} = |\xi|^{2j}U_{2j}\Big(\frac{2|\xi|}{1 + |\xi|^2}\Big).$$

Combining the above, we see that $|\xi|$ solves the equation

$$\frac{1}{c} = \frac{|\xi|^2}{1 - |\xi|^2}\sum_{j=0}^{n-1}|\xi|^{2j}U_{2j}\Big(\frac{2|\xi|}{1 + |\xi|^2}\Big). \tag{4.4}$$

The dependence of λ on $|\xi|$ can be expressed from (4.3) as

$$\lambda = -\frac{(1-|\xi|^2)^4}{|\xi|^2(1+|\xi|^2)^2}. \tag{4.5}$$

From this it is easy to see that there is a one to one correspondence between $\lambda < 0$ and $|\xi| \in (0,1)$. Hence an occurrence of two different solutions $|\xi|$ of (4.4) gives rise to two different eigenvalues of $\Delta^2 + c\delta_n$, which would contradict Lemma 3.2. Therefore there is exactly one solution $|\xi|$ of (4.4) located in $(0,1)$. The formula for $n = 1$ then follows easily from (4.4) and (4.5). ∎

A. Integral identities and asymptotics with Chebyshev polynomials

To compute two integrals with Chebyshev polynomials needed in our proofs, we use a slight modification of equation 8 in 3.631 of [13] which reads

$$\int_0^{\pi/2} \sin^{\nu-1}(\varphi)\cos(2\ell\varphi)\,\mathrm{d}\varphi = \frac{(-1)^\ell \pi\,\Gamma(\nu)}{2^\nu\Gamma\left(\frac{\nu+1}{2}+\ell\right)\Gamma\left(\frac{\nu+1}{2}-\ell\right)} \tag{A.1}$$

for any $\ell \in \mathbb{Z}$ and $\operatorname{Re}\nu > 0$. Indeed, this formula can easily be derived from equation 8 in 3.631 of [13] by using the relation $\sin x = \sin(\pi - x)$ and the fact that $a = 2l \in 2\mathbb{Z}$ therein.

Lemma A.1. *For all $m, n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > -3/2$, one has*

$$\int_{-1}^{1}(1-x)^\alpha\,U_{m-1}(x)\,U_{n-1}(x)\sqrt{1-x^2}\,dx \\ = \frac{\pi}{2^{\alpha+1}}(-1)^{m+n}\left[\binom{2\alpha}{\alpha+m-n} - \binom{2\alpha}{\alpha+m+n}\right],$$

where the generalized binomial number is defined in (2.4). For fixed $m, n \in \mathbb{N}$, the right-hand side of the formula is understood as the respective limit at its removable singularities $\alpha = -1$ and $\alpha = -1/2$.

Proof. Similarly as (2.9) follows from (2.5), we use the substitution $x := \cos\theta$ and (2.6) to get

$$\begin{aligned}\int_{-1}^{1}(1-x)^\alpha\,U_{m-1}(x)\,U_{n-1}(x)\sqrt{1-x^2}\,\mathrm{d}x \\ = 2^\alpha\int_0^{\pi/2}\sin^{2\alpha}(\varphi)\left[\cos(2(m-n)\varphi) - \cos(2(m+n)\varphi)\right]\mathrm{d}\varphi.\end{aligned} \tag{A.2}$$

In the last integral, we apply identity (A.1) twice and find that it is equal to

$$\frac{\pi}{2^{\alpha+1}}(-1)^{m+n}\Big[\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1+m-n)\,\Gamma(\alpha+1-m+n)} \\ - \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1+m+n)\,\Gamma(\alpha+1-m-n)}\Big].$$

When rewritten in terms of the binomial numbers, we arrive at the statement. Note that even though formula (A.1) only applies for $\mathrm{Re}\,\alpha > -1/2$, (with $m, n \in \mathbb{N}$ fixed), a straightforward identity argument between holomorphic functions implies the sought equality for $\mathrm{Re}\,\alpha > -3/2$. To this end, the left-hand side of (A.2) is easily verified to be holomorphic on the half plane $\mathrm{Re}\,\alpha > -3/2$ by dominated convergence. ■

Lemma A.2. *For all $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha < 3/2$, one has*

$$\int_{-1}^{1} \frac{U_{n-1}^2(x)}{(1-x)^\alpha} \sqrt{1-x^2}\, dx = 2^{\alpha-2} \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} \left(1 - \frac{(\alpha)_{2n}}{(1-\alpha)_{2n}}\right) \tan(\pi\alpha),$$

where $(\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1)$ is the Pochhammer symbol. At the removable singularities $\alpha \in \mathbb{Z}/2$, the right-hand side is to be understood as the respective limit.

Proof. As in (2.7), we denote the integral on the left-hand side of the claimed formula by $I_n(\alpha)$. Lemma A.1 applied with $m = n$ and α replaced by $-\alpha$ leads to

$$I_n(\alpha) = \frac{\pi\Gamma(1-2\alpha)}{2^{1-\alpha}} \left(\frac{1}{\Gamma^2(1-\alpha)} - \frac{1}{\Gamma(1-\alpha+2n)\Gamma(1-\alpha-2n)}\right).$$

Repeatedly using the well-known identity $\Gamma(z+1) = z\Gamma(z)$, we arrive at

$$\frac{1}{\Gamma(1-\alpha+2n)\Gamma(1-\alpha-2n)} = \frac{1}{\Gamma^2(1-\alpha)} \frac{(\alpha)_{2n}}{(1-\alpha)_{2n}},$$

and therefore

$$I_n(\alpha) = \pi 2^{\alpha-1} \frac{\Gamma(1-2\alpha)}{\Gamma^2(1-\alpha)} \left(1 - \frac{(\alpha)_{2n}}{(1-\alpha)_{2n}}\right).$$

Applying the reflection identity

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

one further gets

$$\frac{\Gamma(1-2\alpha)}{\Gamma^2(1-\alpha)} = \frac{\sin^2(\pi\alpha)}{\pi\sin(2\pi\alpha)} \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} = \frac{\tan(\pi\alpha)}{2\pi} \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}.$$

The claimed formula now readily follows.

Notice that, for $n \in \mathbb{N}$ fixed, the left-hand side of the claimed identity is analytic for $\mathrm{Re}\,\alpha < 3/2$, while the right-hand side has removable singularities at $\alpha \in -\mathbb{N}_0/2$, as well as at $\alpha = 1/2$ and $\alpha = 1$. The respective formulas at the two positive parameters can be determined as

$$\lim_{\alpha\to 1/2} I_n(\alpha) = \sqrt{2} \sum_{j=1}^{2n} \frac{1}{2j-1} \quad \text{and} \quad \lim_{\alpha\to 1} I_n(\alpha) = \pi n. \qquad ■$$

In the weak coupling regime in Section 4, the needed asymptotic formulae for the Green kernel of the resolvent are a consequence of the following lemma.

Lemma A.3. *For all $n \in \mathbb{N}$, we have the relation*

$$\int_{-1}^{1} \frac{U_{n-1}^2(x)}{2^\alpha(1-x)^\alpha + t} \sqrt{1-x^2}\, dx = \begin{cases} \frac{n^2}{3} \ln \frac{1}{t} + \mathcal{O}(1) & \text{if } \alpha = 3/2, \\ \frac{n^2 \pi}{2\alpha \sin\left(\frac{3\pi}{2\alpha}\right)}\, t^{\frac{3}{2\alpha}-1} + r(t) & \text{if } \alpha > 3/2, \end{cases}$$

as $t \to 0^+$, where $r(t)$ is the decaying remainder

$$r(t) = \begin{cases} \mathcal{O}(1) & \text{if } \alpha \in (3/2, 5/2), \\ \mathcal{O}\left(\ln \frac{1}{t}\right) & \text{if } \alpha = 5/2, \\ \mathcal{O}\left(t^{\frac{5}{2\alpha}-1}\right) & \text{if } \alpha > 5/2. \end{cases}$$

Proof. It follows from a dominated convergence argument that the integral over the subinterval $[-1, 0]$ is bounded as $t \to 0^+$. The (divergent) quantity of interest is thus

$$I(t) := \int_0^1 \frac{U_{n-1}^2(x)}{2^\alpha(1-x)^\alpha + t} \sqrt{1-x^2}\, \mathrm{d}x.$$

By the mean value theorem applied to the function $U_{n-1}^2(x)\sqrt{1+x}$ together with (3.4), there exists a bounded function $\omega\colon [0,1] \to \mathbb{R}$ such that

$$I(t) = n^2\sqrt{2} \int_0^1 \frac{\sqrt{1-x}}{2^\alpha(1-x)^\alpha + t}\, \mathrm{d}x + \int_0^1 \frac{\omega(x)(1-x)^{3/2}}{2^\alpha(1-x)^\alpha + t}\, \mathrm{d}x. \tag{A.3}$$

It remains to study the asymptotic behavior of the above integrals as $t \to 0^+$ (while the first one grows faster and the second one gives only a lower order term). We denote the respective first and second integral by $I_{1,\alpha}(t)$ and $I_{2,\alpha}(t)$. The substitution $2^\alpha(1-x)^\alpha = y^\alpha t$ and the boundedness of ω lead to

$$\begin{aligned} I_{1,\alpha}(t) &= \frac{1}{2\sqrt{2}}\, t^{3/(2\alpha)-1} \int_0^{2t^{-1/\alpha}} \frac{\sqrt{y}}{y^\alpha + 1}\, \mathrm{d}y, \\ |I_{2,\alpha}(t)| &\leq \frac{C}{4\sqrt{2}}\, t^{5/(2\alpha)-1} \int_0^{2t^{-1/\alpha}} \frac{y^{3/2}}{y^\alpha + 1}\, \mathrm{d}y, \end{aligned}$$

where $C := \max_{x\in[0,1]} |\omega(x)|$.

We start by analyzing $I_{1,\alpha}(t)$. For $\alpha > 3/2$, the integral therein converges as $t \to 0^+$ and its remainder can be bounded as follows:

$$\int_{2t^{-1/\alpha}}^{\infty} \frac{\sqrt{y}}{y^\alpha + 1}\, \mathrm{d}y \leq \int_{2t^{-1/\alpha}}^{\infty} y^{1/2-\alpha}\, \mathrm{d}y = \mathcal{O}(t^{1-3/(2\alpha)}).$$

Using equation 2 in 3.241 of [13] to determine the value of the limit integral, we obtain

$$I_{1,\alpha}(t) = \frac{1}{2\sqrt{2}}\, t^{3/(2\alpha)-1}\Big(\frac{\pi}{\alpha \sin\left(\frac{3\pi}{2\alpha}\right)} + \mathcal{O}(t^{1-3/(2\alpha)})\Big), \quad \alpha > \frac{3}{2}.$$

When $\alpha = 3/2$, on the other hand, the integral in $I_{1,3/2}(t)$ diverges as $t \to 0_+$. Indeed, the substitution $z = y^{3/2} + 1$ then gives

$$I_{1,3/2}(t) = \frac{1}{3\sqrt{2}} \int_1^{2\sqrt{2}t^{-1}+1} \frac{1}{z} \, \mathrm{d}z = \frac{1}{3\sqrt{2}} \ln(t^{-1}) + \mathcal{O}(1).$$

Considering $I_{2,\alpha}(t)$, analogous arguments lead to

$$I_{2,\alpha}(t) = \begin{cases} \mathcal{O}(\ln(t^{-1})) & \text{if } \alpha = 5/2, \\ \mathcal{O}\big(t^{\frac{5}{2\alpha}-1}\big) & \text{if } \alpha > 5/2. \end{cases}$$

In case that $\alpha \in [3/2, 5/2)$, we have

$$|I_{2,\alpha}(t)| \le \frac{C}{4\sqrt{2}} t^{5/(2\alpha)-1} \int_0^{2t^{-1/\alpha}} y^{3/2-\alpha} \, \mathrm{d}y = t^{5/(2\alpha)-1} \, \mathcal{O}(t^{1-5/(2\alpha)}) = \mathcal{O}(1).$$

The claim follows by combining the proven asymptotic relations for $I_{1,\alpha}(t)$ and $I_{2,\alpha}(t)$ with (A.3), together with the boundedness of the integral over $[-1, 0]$. ■

In the strong coupling limit in Section 4, the asymptotic regime for large t is needed.

Lemma A.4. *For all $n \in \mathbb{N}$, we have*

$$\int_{-1}^{1} \frac{U_{n-1}^2(x)}{2^\alpha (1-x)^\alpha + t} \sqrt{1-x^2} \, dx = \frac{\pi}{2t}\Big(1 + \mathcal{O}\Big(\frac{1}{t}\Big)\Big), \quad t \to +\infty.$$

Proof. By Taylor's theorem, we have

$$\frac{1}{1 + 2^\alpha (1-x)^\alpha t^{-1}} = 1 + r(t,x), \quad r(t,x) = \mathcal{O}\Big(\frac{1}{t}\Big), \quad t \to +\infty,$$

where the asymptotic relation is uniform in $x \in [-1, 1]$. Using the orthonormality property of the Chebyshev polynomials, the claim then follows from

$$\begin{aligned} &\int_{-1}^{1} \frac{U_{n-1}^2(x)}{2^\alpha (1-x)^\alpha t^{-1} + 1} \sqrt{1-x^2} \, \mathrm{d}x \\ &\qquad = \frac{\pi}{2} + \int_{-1}^{1} U_{n-1}^2(x) \sqrt{1-x^2} \, r(t,x) \, \mathrm{d}x = \frac{\pi}{2} + \mathcal{O}\Big(\frac{1}{t}\Big). \end{aligned}$$ ■

To compute the Green kernel of the bilaplacian, we use a slight extension of equation 2.[6] in 3.613 of [13], namely

$$\int_0^\pi \frac{\cos(l\varphi)}{1 - 2k\cos\varphi + k^2} \, \mathrm{d}\varphi = \frac{\pi k^l}{1-k^2} \tag{A.4}$$

for $l \in \mathbb{N}_0$ and $k \in \mathbb{D}$ (extended by analyticity from the original statement for $k \in [0, 1)$).

Lemma A.5. *For all $m, n \in \mathbb{N}$ and $\mu \in \mathbb{C} \setminus [-4, 4]$, one has*

$$\int_{-1}^{1} \frac{U_{m-1}(x)\,U_{n-1}(x)}{4(1-x)^2-\mu^2}\sqrt{1-x^2}\,dx = \frac{\pi}{2}\,\frac{\xi\eta}{(1-\xi\eta)(\xi-\eta)}\left(\frac{\xi^{m+n}-\xi^{|m-n|}}{\xi-\xi^{-1}} - \frac{\eta^{m+n}-\eta^{|m-n|}}{\eta-\eta^{-1}}\right),$$

where $\xi, \eta \in \mathbb{D} \setminus \{0\}$ are (unique) such that $2+\mu = \xi+\xi^{-1}$ and $2-\mu = \eta+\eta^{-1}$.

Proof. We start by the simple identity

$$\frac{1}{4(1-x)^2-\mu^2} = \frac{1}{2\mu}\left(\frac{1}{2(1-x)-\mu} - \frac{1}{2(1-x)+\mu}\right),$$

which together with the substitution $x := \cos\theta$ and relation (2.6) results in

$$\begin{aligned}\int_{-1}^{1} &\frac{U_{m-1}(x)\,U_{n-1}(x)}{4(1-x)^2-\mu^2}\sqrt{1-x^2}\,\mathrm{d}x \\ &= \frac{1}{2\mu}\int_0^{\pi}\left(\frac{1}{2(1-\cos\theta)-\mu} - \frac{1}{2(1-\cos\theta)+\mu}\right)\sin(m\theta)\sin(n\theta)\,\mathrm{d}\theta \\ &= \frac{1}{4\mu}\int_0^{\pi}\left(\frac{\eta}{1-2\eta\cos\theta+\eta^2} - \frac{\xi}{1-2\xi\cos\theta+\xi^2}\right) \\ &\quad\times[\cos(|m-n|\theta)-\cos((m+n)\theta)]\,\mathrm{d}\theta.\end{aligned}$$

The claim follows from suitably applying (A.4) to each summand above and observing

$$2\mu = \xi+\xi^{-1}-(\eta+\eta^{-1}) = (\xi-\eta)(1-\xi^{-1}\eta^{-1}). \qquad \blacksquare$$

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References

[1] Beckner, W.: Pitt's inequality and the uncertainty principle. *Proc. Amer. Math. Soc.* **123** (1995), no. 6, 1897–1905. Zbl 0842.58077 MR 1254832

[2] Birman, M. Š.: On the spectrum of singular boundary-value problems. *Mat. Sb. (N.S.)* **55/97** (1961), 125–174. Zbl 0104.32601 MR 0142896

[3] Bogdan, K. and Dyda, B.: The best constant in a fractional Hardy inequality. *Math. Nachr.* **284** (2011), no. 5-6, 629–638. Zbl 1217.26025 MR 2663757

[4] Ciaurri, Ó. and Roncal, L.: Hardy's inequality for the fractional powers of a discrete Laplacian. *J. Anal.* **26** (2018), no. 2, 211–225. Zbl 1402.26003 MR 3882021

[5] Devyver, B., Fraas, M. and Pinchover, Y.: Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon. *J. Funct. Anal.* **266** (2014), no. 7, 4422–4489. Zbl 1298.47057 MR 3170212

[6] Devyver, B. and Pinchover, Y.: Optimal L^p Hardy-type inequalities. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **33** (2016), no. 1, 93–118. Zbl 1331.35013 MR 3436428

[7] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G.: *Higher transcendental functions. Vol. II*. Robert E. Krieger Publishing Co., Melbourne, Fla., 1981. MR 0698780

[8] Fernández-Bertolin, A., Roncal, L., Rüland, A. and Stan, D.: Discrete Carleman estimates and three balls inequalities. *Calc. Var. Partial Differential Equations* **60** (2021), no. 6, article no. 239, 28 pp. Zbl 1481.39004 MR 4328433

[9] Frank, R. L.: Eigenvalue bounds for the fractional Laplacian: a review. In *Recent developments in nonlocal theory*, pp. 210–235. De Gruyter, Berlin, 2018. Zbl 1404.35303 MR 3824213

[10] Gerhat, B., Krejčiřík, D. and Štampach, F.: An improved discrete Rellich inequality on the half-line. To appear in *Israel J. Math.*

[11] Gesztesy, F., Littlejohn, L. L., Michael, I. and Wellman, R.: On Birman's sequence of Hardy–Rellich-type inequalities. *J. Differential Equations* **264** (2018), no. 4, 2761–2801. Zbl 1384.26043 MR 3737854

[12] Glazman, I. M.: *Direct methods of qualitative spectral analysis of singular differential operators*. Daniel Davey & Co., New York, 1966. Zbl 0143.36505 MR 0190800

[13] Gradshteyn, I. S. and Ryzhik, I. M.: *Table of integrals, series, and products*. Seventh edition. Elsevier/Academic Press, Amsterdam, 2007. Zbl 1208.65001 MR 2360010

[14] Gupta, S.: *Discrete functional inequalities on lattice graphs*. PhD Thesis, Imperial College London, 2023.

[15] Gupta, S.: Hardy and Rellich inequality on lattices. *Calc. Var. Partial Differential Equations* **62** (2023), no. 3, article no. 81, 18 pp. Zbl 1506.39003 MR 4531775

[16] Gupta, S.: One-dimensional discrete Hardy and Rellich inequalities on integers. *J. Fourier Anal. Appl.* **30** (2024), no. 2, article no. 15, 22 pp. Zbl 07860402 MR 4715384

[17] Hansmann, M. and Krejčiřík, D.: The abstract Birman–Schwinger principle and spectral stability. *J. Anal. Math.* **148** (2022), no. 1, 361–398. Zbl 07637560 MR 4525757

[18] Herbst, I. W.: Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$. *Comm. Math. Phys.* **53** (1977), no. 3, 285–294. Zbl 0375.35047 MR 0436854

[19] Huang, X. and Ye, D.: One-dimensional sharp discrete Hardy–Rellich inequalities. *J. Lond. Math. Soc. (2)* **109** (2024), no. 1, article no. e12838, 26 pp. Zbl 07809011 MR 4680228

[20] Isozaki, H. and Jensen, A.: Continuum limit for lattice Schrödinger operators. *Rev. Math. Phys.* **34** (2022), no. 2, article no. 2250001, 50 pp. Zbl 1492.81085 MR 4386952

[21] Keller, M. and Nietschmann, M.: Optimal Hardy inequality for fractional Laplacians on the integers. *Ann. Henri Poincaré* **24** (2023), no. 8, 2729–2741. Zbl 1546.35033 MR 4612316

[22] Keller, M., Pinchover, Y. and Pogorzelski, F.: An improved discrete Hardy inequality. *Amer. Math. Monthly* **125** (2018), no. 4, 347–350. Zbl 1392.26036 MR 3779222

[23] Keller, M., Pinchover, Y. and Pogorzelski, F.: Optimal Hardy inequalities for Schrödinger operators on graphs. *Comm. Math. Phys.* **358** (2018), no. 2, 767–790. Zbl 1390.26042 MR 3774437

[24] Keller, M., Pinchover, Y. and Pogorzelski, F.: Criticality theory for Schrödinger operators on graphs. *J. Spectr. Theory* **10** (2020), no. 1, 73–114. Zbl 1441.31006 MR 4071333

[25] Krejčiřík, D. and Štampach, F.: A sharp form of the discrete Hardy inequality and the Keller–Pinchover–Pogorzelski inequality. *Amer. Math. Monthly* **129** (2022), no. 3, 281–283. Zbl 1491.26023 MR 4399059

[26] Mityagin, B. and Siegl, P.: Root system of singular perturbations of the harmonic oscillator type operators. *Lett. Math. Phys.* **106** (2016), no. 2, 147–167. Zbl 1330.47023 MR 3451535

[27] Mityagin, B. and Siegl, P.: Local form-subordination condition and Riesz basisness of root systems. *J. Anal. Math.* **139** (2019), no. 1, 83–119. Zbl 1435.47022 MR 4041096

[28] Nakamura, S. and Tadano, Y.: On a continuum limit of discrete Schrödinger operators on square lattice. *J. Spectr. Theory* **11** (2021), no. 1, 355–367. Zbl 1487.47027 MR 4233213

[29] Yafaev, D.: Sharp constants in the Hardy–Rellich inequalities. *J. Funct. Anal.* **168** (1999), no. 1, 121–144. Zbl 0981.26016 MR 1717839

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