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Many partitions of mass assignments

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Abstract. In this paper, extending the recent work of the authors with Calles Loperena and Dimitrijević Blagojević, we give a general and complete treatment of problems of partition of mass assignments with prescribed arrangements of hyperplanes on Euclidean vector bundles. Using a new configuration test map scheme, as well as an alternative topological framework, we are able to reprove known results, extend them to arbitrary bundles as well as to put various types of constraints on the solutions. Moreover, the developed topological methods allow us to give new proofs and extend results of Guth and Katz, Schnider, and Soberón and Takahashi. In this way we place all these results under one "roof".

Dedicated to the memory of Frederick R. Cohen, an exceptional mathematician and an amazing human being.

1. Introduction

Problems of the existence of mass partitions by affine hyperplanes in a Euclidean space have a long and exciting history since the 1930's ham-sandwich theorem of Hugo Steinhaus and Karol Borsuk [27, Prob. 123]. The ham-sandwich theorem claims the existence of a hyperplane which equiparts d given masses in a d-dimensional Euclidean space. For more details about history on the ham-sandwich theorem and its interconnection with the non-existence of antipodal maps between spheres consult [4] or [26]. The followup work by Branko Grünbaum [18], Hugo Hadwiger [21], David Avis [1], and a bit later by Edgar Ramos [30], demonstrated how an increase in complexity of mass partition questions naturally creates even more complicated problems related to the non-existence of equivariant maps. Topological challenges of the Grünbaum–Hadwiger–Ramos hyperplane mass partition problems were discussed recently in [8]. For more information about various types of mass partition problems see the recent survey by Edgardo Roldán-Pensado and Pablo Soberón [31].

In order to motivate a study of partitions of mass assignments over Euclidean vector bundles as a natural extension of classical studies we first briefly recall the original problem. For the sake of brevity, from now on, we write "GHR" for "Grünbaum–Hadwiger–Ramos".

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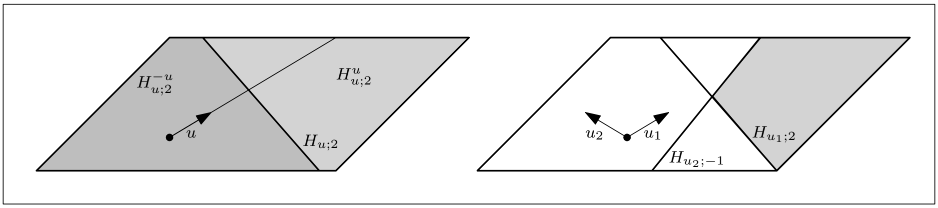


Figure 1. An illustration of an oriented affine hyperplane, associated half-spaces, arrangement of two affine hyperplanes, and an orthant $\mathcal{O}^{\mathcal{H}}_{(u_1,-u_2)}$ where $\mathcal{H} = (H(u_1,2), H(u_2,-1))$.

1.1. What is the GHR problem for masses?

A *mass* in a Euclidean space V is assumed to be a finite Borel measure on V which vanishes on every affine hyperplane.

An *oriented affine hyperplane* $H(u;a)$ in V is given by

- a unit vector $u \in$ V, the unit normal to the associated affine hyperplane $H_{u;a}$, which in addition determines the orientation of the hyperplane, and by
- a scalar $a \in \mathbb{R}$ which determines the distance of the associated affine hyperplane $H_{u;a}$ from the origin in direction u.

The associated affine hyperplane is defined by $H_{u;a} := \{x \in \mathrm{V} \colon \langle x,u\rangle = a\}$. Furthermore, the oriented affine hyperplane $H(u;a)$ defines two closed half-spaces by

$$H^{u}_{u;a} := \big\{x \in \mathrm{V} \colon \langle x,u\rangle - a \geq 0\big\} \quad \text{and} \quad H^{-u}_{u;a} := \big\{x \in \mathrm{V} \colon \langle x,-u\rangle + a \geq 0\big\}.$$

In other words, an oriented affine hyperplane is a triple $H(u;a) = (H_{u;a}, H^{u}_{u;a}, H^{-u}_{u;a})$.

An *arrangement* of k (oriented) *affine hyperplanes* $\mathcal{H}$ in V is an ordered collection $\mathcal{H} = (H(u_1;a_1),\dots,H(u_k;a_k))$ of k oriented affine hyperplanes in V. Such an arrangement $\mathcal{H}$ and a collection of unit normal vectors $(v_1,\dots,v_k) \in \{u_1,-u_1\} \times \cdots \times \{u_k,-u_k\}$ to the elements of the arrangement $\mathcal{H}$ determine an *orthant* as the intersection of the corresponding closed half-spaces:

$$\mathcal{O}^{\mathcal{H}}_{(v_1,\dots,v_k)} := H^{v_1}_{u_1;a_1} \cap \cdots \cap H^{v_k}_{u_k;a_k}.$$

There are $2^k = \mathrm{card}(\{u_1,-u_1\} \times \cdots \times \{u_k,-u_k\})$ orthants determined by the arrangement $\mathcal{H}$. The orthants are not necessarily distinct or non-empty. The arrangement of hyperplanes $\mathcal{H} = (H(u_1;a_1),\dots,H(u_k;a_k))$ is *orthogonal* if $u_r \perp u_s$ for every $1 \leq r < s \leq k$.

Now, we say that an arrangement $\mathcal{H} = (H(u_1;a_1),\dots,H(u_k;a_k))$ in V *equiparts* a collection of masses $\mathcal{M}$ in V if and only if for every mass $\mu \in \mathcal{M}$ and every $(v_1,\dots,v_k) \in \{u_1,-u_1\} \times \cdots \times \{u_k,-u_k\}$ we have the equality:

$$\mu\big(\mathcal{O}^{\mathcal{H}}_{(v_1,\dots,v_k)}\big) = \frac{1}{2^k}\mu(V).$$

Furthermore, a collection of masses $\mathcal{M}$ in V *can be equiparted by an arrangement* of k affine hyperplanes if there exists an arrangement $\mathcal{H}$ of k oriented affine hyperplanes in V which equiparts $\mathcal{M}$.

The GHR problem for masses asks for *the minimal dimension* $d = \Delta(j,k)$ *of a Euclidean space* V *in which every collection* $\mathcal{M}$ *of* j *masses can be equiparted by an arrangement of* k *affine hyperplanes*.

The first few values of the function Δ can be derived from classical results. Indeed, the ham-sandwich theorem [4] implies that $\Delta(d,1) = d$, an argument of Grünbaum [18] says that $\Delta(1,2) = 2$, while the seminal work of Hadwiger [21] yields that $\Delta(2,2) = 3$, $\Delta(1,3) = 3$. Furthermore, Avis and Ramos showed that $\frac{2^k-1}{k} j \le \Delta(j,k)$, while Peter Mani-Levitska, Siniša Vrećica and Rade Živaljević in [25, Thm. 39] proved that $\Delta(j,k) \le j + (2^{k-1}-1)2^{\lfloor \log_2 j \rfloor}$. The list of known values of the function Δ is given in [7].

In our recent paper with Calles Loperena and Dimitrijević Blagojević [6], motivated by the work of Patrick Schnider [32] and Ilani Axelrod-Freed and Soberón [2], we studied an extension of the GHR problem for masses to the problem for mass assignments over Grassmann manifolds.

1.2. What is the GHR problem for mass assignments?

Let $M_+(X)$ be the space of all finite Borel measures on a topological space X equipped with the weak topology. That is the minimal topology on $M_+(X)$ with the property that for every bounded and upper semi-continuous function $f\colon X \to \mathbb{R}$, the induced function $M_+(X) \to \mathbb{R}$ given by $\nu \mapsto \int f d\nu$, is upper semi-continuous. For $X = \mathrm{V}$, a Euclidean space, the subspace of all masses is denoted by $M'_+(\mathrm{V}) \subseteq M_+(\mathrm{V})$.

Let E be a Euclidean vector bundle over a path-connected space B with fibre E_b at $b \in B$. Consider the associated fibre bundle

$$M'_+(E) := \big\{(b,\nu) \mid b \in B,\ \nu \in M'_+(E_b)\big\} \longrightarrow B, \quad (b,\nu) \longmapsto b. \tag{1}$$

The topology on $M'_+(E)$ is defined using the local triviality of E and the topology on fibres we chose. Any cross-section $\mu\colon B \to M'_+(E)$ of the fibre bundle (1) is called a *mass assignment* on the Euclidean vector bundle E. In particular, $\mu(b)$ is a mass on E_b for every $b \in B$.

More generally, let us now write $M_+(E) \to B$ for the locally trivial bundle with fibre at $b \in B$ the space $M_+(E_b)$ of finite Borel measures on the sphere $S(E_b)$. A continuous section μ will be called a *family of (probability) measures* on E if $\mu_b \in M_+(E_b)$ is a (probability) measure for each $b \in B$. In the following we give an illustrative example of a family of probability measures on E.

Example 1.1. Let E be a Euclidean vector bundle over a path-connected space B. Suppose that $X \to B$ is a finite cover embedded fibrewise in E, and suppose that $p : X \to [0,1]$ is a continuous function such that, for each $b \in B$, $\sum_{x \in X_b} p(x) = 1$. For a Borel subset $A \subseteq E_b$, define $\mu_b(A) := \sum_{x \in A \cap X_b} p(x)$. Then μ defines a family of probability measures on $S(E)$.

The GHR problem for mass assignments on a Euclidean vector bundle E over B asks *for all pairs of positive integers* (j,k) *with the property that for every collection of* j *mass assignments* $\mathcal{M} = (\mu_1, \dots, \mu_j)$ *on* E *there exists a point* $b \in B$ *such that the collection of* j *masses* $\mathcal{M}(b) := (\mu_1(b), \dots, \mu_j(b))$ *on* E_b *can be equiparted by an arrangement of* k *affine hyperplanes in* E_b. If we denote by $\Delta(E)$ the set of such pairs (j,k), then the GHR problem for mass assignments on E is a question of describing the set $\Delta(E) \subseteq \mathbb{N}^2$.

Recently, with Calles Loperena and Dimitrijević Blagojević [6], we studied the GHR problem for mass assignment over tautological vector bundles over Grassmann manifolds. In particular, with appropriate reformulation, the result [6, Thm. 1.5] can be stated as follows.

Theorem 1.2. *Let* $d \geq 2$ *and* $\ell \geq 1$ *be integers where* $1 \leq \ell \leq d$, *and let* E^d_ℓ *be the tautological vector bundle over the Grassmann manifold* $\mathrm{G}_\ell(\mathbb{R}^d)$ *of all* ℓ*-dimensional linear subspaces in* $\mathbb{R}^d$. *Then*

$$\big\{(j,k) \in \mathbb{N}^2 : 1 \leq k \leq \ell,\ 2^{\lfloor \log_2 j \rfloor}(2^{k-1} - 1) + j \leq d\big\} \subseteq \Delta(E^d_\ell).$$

It is important to observe that the result of Theorem 1.2 does not really depend on the value of the parameter ℓ. In particular, for $\ell = d$ it recovers the upper bound of Mani-Levitska–Vrećica–Živaljević [25, Thm. 39] for the function Δ.

In this paper, following the ideas of Bárány and Matoušek [3] and Crabb [14], we extend mass assignment partition problems in a Euclidean space by affine hyperplane arrangements to mass assignment partition problems on the unit Euclidean sphere by arrangements of equatorial spheres. Additionally, we will restrict, and therefore simplify, our notions of mass and mass assignment.

1.3. What are the GHR problems on spheres and sphere bundles?

First, we show how the GHR problem for masses in $\mathbb{R}^d$ induces the corresponding mass partition problem on the unit sphere in $\mathbb{R}^{d+1}$.

Let $d \geq 1$ be an integer. Embed $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$ via the embedding $x \mapsto (x, -1)$. In this way $\mathbb{R}^d$ coincides with the tangent space to the unit sphere $S(\mathbb{R}^{d+1}) \cong S^d$ at the point $y_0 := (0, \dots, 0, -1)$. Let $p\colon \mathbb{R}^d \to \Lambda$ be the homeomorphism, between $\mathbb{R}^d$ and the open lower hemisphere $\Lambda := \{y \in S(\mathbb{R}^{d+1}) : \langle y, y_0 \rangle > 0\}$ of the sphere $S(\mathbb{R}^{d+1})$, given by

$$x \mapsto \frac{1}{\sqrt{\|x\|^2 + 1}}(x, -1) \quad \text{for } x \in \mathbb{R}^d.$$

Now, every mass μ on the Euclidean space $\mathbb{R}^d$ induces a measure (mass) μ' on $S(\mathbb{R}^{d+1})$ defined by

$$\mu'(A) := \mu\big(p^{-1}(A \cap \Lambda)\big),$$

where $A \subseteq S(\mathbb{R}^{d+1})$ is an element of the Borel σ-algebra on $S(\mathbb{R}^{d+1})$. In particular, measure μ' vanishes on each equatorial sphere of $S(\mathbb{R}^{d+1})$. Here, an equatorial sphere of $S(\mathbb{R}^{d+1})$ can be always presented as an intersection of $S(\mathbb{R}^{d+1})$ and a unique linear hyperplane in $\mathbb{R}^{d+1}$.

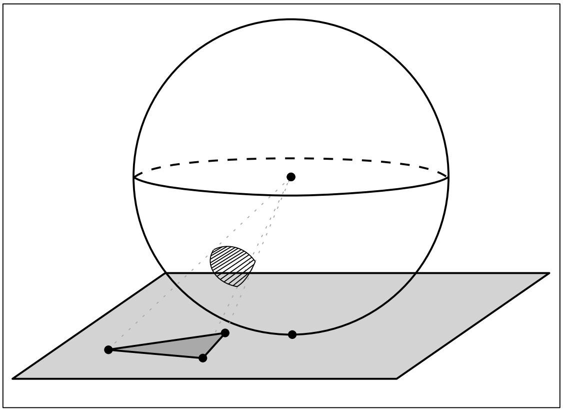


Figure 2. An illustration of a transition of a mass on $\mathbb{R}^2$ into a measure on S^2.

Furthermore, every affine hyperplane H in $\mathbb{R}^d$ is mapped via p to a part of an equatorial sphere of $S(\mathbb{R}^{d+1})$. More precisely,

$$p(H) = \operatorname{span}(H) \cap \Lambda = \big\{\lambda \cdot (x, -1) : \lambda \in \mathbb{R},\ x \in H\big\} \cap \Lambda,$$

where span denotes linear span in $\mathbb{R}^{d+1}$.

Using the transition of masses on $\mathbb{R}^d$ into measures on S^d, and affine hyperplanes in $\mathbb{R}^d$ into equatorial spheres on S^d, we can formulate the GHR problem for masses on a sphere as follows: *determine the minimal dimension $d = \Delta_S(j,k)$ of a unit Euclidean sphere S^d in which every collection of j masses can be equiparted by an arrangement of k equatorial spheres.* Here, the notions of masses and equipartition of masses are naturally extended from the affine to the spherical setup.

Motivated by this spherical extension of the classical problem and with a desire to simplify the treatment of the mass assignments, we restate the GHR problem for mass assignments in the following way.

Let E be a Euclidean vector bundle over a path-connected space B, and let $S(E)$ denote the unit sphere bundle associated to E. Now, *we are looking for all pairs of positive integers (j,k) with the property that for every collection of j continuous real valued functions $\varphi_1, \dots, \varphi_j : S(E) \to \mathbb{R}$, there exists a point $b \in B$ and there exists an arrangement $\mathcal{H}^b = (H_1^b, \dots, H_k^b)$ of k linear hyperplanes in the fibre E_b of E such that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ of the arrangement complement $E_b - (H_1^b \cup \dots \cup H_k^b)$ the following statement holds*

$$\int_{\mathcal{O}' \cap S(E_b)} \varphi_1 = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_1\ , \dots,\ \int_{\mathcal{O}' \cap S(E_b)} \varphi_j = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_j\,.$$

Here integration is assumed to be with the respect to the measure on the sphere $S(E_b)$ induced by the metric. Once again, $\Delta_S(E)$ denotes the set of all such pairs (j,k). Since Euclidean and spherical partition problems are tightly related, we will not make a particular distinction between them. From now on instead of a mass assignment we consider a

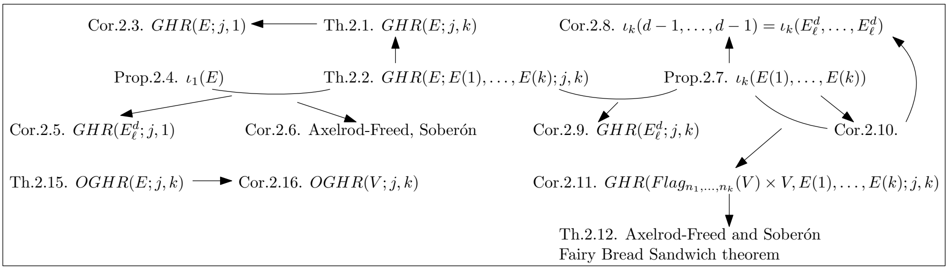


Figure 3. The main results of the paper and connections between them.

real valued continuous function from the sphere bundle, and instead of an affine hyperplane we take a linear hyperplane which induces an equatorial sphere.

2. Statements of the main results

After collecting the first family of results for tautological bundles (Theorem 1.2) it is natural to ask various followup questions:

- Why not consider partitions of mass assignments on arbitrary vector bundles instead of only tautological vector bundles?
- Can we constrain our choice of desired partitions on the given vector bundle by forcing normals of hyperplanes into chosen fixed vector subbundles of the ambient vector bundle?
- What about partitions with pairwise orthogonal hyperplanes, as was considered in the classical case?
- And finally, how can we fit all these questions into a common framework?

In the following we present multiple answers to the question we just asked. The interconnection between the main results of the paper is given in Figure 3.

We begin the list of our results with the full generalisation of [6, Thm. 1.1]. In other words, the old result becomes a special case of the next theorem in the case of tautological vector bundles. For an n-dimensional Euclidean vector bundle E over a compact and connected ENR[1] B and an integer $k \geq 1$ we denote by

$$R_k(B) := H^*(B;\mathbb{F}_2)[x_1,\dots,x_k]$$

the ring of polynomials in k variables $x_1,\dots,x_k$ of degree 1 with coefficients in the cohomology ring of the base space $H^*(B;\mathbb{F}_2)$. Note that by definition an ENR is locally path-connected and so the assumption of connectedness for an ENR is equivalent with the

[1] Euclidean Neighbourhood Retract.

assumption of being path-connected. Classically, we denote by $w_i(E)$, $i \geq 0$, the Stiefel–Whitney classes of the vector bundle E. In addition, we consider the ideal

$$\mathcal{I}_k(E) := \Big(\sum_{s=0}^{n} w_{n-s}(E) x_r^s : 1 \leq r \leq k \Big) \subseteq R_k(B),$$

and the element

$$e_k(B) := \prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k-\{0\}} (\alpha_1 x_1 + \cdots + \alpha_k x_k) \in R_k(B).$$

Now a generalisation of Theorem 1.2, which is proved in Section 4.1, can be stated as follows.

Theorem 2.1. *Let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B, and let $k \geq 1$ and $j \geq 1$ be integers.*

If the element $e_k(B)^j$ does not belong to the ideal $\mathcal{I}_k(E)$, then $(j,k) \in \Delta_S(E)$, or in other words, for every collection of j continuous real valued functions

$$\varphi_1, \dots, \varphi_j : S(E) \to \mathbb{R},$$

there exists a point $b \in B$ and there exists an arrangement $\mathcal{H}^b = (H_1^b, \dots, H_k^b)$ of k linear hyperplanes in the fibre E_b of E such that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ of the arrangement complement $E_b - (H_1^b \cup \cdots \cup H_k^b)$ the following statement holds

$$\int_{\mathcal{O}'\cap S(E_b)} \varphi_1 = \int_{\mathcal{O}''\cap S(E_b)} \varphi_1 \,, \dots, \int_{\mathcal{O}'\cap S(E_b)} \varphi_j = \int_{\mathcal{O}''\cap S(E_b)} \varphi_j \,.$$

The first generalisation of Theorem 2.1 is obtained by a restriction of the family of the arrangements in which we are looking for our partition. Concretely, we ask for the i-th hyperplane in the arrangement to have its normal vector in a specific vector subbundle. For that we modify our setup as follows.

Let $k \geq 1$ be an integer, and let $E(1), \dots, E(k)$ be Euclidean vector bundles over a compact and connected ENR B. Denote by n_i the dimension of the vector bundle $E(i)$ for $1 \leq i \leq k$. We consider the ideal in $R_k(B)$:

$$\mathcal{I}_k\big(E(1), \dots, E(k)\big) := \Big(\sum_{s=0}^{n_r} w_{n_r-s}\big(E(r)\big) x_r^s : 1 \leq r \leq k \Big) \subseteq R_k(B),$$

and set

$$\iota_k\big(E(1), \dots, E(k)\big) := \max\big\{ j : e_k(B)^j \notin \mathcal{I}_k\big(E(1), \dots, E(k)\big)\big\}.$$

Finally, we say that an arrangement of k linear hyperplanes $\mathcal{H}^b = (H_1^b, \dots, H_k^b)$ in the fibre E_b is determined by the collection of vector subbundles $E(1), \dots, E(k)$ if a unit normal of the linear hyperplane H_i^b belongs to the fibre $E(i)_b$, for every $1 \leq i \leq k$.

Now, the generalisation, proved in Section 4.2, says the following.

Theorem 2.2. *Let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B, $k \geq 1$ and $j \geq 1$ integers, and let $E(1), \dots, E(k)$ be vector subbundles of E of dimensions $n_1, \dots, n_k$, respectively.*

If $j \leq \iota_k(E(1), \dots, E(k))$, then for every collection of j continuous real valued functions $\varphi_1, \dots, \varphi_j : S(E) \to \mathbb{R}$, there exists a point $b \in B$ and there exists an arrangement $\mathcal{H}^b = (H_1^b, \dots, H_k^b)$ of k linear hyperplanes in the fibre E_b of E determined by the collection of vector subbundles $E(1), \dots, E(k)$ such that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ of the arrangement complement $E_b - (H_1^b \cup \dots \cup H_k^b)$ the following statement holds

$$\int_{\mathcal{O}' \cap S(E_b)} \varphi_1 = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_1 , \dots , \int_{\mathcal{O}' \cap S(E_b)} \varphi_j = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_j .$$

After a generalisation and an extension of [6, Thm. 1.1], it is natural to ask whether the algebraic criteria from Theorems 2.1 and 2.2 can be substituted by appropriate numerical criteria. In other words, is there an appropriate generalisation of Theorem 1.2 in the case of an arbitrary vector bundle. We start our discussion from the case $k = 1$, the ham-sandwich.

Let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B and let $k = 1$. Since the ideal $\mathcal{I}_1(E) = (\sum_{s=0}^{n} w_{n-s}(E) x_1^s)$ and $e_1(B)^{n-1} = x_1^{n-1} \notin \mathcal{I}_1(E)$ we conclude that

$$\iota_1(E) = \max \left\{ j : x_1^j \notin \mathcal{I}_1(E) \right\} \geq n - 1.$$

The equality $\iota_1(E) = n - 1$ is attained in the case when the base space B is a point. Indeed, when $B = \mathrm{pt}$ then the vector bundle E is a trivial, $w(E) = 1$, and so $\mathcal{I}_1(E) = (x_1^n)$ implying that $\iota_1(E) = \max\{j : x_1^j \notin (x_1^n)\} = n - 1$. We just proved the following ham-sandwich type result for Euclidean vector bundles.

Corollary 2.3. *Let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B.*

If $j = n - 1$ then for every collection $\varphi_1, \dots, \varphi_j : S(E) \to \mathbb{R}$ of j continuous real valued functions, there exists a point $b \in B$ and there exists a hyperplane H^b in E_b such that for the connected components $\mathcal{O}'$ and $\mathcal{O}''$ of the complement $E_b - H^b$ the following statement holds

$$\int_{\mathcal{O}' \cap S(E_b)} \varphi_1 = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_1 , \dots , \int_{\mathcal{O}' \cap S(E_b)} \varphi_j = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_j .$$

The previous result is general and holds for all vector bundles and therefore rather crude because it must contain the classical ham-sandwich theorem. It is natural to ask how the topology of the vector bundle E affects the upper bound for the number of functions we can equipart in a fibre. In other words can we say more about the number $\iota_1(E)$.

Indeed, the following proposition, proved in Section 5.1, explains a connection between the topology of E and the number $\iota_1(E)$.

Proposition 2.4. *Let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B. Then*

$$\iota_1(E) = \max\{j : 0 \neq w_{j-n+1}(-E) \in H^{j-n+1}(B; \mathbb{F}_2)\}.$$

Here, $-E$ denotes an inverse vector bundle of E. This is a vector bundle E', over the base space of E, having the property that the Whitney sum $E \oplus E'$ is a trivial vector bundle over B. In particular, the inverse vector bundle is not uniquely defined. On the other hand the Stiefel–Whitney classes of all inverse vector bundles, of a given vector bundle, do coincide. Finally, for example, the compactness of the base space guarantees the existence of an inverse bundle of a given vector bundle. (Alternatively, we can take $-E$ to be a virtual bundle representing the negative of the class of E in the Grothendieck K-group $KO^0(B)$. Precisely, a virtual bundle is a pair (E_0, E_1) of vector bundles over B, in an appropriate category, and $-E$ is the pair $(0, E)$. The set of isomorphism classes of virtual bundles is precisely the Grothendieck group $KO^0(B)$. Hence, the dimension $\dim(-E) = -\dim E$, and in the K-group $[-E] = -[E] = [E'] - \dim(E \oplus E')$.)

As a special case of the previous result we recover the ham-sandwich result for the tautological vector bundle [6, Cor. 1.2].

Corollary 2.5. *Let $d \geq 2$ and $\ell \geq 1$ be integers where $1 \leq \ell \leq d$, and let E_ℓ^d be the tautological vector bundle over the Grassmann manifold $\mathrm{G}_\ell(\mathbb{R}^d)$ of all ℓ-dimensional linear subspaces in $\mathbb{R}^d$. Then*

$$\iota_1(E_\ell^d) = d - 1.$$

Proof. For the proof we use the fact that the inverse bundle $-E_\ell^d$ can be realised as the orthogonal complement vector bundle $(E_\ell^d)^\perp$. In particular, we have that

$$w(-E_\ell^d) = 1 + w_1\big((E_\ell^d)^\perp\big) + \cdots + w_{d-\ell}\big((E_\ell^d)^\perp\big),$$

where the orthogonal complement is considered inside the trivial vector bundle $\mathrm{G}_\ell(\mathbb{R}^d) \times \mathbb{R}^d$. Since $w_{d-\ell}(-E_\ell^d) = w_{d-\ell}((E_\ell^d)^\perp) \neq 0$ and $w_i(-E_\ell^d) = w_i((E_\ell^d)^\perp) = 0$ for $i \geq d - \ell + 1$, consult for example [23, p. 523], we have from Proposition 2.4 that

$$\iota_1(E_\ell^d) = \max\{j : 0 \neq w_{j-\ell+1}(-E_\ell^d)\} = d - 1. \qquad \blacksquare$$

The following spherical version of the result of Axelrod-Freed–Soberón [2, Thm. 1.3], which was previously conjectured by Schnider [33, Conj. 2.4], is a direct consequence of our Theorem 2.2 and Corollary 2.5.

Corollary 2.6. *Let $d \geq 2$ and $\ell \geq 1$ be integers where $1 \leq \ell \leq d$, and let W be an arbitrary $(\ell - 1)$-dimensional vector subspace of $\mathbb{R}^d$.*

If $j = d - 1$ then for any collection of continuous functions $\varphi_1, \ldots, \varphi_j : S(E_l^d) \to \mathbb{R}$, there exist:

- *$V \in G_\ell(\mathbb{R}^d)$ which contains W, and*
- *$U \in G_{\ell-1}(\mathbb{R}^d)$, which is contained in V*

such that for the connected components $\mathcal{O}'$ and $\mathcal{O}''$ of the complement $V-U$ the following statement holds

$$\int_{\mathcal{O}'\cap S(V)} \varphi_1 = \int_{\mathcal{O}''\cap S(V)} \varphi_1 \,, \dots, \int_{\mathcal{O}'\cap S(V)} \varphi_j = \int_{\mathcal{O}''\cap S(V)} \varphi_j \,.$$

Proof. Consider the vector bundle $E = E(1) = H(W^\perp) \oplus \underline{W}$ over $\mathbb{P}(W^\perp)$ where $\underline{W} = \mathbb{P}(W^\perp) \times W$ is the trivial vector bundle over $\mathbb{P}(W^\perp)$. Recall that here $H(W^\perp)$ is the canonical Hopf line bundle over the projective space $\mathbb{P}(W^\perp)$. According to Theorem 2.2 in the case $k = 1$ we have: if $j \leq \iota_1(E)$, then for any j continuous functions

$$\varphi_1, \dots, \varphi_j \colon S(E) \to \mathbb{R}$$

there exists a line $L \in \mathbb{P}(W^\perp)$ and there exists a linear hyperplane U in $V := L \oplus W$ such that for the connected components $\mathcal{O}'$ and $\mathcal{O}''$ of the complement $V - U$ the following equalities hold:

$$\int_{\mathcal{O}'\cap S(V)} \varphi_1 = \int_{\mathcal{O}''\cap S(V)} \varphi_1 \,, \dots, \int_{\mathcal{O}'\cap S(V)} \varphi_j = \int_{\mathcal{O}''\cap S(V)} \varphi_j \,.$$

Since $w(E) = w(H(W^\perp) \oplus \underline{W}) = w(H(W^\perp))$ and $H(W^\perp) \cong E_1^{d-\ell+1}$ Corollary 2.5 implies that $\iota_1(E) = \iota_1(E_1^{d-\ell+1}) = d-1$. With the assumption $j = d-1 \leq \iota_1(E)$ we conclude the proof of the corollary. ■

Further, if $E(1)$ is an n_1-dimensional vector subbundle of the vector bundle E then

$$\iota_1\big(E(1)\big) \leq \iota_1(E).$$

Indeed, if $E(1)^\perp$ is the orthogonal complement vector bundle of $E(1)$ in E then

$$\begin{aligned} x_1^n &+ w_1(E)x_1^{n-1} + \cdots + w_n(E) \\ &= \big(x_1^{n_1} + w_1\big(E(1)\big)x_1^{n_1-1} + \cdots + w_{n_1}\big(E(1)\big)\big) \\ &\quad \cdot \big(x_1^{n-n_1} + w_1\big(E(1)^\perp\big)x_1^{n-n_1-1} + \cdots + w_{n-n_1}\big(E(1)^\perp\big)\big). \end{aligned}$$

Consequently, $x_1^j \notin \mathcal{I}_1(E(1))$ implies $x_1^j \notin \mathcal{I}_1(E)$.

Recall, that

$$e_k(\mathrm{pt}) = \prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k-\{0\}} (\alpha_1 x_1 + \cdots + \alpha_k x_k) \in R_k(\mathrm{pt}) \cong \mathbb{F}_2[x_1,\dots,x_k].$$

Now, for positive integers $m_1, \dots, m_k$ we define

$$\iota_k(m_1,\dots,m_k) := \max\big\{j : e_k(\mathrm{pt})^j \notin (x_1^{m_1},\dots,x_k^{m_k})\big\}.$$

For example, if $E = \underline{\mathbb{R}^n}$ is a trivial n dimensional real vector bundle over $B = \mathrm{pt}$, then

$$\iota_k(\underline{\mathbb{R}^{n_1}},\dots,\underline{\mathbb{R}^{n_k}}) = \iota_k(n_1,\dots,n_k).$$

Notice that the equality holds for all integers $n \geq \max\{n_1, \ldots, n_k\}$. Indeed, since $w(\underline{\mathbb{R}^{n_1}}) = \cdots = w(\underline{\mathbb{R}^{n_k}}) = 1$, it follows that

$$(x_1^{n_1}, \ldots, x_k^{n_k}) = \mathcal{I}_k(\underline{\mathbb{R}^{n_1}}, \ldots, \underline{\mathbb{R}^{n_k}}).$$

In general, the following inequality always holds

$$\iota_k(n_1, \ldots, n_k) \leq \iota_k\big(E(1), \ldots, E(k)\big).$$

In fact, the condition $e_k(\mathrm{pt})^j \notin (x_1^{n_1}, \ldots, x_k^{n_k})$, for some integer j, implies the existence of a monomial $x_1^{m_1} \cdots x_k^{m_k}$, in the additive presentation of $e_k(\mathrm{pt})^j$ with respect to the monomial base of $\mathbb{F}_2[x_1, \ldots, x_k]$, with the property that $m_1 \leq n_1 - 1, \ldots, m_k \leq n_k - 1$. Since the ideal $\mathcal{I}_k(E(1), \ldots, E(k))$ is generated by polynomials $x_1^{n_i} + w_1(E(i))x_1^{n_i - 1} + \cdots + w_{n_i}(E(i))$, $1 \leq i \leq k$, the existence of the monomial $x_1^{m_1} \cdots x_k^{m_k}$ in the presentation of $e_k(\mathrm{pt})^j$ implies that $e_k(B)^j \notin \mathcal{I}_k(E(1), \ldots, E(k))$.

Actually, we can say more, as the following proposition illustrates. For the proof see Section 5.2.

Proposition 2.7. *Let $k \geq 1$ be an integer, and let $E(1), \ldots, E(k)$ be Euclidean vector bundles over a compact and connected ENR B. Denote by n_i the dimension of the vector bundle $E(i)$ for $1 \leq i \leq k$. If*

$$0 \neq w_{\iota_1(E(1)) - n_1 + 1}\big(-E(1)\big) \cdots w_{\iota_1(E(k)) - n_k + 1}\big(-E(k)\big) \in H^*(B; \mathbb{F}_2),$$

then

$$\iota_k\big(\iota_1\big(E(1)\big) + 1, \ldots, \iota_1\big(E(k)\big) + 1\big) = \iota_k\big(E(1), \ldots, E(k)\big).$$

A direct consequence of the previous proposition, in the case when E is a tautological vector bundle, is the following corollary [6, Lem. 4.1]. For a proof see Section 5.3.

Corollary 2.8. *Let $d \geq 2$, $k \geq 1$, and $\ell \geq 1$ be integers where $1 \leq k \leq \ell \leq d$, and let E_ℓ^d be the tautological vector bundle over the Grassmann manifold $\mathrm{G}_\ell(\mathbb{R}^d)$ of all ℓ-dimensional linear subspaces in $\mathbb{R}^d$. Then*

$$\iota_k(d, \ldots, d) = \iota_k(E_\ell^d, \ldots, E_\ell^d).$$

The next corollary is a spherical version of [6, Thm. 1.4].

Corollary 2.9. *Let $d \geq 2$, $k \geq 1$, and $\ell \geq 1$ be integers where $1 \leq k \leq \ell \leq d$, and let $E = E_\ell^d$ be the tautological vector bundle over the Grassmann manifold $\mathrm{G}_\ell(\mathbb{R}^d)$ of all ℓ-dimensional linear subspaces in $\mathbb{R}^d$.*

If $j = 2^t + r$ where $0 \leq r \leq 2^t - 1$ and $d \geq 2^{t+k-1} + r$, then $(j, k) \in \Delta_S(E_\ell^d)$.

In other words, if $j = 2^t + r$ where $0 \leq r \leq 2^t - 1$ and $d \geq 2^{t+k-1} + r$, then for every collection of j continuous real valued functions $\varphi_1, \ldots, \varphi_j : S(E) \to \mathbb{R}$, there exists a point $b \in B$ and there exists an arrangement $\mathcal{H}^b = (H_1^b, \ldots, H_k^b)$ of k linear hyperplanes in the fibre E_b of E such that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ of the *arrangement complement* $E_b - (H_1^b \cup \cdots \cup H_k^b)$ *the following statement holds*

$$\int_{\mathcal{O}' \cap S(E_b)} \varphi_1 = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_1 \,, \ldots, \int_{\mathcal{O}' \cap S(E_b)} \varphi_j = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_j \,.$$

Proof. From Theorem 2.1 we have that $(j,k) \in \Delta_S(E_\ell^d)$ if

$$e_k(B)^j \notin \mathcal{I}_k(E_\ell^d) = \mathcal{I}_k(E_\ell^d, \ldots, E_\ell^d).$$

Stated differently $(j,k) \in \Delta_S(E_\ell^d)$ if

$$j \le \iota_k(E_\ell^d, \ldots, E_\ell^d) = \iota_k(d, \ldots, d) = \max\big\{j' : e_k(\mathrm{pt})^{j'} \notin (x_1^d, \ldots, x_k^d)\big\}.$$

Here the first equality comes from Corollary 2.8 while the second one is just the definition of $\iota_k(d, \ldots, d)$.

Since $j = 2^t + r$ where $0 \le r \le 2^t - 1$ and $d \ge 2^{t+k-1} + r$, then according to [6, Lem. 4.2] we have that $e_k(\mathrm{pt})^j \notin (x_1^d, \ldots, x_k^d)$. Thus, indeed $j \le \iota_k(E_\ell^d, \ldots, E_\ell^d)$ and the proof of the corollary is complete. ∎

We proceed with the next consequence of Proposition 2.7. In this case the base space of the vector bundle will be the real flag manifold, and so the following statement is an extension of Corollary 2.8. For the relevant background on the real flag manifold, associated canonical vector bundles, and a proof of the corollary see Section 6.1.

Corollary 2.10. *Let* $k \ge 1$ *and* $d \ge 2$ *be integers, and let* $0 = n_0 < n_1 < \cdots < n_{k-1} < n_k < n_{k+1} = d$ *be a strictly increasing sequence of integers. For a real* d*-dimensional vector space* $V = \mathbb{R}^d$ *let* $E_1, \ldots, E_{k+1}$ *denote the canonical vector bundles over the flag manifold* $\mathrm{Flag}_{n_1,\ldots,n_k}(V)$*, with* $\dim(E_i) = n_i - n_{i-1}$ *for* $1 \le i \le k+1$*. Set* $E(i) := \bigoplus_{1 \le r \le i} E_r$ *for all* $1 \le i \le k$*. Then*

$$\iota_k(d, \ldots, d) = \iota_k\big(E(1), \ldots, E(k)\big).$$

The previous corollary, in the language of GHR problem for mass assignments, with the help of Theorem 2.2 and the proof of Corollary 2.9, gives the following consequence. For a proof see Section 6.2.

Corollary 2.11. *Let* $k \ge 1$ *and* $d \ge 2$ *be integers, let* $0 = n_0 < n_1 < \cdots < n_{k-1} < n_k < n_{k+1} = d$ *be a strictly increasing sequence of integers, and let* $V = \mathbb{R}^d$ *be a real* d*-dimensional vector space. Let* $E_1, \ldots, E_{k+1}$ *be the canonical vector bundles over the flag manifold* $\mathrm{Flag}_{n_1,\ldots,n_k}(V)$*, let* $E(i) := \bigoplus_{1 \le r \le i} E_r$ *for all* $1 \le i \le k$*, and let* $E := E(k)$*.*

Assume that $j = 2^t + r$ *is an integer with* $0 \le r \le 2^t - 1$ *and* $d \ge 2^{t+k-1} + r$*. Then for every collection of* j *continuous real valued functions* $\varphi_1, \ldots, \varphi_j \colon S(E) \to \mathbb{R}$*, there exists a point* $b := (W_1, \ldots, W_{k+1}) \in \mathrm{Flag}_{n_1,\ldots,n_k}(V)$ *and there exists an arrangement* $\mathcal{H}^b = (H_1^b, \ldots, H_k^b)$ *of* k *linear hyperplanes in*

$$E_b = \bigoplus_{1 \le r \le k} W_r = W_{k+1}^{\perp}$$

such that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ *of the arrangement complement* $E_b - (H_1^b \cup \cdots \cup H_k^b)$ *the following statements hold*

$$\int_{\mathcal{O}'\cap S(E_b)} \varphi_1 = \int_{\mathcal{O}''\cap S(E_b)} \varphi_1 \,, \ldots, \int_{\mathcal{O}'\cap S(E_b)} \varphi_j = \int_{\mathcal{O}''\cap S(E_b)} \varphi_j \,,$$

and in addition

$$H_1^b \supseteq \bigoplus_{2\le r\le k+1} W_r,\ H_2^b \supseteq \bigoplus_{3\le r\le k+1} W_r \,, \ldots,\ H_k^b \supseteq \bigoplus_{k+1\le r\le k+1} W_{k+1}.$$

Here $(W_1, \ldots, W_{k+1}) \in \mathrm{Flag}_{n_1,\ldots,n_k}(V)$ means that $\dim W_i = n_i - n_{i-1}$ for $1 \le i \le k+1$, and $W_{i'} \perp W_{i''}$ for all $1 \le i' < i'' \le k+1$. For more details on flag manifolds see Section 6.

It should be noticed that once again the numerical assumptions on the parameters (d, j, k) in Corollary 2.11 coincide with the upper bound of Mani-Levitska–Vrećica–Živaljević [25, Thm. 39] for the function Δ, which can be phrased as the inequality

$$\Delta(2^t + r) \le 2^{t+k-1} + r, \quad \text{for } j = 2^t + r \text{ and } 0 \le r \le 2^t - 1.$$

We conclude our collection of results related to flags inside a real vector space with the spherical version of a result by Axelrod-Freed and Soberón [2, Thm. 1.2]. For the so called Fairy Bread Sandwich theorem we give a new proof in Section 6.3 based on the CS/TM scheme presented in Section 3.5.

Theorem 2.12. *Let* $d \ge 1$ *and* $k \ge 1$ *be integers with* $d \ge k$*, and let* $V = \mathbb{R}^{d+1}$ *be a real vector space. Fix a permutation* $(j_k, \ldots, j_d)$ *of the set* $\{k, \ldots, d\}$*, and take an arbitrary collections of functions* $\varphi_{a,b}\colon S(E_{a+1}^{d+1}) \to \mathbb{R}$*,* $k \le a \le d$*,* $1 \le b \le j_a$*, from the sphere bundle of the tautological vector bundle* E_{a+1}^{d+1} *over the Grassmann manifold* $G_{a+1}(V)$ *to the real numbers.*

There exists a flag $(V_k, \ldots, V_d) \in \mathrm{Flag}_{k,\ldots,d}(V)$ *such that for every* $k \le a \le d$ *and every* $1 \le b \le j_a$ *the following statement holds*

$$\int_{\{v\in V_{a+1}:\langle v,u_a\rangle\ge 0\}\cap S(V_{a+1})} \varphi_{a,b} = \int_{\{v\in V_{a+1}:\langle v,u_a\rangle\le 0\}\cap S(V_{a+1})} \varphi_{a,b}.$$

Here the unit vectors $u_k, \ldots, u_d$ *are determined, up to a sign, by the equality* $V_r = \{v \in V_{r+1} : \langle v, u_r\rangle = 0\}$*,* $k \le r \le d$*, and with* $V_{d+1} = V$*, this means that* u_r *is a unit normal vector to* V_r*, considered as a hyperplane inside* V_{r+1}*.*

Returning back to Proposition 2.7 we observe that the numbers $\iota_k(m_1, \ldots, m_k)$, in many cases, imply the existence of equipartitions of mass assignments. Hence, we collect several properties of these numbers with proofs given in Section 7.1.

Proposition 2.13. *Let* $k \ge 1$ *be an integer and let* $m_1, \ldots, m_k$ *be a sequence of positive integers.*

(1) *If* $\iota_{k-1}(m_1, \ldots, m_{k-1}) \ge m$ *and* $m_k \ge 2^{k-1}m + 1$*, then* $\iota_k(m_1, \ldots, m_k) \ge m$*.*

(2) *If* $m_i \geq 2^{i-1}m + 1$ *for all* $1 \leq i \leq k$, *then* $\iota_k(m_1, \dots, m_k) \geq m$.

(3) *If* $m \geq 1$, *then* $\iota_k(m + 1, 2m + 1, 2^2m + 1 \dots, 2^{k-1}m + 1) = m$.

(4) *Let* $m \geq 1$ *and* $1 \leq r \leq k - 1$ *be integers. If*

$$\iota_{k-r}(m_1, \dots, m_{k-r}) \geq m \quad \text{and} \quad \iota_r(m_{k-r+1}, \dots, m_k) \geq 2^{k-r}m,$$

then $\iota_k(m_1, \dots, m_k) \geq m$.

(5) *If* $\iota_{k-1}(m_1, \dots, m_{k-1}) \geq 2m$ *and* $m_k \geq m + 1$, *then* $\iota_k(m_1, \dots, m_k) \geq m$.

(6) *Let* $k = 2$. *The* $m \leq \iota_2(m_1, m_2)$ *if and only if there is an integer* i *such that* $0 \leq i \leq m$, $\binom{m}{i} = 1 \bmod 2$, *and* $2m - m_2 + 1 \leq i \leq m_1 - m - 1$.

(7) *If* $1 \leq r \leq 2^t$, *then* $\iota_2(2^t + 2r, 2^{t+1} + r) \geq 2^t + r - 1$.

Using the fact that $e_k(\mathrm{pt})$ is the top Dickson polynomial in variables $x_1, \dots, x_k$ we can prove even more. For a proof of the proposition which follows see Section 7.2.

Proposition 2.14. *Let* $k \geq 1$ *be an integer and let* $m_1, \dots, m_k$ *be positive integers.*

(1) *If* $0 \leq r \leq 2^t - 1$, $\iota_{k-1}(m_1, \dots, m_{k-1}) \geq 2^t + 2r$ *and* $m_k \geq 2^{t+k-1} + r + 1$, *then* $\iota_k(m_1, \dots, m_k) \geq 2^t + r$.

(2) *If* $0 \leq r \leq 2^t - 1$, $\iota_{k-1}(m_1, \dots, m_{k-1}) \geq 2^{t+1} + r$ *and* $m_k \geq 2^{t+k-1} + r + 1$, *then* $\iota_k(m_1, \dots, m_k) \geq 2^t + r$.

(3) *If* $0 \leq r \leq 2^t - 1$, $m_i \geq 2^{t+k-1} + r + 1$ *for all* $1 \leq i \leq k$, *then* $\iota_k(m_1, \dots, m_k) \geq 2^t + r$.

(4) *If* $\iota_k(m_1, \dots, m_k) \geq m$, *then* $\iota_k(2m_1, \dots, 2m_k) \geq 2m$.

The statement (3) in the previous proposition is equivalent to [6, Lem. 4.2].

We continue with results on partitions by orthogonal arrangements—the orthogonal GHR problem for mass assignments.

First, let us recall the best known results on the orthogonal GHR problem for masses, or more precisely its generalisation, the so called generalised Makeev problem. The question was formulated by Blagojević and Roman Karasev in [9, Sec. 1.2]. For integer parameters $j \geq 1$ and $1 \leq \ell \leq k$, the minimal dimension $d := \Delta(j, \ell : k)$, or $d^{\perp} := \Delta^{\perp}(j, \ell : k)$, of a Euclidean space V such that for every collection $\mathcal{M}$ of j masses in V there exists an arrangement of k affine hyperplanes, or pairwise orthogonal k affine hyperplanes in V, with the property that every subarrangement of ℓ hyperplanes equiparts $\mathcal{M}$. In particular, $\Delta(j, k) = \Delta(j, k : k)$. Blagojević and Karasev gave an algebraic constraint on the parameters j, ℓ, k and the dimensions $\Delta(j, \ell : k)$ and $\Delta^{\perp}(j, \ell : k)$, see [9, Thm. 2.1]. The state of the art results on the generalised Makeev problem are due to Steven Simon [35, Thm. 1.1] and Andres Mejlia, Simon and Jialin Zhang [28, Thm. 1.3 and Thm. 1.5]. For example, Simon in [35, Thm. 1.1] showed that

$$\Delta^{\perp}(2^{q+1}, 2 : 2) = 3 \cdot 2^q + 1, \qquad \Delta^{\perp}(2^{q+1} - 1, 2 : 2) = 3 \cdot 2^q - 1,$$

$$\Delta^{\perp}(2^{q+2} - 2, 2 : 2) = 3 \cdot 2^{q+1} - 2, \qquad \Delta^{\perp}(1, 3 : 3) = 4.$$

Coming back to the mass assignments, let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B, and let $k \geq 1$ be an integer. Recall that we denoted by $R_k(B)$ the cohomology ring $H^*(B;\mathbb{F}_2)[x_1,\dots,x_k]$, and by $e_k(B)$ the cohomology class $\prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k-\{0\}}(\alpha_1 x_1+\cdots+\alpha_k x_k)$. We consider the following ideals in $R_k(B)$

$$\mathcal{J}_k(E) := (f_1,\dots,f_k) \quad \text{and} \quad \mathcal{J}'_k(E) := (\bar{f}_1,\dots,\bar{f}_k),$$

where

$$f_i := \sum_{0\leq r_1+\cdots+r_i\leq n-i+1} w_{n-i+1-(r_1+\cdots+r_i)}(E)\, x_1^{r_1}\cdots x_i^{r_i},$$

and

$$\bar{f}_i := \sum_{0\leq r_1+\cdots+r_k\leq n-i+1} w_{n-i+1-(r_1+\cdots+r_k)}(E)\, x_1^{r_1}\cdots x_k^{r_k},$$

for $1 \leq i \leq k$.

The first result on orthogonal partitions is an analogue of Theorems 2.1 and 2.2. For the proof see Section 8.

Theorem 2.15. *Let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B, and let $k \geq 1$ and $j \geq 1$ be integers. Then the following statements are true:*

(1) $\mathcal{J}_k(E) = \mathcal{J}'_k(E)$.

(2) *If the element $e_k(B)^j$ does not belong to the ideal $\mathcal{J}_k(E) = \mathcal{J}'_k(E)$, then for every collection of j continuous real valued functions $\varphi_1,\dots,\varphi_j\colon S(E)\to\mathbb{R}$, there exists a point $b\in B$ and there exists an orthogonal arrangement $\mathcal{H}^b = (H_1^b,\dots,H_k^b)$ of k linear hyperplanes in the fibre E_b of E such that for every pair of connected components $(\mathcal{O}',\mathcal{O}'')$ of the arrangement complement $E_b - (H_1^b\cup\cdots\cup H_k^b)$ the following equalities hold*

$$\int_{\mathcal{O}'\cap S(E_b)}\varphi_1 = \int_{\mathcal{O}''\cap S(E_b)}\varphi_1,\ \dots,\ \int_{\mathcal{O}'\cap S(E_b)}\varphi_j = \int_{\mathcal{O}''\cap S(E_b)}\varphi_j.$$

The implication [35, Thm. 5.2] of Simon, which says that

$$\Delta(j,l)\leq d \implies \Delta^{\perp}(d-1;j)\leq d-1,$$

has an analogue in the mass assignment world.

Proposition 2.16. *Let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B, and let $k \geq 1$ and $j \geq 1$ be integers. Then, if the element $e_k(B)^{j+1}$ does not belong to the ideal $\mathcal{I}_k(E\oplus\underline{\mathbb{R}})$, the element $e_k(B)^j$ does not belong to the ideal $\mathcal{J}_k(E)$.*

In the previous proposition $\underline{\mathbb{R}}$ denotes the trivial line bundle $B\times\mathbb{R}$. The proof of the statement is postponed to Section 8.

In the case when $B = \mathrm{pt}$ the previous theorem implies directly the result of Blagojević and Karasev [9, Thm. 2.1 and Prop. 3.4] with a better description of the set of generators of the relevant ideal.

Corollary 2.17. *Let V be a Euclidean vector space of dimension n, and let $k \geq 1$ and $j \geq 1$ be integers. If*

$$\begin{aligned} e_k(\mathrm{pt}) := & \prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k-\{0\}} (\alpha_1 x_1 + \cdots + \alpha_k x_k) \\ & \notin \Big(\sum_{r_1+\cdots+r_i=n-i+1} x_1^{r_1}\cdots x_i^{r_i} : 1 \leq i \leq k \Big) \\ & = \Big(\sum_{r_1+\cdots+r_k=n-i+1} x_1^{r_1}\cdots x_k^{r_k} : 1 \leq i \leq k \Big), \end{aligned}$$

then for every collection of j continuous functions $\varphi_1, \dots, \varphi_j \colon S(V) \to \mathbb{R}$, there exists an orthogonal arrangement $\mathcal{H} = (H_1, \dots, H_k)$ of k linear hyperplanes in V such that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ of the arrangement complement $V - (H_1 \cup \cdots \cup H_k)$ the following statement holds

$$\int_{\mathcal{O}'\cap S(V)} \varphi_1 = \int_{\mathcal{O}''\cap S(V)} \varphi_1\,, \dots, \int_{\mathcal{O}'\cap S(V)} \varphi_j = \int_{\mathcal{O}''\cap S(V)} \varphi_j\,.$$

In the case of a vector space we collect some numerical results. For that we denote by

$$\omega_k(n) := \max\Big\{ j : e_k(\mathrm{pt})^j \notin \Big(\sum_{r_1+\cdots+r_k=n-i+1} x_1^{r_1}\cdots x_k^{r_k} : 1 \leq i \leq k \Big) \Big\}.$$

Using a computer algebra system, like `Wolfram Mathematica`, we collect some concrete values of $\omega_k(n)$:

$\omega_k(n)$	**n**	3	4	5	6	7	8	9	10
k									
2		0	1	2	2	3	4	4	5
3		0	0	0	1	1	2	2	3
4		0	0	0	0	0	1	1	1

Using the result of Simon [35, Thm. 5.2] or alternatively our extension, Proposition 2.16, we get the following corollary.

Corollary 2.18. *For all integers $k \geq 1$ and $n \geq 1$ we have*

$$\omega_k(n) \geq \iota_k(n+1, \dots, n+1) - 1.$$

For example, if $0 \leq r \leq 2^t - 1$, $n + 1 \geq 2^{t+k-1} + r + 1$, then $\omega_k(n) \geq 2^t + r - 1$.

3. From a partition problem to a topological question: The CS/TM schemes

In this section, based on the work of Crabb [14], we develop an alternative configuration test map scheme (CS/TM) to the one presented in [6, Sec. 2]. This will be done in two steps, first for the classical GHR mass partition problem, and then for the mass assignment partition problem. The new approach allows us a systematic study of mass assignment partition questions even with addition of constraints.

3.1. The GHR problem for masses

In this part, we reformulate the typical product CS/TM scheme for the classical GHR problem. The reformulation of the scheme naturally gives rise to a convenient CS/TM scheme for the GHR problem for mass assignments.

Let V be a Euclidean vector space of dimension $d \geq 1$. The unit sphere of the vector space V will be denoted by $S(\mathrm{V}) := \{v \in \mathrm{V} : \|v\| = 1\}$ and the corresponding real projective space by $\mathbb{P}(\mathrm{V})$. The associated Hopf line bundle is $H(V) := \{(L, v) \in \mathbb{P}(\mathrm{V}) \times \mathrm{V} : v \in L\}$. In particular, $S(\mathrm{V}) \cong S^{d-1}$ is the space of all oriented 1-dimensional vector subspaces of V and $\mathbb{P}(\mathrm{V}) \cong \mathbb{R}\mathrm{P}^{d-1}$ is the space of all 1-dimensional vector subspaces of V. The canonical homeomorphism $\mathbb{P}(\mathrm{V}) = \mathrm{G}_1(\mathrm{V}) \cong \mathrm{G}_{d-1}(\mathrm{V})$, $L \mapsto L^{\perp}$, identifies the projective space $\mathbb{P}(\mathrm{V})$ with the space of all linear hyperplanes in V, the Grassmann manifold $\mathrm{G}_{d-1}(\mathrm{V})$.

The space of all arrangements of k linear hyperplanes in V can be identified with the product space $\mathbb{P}(\mathrm{V})^{\times k} = \mathbb{P}(\mathrm{V}) \times \cdots \times \mathbb{P}(\mathrm{V})$. On the other hand, the space of all arrangements of k oriented linear hyperplanes in V is the 2^k-fold covering $S(H(\mathrm{V}))^{\times k} = S(H(\mathrm{V})) \times \cdots \times S(H(\mathrm{V}))$ of $\mathbb{P}(\mathrm{V})^{\times k}$, whose total space, in particular, is just the product of spheres $S(\mathrm{V})^{\times k} = S(\mathrm{V}) \times \cdots \times S(\mathrm{V})$. In other words, we have a fibre bundle

$$S\big(H(\mathrm{V})\big)^{\times k} \to \mathbb{P}(\mathrm{V})^{\times k}$$

with a discrete fibre

$$\big(S\big(H(\mathrm{V})\big)^{\times k}\big)_{(L_1,\dots,L_k)} = S(L_1) \times \cdots \times S(L_k)$$

at $(L_1, \dots, L_k) \in \mathbb{P}(\mathrm{V})^{\times k}$. Here, $S(H(\mathrm{V}))$ denotes the sphere bundle of the Hopf line bundle $H(\mathrm{V})$ with fibres homeomorphic to a zero dimensional sphere.

We denote by $A_k(\mathrm{V})$ the 2^k-dimensional real vector bundle over $\mathbb{P}(\mathrm{V})^{\times k}$ with fibre at $(L_1, \dots, L_k) \in \mathbb{P}(\mathrm{V})^{\times k}$ defined to be the vector space $\mathrm{Map}(\prod_{i=1}^{k} S(L_i), \mathbb{R})$ of all maps $\prod_{i=1}^{k} S(L_i) \to \mathbb{R}$. Each vector space $\mathrm{Map}(\prod_{i=1}^{k} S(L_i), \mathbb{R})$ is equipped with the natural $(\mathbb{Z}/2)^k$-action given by the antipodal actions on the 0-dimensional spheres $S(L_1), \dots, S(L_k)$. The vector bundle $A_k(\mathrm{V})$ is isomorphic to the vector bundle

$$q_1^*\big(H(\mathrm{V}) \oplus \underline{\mathbb{R}}\big) \otimes \cdots \otimes q_k^*\big(H(\mathrm{V}) \oplus \underline{\mathbb{R}}\big),$$

where $q_i \colon \mathbb{P}(\mathrm{V})^{\times k} \to \mathbb{P}(\mathrm{V})$ is the projection on the i-th factor, $\underline{\mathbb{R}}$ denotes the trivial line bundle, in this case, over $\mathbb{P}(\mathrm{V})$, and $q_i^*(H(\mathrm{V}) \oplus \underline{\mathbb{R}})$ is the pullback vector bundle. In particular, the vector bundle

$$A_k(\mathrm{V}) \cong q_1^*\big(H(\mathrm{V}) \oplus \underline{\mathbb{R}}\big) \otimes \cdots \otimes q_k^*\big(H(\mathrm{V}) \oplus \underline{\mathbb{R}}\big)$$

has a trivial line subbundle given by all constant maps $\prod_{i=1}^k S(L_i) \to \mathbb{R}$, which we also denote by $\underline{\mathbb{R}}$.

Next, let us consider a continuous function $\varphi\colon S(\mathrm{V}) \to \mathbb{R}$ on the sphere $S(\mathrm{V})$. It induces a section $s_\varphi\colon \mathbb{P}(\mathrm{V})^{\times k} \to A_k(\mathrm{V})$ of the vector bundle $A_k(\mathrm{V})$ which is given by

$$(L_1, \dots, L_k) \mapsto \Big(s_\varphi(L_1, \dots, L_k)\colon \prod_{i=1}^{k} S(L_i) \to \mathbb{R}\Big)$$

for $(L_1, \dots, L_k) \in \mathbb{P}(V)$, where

$$s_\varphi(L_1, \dots, L_k)(v_1, \dots, v_k) := \int_{\mathcal{O}_{v_1,\dots,v_k} \cap S(\mathrm{V})} \varphi$$

for $(v_1, \dots, v_k) \in \prod_{i=1}^k S(L_i)$. Here, $\mathcal{O}_{v_1,\dots,v_k}$ denotes the following intersection of open half-spaces in V:

$$\mathcal{O}_{v_1,\dots,v_k} := \big\{u \in \mathrm{V} : \langle u, v_1\rangle > 0\big\} \cap \cdots \cap \big\{u \in \mathrm{V} : \langle u, v_k\rangle > 0\big\}.$$

Here the integration is with the respect to the measure on the sphere $S(\mathrm{V})$ induced by the metric. Observe that each subset $\mathcal{O}_{v_1,\dots,v_k}$ is actually a (path) connected component of the arrangement complement $\mathrm{V} - (L_1^\perp \cup \cdots \cup L_k^\perp)$.

We have introduced all necessary notions to state and prove the CS/TM scheme theorem for the spherical version of the classical GHR problem. This theorem relates to the similar results in [25, Prop. 6], [10, Prop. 2.2], [8, Prop. 2.1].

Theorem 3.1. *Let* V *be a Euclidean vector space, and let $k \geq 1$ and $j \geq 1$ be integers. If the Euler class of the vector bundle $(A_k(\mathrm{V})/\underline{\mathbb{R}})^{\oplus j}$ does not vanish, then for every collection of j continuous functions $\varphi_1, \dots, \varphi_j\colon S(\mathrm{V}) \to \mathbb{R}$ there exists an arrangement of k linear hyperplanes $H_1, \dots, H_k$ in* V *with the property that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ of the arrangement complement $\mathrm{V} - (H_1 \cup \cdots \cup H_k)$ the following statement holds*

$$\int_{\mathcal{O}' \cap S(\mathrm{V})} \varphi_1 = \int_{\mathcal{O}'' \cap S(\mathrm{V})} \varphi_1\,, \dots, \int_{\mathcal{O}' \cap S(\mathrm{V})} \varphi_j = \int_{\mathcal{O}'' \cap S(\mathrm{V})} \varphi_j\,.$$

In other words,

$$\mathrm{e}\big((A_k(\mathrm{V})/\underline{\mathbb{R}})^{\oplus j}\big) \neq 0 \implies \Delta_S(j,k) \leq \dim(\mathrm{V}).$$

Proof. Let us assume that the Euler class the vector bundle $(A_k(\mathrm{V})/\underline{\mathbb{R}})^{\oplus j}$ does not vanish. Then, in particular, every section of the vector bundle $(A_k(\mathrm{V})/\underline{\mathbb{R}})^{\oplus j}$ has a zero.

Let $\varphi_1, \dots, \varphi_j : S(\mathrm{V}) \to \mathbb{R}$ be an arbitrary collection of j continuous functions on the sphere $S(\mathrm{V})$. Such a collection induces a section $s : \mathbb{P}(\mathrm{V})^{\times k} \to A_k(\mathrm{V})^{\oplus j}$ of the vector bundle $A_k(\mathrm{V})^{\oplus j}$ defined by

$$(L_1, \dots, L_k) \mapsto \Big(s_{\varphi_r}(L_1, \dots, L_k) : \prod_{i=1}^{k} S(L_i) \to \mathbb{R} \Big)_{1 \le r \le j}.$$

Recall that we have already defined functions s_{φ_r}, for $1 \le r \le j$, by

$$s_{\varphi_r}(L_1, \dots, L_k)(v_1, \dots, v_k) = \int_{\mathcal{O}_{v_1,\dots,v_k} \cap S(\mathrm{V})} \varphi_r$$

for $(v_1, \dots, v_k) \in \prod_{i=1}^{k} S(L_i)$.

Let $\Pi : A_k(\mathrm{V})^{\oplus j} \to (A_k(\mathrm{V})/\underline{\mathbb{R}})^{\oplus j}$ denote the map of vector bundles induced by the canonical projection(s). Then the section $\Pi \circ s$ of the vector bundle $(A_k(\mathrm{V})/\underline{\mathbb{R}})^{\oplus j}$ has a zero. Hence, there is a point $(L_1, \dots, L_k) \in \mathbb{P}(\mathrm{V})^{\times k}$ in the base space with the property that $s(L_1, \dots, L_k)$ belongs to the trivial subbundle $\underline{\mathbb{R}}^{\oplus j}$ of the bundle $A_k(\mathrm{V})^{\oplus j}$. In other words

$$\int_{\mathcal{O}' \cap S(\mathrm{V})} \varphi_1 = \int_{\mathcal{O}'' \cap S(\mathrm{V})} \varphi_1 \;, \dots, \; \int_{\mathcal{O}' \cap S(\mathrm{V})} \varphi_j = \int_{\mathcal{O}'' \cap S(\mathrm{V})} \varphi_j$$

for all pairs of the connected components $(\mathcal{O}', \mathcal{O}'')$ of the arrangement complement $\mathrm{V} - (L_1^{\perp} \cup \cdots \cup L_k^{\perp})$. This completes the proof of the theorem. ∎

The non-vanishing of the Euler class $(A_k(\mathrm{V})/\underline{\mathbb{R}})^{\oplus j}$ mod 2 was studied over the years by many authors. For example, Mani-Levitska, Vrećica and Živaljević [25, Thm. 39] gave a sufficient condition for the non-vanishing of the mod 2 Euler class of $(A_k(\mathrm{V})/\underline{\mathbb{R}})^{\oplus j}$, with a complete proof of this result given only now in [6, Lem. 4.3]. It says that: *If* $\dim(\mathrm{V}) \le j + (2^{k-1} - 1)2^{\lfloor \log_2 j \rfloor}$, *then the top Stiefel–Whitney class of the vector bundle* $(A_k(\mathrm{V})/\underline{\mathbb{R}})^{\oplus j}$ *does not vanish.*

Now we focus our attention to the partition problems for mass assignments and the corresponding solution schemes.

3.2. The GHR problem for mass assignments

The scheme we give in this section is derived from the scheme for the spherical version of the classical problem presented in Section 3.1. Due to a transition from a Euclidean space to a sphere, the new scheme differs from the one used in [6, Sec. 2].

Let E be a Euclidean vector bundle over a compact and connected ENR base space B. The associated unit sphere bundle of E is

$$S(E) = \big\{(b, v) : b \in B,\ v \in S(E_b)\big\}.$$

Next, let $\mathbb{P}(E)$ denote the projective bundle of E, that is

$$\mathbb{P}(E) = \big\{(b, L) : b \in B,\ L \in \mathbb{P}(E_b)\big\}.$$

In particular, $S(E)/(\mathbb{Z}/2) \cong \mathbb{P}(E)$. Here the fibrewise antipodal action of the sphere bundle is assumed. Further, let $H(E)$ be the Hopf bundle associated to the vector bundle E. That is the line bundle

$$H(E) := \big\{(b, L, v) : b \in B,\ L \in \mathbb{P}(E_b),\ v \in L\big\}$$

over the projective bundle $\mathbb{P}(E)$.

The space of all arrangements of k linear hyperplanes which belong to one fibre of E is the total space of the pullback

$$\mathbb{P}(E) \times_B \cdots \times_B \mathbb{P}(E) := d^*\big(\mathbb{P}(E) \times \cdots \times \mathbb{P}(E)\big) = d^*\big(\mathbb{P}(E)^{\times k}\big)$$

of the product vector bundle $\mathbb{P}(E)^{\times k}$ via the diagonal embedding $d\colon B \to B^{\times k}$, $x \mapsto (x, \dots, x)$. In other words, there is a pullback diagram

$$\begin{array}{ccc} d^*\big(\mathbb{P}(E)^{\times k}\big) & \xrightarrow{D} & \mathbb{P}(E)^{\times k} \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow{d} & B^{\times k}. \end{array}$$

Let us denote by $\Pi_i\colon \mathbb{P}(E)^{\times k} \to \mathbb{P}(E)$, $(b, (L_1, \dots, L_k)) \mapsto (b, L_i)$, the projection on the i-th factor, and by Θ_i the composition $\Pi_i \circ D\colon d^*(\mathbb{P}(E)^{\times k}) \to \mathbb{P}(E)$, where $1 \le i \le k$.

Now, the space of all arrangements of k oriented linear hyperplanes which belong to one fibre of E is the total space of the pullback

$$S(E) \times_B \cdots \times_B S(E) := d^*\big(S(E) \times \cdots \times S(E)\big) = d^*\big(S(E)^{\times k}\big).$$

The quotient map $d^*(S(E)^{\times k}) \to d^*(\mathbb{P}(E)^{\times k})$, induced by taking orbits of the natural fibrewise free action of $(\mathbb{Z}/2)^k$ on $d^*(S(E)^{\times k})$, is a 2^k-fold cover map with the fibre $S(L_1) \times \cdots \times S(L_k)$ at $(L_1, \dots, L_k) \in \mathbb{P}(E_b)^{\times k}$ for some $b \in$ B. Recall that each sphere $S(L_1), \dots, S(L_k)$ is just a 0-dimensional sphere.

Like in the classical case, the covering $d^*(S(E)^{\times k}) \to d^*(\mathbb{P}(E)^{\times k})$ induces a 2^k-dimensional real vector bundle $A_k(E)$ over $d^*(\mathbb{P}(E)^{\times k})$ with fibre at $(L_1, \dots, L_k) \in \mathbb{P}(E_b)^k$, for some $b \in B$, defined to be the vector space $\mathrm{Map}(\prod_{i=1}^k S(L_i), \mathbb{R})$ of all real valued functions on $\prod_{i=1}^k S(L_i)$. Each fibre is equipped with the natural $(\mathbb{Z}/2)^k$-action given by antipodal actions on the 0-dimensional spheres. There is an isomorphism of vector bundles

$$A_k(E) \cong \Theta_1^*\big(H(E) \oplus \underline{\mathbb{R}}\big) \otimes \cdots \otimes \Theta_k^*\big(H(E) \oplus \underline{\mathbb{R}}\big),$$

where $\underline{\mathbb{R}}$ denotes the trivial line bundle over $\mathbb{P}(E)$, and $\Theta_i^*(H(E) \oplus \underline{\mathbb{R}})$ is the pullback vector bundle. In particular, the vector bundle $A_k(E)$ has a trivial line bundle determined by all constant maps $\prod_{i=1}^k S(L_i) \to \mathbb{R}$.

Let us now consider a continuous function $\varphi\colon S(E) \to \mathbb{R}$. Such a map induces a section $s_\varphi\colon d^*(\mathbb{P}(E)^{\times k}) \to A_k(E)$ of the vector bundle $A_k(E)$ by

$$\big(b,(L_1,\dots,L_k)\big) \mapsto \Big(s_\varphi\big(b,(L_1,\dots,L_k)\big)\colon \prod_{i=1}^{k} S(L_i) \to \mathbb{R}\Big)$$

for $b \in B$ and $(L_1,\dots,L_k) \in \mathbb{P}(E_b)^{\times k}$, where

$$s_\varphi\big(b,(L_1,\dots,L_k)\big)(v_1,\dots,v_k) := \int_{\mathcal{O}_{b,v_1,\dots,v_k} \cap S(E_b)} \varphi$$

for $(v_1,\dots,v_k) \in \prod_{i=1}^{k} S(L_i)$. Here, $\mathcal{O}_{b,v_1,\dots,v_k}$ denotes the subset of E_b defined by

$$\mathcal{O}_{b,v_1,\dots,v_k} := \big\{u \in E_b : \langle u, v_1\rangle > 0\big\} \cap \dots \cap \big\{u \in E_b : \langle u, v_k\rangle > 0\big\}.$$

Once again, the integration is assumed to be with respect to the measure of the sphere $S(E_b)$ induced by the metric on E_b.

Now we can state the CS/TM scheme theorem for the GHR problem for mass assignments, which is analogous to Theorem 3.1.

Theorem 3.2. *Let E be a Euclidean vector bundle over a compact and connected ENR base space B, and let $k \geq 1$ and $j \geq 1$ be integers.*

If the Euler class of the vector bundle $(A_k(E)/\underline{\mathbb{R}})^{\oplus j}$ does not vanish, then for every collection of j continuous functions $\varphi_1,\dots,\varphi_j\colon S(E) \to \mathbb{R}$ there exists a point $b \in B$ and there exists an arrangement of k linear hyperplanes $H_1,\dots,H_k$ in the fibre E_b with the property that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ of the arrangement complement $E_b - (H_1 \cup \dots \cup H_k)$ the following statement holds

$$\int_{\mathcal{O}' \cap S(E_b)} \varphi_1 = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_1\,,\ \dots,\ \int_{\mathcal{O}' \cap S(E_b)} \varphi_j = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_j\,.$$

In other words,

$$\mathrm{e}\left(\left(A_k(E)/\underline{\mathbb{R}}\right)^{\oplus j}\right) \neq 0 \implies (j,k) \in \Delta_S(E).$$

Proof. Our follows in the footsteps of the proof of Theorem 3.1. Assume that the Euler class of the vector bundle $(A_k(E)/\underline{\mathbb{R}})^{\oplus j}$ does not vanish. Consequently, every section of $(A_k(E)/\underline{\mathbb{R}})^{\oplus j}$ has a zero.

Take a collection

$$\varphi_1,\dots,\varphi_j\colon S(E) \to \mathbb{R}$$

of continuous functions on the sphere bundle $S(E)$ and consider the associated section $s = (s_{\varphi_1},\dots,s_{\varphi_j})$ of the vector bundle $(A_k(E)/\underline{\mathbb{R}})^{\oplus j}$.

Denote by $\Pi\colon A_k(E)^{\oplus j} \to (A_k(E)/\underline{\mathbb{R}})^{\oplus j}$ the canonical projection. Then, from the assumption on the Euler class, the section $\Pi \circ s$ of the vectors bundle $(A_k(E)/\underline{\mathbb{R}})^{\oplus j}$ has a zero. In other words, there exists a point $(b,(L_1,\dots,L_k)) \in d^*(\mathbb{P}(E)^{\times k})$ with the property that $s(b,(L_1,\dots,L_k))$ is contained in the trivial vector subbundle $\underline{\mathbb{R}}^{\oplus j}$ of the vector bundle $A_k(E)^{\oplus j}$. This means that for every pair of connected components $(\mathcal{O}',\mathcal{O}'')$ of the arrangement complement $E_b-(L_1^{\perp}\cup\cdots\cup L_k^{\perp})$ the following equalities hold

$$\int_{\mathcal{O}'\cap S(E_b)}\varphi_1=\int_{\mathcal{O}''\cap S(E_b)}\varphi_1\,,\dots,\ \int_{\mathcal{O}'\cap S(E_b)}\varphi_j=\int_{\mathcal{O}''\cap S(E_b)}\varphi_j\,.$$

Hence, we have proved the theorem. ■

3.3. The GHR problem for mass assignments plus constraints

In this section, we extend the CS/TM schemes presented in Section 3.2 to incorporate an additional constraint. More precisely, we require the normals of the hyperplanes to belong to specific, not necessarily equal, vector subbundles.

Fix an integer $k\geq 1$. Let E be an n-dimensional Euclidean vector bundle over a compact and connected ENR B, and let $E(i)$ be a vector subbundle of E, for $1\leq i\leq k$. Following the notation from Section 3.2 we denote by $\mathbb{P}(E(i))$ the projective bundle of $E(i)$, that is

$$\mathbb{P}\big(E(i)\big)=\big\{(b;L):b\in B,\ L\in\mathbb{P}\big(E(i)_b\big)\big\}.$$

In particular, $S(E(i))/(\mathbb{Z}/2)\cong P(E(i))$. Furthermore, let $H(E(i))$ be the Hopf bundle associated to the vector bundle $E(i)$, or in other words

$$H\big(E(i)\big):=\big\{(b,L,v):b\in B,\ L\in\mathbb{P}\big(E(i)_b\big),\ v\in L\big\}.$$

The space of all arrangements of k linear hyperplanes which belong to one fibre of E and are determined by the collection of vector subbundles $E(1),\dots,E(k)$ can be seen as the total space of the pullback vector bundle

$$\mathbb{P}\big(E(1)\big)\times_B\cdots\times_B\mathbb{P}\big(E(k)\big):=d^*\big(\mathbb{P}\big(E(1)\big)\times\cdots\times\mathbb{P}\big(E(k)\big)\big)$$

via the diagonal embedding $d\colon B\to B^k$. We denote by

$$D\colon d^*\big(\mathbb{P}\big(E(1)\big)\times\cdots\times\mathbb{P}\big(E(k)\big)\to\mathbb{P}\big(E(1)\big)\times\cdots\times\mathbb{P}\big(E(k)\big)\big)$$

the pullback map between the bundles. Furthermore, let

$$\Pi_i\colon\mathbb{P}\big(E(1)\big)\times\cdots\times\mathbb{P}\big(E(k)\big)\to\mathbb{P}\big(E(i)\big)$$

be the projection on the i-th factor $(b,(L_1,\dots,L_k))\mapsto(b,L_i)$, and let $\Theta_i:=\Pi_i\circ D$.

The space of all arrangements of k oriented linear hyperplanes which belong to one fibre of E and are given by the collection of vector subbundles $E(1),\dots,E(k)$ is the total space of the pullback

$$S\big(E(1)\big)\times_B\cdots\times_B S\big(E(k)\big):=d^*\big(S\big(E(1)\big)\times\cdots\times S\big(E(k)\big)\big).$$

The quotient map

$$d^*\big(S\big(E(1)\big)\times\cdots\times S\big(E(k)\big)\big)\to d^*\big(\mathbb{P}\big(E(1)\big)\times\cdots\times\mathbb{P}\big(E(k)\big)\big),$$

induced by taking orbits of the natural fibrewise free action of the group $(\mathbb{Z}/2)^k$, is a 2^k-fold cover map with a typical fibre $S(L_1)\times\cdots\times S(L_k)$ where $(L_1,\dots,L_k)\in\mathbb{P}(E(1)_b)\times\cdots\times\mathbb{P}(E(k)_b)$ for some $b\in$ B.

This covering induces a 2^k-dimensional real vector bundle $A_k(E(1),\dots,E(k))$ over $d^*(\mathbb{P}(E(1))\times\cdots\times\mathbb{P}(E(k)))$ with fibre at $(L_1,\dots,L_k)\in\mathbb{P}(E(1)_b)\times\cdots\times\mathbb{P}(E(k)_b)$, for some $b\in B$, defined to be the vector space $\mathrm{Map}(\prod_{i=1}^k S(L_i),\mathbb{R})$ of all real valued functions on $\prod_{i=1}^k S(L_i)$. There is an isomorphism of vector bundles

$$A_k\big(E(1),\dots,E(k)\big)\cong\Theta_1^*\big(H\big(E(1)\big)\oplus\underline{\mathbb{R}(1)}\big)\otimes\cdots\otimes\Theta_k^*\big(H\big(E(k)\big)\oplus\underline{\mathbb{R}(k)}\big),$$

where $\underline{\mathbb{R}(i)}$ denotes the trivial line bundle over $\mathbb{P}(E(i))$, and $\Theta_i^*(H(E(i))\oplus\underline{\mathbb{R}(i)})$ is the pullback vector bundle. In particular, the vector bundle $A_k(E(1),\dots,E(k))$ has a trivial line bundle determined by all constant maps $\prod_{i=1}^k S(L_i)\to\mathbb{R}$, or in other words the vector subbundle $\underline{\mathbb{R}(1)}\otimes\cdots\otimes\underline{\mathbb{R}(k)}$. Clearly, $A_k(E)=A_k(\underbrace{E,\dots,E}_{k\text{ times}})$.

Now we consider a continuous function $\varphi\colon S(E)\to\mathbb{R}$. It induces a section

$$s_\varphi\colon d^*\big(\mathbb{P}\big(E(1)\big)\times\cdots\times\mathbb{P}\big(E(k)\big)\big)\longrightarrow A_k\big(E(1),\dots,E(k)\big)$$

of the vector bundle $A_k(E(1),\dots,E(k))$ by

$$\big(b,(L_1,\dots,L_k)\big)\mapsto\Big(s_\varphi\big(b,(L_1,\dots,L_k)\big)\colon\prod_{i=1}^k S(L_i)\to\mathbb{R}\Big)$$

for $b\in B$ and $(L_1,\dots,L_k)\in\mathbb{P}(E(1)_b)\times\cdots\times\mathbb{P}(E(k)_b)$, where

$$s_\varphi\big(b,(L_1,\dots,L_k)\big)(v_1,\dots,v_k):=\int_{\mathcal{O}_{b,v_1,\dots,v_k}\cap S(E_b)}\varphi$$

for $(v_1,\dots,v_k)\in\prod_{i=1}^k S(L_i)$. Recall, $\mathcal{O}_{b,v_1,\dots,v_k}$ denotes the set:

$$\mathcal{O}_{b,v_1,\dots,v_k}:=\big\{u\in E_b:\langle u,v_1\rangle>0\big\}\cap\cdots\cap\big\{u\in E_b:\langle u,v_k\rangle>0\big\}.$$

The CS/TM scheme theorem for the GHR problem for mass assignments with constraints is as follows.

Theorem 3.3. *Let E be a Euclidean vector bundle over a compact and connected ENR base space B, $k\ge 1$ and $j\ge 1$ integers, and let $E(1),\dots,E(k)$ be vector subbundles of E.*

If the Euler class of the vector bundle $(A_k(E(1),\dots,E(k))/\underline{\mathbb{R}})^{\oplus j}$ does not vanish, then for every collection of j continuous functions $\varphi_1,\dots,\varphi_j\colon S(E)\to\mathbb{R}$ there exists a *point $b \in B$ and there exists an arrangement of k linear hyperplanes $H_1, \dots, H_k$ in the fibre E_b determined by the collection of vector subbundles $E(1), \dots, E(k)$ with the property that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ of the arrangement complement $E_b - (H_1 \cup \dots \cup H_k)$ the following statement holds*

$$\int_{\mathcal{O}' \cap S(E_b)} \varphi_1 = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_1 \,, \dots, \int_{\mathcal{O}' \cap S(E_b)} \varphi_j = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_j \,.$$

Proof. A proof is a slight modification of the proof of Theorem 3.2, so we do not repeat it. ∎

3.4. The orthogonal GHR problem for mass assignments

The scheme for the partitions with orthogonal arrangements is just a "restriction" of the scheme presented in Section 3.2.

For a Euclidean vector bundle over a compact and connected ENR base space B, and integers $k \geq 1$ and $j \geq 1$, we proved the following:

If the Euler class of the vector bundle $(A_k(E)/\underline{\mathbb{R}})^{\oplus j}$ over $d^*(\mathbb{P}(E)^{\times k})$ does not vanish, then for every collection of j continuous functions $\varphi_1, \dots, \varphi_j : S(E) \to \mathbb{R}$ there exists a point $b \in B$ and there exists an arrangement of k linear hyperplanes $H_1, \dots, H_k$ in the fibre E_b with the property that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ of the arrangement complement $E_b - (H_1 \cup \dots \cup H_k)$ holds

$$\int_{\mathcal{O}' \cap S(E_b)} \varphi_1 = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_1 \,, \dots, \int_{\mathcal{O}' \cap S(E_b)} \varphi_j = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_j \,.$$

Since we are interested in partitions by specifically orthogonal arrangements the space of all possible solutions becomes the following subspace of $X_k(E) := d^*(\mathbb{P}(E)^{\times k})$:

$$Y_k(E) := \big\{\big(b, (L_1, \dots, L_k)\big) \in X_k(E) : L_r \perp L_s \text{ for all } 1 \leq r < s \leq k\big\}.$$

In addition, let us denote by q_k the inclusion $Y_k(E) \hookrightarrow X_k(E)$. Thus, the vector bundle we are interested in is the restriction bundle $B_k(E) := A_k(E)|_{Y_k(E)}$. In particular, there is an isomorphism of vector bundles

$$B_k(E) \cong \Psi_1^*\big(H(E) \oplus \underline{\mathbb{R}}\big) \otimes \dots \otimes \Psi_k^*\big(H(E) \oplus \underline{\mathbb{R}}\big),$$

where $\Psi_i = \Theta_i \circ q_k$ for $1 \leq i \leq k$. Recall $H(E)$ and $\underline{\mathbb{R}}$ are here the Hopf line and trivial line bundle over $\mathbb{P}(E)$, respectively.

Now, we get the CS/TM scheme theorem for the GHR problem for mass assignments by orthogonal arrangements directly from the proof of Theorem 3.2.

Theorem 3.4. *Let E be a Euclidean vector bundle over a compact and connected ENR base space B, and let $k \geq 1$ and $j \geq 1$ be integers.*

If the Euler class of the vector bundle $(B_k(E)/\underline{\mathbb{R}})^{\oplus j}$ does not vanish, then for every collection of j continuous functions $\varphi_1, \dots, \varphi_j : S(E) \to \mathbb{R}$ there exists a point $b \in B$

and there exists an orthogonal arrangement of k linear hyperplanes $H_1, \dots, H_k$ in the fibre E_b with the property that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ of the arrangement complement $E_b - (H_1 \cup \dots \cup H_k)$ the following statement holds

$$\int_{\mathcal{O}' \cap S(E_b)} \varphi_1 = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_1 , \dots , \int_{\mathcal{O}' \cap S(E_b)} \varphi_j = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_j .$$

The proof of this result is a copy of the proof of Theorem 3.2 with $Y_k(E)$ in place of $X_k(E)$ and the vector bundle $B_k(E)$ in place of the vector bundle $A_k(E)$.

3.5. The Fairy Bread Sandwich theorem

Fix integers $d \geq 1$ and $k \geq 1$ with $d \geq k$, and let $V = \mathbb{R}^{d+1}$. Let $(j_k, \dots, j_d)$ be a permutation of the set $\{k, \dots, d\}$, and let $\varphi_{a,b}\colon S(E_{a+1}^{d+1}) \to \mathbb{R}$, $k \leq a \leq d$, $1 \leq b \leq j_a$, be a collection of functions from the sphere bundle of the tautological vector bundle E_{a+1}^{d+1} over the Grassmann manifold $G_{a+1}(V)$ to the real numbers.

The space of all potential solutions of the partition problem considered in Theorem 2.12 is the following flag manifold

$$\begin{aligned}
&\mathrm{Flag}_{k,\dots,d}(V)\\
&= \Big\{ (V_k, \dots, V_d) \in \prod_{i=k}^{d} G_i(V) : 0 \subseteq V_k \subseteq \dots \subseteq V_d \subseteq V \Big\}\\
&\cong \big\{ (W_k, \dots, W_{d+1}) \in G_k(V) \times G_1(V)^{d-k+1} : W_{i'} \perp W_{i''} \text{ for all } k \leq i' < i'' \leq d+1 \big\}.
\end{aligned}$$

We used the homeomorphism between these two presentations

$$(W_k, \dots, W_{d+1}) \mapsto \big(W_k, (W_k \oplus W_{k+1}), \dots, (W_k \oplus W_{k+1} \oplus \dots \oplus W_{d-1})\big) \qquad (2)$$

to identify the corresponding elements. More detail on flag manifolds can be found in Section 6.

For every $k+1 \leq i \leq d+1$ we define a 2-dimensional real vector bundle K_i over $\mathrm{Flag}_{k,\dots,d}(V)$ whose fiber over the point

$$(W_k, \dots, W_{d+1}) \overset{(2)}{=} (V_k, V_{k+1}, \dots, V_d) \in \mathrm{Flag}_{k,\dots,d}(V)$$

is the real vector space $\mathrm{Map}(S(W_i), \mathbb{R})$. The vector bundle K_i decomposes into the direct sum

$$K_i \cong E_i \oplus \underline{\mathbb{R}},$$

where E_i, as in Section 6, denotes the canonical line bundle associated to the flag manifold $\mathrm{Flag}_{k,\dots,d}(V)$ and $\underline{\mathbb{R}}$ is the trivial line bundle which corresponds to constant functions.

Take an integer $k+1 \leq i \leq d+1$, and let $\varphi\colon S(E_i^{d+1}) \to \mathbb{R}$ be a continuous real valued function. It induces a section $s_{i,\varphi}$ of K_i defined by

$$(W_k, \dots, W_d) \overset{(2)}{=} (V_k, V_{k+1}, \dots, V_d) \longmapsto \big(s_\varphi(W_k, \dots, W_d)\colon S(W_i) \longrightarrow \mathbb{R}\big),$$

where

$$s_{i,\varphi}(W_k, \dots, W_d)(u) := \int_{\{v \in V_i : \langle v,u\rangle \geq 0\} \cap S(V_i)} \varphi.$$

The section $s_{i,\varphi}$ induces additionally a section $s'_{i,\varphi}$ of the vector bundle E_i by

$$(W_k, \dots, W_d) \overset{(2)}{=} (V_k, V_{k+1}, \dots, V_d) \mapsto \big(s'_{i,\varphi}(W_k, \dots, W_d) \colon S(W_i) \to \mathbb{R}\big),$$

where

$$s'_{i,\varphi}(W_k, \dots, W_d)(u) := s_{i,\varphi}(W_k, \dots, W_d)(u) - s_{i,\varphi}(W_k, \dots, W_d)(-u).$$

Now, the CS/TM scheme theorem for the Fairy Bread Sandwich theorem can be stated as follows.

Theorem 3.5. *Let $d \geq 1$ and $k \geq 1$ be integers with $d \geq k$, and let $V = \mathbb{R}^{d+1}$ be a real vector space. Fix a permutation $(j_k, \dots, j_d)$ of the set $\{k, \dots, d\}$, and take an arbitrary collections of functions $\varphi_{a,b} \colon S(E^{d+1}_{a+1}) \to \mathbb{R}$, $k \leq a \leq d$, $1 \leq b \leq j_a$, from the sphere bundle of the tautological vector bundle E^{d+1}_{a+1} over the Grassmann manifold $G_{a+1}(V)$ to the real numbers.*

If the Euler class of the vector bundle $E := E^{\oplus j_k}_{k+1} \oplus E^{\oplus j_{k+1}}_{k+2} \oplus \cdots \oplus E^{\oplus j_d}_{d+1}$ does not vanish, then there exists a flag $(W_k, \dots, W_d) = (V_k, \dots, V_d) \in \mathrm{Flag}_{k,\dots,d}(V)$ such that for every $k \leq a \leq d$ and every $1 \leq b \leq j_a$ the following equality holds

$$\int_{\{v \in V_{a+1} : \langle v,u_a\rangle \geq 0\} \cap S(V_{a+1})} \varphi_{a,b} = \int_{\{v \in V_{a+1} : \langle v,u_a\rangle \leq 0\} \cap S(V_{a+1})} \varphi_{a,b}.$$

Here the unit vectors $u_k, \dots, u_d$ are determined, up to a sign, by the equality $V_r = \{v \in V_{r+1} : \langle v, u_r\rangle = 0\}$, $k \leq r \leq d$, and with $V_{d+1} = V$. This means that u_r is a unit normal vector to V_r, considered as a hyperplane inside V_{r+1}. In other words, $u_r \in S(W_{r+1})$ for all $k \leq r \leq d$.

Proof. Assume that the Euler class of the vector bundle $E = E^{\oplus j_k}_{k+1} \oplus \cdots \oplus E^{\oplus j_d}_{d+1}$ does not vanish. Hence, every section of E has a zero.

The collections of functions $\varphi_{a,b} \colon S(E^{d+1}_{a+1}) \to \mathbb{R}$, $k \leq a \leq d$, $1 \leq b \leq j_a$ induces a section of the vector bundle E in the following way

$$(W_k, \dots, W_d) \mapsto \bigoplus_{1 \leq b \leq j_k} s'_{k+1,\varphi_{k,b}}(W_k, \dots, W_d) \oplus \cdots \oplus \bigoplus_{1 \leq b \leq j_d} s'_{d+1,\varphi_{d,b}}(W_k, \dots, W_d).$$

Thus, there exists a flag $(W_k, \dots, W_d) = (V_k, \dots, V_d) \in \mathrm{Flag}_{k,\dots,d}(V)$ such that for every $k \leq a \leq d$ and every $1 \leq b \leq j_a$ the following statement holds

$$\begin{aligned} &s'_{a+1,\varphi_{a,b}}(W_k, \dots, W_d)(u) \\ &\quad = \int_{\{v \in V_{a+1} : \langle v,u\rangle \geq 0\} \cap S(V_{a+1})} \varphi_{a,b} - \int_{\{v \in V_{a+1} : \langle v,u\rangle \leq 0\} \cap S(V_{a+1})} \varphi_{a,b} = 0, \end{aligned}$$

for $u \in S(W_{a+1})$. This concludes the proof. ∎

4. Proofs of Theorems 2.1 and 2.2

For the proofs of the theorems we recall and show the following classical fact, see for example [13, Satz und Def. VI.6.4], [24, Thm. 17.2.5 and Def. 17.2.6] and [17, (1.13)].

Lemma 4.1. *Let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B, and let $\mathbb{P}(E)$ denote the associated projective bundle of E. Then there is an isomorphism of $H^*(B;\mathbb{F}_2)$-algebras*

$$H^*(B;\mathbb{F}_2)[x]/\Big(\sum_{s=0}^{n} w_{n-s}(E)\,x^s\Big) \to H^*\big(\mathbb{P}(E);\mathbb{F}_2\big)$$

which maps x to the mod 2 *Euler class of the Hopf line bundle $H(E)$.*

For the proof first recall that for $m \geq 2$

$$H^*\big(\mathbb{P}(\mathbb{R}^m);\mathbb{F}_2\big) = H^*(\mathbb{R}\mathrm{P}^{m-1};\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^m),$$

where $x = \mathrm{e}(H(\mathbb{R}^m))$ is the mod 2 Euler class of the Hopf line bundle $H(\mathbb{R}^m)$. In the case when $m = \infty$ we have

$$H^*\big(\mathbb{P}(\mathbb{R}^\infty);\mathbb{F}_2\big) = H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[x],$$

where $x = \mathrm{e}(H)$ is the mod 2 Euler class of the Hopf line bundle $H := H(\mathbb{R}^\infty)$.

Second, we point out that for an n-dimensional vector bundle E over a compact and connected ENR B we can define its Stiefel–Whitney classes in the following way. Consider, the projections

$$p_1\colon B \times \mathbb{P}(\mathbb{R}^\infty) \to B \quad \text{and} \quad p_2\colon B \times \mathbb{P}(\mathbb{R}^\infty) \to \mathbb{P}(\mathbb{R}^\infty),$$

and the mod 2 Euler class of the vector bundle $p_1^*E \otimes p_2^*H$ which lives in the cohomology

$$H^*\big(B \times \mathbb{P}(\mathbb{R}^\infty);\mathbb{F}_2\big) \cong H^*(B;\mathbb{F}_2) \otimes H^*\big(\mathbb{P}(\mathbb{R}^\infty);\mathbb{F}_2\big) \cong H^*(B;\mathbb{F}_2) \otimes \mathbb{F}_2[x].$$

Hence, there exist classes $w_i \in H^i(B;\mathbb{F}_2)$, $0 \leq i \leq n$, such that

$$\mathrm{e}(p_1^*E \otimes p_2^*H) = \sum_{i=0}^{n} w_i \times x^{n-i}. \tag{3}$$

Here "$\times$" denotes the cohomology cross product; see for example [12, Thm. VI.3.2].

Then we define the i-th Stiefel–Whitney class of E to be w_i for $0 \leq i \leq n$ and 0 otherwise, that is $w_i(E) = w_i$ for $0 \leq i \leq n$ and $w_i(E) = 0$ for $i \geq n+1$; consult for example [24, Thm. 17.2.5 and Def. 17.2.6]. Thus, the relation (3) becomes

$$\mathrm{e}(p_1^*E \otimes p_2^*H) = \sum_{i=0}^{n} w_i(E) \times x^{n-i}. \tag{4}$$

Let us now consider a real line bundle L over a compact ENR B', and let $p'_1\colon B\times B'\to B$ and $p'_2\colon B\times B'\to B'$ be the projections. The line bundle L is isomorphic to a pull-back bundle f^*H of the Hopf line bundle H for some continuous map $f\colon B'\to \mathbb{P}(\mathbb{R}^\infty)$. In particular, the mod 2 Euler class of L is $\mathrm{e}(L)=f^*(t)$. Consequently, first

$$p_1'^*E\otimes p_2'^*L\cong (\mathrm{id}\times f)^*(p_1^*E\otimes p_2^*H). \tag{5}$$

Second, the naturality of the Euler class and the description of the map $\mathrm{id}\times f$ on the level of cohomology imply that

$$\begin{aligned}\mathrm{e}(p_1'^*E\otimes p_2'^*L) &\overset{(5)}{=} (\mathrm{id}\times f)^*\big(\mathrm{e}(p_1^*E\otimes p_2^*H)\big)\\ &\overset{(4)}{=} (\mathrm{id}\times f)^*\Big(\sum_{i=0}^{n} w_i(E)\times x^{n-i}\Big)\\ &= \sum_{i=0}^{n} w_i(E)\times \mathrm{e}(L)^{n-i}.\end{aligned} \tag{6}$$

Now, if $B'=B$, and $d\colon B\to B\times B$ denotes the diagonal embedding, we have that $E\otimes L\cong d^*(p_1'^*E\otimes p_2'^*L)$. Consequently, from (6) and the definition of the cup product [12, Def. VI.4.1], we get

$$\begin{aligned}\mathrm{e}(E\otimes L) &= \mathrm{e}\left(d^*(p_1'^*E\otimes p_2'^*L)\right)\\ &= d^*\Big(\sum_{i=0}^{n} w_i(E)\times \mathrm{e}(L)^{n-i}\Big)\\ &= \sum_{i=0}^{n} w_i(E)\,\mathrm{e}(L)^{n-i}.\end{aligned} \tag{7}$$

Proof of Lemma 4.1. The powers of the mod 2 Euler class of the Hopf line bundle

$$\mathrm{e}\left(H(E)\right)^i\in H^i\left(\mathbb{P}(E);\mathbb{F}_2\right)$$

for $0\le i\le n-1$ when restricted to each fibre $\mathbb{P}(E_b)$, $b\in B$, of the bundle $\mathbb{P}(E)$ form a basis of $H^*(\mathbb{P}(E_b);\mathbb{F}_2)$. By the Leray–Hirsch theorem $H^*(\mathbb{P}(E);\mathbb{F}_2)$ is a free $H^*(B;\mathbb{F}_2)$-module with a basis $1,\mathrm{e}(H(E)),\dots,\mathrm{e}(H(E))^{n-1}$; see [24, Thm. 17.1.1].

We recall that the Hopf line bundle $H(E)$ is defined as

$$H(E)=\big\{(b,L,v): b\in B,\ L\in\mathbb{P}(E_b),\ v\in L\big\}.$$

In particular, it is a subbundle of the pull-back of E along the projection map

$$g\colon \mathbb{P}(E)=\big\{(b,L): b\in B,\ L\in\mathbb{P}(E_b)\big\}\to B,\quad (b,L)\mapsto b.$$

Now, note that the line bundle $H(E)$ can be identified with its dual line bundle $H(E)^*$ by the inner product. So

$$g^*E\otimes H(E)\cong g^*E\otimes H(E)^*\cong \mathrm{Hom}\left(H(E),g^*E\right)$$

is the vector bundle whose sections are linear maps $H(E) \to g^*E$. The inclusion of $H(E)$ into g^*E gives a nowhere-zero cross-section of the vector bundle $g^*E \otimes H(E)$. Consequently, we have that

$$0 = \mathrm{e}\left(g^*E \otimes H(E)\right) \overset{(7)}{=} \sum_{i=0}^{n} w_i(g^*E)\,\mathrm{e}\left(H(E)\right)^{n-i}$$
$$= \sum_{i=0}^{n} g^*\left(w_i(E)\right)\mathrm{e}\left(H(E)\right)^{n-i} = \sum_{i=0}^{n} w_i(E)\cdot \mathrm{e}\left(H(E)\right)^{n-i}.$$

Here "$\cdot$" refers to the $H^*(B;\mathbb{F}_2)$-module structure. Therefore,

$$\mathrm{e}\left(H(E)\right)^{n} = \sum_{i=1}^{n} w_i(E)\cdot \mathrm{e}\left(H(E)\right)^{n-i},$$

which completely determines the structure of $H^*(P(E);\mathbb{F}_2)$ as an $H^*(B;\mathbb{F}_2)$-algebra. ∎

Now we proceed with the proofs of Theorems 2.1 and 2.2.

4.1. Proof of Theorem 2.1

Let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B, and let the integers $k \geq 1$ and $j \geq 1$ be fixed. Assume that $e_k(B)^j$ does not belong to the ideal $\mathcal{I}_k(E)$.

The proof of the theorem relies on the criterion from Theorem 3.2, that is:

$$\mathrm{e}\left(\left(A_k(E)/\underline{\mathbb{R}}\right)^{\oplus j}\right) \neq 0 \Longrightarrow (j,k) \in \Delta_S(E).$$

Observe that the mod 2 Euler class of the vector bundle $(A_k(E)/\underline{\mathbb{R}})^{\oplus j}$, or in other words the top Stiefel–Whitney class, lives in the cohomology of the pullback bundle, that is $H^*(d^*(\mathbb{P}(E)^{\times k});\mathbb{F}_2)$. We will prove that

- $H^*(d^*(\mathbb{P}(E)^{\times k});\mathbb{F}_2) \cong R_k(B)/\mathcal{I}_k(E)$, and
- $w_{(2^k-1)j}((A_k(E)/\underline{\mathbb{R}})^{\oplus j}) = e_k(B)^j + \mathcal{I}_k(E) \in R_k(B)/\mathcal{I}_k(E)$.

Assuming these two claims to be true, the criterion from Theorem 3.2 yields:

$$e_k^j + \mathcal{I}_k(E) \neq \mathcal{I}_k(E) \text{ in } R_k(B)/\mathcal{I}_k(E) \implies (j,k) \in \Delta_S(E).$$

Thus, the proof of Theorem 2.1 is finished, up to a proof of the two facts we listed.

First, we compute the cohomology of the pullback bundle $d^*(\mathbb{P}(E)^{\times k})$ because the Stiefel–Whitney class $w((A_k(E)/\underline{\mathbb{R}})^{\oplus j})$ belongs to $H^*(d^*(\mathbb{P}(E)^{\times k});\mathbb{F}_2)$.

Claim 4.2. *There is an isomorphism of $H^*(B;\mathbb{F}_2)$-algebras*

$$R_k(B)/\mathcal{I}_k(E) = H^*(B;\mathbb{F}_2)[x_1,\dots,x_k]/\left(\sum_{s=0}^{n} w_{n-s}(E)\,x_r^s : 1 \leq r \leq k\right)$$
$$\to H^*\left(d^*\left(\mathbb{P}(E)^{\times k}\right);\mathbb{F}_2\right)$$

mapping x_r to the mod 2 *Euler class of the pullback vector bundle $\Theta_r^*(H(E))$ for all $1 \leq r \leq k$.*

Proof. The proof proceeds by induction on j where $1 \le j \le k$. If $j = 1$, then the statement reduces to Lemma 4.1. Let $j \ge 2$, and assume that there is an isomorphism

$$H^*(B;\mathbb{F}_2)[x_1,\dots,x_{j-1}]/\Big(\sum_{s=0}^{n} w_{n-s}(E)\,x_r^s : 1 \le r \le j-1\Big) \\ \to H^*\big(d^*\big(\mathbb{P}(E)^{\times(j-1)}\big);\mathbb{F}_2\big) \quad (8)$$

which maps each class x_r to the mod 2 Euler class of the pullback vector bundle $\Theta_r^*(H(E))$, where $1 \le r \le j-1$.

The pullback bundle $d^*(\mathbb{P}(E)^{\times(j-1)})$ is a bundle over B with the corresponding projection map $p\colon d^*(\mathbb{P}(E)^{\times(j-1)}) \to B$. Then $d^*(\mathbb{P}(E)^{\times j})$ is isomorphic to the pullback bundle $p^*(P(E)) \cong P(p^*(E))$ over $d^*(\mathbb{P}(E)^{\times(j-1)})$. Recall that $\mathbb{P}(E)$ is the projective bundle associated to E, and therefore a bundle over B. Hence, there is a pullback diagram

$$\begin{array}{ccc} d^*\big(\mathbb{P}(E)^{\times j}\big) \cong p^*\big(\mathbb{P}(E)\big) \cong \mathbb{P}\big(p^*(E)\big) & \longrightarrow & \mathbb{P}(E) \\ \downarrow & & \downarrow \\ d^*\big(\mathbb{P}(E)^{\times(j-1)}\big) & \xrightarrow{\;p\;} & B. \end{array}$$

Consequently, by Lemma 4.1, we get an isomorphism of $H^*(d^*(\mathbb{P}(E)^{\times(j-1)});\mathbb{F}_2)$-algebras

$$H^*\big(d^*\big(\mathbb{P}(E)^{\times(j-1)}\big);\mathbb{F}_2\big)[x_j]/\Big(\sum_{s=0}^{n} w_{n-s}(E)\,x_j^s\Big) \\ \to H^*\big(\mathbb{P}\big(p^*(E)\big);\mathbb{F}_2\big) \cong H^*\big(d^*\big(\mathbb{P}(E)^{\times j}\big);\mathbb{F}_2\big) \quad (9)$$

which maps x_j to the mod 2 Euler class of the Hopf line bundle $H(p^*(E))$.

Now, the induction hypothesis (8) in combination with the isomorphism (9) completes the proof of the claim. ■

Finally, we evaluate the Stiefel–Whitney class $w_{(2^k-1)j}((A_k(E)/\underline{\mathbb{R}})^{\oplus j})$.

Claim 4.3. *The* mod 2 *Euler class of the vector bundle* $(A_k(E)/\underline{\mathbb{R}})^{\oplus j}$ *is equal to:*

$$w_{(2^k-1)j}\big(\big(A_k(E)/\underline{\mathbb{R}}\big)^{\oplus j}\big) \\ = e_k(B)^j + \mathcal{I}_k(E) = \prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k-\{0\}} (\alpha_1x_1+\cdots+\alpha_kx_k)^j + \mathcal{I}_k(E) \in R_k(B)/\mathcal{I}_k(E).$$

Proof. Recall the isomorphism of vector bundles

$$A_k(E) \cong \Theta_1^*\big(H(E)\oplus\underline{\mathbb{R}}\big)\otimes\cdots\otimes\Theta_k^*\big(H(E)\oplus\underline{\mathbb{R}}\big),$$

where $\underline{\mathbb{R}}$ is the trivial line bundle over $\mathbb{P}(E)$, and $\Theta_i^*(H(E)\oplus\underline{\mathbb{R}})$ is a pullback vector bundle. Now the claim follows from the distributivity of the tensor product over the direct sum, the fact that the pullback of a trivial bundle is again a trivial bundle, and the equality $w(\alpha \otimes \beta) = 1 + (w_1(\alpha) + w_1(\beta))$ which holds (only) for line bundles α and β (see [29, Prob. 7 (A)]). Note that $\alpha \otimes \beta$ is also a line bundle. ∎

With the claims verified, the proof of Theorem 2.1 is now complete.

4.2. Proof of Theorem 2.2

The proof we present is an extension of the proof of Theorem 2.1, and therefore it is just outlined. Let $k \geq 1$ and $j \geq 1$ be fixed integers. We consider a Euclidean vector bundle E of dimension n over a compact and connected ENR B, and, in addition, k vector subbundles $E(1), \dots, E(k)$ of η of dimensions $n_1, \dots, n_k$, respectively. Assume that $j \leq \iota_k(E(1), \dots, E(k)) = \max\{j : e_k(B)^j \notin \mathcal{I}_k(E(1), \dots, E(k))\}$.

The proof of the theorem uses the criterion from Theorem 3.3. That is, if the Euler class $\mathrm{e}((A_k(E(1), \dots, E(k))/\underline{\mathbb{R}})^{\oplus j}) \neq 0$ does not vanish, then for every collection of j continuous functions $\varphi_1, \dots, \varphi_j \colon S(E) \to \mathbb{R}$ there exists a point $b \in B$ and there exists an arrangement of k linear hyperplanes $H_1, \dots, H_k$ in the fibre E_b determined by the collection of vector subbundles $E(1), \dots, E(k)$ with the property that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ of the arrangement complement $E_b - (H_1 \cup \dots \cup H_k)$ the following equalities hold

$$\int_{\mathcal{O}' \cap S(E_b)} \varphi_1 = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_1 \,, \dots, \int_{\mathcal{O}' \cap S(E_b)} \varphi_j = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_j \,.$$

The mod 2 Euler class of $(A_k(E(1), \dots, E(k))/\underline{\mathbb{R}})^{\oplus j}$, or in other words the top Stiefel–Whitney class, lives in $H^*(d^*(\mathbb{P}(E(1)) \times \dots \times \mathbb{P}(E(k))); \mathbb{F}_2)$. Therefore, we prove that

- $H^*(d^*(\mathbb{P}(E(1)) \times \dots \times \mathbb{P}(E(k))); \mathbb{F}_2) \cong R_k(B)/\mathcal{I}_k(E(1), \dots, E(k))$, and that
- $w_{(2^k-1)j}((A_k(E(1), \dots, E(k))/\underline{\mathbb{R}})^{\oplus j}) = e_k(B)^j + \mathcal{I}_k(E(1), \dots, E(k))$.

If these two statements are assumed to be true, then the criterion from Theorem 3.3, in combination with the theorem assumption $j \leq \iota_k(E; E(1), \dots, E(k))$, implies that

$$\begin{aligned} w_{(2^k-1)j}\big(\big(A_k\big(E(1), \dots, E(k)\big)/\underline{\mathbb{R}}\big)^{\oplus j}\big) &= e_k^j + \mathcal{I}_k\big(E(1), \dots, E(k)\big) \\ &\neq \mathcal{I}_k\big(E(1), \dots, E(k)\big). \end{aligned}$$

Hence, $\mathrm{e}((A_k(E(1), \dots, E(k))/\underline{\mathbb{R}})^{\oplus j}) \neq 0$, and the proof of Theorem 2.1 is complete. Indeed, the remaining claims are verified in the same way as in the proofs of Claims 4.2 and 4.3.

5. Proofs of Propositions 2.4 and 2.7

We prove the main facts about the integers $\iota_1(E)$ and $\iota_k(E(1), \dots, E(k))$ stated in Propositions 2.4 and 2.7, as well as two related consequences, Corollaries 2.8 and 2.9.

5.1. Proof of Proposition 2.4

Let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B.

Since $k = 1$ we simplify notation by taking $x = x_1$. Hence, $e_1(B) = x$ and $\mathcal{I}_1(E) = (x^n + w_1(E)x^{n-1} + \cdots + w_n(E))$. Set

$$a := \iota_1(E) = \max\big\{j : x^j \notin \mathcal{I}_1(E)\big\}$$

and

$$b := \max\big\{j : 0 \neq w_{j-n+1}(-E) \in H^{j-n+1}(B;\mathbb{F}_2)\big\}.$$

In particular, we have that $w_{b-n+1}(-E) \neq 0$ and that $w_r(-E) = 0$ for all $r \geq b-n+2$. Now, we prove that $a = b$.

Using the Euclidean algorithm in the polynomial ring $R_1(B) = H^*(B;\mathbb{F}_2)[x]$ we have that

$$x^b = \big(x^n + w_1(E)x^{n-1} + \cdots + w_n(E)\big)q + d_{n-1}x^{n-1} + \cdots + d_1x + d_0,$$

where $q \in R_1(B)$, and for $0 \leq i \leq n-1$ the coefficients are given by

$$d_i = w_{b-i}(-E) + w_1(E)w_{b-i-1}(-E) + \cdots + w_{n-i-1}(E)w_{b-n+1}(-E) \in H^*(B;\mathbb{F}_2),$$

as demonstrated by Crabb and Jan Jaworowski [16, Proof of Prop. 4.1]. Since $w_r(-E) \neq 0$ for $r \geq b-n+2$ it follows that

$$d_i = w_{n-i-1}(E)w_{b-n+1}(-E) \quad \text{for } 0 \leq i \leq n-1,$$

and so

$$\begin{aligned} x^b = {} & \big(x^n + w_1(E)x^{n-1} + \cdots + w_n(E)\big)q \\ & + w_{b-n+1}(-E)\big(x^{n-1} + w_1(E)x^{n-2} + \cdots + w_{n-1}(E)\big). \end{aligned}$$

Consequently, from $w_{b-n+1}(-E) \neq 0$ it follows that $x^b \notin \mathcal{I}_1(E)$ and accordingly $b \leq a$.

Let us now assume that $b < a$, or in other words $b-n+2 \leq a-n+1$. Recall that $w_r(-E) \neq 0$ for $r \geq b-n+2$, and in particular for $r \geq a-n+1$. Once again we have

$$x^a = \big(x^n + w_1(E)x^{n-1} + \cdots + w_n(E)\big)q' + d'_{n-1}x^{n-1} + \cdots + d'_1x + d'_0,$$

where

$$d'_i = w_{a-i}(-E) + w_1(E)w_{a-i-1}(-E) + \cdots + w_{n-i-1}(E)w_{a-n+1}(-E) = 0,$$

for all $0 \leq i \leq n-1$. Hence, $x^a \in \mathcal{I}_1(E)$, which contradicts the definition of the integer a. Therefore, $b \geq a$.

We have proved that $a = b$, or in other words that

$$\iota_1(E) = \max\big\{j : 0 \neq w_{j-n+1}(-E) \in H^{j-n+1}(B;\mathbb{F}_2)\big\},$$

as claimed. ■

5.2. Proof of Proposition 2.7

Let $k \geq 1$ be an integer, and let $E(1), \dots, E(k)$ be Euclidean vector bundles over a compact and connected ENR B. Set n_i to be the dimension of the vector bundle $E(i)$ for $1 \leq i \leq k$. Let us denote by

$$\begin{aligned} a_i &:= \iota_1\big(E(i)\big) = \max\big\{j : 0 \neq w_{j-n_i+1}\big(-E(i)\big) \in H^{j-n_i+1}(B;\mathbb{F}_2)\big\},\\ a &:= \iota_k(a_1+1, \dots, a_k+1) = \max\big\{j : e_k(\mathrm{pt})^j \notin (x_1^{a_1+1}, \dots, x_k^{a_k+1})\big\},\\ b &:= \iota_k\big(E(1), \dots, E(k)\big) = \max\big\{j : e_k(B)^j \notin \mathcal{I}_k\big(E(1), \dots, E(k)\big)\big\}, \end{aligned}$$

where $1 \leq i \leq k$, and

$$\mathcal{I}_k\big(E(1), \dots, E(k)\big) = \Bigg(\sum_{s=0}^{n_r} w_{n_r-s}\big(E(r)\big)\, x_r^s : 1 \leq r \leq k\Bigg) \subseteq R_k(B).$$

In the definition of a_i we used the characterisation from Proposition 2.4. In particular, we have that $w_r(-E(i)) = 0$ for all $r \geq a_i - n_i + 2$.

With the notation we just introduced the assumption of the proposition reads

$$w_{a_1-n_1+1}\big(-E(1)\big) \cdots w_{a_k-n_k+1}\big(-E(k)\big) \neq 0,$$

while the claim of the proposition becomes $a = b$.

The main ingredients of our proof of Proposition 2.7 are contained in the next two claims, where the first claim is used for the proof of the second.

Claim 5.1. $x_1^{a_1} \cdots x_k^{a_k} \notin \mathcal{I}_k(E(1), \dots, E(k))$.

Proof. For simplicity set $\mathcal{I} := \mathcal{I}_k(E(1), \dots, E(k))$. Once again we use [16, Proof of Prop. 4.1] and get that for all $1 \leq i \leq k$:

$$x_i^{a_i} = \big(x_i^{n_i} + w_1\big(E(i)\big)x_i^{n_i-1} + \cdots + w_{n_i}\big(E(i)\big)\big) \cdot q_i + d_{n_i-1,i}\,x_i^{n_i-1} + \cdots + d_{0,i},$$

where for $0 \leq s \leq n_i - 1$:

$$\begin{aligned} d_{s,i} = {}& w_{a_i-s}\big(-E(i)\big) + w_1\big(E(i)\big)\, w_{a_i-s-1}\big(-E(i)\big)\\ &+ \cdots + w_{n_i-s-1}\big(E(i)\big)\, w_{a_i-n_i+1}\big(-E(i)\big). \end{aligned}$$

More precisely, since $w_r(-E(i)) = 0$ for all $r \geq a_i - n_i + 2$, we have that

$$d_{s,i} = w_{n_i-s-1}\big(E(i)\big)\, w_{a_i-n_i+1}\big(-E(i)\big).$$

Consequently,

$$x_i^{a_i} + \mathcal{I} = w_{a_i-n_i+1}\big(-E(i)\big)\big(x_i^{n_i-1} + \cdots + w_{n_i-1}\big(E(i)\big)\big) + \mathcal{I},$$

and so

$$x_1^{a_1} \cdots x_k^{a_k} + \mathcal{I} = \prod_{i=1}^{k} w_{a_i-n_i+1}\big(-E(i)\big) \cdot \prod_{i=1}^{k} \big(x_i^{n_i-1} + \cdots + w_{n_i-1}\big(E(i)\big)\big) + \mathcal{I}.$$

From the assumption of the proposition we have that $\prod_{i=1}^{k} w_{a_i-n_i+1}(-E(i)) \neq 0$, and so $x_1^{a_1}\cdots x_k^{a_k} + \mathcal{I} \neq \mathcal{I}$ as claimed. ∎

Claim 5.2.

$$\begin{aligned}(x_1^{a_1+1},\dots,x_k^{a_k+1})\\ &= \ker\big(\mathbb{F}_2[x_1,\dots,x_k] \to H^*(B;\mathbb{F}_2)[x_1,\dots,x_k]/\mathcal{I}_k\big(E(1),\dots,E(k)\big)\big).\end{aligned}$$

Proof. The ring homomorphism we consider

$$h\colon \mathbb{F}_2[x_1,\dots,x_k] \to H^*(B;\mathbb{F}_2)[x_1,\dots,x_k]/\mathcal{I}_k\big(E(1),\dots,E(k)\big)$$

is induced by the coefficient inclusion $\mathbb{F}_2 \hookrightarrow H^0(B;\mathbb{F}_2) \hookrightarrow H^*(B;\mathbb{F}_2)$. Furthermore, denote by $\mathcal{J} := \ker(h)$.

As in the proof of the previous claim we use [16, Proof of Prop. 4.1] and for every $1 \le i \le k$ get that

$$\begin{aligned}x_i^{a_i+1} &= \big(x_i^{n_i} + w_1\big(E(i)\big)x_i^{n_i-1} + \cdots + w_{n_i}\big(E(i)\big)\big)\cdot q_i\\ &\quad + d'_{n_i-1,i}x_i^{n_i-1} + \cdots + d'_{0,i} \in H^*(B;\mathbb{F}_2)[x_1,\dots,x_k].\end{aligned}$$

Here for $0 \le s \le n_i - 1$:

$$\begin{aligned}d'_{s,i} &= w_{a_i+1-s}\big(-E(i)\big) + w_1\big(E(i)\big)\,w_{a_i-s}\big(-E(i)\big)\\ &\quad + \cdots + w_{n_i-s-1}\big(E(i)\big)\,w_{a_i-n_i+2}\big(-E(i)\big).\end{aligned}$$

In this case the fact that $w_r(-E(i)) = 0$ for all $r \ge a_i - n_i + 2$ implies that $d'_{s,i} = 0$ for all $0 \le s \le n-1$ and all $1 \le i \le k$. Consequently, $x_i^{a_i+1} \in \mathcal{I}_k(E(1),\dots,E(k))$, or in other words, $x_i^{a_i+1} \in \mathcal{J}$, for all $1 \le i \le k$. Hence, $(x_1^{a_1+1},\dots,x_k^{a_k+1}) \subseteq \mathcal{J}$.

Assume that

$$0 \neq p = \sum_{(c_1,\dots,c_k)\in C} \alpha_{c_1,\dots,c_k} x_1^{c_1}\cdots x_k^{c_k} \in \mathcal{J} - (x_1^{a_1+1},\dots,x_k^{a_k+1}),$$

where $C \subseteq \mathbb{Z}_{\ge 0}^k$ is a finite set of multi-exponents of the polynomial p, and $\alpha_{c_1,\dots,c_k} \in \mathbb{F}_2$ are the coefficients. After a possible modification of p, by taking away monomials which already belong to the ideal $(x_1^{a_1+1},\dots,x_k^{a_k+1})$, we can assume that the set of exponents satisfies

$$\emptyset \neq C \subseteq [0,a_1]\times\cdots\times[0,a_k].$$

That is, no monomial in the representation of p belongs to $(x_1^{a_1+1},\dots,x_k^{a_k+1})$.

Let $s_k := \min\{s \in \mathbb{Z}_{\ge 0} : \alpha_{c_1,\dots,c_{k-1},s} \neq 0\}$. Then

$$x_k^{a_k-s_k}\,p \in \mathcal{J} - (x_1^{a_1+1},\dots,x_k^{a_k+1})$$

with all monomials having degree of x_k at least a_k. Taking away all monomials in $x_k^{a_k-s_k}p$ which already belong to $(x_1^{a_1+1},\dots,x_k^{a_k+1})$ we get a polynomial which still belongs to

$\mathcal{J} - (x_1^{a_1+1}, \ldots, x_k^{a_k+1})$. Now, repeat the procedure iteratively with variables $x_{k-1}, \ldots x_1$, respectively. At the end we get that

$$x_1^{a_1} \cdots x_k^{a_k} \in \mathcal{J} - (x_1^{a_1+1}, \ldots, x_k^{a_k+1}).$$

We have reached a contradiction with Claim 5.1. In particular, this says, that

$$h(x_1^{a_1} \cdots x_k^{a_k}) \neq 0,$$

or equivalently $x_1^{a_1} \cdots x_k^{a_k} \notin \ker(h) = \mathcal{J}$. ■

Finally, we complete the proof of Proposition 2.7 as follows. According to the definition of a we have that

$$e_k(\mathrm{pt})^a \notin (x_1^{a_1+1}, \ldots, x_k^{a_k+1}) = \ker(h),$$
$$e_k(\mathrm{pt})^{a+1} \in (x_1^{a_1+1}, \ldots, x_k^{a_k+1}) = \ker(h).$$

Consequently,

$$e_k(B)^a + \mathcal{I}_k\big(E(1), \ldots, E(k)\big) = h\big(e_k(\mathrm{pt})^a\big) \neq 0,$$
$$e_k(B)^{a+1} + \mathcal{I}_k\big(E(1), \ldots, E(k)\big) = h\big(e_k(\mathrm{pt})^{a+1}\big) = 0.$$

From the definition of b we conclude that $a = b$, as claimed. This argument completes the proof of Proposition 2.7. ■

5.3. Proof of Corollary 2.8

In order to prove the statement, according to Proposition 2.7, it is enough to check whether $(w_{d-\ell}(-E_\ell^d))^k \neq 0$, because $\iota_1(E_\ell^d) = d - 1$, as demonstrated in Corollary 2.5. Since $k \leq \ell$ it suffices to prove that $(w_{d-\ell}(-E_\ell^d))^\ell \neq 0$. Indeed, the Gambelli's formula ([23, p. 523], [22, Prop. 9.5.37]) implies the equality

$$\big(w_{d-\ell}(-E_\ell^d)\big)^\ell = \det\big(w_{d-\ell+i-j}(-E_\ell^d)\big)_{1\leq i,j\leq \ell} = [d-\ell, d-\ell, \ldots, d-\ell] \neq 0.$$

Here $[d-\ell, d-\ell, \ldots, d-\ell]$ denotes a Schubert class. Note that $w_r(-E_\ell) = 0$ for all $r > d - \ell$, and that we assume $w_r(-E_\ell) = 0$ for $r < 0$. ■

5.4. Proof of Corollary 2.9

From Theorem 2.1, we have that $(j,k) \in \Delta_S(E_\ell^d)$ if $e_k(B)^j \notin \mathcal{I}_k(E_\ell^d) = \mathcal{I}_k(E_\ell^d, \ldots, E_\ell^d)$, or in other words if

$$j \leq \iota_k(E_\ell^d, \ldots, E_\ell^d) = \iota_k(d, \ldots, d) = \max\big\{j' : e_k(\mathrm{pt})^{j'} \notin (x_1^d, \ldots, x_k^d)\big\}.$$

Here the first equality comes from Corollary 2.8 while the second one is just the definition of $\iota_k(d, \ldots, d)$.

Since $j = 2^t + r$ where $0 \leq r \leq 2^t - 1$ and $d \geq 2^{t+k-1} + r$, then according to [6, Lem. 4.2] we have that $e_k(\mathrm{pt})^j \notin (x_1^d, \ldots, x_k^d)$. Thus, indeed $j \leq \iota_k(E_\ell^d, \ldots, E_\ell^d)$ and the proof of the corollary is complete. ■

6. Proofs of Corollaries 2.10, 2.11 and Theorem 2.12

Before going into the proofs we recall the notion of a real flag manifold by introducing it in two equivalent ways. Furthermore we give a description of the cohomology ring with coefficients in $\mathbb{F}_2$.

Let $k \geq 1$ and $d \geq 2$ be integers. Consider a strictly increasing sequence of positive integers $(n_1, \dots, n_k)$ bounded by d, meaning $1 \leq n_1 < \cdots < n_{k-1} < n_k \leq d-1$. Set in addition $n_0 = 0$ and $n_{k+1} = d$.

Let V be a Euclidean vector space of dimension d. The real *flag manifold*, of type $(n_1, \dots, n_k)$, in V is the space $\mathrm{Flag}_{n_1,\dots,n_k}(V)$ of all flags $0 \subseteq V_1 \subseteq \cdots \subseteq V_k \subseteq V$ in V with the property that $\dim(V_i) = n_i$ for every $1 \leq i \leq k$. Alternatively, we can say that $\mathrm{Flag}_{n_1,\dots,n_k}(V)$ is a collection of all $(k+1)$-tuples of vector spaces $(W_1, \dots, W_{k+1})$ with the property that

- $\dim(W_i) = n_i - n_{i-1}$ for all $1 \leq i \leq k+1$, and
- $W_{i'} \perp W_{i''}$ for all $1 \leq i' < i'' \leq k+1$.

In other words

$$\begin{aligned}
&\mathrm{Flag}_{n_1,\dots,n_k}(V)\\
&\quad = \Big\{(V_1, \dots, V_k) \in \prod_{i=1}^{k} G_{n_i}(V) : 0 \subseteq V_1 \subseteq \cdots \subseteq V_k \subseteq V\Big\}\\
&\quad \cong \Big\{(W_1, \dots, W_{k+1}) \in \prod_{i=1}^{k+1} G_{n_i - n_{i-1}}(V) : W_{i'} \perp W_{i''} \text{ for all } 1 \leq i' < i'' \leq k+1\Big\}\\
&\quad \cong \frac{\mathrm{O}(d)}{\mathrm{O}(n_1 - n_0) \times \mathrm{O}(n_2 - n_1) \times \cdots \times \mathrm{O}(n_{k+1} - n_k)}.
\end{aligned}$$

The homeomorphism between these two presentations is given by

$$(W_1, \dots, W_{k+1}) \longmapsto \big(W_1, (W_1 \oplus W_2), \dots, (W_1 \oplus W_2 \oplus \cdots \oplus W_{k-1})\big).$$

The flag manifold $\mathrm{Flag}_{n_1,\dots,n_k}(V)$ is indeed a compact δ-dimensional manifold where $\delta := \sum_{1 \leq i' < i'' \leq k+1} (n_{i'} - n_{i'-1})(n_{i''} - n_{i''-1})$. In the case when $k = d-1$, and consequently $n_i = i$ for all $1 \leq i \leq k = d-1$, the flag manifold $\mathrm{Flag}_{1,2,\dots,d-1}(V)$ is called the *complete flag manifold*. Furthermore, the flag manifold $\mathrm{Flag}_{n_1}(V)$ coincides with the Grassmann manifold $\mathrm{G}_{n_1}(V) \cong \mathrm{G}_{n_1}(\mathbb{R}^d)$.

Over the flag manifold $\mathrm{Flag}_{n_1,\dots,n_k}(V)$ there are $k+1$ vector bundles $E_1, \dots, E_{k+1}$ given by

$$E_i := \big\{\big((W_1, \dots, W_{k+1}), w\big) \in \mathrm{Flag}_{n_1,\dots,n_k}(V) \times V : w \in W_i\big\},$$

where $1 \leq i \leq k+1$. In particular, $E_1 \oplus \cdots \oplus E_{k+1}$ is isomorphic to the trivial vector bundle $\mathrm{Flag}_{n_1,\dots,n_k}(V) \times V$. Now, the classical result of Armand Borel [11, Thm. 11.1] says that

$$H^*\big(\mathrm{Flag}_{n_1,\dots,n_k}(V);\mathbb{F}_2\big) \\ \cong \mathbb{F}_2\big[w_1(E_1),\dots,w_{n_1-n_0}(E_1),\dots,w_1(E_{k+1}),\dots,w_{n_{k+1}-n_k}(E_{k+1})\big]/I_{n_1,\dots,n_k},$$

where the ideal $I_{n_1,\dots,n_k}$ is generated by the identity

$$\big(1+w_1(E_1)+\dots+w_{n_1-n_0}(E_1)\big)\cdots\big(1+w_1(E_{k+1})+\dots+w_{n_{k+1}-n_k}(E_{k+1})\big)=1.$$

In particular, in the case of the complete flag manifold, equivalently when $k=d-1$, we have that

$$H^*\big(\mathrm{Flag}_{1,\dots,d-1}(V);\mathbb{F}_2\big)\cong\mathbb{F}_2\big[w_1(E_1),w_1(E_2),\dots,w_1(E_d)\big]/I_{1,\dots,d-1}. \tag{10}$$

In this case $E_1,\dots,E_d$ are all line bundles. Here, the ideal $I_{1,\dots,d-1}$ is generated by the identity

$$\prod_{i=1}^{d}\big(1+w_1(E_i)\big)=1,$$

which implies that a generating set for $I_{1,\dots,d-1}$ is the set of all elementary symmetric polynomials in $w_1(E_1),w_1(E_2),\dots,w_1(E_d)$ as variables. Thus,

$$I_{1,\dots,d-1}=\big(\sigma_r\big(w_1(E_1),w_1(E_2),\dots,w_1(E_d)\big):1\le r\le d\big), \tag{11}$$

where $\sigma_1,\dots,\sigma_d$ denote elementary symmetric polynomials.

Flag manifolds of different types allow continuous maps between each other induced by a choice of a subflag. In particular, for any type $(n_1,\dots,n_k)$ there is a continuous map

$$\alpha_{n_1,\dots,n_k}\colon \mathrm{Flag}_{1,\dots,d-1}(V)\to\mathrm{Flag}_{n_1,\dots,n_k}(V),$$

given by the selection of a subflag

$$0\subseteq V_1\subseteq V_2\subseteq\dots\subseteq V_{d-1}\subseteq V\mapsto 0\subseteq V_{n_1}\subseteq V_{n_2}\subseteq\dots\subseteq V_{n_k}\subseteq V.$$

An important feature of this map is that the induced map in cohomology

$$\alpha^*_{n_1,\dots,n_k}\colon H^*\big(\mathrm{Flag}_{n_1,\dots,n_k}(V);\mathbb{F}_2\big)\mapsto H^*\big(\mathrm{Flag}_{1,\dots,d-1}(V);\mathbb{F}_2\big)$$

is injective; consult for example [23, pp. 523–524].

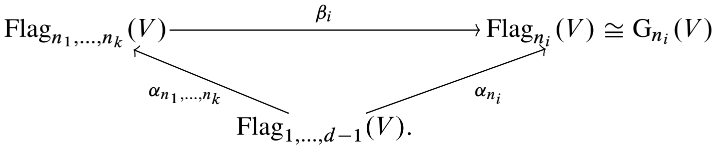
6.1. Proof of Corollary 2.10

We apply Proposition 2.7. Thus, we need to compute first $\iota(E(i))$ for all $1\le i\le k$. From Proposition 2.4 we have that

$$\iota_1\big(E(i)\big)=\max\big\{j: 0\ne w_{j-\dim E(i)+1}\big(-E(i)\big)\in H^*(B;\mathbb{F}_2)\big\},$$

where in our situation $B:=\mathrm{Flag}_{n_1,\dots,n_k}(V)$.

Let $1 \le i \le k$. Consider the following commutative diagram of flag manifolds where all the maps are induced by a selection of the corresponding subflags:



Since the induced maps in cohomology $\alpha^*_{n_1,\ldots,n_k}$ and $\alpha^*_{n_i}$ are injective, it follows that the induced map β_i^* is also injective. Now, from the injectivity of β_i^* and the fact $E(i) = \beta_i^* E^d_{n_i}$, in combination with Corollary 2.5 we have that

$$\iota_1\big(E(i)\big) = \iota_1(E^d_{n_i}) = d - 1.$$

Here, as before, $E^d_{n_i}$ denotes the tautological bundle over the Grassmann manifold $\mathrm{G}_{n_i}(V)$.

To conclude the proof of the corollary we verify the criterion from Proposition 2.7, that is, we prove that the following product does not vanish

$$u := \prod_{i=1}^{k} w_{\iota_1(E(i))-n_i+1}\big(-E(i)\big) = \prod_{i=1}^{k} w_{d-n_i}\big(-E(i)\big) \in H^*(B;\mathbb{F}_2).$$

From the fact that $E(i) \oplus E_{i+1} \oplus \cdots \oplus E_{k+1} = B \times V$ is a trivial vector bundle we get the following equality of total Stiefel–Whitney classes:

$$w\big(-E(i)\big) = w(E_{i+1} \oplus \cdots \oplus E_{k+1}) = w(E_{i+1}) \cdots w(E_{k+1}).$$

Therefore,

$$w_{d-n_i}\big(-E(i)\big) = w_{d-n_i}(E_{i+1} \oplus \cdots \oplus E_{k+1}) = w_{n_{i+1}-n_i}(E_{i+1}) \cdots w_{n_{k+1}-n_k}(E_{k+1}),$$

because $\dim(E_{i+1} \oplus \cdots \oplus E_{k+1}) = d - n_i$ and $\dim(E_r) = n_r - n_{r-1}$ for every $1 \le r \le k+1$. In particular, each Stiefel–Whitney class $w_{n_r-n_{r-1}}(E_r)$ is the mod 2 Euler class $\mathrm{e}(E_r)$ of the vector bundle E_r. We calculate as follows:

$$\begin{aligned} u &= \prod_{i=1}^{k} w_{d-n_i}\big(-E(i)\big) \\ &= \prod_{i=1}^{k} \prod_{r=i+1}^{k+1} w_{n_{i+1}-n_i}(E_{i+1}) \cdots w_{n_{k+1}-n_k}(E_{k+1}) \\ &= w_{n_2-n_1}(E_2) \cdot w_{n_3-n_2}(E_3)^2 \cdots w_{n_{k+1}-n_k}(E_{k+1})^k. \end{aligned}$$

Thus, it remains to show that the class

$$w_{n_2-n_1}(E_2) \cdot w_{n_3-n_2}(E_3)^2 \cdots w_{n_{k+1}-n_k}(E_{k+1})^k$$

does not vanish in $H^*(B;\mathbb{F}_2)$.

For that we apply the homomorphism $\alpha^*_{n_1,\dots,n_k}$ to the class u and land in the cohomology of the complete flag manifold $H^*(\mathrm{Flag}_{1,\dots,d-1}(V);\mathbb{F}_2)$, that is

$$\begin{aligned}\alpha^*_{n_1,\dots,n_k}(u) &= \alpha_{n_1,\dots,n_k}\big(w_{n_2-n_1}(E_2)\cdot w_{n_3-n_2}(E_3)^2\cdots w_{n_{k+1}-n_k}(E_{k+1})^k\big)\\ &= w_1(E_{n_1+1})\cdots w_1(E_{n_2})\\ &\quad\cdot w_1(E_{n_2+1})^2\cdots w_1(E_{n_3})^2\\ &\quad\vdots\\ &\quad\cdot w_1(E_{n_k+1})^k\cdots w_1(E_{n_{k+1}})^k.\end{aligned}$$

The vector bundles on the farthest right-hand side of the last equality are canonical line bundles over the complete flag manifold. Here we used the isomorphisms

$$\alpha^*_{n_1,\dots,n_k}E_2 \cong E_{n_1+1}\oplus\cdots\oplus E_{n_2}\ ,\ \dots,\ \alpha^*_{n_1,\dots,n_k}E_{k+1}\cong E_{n_k+1}\oplus\cdots\oplus E_{n_{k+1}}.$$

Now, we observe that the monomial in the cohomology of the complete flag manifold

$$w_1(E_{n_1+1})\cdots w_1(E_{n_2})\,w_1(E_{n_2+1})^2\cdots w_1(E_{n_3})^2\cdots w_1(E_{n_k+1})^k\cdots w_1(E_{n_{k+1}})^k$$

divides the monomial

$$w_1(E_1)^0 w_1(E_2)^1 w_1(E_3)^2\cdots w_1(E_d)^{d-1}.$$

Thus, in order to prove that $\alpha^*_{n_1,\dots,n_k}(u)\neq 0$ and consequently conclude $u\neq 0$ it suffices to show that

$$\begin{aligned}0\neq w_1(E_1)^0 w_1(E_2)^1 w_1(E_3)^2\cdots w_1(E_d)^{d-1} &\in H^*\big(\mathrm{Flag}_{1,\dots,d-1}(V);\mathbb{F}_2\big)\\ &\cong \mathbb{F}_2\big[w_1(E_1),w_1(E_2),\dots,w_1(E_d)\big]/I_{1,\dots,d-1}.\end{aligned}$$

Recall that the ideal $I_{1,\dots,d-1}=(\sigma_r(w_1(E_1),\dots,w_1(E_d)) : 1\le r\le d)$ is generated by elementary symmetric polynomials. Hence

$$w_1(E_1)^0 w_1(E_2)^1\cdots w_1(E_d)^{d-1}\neq 0 \iff w_1(E_{\pi(1)})^0 w_1(E_{\pi(2)})^1\cdots w_1(E_{\pi(d)})^{d-1}\neq 0$$

for every permutation $\pi\in\mathfrak{S}_d$. For the sake of brevity we prove that

$$w_1(E_d)^0 w_1(E_{d-1})^1\cdots w_1(E_1)^{d-1}\neq 0 \tag{12}$$

in $H^*(\mathrm{Flag}_{1,\dots,d-1}(V);\mathbb{F}_2)$.

The proof of (12) proceeds by induction as follows. First, obviously $w_1(E_1)^{d-1}\neq 0$ in

$$H^*\big(\mathrm{Flag}_1(V);\mathbb{F}_2\big)\cong H^*\big(\mathbb{P}(V);\mathbb{F}_2\big)\cong \mathbb{F}_2[x]/(x^d),$$

where x corresponds to $w_1(E_1)^{d-1}$. Next, let $1\le k\le d-2$ and let

$$w_1(E_k)^{d-k} w_1(E_{k-1})^{d-k+1}\cdots w_1(E_1)^{d-1}\neq 0 \tag{13}$$

in $H^*(\mathrm{Flag}_{1,\dots,k}\,\mathrm{Flag}_{1,\dots,k}(V);\mathbb{F}_2)$. Finally, the map

$$\mathrm{Flag}_{1,\dots,k+1}(V) \to \mathrm{Flag}_{1,\dots,k}(V),$$

given by

$$0 \subseteq V_1 \subseteq \cdots \subseteq V_k \subseteq V_{k+1} \mapsto 0 \subseteq V_1 \subseteq \cdots \subseteq V_k,$$

is the projective bundle of the vector bundle $(E_1 \oplus \cdots \oplus E_k)^\perp$ over the flag manifold $\mathrm{Flag}_{1,\dots,k}(V)$, that is $\mathbb{P}((E_1 \oplus \cdots \oplus E_k)^\perp)$. From Lemma 4.1 we have that

$$\begin{aligned} H^*\big(\mathrm{Flag}_{1,\dots,k+1}(V);\mathbb{F}_2\big) &\cong H^*\big(\mathbb{P}\big((E_1 \oplus \cdots \oplus E_k)^\perp\big);\mathbb{F}_2\big) \\ &\cong H^*\big(\mathrm{Flag}_{1,\dots,k}(V);\mathbb{F}_2\big)[x]/\Big(\sum_{s=0}^{d-k} w_{d-k-s}\, x^s\Big), \end{aligned}$$

where $w_{d-k-s} = w_{d-k-s}(-(E_1 \oplus \cdots \oplus E_k))$ and $x = w_1(E_{k+1})$. Thus, from assumption (13), that is $w_1(E_d)^0 w_1(E_{d-1})^1 \cdots w_1(E_1)^{d-1} \neq 0$ in $H^*(\mathrm{Flag}_{1,\dots,k}(V);\mathbb{F}_2)$ we obtain

$$w_1(E_{k+1})^{d-k-1} w_1(E_k)^{d-k} w_1(E_{k-1})^{d-k+1} \cdots w_1(E_1)^{d-1} \neq 0$$

in $H^*(\mathrm{Flag}_{1,\dots,k+1}(V);\mathbb{F}_2)$. Consequently (12) holds. This concludes the argument and completes the proof of the corollary. ∎

Let us also point out that the non-vanishing of the class u can also be deduced using [15, Rem. 2.8].

6.2. Proof of Corollary 2.11

As in the previous section we assume that $k \geq 1$ and $d \geq 2$ are integers, and that $0 = n_0 < n_1 < \cdots < n_k < n_{k+1} = d$ is a strictly increasing sequence of integers. We take $V = \mathbb{R}^d$ and denote by $E_1, \dots, E_{k+1}$ the canonical vector bundles over the flag manifold $\mathrm{Flag}_{n_1,\dots,n_k}(V)$. Furthermore, $E(i) := \bigoplus_{1 \leq r \leq i} E_r$ for all $1 \leq i \leq k$, and $E := E(k)$. In addition, we assume that $j = 2^t + r$ is an integer, with $0 \leq r \leq 2^t - 1$, and $d = \dim(V) \geq 2^{t+k-1} + r + 1$.

In order to prove the existence of the desired partition we use Theorem 2.2. More precisely, if $j \leq \iota_k(E(1), \dots, E(k))$, then the theorem guarantees the existence of a point $b := (W_1, \dots, W_{k+1})$ in the base space $\mathrm{Flag}_{n_1,\dots,n_k}(V)$ of the vector bundle E and an arrangement $\mathcal{H}^b = (H_1^b, \dots, H_k^b)$ of k linear hyperplanes in the fiber E_b such that for every pair of connected components $(\mathcal{O}', \mathcal{O}'')$ of the arrangement complement $E_b - (H_1^b \cup \cdots \cup H_k^b)$ holds

$$\int_{\mathcal{O}' \cap S(E_b)} \varphi_1 = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_1\,, \dots, \int_{\mathcal{O}' \cap S(E_b)} \varphi_j = \int_{\mathcal{O}'' \cap S(E_b)} \varphi_j\,,$$

and in addition

$$(H_1^b)^\perp \subseteq E(1)_b,\ (H_2^b)^\perp \subseteq E(2)_b\,, \dots, (H_k^b)^\perp \subseteq E(k)_b.$$

Since $E(i)_b = \bigoplus_{1\le r\le i}(E_r)_b = \bigoplus_{1\le r\le i} W_r$ for every $1 \le i \le k$, we have that

$$(H_i^b)^\perp \subseteq E(i)_b \implies H_i^b \supseteq \big(E(i)_b\big)^\perp = \Big(\bigoplus_{1\le r\le i} W_r\Big)^\perp = \bigoplus_{i+1\le r\le k+1} W_r.$$

Hence, for the proof of Corollary 2.11 it suffices to verify that

$$j = 2^t + r \le \iota_k\big(E(1),\dots,E(k)\big)$$

when $d = \dim(V) \ge 2^{t+k-1} + r$.

We have from Corollary 2.10 that $\iota_k(E(1),\dots,E(k)) = \iota_k(d,\dots,d)$, so we need to show that

$$j = 2^t + r \le \iota_k(d,\dots,d) = \max\big\{j' : e_k(\mathrm{pt})^{j'} \notin (x_1^d,\dots,x_k^d)\big\}.$$

Since $d \ge 2^{t+k-1} + r$, using [6, Lem. 4.2], we get that $e_k(\mathrm{pt})^j \notin (x_1^d,\dots,x_k^d)$, and consequently $j \le \iota_k(d,\dots,d)$. This completes the proof of the corollary. ■

6.3. Proof of Theorem 2.12

Fix integers $d \ge 1$ and $k \ge 1$ with $d \ge k$, and let $V = \mathbb{R}^{d+1}$. Let $(j_k,\dots,j_d)$ be a permutation of the set $\{k,\dots,d\}$, and take an arbitrary collections of functions $\varphi_{a,b}\colon S(E_{a+1}^{d+1}) \to \mathbb{R}$, $k \le a \le d$, $1 \le b \le j_a$, from the sphere bundle of the tautological vector bundle E_{a+1}^{d+1} over the Grassmann manifold $G_{a+1}(V)$ to the real numbers.

According to Theorem 3.5, for the existence of the desired partition it suffices to prove the non-vanishing of the Euler class of the vector bundle

$$E = E_{k+1}^{\oplus j_k} \oplus E_{k+2}^{\oplus j_{k+1}} \oplus \cdots \oplus E_{d+1}^{\oplus j_d}.$$

For this we show that the related mod 2 Euler class which lives in the cohomology ring $H^*(\mathrm{Flag}_{k,\dots,d}(V);\mathbb{F}_2)$ is not zero. As already discussed at the beginning of Section 6 we have that

$$w(E) = \big(1 + w_1(E_{k+1})\big)^{j_k} \cdots \big(1 + w_1(E_{d+1})\big)^{j_d}$$

implying that the mod 2 Euler class of E is $\mathrm{e}(E) = w_1(E_{k+1})^{j_k} \cdots w_1(E_{d+1})^{j_d}$. Applying the map $\alpha^*_{k,\dots,d}$, with the usual abuse of notation we have that

$$\alpha^*_{k,\dots,d}\big(\mathrm{e}(E)\big) = w_1(E_{k+1})^{j_k} \cdots w_1(E_{d+1})^{j_d} \ne 0$$

in $H^*(\mathrm{Flag}_{1,\dots,d}(V);\mathbb{F}_2)$, according to (12). Consequently, $\mathrm{e}(E) \ne 0$ and the proof of the theorem is complete.

7. Proofs of Propositions 2.13 and 2.14

In this section, we verify the properties of integers $\iota_k(m_1,\dots,m_k)$ stated in Propositions 2.13 and 2.14.

7.1. Proof of Proposition 2.13

Let $k \geq 1$ be an integer and let $m_1, \ldots, m_k$ be positive integers. Recall that

$$\iota_k(m_1, \ldots, m_k) = \max\left\{j : e_k(\mathrm{pt})^j \notin (x_1^{m_1}, \ldots, x_k^{m_k})\right\},$$

where

$$e_k(\mathrm{pt}) = \prod_{(\alpha_1,\ldots,\alpha_k)\in\mathbb{F}_2^k-\{0\}} (\alpha_1 x_1 + \cdots + \alpha_k x_k) \in R_k(\mathrm{pt}) \cong \mathbb{F}_2[x_1, \ldots, x_k].$$

We prove the claims in the order they are listed.

(1) Assume that $m_k \geq 2^{k-1}m + 1$ and in addition that $\iota_{k-1}(m_1, \ldots, m_{k-1}) \geq m$. Then $e_k(\mathrm{pt})^m \notin (x_1^{m_1}, \ldots, x_{k-1}^{m_{k-1}})$. We transform as follows

$$\begin{aligned} e_k(\mathrm{pt})^m &= e_{k-1}(\mathrm{pt})^m \prod_{(\alpha_1,\ldots,\alpha_{k-1})\in\mathbb{F}_2^{k-1}} (\alpha_1 x_1 + \cdots + \alpha_{k-1}x_{k-1} + x_k)^m \\ &= e_{k-1}(\mathrm{pt})^m \cdot x_k^{2^{k-1}m} + p_{2^{k-1}m-1} \cdot x_k^{2^{k-1}m-1} + \cdots + p_1 \cdot x_k + p_0, \end{aligned}$$

where $p_{2^{k-1}m-1}, \ldots, p_1, p_0 \in \mathbb{F}_2[x_1, \ldots, x_{k-1}]$. Consequently,

$$e_k(\mathrm{pt})^m \notin (x_1^{m_1}, \ldots, x_{k-1}^{m_{k-1}}, x_k^{2^{k-1}m+1}).$$

Since, $m_k \geq 2^{k-1}m + 1$ we have that

$$(x_1^{m_1}, \ldots, x_k^{m_k}) \subseteq (x_1^{m_1}, \ldots, x_k^{2^{k-1}m+1})$$

and thus

$$e_k(\mathrm{pt})^m \notin \left(x_1^{m_1}, \ldots, x_{k-1}^{m_{k-1}}, x_k^{m_k}\right).$$

Therefore, $\iota_k(m_1, \ldots, m_k) \geq m$, as claimed.

(2) We prove the claim by induction on k. For $k = 1$ we assume that $m_1 \geq m + 1$. Then

$$\iota_1(m_1) = \max\left\{j : e_1(\mathrm{pt})^j = x_1^j \notin (x_1^{m_1})\right\} = m_1 - 1 \geq m.$$

Now, assume that the claim holds for $k - 1 \geq 1$, and assume in addition that

$$m_1 \geq 2^{i-1}m + 1 \quad \text{for all } 1 \leq i \leq k.$$

Then from the assumption $\iota_{k-1}(m_1, \ldots, m_{k-1}) \geq m$, and consequently by part (1) of this claim it follows that $\iota_k(m_1, \ldots, m_k) \geq m$.

(3) In this case we have that $m_1 = m + 1, m_2 = 2m + 1, \ldots, m_k = 2^{k-1} + 1$. According to the part (2) of this claim, it follows that

$$\iota_k(m + 1, 2m + 1, 2^2m + 1 \ldots, 2^{k-1}m + 1) \geq m.$$

Now, assume that $\iota_k(m_1,\dots,m_k)\geq r\geq 1$ for some sequence of positive integers $m_1,\dots,m_k$. Hence, $e_k(\mathrm{pt})^r\notin(x_1^{m_1},\dots,x_k^{m_k})$. We expand the transformation from the proof of part (1) of this claim as follows:

$$\begin{aligned}
e_k(\mathrm{pt})^r &= e_{k-1}(\mathrm{pt})^r \prod_{(\alpha_1,\dots,\alpha_{k-1})\in\mathbb{F}_2^{k-1}} (\alpha_1x_1+\dots+\alpha_{k-1}x_{k-1}+x_k)^r\\
&= e_{k-2}(\mathrm{pt})^r\\
&\quad\cdot \prod_{(\alpha_1,\dots,\alpha_{k-2})\in\mathbb{F}_2^{k-2}} (\alpha_1x_1+\dots+x_{k-1})^r \prod_{(\alpha_1,\dots,\alpha_{k-1})\in\mathbb{F}_2^{k-1}} (\alpha_1x_1+\dots+x_k)^r\\
&\;\;\vdots\\
&= x_k^{2^{k-1}r}x_{k-1}^{2^{k-2}r}\cdots x_2^{2r}x_1^r+q.
\end{aligned}$$

Here q is a polynomial whose additive representation in the monomial basis does not contain the monomial $x_k^{2^{k-1}r}x_{k-1}^{2^{k-2}r}\cdots x_2^{2r}x_1^r$. Since, $e_k(\mathrm{pt})^r\notin(x_1^{m_1},\dots,x_k^{m_k})$ we conclude that

$$m_k\geq 2^{k-1}r+1, m_{k-1}\geq 2^{k-2}r+1,\dots,m_1\geq r+1,$$

implying that

$$m_k+m_{k-1}+\dots+m_2+m_1\geq(2^{k-1}+2^{k-2}+\dots+2+1)r+k.$$

In particular,

$$m_k+m_{k-1}+\dots+m_2+m_1\geq(2^k-1)\iota_k(m_1,\dots,m_k)+k.$$

Thus, in the case when $m_1=m+1, m_2=2m+1,\dots,m_k=2^{k-1}+1$, we have that

$$(2^k-1)m+k\geq(2^k-1)\iota_k(m+1,2m+1,2^2m+1,\dots,2^{k-1}m+1)+k,$$

or in other words $m\geq\iota_k(m+1,2m+1,2^2m+1,\dots,2^{k-1}m+1)$.

Hence, we showed that $\iota_k(m+1,2m+1,2^2m+1,\dots,2^{k-1}m+1)=m$, as claimed.

(4) We start with the following transformation

$$\begin{aligned}
e_k(\mathrm{pt})^m &= e_{k-r}(\mathrm{pt})^m \prod_{(\alpha_{k-r+1},\dots,\alpha_k)\in\mathbb{F}_2^r-\{0\}}\ \prod_{(\alpha_1,\dots,\alpha_{k-r})\in\mathbb{F}_2^{k-r}} (\alpha_1x_1+\dots+\alpha_kx_k)^m\\
&= e_{k-r}(\mathrm{pt})^m \prod_{(\alpha_{k-r+1},\dots,\alpha_k)\in\mathbb{F}_2^r-\{0\}}\ \prod_{(\alpha_1,\dots,\alpha_{k-r})\in\mathbb{F}_2^{k-r}} \big((\alpha_1x_1+\dots+\alpha_{k-r}x_{k-r})\\
&\qquad\qquad + (\alpha_{k-r+1}x_{k-r+1}+\dots+\alpha_kx_k)\big)^m.
\end{aligned}$$

Hence,

$$e_k(\mathrm{pt})^m=\underbrace{e_{k-r}(\mathrm{pt})^m \prod_{(\alpha_{k-r+1},\dots,\alpha_k)\in\mathbb{F}_2^r-\{0\}} (\alpha_{k-r+1}x_{k-r+1}+\dots+\alpha_kx_k)^{2^{k-r}m}}_{=p}+q,$$

where the sets of (non-zero) monomials in the additive presentations of the polynomials p and q are disjoint.

The assumptions $\iota_{k-r}(m_1,\dots,m_{k-r}) \geq m$ and $\iota_r(m_{k-r+1},\dots,m_k) \geq 2^{k-r}m$ imply that

$$e_{k-r}(\mathrm{pt})^m \notin \big(x_1^{m_1},\dots,x_{k-r}^{m_{k-r}}\big)$$

and

$$\prod_{(\alpha_{k-r+1},\dots,\alpha_k)\in\mathbb{F}_2^r-\{0\}} (\alpha_{k-r+1}x_{k-r+1}+\cdots+\alpha_k x_k)^{2^{k-r}m} \notin \big(x_{k-r+1}^{m_{k-r+1}},\dots,x_k^{m_k}\big).$$

Therefore, the polynomial p is the witness that

$$e_k(\mathrm{pt})^m \notin \big(x_1^{m_1},\dots,x_{k-r}^{m_{k-r}},x_{k-r+1}^{m_{k-r+1}},\dots,x_k^{m_k}\big),$$

and consequently $\iota_k(m_1,\dots,m_k) \geq m$, as claimed

(5) The polynomial $e_k(\mathrm{pt})^m$ can be presented as follows:

$$e_k(\mathrm{pt})^m = e_{k-1}(\mathrm{pt})^m x_k^m \prod_{(\alpha_1,\dots,\alpha_{k-1})\in\mathbb{F}_2^r-\{0\}} (\alpha_1x_1+\cdots+\alpha_{k-1}x_{k-1}+x_k)^m.$$

Hence the lowest power of x_k in $e_k(\mathrm{pt})^m$ is x_k^m with coefficient $e_{k-1}(\mathrm{pt})^{2m}$.

The assumption $\iota_{k-1}(m_1,\dots,m_{k-1}) \geq 2m$ implies that

$$e_{k-1}(\mathrm{pt})^{2m} \notin (x_1^{m_1},\dots,x_{k-1}^{m_{k-1}}),$$

and since $m_k \geq m+1$ it follows that $e_k(\mathrm{pt})^m \notin (x_1^{m_1},\dots,x_{k-1}^{m_{k-1}},x_k^{m_k})$.

(6) In the case when $k=2$ we have that

$$e_2(\mathrm{pt})^m = \big(x_1x_2(x_1+x_2)\big)^m = \sum_{i=0}^{m}\binom{m}{i}x_1^{m+i}x_2^{2m-i}. \tag{14}$$

If $m \leq \iota(m_1,m_2)$ then $e_2(\mathrm{pt})^m \notin (x_1^{m_1},x_2^{m_2})$. Hence, there exists a non-zero monomial $\binom{m}{i}x_1^{m+i}x_2^{2m-i}$ in the presentation (14) of $e_2(\mathrm{pt})^m$ which does not belong to the ideal $(x_1^{m_1},x_2^{m_2})$. This means, $\binom{m}{i}=1 \bmod 2$, $m+i \leq m_1-1$ and $2m-i \leq m_2-1$ for some integer $0 \leq i \leq m$.

Assume the opposite, that there is an integer $0 \leq i \leq m$ such that $\binom{m}{i}=1 \bmod 2$ and $2m-m_2+1 \leq i \leq m_1-m-1$. Then the polynomial $e_2(\mathrm{pt})^m$ when expressed in the monomial basis has non-zero monomial $\binom{m}{i}x_1^{m+i}x_2^{2m-i}$ which does not belong to the ideal $(x_1^{m_1},x_2^{m_2})$. Consequently,

$$e_2(\mathrm{pt})^m \notin \big(x_1^{m_1},x_2^{m_2}\big).$$

(7) This is a direct consequence of the previous claim with $m=2^t+r-1$, $m_1=2^t+2t$, $m_2=2^{t+1}+r$ and $i=r-1$ because $\binom{2^t+r-1}{r-1}=1 \bmod 2$, and

$$2m-m_2+1 = r-1 \leq i = r-1 \leq m_1-m-1 = r.$$

We have completed the proof of the proposition. ■

7.2. Proof of Proposition 2.14

As before, $k \geq 1$ is an integer and $m_1, \dots, m_k$ are positive integers. In the proof we use the fact that the polynomial $e_k(\mathrm{pt})$ is the top Dickson polynomial in variables $x_1, \dots, x_k$. For more details on Dickson polynomials see for example [38].

(1) Let $D_{k-1}, D_{k-2}, \dots, D_1$ be the Dickson polynomials in variables $x_1, \dots, x_{k-1}$ of degree $2^{k-1}-1, 2^{k-1}-2, \dots, 2^{k-1}-2^{k-2}$, respectively. In particular, $D_{k-1} = e_{k-1}(\mathrm{pt})$. From [38, Prop. 1.1] we have that

$$\begin{aligned} D(x_k) &:= \prod_{(\alpha_1,\dots,\alpha_{k-1}) \in \mathbb{F}_2^{k-1}-\{0\}} (\alpha_1 x_1 + \cdots + \alpha_{k-1}x_{k-1} + x_k) \\ &= x_k^{2^{k-1}-1} + D_1\, x_k^{2^{k-2}-1} + \cdots + D_i\, x_k^{2^{k-1-i}-1} + \cdots + D_{k-2}\, x_k + D_{k-1}. \end{aligned}$$

Here $D(x_k)$ is considered as a polynomial in the polynomial ring $\mathbb{F}_2[x_1, \dots, x_{k-1}][x_k]$, and so $e_k(\mathrm{pt}) = e_{k-1}(\mathrm{pt})x_k D(x_k)$.

Let $0 \leq r \leq 2^t - 1$. We compute in $\mathbb{F}_2[x_1, \dots, x_{k-1}][x_k]$ as follows:

$$\begin{aligned} D(x_k)^{2t+r} &= \big(x_k^{2^{k-1}-1} + \cdots + D_i x_k^{2^{k-1-i}-1} + \cdots + D_{k-2}x_k + D_{k-1}\big)^{2t+r} \\ &= \big(x_k^{2^t(2^{k-1}-1)} + \cdots + D_i^{2^t}\, x_k^{2^t(2^{k-1-i}-1)} + \cdots + D_{k-2}^{2^t} x_k^{2^t} + D_{k-1}^{2^t}\big) \\ &\quad \cdot \big(x_k^{2^{k-1}-1} + \cdots + D_i x_k^{2^{k-1-i}-1} + \cdots + D_{k-2}x_k + D_{k-1}\big)^r. \end{aligned} \tag{15}$$

Then, the coefficient of $x_k^{2^t(2^{k-1}-1)}$ in $D(x_k)^{2t+r}$ is $D_{k-1}^r = e_{k-1}(\mathrm{pt})^r$, obtained as the product of $x_k^{2^t(2^{k-1}-1)}$ from the first factor with D_{k-1}^r from the second factor in (15). Indeed, the only other candidate which might additionally contribute to the coefficient of $x_k^{2^t(2^{k-1}-1)}$ is the product

$$D_1^{2^t}\, x_k^{2^t(2^{k-2}-1)} \cdot x_k^{r(2^{k-1}-1)} = D_1^{2^t}\, x_k^{2^t(2^{k-2}-1)+r(2^{k-1}-1)}$$

when

$$2^t(2^{k-1}-1) = 2^t(2^{k-2}-1) + r(2^{k-1}-1) \iff 2^{t+k-2} = r(2^{k-1}-1).$$

This cannot be, because $0 \leq r \leq 2^t - 1$. Consequently, the coefficient of $x_k^{2^{k-1+t}+r}$ in

$$e_k(\mathrm{pt})^{2t+r} = e_{k-1}(\mathrm{pt})^{2t+r} x_k^{2t+r} D(x_k)^{2t+r}$$

is equal to $e_{k-1}(\mathrm{pt})^{2t+2r}$.

From the assumption $\iota_{k-1}(m_1, \dots, m_{k-1}) \geq 2^t + 2r$ we have that

$$e_{k-1}(\mathrm{pt})^{2t+2r} \notin \big(x_1^{m_1}, \dots, x_{k-1}^{m_{k-1}}\big),$$

and since $m_k \geq 2^{t+k-1} + r + 1$ we conclude that

$$e_k(\mathrm{pt})^{2t+r} \notin \big(x_1^{m_1}, \dots, x_{k-1}^{m_{k-1}}, x_k^{m_k}\big).$$

Thus, $\iota_k(m_1, \dots, m_k) \geq 2^t + r$ as claimed.

(2) The claim follows from the previous instance of the proposition because

$$2^{t+1} + r = 2^t + r + 2^t > 2^t + r + r = 2^t + 2r.$$

(3) The proof is by induction on k for every pair of integers $(2^t, r)$ with $1 \le r \le 2^t - 1$. In the case $k = 1$, the assumption $m_1 \ge 2^t + r + 1$ implies that

$$\iota_1(m_1) = m_1 - 1 \ge 2^t + r + 1 - 1 = 2^t + r.$$

Let us assume that the claim holds for $k - 1 \ge 1$ and every pair of integers $(2^t, r)$ with $1 \le r \le 2^t - 1$ (the induction hypothesis). Take $m_i \ge 2^{t+k-1} + r + 1 = 2^{(t+1)+(k-1)-1} + r + 1$ for all $1 \le i \le k$. Applying the induction hypothesis to the first $k - 1$ inequalities and the pair $(2^{t+1}, r)$ we get that

$$\iota_{k-1}(m_1, \dots, m_k) \ge 2^{t+1} + r.$$

Now, the inequalities $\iota_{k-1}(m_1, \dots, m_k) \ge 2^{t+1} + r + 1$ and $m_k \ge 2^{t+k-1} + r + 1$, and the previous claim of this proposition imply that $\iota_k(m_1, \dots, m_k) \ge 2^t + r$. This completes the proof.

(4) Since $e_k(\mathrm{pt})^2 = \prod_{(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_2^k - \{0\}}(\alpha_1 x_1^2 + \cdots + \alpha_k x_k^2)$, the following equivalence holds

$$e_k(\mathrm{pt})^{2m} \in \left(x_1^{2m_1}, \dots, x_k^{2m_k}\right) \iff e_k(\mathrm{pt})^m \in \left(x_1^{m_1}, \dots, x_k^{m_k}\right).$$

This equivalence implies the claim. ■

8. Proof of Theorem 2.15

Let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B, and let the integers $1 \le k \le n$ and $j \ge 1$ be fixed. We first prove the equality of the ideals and then a criterion for the existence of orthogonal partitions.

8.1. Proof of Part (1)

We prove the equality of the ideals

$$\mathcal{J}_k(E) := (f_1, \dots, f_k) = (\bar{f}_1, \dots, \bar{f}_k) =: \mathcal{J}'_k(E), \tag{16}$$

where

$$f_i := \sum_{0 \le r_1 + \cdots + r_i \le n-i+1} w_{n-i+1-(r_1+\cdots+r_i)}(E)\, x_1^{r_1} \cdots x_i^{r_i},$$

$$\bar{f}_i := \sum_{0 \le r_1 + \cdots + r_k \le n-i+1} w_{n-i+1-(r_1+\cdots+r_k)}(E)\, x_1^{r_1} \cdots x_k^{r_k},$$

for $1 \le i \le k$.

To prove the equality of the ideal we first consider the polynomials

$$X_a[b] := \sum_{r_1+\cdots+r_b=n-a+1} x_1^{r_1}\cdots x_b^{r_b}$$

for $1 \le a \le n+1$ and $1 \le b \le k$. It is straightforward to see that the following equality holds

$$X_a[b+1] = X_a[b] + x_{b+1}\cdot X_{a+1}[b+1]. \tag{17}$$

Indeed, we have that

$$\begin{aligned} X_a[b+1] &:= \sum_{r_1+\cdots+r_b+r_{b+1}=n-a+1} x_1^{r_1}\cdots x_b^{r_b}x_{b+1}^{r_{b+1}} \\ &= \sum_{r_1+\cdots+r_b+0=n-a+1} x_1^{r_1}\cdots x_b^{r_b}x_{b+1}^{0} + x_{b+1}\sum_{r_1+\cdots+r_{b+1}=n-a} x_1^{r_1}\cdots x_{b+1}^{r_{b+1}} \\ &= X_a[b+1] = X_a[b] + x_{b+1}\cdot X_{a+1}[b+1]. \end{aligned}$$

Next, using induction on $\ell \ge 0$, we prove the following identity:

$$X_{c+s}[c+\ell] = \sum_{c\le b\le c+\ell}\Big(\sum_{s_b+\cdots+s_{c+\ell}=b-c} x_b^{s_b}\cdots x_{c+\ell}^{s_{c+\ell}}\Big)X_{b+s}[b]. \tag{18}$$

In case when $\ell = 0$ the equality (18) becomes the identity $X_{c+s}[c] = X_{c+s}[c]$, and so the induction basis is verified. Now, we assume that the equality (18) holds for the given fixed integer $\ell \ge 1$. For the induction step we compute and use the induction hypothesis as follows:

$$\begin{aligned} X_{c+s}[c+\ell+1] &\overset{(17)}{=} X_{c+s}[c+\ell] + x_{c+\ell+1}\cdot X_{c+s+1}[c+\ell+1] \\ &\overset{(18)}{=} \sum_{c\le b\le c+\ell}\Big(\sum_{s_b+\cdots+s_{c+\ell}=b-c} x_b^{s_b}\cdots x_{c+\ell}^{s_{c+\ell}}\Big)X_{b+s}[b] \\ &\quad + x_{c+\ell+1}\cdot X_{c+s+1}[c+\ell+1] \\ &= \sum_{c\le b\le c+\ell}\Big(\sum_{s_b+\cdots+s_{c+\ell}=b-c} x_b^{s_b}\cdots x_{c+\ell}^{s_{c+\ell}}\Big)X_{b+s}[b] \\ &\quad + x_{c+\ell+1}\cdot \sum_{s_1+\cdots+s_{c+\ell+1}=n-c-s} x_1^{s_1}\cdots x_{c+\ell+1}^{s_{c+\ell+1}}. \end{aligned}$$

Gathering two terms on the right-hand side of the previous equality under one sum we get that

$$X_{c+s}[c+\ell+1] = \sum_{c\le b\le c+\ell+1}\Big(\sum_{s_b+\cdots+s_{c+\ell}=b-c} x_b^{s_b}\cdots x_{c+\ell+1}^{s_{c+\ell+1}}\Big)X_{b+s}[b].$$

This completes the induction and the proof of the relation (18).

We proceed with a proof of the equality (16). Observe that for $1 \le i \le k$:

$$f_i = \sum_{0\le s\le n-i+1} w_s(E)X_{s+i}[i] \quad \text{and} \quad \bar{f}_i = \sum_{0\le s\le n-i+1} w_s(E)X_{s+i}[k],$$

and in particular that $f_k = \bar{f}_k$.

Now, using the relation (18) we have that

$$\begin{aligned}
\bar{f}_i &= \sum_{0\le s\le n-i+1} w_s(E)X_{s+i}[k] \\
&\overset{(18)}{=} \sum_{0\le s\le n-i+1} w_s(E)\Big(\sum_{i\le b\le k}\Big(\sum_{s_b+\cdots+s_k=b-i} x_b^{s_b}\cdots x_k^{s_k}\Big)X_{s+b}[b]\Big) \\
&= \sum_{i\le b\le k}\Big(\sum_{s_b+\cdots+s_k=b-i} x_b^{s_b}\cdots x_k^{s_k}\Big)\Big(\sum_{0\le s\le n-i+1} w_s(E)X_{s+b}[b]\Big) \\
&= \sum_{i\le b\le k}\Big(\sum_{s_b+\cdots+s_k=b-i} x_b^{s_b}\cdots x_k^{s_k}\Big)f_b.
\end{aligned}$$

In summary,

$$\bar{f}_r = \sum_{i\le b\le k}\Big(\sum_{s_b+\cdots+s_k=b-i} x_b^{s_b}\cdots x_k^{s_k}\Big)f_b. \tag{19}$$

Hence, $(\bar{f}_1,\ldots,\bar{f}_k) \subseteq (f_1,\ldots,f_k)$.

On the other hand, since $f_k = \bar{f}_k$ we have that $f_k \in \mathcal{J}'_k(E) = (\bar{f}_1,\ldots,\bar{f}_k)$. Now, for $1 \le r \le k-1$ assume that $f_{r+1},\ldots,f_k \in \mathcal{J}'_k(E)$. Then from the equality (19) it follows that

$$\begin{aligned}
\bar{f}_r &= \sum_{r\le b\le k}\Big(\sum_{s_b+\cdots+s_k=b-r} x_b^{s_b}\cdots x_k^{s_k}\Big)f_b \\
&= f_r + \sum_{r+1\le b\le k}\Big(\sum_{s_b+\cdots+s_k=b-r} x_b^{s_b}\cdots x_k^{s_k}\Big)f_b,
\end{aligned}$$

and consequently, by assumption, we have

$$f_r = \bar{f}_r + \sum_{r+1\le b\le k}\Big(\sum_{s_b+\cdots+s_k=b-r} x_b^{s_b}\cdots x_k^{s_k}\Big)f_b \in \mathcal{J}'_k(E).$$

Thus, $(\bar{f}_1,\ldots,\bar{f}_k) \supseteq (f_1,\ldots,f_k)$.

We have completed the proof of the equality (16).

8.2. Proof of Part (2)

For the second part of the theorem assume that the class $e_k(B)^j$ does not belong to the ideal $\mathcal{J}_k(E)$. The proof relies on the criterion from Theorem 3.4. In other words, it suffices to prove that

$$\mathrm{e}\left(\left(B_k(E)/\mathbb{R}\right)^{\oplus j}\right) \neq 0.$$

The mod 2 Euler class of the vector bundle $\big(B_k(E)/\underline{\mathbb{R}}\big)^{\oplus j}$, or in other words the top Stiefel–Whitney class, lives in the cohomology of $H^*(Y_k(E);\mathbb{F}_2)$. We show that

- $H^*(Y_k(E);\mathbb{F}_2) \cong R_k(B)/\mathcal{J}_k(E)$, and that
- $w_{(2^k-1)j}((B_k(E)/\underline{\mathbb{R}})^{\oplus j}) = e_k(B)^j + \mathcal{J}_k(E) \in R_k(B)/\mathcal{J}_k(E)$.

The second claim follows from the first claim, the fact that $B_k(E)$ is the restriction of $A_k(E)$, and the related computation of $w_{(2^k-1)j}((A_k(E)/\underline{\mathbb{R}})^{\oplus j})$ in the proof of Theorem 3.2. Thus we need to prove only the first statement, that is to compute the cohomology ring $H^*(Y_k(E);\mathbb{F}_2)$.

First, we give a description of the space $Y_k(E)$ as a projective bundle at the end of the tower of projective bundles

$$Y_k(E) = \mathbb{P}(E_k) \xrightarrow{p_k} \mathbb{P}(E_{k-1}) \xrightarrow{p_{k-1}} \cdots \xrightarrow{p_2} \mathbb{P}(E_1) \xrightarrow{p_1} B, \tag{20}$$

where $E_1 := E$ and p_1 is the projection. The vector bundles $E_2, \ldots, E_k$ and the maps $p_2, \ldots, p_k$ are defined iteratively as follows.

Let $H(E_1)$ be the Hopf line bundle over $\mathbb{P}(E_1)$, and recall that $p_1 \colon \mathbb{P}(E_1) \to B$ is the projection map. Then $H(E_1)$ is a vector subbundle of the pull-back vector bundle $p_1^* E_1$, and we set

$$E_2 := H(E_1)^{\perp}$$

to be the orthogonal complement of $H(E_1)$ inside $p_1^* E_1$. In particular, E_2 is a $(n-1)$-dimensional vector bundle over $\mathbb{P}(E_1)$. Set $p_2 \colon \mathbb{P}(E_2) \to \mathbb{P}(E_1)$ to be the projection map.

Next, $H(E_2) \oplus p_1^* H(E_1)$ is a vector subbundle of the pull-back vector bundle $(p_2 \circ p_1)^* E_1$, and so we define

$$E_3 := \big(H(E_2) \oplus p_1^* H(E_1)\big)^{\perp},$$

and p_3 to be the projection map $\mathbb{P}(E_3) \to \mathbb{P}(E_2)$.

We continue in the same way. Assume that for $1 \le i \le k-1$, all the vector bundles $E_1, \ldots, E_i$, of dimensions $n, n-1, \ldots, n-i+1$, respectively, and the projection maps $p_1, \ldots, p_i$ are defined. Notice that

$$H(E_i) \oplus p_i^* H(E_{i-1}) \oplus (p_i \circ p_{i-1})^* H(E_{i-1}) \oplus \cdots \oplus (p_i \circ \cdots \circ p_1)^* H(E_1)$$

is a vector subbundle of $(p_i \circ \cdots \circ p_1)^* E_1$. We define the vector bundle E_{i+1} as the orthogonal complement

$$E_{i+1} := \big(H(E_i) \oplus p_i^* H(E_{i-1}) \oplus \cdots \oplus (p_i \circ \cdots \circ p_1)^* H(E_1)\big)^{\perp}. \tag{21}$$

The map p_{i+1} is defined to be the standard projection $\mathbb{P}(E_{i+1}) \to \mathbb{P}(E_i)$. It is clear that $Y_k(E) = \mathbb{P}(E_k)$.

Now, we use the tower of projective bundles (20), Lemma 4.1, as well as the proof of Claim 4.2, to describe the cohomology ring $H^*(Y_k(E);\mathbb{F}_2) = H^*(\mathbb{P}(E_k);\mathbb{F}_2)$.

Since $H^*(Y_k(E);\mathbb{F}_2) = H^*(\mathbb{P}(E_k);\mathbb{F}_2)$ where $\mathbb{P}(E_k)$ is the projective bundle of the $(n-k+1)$-dimensional vector bundle E_k over $\mathbb{P}(E_{k-1})$ from Lemma 4.1 we have that

$$H^*\big(Y_k(E);\mathbb{F}_2\big) \cong H^*\big(\mathbb{P}(E_{k-1});\mathbb{F}_2\big)[x_k]/\Bigg(\sum_{s=0}^{n-k+1} w_{n-k+1-s}(E_k)\, x_k^s\Bigg),$$

where x_k corresponds to mod 2 Euler class of the Hopf line bundle $H(E_k)$. Continuing to apply Lemma 4.1 for the projective bundles $\mathbb{P}(E_{k-1}),\dots,\mathbb{P}(E_1)$ we get the following conclusion

$$\begin{aligned} &H^*\big(Y_k(E);\mathbb{F}_2\big)\\ &\cong H^*(B;\mathbb{F}_2)[x_1,\dots,x_k]/\Bigg(\sum_{s=0}^{n} w_{n-s}(E_1)\, x_1^s,\dots,\sum_{s=0}^{n-k+1} w_{n-k+1-s}(E_k)\, x_k^s\Bigg). \end{aligned} \tag{22}$$

Here x_i, for all $1 \le i \le k$, with a bit of abuse of notation, corresponds to the mod 2 Euler class of the Hopf line bundle $H(E_i)$, or more precisely to the mod 2 Euler class of the pull-back line bundle $(p_k \circ \dots \circ p_{i+1})^* H(E_i)$. Set $f_i := \sum_{s=0}^{n-i+1} w_{n-i+1-s}(E_i)\, x_i^s$ for $1 \le i \le k$. Then

$$H^*\big(Y_k(E);\mathbb{F}_2\big) \cong H^*(B;\mathbb{F}_2)[x_1,\dots,x_k]/\big(f_1,\dots,f_k\big).$$

Now we identify the Stiefel–Whitney classes of the vector bundles $E_1,\dots,E_k$ in terms of the Stiefel–Whitney classes E. Note that $E_1 = E$ by definition, and so $w(E_1) = w(E)$. Next, from the definition (21) of the vector bundles E_i for $2 \le i \le k$, as orthogonal complements, we get that

$$\begin{aligned} w(E_i) &= w\big(-\big(H(E_{i-1}) \oplus p_{i-1}^* H(E_{i-2}) \oplus \dots \oplus (p_{i-1} \circ \dots \circ p_1)^* H(E_1)\big)\big)\\ &= w\big(-H(E_{i-1})\big) \cdot w\big(-p_{i-1}^* H(E_{i-2})\big) \cdots w\big(-(p_{i-1} \circ \dots \circ p_1)^* H(E_1)\big). \end{aligned}$$

From Lemma 4.1 we also know that

$$w\big(H(E_{i-1})\big) = 1 + x_{i-1},\dots, w\big((p_{i-1} \circ \dots \circ p_1)^* H(E_1)\big) = 1 + x_1.$$

Here we assume the expected identifications of the classes $x_1,\dots,x_{i-1}$ along the sequence of isomorphisms given in Lemma 4.1. Combining these last two observations we have that

$$w(E_i) = \frac{1}{1+x_{i-1}} \cdot \frac{1}{1+x_{i-2}} \cdots \frac{1}{1+x_1} = \sum_{r_{i-1} \ge 0} x_{i-1}^{r_{i-1}} \cdot \sum_{r_{i-2} \ge 0} x_{i-2}^{r_{i-2}} \cdots \sum_{r_1 \ge 0} x_1^{r_1},$$

for $2 \le i \le k$. Consequently, we have that

$$\begin{aligned} f_i &= \sum_{s=0}^{n-i+1} w_{n-i+1-s}(E_i)\, x_i^s\\ &= \sum_{0 \le r_1 + \dots + r_i \le n-i+1} w_{n-i+1-(r_1+\dots+r_i)}(E)\, x_1^{r_1} \cdots x_i^{r_i} \end{aligned}$$

for every $1 \le i \le k$.

This finishes the proof of the second claim, and so the proof of Theorem 3.4 is complete.

8.3. Proof of Proposition 2.16

Let E be a Euclidean vector bundle of dimension n over a compact and connected ENR B, and let $k \geq 1$ and $j \geq 1$ be integers.

Consider the composition inclusion

$$Y_k(E) \hookrightarrow X_k(E) \hookrightarrow X_k(E \oplus \underline{\mathbb{R}}).$$

The image, $Y_k(E)$, can be seen as the zero-set of the section s of the vector bundle $A_k(E \oplus \underline{\mathbb{R}})/\underline{\mathbb{R}}$ which is defined as follows.

The fibre of $A_k(E \oplus \underline{\mathbb{R}})/\underline{\mathbb{R}}$ over the point $(b, (L_1, \dots, L_k)) \in X_k(E \oplus \underline{\mathbb{R}})$ decomposes into the direct sum

$$\Big(\bigoplus_{1 \leq i \leq k} L_i\Big) \oplus \Big(\bigoplus_{1 \leq i < j \leq k} L_i \otimes L_j\Big) \oplus \cdots .$$

For every $1 \leq i \leq k$ denote by a_i the dual of the (linear) projection map given by the composition

$$L_i \hookrightarrow E_b \oplus \mathbb{R} \longrightarrow \mathbb{R}.$$

Similarly, for $1 \leq i < j \leq k$ we set $a'_{i,j}$ to be the dual of the (linear) map induced by the inner product

$$L_i \otimes L_j \hookrightarrow (E_b \oplus \mathbb{R}) \otimes (E_b \oplus \mathbb{R}) \longrightarrow \mathbb{R}.$$

Now, define s by $(b, (L_1, \dots, L_k)) \mapsto ((a_i)_{1 \leq i \leq k}, (a'_{i,j})_{1 \leq i < j \leq k}, 0, \dots, 0)$. Hence, the zero-set of the section s is indeed $Y_k(E)$. Additionally, the vector bundle $A_k(E \oplus \underline{\mathbb{R}})/\underline{\mathbb{R}}$ over $X_k(E \oplus \underline{\mathbb{R}})$ restricts to the vector bundle $B_k(E)/\underline{\mathbb{R}}$ over $Y_k(E)$.

Consequently, if the Euler class of $(A_k(E \oplus \underline{\mathbb{R}})/\underline{\mathbb{R}})^{j+1}$ is non-zero, then the Euler class of $(B_k(E)/\underline{\mathbb{R}})^j$ is non-zero. Indeed, see for example [16, Prop. 2.7], which says that if x is any class in the cohomology of $X_k(E \oplus \underline{\mathbb{R}})$ that restricts to zero in the cohomology of the zero-set, in this case $Y_k(E)$, then the product of x with the Euler class of $A_k(E \oplus \underline{\mathbb{R}})/\underline{\mathbb{R}}$ is zero. This concludes the proof of the proposition.

9. Even more main results

In this section, we use methods developed in previous sections to give new proofs and to generalise results of Larry Guth and Nets Hawk Katz [20], Blagojević, Dimitrijević Blagojević and Günter M. Ziegler [5], Schnider [34], and Soberón and Yuki Takahashi [36].

Throughout this section B will be a compact, connected ENR, and E will be a Euclidean real vector bundle of dimension n over B. For an integer $k \geq 1$, $E(1), \dots, E(k)$ will be finite-dimensional non-zero real vector bundles over B with $\dim E(i) = n_i$. As before, we write $S(E(i))$ for the sphere bundle of $E(i)$ with fibre at $b \in B$ the space of oriented 1-dimensional subspaces of $E(i)_b$. Equivalently, $S(E(i))$ is the unit sphere bundle for a chosen Euclidean structure. Also, we shall use V for a Euclidean vector space V, and sometimes see it as a vector bundle over a point.

Recall that $A_k(E(1),\dots,E(k))$ is the 2^k-dimensional real vector bundle over $\mathbb{P}(E(1)) \times_B \cdots \times_B \mathbb{P}(E(k))$ with fibre at $(L_1,\dots,L_k)$, where $L_i \in \mathbb{P}(E(i)_b)$, $b \in B$, the real vector space of all functions $S(L_1) \times \cdots \times S(L_k) \to \mathbb{R}$. As a space of real-valued functions, each fibre of $A_k(E(1),\dots,E(k))$ can be equipped with a partial order by setting

$$f_1 \le f_2 \iff \big(\forall x \in S(L_1) \times \cdots \times S(L_k)\big)\ f_1(x) \le f_2(x)$$

for $f_1, f_2 \in A_k(E(1),\dots,E(k))$. Hence, every finite non-empty subset of functions S has a least upper bound, which we shall denote by $\max(S)$.

9.1. Partitioning by polynomials

Now we give an extension of the results [20, Thm. 4.1], [19, Thm. 0.3] and [5, Thm. 1.3] to the setting of mass assignments over an arbitrary Euclidean vector bundle E. In the case of a vector bundle over a point we recover the original results.

For an integer $d \ge 0$, let $\mathcal{P}^d(E)$ denote the real vector bundle of dimension $\binom{n+d-1}{d}$ over B with fibre at $b \in B$ the vector space of homogeneous polynomial functions $v\colon E_b \to \mathbb{R}$ of degree d. It is the dual $(S^d E)^*$ of the vector bundle obtained from the d-th symmetric power of E. If $d = 1$, we can identify $\mathcal{P}^1(E) = E^*$ with E using the inner product.

In the following, the crucial property of polynomial functions that we shall need is that for a non-zero homogeneous polynomial function $v \in \mathcal{P}^d(V)$, the zero-set

$$Z(v) = \big\{x \in S(V) \mid v(x) = 0\big\}$$

is null with respect to the Lebesgue measure on the Riemannian manifold $S(V)$. It follows that, for any $\varepsilon > 0$, there is an open neighbourhood of $Z(v)$ in the sphere $S(V)$ with volume less than ε, consult [37].

Now we extend our discussion from Section 3.3. Assume that $E(i) \subseteq \mathcal{P}^{d(i)}(E)$ is a vector subbundle of the vector bundle of homogeneous polynomial functions of degree $d(i) \ge 1$. For $b \in B$, $(L_1,\dots,L_k) \in \mathbb{P}(E(1)_b) \times \cdots \times \mathbb{P}(E(k)_b)$, and $(v_1,\dots,v_k) \in S(L_1) \times \cdots \times S(L_k)$, let us define an analogue of an orthant by

$$\mathcal{A}_{b;v_1,\dots,v_k} := \big\{u \in S(E_b) \mid v_1(u) > 0,\ \dots,\ v_k(u) > 0\big\}.$$

We note that any real continuous function on the sphere bundle $\varphi\colon S(E) \to \mathbb{R}$ restricts to a function $\varphi_b\colon S(E_b) \to \mathbb{R}$ which can be integrated over the set $\mathcal{A}_{b;v_1,\dots,v_k}$.

The first generalisation of [20, Thm. 4.1], and also at the same time extension of our Theorem 2.2, can be stated as follows.

Theorem 9.1. *Under the hypotheses in the text, for an integer $j \ge 1$, given continuous functions $\varphi_1,\dots,\varphi_j\colon S(E) \to \mathbb{R}$ assume that the $\mathbb{F}_2$-cohomology Euler class*

$$e\big(A_k\big(E(1),\dots,E(k)\big)/\underline{\mathbb{R}}\big)^j \in H^{(2^k-1)j}\big(\mathbb{P}\big(E(1)\big) \times_B \cdots \times_B \mathbb{P}\big(E(k)\big);\mathbb{F}_2\big)$$

of the vector bundle $\underline{\mathbb{R}}^j \otimes (A_k(E(1),\dots,E(k))/\underline{\mathbb{R}}) \cong (A_k(E(1),\dots,E(k))/\underline{\mathbb{R}})^{\oplus j}$ is non-zero.

Then there exists a point $b \in B$ and lines $L_i \in \mathbb{P}(E(i)_b)$, $1 \le i \le k$, such that, for each $1 \le \ell \le j$, the function

$$S(L_1) \times \cdots \times S(L_k) \to \mathbb{R}, \quad (v_1, \ldots, v_k) \mapsto \int_{\mathcal{A}_{b;\, v_1,\ldots,v_k}} (\varphi_\ell)_b$$

is constant.

Proof. As in the Section 3.3, we define for any continuous function $\varphi\colon S(E) \to \mathbb{R}$ a section s_φ of the vector bundle $A_k(E(1), \ldots, E_k)$ by

$$s_\varphi\big(b, (L_1, \ldots, L_k)\big)(v_1, \ldots, v_k) := \int_{\mathcal{A}_{b;\, v_1,\ldots,v_k}} \varphi_b.$$

Continuity of s_φ follows from the fact that zero sets of polynomial functions are sets of Lebesgue measure zero on the sphere $S(V)$. The proof then follows the pattern of the arguments in the proof of Theorem 3.3. ∎

The result remains true if the functions φ_ℓ are only assumed to be integrable in an appropriate sense. Form the locally trivial bundle $L^1_B(S(E); \mathbb{R}) \to B$ with fibre at $b \in B$ the Banach space $L^1(S(E_b); \mathbb{R})$ of all absolutely Lebesgue integrable functions $S(E_b) \to \mathbb{R}$. If φ is a section of this Banach bundle, then we can integrate $\varphi_b \in L^1(S(E_b); \mathbb{R})$ and the associated section s_φ is continuous. Next, we extend our results to probability measures. Let us write $M_+(S(E)) \to B$ for the locally trivial bundle with fibre at $b \in B$ the space $M_+(S(E_b))$ of all finite Borel measures on the sphere $S(E_b)$, see Section 1.2. A continuous section μ of $M_+(S(E))$ will be called a *family of probability measures* on $S(E)$ if $\mu_b \in M_+(S(E_b))$ is a probability measure for each $b \in B$. In this more general context the zero set of a polynomial function can have positive measure.

Now, for each $b \in B$ and every $(L_1, \ldots, L_k) \in \mathbb{P}(E(1)_b) \times \cdots \times \mathbb{P}(E(k)_b)$, we have 2^k non-negative real numbers $\mu_b(\mathcal{A}_{b;v_1,\ldots,v_k}) \in \mathbb{R}$, $(v_1, \ldots, v_k) \in S(L_1) \times \cdots \times S(L_k)$, (the measures of generalised orthants) with sum less than or equal to 1 (the measure of a zero set can be positive).

The following proposition allows us to transfer our more general setup in the previously developed topological framework.

Proposition 9.2. *Assume that for an integer $j \ge 1$ there exist families of probability measures $\mu_1, \ldots, \mu_j$ on $S(E)$ with the property that, for each $b \in B$ and every $(L_1, \ldots, L_k) \in \mathbb{P}(E(1)_b) \times \cdots \times \mathbb{P}(E(k)_b)$, there is $(v_1, \ldots, v_k) \in S(L_1) \times \cdots \times S(L_k)$ and some ℓ such that $(\mu_\ell)_b(\mathcal{A}_{b;v_1,\ldots,v_k}) > 1/2^k$.*

Then the vector bundle $\underline{\mathbb{R}}^j \otimes (A_k(E(1), \ldots, E(k))/\underline{\mathbb{R}}) \cong (A_k(E(1), \ldots, E(k))/\underline{\mathbb{R}})^{\oplus j}$ has a nowhere zero section.

Proof. For a fixed integer $1 \le \ell \le j$, consider the set of points

$$U_\ell := \big\{x = (b; L_1, \ldots, L_k) \in \mathbb{P}\big(E(1)\big) \times_B \cdots \times_B \mathbb{P}\big(E(k)\big) :$$
$$\big(\exists (v_1, \ldots, v_k) \in S(L_1) \times \cdots \times S(L_k)\big)\ (\mu_\ell)_b(\mathcal{A}_{b;v_1,\ldots,v_k}) > 1/2^k\big\},$$

which is an open subspace of the base space $X := \mathbb{P}(E(1)) \times_B \cdots \times_B \mathbb{P}(E(k))$. From the assumption it follows that $U_1, \ldots, U_j$ forms an open cover of the base space X.

Using the local triviality of the vector bundles, for every point $x \in X$ we can manufacture a (continuous) section s_ℓ^x of $A_k(E(1), \ldots, E(k))$ and an open neighbourhood U_ℓ^x of x such that for each $x' = (b'; L'_1, \ldots, L'_k) \in X$ the following holds

(i) $s_\ell^x(x')(v'_1, \ldots, v'_k) \in [0, 1]$, for all $(v'_1, \ldots, v'_k) \in S(L'_1) \times \cdots \times S(L'_k)$;

(ii) if $s_\ell^x(x')(v'_1, \ldots, v'_k) = 1$, then $(\mu_\ell)_{b'}(\mathcal{A}_{b';v'_1,\ldots,v'_k}) > 1/2^k$;

(iii) if $x' \in U_\ell^x$, then there is some $(v'_1, \ldots, v'_k)$ such that $s_\ell^x(x')(v'_1, \ldots, v'_k) = 1$.

Since X is compact, and $U_1, \ldots, U_j$ forms an open cover of X it can be refined to a compact cover $K_1, \ldots, K_j$ of X with the property that $K_\ell \subseteq U_\ell$ for $1 \le \ell \le j$.

Now, for each ℓ, we can choose a finite subset $S_\ell \subseteq U_\ell$ such that $K_\ell \subseteq \bigcup_{x \in S_\ell} U_\ell^x$. This allows as to define a continuous section s_ℓ of $A_k(E(1), \ldots, E(k))$ as $s_\ell := \max\{s_\ell^x : x \in S_\ell\}$. Here the maximum is taken with respect to the partial order on the space of real valued functions which was introduced at the beginning of this section. The properties (i), (ii) and (iii) ensure that at each point $x \in K_\ell$ at least one of the 2^k components of $s_\ell(x)$ is equal to 1, but that not all are equal to 1. Thus, the associated section $\bar{s}_\ell$ of $A_k(E(1), \ldots, E(k))/\underline{\mathbb{R}}$ has no zeros in K_ℓ. The sum $(\bar{s}_1, \ldots, \bar{s}_j)$ is a nowhere zero section of $\underline{\mathbb{R}}^j \otimes (A_k(E(1), \ldots, E(k))/\underline{\mathbb{R}})$. ■

Now a generalisation of Theorem 9.1 can be stated as follows.

Theorem 9.3. *Under the hypotheses in the text, suppose that for an integer $j \ge 1$, $\mu_1, \ldots, \mu_j$ are families of probability measures on the sphere bundle $S(E)$. If the $\mathbb{F}_2$-cohomology Euler class*

$$e\big(A_k\big(E(1), \ldots, E(k)\big)/\underline{\mathbb{R}}\big)^j \in H^{(2^k-1)j}\big(\mathbb{P}\big(E(1)\big) \times_B \cdots \times_B \mathbb{P}\big(E(k); \mathbb{F}_2\big)\big)$$

of the vector bundle $\underline{\mathbb{R}}^j \otimes (A_k(E(1), \ldots, E(k))/\underline{\mathbb{R}})$ is non-zero, then there exists a point $b \in B$ and lines $L_i \in \mathbb{P}(E(i)_b)$, $1 \le i \le k$, such that, for each $1 \le \ell \le j$ and every $(v_1, \ldots, v_k) \in S(L_1) \times \cdots \times S(L_k)$,

$$\mu_\ell(\mathcal{A}_{b;v_1,\ldots,v_k}) \le \frac{1}{2^k}.$$

Proof. Since the Euler class is non-zero, every section of the vector bundle has a zero. So the assertion follows from Proposition 9.2. ■

As an application we give a spherical version of a generalisation in [5, Thm. 1.3] of [20, Thm. 4.1].

Corollary 9.4. *Let V be a real vector space of dimension n. There is a constant C_n with the property that for integers $d \ge 1$ and $j \ge 1$, and probability measures $\mu_1, \ldots, \mu_j$ on the sphere $S(V)$, there exists a non-zero homogeneous polynomial function v of degree d on $S(V)$ such that for each component $\mathcal{O}$ of the complement in $S(V)$ of the zero set of v*

$$\mu_1(\mathcal{O}) < C_n \cdot \frac{j}{d^{n-1}}, \ldots, \mu_j(\mathcal{O}) < C_n \cdot \frac{j}{d^{n-1}}.$$

Proof. Consider probability measures $\mu_1, \dots, \mu_j$ and fix an integer $k \geq 1$. We shall apply Theorem 9.3 with $B = \mathrm{pt}$ a point, $V = \mathbb{R}^n$ and $E(i) = V(i) \subseteq \mathcal{P}^{d(i)}(V)$ a vector subspace of dimension $n_i \leq \binom{n+d(i)-1}{d(i)}$.

Let $r(i) \geq 1$ be the least positive integer such that $r(i)^{n-1} > 2^{i-1} j$ and set

$$d(i) = (n-1)r(i) \quad \text{and} \quad n_i = r(i)^{n-1}.$$

Take $V(i)$ to be the $n_i = r(i)^{n-1}$-dimensional space of polynomials with basis, in terms of the standard coordinate functions ξ_i, the monomials

$$(\xi_1^{s_1}\xi_2^{r(i)-s_1})(\xi_2^{s_2}\xi_3^{r(i)-s_2})\cdots(\xi_{n-1}^{s_{n-1}}\xi_n^{r(i)-s_{n-1}}),$$

where $0 \leq s_1, \dots, s_{n-1} < r(i)$.

It follows from Proposition 2.13 that $\iota_k(n_1, \dots, n_k) \geq j$. By Theorem 9.3, there exist homogeneous polynomials $v_1, \dots, v_k$ of degree $d(1), \dots, d(k)$, respectively, such that $\mu_\ell(\mathcal{A}_{\mathrm{pt};v_1,\dots,v_k}) \leq 1/2^k$ for all $1 \leq \ell \leq j$. The product $v_1 \cdots v_k$ has degree $d_k = d(1) + \cdots + d(k)$ and each component of the complement of its zero-set is contained in some $\mathcal{A}_{\mathrm{pt};v_1,\dots,v_j}$. Since

$$2^{\frac{i-1}{n-1}} \cdot j^{\frac{1}{n-1}} < r(i) \leq 2 \cdot 2^{\frac{i-1}{n-1}} \cdot j^{\frac{1}{n-1}},$$

it follows that

$$\begin{aligned} d_k &= d(1) + \cdots + d(k) \leq (n-1)\big(r(1) + \cdots + r(k)\big) \\ &\leq 2(n-1) \cdot j^{\frac{1}{n-1}} \cdot \sum_{i=1}^{k} 2^{\frac{i-1}{n-1}} = 2(n-1) \cdot j^{\frac{1}{n-1}} \cdot \frac{2^{\frac{k}{n-1}} - 1}{2^{\frac{1}{n-1}} - 1}. \end{aligned}$$

So $d_k^{n-1} < C_n' 2^k j$, where $C_n' = \left(\frac{2(n-1)}{2^{\frac{1}{n-1}}-1}\right)^{n-1}$.

Now (d_k) is a strictly increasing sequence. If k is chosen so that $d_k \leq d < d_{k+1}$, then $1/2^{k+1} < C_n' j / d_{k+1}^{n-1}$, and so

$$\frac{1}{2^k} < C_n \cdot \frac{j}{d_{k+1}^{n-1}} \leq C_n \cdot \frac{j}{d^{n-1}},$$

where $C_n = 2C_n'$. We can multiply $v_1 \cdots v_k$ by any non-zero polynomial of degree $d - d_1 \cdots d_k$ to produce the required polynomial of degree d. ∎

9.2. Partitioning by affine functions

In this section we give an extension of our results on the spherical GHR problem for the mass assignments to the broader class of partitions by caps which are not necessarily hemispheres.

Let V be an n-dimensional real vector space with $n \geq 2$. Using the inner product we can identify the vector space $\mathbb{R} \oplus V$ with the $(n+1)$-dimensional vector space of affine functions $V \to \mathbb{R}$ where the pair $(t, w) \in \mathbb{R} \oplus V$ determines the function $u \mapsto t + \langle u, w \rangle$.

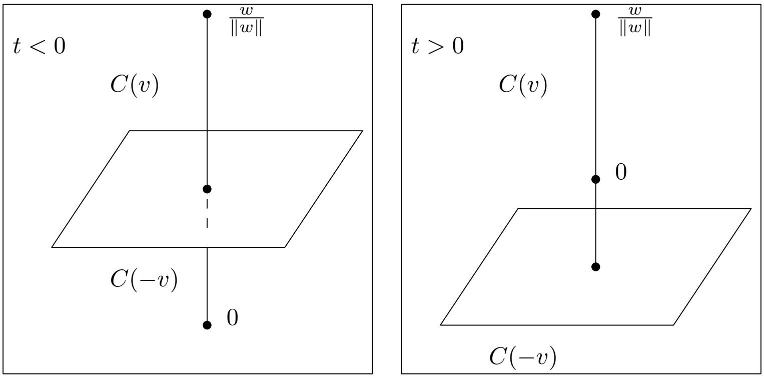


Figure 4. The halfspaces defining the caps.

A unit vector $v = (t, w) \in S(\mathbb{R} \oplus V)$ decomposes the sphere $S(V)$ as the union $S(V) = C(v) \cup C(-v)$ of two *caps*:

$$C(v) = \{u \in S(V) : \langle u, w\rangle \geq -t\} \quad \text{and} \quad C(-v) = \{u \in S(V) : \langle u, w\rangle \leq -t\}$$

with intersection $\{u \in S(V) : \langle u, w\rangle = -t\}$. For an illustration see Figure 4. If $t = 0$, the caps are hemispheres. If $t > \|w\|$, then $C(-v) = \emptyset$; if $t < -\|w\|$, then $C(v) = \emptyset$. If $t = \|w\|$, then $C(-v)$ is the single point $-w/\|w\|$, and, if $t = -\|w\|$, $C(v) = \{w/\|w\|\}$. The intersection $C(v) \cap C(-v)$, if $|t| < \|w\|$ is a sphere of dimension $n - 2$ (if $n > 1$, which we now assume).

Now suppose that each vector bundle $E(i)$ is a subbundle of $\underline{\mathbb{R}} \oplus E$, regarding a vector $v \in E(i)_b \subseteq \mathbb{R} \oplus E_b$ in the fibre at $b \in B$ as an affine linear function $E_b \to \mathbb{R}$. For a point $b \in B$, a collection of lines $(L_1, \dots, L_k) \in \mathbb{P}(E(1)_b) \times \cdots \times \mathbb{P}(E(k)_b)$, and a collection of vectors $(v_1, \dots, v_k) \in S(L_1) \times \cdots \times S(L_k)$, we define another analogue of an orthant by

$$\mathcal{C}_{b;v_1,\dots,v_k} := \{u \in S(E_b) : v_1(u) > 0, \dots, v_k(u) > 0\}.$$

The corresponding equipartition theorem is proved in the usual way by constructing a section of the vector bundle $\underline{\mathbb{R}}^j \otimes (A_k(E(1), \dots, E(k))/\underline{\mathbb{R}})$.

Theorem 9.5. *Under the hypotheses in the text, suppose that for an integer $j \geq 1$ the function $\varphi_1, \dots, \varphi_j : S(E) \to \mathbb{R}$ are continuous. If the $\mathbb{F}_2$-cohomology Euler class*

$$e\big(A_k\big(E(1), \dots, E(k)\big)/\underline{\mathbb{R}}\big)^j \in H^{(2^k-1)j}\big(\mathbb{P}\big(E(1)\big) \times_B \cdots \times_B \mathbb{P}\big(E(k); \mathbb{F}_2\big)\big)$$

of the vector bundle $\underline{\mathbb{R}}^j \otimes (A_k(E(1), \dots, E(k))/\underline{\mathbb{R}})$ is non-zero, then there exists a point $b \in B$ and there exist lines $L_i \in \mathbb{P}(E(i)_b)$, $1 \leq i \leq k$, such that, for each $1 \leq \ell \leq j$, the function

$$S(L_1) \times \cdots \times S(L_k) \to \mathbb{R}, \quad (v_1, \dots, v_k) \mapsto \int_{\mathcal{C}_{b;v_1,\dots,v_k}} (\varphi_\ell)_b$$

is constant.

9.3. Partitioning by spherical wedge

Next we describe an extension of the results of Schnider [34] and Soberón and Takahashi [36].

Let V be a vector space of dimension $n \geq 3$, and let $U \subseteq V$ be a vector subspace of dimension $m \geq 2$. Then $V = U \oplus U^{\perp}$ is the direct sum of U and its orthogonal complement $U^{\perp}$ and the unit sphere $S(V) = S(U \oplus U^{\perp})$ is the join $S(U) * S(U^{\perp})$. To be precise, we also think of the join as the space

$$S(V) = \big\{\cos(\theta)x + \sin(\theta)y \mid x \in S(U),\ y \in S(U^{\perp}),\ 0 \leq \theta \leq \pi/2\big\}.$$

Just as in the previous section, given $v = (t, w) \in S(\mathbb{R} \oplus U)$, we have the decomposition of the sphere $S(U)$ as $C(v) \cup C(-v)$, where

$$C(v) = \big\{u \in S(U) : \langle u, w\rangle \geq -t\big\} \quad \text{and} \quad C(v) = \big\{u \in S(U) : \langle u, w\rangle \geq -t\big\}.$$

This leads to a decomposition of the bigger sphere $S(V)$ as the union $W(v, U) \cup W(-v, U)$ of two *wedges*

$$\begin{aligned} W(v, U) &= C(v) * S(U^{\perp}) \\ &= \big\{\cos(\theta)u + \sin(\theta)y : u \in S(U),\ y \in S(U^{\perp}),\ \langle u, w\rangle \geq -t,\ 0 \leq \theta \leq \pi/2\big\} \end{aligned}$$

and

$$\begin{aligned} W(-v, U) &= C(-v) * S(U^{\perp}) \\ &= \big\{\cos(\theta)u + \sin(\theta)y : u \in S(U),\ y \in S(U^{\perp}),\ \langle u, w\rangle \leq -t,\ 0 \leq \theta \leq \pi/2\big\}. \end{aligned}$$

The intersection $W(v, U) \cap W(-v, U)$ is $S(U^{\perp})$ if $|t| > \|w\|$, a disc of dimension $n - m$ if $|t| = \|w\|$, and a sphere of dimension $n - 2$ if $|t| < \|w\|$. (The subspace $\{rx : r \geq 0,\ x \in W(v, U)\}$ of V is an *m-cone* in the sense of [34].)

For example, take $U = \mathbb{R}^2$, $U^{\perp} = \mathbb{R}$, $V = \mathbb{R}^2 \oplus \mathbb{R}$, so that $m = 2$, $n = 3$. The wedges $W(v, U)$, where $v = (t, w) \in S(\mathbb{R}^2 \oplus \mathbb{R})$ with $|t| < \|w\|$, are the subsets

$$\big\{\big(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), \sin(\theta)\big) \in S(\mathbb{R}^2 \oplus \mathbb{R}) : \alpha \leq \phi \leq \beta,\ -\pi/2 \leq \theta \leq \pi/2\big\},$$

where $0 \leq \alpha < \beta < 2\pi$.

Now suppose that $F(i) \subseteq E$ is a vector subbundle of dimension $m_i \geq 2$, for every $1 \leq i \leq k$, and that $E(i)$ is a subbundle of $\underline{\mathbb{R}} \oplus F(i)$ of dimension $n_i \leq m_i + 1$. For a point $b \in B$, lines $(L_1, \ldots, L_k) \in \mathbb{P}(E(1)_b) \times \cdots \times \mathbb{P}(E(k)_b)$, and vectors $(v_1, \ldots, v_k) \in S(L_1) \times \cdots \times S(L_k)$, we write

$$\mathcal{W}_{b;v_1,\ldots,v_k} := \bigcap_{i=1}^{k} \big(S(E_b) - W\big(-v_i, F(i)_b\big)\big)$$

as the intersection of the open subsets $S(E_b) - W(-v_i, F(i)_b) \subseteq W(v_i, E(i)_b)$.

As in all the previous partition problems for mass assignments we derive the following result in almost identical manner.

Theorem 9.6. *Under the hypotheses in the text, suppose that $j \geq 1$ is an integer, the functions $\varphi_1, \dots, \varphi_j : S(E) \to \mathbb{R}$ are continuous, and $j \leq \iota_k(E(1), \dots, E(k))$.*

Then there exists a point $b \in B$ and there exist lines $L_i \in \mathbb{P}(E(i)_b)$, $1 \leq i \leq k$, such that, for each $1 \leq \ell \leq j$, the function

$$S(L_1) \times \cdots \times S(L_k) \to \mathbb{R}, \quad (v_1, \dots, v_k) \mapsto \int_{W_{b;\, v_1, \dots, v_k}} (\varphi_\ell)_b$$

is constant.

In the special case of a vector bundle over a point we get the following corollary.

Corollary 9.7. *Suppose that $j \geq 1$ is an integer, $\varphi_1, \dots, \phi_j : S(V) \to \mathbb{R}$ are continuous functions and that $j \leq \iota_k(n+1, \dots, n+1)$. Let $m_1, \dots, m_k$ be integers in the range $2 \leq m_i \leq n$.*

Then there exist vector subspaces $U_1, \dots, U_k \subseteq V$ with $\dim(U_i) = m_i$ and lines $L_i \in \mathbb{P}(\mathbb{R} \oplus U_i)$, $1 \leq i \leq k$, such that for each $1 \leq \ell \leq j$, the function

$$S(L_1) \times \cdots \times S(L_k) \to \mathbb{R}, \quad (v_1, \dots, v_k) \mapsto \int_{W(v_1, U_1) \cap \cdots \cap W(v_k, U_k)} \varphi_\ell$$

is constant.

Proof. Take B to be the product $G_{m_1}(V) \times \cdots \times G_{m_k}(V)$ of Grassmann manifolds and $F(i)$ to be the canonical m_i-dimensional bundle over the i-th factor. Apply Theorem 9.6 with $n_i = m_i + 1$ and $E(i) = \underline{\mathbb{R}} \oplus F(i)$. Indeed, since $\iota_1(\underline{\mathbb{R}} \oplus F(i)) = n$, we have that

$$\iota_k\big(E(1), \dots, E(k)\big) = \iota_k(n+1, \dots, n+1)$$

by Proposition 2.7. ∎

Remark 9.8. The previous Corollary 9.7 can be sharpened by restricting the base space in the following way. Replace the Grassmann manifolds $G_{m_i}(V)$, where $V = \mathbb{R}^n$, by the its subspace $\mathbb{P}(\mathbb{R}^{n-m_i+1})$, embedded by taking the direct sum of a line in $\mathbb{R}^{n-m_i+1}$ with $\mathbb{R}^{m_i-1}$ to get a subspace of $\mathbb{R}^n = \mathbb{R}^{n-m_i+1} \oplus \mathbb{R}^{m_i-1}$ of dimension m_i. Then the vector bundle $E(i)$ restricts to $\underline{\mathbb{R}}^{m_i} \oplus H_i$ where H_i is the Hopf line bundle $H(\mathbb{R}^{n-m_i+1})$. So $\iota_1(\underline{\mathbb{R}}^{m_i} \oplus H_i) = n$, because $w_{n-m_i}(-H_i) \neq 0$.

To illustrate the conditions in Corollary 9.7 we spell out the special case $n = 3$, $j = 3$, $k = 1$, $m_1 = 2$, for which $\iota_1(3+1) = 3$. Suppose that $\varphi_1, \varphi_2, \varphi_3 : S^2 = S(\mathbb{R}^3) \to \mathbb{R}$ are continuous functions. Then there is a wedge $W \subseteq \mathbb{R}^3$, specified by a plane U through the origin in $\mathbb{R}^3$ and $(t, w) \in S(\mathbb{R} \oplus U)$, such that

$$\int_W \phi_1 = \frac{1}{2} \int_{S^2} \phi_1, \quad \int_W \phi_2 = \frac{1}{2} \int_{S^2} \phi_2, \quad \int_W \phi_3 = \frac{1}{2} \int_{S^2} \phi_3.$$

Furthermore, the $k = 1$ case of Corollary 9.7 gives [34, Thm. 8], and also the spherical version of [36, Thm. 1.2 and Thm. 3.2].

Corollary 9.9. *Suppose that $\varphi_1, \dots, \varphi_n \colon S(\mathbb{R}^n) \to \mathbb{R}$ are continuous functions and m is an integer, $2 \le m \le n$. Write $V = \mathbb{R}^n$ and $V' = \mathbb{R}^{m-1} \subseteq \mathbb{R}^{n-m+1} \oplus \mathbb{R}^{m-1} = V$.*

Then there exists a vector subspace $U \subseteq V$ of dimension m containing the subspace V' and a vector $v \in S(\mathbb{R} \oplus U)$ such that

$$\int_{W(v,U)} \varphi_\ell = \frac{1}{2} \int_{S(V)} \phi_l = \int_{W(-v,U)} \varphi_\ell$$

for $\ell = 1, \dots, n$.

Proof. We just need to recall that $\iota_1(n+1) = n$. The sharpening, to give the restriction that U should contain V', is given by Remark 9.8. ∎

The connection between the affine and spherical cases was discussed in Section 1.3. We explain how [36, Thm. 1.2] can be deduced from the case $m = 2$ of our Corollary 9.9.

Corollary 9.10. *For an integer $n \ge 2$, suppose that $\psi_1, \dots, \psi_n \colon \mathbb{R}^{n-1} \to \mathbb{R}$ are continuous functions with compact support with the n integrals $\int_{\mathbb{R}^{n-1}} \psi_\ell$, $1 \le \ell \le n$, not all equal to zero.*

Then there exist two distinct parallel hyperplanes in $\mathbb{R}^{n-1}$ such that the closed region S sandwiched between them satisfies

$$\int_S \psi_l = \frac{1}{2} \int_{\mathbb{R}^{n-1}} \psi_\ell,$$

for all $1 \le \ell \le n$.

(Note that if all the integrals $\int_{\mathbb{R}^{n-1}} \phi_\ell$ are zero, then there is a trivial statement for any two coinciding hyperplanes.)

Proof. Consider the diffeomorphism

$$\pi \colon \Lambda = \big\{(x,y) \in S(\mathbb{R}^{n-1} \oplus \mathbb{R}) : y > 0\big\} \to \mathbb{R}^{n-1}, \quad (x,y) \mapsto \frac{x}{y},$$

which maps intersections of linear subspaces of $\mathbb{R}^{n-1} \oplus \mathbb{R}$ with Λ to affine subspaces of $\mathbb{R}^{n-1}$. Each density ψ_ℓ lifts to a density φ_ℓ on $S(\mathbb{R}^{n-1} \oplus \mathbb{R})$ with support in the open upper hemisphere Λ. (To be precise, $\varphi(x,y) = y^n \psi(x/y)$.)

Let $U \subseteq \mathbb{R}^{n-1} \oplus \mathbb{R} = V$ be a 2-dimensional vector subspace and $v \in S(\mathbb{R} \oplus U)$ a vector as provided by Corollary 9.9 when $m = 2$. Since some $\int_{\mathbb{R}^{n-1}} \psi_\ell$ is non-zero, both $S(V) - W(-v,U)$ and $S(V) - W(v,U)$ have to be non-empty. The intersection $W(v,U) \cap W(-v,U)$ is, therefore, the union of two discs

$$\{a\} * S(U^\perp) \subseteq S(\mathbb{R} \cdot a \oplus U^\perp) \quad \text{and} \quad \{b\} * S(U^\perp) \subseteq S(\mathbb{R} \cdot b \oplus U^\perp)$$

meeting in $S(U^\perp)$. Here $a, b \in S(U)$.

The image of the intersection $\pi(W(v,U) \cap W(-v,U) \cap \Lambda)$ is the union of two affine hyperplanes meeting in $\pi(S(U^\perp) \cap \Lambda)$.

We can prescribe the subspace V' in Corollary 9.9 to be the line $0 \oplus \mathbb{R} \subseteq \mathbb{R}^{n-1} \oplus \mathbb{R}$. In that case, $S(U^\perp) \cap \Lambda$ is empty, and the two hyperplanes are parallel. ∎

10. Concluding remarks: real flag manifolds

In the final section we make some further remarks on particular arguments used in the proofs of our results.

For a Euclidean vector space V of dimension n and integers $0 = n_0 < n_1 < \cdots < n_k < n$, let $B := \mathrm{Flag}_{n_1,\ldots,n_k}(V)$ be the manifold of flags $(V_*) : 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k \subseteq V$ with $\dim V_i = n_i$. The canonical bundles of dimension n_i over B are denoted by $E(i)$, as in the statement of Corollary 2.10. Write E for the trivial bundle over B with fibre V.

Proposition 10.1. *The* $\mathbb{F}_2$*-Euler classes satisfy*

$$\prod_{i=1}^{k} \mathrm{e}\left(E/E(i)\right)^{n_i-n_{i-1}} \neq 0 \in H^d(B;\mathbb{F}_2) = \mathbb{F}_2,$$

where $n_0 = 0$ *and the dimension* d *is equal to* $\sum_{i=1}^{k}(n-n_i)(n_i-n_{i-1})$.

Proof. Let $(U_*) : 0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_k$ be a fixed flag in V. The general linear group $G = \mathrm{GL}(V)$ acts transitively on B. If $H \leq G$ denotes the stabiliser of (U_*), we have a map $\pi\colon G \to B$ defined by $\pi(g) = (gU_*)$ which describes B as the homogeneous space G/H.

The derivative of π at $1 \in G$ is a map from the Lie algebra $\mathfrak{g} = \mathrm{End}(V)$ onto the tangent space of B at (U_*) with kernel the Lie algebra $\mathfrak{h}$ of H, that is, the space of endomorphisms a of V such that $a(U_i) \subseteq U_i$ for all i. The tangent bundle of B is the quotient of the trivial Lie algebra bundle $B \times \mathrm{GL}(V) = \mathrm{GL}(F)$ by the subbundle with fibre at $(V_*) \in B$ the quotient of $\mathrm{End}(V)$ by the Lie subalgebra, $\mathfrak{h}(V_*)$ of endomorphisms that preserve the flag. Using the inner product, we can express $\mathfrak{h}(V_*)$ as $\bigoplus_{i=1}^{k} \mathrm{Hom}(V_i^{\perp}, V_i \cap (V_{i-1}^{\perp}))$, which has dimension $\sum_{i=1}^{k}(n-n_i)(n_i-n_{i-1})$.

Now consider the vector bundle E', defined as a quotient of $B \times \mathrm{End}(V)$, with the fibre at $(V_*) \in B$ the quotient of $\mathfrak{g} = \mathrm{End}(V)$ by the vector subspace $\mathfrak{h}(V_*, U_*)$ of maps $a\colon V \to V$ such that $a(V_i) \subseteq U_i$ for $i = 1, \ldots, k$. In metric terms,

$$E' = \bigoplus_{i=1}^{k} \mathrm{Hom}\left(F(i)^{\perp}, U_i \cap (U_{i-1}^{\perp})\right),$$

and its Euler class $\mathrm{e}(E')$ is equal to $\prod_{i=1}^{k} e(E(i)^{\perp})^{n_i-n_{i-1}} \in H^d(B;\mathbb{F}_2)$.

The vector bundle E' over the closed connected d-dimensional manifold B has the same dimension d. We shall prove that $\mathrm{e}(E')$ is non-zero by writing down a smooth section s of E' with exactly one zero and checking that the (mod 2) degree of that zero is equal to 1.

The section s is defined to have the value at (V_*) given, modulo $\mathfrak{h}(V_*, U_*)$, by the identity endomorphism $1 \in \mathrm{End}(V)$. At a zero of s, $V_i \subseteq U_i$ for all i, that is, $V_* = U_*$. At this zero, the tangent space of B coincides with the fibre of E', and we shall show that the derivative of s is the identity endomorphism of $\mathfrak{g}/\mathfrak{h}$.

To do this, we lift from $B = G/H$ to G by the projection π. The pullback $\pi^* E'$ is trivialised by the isomorphism

$$G \times (\mathfrak{g}/\mathfrak{h}) \to E'$$

taking $(g, a + \mathfrak{h})$, where $g \in \mathrm{GL}(V)$, $a \in \mathrm{End}(V)$, to $((gU_*), ag^{-1} + \mathfrak{h}(gU_*))$. And the section s lifts to the map

$$G \to \mathfrak{g}/\mathfrak{h} : g \mapsto g + \mathfrak{h},$$

for which the derivative at 1 is, transparently, the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. This completes the proof. ■

Writing the quotient $E/E(i) = E(i)^{\perp}$ as the direct sum $\bigoplus_{j=i}^{k}(E(j+1)/E(j))$, where $E(k+1) = E$, we can reformulate Proposition 10.1 as follows.

Corollary 10.2. *The product of Euler classes*

$$\prod_{i=1}^{k} \mathrm{e}\left(E(i+1)/E(i)\right)^{n_i} \in H^d(B; \mathbb{F}_2)$$

is non-zero.

This Corollary 10.2 connects with previously given arguments in the following way:

- The case $k = 1$, shows that $\mathrm{e}(E(1)^{\perp})^{n-n_1} \neq 0$, and in particular $\mathrm{e}(E(1)^{\perp}) \neq 0$, as used in the proof of Corollary 2.5.
- The statement $\mathrm{e}(E(1)^{\perp})^{n-n_1} \neq 0$ is the result needed in Section 5.3 for the proof of Corollary 2.8.
- For general k, we have in particular that $\prod_{i=1}^{k} \mathrm{e}(E(i)^{\perp}) \neq 0$. This is what is required in Section 6.1 to prove Corollary 2.10.
- If $n_i = n - k + i - 1$ (that is, $n_1 = n - k, n_2 = n - k + 1, \ldots, n_k = n - 1$), then

$$\mathrm{e}\left(E(1)^{\perp}\right)^{k} \mathrm{e}\left(E(2)^{\perp}\right)^{k-1} \cdots \mathrm{e}\left(E(k)^{\perp}\right)^{1} \neq 0.$$

  This is what is needed in Section 6.3 to prove Theorem 2.12. It shows directly that $\mathrm{e}(E_{k+1})^{k} \cdots \mathrm{e}(E_{d+1})^{d} \neq 0$. The permutation symmetry of the cohomology then gives

$$\mathrm{e}(E_{k+1})^{j_k} \cdots \mathrm{e}(E_{d+1})^{j_d} \neq 0.$$

Thus, the different arguments we offered in the proofs can be seen as direct consequences of Corollary 10.2. To the best of our knowledge these implications were not known until now, and we believe it was worth explaining these connections.

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