Doc. Math. 30 (2025), 105–181
DOI 10.4171/DM/990

Mixed ℓ-adic complexes for schemes over number fields

Sophie Morel

Abstract. If X is a variety over a number field, Annette Huber has defined a category of "horizontal" (or "almost everywhere unramified") ℓ-adic complexes and ℓ-adic perverse sheaves on X. For such objects, the notion of weights makes sense (in the sense of Deligne), just as in the case of varieties over finite fields. However, contrary to what happens in that last case, mixed perverse sheaves (or mixed locally constant sheaves) on X do not have a weight filtration in general, even when X is a point. The goal of this paper is to show how to avoid this problem by working directly in the derived category of the abelian category of perverse sheaves that do admit a weight filtration. As an application, the method of the author to calculate the intermediate extension of a pure perverse sheaf using weight truncations apply over any finitely generated field, and not just over a finite field.

Contents

1. Introduction

Let k be a field of finite type over its prime subfield, let X be a separated scheme of finite type over k, and let ℓ be a prime number invertible in k. In her article [14] Annette Huber introduced a category $\mathrm{D}^b_m(X) = \mathrm{D}^b_m(X, E)$ of mixed horizontal ℓ-adic sheaves on X, where E is an algebraic extension of $\mathbb{Q}_\ell$. The idea of [14] is to consider the category of ℓ-adic complexes on X that extend to a constructible ℓ-adic complex on a model $\mathcal{X}$ of X over a normal scheme $\mathcal{U}$ of finite type over $\mathbb{Z}$ and with field of fractions k; we also want the morphisms between complexes to extend to $\mathcal{X}$. There is a natural definition of

Mathematics Subject Classification 2020: 14F43 (primary); 11G35, 18G80 (secondary).
Keywords: perverse sheaves, Deligne weights, varieties over number fields.

weights (in the sense of Deligne's [9]) on such complexes, by considering their restriction to the fibers of $\mathcal{X}$ over closed points of $\mathcal{U}$. So we have a notion of pure sheaves, and mixed complexes are defined (as in [9]) as those complexes whose cohomology sheaves have a filtration with pure quotients.

By [14, Sections 2 and 3], the 6 operations (usual and exceptional direct and inverse images, tensor products and internal Homs) exist on these categories of complexes. Moreover, it is shown in [14, Theorem 2.5 and Proposition 3.2] that the category $\mathrm{D}^b_m(X)$ has a (self-dual) perverse t-structure, whose heart $\mathrm{Perv}_m(X)$ is called the category of horizontal mixed perverse sheaves on X.

Also, the results of Chapters 4 and 5 of Beilinson–Bernstein–Deligne's book [6] about the t-exactness (or perverse cohomological amplitude) of the 6 functors, and the way these 6 functors affect weights, can be extended to our situation thanks to Deligne's generic base change theorem (SGA 4 1/2 [Th. finitude] Section 2), see for example [14, Proposition 3.4 and Corollary 3.5].

Finally, there is a notion of weight filtration on an object of $\mathrm{Perv}_m(X)$ (see [14, Definition 3.7]); it is an increasing filtration whose quotients are pure perverse sheaves of increasing weights. This filtration is unique if it exists [14, Lemma 3.8], but unfortunately it does not always exist, unless k is a finite field. As noted in the remark below [14, Lemma 3.8], the category of horizontal mixed perverse sheaves on X admitting a weight filtration is a full abelian subcategory $\mathrm{Perv}_{mf}(X)$ of $\mathrm{Perv}_m(X)$ which is stable by subquotients, but it is not stable by extensions.

As a consequence, if we start from a horizontal mixed perverse sheaf that does have a weight filtration and apply some sheaf operations, then it is not clear that the perverse cohomology sheaves of the resulting mixed complex will still have weight filtrations. (Although we would certainly expect that to be the case.) For example, this is a problem if we want to generalize the arguments of [20], that gives among other things a formula for the intersection complex of X.

The goal of this paper is to give a solution to this problem, inspired by Beilinson's theorem that, if k is a finite field, then the derived category of $\mathrm{Perv}_m(X)$ is canonically equivalent to $\mathrm{D}^b_m(X)$ (see [3, 4]; note that Beilinson's result is more general). Beilinson also gives a way to reconstruct the derived direct image functors from their perverse versions, and formulas adapted to perverse sheaves for the unipotent nearby and vanishing cycles functors. Building on this, Morihiko Saito has shown in [22, 23] how to recover the other operations (inverse images, tensor products and internal Homs) using only perverse sheaves.

In this paper, we will follow the ideas of Beilinson and M. Saito to construct all the sheaf operations on the bounded derived categories of the categories $\mathrm{Perv}_{mf}(X)$. The main point, which is taken as an axiom in [23], is the fact that these categories are stable by perverse direct images; in Section 6.3, we show how to deduce it from Deligne's weight-monodromy theorem. Another difficulty is to state all the compatibilities that the sheaf operations should satisfy. We have chosen to use the formalism of crossed functors ("foncteurs croisés"), originally due to Deligne and developed by Voevodsky and

Ayoub. In order to check that the constructions of Beilinson and M. Saito do fit into this formalism, we have had to rewrite some of them. (Another reason is that the categories $\mathrm{Perv}_{mf}(X)$ satisfy assumptions that are slightly different from the axioms of [23], and so certain proofs become simpler, and at least one proof has to be totally changed. However, most of the constructions are very similar to the ones in [23].)

Here is a quick description of the different parts of the paper. Section 1 is the introduction, and Section 2 contains reminders about ℓ-adic perverse sheaves, the realization functor and a quick summary of the beginning of Huber's article [14], in particular the definition of the main object of study $\mathrm{Perv}_{mf}(X)$. In Section 3, we state the main results of the paper, first informally and then using the language of crossed functors. Section 4 gives a list of functors that obviously preserve the categories $\mathrm{Perv}_{mf}(X)$. Section 5 contains reminders about Beilinson's construction of unipotent nearby and vanishing cycles. In Section 6, we state the form of Deligne's weight-monodromy theorem that we will use, and deduce the crucial fact that perverse direct images also preserve the categories $\mathrm{Perv}_{mf}(X)$; we also give an application to complexes with support in a closed subscheme, that was already noted in [4, Section 2.2.1] and [23, Theorem 5.6]. Section 7 gives the proof of the first main theorem (Theorem 3.2.4, concerning the existence of the four operations f^*, f_*, $f_!$, $f^!$), and Section 8 gives the proof of the second main theorem (Theorem 3.2.12, about the existence of tensor products and internal Homs). Section 9 shows how the results of this article imply that we can extend the formalism of weight truncation functors defined in [20]. Finally, the appendix gives a review of filtered derived categories and f-categories (i.e., filtered triangulated categories), with a focus on the compatibility between the realization functor and derived functors.

Here are some conventions that will be used throughout the paper:

- As we are considering sheaves for the étale topology or proétale topology, we only care about schemes up to universal homeomorphism. So we will allow ourselves to specify a closed subscheme of a scheme X by giving only the underlying closed subset.
- We are mostly interested in the triangulated versions of the sheaf operations, so we will denote them without the usual "R"'s or "L"'s. For example, the derived direct image functors will simply be denoted by f_*, and we will similarly write f^*, $f_!$ and $f^!$ for the other direct and image inverse functors, seen as functors between the triangulated categories of complexes of sheaves. The only exception we will make is for the functor $R\,\mathrm{Hom}$ (in an abelian category), in order to distinguish it from Hom.
- All the schemes will be assumed to be excellent and separated, and all the morphisms will be assumed to be of finite type. (We are only interested in schemes that are of finite type over $\mathbb{Z}$ or over a field, and these schemes are automatically excellent.) If we write "scheme over k", where k is a field, we will mean "separated field of finite type over k". Also, the letter ℓ always stands for a prime number invertible over all the schemes considered.
- If $\mathcal{C}$ is an additive category, we will denote by $\mathrm{C}(\mathcal{C})$ (resp. $\mathrm{C}^+(\mathcal{C})$, $\mathrm{C}^-(\mathcal{C})$, $\mathrm{C}^b(\mathcal{C})$) the additive category of (cohomological) complexes of objects of $\mathcal{C}$ (resp. of bounded

below complexes, bounded above complexes, bounded complexes); we also denote by $\mathrm{K}(\mathscr{C})$ (resp. $\mathrm{K}^+(\mathscr{C})$, $\mathrm{K}^-(\mathscr{C})$, $\mathrm{K}^b(\mathscr{C})$) the corresponding triangulated categories of complexes modulo homotopy.

- If $\mathscr{A}$ is a quasi-abelian category (for example of abelian category), we denote by $\mathrm{D}(\mathscr{A})$ (resp. $\mathrm{D}^+(\mathscr{A})$, $\mathrm{D}^-(\mathscr{A})$, $\mathrm{D}^b(\mathscr{A})$) the derived category (resp. bounded below derived category, bounded above derived category, bounded derived) of $\mathscr{A}$, defined as the quotient of $\mathrm{K}(\mathscr{A})$ (resp. $\mathrm{K}^+(\mathscr{A})$, $\mathrm{K}^-(\mathscr{A})$, $\mathrm{K}^b(\mathscr{A})$) by the triangulated subcategory of exact complexes.

2. Horizontal perverse sheaves

In this section, we recall the definition of ℓ-adic constructible complexes and ℓ-adic perverse sheaves, the construction of the realization functor from the bounded derived category of perverse sheaves to the category of constructible complexes, and finally the definition by A. Huber of horizontal ℓ-adic complexes and horizontal perverse sheaves, as well as the mixed versions.

2.1. ℓ-adic complexes

Let X be a scheme and E be an algebraic extension of $\mathbb{Q}_\ell$. If we want to stay in the familiar framework of triangulated categories (and avoid ∞-categories), there are two approaches to the category of bounded constructible étale E-complexes on X that work at the level of generality that we need: Ekedahl's approach in [10] (see also Fargues's paper [11] for some complements) and the Bhatt–Scholze definition via the proétale site in [7]. The second works in a more general setting, and it is known to be equivalent to the first when they both apply. While we could make the constructions that we need work with both approaches, we will mostly stick to the Bhatt–Scholze approach, because it makes the homological algebra simpler.

Remark 2.1.1. In his article [10], Ekedahl makes the assumption that the scheme X is of finite type over a regular scheme of dimension ≤ 1. The reason for this is that the necessary theorems for torsion étale sheaves were only available in this setting at the time. Since then, Gabber has proved the finiteness theorem (see [17, Exposé XIII]), the absolute purity theorem (see [12] or [17, Theorem XVI.3.1.1]) and the existence of a dualizing complex (see [17, Exposé XVII]) in the more general setting considered here, so the results of [10] extend to this setting.

Let us review quickly the construction of Bhatt and Scholze via the proétale site $X_{\text{proét}}$ of X (see [7]): The category $\mathrm{D}^b_c(X, E)$ is defined as a the full subcategory of the category $\mathrm{D}(X_{\text{proét}}, E)$ of sheaves of E-modules on $X_{\text{proét}}$ [7, Definition 4.1.1] whose objects are bounded complexes with constructible cohomology sheaves, where a proétale sheaf $\mathscr{F}$ of E-vector spaces is called *constructible* if X has a finite stratification $(Z_i)_{i\in I}$ by locally closed subschemes such that each $\mathscr{F}_{|Z_i}$ is lisse, i.e., locally (in the proétale topology) free of finite rank; see [7, Definitions 6.8.6 and 6.8.8]. By [7, Propositions 5.5.4, 6.6.11, 6.8.11 and 6.8.14], this category is canonically equivalent to the one defined by Ekedahl if X

satisfies condition (A) or (B) of [7, Definition 5.5.1].[1] This point of view is conceptually simpler and has the advantage that direct images, tensor products and internal Homs are the restriction of actual derived functors on the categories $\mathrm{D}(X_{\text{proét}}, E)$.

The six operations on the categories $\mathrm{D}^b_c(X, E)$ (direct and inverses images, direct images with proper support, exceptional inverse images, derived tensor products and derived internal Homs) are constructed in [7, Sections 6.7 and 6.8]. Suppose that we are given a dimension function δ on X (see [17, Définition XVII.2.1.1]), and let K_X be the corresponding dualizing complex on X. By this, we mean a potential dualizing complex on X for the dimension function δ (see [17, Définition XVII.2.1.2]); this is known to be unique up to unique isomorphism [17, Théorème XVII.5.1.1] and to be a dualizing complex [17, Théorème XVII.6.1.1]. We then denote by $D_X = \underline{\mathrm{Hom}}_X(., K_X)$ the duality functor defined by K_X; it satisfies all the usual properties, see [7, Lemma 6.7.20].

The category $\mathrm{D}^b_c(X, E)$ has a canonical t-structure, whose heart is the category of constructible sheaves $\mathrm{Sh}_c(X, E)$ (this is automatic if we use [7, Definition 6.8.8] for $\mathrm{D}^b_c(X, E)$). This category has a full abelian subcategory stable by extensions $\mathcal{L}oc(X, E)$, the category of lisse sheaves (or locally constant sheaves, or local systems), see [7, Definition 6.8.3]. We will only use the category $\mathcal{L}oc(X, E)$ if X is connected regular; in that case (and more generally if X is geometrically unibranch), this category is equivalent to the category of continuous representations of the étale fundamental group $\pi^{\text{ét}}(X)$ of X on finite-dimensional E-vector spaces (see [7, Lemmas 7.4.7 and 7.4.10 and Remark 7.4.8]; the equivalence is given by taking stalks at a geometric point of X). In particular, if X is smooth of relative dimension d over a field k and if we use the dimension function $\delta\colon x \mapsto \dim(\overline{\{x\}})$ (see the beginning of Section 2.2), then for every $\mathcal{L} \in \mathrm{Ob}(\mathcal{L}oc(X, E))$ corresponding to a representation of $\pi^{\text{ét}}(X)$, if we denote by $\mathcal{L}^\vee$ the lisse sheaf corresponding to the dual representation, then $D_X(\mathcal{L}) \simeq \mathcal{L}^\vee(-d)[-2d]$. Indeed, we have $K_X = \underline{E}_X(-d)[-2d]$, hence $\mathrm{H}^0\,\underline{\mathrm{Hom}}_X(\mathcal{L}, K_X) = \mathcal{L}^\vee(-d)[-2d]$, and all the $\mathrm{H}^i\,\underline{\mathrm{Hom}}_X(\mathcal{L}, K_X)$ for $i \geq 1$ vanish by [19, Exercise III.1.31] (and [7, Lemma 6.7.13]).

2.2. Perverse sheaves

In this section, we assume that X satisfies the conditions of [17, Corollaire XIV.2.4.4] (for example, X is of finite type over $\mathbb{Z}$ or over a field) and we fix the dimension function δ on X defined by $\delta(x) = \dim(\overline{\{x\}})$. As explained in 2.1, it determines a dualizing complex K_X and a duality functor D_X. We define two full subcategories ${}^p\mathrm{D}^{\leq 0}$ and ${}^p\mathrm{D}^{\geq 0}$ of $\mathrm{D}^b_c(X, E)$ by the following formulas:

$$
\begin{aligned}
{}^p\mathrm{D}^{\leq 0} &= \{K \in \mathrm{Ob}\,\mathrm{D}^b_c(X, E) \mid \forall x \in X,\ \forall i > -\delta(x),\ \mathrm{H}^i(i_x^* K) = 0\},\\
{}^p\mathrm{D}^{\geq 0} &= \{K \in \mathrm{Ob}\,\mathrm{D}^b_c(X, E) \mid \forall x \in X, \forall i < -\delta(x), \mathrm{H}^i(i_x^! K) = 0\},
\end{aligned}
$$

[1] Technically, Ekedahl only defines the category $\mathrm{D}^b_c(X, \mathcal{O}_E)$ for a finite extension E of $\mathbb{Q}_\ell$, so to be precise, we should say that the category $\mathrm{D}^b_c(X, E)$ of Bhatt–Scholze is canonically equivalent to the inverse 2-limit over all finite subextensions of E of the tensor product over $\mathcal{O}_E$ of E and of the category of constructible $\mathcal{O}_E$-complexes defined by Ekedahl.

where, for every point x of X (not necessarily closed), we denote the inclusion $x \to X$ by i_x. This is a t-structure on $\mathrm{D}^b_c(X, E)$ for the same reasons as in [6, Sections 2.2.9–2.2.19]. We consider couples $(\mathcal{S}, \mathcal{L})$, where where $\mathcal{S}$ is a finite stratification of X by locally closed connected regular subschemes, and $\mathcal{L}$ is the data, for each stratum Z of $\mathcal{S}$, of a finite set $\mathcal{L}(Z)$ of lisse sheaves on Z, such that condition (c) of [6, Section 2.2.9] is satisfied. We denote by $\mathrm{D}_{(\mathcal{S},\mathcal{L})}(X, E)$ the full subcategory of $\mathrm{D}^b_c(X, E)$ whose objects are the complexes K such that, for each stratum Z of $\mathcal{S}$ and each $i \in \mathbb{Z}$, $\mathrm{H}^i K_{|Z}$ is isomorphic to an element of $\mathcal{L}(Z)$. The categories $({}^p\mathrm{D}^{\leq 0}, {}^p\mathrm{D}^{\geq 0})$ induces a t-structure on $\mathrm{D}_{(\mathcal{S},\mathcal{L})}(X, E)$ by gluing, as in [6, Section 1.4]. Then we note that the category $\mathrm{D}^b_c(X, E)$ is the filtered inductive limit of its subcategories $\mathrm{D}_{(\mathcal{S},\mathcal{L})}(X, E)$, and that the t-structures are compatible thanks to the purity theorem ([12] or [17, Theorem XVI.3.1.1]).

We will call the t-structure $({}^p\mathrm{D}^{\leq 0}, {}^p\mathrm{D}^{\geq 0})$ the *perverse t-structure* on $\mathrm{D}^b_c(X, E)$, and denote its heart by $\mathrm{Perv}(X, E)$. This is the category of *perverse sheaves* on X (with coefficients in E). We denote the associated cohomology functor by

$$ {}^p\mathrm{H}^i \colon \mathrm{D}^b_c(X, E) \to \mathrm{Perv}(X, E). $$

Let us list the exactness properties of the (derived) sheaf operations for this t-structure.

Suppose that we have a flat morphism of finite type $X \to S$. The following proposition is an immediate consequence of the definitions.

Proposition 2.2.1. *Let $u \colon T \to S$ be an étale map (resp. the inclusion of the generic point of S), and consider the functor $u^* \colon \mathrm{D}^b_c(X, E) \to \mathrm{D}^b_c(X \times_S T, E)$.*

Then u^* (resp. $u^*[-\dim S]$) is t-exact.

Then we recall the properties proved in [6, Sections 4.1 and 4.2].

Proposition 2.2.2. *Let $f \colon X \to Y$ be a finite type morphism. Then:*

(i) *The functors D_X and D_Y are t-exact.*

(ii) *If f is affine, then f_* is right t-exact and $f_!$ is left t-exact.*

(iii) *If the dimension of the fibers of f is $\leq d$, then f_* (resp. $f_!$, resp. f^*, resp. $f^!$) is of perverse cohomological amplitude $\geq -d$ (resp. $\leq d$, resp. $\leq d$, resp. $\geq -d$).*

(iv) *If f is quasi-finite and affine, then f_* and $f_!$ are t-exact.*

(v) *If f is smooth of relative dimension d, then $f^! \simeq f^*[2d](d)$, and $f^*[d]$ and $f^![-d]$ are t-exact. In particular, if f is étale, then $f^* = f^!$ is t-exact.*

(vi) *The external tensor product $\boxtimes \colon \mathrm{D}^b_c(X, E) \times \mathrm{D}^b_c(Y, E) \to \mathrm{D}^b_c(X \times Y, E)$ is t-exact.*

(vii) *The Tate twist functor $K \mapsto K(1)$ is t-exact.*

Remember that the external tensor product of $K \in \mathrm{Ob}\,\mathrm{D}^b_c(X, E)$ and $L \in \mathrm{D}^b_c(Y, E)$ is the object $K \boxtimes L$ of $\mathrm{D}^b_c(X \times Y, E)$ defined by

$$ K \boxtimes L = (\mathrm{pr}_X^* K) \otimes (\mathrm{pr}_Y^* L), $$

where $\mathrm{pr}_X \colon X \times Y \to X$ and $\mathrm{pr}_Y \colon X \times Y \to Y$ are the two projections (see SGA 5 III 1.5, where $K \boxtimes L$ is denoted $K \otimes_{\mathrm{Spec}\, k} L$).

Proof. For (i), note that, by [10, Theorem 6.3 (iii)], for all $K, L \in \operatorname{Ob} \mathrm{D}^b_c(X, E)$ and every $x \in X$, we have a canonical isomorphism

$$i_x^! \, \underline{\mathrm{Hom}}_X(K, L) \simeq \underline{\mathrm{Hom}}_x(i_x^* K, i_x^! L).$$

Applying this to $L = K_X$ and using the isomorphisms $i_x^! K_X \simeq E(\delta(x))[2\delta(x)]$ that are part of the definition of a potential dualizing complex (see [17, Définition XVII.2.1.2]), we see that $K \in \operatorname{Ob}(Dp^{\leq 0})$ if and only if $D_X(K) \in \operatorname{Ob}({}^p\mathrm{D}^{\geq 0})$. As $D_X^2 \simeq \mathrm{id}_{\mathrm{D}^b_c(X,E)}$, this also implies that $K \in \operatorname{Ob}({}^p\mathrm{D}^{\geq 0})$ if and only if $D_X(K) \in \operatorname{Ob}({}^p\mathrm{D}^{\leq 0})$, and we are done.

Point (ii) is proved exactly as [6, Théorème 4.1.1], as soon as we have the analogue of Artin's vanishing theorem, which is proved in [17, Exposé XV]. Point (iii) is proved exactly as [6, Section 4.2.4], and (iv) follows from (ii) and (iii). To prove point (v), it suffices to prove the isomorphism $f^! \simeq f^*[2d](d)$; but this is SGA 4 XVIII 3.2.5. Point (vi) is proved as in [6, Proposition 4.2.8]. Finally, point(vii) follows from (vi), because $K(1)$ is the exterior tensor product of the complex K on X and of the perverse sheaf $\mathbb{Q}_\ell(1)$ on $\operatorname{Spec} k$. ∎

We define the intermediate extension functor as in [6]: if $j : U \to X$ is a locally closed immersion and K is an object of $\operatorname{Perv}(U, E)$, then

$$j_{!*}K = \Im({}^p\mathrm{H}^0 j_! K \to {}^p\mathrm{H}^0 j_* K).$$

The methods of [6, Section 4.3] adapt immediately to our case and give the following result.

Theorem 2.2.3. *The category* $\operatorname{Perv}(X, E)$ *is Artinian and Noetherian, that is, all its objects have finite length. Moreover, the simple objects are of the form* $j_{!*}L[d]$*, where* $j : Z \to X$ *is a locally closed immersion, the subscheme* Z *is connected regular of dimension* d*, and* L *is a lisse sheaf on* Z *corresponding to an irreducible representation of* $\pi_1^{\text{ét}}(Z)$ *(we call such a* L *a* simple *lisse sheaf).*

2.3. Perverse t-structures and the realization functor

Fix schemes X, Y as in Section 2.2. By Theorem A.2.3 and Section A.4, we have a triangulated *realization functor* $\mathrm{real} : \mathrm{D}^b \operatorname{Perv}(X, E) \to \mathrm{D}^b_c(X, E)$ extending the inclusion $\operatorname{Perv}(X, E) \subset \mathrm{D}^b_c(X, E)$, and similarly for $\mathrm{D}^b_c(Y, E)$.

The proof of the following proposition is explained in Section A.4.

Proposition 2.3.1. *Let* $f : X \to Y$ *be a finite type morphism.*

(i) *If* f *is quasi-finite and affine, then we have commutative diagrams (up to natural isomorphism)*

$$\begin{array}{ccc} \mathrm{D}^b(\operatorname{Perv}(X,E)) & \xrightarrow{f_*} & \mathrm{D}^b(\operatorname{Perv}(Y,E)) \\ \downarrow{\scriptstyle \mathrm{real}} & & \downarrow{\scriptstyle \mathrm{real}} \\ \mathrm{D}^b_c(X,E) & \xrightarrow[f_*]{} & \mathrm{D}^b_c(Y,E), \end{array} \qquad \begin{array}{ccc} \mathrm{D}^b(\operatorname{Perv}(X,E)) & \xrightarrow{f_!} & \mathrm{D}^b(\operatorname{Perv}(Y,E)) \\ \downarrow{\scriptstyle \mathrm{real}} & & \downarrow{\scriptstyle \mathrm{real}} \\ \mathrm{D}^b_c(X,E) & \xrightarrow[f_!]{} & \mathrm{D}^b_c(Y,E). \end{array}$$

(ii) *If f is smooth and of relative dimension d, then we have a commutative diagram (up to natural isomorphism)*

$$\begin{array}{ccc} \mathrm{D}^b\big(\mathrm{Perv}(Y,E)\big) & \xrightarrow{f^*[d]} & \mathrm{D}^b\big(\mathrm{Perv}(X,E)\big) \\ \Big\downarrow{\scriptstyle \mathrm{real}} & & \Big\downarrow{\scriptstyle \mathrm{real}} \\ \mathrm{D}^b_c(Y,E) & \xrightarrow[f^*[d]]{} & \mathrm{D}^b_c(X,E). \end{array}$$

(iii) *We have a commutative diagram (up to natural isomorphism)*

$$\begin{array}{ccc} \mathrm{D}^b\big(\mathrm{Perv}(X,E)\big)^{\mathrm{op}} & \xrightarrow{D_X} & \mathrm{D}^b\big(\mathrm{Perv}(X,E)\big) \\ \Big\downarrow{\scriptstyle \mathrm{real}} & & \Big\downarrow{\scriptstyle \mathrm{real}} \\ \mathrm{D}^b_c(X,E)^{\mathrm{op}} & \xrightarrow[D_X]{} & \mathrm{D}^b_c(X,E). \end{array}$$

We need one last compatibility. Suppose that we have a flat morphism of finite type $X \to S$. If $T \to S$ is étale (resp. the inclusion of the generic point of S), $u\colon X_T \to X$ is its base change to X and $u^*\colon \mathrm{D}^b_c(X,E) \to \mathrm{D}^b_c(X_T,E)$ is the restriction functor, then u^* (resp. $u^*[-\dim S]$) is t-exact by Proposition 2.2.1. The following result is proved exactly as Proposition 2.3.1 (see Section A.4).

Proposition 2.3.2. *In the situation above, we have a commutative diagram (up to natural isomorphism)*

$$\begin{array}{ccc} \mathrm{D}^b\big(\mathrm{Perv}(X,E)\big) & \xrightarrow{u^*[a]} & \mathrm{D}^b\big(\mathrm{Perv}(X_T,E)\big) \\ \Big\downarrow{\scriptstyle \mathrm{real}} & & \Big\downarrow{\scriptstyle \mathrm{real}} \\ \mathrm{D}^b_c(X,E) & \xrightarrow[u^*[a]]{} & \mathrm{D}^b_c(X_T,E), \end{array}$$

where $a = 0$ is $T \to S$ is étale and $a = -\dim S$ if $T \to S$ is the inclusion of the generic point of S.

2.4. Horizontal constructible complexes

From now on, we fix a field k of finite type over its prime field (in other words, k is the field of fractions of an integral scheme of finite type over $\mathbb{Z}$) and an algebraic extension E of $\mathbb{Q}_\ell$. We will consider separated schemes of finite type over k and denote by them by capital Roman letters such as X, Y, U etc.

We will recall some constructions and results of [14, Sections 1–3]. In this article, Huber assumes that k is a number field, but, as she notes herself in the remark after proposition 2.3, this is not really necessary and all her constructions extend to the more general situation considered here, either by Deligne's generic constructibility theorem (SGA 4 1/2 [Th. finitude]) or by Gabber's finiteness results [17].

Let $\mathcal{U}$ be the set (ordered by inclusion) of $\mathbb{Z}$-subalgebras $A \subset k$ that are regular and of finite type over $\mathbb{Z}$ and such that k is the field of fractions of A. By a theorem of Nagata (see EGA IV 6.12.6), if $\mathcal{B}$ is an integral scheme of finite type over $\mathbb{Z}$, then the regular locus of $\mathcal{B}$ is open in $\mathcal{B}$. Hence $k = \varinjlim_{A \in \mathcal{U}} A$. So we are in the situation of EGA IV 8 and can use the results of this reference.

If $A \in \mathcal{U}$, we say that a scheme over Spec A is *horizontal* if it is flat and of finite type over A. Let X be a scheme over k. We denote by $\mathcal{U}X$ the category of triples $(A, \mathcal{X}, u)$, where $A \in \mathcal{U}$, $\mathcal{X}$ is a horizontal scheme over A and u is an isomorphism of k-schemes $X \xrightarrow{\sim} \mathcal{X} \otimes_A k$; we will often omit u from the notation. A morphism $(A, \mathcal{X}, u) \to (A', \mathcal{X}', u')$ is an inclusion $A \subset A'$ and an open embedding $f\colon \mathcal{X}' \to \mathcal{X} \otimes_A A'$ such that $u' = u \circ f$. Then we have a canonical isomorphism (given by the entry u of the triples)

$$X \xrightarrow{\sim} \varinjlim_{(A,\mathcal{X}) \in \operatorname{Ob} \mathcal{U}X} \mathcal{X} \otimes_A k.$$

If $f\colon (A, \mathcal{X}) \to (A', \mathcal{X}')$ is a morphism in $\mathcal{U}X$, then it induces an exact functor

$$\mathrm{D}^b_c(\mathcal{X}, E) \to \mathrm{D}^b_c(\mathcal{X} \otimes_A A', E) \xrightarrow{f^*} \mathrm{D}^b_c(\mathcal{X}', E),$$

where the first functor is the restriction functor along the open embedding $\mathcal{X} \otimes_A A' \to \mathcal{X}$.

Definition 2.4.1. (See [14, Definition 1.2]). Let X be a scheme over k. We define the category $\mathrm{D}^b_h(X, E)$ by

$$\mathrm{D}^b_h(X, E) = 2- \varinjlim_{(A,\mathcal{X}) \in \operatorname{Ob} \mathcal{U}X} \mathrm{D}^b_c(\mathcal{X}, E).$$

We call this category the *category of bounded constructible horizontal E-complexes of sheaves on X.*

Note that we could also define versions of these categories with coefficients in $\mathcal{O}_E$ (if E is a finite extension of $\mathbb{Q}_\ell$), as Huber does. But we will only be interested in this article in the category $\mathrm{D}^b_h(X, E)$.

As in the remark following [14, Definition 1.2], we see that these categories are triangulated and have a tautological t-structure (induced by the tautological t-structure on the categories $\mathrm{D}^b_c(\mathcal{X}, E)$), and that all the properties of [10, Theorem 6.3] carry over. The heart of the canonical t-structure will be denoted by $\mathrm{Sh}_h(X, E)$, and we will call its objects *horizontal constructible sheaves* on X.

We denote by $\eta^*\colon \mathrm{D}^b_h(X, E) \to \mathrm{D}^b_c(X, E)$ the exact functor induced by the restriction functors

$$\mathrm{D}^b_c(\mathcal{X}, E) \to \mathrm{D}^b_c(\mathcal{X} \otimes_A k, E) \xrightarrow{u^*} \mathrm{D}^b_c(X, E), \quad \text{for } (A, \mathcal{X}, u) \in \operatorname{Ob} \mathcal{U}X.$$

Proposition 2.4.2. *The following hold:*

(i) *The functor η^* is fully faithful on the heart of the tautological t-structure.*

(ii) *If* $\mathscr{F}, \mathscr{G} \in \mathrm{Ob}(\mathrm{Sh}_h(X, E))$, *then*

$$\eta^*\colon \mathrm{Ext}^1_{\mathrm{D}^b_h(X,E)}(\mathscr{F}, \mathscr{G}) \to \mathrm{Ext}^1_{\mathrm{D}^b_c(X,E)}(\eta^*\mathscr{F}, \eta^*\mathscr{G})$$

is injective.

Proof. The first point is [14, Proposition 1.3]. We prove the second point. Let $(A, \mathfrak{X}, u) \in \mathrm{Ob}(\mathcal{U}X)$ such that $\mathscr{F}$ and $\mathscr{G}$ come from objects K and L of $\mathrm{D}^b_c(\mathfrak{X}, E)$. We use u to identify X and $\mathfrak{X} \otimes_A k$. The constructible sheaves $\bigoplus_{i\neq 0} \mathrm{H}^i K$ and $\bigoplus_{i\neq 0} \mathrm{H}^i L$ on $\mathfrak{X}$ are supported on a closed subset disjoint from X, so, after shrinking $\mathfrak{X}$, we may assume that K and L are constructible sheaves on $\mathfrak{X}$. By definition of $\mathrm{D}^b_h(X, E)$, we have

$$\mathrm{Ext}^1_{\mathrm{D}^b_h(X,E)}(\mathscr{F}, \mathscr{G}) = \varinjlim_{A\subset A'\in\mathcal{U}} \mathrm{Ext}^1_{\mathrm{D}^b_c(\mathfrak{X}\otimes_A A',E)}(K_{|\mathfrak{X}\otimes_A A'}, L_{|\mathfrak{X}\otimes_A A'}).$$

Let $A' \supset A$ be an element of $\mathcal{U}$. The category $\mathrm{D}^b_c(\mathfrak{X} \otimes_A A', E)$ is a full subcategory of $\mathrm{D}((\mathfrak{X} \otimes_A A')_{\text{proét}}, E)$, the derived category of the category of E-modules on the proétale site of $\mathfrak{X} \otimes_A A'$, so the groups $\mathrm{Ext}^1_{\mathrm{D}^b_c(\mathfrak{X}\otimes_A A',E)}$ parametrize extensions in this category of E-modules (see Section 3.2 of Chapter III of Verdier's book [30]). But $\mathrm{Sh}_c(\mathfrak{X} \otimes_A A', E)$ is a Serre subcategory of the category of all sheaves of E-modules (see [7, Proposition 6.8.11]), so $\mathrm{Ext}^1_{\mathrm{D}^b_c(\mathfrak{X}\otimes_A A',E)}(K_{|\mathfrak{X}\otimes_A A'}, L_{|\mathfrak{X}\otimes_A A'})$ is the group of equivalence classes of extensions of $K_{|\mathfrak{X}\otimes_A A'}$ by $L_{|\mathfrak{X}\otimes_A A'}$ in $\mathrm{Sh}_c(\mathfrak{X} \otimes_A A', E)$. We have a similar statement for $\mathrm{Ext}^1_{\mathrm{D}^b_c(X,E)}(\eta^*\mathscr{F}, \eta^*\mathscr{G})$.

Now let $c \in \mathrm{Ext}^1_{\mathrm{D}^b_h(X,E)}(\mathscr{F}, \mathscr{G})$, and suppose that its image in $\mathrm{Ext}^1_{\mathrm{D}^b_c(X,E)}(\eta^*\mathscr{F}, \eta^*\mathscr{G})$ is 0. There exists an element $A' \supset A$ of $\mathcal{U}$ and an extension

$$0 \to L_{|\mathfrak{X}\otimes_A A'} \to M \to K_{|\mathfrak{X}\otimes_A A'} \to 0$$

in $\mathrm{Sh}_c(\mathfrak{X} \otimes_A A', E)$ whose class is c. The hypothesis on c says that the restriction of this extension to X is split. But, by point (i), this implies that there exists an element $A'' \supset A'$ of $\mathcal{U}$ such that the restriction of the extension to $\mathfrak{X} \otimes_A A''$ already splits, which means that $c = 0$. ■

2.5. Horizontal perverse sheaves

In this section, we define the perverse t-structure on $\mathrm{D}^b_h(X, E)$. Note that, by Proposition 1.4 of Giral's article [13], all the rings A in $\mathcal{U}$ have the same Krull dimension c, which is the transcendence degree of k over its prime field if k is of positive characteristic, and 1 plus this transcendence degree if k is of characteristic 0.

If $(A, \mathfrak{X}, u) \in \mathrm{Ob}\,\mathcal{U}X$, then we consider the perverse t-structure on $\mathrm{D}^b_c(\mathfrak{X}, E)$ defined in Section 2.2. Then the functor $u^*[-c]\colon \mathrm{D}^b_c(\mathfrak{X}, E) \to \mathrm{D}^b_c(X, E)$ is t-exact by Proposition 2.2.1. Also, for every morphism $f\colon (A, \mathfrak{X}, u) \to (A', \mathfrak{X}', u')$ in $\mathcal{U}X$, the restriction functor $f^*\colon \mathrm{D}^b_c(\mathfrak{X}', E) \to \mathrm{D}^b_c(\mathfrak{X}, E)$ is t-exact by the same proposition. By taking the limit of the shift by c of the t-structures on the $\mathrm{D}^b_c(\mathfrak{X}, E)$, we get a t-structure on $\mathrm{D}^b_h(X, E)$

such that $\eta^*\colon \mathrm{D}^b_h(X, E) \to \mathrm{D}^b_c(X, E)$ is t-exact. We denote the heart of this t-structure by $\mathrm{Perv}_h(X, E)$ and call it the category of *horizontal perverse sheaves* on X. We still denote the perverse cohomology functors by ${}^p\mathrm{H}^i\colon \mathrm{D}^b_h(X, E) \to \mathrm{Perv}_h(X, E)$. We also have a realization functor real: $\mathrm{D}^b\,\mathrm{Perv}_h(X, E) \to \mathrm{D}^b_h(X, E)$ extending the inclusion $\mathrm{Perv}_h(X, E) \subset \mathrm{D}^b_h(X, E)$, by Theorem A.2.3 and Section A.4.

Remark 2.5.1. This is not Huber's construction. Let us recall her construction and compare it with ours. Let $A \in \mathcal{U}$ and let $\mathcal{X}$ be a horizontal scheme over A. As in [14, Lemma 2.1], we say that a stratification of $\mathcal{X}$ is *horizontal* if all its strata are smooth over A. Suppose that $E/\mathbb{Q}_\ell$ is finite. If S is a horizontal stratification of $\mathcal{X}$ and L is the data of a set of irreducible lisse $\mathcal{O}_E$-sheaves on every stratum of S satisfying condition (c) of [14, Definition 2.2], we get as in [14, Definition 2.2 and Lemma 2.4] a full subcategory $D^b_{(S,L)}(\mathcal{X}, \mathcal{O}_E)$ of (S, L)-constructible objects in $\mathrm{D}^b_c(\mathcal{X}, \mathcal{O}_E)$, and it has a self-dual perverse t-structure, whose heart we will denote by $\mathrm{Perv}_{(S,L)}(\mathcal{X}, \mathcal{O}_E)$. Because the strata S are smooth over Spec A, lisse sheaves on them are perverse for our t-structure on $\mathrm{D}^b_c(\mathcal{X}, \mathcal{O}_E)$ when placed in degree $-\dim(A) = -c$, so Huber's t-structure is the shift by c of the one induced by our perverse t-structure on $\mathrm{D}^b_c(\mathcal{X}, \mathcal{O}_E)$.

By [14, Proposition 2.3], the category $\mathrm{D}^b_h(X, \mathcal{O}_E)$ is the 2-colimit of the categories $D^b_{(S,L)}(\mathcal{X}, \mathcal{O}_E)$ over all $(A, \mathcal{X}) \in \mathrm{Ob}\,\mathcal{U}X$ and couples (S, L) as before, and by [14, Theorem 2.5], the perverse t-structure goes to the limit and induces a t-structure on $\mathrm{D}^b_h(X, \mathcal{O}_E)$. This is the t-structure that is used in [14], and, by the observation of the previous paragraph, it coincides with the one that we defined at the beginning of this section.

Huber's definition has the advantage that we can apply Deligne's generic base change theorem (from SGA 4 1/2 [Th. finitude]) to deduce statements for horizontal perverse sheaves from statements for perverse sheaves on schemes over finite fields proved in [6, Sections 4 and 5].

For example, we see as in [14, Theorem 2.7] that the six operations have the usual exactness properties with respect to the perverse t-structure (which means the properties of [6, Sections 4.1 and 4.2]), that the category $\mathrm{Perv}_h(X, E)$ is Artinian and Noetherian, and that we have the same description of its simple objects as in [6, Theorem 4.3.1].[2]

The following result is a slight generalization of the first part of [14, Lemma 2.12].

Proposition 2.5.2. *The following hold:*

(i) *The functor* $\eta^*\colon \mathrm{Perv}_h(X, E) \to \mathrm{Perv}(X, E)$ *is fully faithful, and its essential image is the full category of perverse sheaves on* X *that extend to a constructible complex on some* $\mathcal{X}$, *for* $(A, \mathcal{X}) \in \mathrm{Ob}\,\mathcal{U}X$.

(ii) *For every* $K, L \in \mathrm{Ob}\,\mathrm{Perv}(X, E)$, *the morphism*

$$\mathrm{Ext}^1_{\mathrm{D}^b_h(X,E)}(K, L) \to \mathrm{Ext}^1_{\mathrm{D}^b_c(X,E)}(\eta^* K, \eta^* L)$$

induced by η *is injective.*

[2] But we could also have proved all these statements from our definition.

Proof. If the category where we take the Ext^i is clear from context, we omit it in this proof. Also, we omit the coefficients E in the notation.

We prove both points by Noetherian induction on X. If $\dim X = 0$, then the perverse t-structure on $\mathrm{D}^b_c(X, E)$ is the usual t-structure, so both points follow from Proposition 2.4.2 (which is an easy consequence of [14, Proposition 1.3]).

Suppose that $\dim X > 0$, and let $K, L \in \mathrm{Ob}\,\mathrm{Perv}_h(X)$. [14, Lemma 2.12] says that the map $\mathrm{Hom}(K, L) \to \mathrm{Hom}(\eta^* K, \eta^* L)$ is injective, and we want to show that it is also surjective. We show this by induction on the sum of the lengths of K and L.

Suppose first that K and L are both simple. Then we have smooth connected locally closed subschemes $k_1\colon Y_1 \to X$ and $k_2\colon Y_2 \to X$ and horizontal locally constant sheaves $\mathscr{L}_1$ on Y_1 and $\mathscr{L}_2$ on Y_2 such that $K = k_{1!*}\mathscr{L}_1[\dim Y_1]$ and $L = k_{2!*}\mathscr{L}_2[\dim Y_2]$. We have $\mathrm{Hom}(K, L) = 0$ if $K \not\simeq L$, and $\mathrm{Hom}(K, K) = \mathrm{Hom}_{\mathrm{D}^b_h(X)}(\mathscr{L}_1, \mathscr{L}_1)$. In particular, by [14, Proposition 1.3], $\eta^* K$ and $\eta^* L$ are also simple, and $\mathrm{Hom}(K, L) \xrightarrow{\sim} \mathrm{Hom}(\eta^* K, \eta^* L)$, proving the first point.

We prove the second point. Let $Z = \overline{Y}_1 \cap \overline{Y}_2$, and denote by $i : Z \to X$ and $j : X - Z \to X$ the inclusions. We have an exact triangle

$$R\,\mathrm{Hom}(i^* K, i^! L) \to R\,\mathrm{Hom}(K, L) \to R\,\mathrm{Hom}(j^* K, j^* L) \xrightarrow{+1} .$$

As $j^* K$ and $j^* L$ are perverse with disjoint supports on $X - Z$, $R\,\mathrm{Hom}(j^* K, j^* L) = 0$, so we get isomorphisms $\mathrm{Ext}^i(i^* K, i^! L) \xrightarrow{\sim} \mathrm{Ext}^i(K, L)$ for every $i \in \mathbb{Z}$. We have a similar result for $\eta^* K$ and $\eta^* L$.

If $\overline{Y}_1$ is not contained is $\overline{Y}_2$, then Z is a proper closed subset of $\overline{Y}_1$, so $i^* K$ and $i^*\eta^* K$ are concentrated in perverse degree ≤ -1. As $i^! L$ and $i^!\eta^* L$ are concentrated in perverse degree ≥ 0, we get

$$\mathrm{Ext}^1(K, L) = \mathrm{Hom}({}^p\mathrm{H}^{-1} i^* K, {}^p\mathrm{H}^0 i^! L),$$
$$\mathrm{Ext}^1(\eta^* K, \eta^* L) = \mathrm{Hom}({}^p\mathrm{H}^{-1} i^* \eta^* K, {}^p\mathrm{H}^0 i^! \eta^* L),$$

so the second point follows from the induction hypothesis applied to Z.

If $\overline{Y}_2$ is not contained is $\overline{Y}_1$, then Z is a proper closed subset of $\overline{Y}_2$, so $i^! L$ and $i^!\eta^* L$ are concentrated in perverse degree ≥ 1. As $i^* K$ and $i^*\eta^* K$ are concentrated in perverse degree ≤ 0, we get

$$\mathrm{Ext}^1(K, L) = \mathrm{Hom}({}^p\mathrm{H}^0 i^* K, {}^p\mathrm{H}^1 i^! L),$$
$$\mathrm{Ext}^1(\eta^* K, \eta^* L) = \mathrm{Hom}({}^p\mathrm{H}^0 i^* \eta^* K, {}^p\mathrm{H}^1 i^! \eta^* L),$$

so the second point again follows from the induction hypothesis applied to Z.

Finally, suppose that $\overline{Y}_1 = \overline{Y}_2$. Then $i^* K$ and $i^! L$ are perverse and simple, and we may assume that $Y_1 = Y_2$. Let b be the inclusion of the open subscheme Y_1 of Z, and a be the inclusion of its complement. As before, we have an exact triangle

$$R\,\mathrm{Hom}(a^* i^* K, a^! i^! L) \to R\,\mathrm{Hom}(i^* K, i^! L) \to R\,\mathrm{Hom}(b^* i^* K, b^* i^! L) \xrightarrow{+1} .$$

As i^*K and $i^!L$ are simple of support Z, we know that a^*i^*K is concentrated in perverse degree ≤ -1 and that $a^!i^!L$ is concentrated in perverse degree ≥ 1, so we get an injective map

$$\mathrm{Ext}^1(i^*K, i^!L) \to \mathrm{Ext}^1(b^*i^*K, b^*i^!L) = \mathrm{Ext}^1_{\mathrm{D}^b_h(X)}(\mathcal{L}_1, \mathcal{L}_2).$$

We have a similar calculation for $i^*\eta^*K$ and $i^!\eta^*L$, and so the second point follows from Proposition 2.4.2. This finishes the proof in the case where K and L are both simple.

Now suppose that we have an exact sequence $0 \to K_1 \to K \to K_2 \to 0$, and that we know the result for the pairs (K_1, L) and (K_2, L). We show it for (K, L). Write $K' = \eta^*K$, $L' = \eta^*L$ etc. We have commutative diagrams with exact rows

$$\begin{array}{ccccccc}
0 \to \mathrm{Hom}(K_2, L) & \longrightarrow & \mathrm{Hom}(K, L) & \longrightarrow & \mathrm{Hom}(K_1, L) & \longrightarrow & \mathrm{Ext}^1(K_2, L) \\
\downarrow \wr & & \downarrow & & \downarrow \wr & & \downarrow \\
0 \to \mathrm{Hom}(K'_2, L') & \to & \mathrm{Hom}(K', L') & \to & \mathrm{Hom}(K'_1, L') & \to & \mathrm{Ext}^1(K'_2, L'),
\end{array}$$

$$\begin{array}{ccccccc}
\mathrm{Hom}(K_1, L) & \longrightarrow & \mathrm{Ext}^1(K_2, L) & \longrightarrow & \mathrm{Ext}^1(K, L) & \longrightarrow & \mathrm{Ext}^1(K_1, L) \\
\downarrow \wr & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(K'_1, L') & \to & \mathrm{Ext}^1(K'_2, L') & \to & \mathrm{Ext}^1(K', L') & \to & \mathrm{Ext}^1(K'_1, L')
\end{array}$$

so both points follow from the five lemma.

The case where we have an exact sequence $0 \to L_1 \to L \to L_2 \to 0$ such that the result is known for (K, L_1) and (K, L_2) is treated in the same way. ∎

2.6. Mixed perverse sheaves

The key observation in order to define weights in our setting, if $A \in \mathcal{U}$, then, as A is a $\mathbb{Z}$-algebra of finite type, the residue fields of closed points of Spec A are finite, so we can use the theory of [6, Chapter 5] as in [14, Section 3] to define categories $\mathrm{D}^b_m(X, E)$ of mixed horizontal complexes. Once we have defined what it means for a horizontal constructible sheaf to be punctually pure of a certain weight, the definition proceeds as in [6, Section 5.1.5]. If $\mathcal{F} \in \mathrm{Ob}\,\mathrm{Sh}_h(X, E)$ and $w \in \mathbb{Z}$, we say that $\mathcal{F}$ is *punctually pure of weight* w if there exists $(A, \mathcal{X}) \in \mathrm{Ob}\,\mathcal{U}X$ and $\mathcal{F}' \in \mathrm{Ob}\,\mathrm{Sh}_c(\mathcal{X}, E)$ a constructible sheaf extending $\mathcal{F}$ such that, for every closed point x of Spec A, $\mathcal{F}'_{|\mathcal{X}_x}$ is punctually pure of weight w in the sense of [6, Section 5.1.5] (that is, of Deligne's [9]). We say that $\mathcal{F}$ is *mixed* if it has a filtration whose graded pieces are punctually pure of some weight.

We denote by $\mathrm{D}^b_m(X, E)$ the full subcategory of *mixed complexes* in $\mathrm{D}^b_h(X, E)$; the objects of $\mathrm{D}^b_m(X, E)$ are complexes K such that all the (usual) cohomology sheaves of K are mixed.

By [14, Proposition 3.2], these subcategories are stable by the 6 operations and inherit a perverse t-structure from $\mathrm{D}^b_h(X, E)$. We denote the heart of this t-structure by $\mathrm{Perv}_m(X, E)$; it is a full subcategory of $\mathrm{Perv}_h(X, E)$, stable by subquotients and extensions. All the compatibilities between the six operations (and the intermediate extension functor) and weights that are proved in [6, Chapter 5] and [9] remain true, see [14, Definition 3.3 to Corollary 3.6]. Also, the realization functor real: $D^b \operatorname{Perv}_h(X, E) \to \mathrm{D}^b_h(X, E)$ restricts to a functor

$$\text{real}\colon D^b \operatorname{Perv}_m(X, E) \to \mathrm{D}^b_h(X, E),$$

whose essential image is contained in $\mathrm{D}^b_m(X, E)$ by definition of $\mathrm{D}^b_m(X, E)$.

Let us introduce weight filtrations, following [14, Definition 3.7].

Definition 2.6.1. Let $K \in \operatorname{Ob}\operatorname{Perv}_m(X, E)$. A *weight filtration* on K is a separated exhaustive ascending filtration W on K (in the abelian category $\operatorname{Perv}_m(X, E)$) such that $\operatorname{Gr}^W_k K$ is pure of weight k for every $k \in \mathbb{Z}$.

As the abelian category $\operatorname{Perv}_m(X, E)$ is Artinian and Noetherian, such a filtration is automatically finite. Note also that morphisms in $\operatorname{Perv}_m(X, E)$ are strictly compatible with weight filtrations [14, Lemma 3.8], so in particular a weight filtration is unique if it exists.

Definition 2.6.2. Let $\operatorname{Perv}_{mf}(X, E)$ be the full subcategory of $\operatorname{Perv}_m(X, E)$ whose objects are mixed horizontal perverse sheaves admitting a weight filtration.

This subcategory is clearly stable by subquotients in $\operatorname{Perv}_m(X, E)$, but it is not stable by extensions (even if $X = \operatorname{Spec} k$), see the warning before [14, Proposition 3.4].

Finally, the following conservativity result will be very useful.

Proposition 2.6.3. *The following hold:*

(i) *The functor* $\eta^*\colon \mathrm{D}^b_h(X, E) \to \mathrm{D}^b_c(X, E)$ *is conservative.*

(ii) *The realization functor* $\mathrm{D}^b \operatorname{Perv}_h(X, E) \to \mathrm{D}^b_h(X, E)$ *is conservative.*

(iii) *The obvious functor* $\mathrm{D}^b \operatorname{Perv}_{mf}(X, E) \to \mathrm{D}^b \operatorname{Perv}_h(X, E)$ *is conservative.*

Proof. In all three cases, we have t-structures on the source and target for which the functors are t-exact and such that the family of cohomology functors for the t-structure is conservative (the perverse t-structure on $\mathrm{D}^b_h(X, E)$ and $\mathrm{D}^b_c(X, E)$, and the canonical t-structure on the derived categories). So it suffices to check that the functors on the hearts are conservative. But these functors are all faithful and exact (in fact, they are all fully faithful), so they are conservative. ■

3. Main theorems

From now on, we will fix the algebraic extension E of $\mathbb{Q}_\ell$ and omit it in the notation.

3.1. Informal statement

Informally, the main theorems say that the sheaves operations (f_*, f^*, $f_!$ and $f^!$, $\underline{\operatorname{Hom}}$, $\otimes$, Poincaré–Verdier duality, unipotent nearby and vanishing cycles) lift to the categories

$\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$ in a way that is compatible with the realization functors $\mathrm{D}^b\,\mathrm{Perv}_{mf}(X) \to \mathrm{D}^b_m(X)$, and that all the relations between these functors that are true in the categories $\mathrm{D}^b_m(X)$ are still true in the categories $\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$.

A convenient way to say this is to use the formalism introduced in Ayoub's thesis [1] (and in his article [2]). Then Theorem 3.2.4 says that the four operations f_*, f^*, $f_!$ and $f^!$ exist and satisfy all the expected adjunctions and compatibilities, and Theorem 3.2.12 asserts the existence and properties of the derived internal Homs and derived tensor products. The stability of the categories Perv_{mf} under the perverse direct image functors is proved in Section 6.3, and the unipotent vanishing cycles are constructed in Section 5.2 (see Corollary 6.3.3).

3.2. Formal statement

We denote by $\mathbf{Sch}/k$ the category of schemes over k (always assumed to be separated of finite type, as before) and by $\mathfrak{TR}$ the 2-category of triangulated categories.

The notion of a formalism of the four operations $(f^*, f_*, f_!, f^!)$ has been axiomatized by Deligne, Voevodsky and Ayoub, under the name of "foncteur croisé".[3] We will follow Ayoub's presentation.

Definition 3.2.1. (See [1, Definition 1.2.12].)[4] A *crossed functor* ("foncteur croisé") on $\mathbf{Sch}/k$ with values in $\mathfrak{TR}$ (relatively to the class of cartesian squares) is a quadruple of 2-functors $H = (H^*, H_*, H_!, H^!)\colon \mathbf{Sch}/k \to \mathfrak{TR}$, such that:

(0) for every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, we have $H_*(X) = H_!(X) = H^*(X) = H^!(X)$ (we denote this triangulated category by $H(X)$);

(1) the functors H_*, $H_!$ are covariant, and the functors H^*, $H^!$ are contravariant;

(2) the functor H^* is a global left adjoint of H_*;

(3) the functor $H^!$ is a global right adjoint of $H_!$;

together with the data of exchange structures of type $\swarrow$ on the couples $(H_*, H_!)$ and $(H^*, H^!)$ (see [1, Definition 1.2.1]), i.e., for every cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[g]{} & Y \end{array}$$

in $\mathbf{Sch}/k$, we have morphisms of functors

$$H_!(f) \circ H_*(g') \to H_*(g) \circ H_!(f') \quad \text{and} \quad H^*(g') \circ H^!(f) \to H^!(f') \circ H^*(g)$$

compatible with horizontal and vertical composition of squares.

[3]There are other approaches, but this particular one seems better suited to our situation. For example, using derivators is complicated by the fact that it is difficult to make sense of the notion of "perverse sheaf over a diagram of schemes", because inverse image functors typically do not preserve perverse sheaves.

[4]Note that we take the two categories $\mathscr{C}_1$ and $\mathscr{C}_2$ of this reference to be equal to $\mathbf{Sch}/k$.

This data is moreover required to satisfy the following condition: For every cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[g]{} & Y \end{array}$$

in **Sch**$/k$, the morphisms $H^*(g) \circ H_!(f) \to H_!(f') \circ H^*(g')$ and $H_!(f') \circ H^*(g') \to H^*(g) \circ H_!(f)$ formally constructed using the exchange structures and adjunctions (see the beginning of [1, Section 1.2.4]) are isomorphisms and inverses of each other; equivalently, we could required that the morphisms $H^!(f) \circ H_*(g') \to H_*(g) \circ H^!(f')$ and $H_*(g) \circ H^!(f') \to H^!(f) \circ H_*(g')$ are isomorphisms and inverses of each other.

Definition 3.2.2. (See [2, Definition 3.1 and Theorem 3.4].) Suppose that we have two crossed functors $H_1, H_2 \colon \mathbf{Sch}/k \to \mathfrak{TR}$. A *morphism of crossed functors* $R \colon H_1 \to H_2$ is the following data:

(1) For every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, a triangulated functor $R_X \colon H_1(X) \to H_2(X)$.

(2) For every $f \colon X \to Y$ in **Sch**$/k$, invertible natural transformations

$$\theta_f \colon H_2^*(f) \circ R_Y \xrightarrow{\sim} R_X \circ H_1^*(f),$$
$$\gamma_f \colon R_Y \circ H_{1,*}(f) \xrightarrow{\sim} H_{2,*}(f) \circ R_X,$$
$$\rho_f \colon H_{2,!}(f) \circ R_X \xrightarrow{\sim} R_Y \circ H_{2,!}(f),$$
$$\xi_f \colon R_X \circ H_2^!(f) \xrightarrow{\sim} H_2^!(f) \circ R_Y.$$

We require these transformations to satisfy the compatibility conditions spelled out in Section 3 of Ayoub's paper [2].

Example 3.2.3. For $a \in \{c, h, m\}$, we have a crossed functor

$$H_a = (H_a^*, H_{a,*}, H_{a,!}, H_a^!) \colon \mathbf{Sch}/k \to \mathfrak{TR}$$

defined in the following way:

- For every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$,

$$H_a^*(X) = H_{a,*}(X) = H_{a,!} = (X) = H_a^!(X) = \mathrm{D}_a^b(X).$$

- For every $f \colon X \to Y$ in **Sch**$/k$, we have $H_a^*(f) = f^*$, $H_{a,*}(f) = f_*$, $H_{a,!}(f) = f_!$ and $H_a^!(f) = f^!$.

  Moreover, we have morphisms of crossed functors $H_m \to H_h \to H_c$.

Then our first main result is the following theorem.

Theorem 3.2.4. *There exists a crossed functor*

$$H_{mf} = (H_{mf}^*, H_{mf,*}, H_{mf,!}, H_{mf}^!) \colon \mathbf{Sch}/k \to \mathfrak{TR}$$

and a morphism of crossed functors $R\colon H_{mf} \to H_m$ *such that, for every* $X \in \mathrm{Ob}(\mathbf{Sch}/k)$*,* $H_{mf}(X) = \mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$ *and* $R_X\colon H_{mf}(X) = \mathrm{D}^b\,\mathrm{Perv}_{mf}(X) \to H_m(X) = \mathrm{D}^b_m(X)$ *is the composition of the obvious functor* $\mathrm{D}^b\,\mathrm{Perv}_{mf}(X) \to \mathrm{D}^b\,\mathrm{Perv}_m(X)$ *and of the realization functor of Section 2.5.*

Moreover, the functor R_X *is conservative for every* k*-scheme* X*, and we have for every morphism* f *in* $\mathbf{Sch}/k$ *a natural transformation* $H_{mf,!}(f) \to H_{mf,*}(f)$*, which is an isomorphism if* f *is proper.*

To prove this, we will follow the same strategy as in [1, Chapter 1] and [2, Section 3], and deduce the existence of the crossed functor and of the natural transformation $f_! \to f_*$ from that of a stable homotopic 2-functor (see Definition 3.2.5).

We note that the conservativity of R_X follows immediately from Proposition 2.6.3, and then the fact that $f_! \to f_*$ is an isomorphism for f proper follows from the conservativity of the functors R_X.

Definition 3.2.5. (See [1, Definition 1.4.1].) Let $H^*\colon \mathbf{Sch}/k \to \mathfrak{TR}$ be a contravariant 2-functor. For $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, we write $H^*(X) = H(X)$, and for f a morphism of $\mathbf{Sch}/k$, we also denote the 1-functor $H^*(f)$ by f^*. We assume that H^* is strictly unital, i.e., for every morphism $f\colon X \to Y$ in $\mathbf{Sch}/k$, the connection isomorphisms $(f \circ \mathrm{id}_X)^* \simeq f^*$ and $(\mathrm{id}_Y \circ f)^* \simeq f^*$ are the identity.

We say that H^* is a *stable homotopic* 2*-functor* if it satisfies the following conditions:

(1) $H(\varnothing) = 0$.

(2) For every $f\colon X \to Y$ in $\mathbf{Sch}/k$, the functor $f^*\colon H(Y) \to H(X)$ admits a right adjoint f_*. Moreover, if f is a locally closed immersion, then the counit $f^* f_* \to \mathrm{id}_{H(X)}$ is an isomorphism.

(3) If $f\colon X \to Y$ is a smooth morphism in $\mathbf{Sch}/k$, then the functor f^* admits a left adjoint $f_\sharp$. Moreover, if we have a cartesian square:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[g]{} & Y \end{array}$$

with f smooth, then the exchange morphism $f'_\sharp g'^* \to g^* f_\sharp$ (defined formally using the adjunctions, see [1, Section 1.4.5]) is an isomorphism.

(4) If $j\colon U \to X$ and $i\colon Z \to X$ are complementary open and closed immersions in $\mathbf{Sch}/k$, then the pair (j^*, i^*) is conservative.

(5) If $X \in \mathrm{Ob}(\mathbf{Sch}/R)$ and $p\colon \mathbb{A}^1_X \to X$ is the canonical projection, then the unit $\mathrm{id}_X \to p_* p^*$ is an isomorphism.

(6) With the notation of (5), if $s\colon X \to \mathbb{A}^1_X$ is the zero section, then $p_\sharp s_*\colon H(X) \to H(X)$ is an equivalence of categories.

Definition 3.2.6. (See [2, Definition 3.1].) Let $H_1^*, H_2^*: \mathbf{Sch}/k \to \mathfrak{TR}$ be two stable homotopic 2-functors. A *morphism of stable homotopic 2-functors* $R: H_1^* \to H_2^*$ is the data of:

(1) For every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, a triangulated functor $R_X: H_1(X) \to H_2(X)$.

(2) For every $f: X \to Y$ in $\mathbf{Sch}/k$, an invertible natural transformation

$$\theta_f: f^* \circ R_Y \xrightarrow{\sim} R_X \circ f^*.$$

We require that this data satisfy the following compatibility conditions:

(A) The natural transformations are compatible with the composition of morphisms in $\mathbf{Sch}/k$.

(B) If f is smooth, then the natural transformation $f_\sharp \circ R_X \to R_Y \circ f_\sharp$ (obtained using the adjunction and θ_f^{-1}) is invertible.

Example 3.2.7. The crossed functors of Example 3.2.3 define (by forgetting part of the data) three stable homotopic 2-functors H_m^*, H_h^* and H_c^*, and morphisms $H_m^* \to H_h^* \to H_c^*$.

Theorem 3.2.4 now follows immediately from the following two results (the first one is a consequence of several theorems of Ayoub and is also used to construct the four operations on the triangulated categories of Voevodsky motives, and the second one is the main technical result of this paper).

Theorem 3.2.8. *The following hold:*

(i) (See [1, Scholie 1.4.2].) *Let $H^*: \mathbf{Sch}/k \to \mathfrak{TR}$ be a stable homotopic 2-functor. Then H^* extends to a crossed functor $\mathbf{Sch}/k \to \mathfrak{TR}$.*

(ii) (See [2, Theorems 3.4 and 3.7].) *Let $H_1^*, H_2^*: \mathbf{Sch}/k \to \mathfrak{TR}$ be two stable homotopic 2-functors and $R: H_1^* \to H_2^*$ be a morphism. Let $H_1, H_2: \mathbf{Sch}/k \to \mathfrak{TR}$ be crossed functors extending H_1^*, H_2^* as in (i). Then R extends to a morphism of crossed functors from H_1 to H_2.*

Theorem 3.2.9. *There exists a stable homotopic 2-functor $H_{mf}^*: \mathbf{Sch}/k \to \mathfrak{TR}$ and a morphism of stable homotopic 2-functors $R: H_{mf}^* \to H_m^*$ such that, for every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, $H_{mf}(X) = \mathrm{D}^b \operatorname{Perv}_{mf}(X)$ and $R_X: H_{mf}(X) \to H_m(X)$ is the same functor as in Theorem 3.2.4.*

Proof. The construction of the 2-functor H_{mf}^* is given in Corollary 7.2.4. Let us check that it is a stable homotopic 2-functor. Property (1) is obvious. The fact that $H_{mf}^*(f)$ admits a right adjoint for every f follows from the definition of H_{mf}^* as a global left adjoint, and the last part of property (2) follows from Corollary 6.4.2 and Proposition 7.1.7. The fact that f^* admits a left adjoint for f smooth is proved in Proposition 7.3.2 (iv), and the last part of property (3) as well as properties (4) and (5) follow from the conservativity of the realization functor. Finally, let Y be a k-scheme, let $p: \mathbb{A}_Y^1 \to Y$ be the canonical projection and $s: Y \to \mathbb{A}_Y^1$ be the zero section. By Proposition 7.3.2 (v), we have a natural isomorphism $s_! \xrightarrow{\sim} s_*$. So we get a natural isomorphism

$$p_\sharp s_* = p_![2](1)s_* \xrightarrow{\sim} p_! s_![2](1) \simeq (ps)_![2](1) = \mathrm{id}[2](1),$$

which shows that $p_\sharp s_* \colon \mathrm{D}^b\mathrm{Perv}_{mf}(Y) \to \mathrm{D}^b\mathrm{Perv}_{mf}(Y)$ is an equivalence of categories. ■

Finally, we show the existence of tensor products and internal Homs on the categories $\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$.

Definition 3.2.10. We make the following definitions:

(i) (See [1, Definition 2.3.1].) A *unitary symmetric monoïdal* stable homotopic 2-functor is a stable homotopic 2-functor H^* that takes its values in the 2-category of symmetric monoïdal unitary triangulated categories, that is, that associates to every $X \in \mathrm{Ob}\,\mathbf{Sch}/k$ a unitary symmetric monoïdal category $(H(X), \otimes_X, \mathbf{1}_X)$ and such that:

  (a) For every morphism $f\colon X \to Y$ in $\mathbf{Sch}/k$, the functor f^* is unitary monoïdal.

  (b) (Projection formula.) If $f\colon X \to Y$ is smooth, $K \in \mathrm{Ob}\,H(Y)$ and $L \in \mathrm{Ob}\,H(X)$, then the functorial map

$$p\colon f_\sharp\big(f^*(K) \otimes_Y L\big) \to K \otimes_X f_\sharp(L)$$

  constructed in [1, Proposition 2.1.97] is an isomorphism.

(ii) (See [2, Definition 3.2].) Let H_1^* and H_2^* be two symmetric monoïdal unitary stable homotopic 2-functors. Then a morphism of symmetric monoïdal unitary stable homotopic 2-functors from H_1^* to H_2^* is a morphism of stable homotopic 2-functors $R\colon H_1^* \to H_2^*$ such that:

  (a) For every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, the functor R_X is monoïdal unitary.

  (b) For every morphism of k-schemes f, the natural transformation θ_f is a morphism of monoïdal unitary functors.

(iii) (See [1, Definition 2.3.50]). If H^* is as in (i), we say that H^* is *closed* if, for every $X \in \mathrm{Ob}\,\mathbf{Sch}/k$, the symmetric monoïdal category $(H(X), \otimes_X)$ is closed; this means that, for every object K of $H(X)$, the endofunctor $K \otimes_X \cdot$ of $H(X)$ admits a right adjoint, that will be denoted by $\underline{\mathrm{Hom}}_X(K, \cdot)$.

Example 3.2.11. The stable homotopic 2-functors H_m^*, H_h^* and H_c^* are all closed symmetric monoïdal unitary (for the derived tensor product), and the morphisms $H_m^* \to H_h^* \to H_c^*$ are morphisms of symmetric monoïdal unitary stable homotopic 2-functors.

Our last result is the following.

Theorem 3.2.12. *There exists a structure of closed symmetric monoïdal unitary stable homotopic* 2*-functor on* H_{mf}^* *such that* $R\colon H_{mf}^* \to H_m^*$ *is a morphism of symmetric monoïdal unitary stable homotopic* 2*-functors.*

Moreover, for every k-scheme X, the functorial map

$$R_X \, \underline{\mathrm{Hom}}_{\mathrm{D}^b \, \mathrm{Perv}_{mf}(X)}(\cdot,\cdot) \to \underline{\mathrm{Hom}}_{\mathrm{D}^b_m(X)} \left(R_X(\cdot), R_X(\cdot)\right)$$

of [2, (3.1)] *is an isomorphism.*

Proof. This theorem is proved in Section 8. More precisely, the bifunctors $\otimes_X$ and $\underline{\mathrm{Hom}}$ are constructed in Section 8, and all their properties are proved there except for condition (i)(b) of Definition 3.2.10. But this last condition follows from the fact that the functor R_X is conservative (and that the analogous result is true in $\mathrm{D}^b_m(X)$). ■

4. Easy stabilities

The proof of Theorem 3.2.9 will require us to show that the full subcategories $\mathrm{Perv}_{mf}(X) \subset \mathrm{Perv}_m(X)$ are preserved by a certain number of sheaf operations. Here we list the easier such results.

Proposition 4.1. *Let $f: X \to Y$ be a morphism of k-schemes.*

(i) *If f is smooth of relative dimension d, then the exact functor $f^*[d]: \mathrm{Perv}_m(Y) \to \mathrm{Perv}_m(X)$ sends $\mathrm{Perv}_{mf}(Y)$ to $\mathrm{Perv}_{mf}(X)$.*

(ii) *If f is proper, then, for every $k \in \mathbb{Z}$, the functor ${}^p\mathrm{H}^k f_*: \mathrm{Perv}_m(X) \to \mathrm{Perv}_m(Y)$ sends $\mathrm{Perv}_{mf}(X)$ to $\mathrm{Perv}_{mf}(Y)$.*

Proof. Point (i) follows from the fact that the functor $f^*[d]$ is exact (see Proposition 2.2.2) and sends pure perverse sheaves to pure perverse sheaves (by [6, Stabilités 5.1.14]). Point (ii) is Proposition 3.9 of Huber's paper [14]. (This proposition is stated for f smooth, but its proof doesn't use the smoothness of f.) ■

Proposition 4.2. *Let $X, Y \in \mathrm{Ob}(\mathbf{Sch}/k)$.*

(i) *The Poincaré–Verdier duality functor $D_X: \mathrm{Perv}_m(X)^{\mathrm{op}} \to \mathrm{Perv}_m(X)$ sends the full subcatgeory $\mathrm{Perv}_{mf}(X)^{\mathrm{op}}$ to $\mathrm{Perv}_{mf}(X)$.*

(ii) *The external tensor product functor $\boxtimes: \mathrm{Perv}_m(X) \times \mathrm{Perv}_m(Y) \to \mathrm{Perv}_m(X \times Y)$ sends $\mathrm{Perv}_{mf}(X) \times \mathrm{Perv}_{mf}(Y)$ to $\mathrm{Perv}_{mf}(X \times Y)$.*

(iii) *The Tate twist functor $(1): \mathrm{Perv}_m(X) \to \mathrm{Perv}_m(X)$, $K \mapsto K(1)$ sends $\mathrm{Perv}_{mf}(X)$ to $\mathrm{Perv}_{mf}(X)$.*

Proof. This follows from the fact all these functors are exact (see Proposition 2.2.2) and send pure perverse sheaves to pure perverse sheaves (see [6, Stabilités 5.1.14]). ■

In particular, by deriving trivially the functors above, we get:

(i) For every $X \in \mathrm{Ob}(\mathbf{Sch}/k)$, an exact functor $D_X: \mathrm{D}^b\mathrm{Perv}^{\mathrm{op}}_{mf}(X) \to \mathrm{D}^b\mathrm{Perv}_{mf}(X)$ and an isomorphism $D_X^2 \simeq \mathrm{id}$, and also an exact functor from $\mathrm{D}^b \, \mathrm{Perv}_{mf}(X)$ to itself sending K to $K(1)$.

(ii) For every $X, Y \in \mathrm{Ob}(\mathbf{Sch}/k)$, an exact functor $\boxtimes$ from $\mathrm{D}^b \operatorname{Perv}_{mf}(X) \times \mathrm{D}^b \operatorname{Perv}_{mf}(Y)$ to $\mathrm{D}^b \operatorname{Perv}_{mf}(X \times Y)$, satisfying the same properties of commutativity and associativity as the external tensor product on the categories D^b_c.

Moreover, these functors correspond to the usual ones on $\mathrm{D}^b_m(X)$ by the realization functor (by Proposition 2.3.1).

Note that, by [6, Proposition 3.2.2 and Theorem 3.2.4], the 2-functor $X \mapsto \operatorname{Perv}(X)$ is a stack for the étale topology on X. We have the following easy result.

Proposition 4.3. *The categories* $\operatorname{Perv}_h(U)$ *(resp.* $\operatorname{Perv}_m(U)$*, resp.* $\operatorname{Perv}_{mf}(U)$*) define a substack of* $X \mapsto \operatorname{Perv}(X)$.

Proof. As $\operatorname{Perv}_h(U)$ (resp. $\operatorname{Perv}_m(U)$, resp. $\operatorname{Perv}_{mf}(U)$) is a full subcategory of $\operatorname{Perv}(U)$ for every U, we only need to show the following fact: If K is an object of $\operatorname{Perv}(X)$ and if there exists an étale cover $(u_i\colon U_i \to X)_{i\in I}$ of X such that $u_i^* K$ is in $\operatorname{Perv}_h(U_i)$ (resp. $\operatorname{Perv}_m(U_i)$, resp. $\operatorname{Perv}_{mf}(U_i)$) for every $i \in I$, then K is in $\operatorname{Perv}_h(X)$ (resp. $\operatorname{Perv}_m(X)$, resp. $\operatorname{Perv}_{mf}(X)$).

We first treat the case of Perv_h and Perv_m. We may assume that I is finite and that the U_i are affine. For all $i, j \in I$, we denote the fiber product of u_i and u_j by $u_{ij}\colon U_i \times_X U_j \to X$. Then, as Perv is a stack, we have an exact sequence in $\operatorname{Perv}(X)$:

$$0 \to K \to \bigoplus_{i\in I} u_{i*}u_i^* K \to \bigoplus_{i,j\in I} u_{ij*}u_{ij}^* K.$$

As the last two terms are in $\operatorname{Perv}_h(X)$ (resp. $\operatorname{Perv}_m(X)$) by assumption, and as $\operatorname{Perv}_h(X)$ (resp. $\operatorname{Perv}_m(X)$) is a full abelian subcategory of $\operatorname{Perv}(X)$ by Proposition 2.5.2, the perverse sheaf K is also an object of $\operatorname{Perv}_h(X)$ (resp. $\operatorname{Perv}_m(X)$).

We now treat the case of $\operatorname{Perv}_{mf}(X)$. Let $a \in \mathbb{Z}$. We need to construct a subobject L of K such that L is of weight $\leq a$ and K/L is of weight $> a$. For every $i \in I$, we set $L_i = W_a(u_i^* K)$, where W is the weight filtration on K_i. By the uniqueness of the weight filtration, the L_i glue to a subobject L of K. As we can test weights on an étale cover of X (for example by [6, Theorem 5.2.1 and 5.1.14 (iii)]), this L satisfies the required conditions. ∎

Lemma 4.4. *Let* $i\colon Y \to X$ *be a closed immersion, and let* $K \in \operatorname{Ob}\operatorname{Perv}_m(Y)$. *Then* K *is in* $\operatorname{Perv}_{mf}(Y)$ *if and only if* i_*K *is in* $\operatorname{Perv}_{mf}(X)$.

Proof. If K is in $\operatorname{Perv}_{mf}(Y)$, then i_*K is in $\operatorname{Perv}_{mf}(X)$ by Proposition 4.1 (ii).

Conversely, assume that i_*K is in $\operatorname{Perv}_{mf}(X)$. Let $a \in \mathbb{Z}$. We want to show that there exists a subobject K' of K (in $\operatorname{Perv}_m(Y)$) such that K' is of weight $\leq a$ and K/K' is of weight $\geq a+1$. By the assumption, there exists a subobject $L' \subset i_*K$ (in $\operatorname{Perv}_m(X)$) such that L' is of weight $\leq a$ and $L'' := (i_*K)/L'$ is of weight $\geq a+1$. Let j be the inclusion of the complement of Y in X. Then the functor j^* is t-exact, so, applying j^* to the exact sequence $0 \to L' \to i_*K \to L'' \to 0$, we get an exact sequence $0 \to j^*L' \to 0 \to j^*L'' \to 0$ of mixed perverse sheaves on $X - Y$. This implies that $j^*L' =$

$j^*L'' = 0$, so the adjunction morphisms $i_*i^!L' \to L' \to i_*i^*L'$ and $i_*i^!L'' \to L'' \to i_*i^*L''$ are isomorphisms. In particular, the mixed complexes $i^*L' = i^!L'$ and $i^*L'' = i^!L''$ are perverse. Let $K' = i^*L'$. We have just seen that K' is perverse, and the weights of K' are $\leq a$ (see the remark after [14, Definition 3.3]). Also, we have an exact triangle

$$K' = i^*L' \to K \to i^*L'' = i^!L'' \xrightarrow{+1},$$

which is actually an exact sequence in $\mathrm{Perv}_m(Y)$, so the canonical map $K' \to K$ is injective, and $K/K' \simeq i^!L''$, which is of weight $\geq a+1$ (by the same remark in [14]). ■

5. Beilinson's construction of unipotent nearby cycles

In this section, we review Beilinson's construction of the unipotent nearby and vanishing cycles functors from [3]. There are two reasons to do this:

(1) We will want to define nearby cycles for horizontal perverse sheaves, and to apply known theorems (about weights for example). The easiest way to do this is to use Deligne's generic base change theorem, but this might cause technical problems if we use the original construction of nearby cycles (from SGA 7 I and XIII), which involves direct images by morphisms that are not of finite type.

(2) We will need some of Beilinson's auxiliary functors anyway to construct a left adjoint of i_* for i a closed immersion.

All the proofs of the results in this section can be found in [3] (see also [21]).

5.1. Unipotent nearby cycles

Fix a base field k, let X be a k-scheme, and let $f: X \to \mathbb{A}^1_k$ be a morphism. We write $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$, $U = X \times_{\mathbb{A}^1} \mathbb{G}_m \xrightarrow{j} X$ and $Y = X \times_{\mathbb{A}^1} \{0\} \xrightarrow{i} X$.

We have an exact sequence

$$1 \to \pi_1^{\mathrm{geom}}(\mathbb{G}_m, 1) \to \pi_1(\mathbb{G}_m, 1) \to \mathrm{Gal}(\bar{k}/k) \to 1,$$

which is split by the morphism coming from the unit section of $\mathbb{G}_m$. If k is of characteristic 0, then $\pi_1^{\mathrm{geom}}(\mathbb{G}_m, 1) \simeq \widehat{\mathbb{Z}}(1)$; if k is of characteristic $p > 0$, then $\pi_1^{\mathrm{geom},(p)}(\mathbb{G}_m, 1) \simeq \widehat{\mathbb{Z}}^{(p)}(1)$. In both cases, we get a projection $t_\ell: \pi_1^{\mathrm{geom}}(\mathbb{G}_m, 1) \to \mathbb{Z}_\ell(1)$. We also denote by $\chi: \mathrm{Gal}(\bar{k}/k) \to \widehat{\mathbb{Z}}_\ell$ the ℓ-adic cyclotomic character.

Let $\Psi_f: \mathrm{D}^b_c(U) \to \mathrm{D}^b_c(Y_{\bar{k}})$ and $\Phi_f: \mathrm{D}^b_c(X) \to \mathrm{D}^b_c(Y_{\bar{k}})$ be the nearby and vanishing cycles functors defined in SGA 7 Exposé XVIII, shifted by -1 so that they will be t-exact for the perverse t-structure. (See Corollary 4.5 of Illusie's [16], and note that the dimension function we use on U is shifted by $+1$ when compared with Illusie's dimension function.) We denote by T a topological generator of $\pi_1^{\mathrm{geom}}(\mathbb{G}_m, 1)$ or $\pi_1^{\mathrm{geom},(p)}(\mathbb{G}_m)$ (depending on the characteristic of k). We have a functorial exact triangle $\Psi_f \xrightarrow{T-1} \Psi_f \to i^*j_* \xrightarrow{+1}$.

Proposition 5.1.1. *There exists a functorial T-equivariant direct sum decomposition $\Psi_f = \Psi_f^u \oplus \Psi_f^{nu}$ such that, for every $K \in \mathrm{D}_c^b(U)$, $T-1$ acts nilpotently on $\Psi_f^u(K)$ and invertibly on $\Psi_f^{nu}(K)$.*

In particular, the functorial exact triangle $\Psi_f \xrightarrow{T-1} \Psi_f \to i^*j_* \xrightarrow{+1}$ induces a functorial exact triangle $\Psi_f^u \xrightarrow{T-1} \Psi_f^u \to i^*j_* \xrightarrow{+1}$.

The functor Ψ_f^u is called the *unipotent nearby cycles functor*.

Proof. It suffices to prove that, for every $K \in \mathrm{D}_c^b(U)$, there exists a nonzero polynomial P (with coefficients in the coefficient field E that we are using for the categories D_c^b) such that $P(T)$ acts by 0 on $\Psi_f(K)$. (The rest is standard linear algebra.) As we know that Ψ_f sends $\mathrm{D}_c^b(X)$ to $\mathrm{D}_c^b(Y_{\bar{k}})$ (i.e., preserves constructibility), this follows from the fact that, for every $L \in \mathrm{D}_c^b(Y_{\bar{k}})$, the ring of endomorphisms of L is finite-dimensional (over the same coefficient field E). To prove this fact, we use induction on the dimension of X to reduce to the case where the cohomology sheaves of L are local systems, and then it is trivial. ∎

Let $K \in \mathrm{D}_c^b(U)$. Then $T\colon \Psi_f^u K \to \Psi_f^u K$ is unipotent, so there exists a unique nilpotent $N\colon \Psi_f^u K \to \Psi_f^u K(-1)$ such that $T = \exp(t_\ell(T)N)$ on $\Psi_f^u K$. The operator N is usually called the "logarithm of the unipotent part of the monodromy". We get a functorial exact triangle $\Psi_f^u \xrightarrow{N} \Psi_f^u(-1) \to i^*j_* \xrightarrow{+1}$.

5.2. Beilinson's construction

Now we introduce the unipotent local systems that are used in Beilinson's construction of Ψ_f^u.

Definition 5.2.1. For every $i \geq 0$, we define a E-local system $\mathcal{L}_i$ on $\mathbb{G}_m$ in the following way: the stalk $\mathcal{L}_{i,1}$ of $\mathcal{L}_i$ at $1 \in \mathbb{G}_m(k)$ is the E-vector space E^{i+1}, on which an element $u \rtimes \sigma$ of $\pi_1(\mathbb{G}_m, 1) \simeq \widehat{\mathbb{Z}}(1) \rtimes \mathrm{Gal}(\bar{k}/k)$ acts by $\exp(t_\ell(u)N)\,\mathrm{diag}(1, \chi(\sigma)^{-1}, \dots, \chi(\sigma)^{-i})$, where $\mathrm{diag}(x_0, \dots, x_i)$ is the diagonal matrix with diagonal entries $x_0, \dots, x_i$ and N is the Jordan block

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}.$$

If $i \leq j$, we have an obvious injection $\alpha_{i,j}\colon \mathcal{L}_i \to \mathcal{L}_j$ and an obvious surjection $\beta_{j,i}\colon \mathcal{L}_j \to \mathcal{L}_i(i-j)$.

Note that $\mathcal{L}_i^\vee \simeq \mathcal{L}_i(i)$, so (by the calculation at the end of Section 2.1) we have

$$D_U(\mathcal{L}_i) \simeq \mathcal{L}_i(i-1)[-2],$$

and $D_U(\alpha_{i,j})$ corresponds by this isomorphism to $\beta_{j,i}(j-1)[-2]$.

Notation 5.2.2. If $\mathcal{L}$ is a lisse sheaf on $\mathbb{G}_m$ and K is a perverse sheaf on U, then the complex $K \otimes f^*\mathcal{L}$ is also perverse. We denote it by $K \otimes \mathcal{L}$.

We start with the construction of Ψ_f^u.

Proposition 5.2.3. *Let $K \in \operatorname{Ob}\operatorname{Perv}(U)$.*

(i) *For every $a \in \mathbb{N}$, we have a canonical isomorphism*

$$\begin{aligned} i_* \operatorname{Ker}(N^{a+1}, \Psi_f^u K) &\xrightarrow{\sim} \operatorname{Ker}\big(j_!(K \otimes \mathcal{L}_a) \to j_*(K \otimes \mathcal{L}_a)\big) \\ &= i_* {}^p\mathrm{H}^{-1} i^* j_*(K \otimes \mathcal{L}_a). \end{aligned}$$

In particular, if a is big enough, we get an isomorphism

$$i_* \Psi_f^u K \xrightarrow{\sim} i_* {}^p\mathrm{H}^{-1} i^* j_*(K \otimes \mathcal{L}_a).$$

(ii) *For every $a \in \mathbb{N}$ such that $N^{a+1} = 0$ on $\Psi_f^u K$, the following diagram is commutative:*

$$\begin{array}{ccccccc} 0 \longrightarrow & i_*\Psi_f^u K & \longrightarrow & j_!(K \otimes f^*\mathcal{L}_a) & \longrightarrow & j_*(K \otimes f^*\mathcal{L}_a) \\ & \| & & \downarrow{\scriptstyle \alpha_{a,a+1}} & & \downarrow{\scriptstyle \alpha_{a,a+1}} \\ 0 \longrightarrow & i_*\Psi_f^u K & \longrightarrow & j_!(K \otimes f^*\mathcal{L}_{a+1}) & \longrightarrow & j_*(K \otimes f^*\mathcal{L}_{a+1}) \\ & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle \beta_{a,a+1}} & & \downarrow{\scriptstyle \beta_{a,a+1}} \\ 0 \longrightarrow & i_*\Psi_f^u K(-1) & \longrightarrow & j_!(K \otimes f^*\mathcal{L}_a)(-1) & \longrightarrow & j_*(K \otimes f^*\mathcal{L}_a)(-1) \end{array}$$

(iii) *Let $a, b \in \mathbb{N}$ be such that $N^{a+1} = N^{b+1} = 0$ on $\Psi_f^u K$. Then there is a canonical isomorphism*

$$\begin{aligned} &\operatorname{Ker}\big(j_!(K \otimes f^*\mathcal{L}_b) \to j_*(K \otimes f^*\mathcal{L}_b)\big)(-a-1) \\ &\qquad\xrightarrow{\sim} \operatorname{Coker}\big(j_!(K \otimes f^*\mathcal{L}_a) \to j_*(K \otimes f^*\mathcal{L}_a)\big) \end{aligned}$$

induced by the connecting map coming from the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & j_!(K\otimes f^*\mathcal{L}_a) & \xrightarrow{\alpha_{a,a+b+1}} & j_!(K\otimes f^*\mathcal{L}_{a+b+1}) & \xrightarrow{\beta_{b,a+b+1}} & j_!(K\otimes f^*\mathcal{L}_b)(-a-1) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & j_*(K\otimes f^*\mathcal{L}_a) & \xrightarrow{\alpha_{a,a+b+1}} & j_*(K\otimes f^*\mathcal{L}_{a+b+1}) & \xrightarrow{\beta_{a,a+b+1}} & j_*(K\otimes f^*\mathcal{L}_b)(-a-1) & \longrightarrow 0 \end{array}$$

Moreover, the morphism

$${}^p\mathrm{H}^0 i^* j_*(K \otimes \mathcal{L}_a) \to {}^p\mathrm{H}^0 i^* j_*(K \otimes \mathcal{L}_{a+b+1})$$

induced by $\alpha_{a,a+b+1}$ is zero.

Note in particular that we can use this construction to see Ψ_f^u as a functor from $\mathrm{Perv}(U)$ to $\mathrm{Perv}(Y)$ (and not just to the category of $\mathrm{Gal}(\bar{k}/k)$-equivariant objects in $\mathrm{Perv}(Y)$).

Corollary 5.2.4. *For every $K \in \mathrm{Perv}(U)$, we have a canonical isomorphism $D(\Psi_f^u K) \simeq \Psi_f^u(DK)(-1)$.*

Corollary 5.2.5. *For every $a \in \mathbb{N}$, we define a functor*

$$C_a^\bullet : \mathrm{Perv}(U) \to C^{[0,1]}\big(\mathrm{Perv}(X)\big)$$

(where the second category is the category of complexes concentrated in degrees 0 and 1) by

$$C_a^\bullet(K) = \big(j_!(K \otimes \mathcal{L}_a) \to j_*(K \otimes \mathcal{L}_a)\big).$$

With the transition morphisms given by the $\alpha_{a,b}$, the family $(C_a)_{a \geq 0}$ becomes an inductive system of functors.

Then we have canonical isomorphisms

$$i_*\Psi_f^u \simeq \varinjlim_{a\in\mathbb{N}} \mathrm{H}^0(C_a^\bullet), \quad 0 = \varinjlim_{a\in\mathbb{N}} \mathrm{H}^1(C_a^\bullet).$$

Remark 5.2.6. If we use the Ind-category $\mathrm{Ind}(\mathrm{Perv}(X))$ of $\mathrm{Perv}(X)$ (see for example Chapter 6 of Kashiwara and Schapira's book [18], and Theorem 8.6.5 of the same book for the fact that this category is abelian), then we can reformulate this corollary in the following way: We have a canonical isomorphism

$$i_*\Psi_f^u \xrightarrow{\sim} \varinjlim_{a\in\mathbb{N}} C_a^\bullet$$

of functors $\mathrm{Perv}(X) \to \mathrm{D}^b \mathrm{Ind}(\mathrm{Perv}(X))$. Note that, by [18, Theorem 15.3.1], the obvious functor $\mathrm{D}^b \mathrm{Perv}(X) \to \mathrm{D}^b \mathrm{Ind}(\mathrm{Perv}(X))$ is fully faithful (and its essential image is the full subcategory of complexes with all their cohomology objects in $\mathrm{Perv}(X)$). So $\varinjlim_{a\in\mathbb{N}} C_a^\bullet$ factors through the category $\mathrm{D}^b \mathrm{Perv}(X)$.

We now give the definition of the maximal extension functor.

Let $K \in \mathrm{Ob}\,\mathrm{Perv}(U)$. For each $a \geq 1$, we have a commutative diagram:

$$\begin{array}{ccc} j_!(K \otimes f^*\mathcal{L}_a) & \longrightarrow & j_*(K \otimes f^*\mathcal{L}_a) \\ \downarrow{\scriptstyle \beta_{a,a+1}} & & \downarrow{\scriptstyle \beta_{a,a+1}} \\ j_!(K \otimes f^*\mathcal{L}_{a-1})(-1) & \longrightarrow & j_*(K \otimes f^*\mathcal{L}_{a-1})(-1) \end{array}$$

We write

$$\gamma_{a,a-1} : j_!(K \otimes f^*\mathcal{L}_a) \to j_*(K \otimes f^*\mathcal{L}_{a-1})(-1)$$

for the diagonal map in this diagram.

Proposition 5.2.7. *We use the notation of the previous paragraph.*

(i) *For $a \in \mathbb{N}$ big enough, the (injective) map from* $\mathrm{Ker}(\gamma_{a,a-1})$ *to* $\mathrm{Ker}(\gamma_{a+1,a})$ *induced by $\alpha_{a,a+1}\colon j_!(K \otimes f^*\mathcal{L}_a) \to j_!(K \otimes f^*\mathcal{L}_{a+1})$ is an isomorphism. We write $\Xi_f K$ for the direct limit of the* $\mathrm{Ker}(\gamma_{a,a-1})$*. This defines a left exact functor from* $\mathrm{Perv}(U)$ *to* $\mathrm{Perv}(X)$*, called the* maximal extension functor.
Moreover, if a and b are big enough, then the map

$$\mathrm{Coker}(\gamma_{a,a-1}) \to \mathrm{Coker}(\gamma_{a+b,a+b-1})$$

induced by $\alpha_{a-1,a+b-1}(-1)$ is zero. In particular, we have

$$\varinjlim_{a} \mathrm{Coker}(\gamma_{a-1,a}) = 0,$$

and so Ξ_f is exact.

(ii) *We have a functorial isomorphism $D_X \circ \Xi_f \simeq \Xi_f \circ D_U$ and two functorial exact sequences*

$$0 \to j_! \xrightarrow{\alpha} \Xi_f \to i_*\Psi_f^u(-1) \to 0$$

and

$$0 \to i_*\Psi_f^u \to \Xi_f \xrightarrow{\beta} j_* \to 0,$$

dual of each other, in which the maps are the obvious ones. For example, in the first sequence, the map $j_!K \to \Xi_f K$ is induced by the injection

$$\alpha_{0,a}\colon j_!K = j_!(K \otimes f^*\mathcal{L}_0) \to j_!(K \otimes f^*\mathcal{L}_a),$$

and the map $\Xi_f \to i_*\Psi_f^u(-1)$ is induced by the commutative square

$$\begin{array}{ccc} j_!(\cdot \otimes f^*\mathcal{L}_a) & \xrightarrow{\gamma_{a,a-1}} & j_*(\cdot \otimes f^*\mathcal{L}_{a-1})(-1) \\ \downarrow{\scriptstyle \beta_{a,a-1}} & & \downarrow{\scriptstyle \mathrm{id}} \\ j_!(\cdot \otimes f^*\mathcal{L}_{a-1})(-1) & \longrightarrow & j_*(\cdot \otimes f^*\mathcal{L}_{a-1})(-1) \end{array}$$

Remark 5.2.8. As in Remark 5.2.6, we can deduce from (i) of the proposition a natural isomorphism

$$\Xi_f K \xrightarrow{\sim} \varinjlim_{a \in \mathbb{N}} \left(j_!(K \otimes f^*\mathcal{L}_a) \xrightarrow{\gamma_{a,a-1}} j_*(K \otimes \mathcal{L}_{a-1})(-1) \right)$$

in $\mathrm{D}^b\,\mathrm{Ind}(\mathrm{Perv}(X))$.

The next functor that we construct is the *unipotent vanishing cycles functor* Φ_f^u. It is not very hard to show that this functor is isomorphic to the direct summand of the usual vanishing cycles functor on which the monodromy acts unipotently, but we will not need this, so we will just use the following proposition as the definition of Φ_f^u.

Proposition 5.2.9. *The following hold:*

(i) *The complex of exact endofunctors of* $\mathrm{Perv}(X)$ *defined by*

$$j_! j^* \xrightarrow{\alpha+\eta} \Xi_f j^* \oplus \mathrm{id} \xrightarrow{\beta-\varepsilon} j_* j^*$$

in degrees -1, 0 *and* 1, *where* $\eta\colon j_! j^* \to \mathrm{id}$ *is the counit of the adjunction* $(j_!, j^*)$ *and* $\varepsilon\colon \mathrm{id} \to j_* j^*$ *is the unit of the adjunction* (j^*, j_*), *has its cohomology concentrated in degree* 0 *and with support in* Y. *We define an exact functor* $\Phi_f^u\colon \mathrm{Perv}(X) \to \mathrm{Perv}(Y)$ *by setting* $i_*\Phi_f^u$ *to be the* H^0 *of this complex.*

(ii) *We denote by* $\mathrm{can}\colon \Psi_f^u j^* K \to \Phi_f^u K$ *the functorial map defined by*

$$i_*\Psi_f^u j^* K \to \Xi_f j^* K,$$

and by $\mathrm{var}\colon \Phi_f^u K \to \Psi_f^u j^* K(-1)$ *the functorial map defined by*

$$\Xi_f j^* K \to \Psi_f^u K(-1).$$

Then $\mathrm{var} \circ \mathrm{can} = N$ *and* $\mathrm{can}(-1) \circ \mathrm{var} = N$.

(iii) *We have a functorial isomorphism* $D \circ \Phi_f^u \simeq \Phi_f^u \circ D$, *and the duality exchanges* can *and* var.

(iv) *There are canonical isomorphisms* $\mathrm{Ker}(\mathrm{can}) = {}^p\mathrm{H}^{-1} i^* K$ *and* $\mathrm{Coker}(\mathrm{can}) = {}^p\mathrm{H}^0 i^* K$. *Dually, we have canonical isomorphisms* $\mathrm{Ker}(\mathrm{var}) = {}^p\mathrm{H}^0 i^! K$ *and* $\mathrm{Coker}(\mathrm{var}) = {}^p\mathrm{H}^1 i^! K$.

Finally, we will need the functor that M. Saito calls Ω_f.

Proposition 5.2.10. *The functor* $\beta + \varepsilon\colon \Xi_f j^* \oplus \mathrm{id} \to j_* j^*$ *is surjective. Its kernel* Ω_f *is an exact endofunctor of* $\mathrm{Perv}(X)$, *and we have functorial exact sequences*

$$0 \to j_! j^* \xrightarrow{\alpha-\eta} \Omega_f \to i_*\Phi_f^u \to 0$$

and

$$0 \to i_*\Psi_f^u j^* \to \Omega_f \to \mathrm{id} \to 0,$$

in which the unmarked maps are the obvious ones.

6. Nearby cycles and mixed perverse sheaves

The goal of this section is to show that the functor of unipotent nearby cycles preserves the categories $\mathrm{Perv}_{mf}(X)$ and to deduce that these categories are also preserved by the functors ${}^p\mathrm{H}^k f_*$, for every morphism f of $\mathbf{Sch}/k$. The main tool is Deligne's weight-monodromy theorem from [9].

We also give an application to the direct image functor by a closed immersion i, which then allows us to construct the functor i^* on the categories $\mathrm{D}^b\,\mathrm{Perv}_{mf}$.

6.1. Nearby cycles on horizontal perverse sheaves

We assume again that k is a field that is finitely generated over its prime field. Let X be a k-scheme and $f: X \to \mathbb{A}^1$ be a morphism. We write $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$, $U = X \times_{\mathbb{A}^1} \mathbb{G}_m \xrightarrow{j} X$ and $Y = X \times_{\mathbb{A}^1} \{0\} \xrightarrow{i} X$.

In Section 5, we have constructed exact functors:

(1) $\Psi_f^u : \mathrm{Perv}(U) \to \mathrm{Perv}(Y)$ (see Proposition 5.1.1);

(2) $\Phi_f^u : \mathrm{Perv}(X) \to \mathrm{Perv}(Y)$ (see Proposition 5.2.9 (i));

(3) $\Xi_f : \mathrm{Perv}(U) \to \mathrm{Perv}(X)$ (see Proposition 5.2.3 (i)) and

(4) $\Omega_f : \mathrm{Perv}(X) \to \mathrm{Perv}(X)$ (see Proposition 5.2.10).

The next proposition says that all these functors respect the full subcategories of horizontal perverse sheaves (resp. mixed perverse sheaves).

Proposition 6.1.1. *For $? \in \{h, m\}$, the functor Ψ_f^u (resp. Φ_f^u, Ξ_f, Ω_f) sends the full subcategory* $\mathrm{Perv}_?(U)$ *(resp.* $\mathrm{Perv}_?(X)$, $\mathrm{Perv}_?(U)$, $\mathrm{Perv}_?(X)$*) of* $\mathrm{Perv}(U)$ *(resp.* $\mathrm{Perv}(X)$, $\mathrm{Perv}(U)$, $\mathrm{Perv}(X)$*) to the full subcategory* $\mathrm{Perv}_?(Y)$ *(resp.* $\mathrm{Perv}_?(Y)$, $\mathrm{Perv}_?(X)$, $\mathrm{Perv}_?(X)$*) of* $\mathrm{Perv}(Y)$ *(resp.* $\mathrm{Perv}(Y)$, $\mathrm{Perv}(X)$, $\mathrm{Perv}(X)$*).*

Proof. Let $K \in \mathrm{Perv}_h(U)$ (resp. $\mathrm{Perv}_m(U)$). As the lisse sheaves $\mathcal{L}_a$ on $\mathbb{G}_m$ of Definition 5.2.1 are horizontal and mixed, the perverse sheaves $K \otimes f^*\mathcal{L}_a$ are horizontal (resp. mixed), so the perverse sheaves ${}^p\mathrm{H}^{-1} i^* j_*(K \otimes \mathcal{L}_a)$ are all horizontal (resp. mixed). As $\Psi_f^u(K) \xrightarrow{\sim} {}^p\mathrm{H}^{-1} i^* j_*(K \otimes \mathcal{L}_a)$ for a big enough (Proposition 5.2.3 (i)), we deduce that $\Psi_f^u(K)$ is in $\mathrm{Perv}_h(Y)$ (resp. $\mathrm{Perv}_m(Y)$). The statement for Ξ_f, Φ_f^u and Ω_f follows immediately from their definition, once we know that the $\mathcal{L}_a$ are horizontal and mixed. ∎

As only finite type schemes and constructible complexes are involved in the definition of Ψ_f^u if we use the alternative definition given by Proposition 5.2.3 (i), we can use Deligne's generic base change to compare our situation with the situation over closed points of some ring $A \in \mathcal{U}$. We will see an example of this in the next section.

6.2. The relative monodromy filtration

We recall the definition of the relative monodromy filtration, due to Deligne.

Proposition 6.2.1. *(See* [9, Propositions 1.6.1 and 1.6.13]*.) Let K be an object in some abelian category, and suppose that we have a finite increasing filtration W on K and a nilpotent endomorphism N of K. Then there exists at most one finite increasing filtration M on K such that $N(M_i) \subset M_{i-2}$ for every $i \in \mathbb{Z}$ and that, for every $k \in \mathbb{N}$ and every $i \in \mathbb{Z}$, the morphism N^k induces isomorphisms*

$$\mathrm{Gr}^M_{i+k} \mathrm{Gr}^W_i K \xrightarrow{\sim} \mathrm{Gr}^M_{i-k} \mathrm{Gr}^W_i K.$$

Moreover, if W is trivial (that is, if there exists $i \in \mathbb{Z}$ such that $\mathrm{Gr}^W_i K = K$), then the filtration M always exists.

The filtration M is called the *monodromy filtration on K relative to the filtration W*. If W is trivial, it is simply called the *monodromy filtration on K*.

We will use the following theorem, which is a close relative of Theorem 1.8.4 of Deligne's Weil II paper [9].

Theorem 6.2.2. *Let $K \in \operatorname{Ob}\operatorname{Perv}_{mf}(U)$, and let W be the weight filtration on K. Then the monodromy filtration M on $\Psi_f^u K$ relative to the filtration $\Psi_f^u W$ exists, and $\operatorname{Gr}_i^M \Psi_f^u K$ is pure of weight $i-1$ for every $i \in \mathbb{Z}$. In particular, $\Psi_f^u K$ is an object of $\operatorname{Perv}_{mf}(Y)$.*

Before giving the proof of the theorem, we state and prove a lemma.

Lemma 6.2.3. *In the situation of the theorem, suppose that K is pure. Then the monodromy filtration M on $\Psi_f^u K$ (which always exists) is such that $\operatorname{Gr}_i^M \Psi_f^u K$ is pure of weight $i-1$ for every $i \in \mathbb{Z}$.*

Proof. Let w be the weight of K. Let $(A, \mathcal{X}, u)$ be an object of $\mathcal{U}X$ such that K comes by restriction from a shifted perverse sheaf $\mathcal{K}[-d]$ on $\mathcal{X}$, where $d = \dim \operatorname{Spec} A$. Fix $a \in \mathbb{N}$ such that $N^{a+1} = 0$ on $\Psi_f^u K$. After shrinking $\operatorname{Spec} A$ and $\mathcal{X}$ if necessary, we may assume that:

- The morphism $f: X \to \mathbb{A}_k^1$ extends to a morphism $F: \mathcal{X} \to \mathbb{A}_A^1$. We write
$$\mathcal{U} = \mathcal{X} \times_{\mathbb{A}_A^1} \mathbb{G}_{A,m} \xrightarrow{J} \mathcal{X} \quad \text{and} \quad \mathcal{Y} = \mathcal{X} \times_{\mathbb{A}_A^1} \{0\} \xrightarrow{I} \mathcal{X}.$$
- The lisse sheaves $\mathcal{L}_0, \ldots, \mathcal{L}_{a+1}$ all extend to $\mathbb{G}_{m,A}$. (In fact we can get all the $\mathcal{L}_b$ as soon as we have $\mathcal{L}_1$, because they are the symmetric powers of $\mathcal{L}_1$.)
- For every closed point x of $\operatorname{Spec} A$, the restriction of $\mathcal{K}$ to $\mathcal{X}_x$ is still perverse, and it is pure of weight $w + d$.
- The formation of the complexes $J_!(\mathcal{K} \otimes \mathcal{L}_b)$ and $J_*(\mathcal{K} \otimes \mathcal{L}_b)$, for $b \in \{a, a+1\}$, is compatible with every base change $x \to \operatorname{Spec} A$, where x is a closed point of $\operatorname{Spec} A$. Moreover, if $\mathcal{L}$ is any subquotient of ${}^p\mathrm{H}^{-1} I^* J_*(\mathcal{K} \otimes \mathcal{L}_a)$ (in the category $\operatorname{Perv}(\mathcal{X}, E)$), then its restrictions to the fibers of $\mathcal{X}$ above all the closed points of $\operatorname{Spec} A$ are still perverse.

Indeed, the first two points are standard, and the last two follow from Deligne's generic base change theorem (see SGA 4 1/2, [Th. finitude], Théorème 1.9) and from the purity theorem.

Let $\mathcal{K}' = {}^p\mathrm{H}^{-1} I^* J_*(\mathcal{K} \otimes \mathcal{L}_a)$, and let M the monodromy filtration on $\mathcal{K}'$ induced by N. By the conditions above (and (i) of Proposition 5.2.3), for every closed point x of $\operatorname{Spec} A$, the restriction of $\mathcal{K}'$ to $\mathcal{X}_x$ is a subobject of $\Psi_f^u \mathcal{K}_x$, and the restriction of M is the monodromy filtration. The result about the weights of the graded pieces of the monodromy filtration over the spectrum of a finite field (such as x) is known by Theorem 5.1.2 of Beilinson and Bernstein's paper [5] (where it is attributed to Gabber). So we get the conclusion by definition of the weights on horizontal sheaves. ∎

Proof of Theorem 6.2.2. We reason by induction on the length of the filtration W. If K is pure (i.e., if W is trivial), then the conclusion of the theorem is proved in Lemma 6.2.3.

Now assume that W is of length ≥ 2, and that we know the result for every object of $\mathrm{Perv}_{mf}(U)$ with a shorter weight filtration. Let $a \in \mathbb{Z}$ be such that $W_a K = K$ and $\mathrm{Gr}_a^W K \neq 0$. By the induction hypothesis, we know the theorem for $W_{a-1}K$ and $\mathrm{Gr}_a^W K$. Write $L = \Psi_f^u K$, and let F be the filtration $\Psi_f^u W$ on L. By Theorem 2.20 of Steenbrink and Zucker's paper [28], the filtration M exists if and only if, for every $i \geq 1$, we have:

$$N^i(L) \cap F_{a-1}L(-i) \subset N^i(F_{a-1}L) + M_{a-i-1}F_{a-1}L(-i).$$

This is equivalent to saying that $(N^i(L) \cap F_{a-1}L(-i))/N^i(F_{a-1}L)$ is included in

$$\big[M_{a-i-1}F_{a-1}L/\big(M_{a-i-1}F_{a-1}L \cap N^i(F_{a-1}L)\big)\big](-i).$$

As the filtration M on $F_{a-1}L$ is the weight filtration up to a shift, the inclusion above is also equivalent to the fact that $(N^i(L) \cap F_{a-1}L(-i))/N^i(F_{a-1}L)$ is of weight $\leq a + i - 2$. Observe that the perverse sheaf $(N^i(L) \cap F_{a-1}L(-i))/N^i(F_{a-1}L)$ is the kernel of the map

$$F_{a-1}L(-i)/N^i(F_{a-1}L) \to L(-i)/N^i(L),$$

so applying the snake lemma to the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{a-1}L & \longrightarrow & L & \longrightarrow & \mathrm{Gr}_a^F L & \longrightarrow & 0 \\ & & \downarrow N^i & & \downarrow N^i & & \downarrow N^i & & \\ 0 & \longrightarrow & F_{a-1}L(-i) & \longrightarrow & L(-i) & \longrightarrow & \mathrm{Gr}_a^F L(-i) & \longrightarrow & 0 \end{array}$$

gives a surjection

$$\mathrm{Ker}\big(N^i : \mathrm{Gr}_a^F L \to \mathrm{Gr}_a^F L(-i)\big) \to \big(N^i(L) \cap F_{a-1}L(-i)\big)/N^i(F_{a-1}L).$$

But as $\mathrm{Gr}_a^F L = \Psi_f^u \mathrm{Gr}_a^W K$, we know by [9, Section 1.6.4] that $\mathrm{Ker}(N^i : \mathrm{Gr}_a^F L \to \mathrm{Gr}_a^F L(-i))$ is of weight $\leq a + i - 2$ (or more correctly, we can deduce this from the result we cited and Deligne's generic base change theorem, as in the proof of Lemma 6.2.1), and hence all its quotients are. This proves the existence of the filtration M on L.

Finally, we prove that $\mathrm{Gr}_i^M L$ is pure of weight $i - 1$ for every $i \in \mathbb{Z}$. The two properties defining M in Proposition 6.2.1 stay true if we intersect M with $F_{a-1}L$ or take the quotient filtration in $\mathrm{Gr}_a^F L$, so this gives the relative monodromy filtration on $F_{a-1}L$ and $\mathrm{Gr}_a^F L$ (by the uniqueness statement). Hence we get exact sequences

$$0 \to \mathrm{Gr}_i^M F_{a-1}L \to \mathrm{Gr}_i^M L \to \mathrm{Gr}_i^M \mathrm{Gr}_a^F L \to 0,$$

and so the fact that $\mathrm{Gr}_i^M L$ is pure of weight $i-1$ follows from the induction hypothesis. ∎

6.3. Cohomological direct image functors and weights

Corollary 6.3.1. *Let $f : X \to Y$ be a morphism of k-schemes. Then the functors ${}^p\mathrm{H}^i f_*$ and ${}^p\mathrm{H}^i f_!$ from* $\mathrm{Perv}_m(X)$ *to* $\mathrm{Perv}_m(Y)$ *send* $\mathrm{Perv}_{mf}(X)$ *to* $\mathrm{Perv}_{mf}(Y)$.

Proof. As Poincaré–Verdier duality exchanges ${}^p\mathrm{H}^i f_*$ and ${}^p\mathrm{H}^{-i} f_!$ and preserves the categories Perv_{mf}, it suffices to treat the case of ${}^p\mathrm{H}^i f_*$.

By Nagata's compactification theorem (see for example Conrad's paper [8]), we can write $f = gj$, with $j: X \to X'$ an open embedding and $g: X' \to Y$ proper. After replacing X' by the blowup of $X' - X$ in X', we may assume that the ideal of $X' - j(X)$ is invertible. Then j is affine, so j_* is t-exact, so we have ${}^p\mathrm{H}^i f_* = ({}^p\mathrm{H}^i g_*) \circ j_*$ for every $i \in \mathbb{Z}$. By Proposition 4.1 (ii), it suffices to prove the corollary for j. By Proposition 4.3, we may assume that X' is affine, and hence that there exists $h \in \mathcal{O}(X')$ generating the ideal of $X' - j(X)$.

So we see that it is enough to prove the corollary in the following situation: there exists $h: Y \to \mathbb{A}^1_k$ such that $f = j$ is the inclusion of $X := h^{-1}(\mathbb{G}_m)$ in Y. Let $i: Y - X \to X$ be the inclusion of the complement. Let K be an object of $\mathrm{Perv}_{mf}(X)$, and denote by W its weight filtration. Let $a \in \mathbb{Z}$. We want to find a subobject L of j_*K such that L is of weight $\leq a$ and j_*K/L is of weight $> a$. (This clearly implies that j_*K has a weight filtration.)

If $W_a K = 0$, then K is of weight $> a$, so j_*K is of weight $> a$, and we take $L = 0$.

If $W_a K = K$, then K is of weight $\leq a$, so $j_{!*}K$ is of weight $\leq a$ by [6, Corollary 5.4.3]. So it is enough to find a subobject L' of weight $\leq a$ of $j_*K/j_{!*}K$ such that $(j_*K/j_{!*}K)/L'$ is of weight $> a$. But we know that $j_*K/j_{!*}K = i_*{}^p\mathrm{H}^0 i^* j_*K$ (by [6, equation (4.1.11.1)]), which is a quotient of $i_*\Psi^u_f K(-1)$. As $\Psi^u_f K$ has a weight filtration by Theorem 6.2.2, so does $j_*K/j_{!*}K$, and we can find a L' with the desired properties.

Suppose that $0 \neq W_a K \neq K$, and let $K' = W_a K$ and $K'' = K/W_a K$. By the previous paragraph, there exists a subobject L' of weight $\leq a$ of j_*K' such that j_*K'/L' is of weight $> a$. As K'' is of weight $> a$, so is j_*K''. Using the exact sequence

$$0 \to j_*K' \to j_*K \to j_*K'' \to 0,$$

we see that j_*K/L' is also of weight $> a$, so we can take $L = L'$. ∎

Corollary 6.3.2. *Let $j: U \to X$ be an affine open embedding. Denote by*

$$j^*: D^b\,\mathrm{Perv}_{mf}(X) \to D^b\,\mathrm{Perv}_{mf}(U) \quad \text{and} \quad j_*: D^b\,\mathrm{Perv}_{mf}(U) \to D^b\,\mathrm{Perv}_{mf}(X)$$

the derived functors of the exact functors

$$j^*: \mathrm{Perv}_{mf}(X) \to \mathrm{Perv}_{mf}(U) \quad \text{and} \quad j_*: \mathrm{Perv}_{mf}(U) \to \mathrm{Perv}_{mf}(X).$$

Then these derived functors (j^*, j_*) form a pair of adjoint functors.

Proof. By [27, Corollary 8.12], it suffices to prove that the underived functors form a pair of adjoint functors. But, once we know that both functors preserve the full subcategories $\mathrm{Perv}_{mf} \subset \mathrm{Perv}_m$, this follows from the adjunction for the categories Perv_m. ∎

Corollary 6.3.3. *The exact functors Ψ^u_f, Φ^u_f, Ξ_f and Ω_f of Section 5.2 preserve the full subcategories of mixed perverse sheaves with weight filtrations.*

Proof. We already know the result for Ψ_f^u, by Theorem 6.2.2.

Suppose that $K \in \mathrm{Perv}_{mf}(U)$. Then $K \otimes f^*\mathcal{L}_i$ is in $\mathrm{Perv}_{mf}(U)$ for every $i \geq 0$. Indeed, if we denote by W the weight filtration on K, then we get a weight filtration on $K \otimes f^*\mathcal{L}_i$ by setting

$$W_a(K \otimes f^*\mathcal{L}_i) = \sum_{0 \leq j \leq i} (W_{a-2j}K) \otimes f^*\mathcal{L}_j.$$

By Corollary 6.3.1, we see that $j_!(K \otimes f^*\mathcal{L}_i)$ and $j_*(K \otimes f^*\mathcal{L}_i)$ are in $\mathrm{Perv}_{mf}(X)$ for every $i \geq 0$. By definition of Ξ_f, this implies that $\Xi_f K \in \mathrm{Perv}_{mf}(X)$. The conclusion for Ω_f then follows from its construction in Proposition 5.2.10. Finally, by the construction in Propositions 5.2.7, the functor $i_*\Phi_f$ is a subquotient of $\Xi_f j^* \oplus \mathrm{id}$. As $\mathrm{Perv}_{mf}(X)$ is stable by subquotients in $\mathrm{Perv}_m(X)$, the functor $i_*\Phi_f$ sends $\mathrm{Perv}_{mf}(X)$ to itself. By Lemma 4.4, this implies that Φ_f sends $\mathrm{Perv}_{mf}(X)$ to $\mathrm{Perv}_{mf}(Y)$. ∎

6.4. Direct and inverse image by a closed immersion

Let X be a k-scheme and $Y \xrightarrow{i} X$ be a closed subscheme of X. We denote by $D^b_Y \mathrm{Perv}_{mf}(X)$ the full subcategory of $D^b \mathrm{Perv}_{mf}(X)$ whose objects are the complexes K such that the support of $\mathrm{H}^i K \in \mathrm{Perv}_{mf}(X)$ is contained in Y for every $i \in \mathbb{Z}$. The exact functor $i_* \colon \mathrm{Perv}_{mf}(Y) \to \mathrm{Perv}_{mf}(X)$ induces a functor $i_* \colon D^b \mathrm{Perv}_{mf}(Y) \to D^b \mathrm{Perv}_{mf}(X)$, whose image is obviously in contained in $D^b_Y \mathrm{Perv}_{mf}(X)$.

Corollary 6.4.1. *With notation as above, the functor* $i_* \colon D^b \mathrm{Perv}_{mf}(Y) \to D^b_Y \mathrm{Perv}_{mf}(X)$ *is an equivalence of categories.*

We have a similar equivalence $\mathrm{D}^b_m(Y) \xrightarrow{\sim} \mathrm{D}^b_{m,Y}(X)$*, where* $\mathrm{D}^b_{m,Y}(X)$ *is the full subcategory of objects* K *of* $\mathrm{D}^b_m(X)$ *such that* ${}^p\mathrm{H}^i K$ *is in* $i_* \mathrm{Perv}_m(Y)$ *for every* $i \in \mathbb{Z}$.

Moreover, we can choose inverses $(i_*)^{-1}$ *of these equivalences such that the following diagram commutes:*

$$\begin{array}{ccccc}
D^b \mathrm{Perv}_{mf}(Y) & \xrightarrow{i_*} & D^b_Y \mathrm{Perv}_{mf}(X) & \xrightarrow{(i_*)^{-1}} & D^b \mathrm{Perv}_{mf}(Y) \\
\downarrow R_Y & & \downarrow R_X & & \downarrow R_Y \\
\mathrm{D}^b_m(Y) & \xrightarrow[i_*]{} & \mathrm{D}^b_{m,Y}(X) & \xrightarrow[(i_*)^{-1}]{} & \mathrm{D}^b_m(Y)
\end{array}$$

where the functors R_X *and* R_Y *are defined in Theorem 3.2.4.*

Proof. We prove the first statement. It suffices to prove that, for all $K, L \in \mathrm{Ob}\,\mathrm{Perv}_{mf}(Y)$ and every $n \in \mathbb{Z}$, the functor i_* induces an isomorphism

$$\mathrm{Hom}_{\mathrm{D}^b \mathrm{Perv}_{mf}(Y)}\big(K, L[n]\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}^b \mathrm{Perv}_{mf}(X)}\big(i_*K, i_*L[n]\big).$$

Note that both of these Hom groups are 0 for $n < 0$, so we only need to consider the case $n \geq 0$. Fix $K \in \mathrm{Ob}\,\mathrm{Perv}_{mf}(Y)$. The families of functors $(L \mapsto \mathrm{Hom}_{\mathrm{D}^b \mathrm{Perv}_{mf}(Y)}(K, L[n]))_{n \geq 0}$

and $(L \mapsto \mathrm{Hom}_{\mathrm{D}^b \mathrm{Perv}_{mf}(X)}(i_* K, i_* L[n]))_{n \geq 0}$ are δ-functors from $\mathrm{Perv}_{mf}(Y)$ to the category of abelian groups (in the sense of Definition [29, Tag 010Q]), and i_* induces a morphism between these δ-functors (see Definition [29, Tag 010R]). We want to show that this morphism is an isomorphism. We know that

$$i_* \colon \mathrm{Hom}_{\mathrm{Perv}_{mf}(Y)}(K, L) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Perv}_{mf}(X)}(i_* K, i_* L)$$

is an isomorphism for every $L \in \mathrm{Ob}\,\mathrm{Perv}_{mf}(Y)$ (because this is true in the categories Perv_m). Moreover, it follows easily from the Yoneda description of the extension groups in the derived category (see Section 3.2 of Chapter III of Verdier's book [30] or Lemma [29, Tag 06XU]) that the first of the two δ-functors introduced above is effacable, i.e., satisfies the hypothesis of Lemma [29, Tag 010T], and hence is a universal δ-functor (see Definition [29, Tag 010S]). By Lemma [29, Tag 010U] (and Lemma [29, Tag 010T] again), it suffices to prove that the second δ-functor is also effacable. So we want to prove that, for all $K, L \in \mathrm{Ob}\,\mathrm{Perv}_{mf}(Y)$, every $n \geq 1$ and every $u \in \mathrm{Hom}_{\mathrm{D}^b \mathrm{Perv}_{mf}(Y)}(K, L[n])$, there exists an injective morphism $L \to L'$ in $\mathrm{Perv}_{mf}(Y)$ such that the image of u in $\mathrm{Hom}_{D^b \mathrm{Perv}_{mf}(X)}(i_* K, i_* L'[n])$ is 0.

Let $(U_a)_{a \in A}$ be a finite affine cover of X. For every $a \in A$, we have a cartesian diagram of immersions

$$\begin{array}{ccc} Y & \xrightarrow{\ i\ } & X \\ {\scriptstyle j'_a}\uparrow & & \uparrow{\scriptstyle j_a} \\ Y \cap U_\alpha & \xrightarrow[i_a]{} & U_\alpha \end{array}$$

As j'_α and j_α are affine, the functors $j'_{\alpha *}$ and $j_{\alpha *}$ are t-exact. Let $L \in \mathrm{Ob}\,\mathrm{Perv}_{mf}(Y)$. By Corollary 6.3.1, the isomorphisms $i_* j'_{a*} j'^*_a L \simeq j_{a*} j^*_a i_* L$ and $j^*_a i_* L \simeq i_{a*} j'^*_a L$ in $\mathrm{Perv}_m(X)$ and $\mathrm{Perv}_m(U_\alpha)$ are isomorphisms of objects of $\mathrm{Perv}_{mf}(X)$ and $\mathrm{Perv}_{mf}(U_\alpha)$. Using this and Corollary 6.3.2 we get for $K, L \in \mathrm{Ob}\,\mathrm{Perv}_{mf}(Y)$ and $n \in \mathbb{Z}$ a canonical isomorphism

$$\mathrm{Hom}_{D^b \mathrm{Perv}_{mf}(X)}\left(i_* K, i_* j'_{a*} j'^*_a L[n]\right) = \mathrm{Hom}_{D^b \mathrm{Perv}_{mf}(U_a)}\left(i_{a*} j'^*_a K, i_{a*} j'^*_a L[n]\right).$$

As we have an injective morphism $L \to \bigoplus_{a \in A} j_{a*} j^*_a L$ in $\mathrm{Perv}_{mf}(Y)$ (by Corollary 6.3.1 again), this reduces the corollary to the case where X is affine.

Now suppose that X is affine. By an easy induction on the number of generators of the ideal of Y, we may assume that this ideal only has one generator, i.e., that there exists a function $f \colon X \to \mathbb{A}^1$ such that $Y = X \times_{\mathbb{A}^1} \{0\}$. The exact functor $\Phi^u_f \colon \mathrm{Perv}_{mf}(X) \to \mathrm{Perv}_{mf}(Y)$ induces a functor

$$\Phi^u_f \colon D^b_Y \mathrm{Perv}_{mf}(X) \to D^b \mathrm{Perv}_{mf}(Y),$$

and we have $\Phi^u_f \circ i_* \simeq \mathrm{id}_{D^b \mathrm{Perv}_{mf}(Y)}$. Let us show that $i_* \circ \Phi^u_f \simeq \mathrm{id}_{D^b_Y \mathrm{Perv}_{mf}(X)}$, which will finish the proof. By Proposition 5.2.10 and Corollary 6.3.3 we have two exact sequences of exact endofunctors of $\mathrm{Perv}_{mf}(X)$:

$$0 \to j_! j^* \to \Omega_f \to i_* \Phi^u_f \to 0, \quad 0 \to i_* \Psi^u_f j^* \to \Omega_f \to \mathrm{id} \to 0,$$

where $j: X - Y \to X$ is the inclusion. Note that the restriction of the functor j^* from $D^b \operatorname{Perv}_{mf}(X)$ to $D^b \operatorname{Perv}_{mf}(U)$ to the full subcategory $D^b_Y \operatorname{Perv}_{mf}(X)$ is zero. Hence the exact sequences above induces isomorphisms of endofunctors of $D^b_Y \operatorname{Perv}_{mf}(X)$:

$$i_* \Phi^u_f \overset{\sim}{\leftarrow} \Omega_f \overset{\sim}{\rightarrow} \mathrm{id}.$$

The proof of the second equivalence of categories is similar, except that we don't need to use the Yoneda description to show that the Ext groups in $\mathrm{D}^b_m(Y)$ define a δ-functor.

The last statement of the Corollary follows from the fact that we have isomorphisms

$$R_Y \circ \Phi^u_f \simeq \Phi^u_f \circ R_X, \quad R_X \circ \Omega_f \simeq \Omega_f \circ R_X. \qquad \blacksquare$$

Corollary 6.4.2. *Let $i: X \to Y$ be a closed immersion. Denote by $i_*: D^b \operatorname{Perv}_{mf}(X) \to D^b \operatorname{Perv}_{mf}(Y)$ the derived functor of the exact functor $i_*: \operatorname{Perv}_{mf}(X) \to \operatorname{Perv}_{mf}(Y)$.*

Then this functor i_* admits a left adjoint $i^*: D^b \operatorname{Perv}_{mf}(Y) \to D^b \operatorname{Perv}_{mf}(X)$, and the counit $i^* i_* \to \mathrm{id}$ of this adjunction is an isomorphism. Moreover, we have an invertible natural transformation $\theta_i: i^* \circ R_Y \overset{\sim}{\rightarrow} R_X \circ i^*$.

Finally, if $i': Y \to Z$ is another closed immersion, then the following diagram is commutative:

$$\begin{array}{ccccc} R_X \circ i^* i'^* & \xrightarrow{\theta_i} & i^* \circ R_Y \circ i' & \xrightarrow{\theta_{i'}} & i^* i'^* \circ R_Z \\ \downarrow \wr & & & & \downarrow \wr \\ R_X \circ (i'i)^* & & \xrightarrow[\theta_{i'i}]{} & & (i'i)^* \circ R_Z \end{array}$$

where the vertical maps come from the composition isomorphisms $i'_* i_* \simeq (i'i)_*$ and the uniqueness of the adjoint.

Proof. By Corollary 6.4.1, we have an equivalence of categories $i_*: D^b \operatorname{Perv}_{mf}(X) \overset{\sim}{\rightarrow} D^b_X \operatorname{Perv}_{mf}(Y)$, where $D^b_X \operatorname{Perv}_{mf}(Y)$ is the full subcategory of $D^b \operatorname{Perv}_{mf}(Y)$ whose objects are complexes K such that the support of $H^i K \in \operatorname{Perv}_{mf}(Y)$ is contained in X for every $i \in \mathbb{Z}$. So, to show that $i_*: D^b \operatorname{Perv}_{mf}(X) \to D^b \operatorname{Perv}_{mf}(Y)$ admits a left adjoint, it suffices to show that the inclusion $\alpha: D^b_X \operatorname{Perv}_{mf}(Y) \to D^b \operatorname{Perv}_{mf}(Y)$ admits a left adjoint. Let $j: Y - X \to Y$ be the inclusion. Then we have an exact triangle

$$j_! j^* \to \mathrm{id} \to i_* i^* \xrightarrow{+1}$$

of endofunctors of $\mathrm{D}^b_m(Y)$, and we can make sense of the first two terms in $D^b \operatorname{Perv}_{mf}(Y)$, so we will try to construct the left adjoint of α as their cone.

More precisely, let $(U_i)_{i \in I}$ be a finite open affine cover of $U := Y - X$. For every $J \subset I$, we denote by $j_J: \bigcap_{i \in J} U_i \to X$ the inclusion. As X is separated, all the finite intersections of U_i's are affine, so the morphism j_J is affine for every $J \subset I$. If $K \in \operatorname{Ob} \operatorname{Perv}_{mf}(X)$, we denote by $D^\bullet(K)$ the complex of $\operatorname{Perv}_{mf}(X)$ defined by $D^{-r}(K) = \bigoplus_{|J|=r} j_{J!} j_J^* K$ if $r \geq 1$, $D^0(K) = K$ and $D^r(K) = 0$ if $r \geq 1$, where the maps

$$D^{-r-1}(K) \to D^{-r}(K), \quad r \geq 0,$$

are alternating sums of adjunction morphisms. Note that we have a morphism of complexes $K \to D^\bullet(K)$, where K is in degree 0. Also, there is a canonical morphism $D^{-1}(K) \to j_! j^* K$, which induces an isomorphism $D^{\leq -1}(K) \xrightarrow{\sim} j_! j^* K[1]$ in $D^b \operatorname{Perv}_{mf}(Y)$, so we get a quasi-isomorphism $R_Y(D^\bullet(K)) \xrightarrow{\sim} i_* i^* K$ in $\mathrm{D}^b_m(Y)$. In particular, $D^\bullet(K)$ is in $D^b_X \operatorname{Perv}_{mf}(Y)$. Note that the construction of $D^\bullet(K)$ is functorial in K, so we can define a functor $\beta\colon D^b \operatorname{Perv}_{mf}(Y) \to D^b_X \operatorname{Perv}_{mf}(Y)$ by sending a complex K to the total complex of the double complex $D^\bullet(K)$.

Let us show that β is left adjoint to α. For every complex K of objects of $\operatorname{Perv}_{mf}(Y)$, the morphism of double complexes $K \to D^\bullet(K)$ induces a morphism $\varepsilon_K\colon K \to \alpha\beta(K)$ in $D^b \operatorname{Perv}_{mf}(Y)$. If moreover K is in $D^b_X \operatorname{Perv}_{mf}(X)$, then $K \to D^\bullet(K)$ is a quasi-isomorphism, so we get an isomorphism $\eta_K\colon \beta\alpha(K) \xrightarrow{\sim} K$. Moreover, the morphism

$$\alpha(K) \xrightarrow{\varepsilon_K \alpha} \alpha\beta\alpha(K) \xrightarrow{\alpha\eta_K} \alpha(K)$$

is clearly the identity of $\alpha(K)$. So we have constructed the unit and counit of the adjunction, and shown that the counit is an isomorphism.

To construct the isomorphism θ_i, we use the isomorphism $R_Y \circ V \xrightarrow{\sim} i_* i^* \circ R_X$ constructed above and the last statement of Corollary 6.4.1. The last statement is also easy to check. ∎

7. Construction of the stable homotopic 2-functor H_{mf}

7.1. Direct images

If $f\colon X \to Y$ is a morphism of k-schemes, we write ${}^0 f_*$ for ${}^p\mathrm{H}^0 f_*$. Remember that if f is affine, then f_* is right t-exact for the perverse t-structure by [6, Theorem 4.1.1].

In this section, we want to prove the following result.

Proposition 7.1.1. *There exists a 2-functor $H_{mf,*}\colon \mathbf{Sch}/k \to \mathfrak{TR}$ such that $H_{mf,*}(X) = D^b \operatorname{Perv}_{mf}(X)$ for every $X \in \operatorname{Ob}(\mathbf{Sch}/k)$ and a natural transformation $R\colon H_{mf,*} \to H_{m,*}$ (with the notation of Example 3.2.3) such that:*

- (a) *for every $X \in \operatorname{Ob}(\mathbf{Sch}/k)$, the functor $R_X\colon \mathrm{D}^b \operatorname{Perv}_{mf}(X) \to \mathrm{D}^b_m(X)$ is the composition of the obvious functor $\mathrm{D}^b \operatorname{Perv}_{mf}(X) \to \mathrm{D}^b \operatorname{Perv}_m(X)$ and of the realization functor $\mathrm{D}^b \operatorname{Perv}_m(X) \to \mathrm{D}^b_m(X)$ (see Section 2.6);*
- (b) *for every morphism $f\colon X \to Y$, the natural transformation $\gamma_f\colon R_Y \circ H_{mf,*}(f) \to H_{m,*}(f) \circ R_X$ is an isomorphism.*

The proof of the proposition will occupy most of this section. The main ingredients are:

- Beilinson's version of the "basic lemma" that provides f_*-acyclic perverse sheaves for $f\colon X \to Y$ an affine morphism. (See Theorem 7.1.2.)
- The fact that the functors ${}^p\mathrm{H}^k f_*$ (and in particular ${}^0 f_*$) preserve the categories Perv_{mf}. (See Corollary 6.3.1.)
- Čech resolutions for finite open affine coverings.

We first review Beilinson's basic lemma.

Theorem 7.1.2. *Let $f: X \to Y$ be a morphism of k-schemes, let $(a_i: U_i \to X)_{i \in I}$ be a finite family of open affine subschemes of X and let $K \in \mathrm{Ob}\,\mathrm{Perv}(X)$. Then there exists a finite family $(b_j: V_j \to X)_{j \in J}$ of open affine subschemes of X such that:*

(i) *The canonical morphism $\bigoplus_{j \in J} b_{j!} b_j^* K \to K$ is surjective.*

(ii) *For every $i \in I$, the object $\bigoplus_{j \in J} a_i^* b_{j!} b_j^* K$ is $(f \circ a_i)_*$-acyclic.*

The theorem follows from the proof of [4, Lemma 3.3].

Note also that, as the categories Perv_h, Perv_m and Perv_{mf} are stable by the functors j^* and $j_!$ for j an open affine embedding (by Corollary 6.3.1 for Perv_{mf}), if K is in one of these categories, so is $\bigoplus_{j \in J} b_{j!} b_j^* K$.

We now turn to the proof of Proposition 7.1.1.

Following [4, Section 3.4], we explain how to reconstruct the functor f_* from ${}^0 f_*$. More precisely, we want to apply Proposition A.3.2 to $\mathcal{D} = \mathrm{D}^b_m(X)$ and $\mathcal{D}' = \mathrm{D}^b_m(Y)$ with the perverse t-structures, $\mathrm{DF} = \mathrm{DF}^b_m(X, E)$ and $\mathrm{DF}' = \mathrm{DF}^b_m(Y, E)$ (see Section 2.6) with the unique t-structures compatible with the perverse t-structures (see Proposition A.2.2), and $T = f_*$.

To check condition (a) of that proposition, remember that $\mathrm{DF}^b_m(X, E)$ is by definition a full subcategory of the triangulated category $\mathrm{DF}^b_h(X, E)$ constructed in Section 2.4; we have

$$\mathrm{DF}^b_h(X, E) = 2 - \varinjlim_{(A, \mathfrak{X}) \in \mathrm{Ob}\,\mathcal{U}X} \mathrm{DF}^b_c(\mathfrak{X}, E),$$

and each $\mathrm{DF}^b_c(\mathfrak{X}, E)$ is a full subcategory of $\mathrm{DF}^b(\mathfrak{X}_{\text{proét}}, E)$. Of course, we have similar statements for $\mathrm{DF}^b_m(Y, E)$. If we choose $(A, \mathfrak{X}) \in \mathrm{Ob}\,\mathcal{U}X$ and $(A, \mathfrak{Y}) \in \mathrm{Ob}\,\mathcal{U}Y$ such that f extends to an A-morphism $f: \mathfrak{X} \to \mathfrak{Y}$, then Proposition A.1.11 and Section A.4 give an f-lifting $f_{F,*}: \mathrm{DF}^b(\mathfrak{X}_{\text{proét}}, E) \to \mathrm{DF}^b(\mathfrak{Y}_{\text{proét}}, E)$ of $f_*: \mathrm{D}^b(\mathfrak{X}_{\text{proét}}, E) \to \mathrm{D}^b(\mathfrak{Y}_{\text{proét}}, E)$. Taking the limit of these f-liftings, we get an f-lifting $\mathrm{DF}^b_h(X, E) \to \mathrm{DF}^b_h(Y, E)$ of

$$f_*: \mathrm{D}^b_h(X, E) \to \mathrm{D}^b_h(Y, E),$$

and it is easy to check that it sends $\mathrm{DF}^b_m(X, E)$ to $\mathrm{DF}^b_m(Y, E)$.

We check condition (b). Let $\mathcal{I}_m(f)$ be the full subcategory of f_*-acyclic objects in $\mathrm{Perv}_m(X)$. Let $\mathcal{U} = (a_i: U_i \to X)_{i \in I}$ be a finite covering of X by open affine subschemes. We denote by $\mathcal{I}_m(f, \mathcal{U})$ the full subcategory of $\mathrm{Perv}_m(X)$ whose objects are mixed perverse sheaves that are $(f \circ a_i)_* a_i^*$-acyclic for every $i \in I$; equivalently, $\mathcal{I}_m(f, \mathcal{U})$ is the full subcategory of $\bigoplus_{i \in I} (f \circ a_i)_* a_i^*$-acyclic objects. As all the functors $a_{i*} a_i^*$ are t-exact, we have $\mathcal{I}_m(f) \subset \mathcal{I}_m(f, \mathcal{U})$. By Remark A.3.4 (applied to the triangulated categories $\mathrm{K}^b(\mathrm{Perv}_m(X)) \supset \mathrm{K}^b(\mathcal{I}_m(f, \mathcal{U})) \supset \mathrm{K}^b(\mathcal{I}_m(f))$), to check condition (b) of Proposition A.3.2, it suffices to prove the following two statements:

(1) For every $A^\bullet \in \mathrm{Ob}\,\mathrm{C}^b(\mathrm{Perv}_m(X))$, there exists a quasi-isomorphism $B^\bullet \to A^\bullet$ with $B^\bullet$ in $\mathrm{C}^b(\mathcal{I}_m(f, \mathcal{U}))$.

(2) For every $B^\bullet \in \mathrm{Ob}\,\mathrm{C}^b(\mathcal{I}_m(f, \mathcal{U}))$, there exists a quasi-isomorphism $B^\bullet \to C^\bullet$ with $C^\bullet$ in $\mathrm{C}^b(\mathcal{I}_m(f))$.

We prove (1). Let $A^\bullet \in \operatorname{Ob} \mathrm{C}^b(\mathrm{Perv}_m(X))$, and let $N \in \mathbb{N}$ such that $A^r = 0$ for $r \notin [-N, N]$. We choose a finite family $(b_j : V_j \to X)_{j\in J}$ of open affine subschemes of X as in Theorem 7.1.2, for the fixed f, the family $(U_{i_0} \cap \cdots \cap U_{i_p})_{p\in\mathbb{N}, i_0,\dots,i_p\in I}$ of open affine subschemes of X and the perverse sheaf $K = \bigoplus_{-N\le r\le N} A^r$. Let $A_0^\bullet \in \operatorname{Ob} \mathrm{C}^b(\mathrm{Perv}_m(X))$ be the complex obtained by applying the exact functor $\bigoplus_{j\in J} b_{j!} b_j^*$ to $A^\bullet$. Then $A_0^\bullet$ is a complex of objects of $\mathcal{I}_m(f, \mathcal{U})$ such that $A_0^r = 0$ for $r \notin [-N, N]$, and we have a morphism of complexes $A_0^\bullet \to A^\bullet$ that is surjective in each degree. By iterating this procedure, we get an exact sequence $\cdots \to A_2^\bullet \to A_1^\bullet \to A_0^\bullet \to A^\bullet \to 0$ in $\mathrm{C}^b(\mathrm{Perv}_m(X))$ such that each $A_i^\bullet$ is a complex of objects of $\mathcal{I}_m(f, \mathcal{U})$ concentrated in degrees $[-N, N]$. Moreover, by Lemma A.3.1 (v) (whose hypothesis is satisfied thanks to [6, Sections 4.2.3 and 4.2.4]), we may without losing these properties assume that $A_i^\bullet = 0$ for i big enough. We can take for $B^\bullet$ the total complex of the double complex $A_\bullet^\bullet$.

We prove (2). For every $I' \subset I$, we denote by $a_{I'} : U_{I'} := \bigcap_{i\in I'} U_i \to X$ the inclusion; if $I' = \{i_0, \dots, i_r\}$, we also write $U_{I'} = U_{i_0,\dots,i_r}$ and $a_{I'} = a_{i_0,\dots,i_r}$. As X is separated, all the finite intersections of U_i's are affine. We first define the functor $\check{\mathrm{C}}^\bullet_{\mathcal{U}} : \mathrm{Perv}_m(X) \to \mathrm{C}^b(\mathrm{Perv}_m(X))$ sending a mixed perverse sheaf to its Čech resolution. If $K \in \operatorname{Ob} \mathrm{Perv}_m(X)$, we set

$$\check{\mathrm{C}}^r_{\mathcal{U}}(K) = \bigoplus_{i_0,\dots,i_r\in I} a_{i_0,\dots,i_r *} a^*_{i_0,\dots,i_r} K.$$

The differential $d^r : \check{\mathrm{C}}^r_{\mathcal{U}}(\mathcal{F}) \to \check{\mathrm{C}}^{r+1}_{\mathcal{U}}(\mathcal{F})$ is an alternating sum of adjunction morphisms. More precisely, for $i_0, \dots, i_r \in I$, the restriction to $a_{i_0,\dots,i_r *} a^*_{i_0,\dots,i_r} K$ of d^r is the sum over $i \in I$ and $0 \le s \le r+1$ of $(-1)^s$ times the adjunction morphism $a_{i_0,\dots,i_r *} a^*_{i_0,\dots,i_r} K \to a_{i_0,\dots,i_{s-1},i,i_s,\dots,i_r *} a^*_{i_0,\dots,i_{s-1},i,i_s,\dots,i_r} K$ coming from the inclusion of $U_{i_0,\dots,i_r,i}$ into $U_{i_0,\dots,i_r}$. This defines an exact functor

$$\check{\mathrm{C}}^\bullet_{\mathcal{U}} : \mathrm{Perv}_m(X) \to \mathrm{C}^+\big(\mathrm{Perv}_m(X)\big).$$

We also have a natural morphism $K \to \check{\mathrm{C}}^\bullet_{\mathcal{U}}(K)$, for every $K \in \operatorname{Ob} \mathrm{Perv}_m(K)$, given by the sum of the adjunction morphisms.

We need a variant that takes its values in the category of bounded complexes, called the *alternating Čech complex* (see [29, Section 01FG]). Let $K \in \operatorname{Ob} \mathrm{Perv}_m(X)$. For every $r \in \mathbb{N}$, we have an action of the symmetric group $\mathfrak{S}_{r+1}$, seen as the group of permutations of $\{0, 1, \dots, r\}$, on the perverse sheaf $\check{\mathrm{C}}^r_{\mathcal{U}}(K)$: if $\sigma \in \mathfrak{S}_{r+1}$ and $i_0, \dots, i_r \in I$, then this action sends the component $a_{i_0,\dots,i_r *} a^*_{i_0,\dots,i_r} K$ to

$$a_{i_{\sigma^{-1}(0)},\dots,i_{\sigma^{-1}(r)} *} a^*_{i_{\sigma^{-1}(0)},\dots,i_{\sigma^{-1}(r)}} K = a_{i_0,\dots,i_r *} a^*_{i_0,\dots,i_r} K,$$

and it acts by $(-1)^\sigma \mathrm{id}$. We denote by $\check{\mathrm{C}}^r_{\mathrm{alt},\mathcal{U}}(K) \subset \check{\mathrm{C}}^r_{\mathcal{U}}(K)$ the invariants of this action. This defines a subcomplex $\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(K)$ of $\check{\mathrm{C}}^\bullet(K)$. Also, as a perverse sheaf on X, $\check{\mathrm{C}}^r_{\mathrm{alt},\mathcal{U}}(K)$ is isomorphic to $\bigoplus_{I'\subset I, |I'|=r+1} a_{I' *} a^*_{I'} K$; in particular, we have $\check{\mathrm{C}}^1_{\mathrm{alt},\mathcal{U}}(K) = \check{\mathrm{C}}^1_{\mathcal{U}}(K)$, so the morphism $K \to \check{\mathrm{C}}^\bullet_{\mathcal{U}}(K)$ factors through a morphism $K \to \check{\mathrm{C}}^1_{\mathrm{alt},\mathcal{U}}(K)$. Another consequence of this observation is that the subfunctor $\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}$ of $\check{\mathrm{C}}^\bullet_{\mathcal{U}}$ is still exact, and that it takes its values in the full subcategory $\mathrm{C}^b(\mathrm{Perv}_m(X))$ of $\mathrm{C}^+(\mathrm{Perv}_m(X))$.

Lemma 7.1.3. *Let $K \in \operatorname{Ob}\operatorname{Perv}_m(X)$.*

(i) *The morphism of complexes $K \to \check{C}^\bullet_{\mathcal{U}}(K)$ is a quasi-isomorphism.*

(ii) *The inclusion $\check{C}^\bullet_{\mathrm{alt},\mathcal{U}}(K) \to \check{C}^\bullet_{\mathcal{U}}(K)$ is a homotopy equivalence.*

(iii) *The morphism of complexes $K \to \check{C}^\bullet_{\mathrm{alt},\mathcal{U}}(K)$ is a quasi-isomorphism.*

Proof. Point (iii) follows from (i) and (ii), and (ii) is proved as [29, Lemma 01FM]. To prove (i), we can restrict the complexes to every open subset of an open covering of X, for example the covering $\mathcal{U}$. So we may assume that X is equal to one of the U_i. In that case the complex $K \to \check{C}^\bullet_{\mathcal{U}}(K)$ (with K in degree -1) is homotopy equivalent to 0, see for example the proof of [29, Lemma 0G6S]. ∎

Now let $B^\bullet \in \operatorname{Ob} C^b(\mathcal{I}_m(f, \mathcal{U}))$. The double complex $\check{C}^\bullet_{\mathrm{alt},\mathcal{U}}(B^\bullet)$ is bounded, and we denote its total complex by $C^\bullet$. By the definition of $\mathcal{I}_m(f, \mathcal{U})$, this is a complex of f_*-acyclic objects, and by Lemma 7.1.3 (iii), the canonical morphism $B^\bullet \to C^\bullet$ is a quasi-isomorphism. This gives the conclusion of (2).

Proposition A.3.2 now says that, if $T\colon D^b \operatorname{Perv}_m(X) \to D^b \operatorname{Perv}_m(Y)$ is the composition of a quasi-inverse of the equivalence $K^b(\mathcal{I}_m(f))/N^b(\mathcal{I}_m(f))$ (where, for every additive subcategory $\mathcal{C}$ of $\operatorname{Perv}(X)$, we denote by $N^b(\mathcal{C})$ the full category of exact complexes in $K^b(\mathcal{C})$) and of the functor

$$K^b\big(\mathcal{I}_m(f)\big)/N^b\big(\mathcal{I}_m(f)\big) \xrightarrow{K^b({}^0 f_*)} D^b \operatorname{Perv}(Y),$$

then T is well-defined and we have $\mathrm{real} \circ T \simeq f_* \circ \mathrm{real}$.

Note also that the proofs of statements (1) and (2), and the construction and properties of the Čech complex and alternating Čech complex, would work just as well for the categories Perv_{mf}, because we know that, for g a morphism of k-schemes, the functors ${}^p\mathrm{H}^k g_*$ preserve the subcategories Perv_{mf} (Corollary 6.3.1). Denote by $\mathcal{I}_{mf}(f)$ the full subcategory of f_*-acyclic objects in $\operatorname{Perv}_{mf}(X)$. We get in particular that the obvious functor $K^b(\mathcal{I}_{mf}(f))/N^b(\mathcal{I}_{mf}(f)) \to D^b \operatorname{Perv}_{mf}(X)$ is an equivalence of categories and that $K^b({}^0 f_*)$ sends objects of $N^b(\mathcal{I}_{mf}(X))$ to exact complexes, so we can make the following definition.

Definition 7.1.4. Let $f\colon X \to Y$. We define the functor

$$H_{mf,*}(f)\colon D^b \operatorname{Perv}_{mf}(X) \to D^b_{mf} \operatorname{Perv}(Y)$$

to be the composition

$$D^b \operatorname{Perv}_{mf}(X) \to K^b\big(\mathcal{I}_{mf}(f)\big)/N^b\big(\mathcal{I}_{mf}(f)\big) \xrightarrow{K^b({}^0 f_*)} D^b \operatorname{Perv}_{mf}(Y),$$

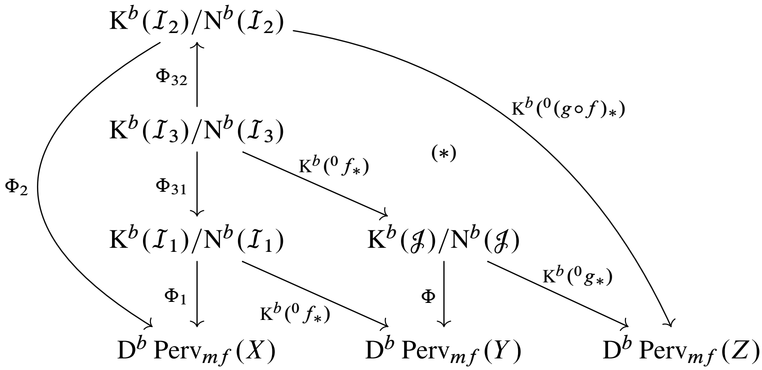
where the first functor is right adjoint to the obvious functor $K^b(\mathcal{I}_{mf}(f))/N^b(\mathcal{I}_{mf}(f)) \to D^b \operatorname{Perv}_{mf}(X)$.[5]

[5] If a functor F is an equivalence of categories, then it admits a right adjoint, and any right adjoint is a quasi-inverse. We use a right adjoint instead of an arbitrary quasi-inverse, because a right adjoint is unique up to unique isomorphism.

It remains to show that Definition 7.1.4 does give a 2-functor from **Sch**$/k$ to $\mathfrak{T}\mathfrak{R}$, that is, to construct connection morphisms. So suppose that we have two morphisms of k-schemes $f\colon X \to Y$ and $g\colon Y \to Z$. We want to lift the connection isomorphism

$$g_* \circ f_* \simeq (g \circ f)_*\colon \mathrm{D}^b_m(X) \to \mathrm{D}^b_m(Z)$$

to a natural isomorphism $H_{mf,*}(g) \circ H_{mf,*}(f) \xrightarrow{\sim} H_{mf,*}(g \circ f)$. Let us write $\mathcal{I}_1 = \mathcal{I}_{mf}(f)$, $\mathcal{I}_2 = \mathcal{I}_{mf}(g \circ f)$, $\mathcal{I}_3 = \mathcal{I}_1 \cap \mathcal{I}_2 = \mathcal{I}_{mf}(f \times (g \circ f))$ and $\mathcal{J} = \mathcal{I}_{mf}(g)$. We consider the following diagram:



All the vertical maps in it are equivalences, and the diagram commutes, except for triangle $(*)$ that only commutes up to a natural isomorphism coming from the connection isomorphism $(g \circ f)_* \simeq g_* \circ f_*$.

We also set $\Phi_3 = \Phi_1 \circ \Phi_{31} = \Phi_2 \circ \Phi_{32}$. For every arrow $\Phi_?$, we denote by $\Psi_?$ a right adjoint of $\Phi_?$. Then we have natural transformations

$$\begin{aligned}
H_{mf,*}(g \circ f) = \mathrm{K}^b\big({}^0(g \circ f)_*\big) \circ \Psi_2 &\leftarrow \mathrm{K}^b\big({}^0(g \circ f)_*\big) \circ \Phi_{32} \circ \Psi_{32} \circ \Psi_2 \\
&\simeq \mathrm{K}^b\big({}^0(g \circ f)_*\big) \circ \Phi_{32} \circ \Psi_3 \\
&\overset{(*)}{\simeq} \mathrm{K}^b({}^0 g_*) \circ \mathrm{K}^b({}^0 f_*) \circ \Psi_3 \\
&\to \mathrm{K}^b({}^0 g_*) \circ \Psi \circ \Phi \circ \mathrm{K}^b({}^0 f_*) \circ \Psi_3 \\
&= H_{mf,*}(g) \circ \mathrm{K}^b({}^0 f_*) \circ \Phi_{31} \circ \Psi_3 \\
&\simeq H_{mf,*}(g) \circ \mathrm{K}^b({}^0 f_*) \circ \Phi_{31} \circ \Psi_{31} \circ \Psi_1 \\
&\to H_{mf,*}(g) \circ \mathrm{K}^b({}^0 f_*) \circ \Psi_1 \\
&= H_{mf,*}(g) \circ H_{mf,*}(f).
\end{aligned}$$

Moreover, all these natural transformations are isomorphisms: the ones marked $\leftarrow$ and $\rightarrow$ are units or counits of adjunctions between equivalences of categories, isomorphism (*) comes from the connection isomorphism $(g \circ f)_* \simeq g_* \circ f_*$, and the two remaining isomorphisms come from the uniqueness of right adjoints. This gives the desired connection isomorphism, and we check the cocycle condition in a similar way. This finishes the proof of Proposition 7.1.1.

Remark 7.1.5. Suppose that $f: X \to Y$ is an affine morphism of k-schemes. By Theorem 7.1.2 (and Corollary 6.3.1), the category $\mathcal{I}_{mf}(f_*)$ of f_*-acyclic objects in $\mathrm{Perv}_{mf}(X)$ is cogenerating. So, by Remark A.3.3, the functor ${}^0 f_*: \mathrm{Perv}_{mf}(X) \to \mathrm{Perv}_{mf}(Y)$ has a left derived functor, and the functor $H_{mf,*}(f)$ is the restriction of this derived functor to $\mathrm{D}^b \mathrm{Perv}_{mf}(X)$. In particular, if f is quasi-finite and affine, then

$$f_*: \mathrm{Perv}_{mf}(X) \to \mathrm{Perv}_{mf}(Y)$$

is exact, and $H_{mf,*}(f)$ is its obvious extension to the bounded derived categories.

From now on, for $f: X \to Y$ a morphism of k-schemes, we will also denote the functor $H_{mf,*}(f)$ by f_* if there is no risk of confusion.

Proposition 7.1.6. *The functor $\boxtimes$ from Proposition 4.2 induces a natural isomorphism between the 2-functors*

$$H_{mf,*} \times H_{mf,*}: \mathbf{Sch}_k \times \mathbf{Sch}_k \to \mathfrak{TR}$$

and

$$\mathbf{Sch}_k \times \mathbf{Sch}_k \xrightarrow{\times} \mathbf{Sch}_k \xrightarrow{H_{mf,*}} \mathfrak{TR}$$

(where the first arrow sends (X, Y) to $X \times Y$).

In other words, if $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ are morphisms in $\mathbf{Sch}/k$ *and $K_1 \in D^b \mathrm{Perv}_{mf}(X_1)$, $K_2 \in D^b \mathrm{Perv}_{mf}(X_2)$, then we have an isomorphism*

$$(f_1 \times f_2)_*(K_1 \boxtimes K_2) \xrightarrow{\sim} (f_{1*}K_1) \boxtimes (f_{2*}K_2)$$

functorial in K_1 and K_2 and compatible with the composition of arrows in $\mathbf{Sch}/k$.

Proof. On the categories D^b_c, we have canonical isomorphisms (see SGA 5 III 1.6). These induce similar isomorphisms in the categories D^b_h and D^b_m.

By the construction of f_* (see Definition 7.1.4), we only need to show the statement of the proposition for the functors ${}^0 f_*$ between the categories $\mathrm{Perv}_{mf}(X)$. But then it is an immediate consequence of the similar result for the categories $\mathrm{Perv}(X)$, which follows from the result recalled at the beginning of the proof and from the t-exactness of the external tensor product (see Proposition 2.2.2). ∎

Proposition 7.1.7. *Let $j: U \to X$ be an open embedding. Denote by*

$$j^*: D^b \mathrm{Perv}_{mf}(X) \to D^b \mathrm{Perv}_{mf}(U)$$

the derived functor of the exact functor $j^*: \mathrm{Perv}_{mf}(X) \to \mathrm{Perv}_{mf}(U)$.

Then this functor j^* is left adjoint to the functor $j_*: D^b \mathrm{Perv}_{mf}(U) \to D^b \mathrm{Perv}_{mf}(X)$, and the counit map $j^* j_* \to \mathrm{id}$ is an isomorphism.

Proof. Let $\mathcal{U} = (U_i)_{i \in I}$ be a finite open affine cover of U. For every $J \subset I$, we denote by $a_J : \bigcap_{i \in J} U_i \to U$ the inclusion. As U is separated, all the finite intersections of U_i's are affine, so the morphisms a_J and $j \circ a_J$ are affine for every $J \subset I$. If $K \in \operatorname{Ob}\operatorname{Perv}_{mf}(U)$, we denote by $\check{\mathrm{C}}^{\bullet}_{\mathrm{alt},\mathcal{U}}(K)$ the alternating Čech complex of K associated to the covering $(U_i)_{i \in I}$ (defined after Lemma 7.1.3), so that

$$\check{\mathrm{C}}^{r}_{\mathrm{alt},\mathcal{U}}(K) = \bigoplus_{|J|=r+1} a_{J*}a_J^* K.$$

The canonical morphism $K \to \check{\mathrm{C}}^{\mathcal{U}}_{\mathrm{alt},\bullet}(K)$ is a quasi-isomorphism (Lemma 7.1.3 (iii)), and all the $\check{\mathrm{C}}^{r}_{\mathrm{alt},\mathcal{U}}(K)$ are j_*-acyclic (indeed, as $j \circ a_J$ is affine for every $J \subset I$, the complex $j_*(a_{J*}a_J^* K)$ is perverse, and so $j_*\check{\mathrm{C}}^{r}_{\mathrm{alt},\mathcal{U}}(K)$ is perverse).

As the alternating Čech complex is a functor $\operatorname{Perv}_{mf}(U) \to \mathrm{C}^b(\operatorname{Perv}_{mf}(U))$, we have by Definition 7.1.4 an isomorphism of functors between j_* and the functor sending K to the the total complex of the double complex $j_*\check{\mathrm{C}}^{\bullet}_{\mathrm{alt},\mathcal{U}}(K)$. So j^*j_* is isomorphic to the functor sending K to the total complex of $\check{\mathrm{C}}^{\bullet}_{\mathrm{alt},\mathcal{U}}(K)$. Let $\alpha\colon \mathrm{id} \to j^*j_*$ the map corresponding to the natural transformation $K \to \operatorname{Tot}\check{\mathrm{C}}^{\bullet}_{\mathrm{alt},\mathcal{U}}(K)$. This is an isomorphism of endofunctors of $\mathrm{D}^b \operatorname{Perv}_{mf}(U)$ by Lemma 7.1.3 (iii), and we set $\varepsilon = \alpha^{-1}$; this is the counit of the adjunction.

Now we construct the unit $\eta\colon \mathrm{id} \to j_*j^*$. As j^* is exact, the functor j^*j_* is isomorphic to the functor sending $L \in \operatorname{Ob}\mathrm{D}^b \operatorname{Perv}_{mf}(X)$ to the total complex of the double complex $j_*\check{\mathrm{C}}^{\bullet}_{\mathrm{alt},\mathcal{U}}(j^*L)$. If L is a finite complex of objects of $\operatorname{Perv}_{mf}(X)$, then

$$j_*\check{\mathrm{C}}^{0}_{\mathrm{alt},\mathcal{U}}(j^*L) = \bigoplus_{i \in I}(j \circ a_i)_*(j \circ a_i)^*L;$$

taking the sum of the unit morphisms of the adjunctions $((j \circ a_i)^*, (j \circ a_i)_*)$, we get a morphism $L \to j_*\check{\mathrm{C}}^{0}_{\mathrm{alt},\mathcal{U}}(j^*L)$ in $\mathrm{C}^b(\operatorname{Perv}_m(X))$, hence in its full subcategory $\mathrm{C}^b(\operatorname{Perv}_{mf}(X))$. Also, it is easy to see that the composition

$$L \to j_*\check{\mathrm{C}}^{0}_{\mathrm{alt},\mathcal{U}}(j^*L) \to j_*\check{\mathrm{C}}^{1}_{\mathrm{alt},\mathcal{U}}(j^*L)$$

is 0 in $\mathrm{C}^b(\operatorname{Perv}_{mf}(X))$. So we get a morphism from L to the total complex of the double complex $j_*\check{\mathrm{C}}^{\bullet}_{\mathrm{alt},\mathcal{U}}(j^*L)$, which induces $\eta\colon \mathrm{id} \to j_*j^*$.

To finish the proof, it suffices to show that, for every K in $D^b \operatorname{Perv}_{mf}(U)$ and every L in $D^b \operatorname{Perv}_{mf}(X)$, the composition

$$j_*K \xrightarrow{\eta j_*} j_*j^*j_*K \xrightarrow{j_*\varepsilon} j_*K$$

is the identity of j_*K and the composition

$$j^*L \xrightarrow{j^*\eta} j^*j_*j^*L \xrightarrow{\varepsilon j^*} j^*L$$

is the identity of j^*L.

Let $K \in \mathrm{C}^b(\mathrm{Perv}_{mf}(U))$. We denote by s the canonical morphism from K to the total complex of $\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(K)$. Using the canonical isomorphism between $j^* \operatorname{Tot} j_*\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(K)$ and $\operatorname{Tot}\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(K)$, we get a commutative diagram

$$\begin{array}{ccccc}
j_*K & \xrightarrow{\eta j_*} & j_*j^*j_*K & \xrightarrow{j_*\varepsilon} & j_*K \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\operatorname{Tot} j_*\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(K) & \xrightarrow[\operatorname{Tot} j_*\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(s)]{} & \operatorname{Tot} j_*\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}\big(\operatorname{Tot}\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(K)\big) & \xleftarrow[\operatorname{Tot} j_*\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(s)]{} & \operatorname{Tot} j_*\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(K)
\end{array}$$

This gives the first statement.

Let $L \in \mathrm{C}^b(\mathrm{Perv}_{mf}(X))$. We denote by s the canonical morphism $j^*L \to \check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(j^*L)$, and by s' the morphism from L to $\operatorname{Tot} j_*\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(j^*L)$ from the construction of η. We have a canonical isomorphism $j^* \operatorname{Tot} j_*\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(j^*L) = \operatorname{Tot}\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(j^*L)$, and this identifies j^*s' and s. So we get a commutative diagram

$$\begin{array}{ccccc}
j^*L & \xrightarrow{j^*\eta} & j^*j_*j^*L & \xrightarrow{\varepsilon j^*} & j^*L \\
& {}_{s}\searrow & \downarrow\wr & \swarrow_{s} & \\
& & \operatorname{Tot}\check{\mathrm{C}}^\bullet_{\mathrm{alt},\mathcal{U}}(j^*L) & &
\end{array}$$

This gives the second statement. ■

7.2. Inverse images

In this section, we construct the inverse images functors as the left adjoints of the direct image functors of Proposition 7.1.1.

First we treat a particular case. For every smooth equidimensional k-scheme X, we denote by $\mathbf{1}_X$ the constant sheaf on X, seen as an object of $\mathrm{Perv}_{mf}(X)[-\dim X]$.

Proposition 7.2.1. *Let $X, Y \in \mathrm{Ob}(\mathbf{Sch}/k)$, and suppose that X is smooth equidimensional. Let $p\colon X \times Y \to Y$ be the second projection.*

Then the functor $p_*\colon D^b\,\mathrm{Perv}_{mf}(X \times Y) \to D^b\,\mathrm{Perv}_{mf}(Y)$ admits a left adjoint p^*, which is given by $K \mapsto \mathbf{1}_X \boxtimes K$.

In particular, we get a natural isomorphism $\theta_p\colon p^* \circ R_Y \xrightarrow{\sim} R_{X\times Y} \circ p^*$.

Proof. Let p^* be as in the statement. It suffices to construct natural morphisms $\eta\colon \mathrm{id} \to p_*p^*$ and $\varepsilon\colon p^*p_* \to \mathrm{id}$ whose images by R are the unit and counit of the adjunction in the categories $D^b\,\mathrm{Perv}_m$, and such that

$$p_* \xrightarrow{\eta p_*} p_*p^*p_* \xrightarrow{p_*\varepsilon} p_*$$

is the identity. (As $R_{X\times Y}$ is conservative, we will automatically get the fact that $p^* \xrightarrow{\varepsilon p^*} p^*p_*p^* \xrightarrow{p^*\eta} p^*$ is an isomorphism.)

Let

$$a_X : X \to \operatorname{Spec} k$$

be the structural map. Note that, as $a_{X*}\mathbf{1}_X$ is an object of $\mathrm{D}^{\geq 0}\,\mathrm{Perv}_{mf}(\operatorname{Spec} k)$, we have

$$\begin{aligned}\operatorname{Hom}_{\mathrm{D}^b\,\mathrm{Perv}_{mf}(\operatorname{Spec} k)}(\mathbf{1}_{\operatorname{Spec} k}, a_{X*}\mathbf{1}_X) &= \operatorname{Hom}_{\mathrm{Perv}_{mf}(\operatorname{Spec} k)}(\mathbf{1}_{\operatorname{Spec} k}, \mathrm{H}^0 a_{X*}\mathbf{1}_X)\\ &= \operatorname{Hom}_{\mathrm{Perv}(\operatorname{Spec} k)}(E, \mathrm{H}^0 a_{X*}\underline{E}_X).\end{aligned}$$

So the canonical morphism $E \to \mathrm{H}^0 a_{X*}\underline{E}_X$ (coming from the unit of the adjunction (a_X^*, a_{X*})) gives a morphism $u_X : \mathbf{1}_{\operatorname{Spec} k} \to a_{X*}\mathbf{1}_X$ in $\mathrm{D}^b\,\mathrm{Perv}_{mf}(\operatorname{Spec} k)$).

If $K \in \operatorname{Ob}(\mathrm{D}^b\,\mathrm{Perv}_{mf}(Y))$, then we have a morphism

$$K = \mathbf{1}_{\operatorname{Spec} k} \boxtimes K \xrightarrow{u_X \boxtimes \mathrm{id}} (a_{X*}\mathbf{1}_X) \boxtimes K \simeq p_*(\mathbf{1}_X \boxtimes K) = p_*p^*K,$$

where the third arrow is the isomorphism of Proposition 7.1.6. This morphism is an isomorphism because its image by R_Y is an isomorphism, and we denote it by η.

Now we want to construct ε. Consider the commutative diagram

$$\begin{array}{ccccc} X \times Y & \xleftarrow{q_2} & X \times X \times Y & \xleftarrow{i} & X \times Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q_1} & & \\ Y & \xleftarrow[p]{} & X \times Y & & \end{array}$$

where $q_1 = \mathrm{id}_X \times p$, $q_2 = a_X \times \mathrm{id}_{X \times Y}$ and i is the product of the diagonal embedding of X and of id_Y. Note that $q_1 i = q_2 i = p$. Using Proposition 7.1.6, we get an isomorphism

$$p^*p_*K = \mathbf{1}_X \boxtimes (p_*K) \simeq q_{1*}(\mathbf{1}_X \boxtimes K) = q_{1*}q_2^*K.$$

As i is a closed immersion, we know (by Corollary 6.4.2) that the functor i_* has a left adjoint i^*. This and the functoriality of $H_{mf,*}$ gives a morphism

$$q_{1*}q_2^*K \to q_{1*}i_*i^*q_2^*K \simeq q_{2*}i_*i^*q_2^*K.$$

Note also that using the unit of (i^*, i_*) and the analogue of the natural transformation η for q_2 instead of p, we get a morphism

$$K \xrightarrow{\sim} q_{2*}q_2^*K \to q_{2*}i_*i^*q_2^*K,$$

which is an isomorphism because its image by $R_{X \times Y}$ is an isomorphism. Putting all these together gives

$$\varepsilon : p^*p_*K \xrightarrow{\sim} q_{1*}q_2^*K \to q_{2*}i_*i^*q_2^*K \simeq K.$$

It is clear from the construction that the images of η and ε by R are the unit and the counit of the adjunction (p^*, p_*) in D^b_m. So we just need to show that

$$p_* \xrightarrow{\eta p_*} p_*p^*p_* \xrightarrow{p_*\varepsilon} p_*$$

is the identity. This follows from the fact that we get this composition by following the outside of the commutative diagram below in the clockwise direction (where the two arrows marked “adj” come from the unit of the adjunction (i^*, i_*)):

$$\begin{array}{ccccccc}
 & & & & p_*(\mathbf{1}_X \boxtimes (p_*K)) & = & p_*p^*p_*K \\
 & & & & \downarrow \wr & & \\
\mathbf{1}_{\operatorname{Spec} k} \boxtimes (p_*K) & \xrightarrow{u_X} & (a_{X*}\mathbf{1}_X) \boxtimes (p_*K) & \xleftarrow{\sim} & p_*q_{1*}(\mathbf{1}_X \boxtimes K) & = & p_*q_{1*}q_2^*K \\
\wr\!\!\diagup & & & \diagup\!\sim & & & \downarrow \text{adj} \\
p_*K \xleftarrow{\sim} p_*(\mathbf{1}_{\operatorname{Spec} k} \boxtimes K) & & & & & & p_*q_{1*}i_*i^*q_2^*K \\
\downarrow u_X & & & & & & \downarrow \wr \\
p_*((a_{X*}\mathbf{1}_X) \boxtimes K) & \xrightarrow{\sim} & p_*q_{2*}(\mathbf{1}_X \boxtimes K) & = & p_*q_{2*}q_2^*K & \xrightarrow[\text{adj}]{} & p_*q_{2*}i_*i^*q_2^*K
\end{array} \quad \blacksquare$$

Having at our disposal the constant sheaf on X was very important when constructing the inverse image of the second projection $X \times Y \to Y$. Now, in order to generalize this construction to the case when X is not necessarily smooth, we want to construct (and characterize) the analogue in $\mathrm{D}^b \operatorname{Perv}_{mf}(X)$ of the constant sheaf $\underline{E}_X$. Note that this is not totally obvious in this context because, if X is not smooth, then the constant sheaf is not perverse (or shifted perverse) in general.

For every k-scheme X, we denote by $a_X \colon X \to \operatorname{Spec} k$ the structural morphism. We also denote by $\mathbf{1}_{\operatorname{Spec} k}$ the constant sheaf with value E on $\operatorname{Spec} k$, seen as an object of $\operatorname{Perv}_{mf}(\operatorname{Spec} k)$.

Corollary 7.2.2. *For every k-scheme X, the functor $\mathrm{D}^b \operatorname{Perv}_{mf}(X) \to \mathrm{Sets}$ (where Sets is the category of sets), $K \mapsto \operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(\operatorname{Spec} k)}(\mathbf{1}_{\operatorname{Spec} k}, a_{X*}K)$, is representable.*

Moreover, if $(\mathbf{1}_X, u_X \colon \mathbf{1}_{\operatorname{Spec} k} \to a_{X*}\mathbf{1}_X)$ represents this functor, then there is an isomorphism $R_X(\mathbf{1}_X) \simeq \underline{E}_X$ that makes the following diagram commute:

$$\begin{array}{ccccc}
R_{\operatorname{Spec} k}(\mathbf{1}_{\operatorname{Spec} k}) & = & \underline{E}_{\operatorname{Spec} k} & \xrightarrow{\text{adj}} & a_{X*}\underline{E}_X \\
\downarrow R_{\operatorname{Spec} k}(u_X) & & & & \downarrow \wr \\
R_{\operatorname{Spec} k}(a_{X*}\mathbf{1}_X) & & \xrightarrow[\gamma_{a_X}]{} & & a_{X*}R_X(\mathbf{1}_X)
\end{array}$$

where the arrow marked “adj” is the unit of the adjunction (a_X^*, a_{X*}).

Note that the couple $(\mathbf{1}_X, u_X)$ is unique up to unique isomorphism if it exists.

Proof. First note that, thanks to Corollary 6.4.2 and Proposition 7.1.7, if $h \colon Z \to X$ is an open embedding or a closed embedding and the result is true for X, then it is also true for Z, and moreover we have a canonical isomorphism $\mathbf{1}_Z \simeq h^*\mathbf{1}_X$. Moreover, if X is smooth, then the result follows immediately from Proposition 7.2.1. In particular, we get the result for X affine, because in that case X is a closed subscheme of some $\mathbb{A}^n$.

For a general k-scheme X, we chose a finite open cover $X = \bigcup_{i=1}^{n} U_i$ such that the result is known for every U_i. (For example, we can take a finite affine open cover.) We want to show that this implies the result for X. We reduce to the case $n = 2$ by an easy induction on n. Let $j_1\colon U_1 \to X$, $j_2\colon U_2 \to X$ and $j_{12}\colon U_1 \cap U_2 \to X$ be the inclusions. By the uniqueness statement of the corollary, we have canonical isomorphisms $\mathbf{1}_{U_i|U_1\cap U_2} \simeq \mathbf{1}_{U_1\cap U_2}$ for $i = 1, 2$ that identify u_{U_i} and $u_{U_1\cap U_2}$, so, using Proposition 7.1.7, we get morphisms $v_i\colon j_{i*}\mathbf{1}_{U_i} \to j_{12,*}\mathbf{1}_{U_1\cap U_2}$, $i = 1, 2$. Complete $v := v_1 \oplus (-v_2)$ into an exact triangle

$$(*) \quad K \to j_{1*}\mathbf{1}_{U_1} \oplus j_{2*}\mathbf{1}_{U_2} \xrightarrow{v} j_{12*}\mathbf{1}_{U_1\cap U_2} \xrightarrow{+1}$$

Applying a_{X*}, we get a triangle

$$(**) \quad a_{X*}K \to a_{U_1,*}\mathbf{1}_{U_1} \oplus a_{U_2,*}\mathbf{1}_{U_2} \xrightarrow{a_{X*}v} a_{U_1\cap U_2,*}\mathbf{1}_{U_1\cap U_2} \xrightarrow{+1}$$

Consider the morphism $u_{U_1} \oplus u_{U_2}\colon \mathbf{1}_{\operatorname{Spec} k} \to a_{U_1,*}\mathbf{1}_{U_1} \oplus a_{U_2,*}\mathbf{1}_{U_2}$. Composing it by $a_{X*}v$ gives 0, by definition of v, so it comes from a map $u_X\colon \mathbf{1}_{\operatorname{Spec} k} \to a_{X*}K$. Also, as $a_{U_1\cap U_2,*}\mathbf{1}_{U_1\cap U_2}$ is concentrated in degree ≥ 0, we have

$$\operatorname{Hom}_{D^b \operatorname{Perv}_{mf}(\operatorname{Spec} k)}\left(\mathbf{1}_{\operatorname{Spec} k}, a_{U_1\cap U_2,*}\mathbf{1}_{U_1\cap U_2}[-1]\right) = 0,$$

and so the map u_X is uniquely determined.

Now we show that (K, u_X) represents the functor of the statement. For every $L \in \operatorname{Ob}(D^b \operatorname{Perv}_{mf}(X))$, the map $u_X\colon \mathbf{1}_{\operatorname{Spec} k} \to a_{X*}K$ induces a morphism

$$\begin{aligned}\operatorname{Hom}_{D^b \operatorname{Perv}_{mf}(X)}(K, L) &\to \operatorname{Hom}_{D^b \operatorname{Perv}_{mf}(\operatorname{Spec} k)}(a_{X*}K, a_{X*}L)\\ &\to \operatorname{Hom}_{D^b \operatorname{Perv}_{mf}(\operatorname{Spec} k)}(\mathbf{1}_{\operatorname{Spec} k}, a_{X*}L),\end{aligned}$$

and we must show that this is an isomorphism. Suppose that we can prove this if one of the adjunction maps $L \to j_{1*}j_1^*L$, $L \to j_{2*}j_2^*L$ or $L \to j_{12*}j_{12}^*L$ is an isomorphism, then we are done. Indeed, for a general L, we have an exact triangle $L \to j_{1*}j_1^*L \oplus j_{2*}j_2^*L \to j_{12*}j_{12}^*L \xrightarrow{+1}$, and we use the five lemma.

Suppose that the adjunction map $L \to j_{1*}j_1^*L$ is an isomorphism. Applying j_1^* to the triangle $(*)$ and noting that $j_1^*j_{2*}\mathbf{1}_{U_2} \to j_1^*j_{12*}\mathbf{1}_{U_1\cap U_2}$ is an isomorphism, we get an isomorphism $j_1^*K \xrightarrow{\sim} \mathbf{1}_{U_1}$. We denote by c the base change morphism $a_{X*} \to a_{U_1*}j_1^*$. Applying c to the entries of the triangle $(**)$, we get a commutative diagram

$$\begin{array}{ccccccc} a_{U_1*}j_1^*K & \longrightarrow & a_{U_1,*}j_1^*j_{1*}\mathbf{1}_{U_1} \oplus a_{U_1,*}j_1^*j_{2*}\mathbf{1}_{U_2} & \longrightarrow & a_{U_1*}j_1^*j_{12*}\mathbf{1}_{U_1\cap U_2} & \xrightarrow{+1} \\ \uparrow & & \uparrow & & \uparrow & \\ a_{X*}K & \longrightarrow & a_{U_1,*}\mathbf{1}_{U_1} \oplus a_{U_2,*}\mathbf{1}_{U_2} & \longrightarrow & a_{U_1\cap U_2,*}\mathbf{1}_{U_1\cap U_2} & \xrightarrow{+1} \end{array}$$

The morphism $a_{U_1,*}j_1^*j_{2*}\mathbf{1}_{U_2} \to a_{U_1*}j_1^*j_{12*}\mathbf{1}_{U_1\cap U_2}$ in the first row of this diagram is an isomorphism, so we get an isomorphism $a_{U_1*}j_1^*K \to a_{U_1,*}j_1^*j_{1*}\mathbf{1}_{U_1} \simeq a_{U_1*}\mathbf{1}_{U_1}$ (which is just the image by a_{U_1*} of the isomorphism $j_1^* K \xrightarrow{\sim} \mathbf{1}_{U_1}$ of the beginning of this paragraph). By this isomorphism, the map $c_K \circ u_X \colon \mathbf{1}_{\operatorname{Spec} k} \to a_{U_1*} j_1^* K$ corresponds to the composition of

$$(c_{j_{1*}\mathbf{1}_{U_1}} \oplus c_{j_{2*}\mathbf{1}_{U_2}}) \circ (u_{U_1} \oplus u_{U_2}) \colon \mathbf{1}_{\operatorname{Spec} k} \to a_{U_1,*} j_1^* j_{1*} \mathbf{1}_{U_1} \oplus a_{U_1,*} j_1^* j_{2*} \mathbf{1}_{U_2}$$

and of the first projection. In other words, we get a commutative diagram:

$$\begin{array}{ccc} \mathbf{1}_{\operatorname{Spec} k} & \xrightarrow{u_X} & a_{X*} K \\ {\scriptstyle u_{U_1}} \downarrow & & \downarrow {\scriptstyle c_K} \\ a_{U_1*} \mathbf{1}_{U_1} & \xleftarrow[\sim]{} & a_{U_1*} j_1^* K \end{array}$$

Consider the following diagram (where all the Hom groups are taken in the appropriate $\mathrm{D}^b \operatorname{Perv}_{mf}$ category):

$$\begin{array}{ccccc} \operatorname{Hom}(K, j_{1*} j_1^* L) & \xrightarrow{a_{X*}} & \operatorname{Hom}(a_{X*} K, a_{X*} j_{1*} j_1^* L) & \xrightarrow{(-)\circ u_X} & \operatorname{Hom}(\mathbf{1}_{\operatorname{Spec} k}, a_{U_1*} j_1^* L) \\ \wr \downarrow {\scriptstyle j_1^*} & & \uparrow {\scriptstyle (-)\circ c_K} & & \| \\ \operatorname{Hom}(j_1^*, j_1^* L) & \xrightarrow{a_{U_1*}} & \operatorname{Hom}(a_{U_1*} j_1^* K, a_{U_1*} j_1^* L) & & \\ \wr \uparrow & & \wr \uparrow & & \\ \operatorname{Hom}(\mathbf{1}_{U_1}, j_1^* L) & \xrightarrow[a_{U_1*}]{} & \operatorname{Hom}(a_{U_1*} \mathbf{1}_{U_1}, a_{U_1*} j_1^* L) & \xrightarrow[(-)\circ u_{U_1}]{} & \operatorname{Hom}(\mathbf{1}_{\operatorname{Spec} k}, a_{U_1*} j_1^* L) \end{array}$$

We have just seen that the right rectangle of this diagram is commutative. It is also easy to see that the two squares on the left are commutative, so the whole diagram commutes. As the composition of the two bottom horizontal arrows is an isomorphism by assumption, the composition of the two top horizontal arrows is also an isomorphism, which is what we wanted to prove.

The case where $L \to j_{2*} j_2^* L$ (resp. $L \to j_{12*} j_{12}^* L$) is an isomorphism is similar. This finishes the proof of the first statement of the corollary. The second statement of the corollary follows easily from the explicit definition of u_X. ∎

Now that we have the object $\mathbf{1}_X$, the proof of the following corollary is exactly the same as the proof of Proposition 7.2.1.

Corollary 7.2.3. *Let $X, Y \in \operatorname{Ob}(\mathbf{Sch}/k)$, and let $p \colon X \times Y \to Y$ be the second projection.*

Then the functor $p_* \colon \mathrm{D}^b \operatorname{Perv}_{mf}(X \times Y) \to \mathrm{D}^b \operatorname{Perv}_{mf}(Y)$ admits a left adjoint p^*, which is given by $K \mapsto \mathbf{1}_X \boxtimes K$.

Corollary 7.2.4. *The 2-functor $H_{mf,*} \colon \mathbf{Sch}/k \to \mathfrak{TR}$ of Proposition 7.1.1 admits a global left adjoint in the sense of* [1, Definition 1.1.18].

In particular, we get a uniquely determined 2-functor $H_{mf}^* \colon \mathbf{Sch}/k \to \mathfrak{TR}$ such, for every morphism $f \colon X \to Y$ in $\mathbf{Sch}/k$, the functor $H_{mf}^*(f)$ from $\mathrm{D}^b \operatorname{Perv}_{mf}(Y)$ to $\mathrm{D}^b \operatorname{Perv}_{mf}(X)$ is a left adjoint of $H_{mf,*}(f) \colon \mathrm{D}^b \operatorname{Perv}_{mf}(X) \to \mathrm{D}^b \operatorname{Perv}_{mf}(Y)$.

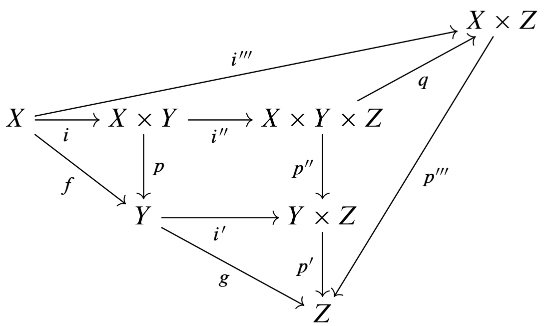
Moreover, for every morphism of k-schemes $f: X \to Y$, we have an invertible natural transformation $\theta_f: H_m^*(f) \circ R_Y \xrightarrow{\sim} R_X \circ H_{mf}^*(f)$, and this is compatible with the composition of morphisms in **Sch**/k.

Proof. By [1, Proposition 1.1.17], to show the first statement, it suffices to show that, for every $f: X \to Y$ in **Sch**/k, the functor $H_{mf,*}(f): \mathrm{D}^b\,\mathrm{Perv}_{mf}(X) \to \mathrm{D}^b\,\mathrm{Perv}_{mf}(Y)$ admits a left adjoint. We factor f as $X \xrightarrow{i} X \times Y \xrightarrow{p} Y$, where $i = \mathrm{id}_X \times f$ and p is the second projection. The first map is a closed embedding, so it admits a left adjoint by Corollary 6.4.2, and the second map admits a left adjoint by Corollary 7.2.3. The natural transformation θ_i and θ_p are also constructed in these corollaries, and we take θ_f equal to:

$$R_X \circ f^* = R_X \circ i^* p^* \xrightarrow{\theta_i} i^* \circ R_{X\times Y} \circ p^* \xrightarrow{\theta_p} i^* p^* \circ R_Y \simeq f^* \circ R_Y.$$

By a slight abuse of notation, we will write that $\theta_f = \theta_p \circ \theta_i$.

Suppose that we are given a second morphism $g: Y \to Z$, and that we are trying to prove the compatibility between θ_f, θ_g and θ_{gf}. Consider the commutative diagram:



where $i' = \mathrm{id}_Y \times g$, $i''(x, y) = (x, y, g(y))$, $i''' = \mathrm{id}_X \times (gf)$ and p', p'', p''', q are the obvious projections. Then $\theta_g = \theta_{p'} \circ \theta_{i'}$ and $\theta_{gf} = \theta_{p'''} \circ \theta_{i'''}$. So it suffices to prove that:

(a) $\theta_q \circ \theta_{i''i} = \theta_{i'''}$;

(b) $\theta_{i''i} = \theta_{i''}\theta_i$;

(c) $\theta_{p''} \circ \theta_q = \theta_{p'} \circ \theta_{p''}$;

(d) $\theta_{p''} \circ \theta_{i''} = \theta_{i'} \circ \theta_p$.

Point (b) follows from Corollary 6.4.2 and point (c) from the explicit formula for the inverse image of a projection in Corollary 7.2.3. The other two compatibilities can easily be proved directly. ∎

Finally, we have the following.

Proposition 7.2.5. *The functor* ⊠ *of Proposition 4.2 induces a natural isomorphism from the 2-functor*

$$H_{mf}^* \times H_{mf}^*: \mathbf{Sch}_k \times \mathbf{Sch}_k \to \mathfrak{TR}$$

to the 2*-functor*

$$\mathbf{Sch}_k \times \mathbf{Sch}_k \xrightarrow{\times} \mathbf{Sch}_k \xrightarrow{H^*_{mf}} \mathfrak{TR},$$

where the first arrow sends (X, Y) *to* $X \times Y$*.*

In other words, if $f_1 \colon X_1 \to Y_1$ *and* $f_2 \colon X_2 \to Y_2$ *are morphisms in* $\mathbf{Sch}/k$ *and* $L_1 \in \mathrm{D}^b \operatorname{Perv}_{mf}(Y_1)$*,* $L_2 \in \mathrm{D}^b \operatorname{Perv}_{mf}(Y_2)$*, then we have an isomorphism*

$$(f_1 \times f_2)^*(L_1 \boxtimes L_2) \xrightarrow{\sim} (f_1^* L_1) \boxtimes (f_2^* L_2)$$

functorial in L_1 *and* L_2 *and compatible with the composition of arrows in* $\mathbf{Sch}/k$*.*

Proof. By the construction of the functors f^* above, we only need to show the statement when f_1 and f_2 are both closed immersions, or when they are both projections. If f_1 and f_2 are both projections, the result is obvious. If they are both closed immersions, the result follows from the construction in the proof of Corollary 6.4.2 and from Proposition 7.1.6. ■

7.3. Poincaré–Verdier duality

Just as in Sections 7.1 and 7.2, we can prove the following result.

Proposition 7.3.1. *There exists a* 2*-functor* $H_{mf,!} \colon \mathbf{Sch}/k \to \mathfrak{TR}$ *satisfying* $H_{mf,!}(X) = \mathrm{D}^b \operatorname{Perv}_{mf}(X)$ *for every* $X \in \operatorname{Ob}(\mathbf{Sch}/k)$ *and a natural transformation* $R \colon H_{mf,!} \to H_{m,!}$ *(with the notation of Example 3.2.3) such that:*

- (a) *for every* $X \in \operatorname{Ob}(\mathbf{Sch}/k)$*, the functor* $R_X \colon \mathrm{D}^b \operatorname{Perv}_{mf}(X) \to \mathrm{D}^b_m(X)$ *is the functor of Theorem 3.2.4;*
- (b) *for every morphism* $f \colon X \to Y$*, the natural transformation* $\rho_f \colon R_Y \circ H_{mf,!}(f) \to H_{m,!}(f) \circ R_X$ *is an isomorphism.*

This functor satisfies the same compatibility with $\boxtimes$ *as in Proposition 7.1.6, and it admits a global right adjoint* $H^!_{mf}$*.*

Moreover, by Proposition 4.2, we have an exact contravariant endofunctor D_X of the category $\mathrm{D}^b \operatorname{Perv}_{mf}(X)$ together with an isomorphism

$$D_X^2 \simeq \mathrm{id}, \quad \text{for every } X \in \operatorname{Ob}(\mathbf{Sch}/k).$$

Proposition 7.3.2. *Let* $f \colon X \to Y$ *be a morphism of* k*-schemes.*

(i) *We have a natural isomorphism* $\alpha_f \colon f_* \xrightarrow{\sim} D_Y \circ f_! \circ D_X$ *such that, if* $g \colon Y \to Z$ *is another morphism of* k*-schemes, then the isomorphism* $\alpha_{gf} \colon (gf)_* \simeq D_Z (gf)_! D_X$ *is equal to the isomorphism*

$$(gf)_* \simeq g_* f_* \xrightarrow{\alpha_g \alpha_f} D_Z g_! D_Y D_Y f_! D_X \simeq D_Z g_! f_! D_X \simeq D_Z (gf)_! D_X$$

where the first and fourth arrows are given by the composition isomorphisms of the 2*-functors* $H_{mf,*}$ *and* $H_{mf,!}$*, and the third arrow is given by the isomorphism* $D_Y^2 \simeq \mathrm{id}$*.*

(ii) *We have a natural isomorphism* $\beta_f : f^* \xrightarrow{\sim} D_X \circ f^! \circ D_Y$, *where* $f^! = H^!_{mf}(f)$, *such that, if* $g : Y \to Z$ *is another morphism of* k*-schemes, then the isomorphism* $\beta_{gf}(gf)^* \simeq D_X(gf)^! D_Z$ *is equal to the isomorphism*

$$(gf)^* \simeq f^* g^* \xrightarrow{\beta_f \beta_g} D_X f^! D_Y D_Y g^! D_Z \simeq D_X f^! g^! D_Z \simeq D_X (gf)^! D_Z$$

where the first and fourth arrows are given by the composition isomorphisms of the 2*-functors* H^*_{mf} *and* $H^!_{mf}$, *and the third arrow is given by the isomorphism* $D_Y^2 \simeq \mathrm{id}$.

(iii) *If* f *is smooth and purely of relative dimension* d, *then we have an natural isomorphism* $f^![-d] \simeq f^*[d](d)$ *of functors* $\mathrm{D}^b \operatorname{Perv}_{mf}(Y) \to \mathrm{D}^b \operatorname{Perv}_{mf}(X)$.

(iv) *If* f *is smooth and purely of relative dimension* d, *then the functor* f^* *from the category* $\mathrm{D}^b \operatorname{Perv}_{mf}(Y)$ *to* $\mathrm{D}^b \operatorname{Perv}_{mf}(X)$ *admits a left adjoint* $f_\sharp$.

(v) *If* $i : X \to Y$ *is a closed immersion, then we have a natural isomorphism* $i_! \xrightarrow{\sim} i_*$.

Proof. Point (iii) follows from the fact that both functors are t-exact and that such an isomorphism exists in the category of functors $\operatorname{Perv}_{mf}(Y) \to \operatorname{Perv}_{mf}(X)$ (because it does for mixed perverse sheaves and the categories Perv_{mf} are full subcategories of the categories of mixed perverse sheaves).

Point (iv) follows from point (iii): take $f_\sharp = f_![2d](d)$.

Point (v) is proved like point (iii): both functors are t-exact, and the natural isomorphism exists when we see $i_!$ and i_* as functors from $\operatorname{Perv}_m(X)$ to $\operatorname{Perv}_m(Y)$.

Let us prove (i). By the construction of f_* in Section 7.1 (and point (iii) applied to inverse images by open immersions) it suffices to prove the analogous result for the functors ${}^p\mathrm{H}^0 f_*$ and ${}^p\mathrm{H}^0 f_!$ if f is affine. But then this follows from the case of the categories $\mathrm{D}^b_c(X)$.

Point (ii) now follows from (i) and from the uniqueness of adjoint functors. ∎

8. Tensor products and internal Homs

Definition 8.1. Let X be a k-scheme. We denote by $\Delta_X : X \to X \times X$ the diagonal embedding. We define a functor $\otimes_X : (\mathrm{D}^b \operatorname{Perv}_{mf}(X))^2 \to \mathrm{D}^b \operatorname{Perv}_{mf}(X)$ by $K \otimes_X L = \Delta_X^*(K \boxtimes L)$.

Note that it follows from Proposition 7.2.5 that, for every morphism of k-schemes $f : X \to Y$ and all $K, L \in \operatorname{Ob} \mathrm{D}^b \operatorname{Perv}_{mf}(Y)$, we have a canonical isomorphism

$$f^*(K \otimes_Y L) \simeq (f^*K) \otimes_X (f^*L).$$

Proposition 8.2. *The operation* $\otimes_X$ *defined above makes* $\mathrm{D}^b \operatorname{Perv}_{mf}(X)$ *into a symmetric monoïdal triangulated category. Also, the object* $\mathbf{1}_X$ *constructed in Corollary* 7.2.2 *is a unit for* $\otimes_X$, *and the functor* $R_X : \mathrm{D}^b \operatorname{Perv}_{mf}(X) \to \mathrm{D}^b_m(X)$ *is symmetric monoïdal unitary.*

Proof. The first statement follows easily from the commutativity and associativity of $\boxtimes$ (which in turn follows from the similar statement in $\mathrm{D}^b_m(X)$, as $\boxtimes$ is exact). Moreover, for every $K \in \mathrm{Ob}\,\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$, if $p\colon X \times X \to X$ is the second projection, then:

$$K \otimes_X \mathbf{1}_X = \Delta_X^*(K \boxtimes \mathbf{1}_X) = \Delta_X^* p^* K \simeq (p\Delta_X)^* K \simeq K$$

because $p\Delta_X = \mathrm{id}_X$. This proves the second statement. Finally, the fact that R_X is monoïdal follows from the fact that it preserves $\boxtimes$, and the last statement of Corollary 7.2.2 (i.e., the isomorphism $R_X(\mathbf{1}_X) \simeq \underline{E}_X$) implies that R_X is unitary. ∎

The main result of this section is the following.

Proposition 8.3. *For every k-scheme X and every $K \in \mathrm{Ob}\,\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$, the endofunctor $K \otimes_X \cdot$ of $\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$ has a right adjoint $\underline{\mathrm{Hom}}(K, \cdot)$, given by $L \mapsto D_X(K \otimes_X D_X(L))$. Moreover, for all $K, L, M \in \mathrm{Ob}\,\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$, if we set $K' = R_X(K)$, $L' = R_X(L)$ and $M' = R_X(M)$, then we have a commutative diagram*

$$\begin{array}{ccc} R\,\mathrm{Hom}_{\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, M) & \xrightarrow{\sim} & R\,\mathrm{Hom}_{\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)}\big(L, D_X(K \otimes_X D_X M)\big) \\ {\scriptstyle R_X}\downarrow & & \downarrow{\scriptstyle R_X} \\ R\,\mathrm{Hom}_{\mathrm{D}^b_m(X)}(K' \otimes_X L', M') & \xrightarrow[\sim]{} & R\,\mathrm{Hom}_{\mathrm{D}^b_m(X)}\big(L', D_X(K' \otimes_X D_X M')\big) \end{array}$$

where the horizontal arrows are the adjunction isomorphisms (see Lemma 8.4 for the natural isomorphism $D_X(K' \otimes_X D_X M') \simeq \underline{\mathrm{Hom}}_X(K', M')$).

In the lemmas that follow, we will denote the structural morphism $X \to \mathrm{Spec}\,k$ by a. Remember that we write $K_X = a^! \underline{E}_{\mathrm{Spec}\,k}$ for the dualizing complex in $\mathrm{D}^b_h(X)$. This is an object of $\mathrm{D}^b_m(X)$ (because $\underline{E}_{\mathrm{Spec}\,k}$ clearly is a mixed complex, and $a^!$ preserves mixed complexes).

Lemma 8.4. *In the category $\mathrm{D}^b_h(X)$, we have a canonical isomorphism, functorial in K and L:*

$$\underline{\mathrm{Hom}}_X(K \otimes_X L, K_X) \simeq \underline{\mathrm{Hom}}_X\big(K, D_X(L)\big).$$

Moreover, these complexes are concentrated in perverse degree ≥ 0 if K and L are perverse.

In particular, if we replace L by $D_X(L)$, we get a natural isomorphism

$$\underline{\mathrm{Hom}}_X(K, L) \simeq D_X\big(K \otimes_X D_X(L)\big),$$

which explains the definition of the internal Hom given in Proposition 8.3.

Proof. By [10, Theorem 6.3 (ii)] (see also the remark following [14, Definition 1.2] for the extension of this to the category $\mathrm{D}^b_h(X)$), we have a natural isomorphism

$$\underline{\mathrm{Hom}}_X(K \otimes_X L, M) = \underline{\mathrm{Hom}}_X\big(K, \underline{\mathrm{Hom}}_X(L, M)\big)$$

for all $K, L, M \in \mathrm{Ob}\,\mathrm{D}^b_h(X)$. Applying this to $M = K_X$ gives the desired isomorphism.

If K and L are perverse, then the complex $K \otimes_X L$ is concentrated in perverse degree ≤ 0 (because it is equal by definition to $\Delta_X^*(K \boxtimes L)$, where $\Delta_X \colon X \to X \times X$ is the diagonal morphism, and Δ_X^* is right t-exact), so its dual $\underline{\mathrm{Hom}}_X(K \otimes_X L, K_X)$ is concentrated in perverse degree ≥ 0. ∎

Lemma 8.5. *If $K, L \in \mathrm{Ob}\,\mathrm{Perv}_h(X)$, then the complex $a_!(K \otimes_X L) \in \mathrm{D}_h^b(\mathrm{Spec}\,k)$ is concentrated in degree ≤ 0, and so the adjunction $(a_!, a^!)$ gives a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{D}_h^b(X)}(K \otimes_X L, K_X) \simeq \mathrm{Hom}_{\mathrm{Perv}_h(\mathrm{Spec}\,k)}\left(\mathrm{H}^0\left(a_!(K \otimes_X L)\right), \underline{E}_{\mathrm{Spec}\,k}\right)$$

and equalities

$$\mathrm{Ext}^i_{\mathrm{D}_h^b(X)}(K \otimes_X L, K_X) = 0$$

for every $i < 0$.

We could also have deduced the vanishing of $\mathrm{Ext}^i_{\mathrm{D}_h^b(X)}(K \otimes_X L, K_X)$ for $i < 0$ from the adjunction isomorphism

$$\mathrm{Ext}^i_{\mathrm{D}_h^b(X)}(K \otimes_X L, K_X) = \mathrm{Ext}^i\left(K, D_X(L)\right).$$

(But we will not be able to do this in the next lemma, which is the analogous statement in $\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$.)

Proof. We have

$$a_!(K \otimes_X L) \simeq D_{\mathrm{Spec}\,k}\left(a_* D_X(K \otimes_X L)\right) \simeq D_{\mathrm{Spec}\,k}\left(a_* \underline{\mathrm{Hom}}_X\left(K, D_X(L)\right)\right),$$

where the second isomorphism comes from Lemma 8.4. So it suffices to show that

$$a_* \underline{\mathrm{Hom}}_X\left(K, D_X(L)\right) = R\,\mathrm{Hom}_{\mathrm{D}_h^b(X)}\left(K, D_X(L)\right)$$

is concentrated in degree ≥ 0. As K and $D_X(L)$ are perverse, this just follows from the definition of a t-structure.

Now, using the adjunction $(a_!, a^!)$ and the fact that $K_X = a^! \underline{E}_{\mathrm{Spec}\,k}$, we get a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{D}_h^b(X)}(K \otimes_X L, K_X) = \mathrm{Hom}_{\mathrm{D}_h^b(\mathrm{Spec}\,k)}\left(a_!(K \otimes_X L), \underline{E}_{\mathrm{Spec}\,k}\right).$$

The second statement follows from this and from the fact that $a_!(K \otimes_X L)$ is concentrated in degree ≤ 0. ∎

Lemma 8.6. *If $K, L \in \mathrm{Ob}\,\mathrm{Perv}_{mf}(X)$, then the complex $a_!(K \otimes_X L) \in \mathrm{D}^b\,\mathrm{Perv}_{mf}(\mathrm{Spec}\,k)$ is concentrated in degree ≤ 0, and so the adjunction $(a_!, a^!)$ gives a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, a^! \mathbf{1}_{\mathrm{Spec}\,k}) \simeq \mathrm{Hom}_{\mathrm{Perv}_{mf}(\mathrm{Spec}\,k)}\left(\mathrm{H}^0\left(a_!(K \otimes_X L)\right), \mathbf{1}_{\mathrm{Spec}\,k}\right)$$

and equalities

$$\mathrm{Ext}^i_{\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)}(K \otimes_X L, a^! \mathbf{1}_X) = 0$$

for every $i < 0$.

Proof. We have

$$R_X\big(a_!(K \otimes_X L)\big) \simeq a_!\big(R_X(K) \otimes_X R_X(L)\big),$$

so $R_X(a_!(K \otimes_X L))$ is concentrated in degree ≤ 0 by Lemma 8.5. The first statement follows from the conservativity of R_X. The second statement is proved exactly as the second statement of Lemma 8.5, using the adjunction $(a_!, a^!)$ in the categories $\mathrm{D}^b \operatorname{Perv}_{mf}$. ■

Lemma 8.7. *Let* $K, L \in \operatorname{Ob}\operatorname{Perv}_{mf}(X)$*, write* $K' = R_X(K)$*,* $L' = R_X(L)$*. Then the morphism*

$$R_X : \operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}\big(K \otimes_X L, D_X(\mathbf{1}_X)\big) \to \operatorname{Hom}_{\mathrm{D}^b_h(X)}(K' \otimes_X L', K_X)$$

is an isomorphism. In particular, there exists a unique isomorphism

$$\alpha_{K,L} : \operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}\big(K \otimes_X L, D_X(\mathbf{1}_X)\big) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{Perv}_{mf}(X)}\big(K, D_X(L)\big)$$

making the following diagram commute

$$\begin{array}{ccc} \operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}\big(K \otimes_X L, D_X(\mathbf{1}_X)\big) & \xrightarrow{\alpha_{K,L}} & \operatorname{Hom}_{\operatorname{Perv}_{mf}(X)}\big(K, D_X(L)\big) \\ \downarrow R_X & & \downarrow R_X \\ \operatorname{Hom}_{\mathrm{D}^b_h(X)}(K' \otimes_X L', K_X) & \xrightarrow[\sim]{} & \operatorname{Hom}_{\operatorname{Perv}_h(X)}\big(K', D_X(L')\big) \end{array}$$

where the bottom isomorphism comes from applying the functor $\mathrm{H}^0(X, .)$ *to the isomorphism of Lemma* 8.4.

Proof. As $\operatorname{Perv}_{mf}(X)$ is a full subcategory of $\operatorname{Perv}_h(X)$, the morphism

$$R_X : \operatorname{Hom}_{\operatorname{Perv}_{mf}(X)}\big(K, D_X(L)\big) \to \operatorname{Hom}_{\operatorname{Perv}_h(X)}\big(K', D_X(L')\big)$$

is an isomorphism. So we just need to show that

$$R_X : \operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}\big(K \otimes_X L, D_X(\mathbf{1}_X)\big) \to \operatorname{Hom}_{\mathrm{D}^b_h(X)}(K' \otimes_X L', K_X)$$

is an isomorphism. By Lemmas 8.5 and 8.6, we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}(K \otimes_X L, a^! \mathbf{1}_{\operatorname{Spec} k}) & \xrightarrow{\sim} & \operatorname{Hom}_{\operatorname{Perv}_{mf}(\operatorname{Spec} k)}\big(\mathrm{H}^0\big(a_!(K \otimes_X L)\big), \mathbf{1}_{\operatorname{Spec} k}\big) \\ \downarrow R_X & & \downarrow R_{\operatorname{Spec} k} \\ \operatorname{Hom}_{\mathrm{D}^b_h(X)}(K' \otimes_X L', K_X) & \xrightarrow[\sim]{} & \operatorname{Hom}_{\operatorname{Perv}_h(\operatorname{Spec} k)}\big(\mathrm{H}^0\big(a_!(K' \otimes_X L')\big), \underline{E}_{\operatorname{Spec} k}\big) \end{array}$$

The right vertical map in this diagram is an isomorphism because $\operatorname{Perv}_{mf}(\operatorname{Spec} k)$ is a full subcategory of $\operatorname{Perv}_h(\operatorname{Spec} k)$, so the left vertical map is also an isomorphism. ■

We use the formalism of filtered derived categories, which is recalled at the beginning of Section A.1. Let $\mathscr{A}$ be an abelian category, and let $\mathrm{DF}^b(\mathscr{A})$ be its bounded filtered derived category. Let us recall the spectral sequence of [6, equation (3.1.3.4)]: If K and L are two objects of $\mathrm{DF}^b(A)$, then we have a spectral sequence

$$E_1^{pq} = \bigoplus_{j-i=p} \mathrm{Ext}^{p+q}_{\mathrm{D}(\mathcal{A})}(\mathrm{Gr}^i_F K, \mathrm{Gr}^j_F L) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{D}(\mathcal{A})}\big(\omega(K), \omega(L)\big).$$

Remember that $\omega\colon \mathrm{DF}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{A})$ is the functor that forgets the filtration.

Lemma 8.8. *Let $K_1^\bullet$, $K_2^\bullet$ be bounded complexes of objects of* $\mathrm{Perv}_h(X)$*, and let K_1, K_2 be their images by* real: $\mathrm{D}^b\,\mathrm{Perv}_h(X) \to \mathrm{D}^b_h(X)$. *Then, for every object L of $\mathrm{D}^b_h(X)$, we have a spectral sequence*

$$E_1^{pq} = \bigoplus_{a-b=-p} \mathrm{Ext}^q_{\mathrm{D}^b_h(X)}\big(K_1^a \otimes_X D_X(K_2^b), L\big) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{D}^b_h(X)}\big(K_1 \otimes_X D_X(K_2), L\big).$$

Proof. By definition of the category $\mathrm{D}^b_h(X)$, it suffices to prove the statement in $\mathrm{D}^b_c(\mathcal{X})$, where $(A, \mathcal{X}, u)$ is an object of $\mathcal{U}X$ such that all the K_r^i (resp. L) extend to shifts of objects of $\mathrm{Perv}(\mathcal{X})$ (resp. $\mathrm{D}^b_c(\mathcal{X})$), that we will denote by the same letters.

Remember the construction of the realization functor $\mathrm{D}^b\,\mathrm{Perv}(\mathcal{X}) \to \mathrm{D}^b_c(\mathcal{X})$ in Section A.2. We consider the full subcategory $\mathrm{DF}_{\text{bête}}(\mathcal{X})$ of objects A of $\mathrm{DF}^b(\mathcal{X}_{\text{proét}})$ such that $\mathrm{Gr}^i_F A[i]$ is in $\mathrm{Perv}(\mathcal{X})$ for every $i \in \mathbb{Z}$ and 0 for $|i|$ big enough. We have an equivalence $G\colon \mathrm{DF}_{\text{bête}}(\mathcal{X}) \to C^b(\mathrm{Perv}(\mathcal{X}))$ (see [6, Section 3.1.7] or Theorem A.2.3), and real is induced by $\omega \circ G^{-1}\colon C^b(\mathrm{Perv}(\mathcal{X})) \to \mathrm{D}^b_c(\mathcal{X})$.

Let $\Delta\colon X \to X \times X$ be the diagonal morphism. As $K_1 \otimes_X D_X(K_2)$ is equal to $\Delta^*(K_1 \boxtimes D_X(K_2))$, we have a canonical isomorphism

$$R\,\mathrm{Hom}_{\mathrm{D}^b_c(\mathcal{X})}\big(K_1 \otimes_X D_X(K_2), L\big) = R\,\mathrm{Hom}_{\mathrm{D}^b_c(\mathcal{X}\times\mathcal{X})}\big(K_1 \boxtimes D_X(K_2), \Delta_* L\big).$$

Let $M = G^{-1}(K_1^\bullet \boxtimes D_X(K_2^\bullet)) \in \mathrm{Ob}\,\mathrm{DF}_{\text{bête}}(\mathcal{X} \times \mathcal{X})$. We can also see $\Delta_* L$ as an object of $\mathrm{DF}((\mathcal{X} \times \mathcal{X})_{\text{proét}})$ (because, for any abelian category $\mathcal{A}$, the category $\mathrm{D}^b(\mathcal{A})$ is canonically equivalent to the full subcategory of $A \in \mathrm{Ob}\,\mathrm{DF}^b(\mathcal{A})$ such that $\mathrm{Gr}^i_F A = 0$ for $i \neq 0$). Using the spectral sequence recalled before the statement of the lemma (and (iii) of Proposition 2.3.1), we get a spectral sequence

$$E_1^{pq} = \bigoplus_{-i=p} \mathrm{Ext}^{p+q}_{\mathrm{D}^b_c(\mathcal{X}\times\mathcal{X})}\big(\mathrm{Gr}^i_F(M), \Delta_* L\big) \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{D}^b_c(\mathcal{X})}\big(K_1 \otimes_X D_X(K_2), L\big).$$

For every $i \in \mathbb{Z}$, we have

$$\mathrm{Gr}^i_F M = \bigoplus_{a+b=i} K_1^a[-a] \boxtimes D_X(K_2^{-b})[-b] = \bigoplus_{a-b=i} \big(K^a \boxtimes D_X(K^b)\big)[-i].$$

So

$$\begin{aligned} E_1^{pq} &= \bigoplus_{a-b=-p} \mathrm{Ext}^{p+q}_{\mathrm{D}^b_c(\mathcal{X}\times\mathcal{X})}\big(K_1^a \boxtimes D_X(K_2^b), \Delta_*[-p]\big) \\ &= \bigoplus_{a-b=-p} \mathrm{Ext}^{p}_{\mathrm{D}^b_c(\mathcal{X})}\big(K^a \otimes_X D_X(K^b), L\big). \end{aligned}$$

The statement of the lemma now follows by taking the limit over A', with $A \subset A' \in \mathcal{U}$. ∎

Lemma 8.9. *Let $K_1^\bullet, K_2^\bullet$ be bounded complexes of objects of* $\mathrm{Perv}_{mf}(X)$*, and let K_1, K_2 be their images by the canonical functor* $\mathrm{C}^b\,\mathrm{Perv}_{mf}(X) \to \mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$*. Then, for every object L of* $\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$*, we have a spectral sequence*

$$E_1^{pq} = \bigoplus_{a-b=-p} \mathrm{Ext}^q_{\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)}\big(K_1^a \otimes_X D_X(K_2^b), L\big)$$
$$\Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)}\big(K_1 \otimes_X D_X(K_2), L\big).$$

Moreover, the functor R_X induces a morphism of spectral sequences from this spectral sequence to the one of Lemma 8.8.

Proof. The proof is exactly the same as for Lemma 8.8, except that we work in the filtered derived category $\mathrm{DF}^b(\mathrm{Perv}_{mf}(X \times X))$. The last statement is obvious. ∎

Notation 8.10. Let $K \in \mathrm{Ob}\,\mathrm{D}_h^b(X)$. We denote by ι_K the evaluation morphism

$$K \otimes_X D_X(K) = K \otimes_X \underline{\mathrm{Hom}}_X(K, K_X) \to K_X.$$

This morphism satisfies the following naturality property: If $u\colon K \to L$ is a morphism in $\mathrm{D}_h^b(X)$, then we get a commutative square

$$\begin{array}{ccc} K \otimes_X D_X(L) & \xrightarrow{\mathrm{id}\otimes_X D_X(u)} & K \otimes_X D_X(K) \\ {\scriptstyle u\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \iota_K} \\ L \otimes_X D_X(L) & \xrightarrow[\iota_L]{} & K_X \end{array}$$

Lemma 8.11. *Let $K_1^\bullet, K_2^\bullet$ be bounded complexes of objects of* $\mathrm{Perv}_h(X)$*, and let K_1, K_2 be their images by the functor* real: $\mathrm{D}^b\,\mathrm{Perv}_h(X) \to \mathrm{D}_h^b(X)$*. Let*

$$E_1^{pq} \Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{D}_h^b(X)}\big(K_1 \otimes_X D_X(K_2), K_X\big)$$

be the spectral sequence of Lemma 8.8 for $L = K_X$.

Then $E_1^{pq} = 0$ if $q < 0$. Moreover, if $K_1^\bullet = K_2^\bullet = K^\bullet$ and we write $K = K_1$, then the element $\sum_{a\in\mathbb{Z}} \iota_{K^a}$ of E_1^{00} is in $\mathrm{Ker}(E_1^{00} \to E_1^{10}) = E_2^{00}$*, and the element ι_K of* $\mathrm{Hom}_{\mathrm{D}_h^b(X)}(K \otimes_X D_X(K), K_X) \supset E_\infty^{00}$ *is the image of $\sum_{a\in\mathbb{Z}} \iota_{K^a}$ by the map $E_2^{00} \to E_\infty^{00}$.*

Proof. We have

$$E_1^{pq} = \bigoplus_{a-b=-p} \mathrm{Ext}^q_{\mathrm{D}_h^b(X)}\big(K_1^a \otimes_X D_X(K_2^b), K_X\big).$$

As all the K_1^a and $D_X(K_2^b)$ are perverse, this is 0 for $q < 0$ by Lemma 8.5. This implies that $E_2^{00} = \mathrm{Ker}(E_1^{00} \to E_1^{10})$ and that $E_r^{pq} = 0$ for any $r \geq 1$ and any $q < 0$, so $E_\infty^{pq} = 0$ for $q < 0$. In particular, we get that E_∞^{00} is a quotient of E_2^{00} and that E_∞^{00} injects in $\mathrm{Hom}_{\mathrm{D}_h^b(X)}(K_1 \otimes_X D_X(K_1), K_X)$. The last statement now follows from the construction of the spectral sequence (and (iii) of Proposition 2.3.1). ∎

Lemma 8.12. *Let $K^\bullet$ be a bounded complex of objects of* $\mathrm{Perv}_{mf}(X)$*, and let K be its image by the obvious functor* $C^b\,\mathrm{Perv}_{mf}(X) \to \mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$*. Then there exists a unique morphism* $\iota_{K^\bullet}\colon K \otimes_X D_X(K) \to a^!\mathbf{1}_{\mathrm{Spec}\,k}$ *satisfying the following conditions:*

(a) *The image of $\iota_{K^\bullet}$ by R_X is the morphism $\iota_{R_X(K)}$ of* 8.10.

(b) *The analogue of Lemma* 8.11 *holds if we use the spectral sequence of Lemma* 8.9.

Moreover, if $u^\bullet\colon K^\bullet \to L^\bullet$ is a morphism in $\mathrm{C}^b\,\mathrm{Perv}_{mf}(X)$ *and if $u\colon K \to L$ is its image in* $\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$*, then the following square commutes:*

$$\begin{array}{ccc} K \otimes_X D_X(L) & \xrightarrow{\mathrm{id}\otimes_X D_X(u)} & K \otimes_X D_X(K) \\ {\scriptstyle u\otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \iota_{K^\bullet}} \\ L \otimes_X D_X(L) & \xrightarrow[\iota_{L^\bullet}]{} & K_X \end{array} \tag{*}$$

Proof. Let $K'^\bullet = R_X(K^\bullet)$ and $K' = R_X(K)$. If $K^\bullet$ is concentrated in degree 0, then, by Lemma 8.7, the morphism

$$R_X\colon \mathrm{Hom}_{\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)}\big(K \otimes_X D_X(K), D_X(\mathbf{1}_X)\big) \to \mathrm{Hom}_{\mathrm{D}^b_h(X)}\big(K' \otimes_X D_X(K'), K_X\big)$$

is an isomorphism. So condition (a) forces us to take $\iota_K = R_X^{-1}(\iota_{K'})$, and condition (b) is trivial in this case.

We now construct $\iota_{K^\bullet}$ in the general case. The spectral sequence of Lemma 8.9 for $L = a^!\mathbf{1}_X$ is

$$\begin{aligned} E_1^{pq} &= \mathrm{Ext}^q_{\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)}\big(K^a \otimes_X D_X(K^b), a^!\mathbf{1}_X\big) \\ &\Rightarrow \mathrm{Ext}^{p+q}_{\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)}\big(K \otimes_X D_X(K), a^!\mathbf{1}_X\big). \end{aligned}$$

We have $E_1^{pq} = 0$ for $q < 0$ by Lemma 8.6. As in the proof of Lemma 8.11, this implies that

$$E_2^{00} = \mathrm{Ker}(E_1^{00} \to E_1^{10})$$

surjects to E_∞^{00} and that E_∞^{00} injects in $\mathrm{Hom}_{\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)}(K \otimes_X D_X(K), a^!\mathbf{1}_X)$. By condition (b), the element $\iota_K \in \mathrm{Hom}_{\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)}(K \otimes_X D_X(K), a^!\mathbf{1}_X)$ that we want to construct must be the image of $\sum_{a\in\mathbb{Z}} \iota_{K^a} \in E_1^{00}$. As ι_{K^a} exists and is uniquely determined by the first case, it suffices to show that $\sum_{a\in\mathbb{Z}} \iota_{K^a} \in \mathrm{Ker}(E_1^{00} \to E_1^{10})$. Indeed, condition (a) will then follow from the fact that R_X induces a morphism between the spectral sequences of Lemmas 8.8 and 8.9 (and from Lemma 8.11). We denote by $E_1^{pq}(K')$ the spectral sequence of Lemma 8.8 for $K'^\bullet$. Then we have a commutative diagram

$$\begin{array}{ccc} E_1^{00} & \longrightarrow & E_1^{10} \\ {\scriptstyle R_X}\downarrow & & \downarrow{\scriptstyle R_X} \\ E_1^{00}(K') & \longrightarrow & E_1^{10}(K') \end{array}$$

By Lemma 8.7, the vertical maps in this diagram are isomorphisms. By Lemma 8.11, the image by R_X of $\sum_{a\in\mathbb{Z}} \iota_{K^a} \in E_1^{00}$, which is $\sum_{a\in\mathbb{Z}} \iota_{K'^a}$ by construction of the ι_{K^a}, is in $\operatorname{Ker}(E_1^{00}(K') \to E_1^{10}(K'))$. So $\sum_{a\in\mathbb{Z}} \iota_{K^a}$ is in $\operatorname{Ker}(E_1^{00} \to E_1^{10})$, and we are done.

We prove the last statement. If $K^\bullet$ and $L^\bullet$ are both concentrated in degree 0, then K and L are perverse, so the commutativity of (*) follows from the commutativity of the square we get applying R_X and from Lemma 8.7. We treat the general case. For $K_1^\bullet, K_2^\bullet \in \operatorname{Ob} \mathrm{C}^b \operatorname{Perv}_{mf}(X)$, we denote by ${}_{K_1^\bullet,K_2^\bullet}E_\bullet^{\bullet,\bullet}$ the spectral sequence of Lemma 8.9 for $K_1^\bullet$, $K_2^\bullet$ and K_X. The morphisms $u^\bullet$ and $D_X(u^\bullet)$ induce morphisms of spectral sequences

$$ {}_{L^\bullet,L^\bullet}E_\bullet^{\bullet,\bullet} \to {}_{K^\bullet,L^\bullet}E_\bullet^{\bullet,\bullet} \leftarrow {}_{K^\bullet,K^\bullet}E_\bullet^{\bullet,\bullet} $$

that are compatible with the morphisms

$$ \operatorname{Ext}^\bullet_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}(L,L) \xrightarrow{u^*} \operatorname{Ext}^\bullet_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}(K,L) \xleftarrow{u_*} \operatorname{Ext}^\bullet_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}(K,K). $$

So we get the commutative diagram (**), where $\mathrm{H}(\cdot)$ means $\operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}(\cdot, K_X)$. We have $\iota_{K^\bullet} \in \mathrm{H}(K \otimes_X D_X(K))$, and its image in $\mathrm{H}(K \otimes_X D_X(L))$ by the arrow in the last row of (**) is $\iota_{K^\bullet} \circ (\mathrm{id} \otimes_X D_X(u))$. Similarly, we have $\iota_{L^\bullet} \in \mathrm{H}(L \otimes_X D_X(L))$, and its image in $\mathrm{H}(K \otimes_X D_X(L))$ by the arrow in the last row of (**) is $\iota_{L^\bullet} \circ (u \otimes_X \mathrm{id})$. On the other hand, during the construction of $\iota_{K^\bullet}$, we proved that the element $\sum_{a\in\mathbb{Z}} \iota_{K^a}$ of ${}_{K^\bullet,K^\bullet}E_1^{00}$ comes from a (unique) element $e_{K^\bullet}$ of ${}_{K^\bullet,K^\bullet}E_2^{00}$, and that $\iota_{K^\bullet}$ is the image of $e_{K^\bullet}$ in $H(K \otimes_X D_X(K))$; we have a similar statement for $\iota_{L^\bullet}$. By the commutativity of (**), it suffices to prove that the images of $\sum_{a\in\mathbb{Z}} \iota_{K^a} \in {}_{K^\bullet,K^\bullet}E_1^{00}$ and $\sum_{a\in\mathbb{Z}} \iota_{L^a} \in {}_{L^\bullet,L^\bullet}E_1^{00}$ in ${}_{K^\bullet,L^\bullet}E_1^{00}$ by the maps of the second row of (**) are equal. But the image of the first element is $\sum_{a\in\mathbb{Z}} \iota_{K^a} \circ (\mathrm{id} \otimes_X D_X(u^a))$, and the image of the second element is $\sum_{a\in\mathbb{Z}} \iota_{L^a} \circ (u^a \otimes_X \mathrm{id})$. So the equality of these images follows from the case where $K^\bullet$ and $L^\bullet$ are concentrated in degree 0, which we already treated.

$$ \begin{array}{ccccc}
\bigoplus_{a\in\mathbb{Z}} \mathrm{H}\big(L^a \otimes_X D_X(L^a)\big) & \to & \bigoplus_{a\in\mathbb{Z}} \mathrm{H}\big(K^a \otimes_X D_X(L^a)\big) & \leftarrow & \bigoplus_{a\in\mathbb{Z}} \mathrm{H}\big(K^a \otimes_X D_X(K^a)\big) \\
\| & & \| & & \| \\
{}_{L^\bullet,L^\bullet}E_1^{00} & \longrightarrow & {}_{K^\bullet,L^\bullet}E_1^{00} & \longleftarrow & {}_{K^\bullet,K^\bullet}E_1^{00} \\
\uparrow & & \uparrow & & \uparrow \\
{}_{L^\bullet,L^\bullet}E_2^{00} & \longrightarrow & {}_{K^\bullet,L^\bullet}E_2^{00} & \longleftarrow & {}_{K^\bullet,K^\bullet}E_2^{00} \\
\downarrow & & \downarrow & & \downarrow \\
{}_{L^\bullet,L^\bullet}E_\infty^{00} & \longrightarrow & {}_{K^\bullet,L^\bullet}E_\infty^{00} & \longleftarrow & {}_{K^\bullet,K^\bullet}E_\infty^{00} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}\big(L \otimes_X D_X(L)\big) & \longrightarrow & \mathrm{H}\big(K \otimes_X D_X(L)\big) & \longleftarrow & \mathrm{H}\big(K \otimes_X D_X(K)\big)
\end{array} \qquad (**) $$

■

The last statement of Lemma 8.12 implies in particular that, for $K \in \operatorname{Ob} \mathrm{D}^b \operatorname{Perv}_{mf}(X)$, the morphism $\iota_{K^\bullet}\colon K \otimes_X D_X(K) \to K_X$ does not depend on the lift $K^\bullet$ of K to an object of $\mathrm{C}^b \operatorname{Perv}_{mf}(X)$. We denote this morphism by ι_K.

Lemma 8.13. *For $K, L \in \operatorname{Ob} \mathrm{D}^b \operatorname{Perv}_{mf}(X)$, we define a morphism*

$$u_{K,L}\colon R\operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}\big(L, D_X(K)\big) \to R\operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}(K \otimes_X L, a^! \mathbf{1}_{\operatorname{Spec} k})$$

as the composition of

$$K \otimes_X (.)\colon R\operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}\big(L, D_X(K)\big) \to R\operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}\big(K \otimes_X L, K \otimes_X D_X(K)\big)$$

and of

$$\iota_{K*}\colon R\operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}\big(K \otimes_X L, K \otimes_X D_X(K)\big) \to R\operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}(K \otimes_X L, a^! \mathbf{1}_{\operatorname{Spec} k}).$$

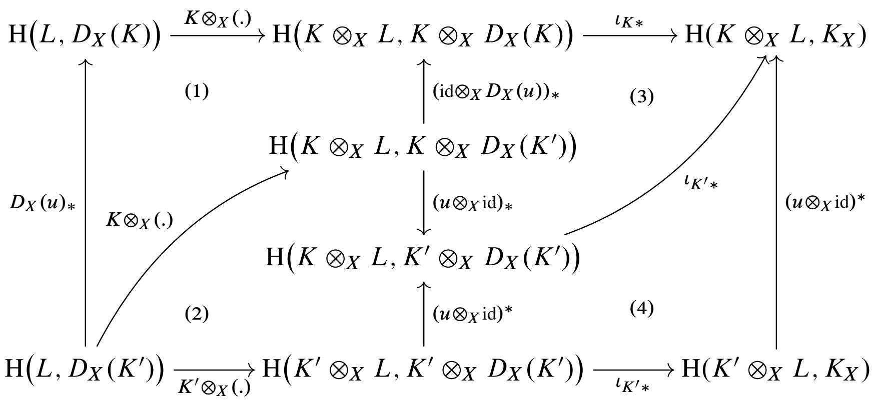
Then this morphism is natural in K and L, its image by R_X is the adjunction morphism

$$R\operatorname{Hom}_{\mathrm{D}^b_h(X)}\big(R_X(L), D_X(R_X(K))\big) = R\operatorname{Hom}_{\mathrm{D}^b_h(X)}\big(R_X(K) \otimes_X R_X(L), K_X\big),$$

and it is an isomorphism.

Proof. The naturality in L is obvious and the second statement follows from property (a) of Lemma 8.12.

We prove the naturality in K. Let $u\colon K \to K'$ be a morphism in $\mathrm{D}^b \operatorname{Perv}_{mf}(X)$. We consider the diagram, where we write H for $R\operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf} \operatorname{Perv}(X)}$



Squares (1), (2) and (4) in this diagram clearly commute, and square (3) commutes by the last statement of Lemma 8.12. So the exterior rectangle commutes, which is what we needed to prove.

We turn to the proof of the third statement, i.e., that $u_{K,L}$ is an isomorphism. We first prove it in the case where X is smooth and connected and $K = \mathcal{L}$, $L = \mathcal{M}$ are lisse sheaves on X. Let $d = \dim(X)$. Then we have $a^!\mathbf{1}_{\operatorname{Spec} k} = \mathbf{1}_X[2d](d)$ (by Proposition 2.4.2 (i)) and $D_X(\mathscr{L}) = \mathscr{L}^*[2d](d)$, where $\mathscr{L}^* = \underline{\operatorname{Hom}}(\mathscr{L}, \underline{E}_X)$ is the dual locally constant sheaf (by the calculation at the end of Section 2.1 and Proposition 2.5.2). So $u_{\mathscr{L},\mathscr{M}}$ is a morphism

$$R\operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}(\mathscr{M}, \mathscr{L}^*) \to R\operatorname{Hom}_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}(\mathscr{L} \otimes_X \mathscr{M}, \mathbf{1}_X),$$

and the morphism $\iota_{\mathscr{L}}\colon \mathscr{L} \otimes D_X(\mathscr{L}) \to a^!\mathbf{1}_{\operatorname{Spec} k}$ of Lemma 8.12 is just the the canonical morphism $\mathscr{L} \otimes_X \mathscr{L}^* \to \mathbf{1}_X$, shifted by $2d$ and twisted by d (we see this easily from conditions (a) and (b) of Lemma 8.12, as $\mathscr{L}$ is perverse up to a shift). We will use the Yoneda description of the Ext^k groups, as in Section 3.2 of Chapter III of Verdier's book [30]. The definition of $u_{\mathscr{L},\mathscr{M}}$ gives the following formula for the image of a class c in

$$\operatorname{Ext}^i_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}(\mathscr{M}, \mathscr{L}^*) = \operatorname{Ext}^i_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}\left(\mathscr{M}[d], \mathscr{L}^*[d]\right)$$

Choose an exact sequence in $\operatorname{Perv}_{mf}(X)$ representing c, say:

$$0 \to \mathscr{L}^*[d] \to K_{i-1} \to \cdots \to K_0 \to \mathscr{M}[d] \to 0.$$

Tensoring this sequence by $\mathscr{L}$, we still get an exact sequence in $\operatorname{Perv}_{mf}(X)$:

$$0 \to \mathscr{L} \otimes_X \mathscr{L}^*[d] \to \mathscr{L} \otimes_X K_{i-1} \to \cdots \to \mathscr{L} \otimes_X K_0 \to \mathscr{L} \otimes_X \mathscr{M}[d] \to 0.$$

Then $u_{\mathscr{L},\mathscr{M}}(c)$ is represented by the exact sequence

$$0 \to \mathbf{1}_X[d] \to K'_{i-1} \to \mathscr{L} \otimes_X K_{i-2} \to \cdots \to \mathscr{L} \otimes_X K_0 \to \mathscr{L} \otimes_X \mathscr{M}[d] \to 0,$$

where K'_{i-1} is the amalgamated sum

$$\mathbf{1}_X[d] \oplus_{\mathscr{L} \otimes_X \mathscr{L}^*[d]} (\mathscr{L} \otimes_X K_{i-1})$$

with the morphism $\mathscr{L} \otimes_X \mathscr{L}^*[d] \to \mathbf{1}_X[d]$ being the shift of the obvious one. We want to show that $u_{\mathscr{L},\mathscr{M}}$ is bijective, so it suffices to construct its inverse. Suppose that c' is an element of

$$\operatorname{Ext}^i_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}(\mathscr{L} \otimes_X \mathscr{M}, \mathbf{1}_X) = \operatorname{Ext}^i_{\mathrm{D}^b \operatorname{Perv}_{mf}(X)}\left(\mathscr{L} \otimes_X \mathscr{M}[d], \mathbf{1}_X[d]\right),$$

and choose an exact sequence in $\operatorname{Perv}_{mf}(X)$ representing c', say:

$$0 \to \mathbf{1}_X[d] \to L_{i-1} \to \cdots \to L_0 \to \mathscr{L} \otimes_X \mathscr{M}[d] \to 0.$$

Tensoring this sequence by $\mathscr{L}^*$, we still get an exact sequence in $\operatorname{Perv}_{mf}(X)$:

$$0 \to \mathscr{L}^*[d] \to \mathscr{L}^* \otimes_X L_{i-1} \to \cdots \to \mathscr{L}^* \otimes_X L_0 \to \mathscr{L}^* \otimes_X \mathscr{L} \otimes_X \mathscr{M}[d] \to 0.$$

We send c' to the element of $\operatorname{Ext}^i_{\operatorname{Perv}_{mf}(X)}(\mathscr{M}[d], \mathscr{L}^*[d])$ represented by the exact sequence

$$0 \to \mathscr{L}^*[d] \to \mathscr{L}^* \otimes_X L_{i-1} \to \cdots \to \mathscr{L}^* \otimes_X L_1 \to L'_0 \to \mathscr{M}[d] \to 0,$$

where L_0' is the fiber product

$$(\mathcal{L}^* \otimes_X L_0) \times_{\mathcal{M}[d]} \big(\mathcal{L}^* \otimes_X \mathcal{L} \otimes_X \mathcal{M}[d]\big)$$

with the morphism $\mathcal{L}^* \otimes_X \mathcal{L} \otimes_X \mathcal{M}[d] \to \mathcal{M}[d]$ coming from $\mathcal{L}^* \otimes_X \mathcal{L} \to \mathbf{1}_X$ by tensoring by $\mathcal{M}[d]$. This is clearly the inverse of $u_{\mathcal{L},\mathcal{M}}$.

Now we show that the morphism $u_{K,L}$ is an isomorphism for all objects K and L of $\mathrm{D}^b \operatorname{Perv}_{mf}(X)$. Note the following two reductions: First, using the fact that all the functors are triangulated and the five lemma, we see that if we have an exact triangle

$$K' \to K \to K'' \xrightarrow{+1}$$

such that the result is true for (K', L) and (K'', L), then the result if true for (K, L). There is a similar statement for the second variable L. So it suffices to prove the result for K and L concentrated in perverse degree 0, and we may also assume that K and L are simple perverse sheaves. Second, suppose that we have a closed immersion $i\colon Y \to X$, and let $j\colon U := X - Y \to X$ be the complementary open immersion. Then we have a commutative diagram whose columns are distinguished triangles (all the $R\operatorname{Hom}$s are taken in the appropriate category $\mathrm{D}^b \operatorname{Perv}_{mf}(Z)$, with $Z \in \{X, U, Y\}$):

$$\begin{array}{ccc}
R\operatorname{Hom}(i^*L, i^! D_X K) & \xrightarrow{i^* u_{K,L}} & R\operatorname{Hom}\big(i^*(K \otimes_X L), i^! a^! \mathbf{1}_{\operatorname{Spec} k}\big) \\
\downarrow & & \downarrow \\
R\operatorname{Hom}(L, D_X K) & \xrightarrow{u_{K,L}} & R\operatorname{Hom}(K \otimes_X L, a^! \mathbf{1}_{\operatorname{Spec} k}) \\
\downarrow & & \downarrow \\
R\operatorname{Hom}(j^*L, j^* D_X K) & \xrightarrow{j^* u_{K,L}} & R\operatorname{Hom}\big(j^*(K \otimes_X L), j^* a^! \mathbf{1}_{\operatorname{Spec} k}\big) \\
{\scriptstyle +1}\downarrow & & {\scriptstyle +1}\downarrow
\end{array}$$

Moreover, using the compatibility of $\otimes_X$ with inverse images and point (ii) of Proposition 7.3.2, we get isomorphisms:

$$R\operatorname{Hom}\big(i^*(K \otimes_X L), i^! a^! \mathbf{1}_{\operatorname{Spec} k}\big) \simeq R\operatorname{Hom}\big((i^*K) \otimes_Y (i^*L), a_Y^! \mathbf{1}_{\operatorname{Spec} k}\big)$$

and

$$R\operatorname{Hom}\big(j^*(K \otimes_X L), j^* a^! \mathbf{1}_{\operatorname{Spec} k}\big) \simeq R\operatorname{Hom}\big((j^*K) \otimes_U (j^*L), a_U^! \mathbf{1}_{\operatorname{Spec} k}\big),$$

where $a_Y = a \circ i$ and $a_U = a \circ j$. It is easy to see that these isomorphisms identify $i^* u_{K,L}$ (resp. $j^* u_{K,L}$) with u_{i^*K, i^*L} (resp. u_{j^*K, j^*L}). So the result for X follows from the result for Y and U.

Using the two reductions above and Noetherian induction on X, we can reduce to the case where X is smooth and K and L are both shifts of locally constant sheaves on X. But this case has already been treated in the first part of the proof. ∎

Proof of Proposition 8.3. We have to construct an isomorphism

$$R\operatorname{Hom}_{\mathrm{D}^b\operatorname{Perv}_{mf}(X)}(K\otimes_X L, M)\xrightarrow{\sim} R\operatorname{Hom}_{\mathrm{D}^b\operatorname{Perv}_{mf}(X)}\big(L, D_X(K\otimes_X D_X M)\big)$$

functorial in $K, L, M\in\operatorname{Ob}\mathrm{D}^b\operatorname{Perv}_{mf}(X)$ and compatible (via R_X) with the adjunction morphism in $\mathrm{D}^b_h(X)$. But such an isomorphism is given by

$$u_{L,K\otimes_X D_X M}\circ u^{-1}_{K\otimes_X L, D_X M},$$

where $u_{.,.}$ is constructed in Lemma 8.13. ∎

9. Weight filtration on complexes

The goal of this section is to generalize the results of [20, Section 3], and in particular the formula for the intermediate extension of a pure perverse sheaf, to the categories $\operatorname{Perv}_{mf}(X)$ and their derived categories. This was the original motivation for considering the categories $\mathrm{D}^b\operatorname{Perv}_{mf}(X)$.

Definition 9.1. Let X be a k-scheme. For every $a\in\mathbb{Z}\cup\{\pm\infty\}$, we denote by ${}^w\mathrm{D}^{\leq a}(X)$ (resp. ${}^w\mathrm{D}^{\geq a}(X)$) the full subcategory of $\mathrm{D}^b\operatorname{Perv}_{mf}(X)$ whose objects are the complexes K such that, for every $i\in\mathbb{Z}$, $\mathrm{H}^i K\in\operatorname{Perv}_{mf}(X)$ is of weight $\leq a$ (resp. $\geq a$).

Note that ${}^w\mathrm{D}^{\leq a}(X)$ and ${}^w\mathrm{D}^{\geq a}(X)$ are triangulated subcategories of $\mathrm{D}^b\operatorname{Perv}_{mf}(X)$.

Proposition 9.2. *Let $K, L\in\operatorname{Ob}\operatorname{Perv}_{mf}(X)$. Suppose that there exists $a\in\mathbb{Z}$ such that K is of weight $\leq a$ and L is of weight $\geq a+1$. Then we have, for every $i\in\mathbb{Z}$,*

$$\operatorname{Ext}^i_{\operatorname{Perv}_{mf}(X)}(K, L) = 0.$$

For categories like that of mixed Hodge modules, this result follows from [23, Lemma 6.9], but M. Saito assumes (and uses) the fact that pure objects are semisimple, which is false in our case.

Proof. We obviously have $\operatorname{Ext}^i_{\operatorname{Perv}_{mf}(X)}(K, L) = 0$ if $i<0$, and $\operatorname{Hom}_{\operatorname{Perv}_{mf}(X)}(K, L) = 0$ because the weights of K and L are disjoint. We denote by W the weight filtration on objects of $\operatorname{Perv}_{mf}(X)$. For every $b\in\mathbb{Z}$, we get an endofunctor W_b of $\operatorname{Perv}_{mf}(X)$, which is exact because weight filtrations are strictly compatible with morphisms in $\operatorname{Perv}_{mf}(X)$ (by [14, Lemma 3.8]).

As in the proof of Proposition 8.3, we will use the Yoneda description of the Ext^k groups (see Section 3.2 of Chapter III of Verdier's book [30]). Let $i\geq 1$ and let α be a class in $\operatorname{Ext}^i_{\operatorname{Perv}_{mf}(X)}(K, L)$. Choose an exact sequence

$$0\to L\xrightarrow{u_i} M_{i-1}\xrightarrow{u_{i-1}}\cdots\xrightarrow{u_1} M_0\xrightarrow{u_0} K\to 0$$

in $\operatorname{Perv}_{mf}(X)$ that represents α. Applying W_a to this exact sequence and using the fact that $W_a K = K$ and $W_a L = 0$, we get a morphism of exact sequences

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & L & \xrightarrow{u_i} & M_{i-1} & \xrightarrow{u_{i-1}} & \cdots & \xrightarrow{u_1} & M_0 & \xrightarrow{u_0} & K & \longrightarrow & 0 \\ & & {\scriptstyle u+\mathrm{can}}\Big\| & & \uparrow{\scriptstyle \mathrm{can}} & & & & \uparrow{\scriptstyle \mathrm{can}} & & \Big\| & & \\ 0 & \longrightarrow & L & \xrightarrow{\mathrm{id}_L+0} & L \oplus W_a M_{i-1} & \xrightarrow{0+u_{i-1}} & \cdots & \xrightarrow{u_1} & W_a M_0 & \xrightarrow{u_0} & K & \longrightarrow & 0 \end{array}$$

where can: $W_a \to \mathrm{id}$ is the canonical inclusion. So the class α is also represented by the second row of this diagram, hence it is trivial. ■

Corollary 9.3. *For every $a \in \mathbb{Z} \cup \{\pm\infty\}$, the pair $({}^w\mathrm{D}^{\leq a}, {}^w\mathrm{D}^{\geq a+1})$ is a t-structure on* $\mathrm{D}^b\,\mathrm{Perv}_{mf}(X)$.

We denote by $w_{\leq a}$ and $w_{\geq a+1}$ the truncation functors for this t-structure. They extend the exact functors $K \mapsto W_a K$ and $K \mapsto K/W_a K$ on $\mathrm{Perv}_{mf}(X)$.

Proof. Once we have the vanishing result of Proposition 9.2, the proofs of [20, Lemmas 3.2.1 and 3.2.2] apply without modification. ■

Corollary 9.4. *The results of* [20, Sections 3 and 5.1] *are still true in our situation. In particular, if $j: U \to X$ is an open immersion of k-schemes and $K \in \mathrm{Ob}\,\mathrm{Perv}_{mf}(U)$ is pure of weight a, then the canonical morphisms*

$$w_{\geq a} j_! K \to j_{!*} K \to w_{\leq a} j_* K$$

are isomorphisms.

Proof. The proofs of [20] apply without modification. ■

A. Filtered derived categories and f-categories

Let $\mathcal{D}$ be a triangulated category and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure on $\mathcal{D}$ with heart $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. Under appropriate hypotheses, the inclusion $\mathcal{C} \to \mathcal{D}$ extends to a triangulated functor real: $\mathrm{D}^b(\mathcal{C}) \to \mathcal{D}$ called the *realization functor*; the first goal of this appendix is to explain how to construct this extension (see Theorem A.2.3).

Now suppose that $\mathcal{D}'$ is another triangulated category, that $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$ is a t-structure with heart $\mathcal{C}'$ and that $T: \mathcal{D} \to \mathcal{D}'$ is a triangulated functor. Under appropriate hypotheses (see Theorem A.2.3), we will have realization functors real: $\mathrm{D}^b(\mathcal{C}) \to \mathcal{D}$ and real: $\mathrm{D}^b(\mathcal{C}') \to \mathcal{D}'$. The second goal of this appendix is, supposing that there are "enough" T-acyclic objects in $\mathcal{C}$, to extend $\mathrm{H}^0 T: \mathcal{C} \to \mathcal{C}'$ to a triangulated functor $DT: \mathrm{D}^b(\mathcal{C}) \to \mathrm{D}^b(\mathcal{C}')$ such that real $\circ DT \simeq T \circ$ real; the functor DT should be the restriction to $\mathrm{D}^b(\mathcal{C})$ of the derived functor of $\mathrm{H}^0 T$ if T is left or right t-exact. For precise statements, see Proposition A.3.2 and Remarks A.3.3 and A.3.4.

The technical tool that we need for the two goals above is the formalism of f-categories over triangulated categories (introduced by Beilinson [4, Appendix A]), which is inspired by the properties of filtered derived categories. So we start with a review of this formalism.

A.1. Review of f-categories and f-functors

Filtered derived categories. We recall the definition of filtered derived categories, as they are the motivation for the definition of f-categories, and also the main example of this formalism.

Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{Fil}(\mathcal{A})$ category of filtered objects of $\mathcal{A}$, where filtrations are assumed to be decreasing; it has a full subcategory $\mathrm{Fil}^f(\mathcal{A})$, whose objects are filtered objects of $\mathcal{A}$ whose filtration is finite, which means in particular that it is separated and exhaustive. Both $\mathrm{Fil}(\mathcal{A})$ and $\mathrm{Fil}^f(\mathcal{A})$ are quasi-abelian categories. We denote by $\mathrm{D}(\mathrm{Fil}^f(\mathcal{A}))$ (resp. $\mathrm{D}^+(\mathrm{Fil}^f(\mathcal{A}))$, $\mathrm{D}^-(\mathrm{Fil}^f(\mathcal{A}))$, $\mathrm{D}^b(\mathrm{Fil}^f(\mathcal{A}))$) the unbounded (resp. bounded above, bounded below, bounded) derived category of $\mathrm{Fil}^f(\mathcal{A})$ (see for example Definition 2.3 of Schapira and Schneiders's [24], or [29, Definition 05S2] and [29, Definition 05S4]; note that the definitions of the variously bounded derived categories in [24] and in the Stacks Project are a priori different, but they define the same subcategories by [29, Lemma 05S5]). Following the convention of [6, Section 3.1] and Chapter V of Illusie's [15], we define the the *filtered derived category* $\mathrm{DF}(\mathcal{A})$ (resp. the *bounded above filtered derived category* $\mathrm{DF}^+(\mathcal{A})$, etc) of $\mathcal{A}$ to be the full subcategory of $\mathrm{D}(\mathrm{Fil}^f(\mathcal{A}))$ (resp. $\mathrm{D}^+(\mathrm{Fil}^f(\mathcal{A}))$ etc) of complexes whose filtration is finite, and not just finite in each degree.

As the filtration on a finite complex of objects of $\mathrm{Fil}^f(\mathcal{A})$ is always finite, we have $\mathrm{DF}^b(\mathcal{A}) = \mathrm{D}^b(\mathrm{Fil}^f(\mathcal{A}))$, so the definitions of [29, Section 05RX], [6, 15] coincide for the bounded filtered derived category.

We denote objects of $\mathrm{Fil}^f(\mathcal{A})$ and $\mathrm{D}(\mathrm{Fil}^f(\mathcal{A}))$ by $(K, F^\bullet)$, or often just K.

We have additive functors:

- $s\colon \mathrm{Fil}^f(\mathcal{A}) \to \mathrm{Fil}^f(\mathcal{A})$, $(K, F^\bullet) \mapsto (K, F^{\bullet-1})$;
- $\mathrm{Fil}^r\colon \mathrm{Fil}^f(\mathcal{A}) \to \mathrm{Fil}^f(\mathcal{A})$ sending $(K, F^\bullet)$ to F^r with the filtration induced by $F^\bullet$;
- $(.)/\mathrm{Fil}^r\colon \mathrm{Fil}^f(\mathcal{A}) \to \mathrm{Fil}^f(\mathcal{A})$ sending $(K, F^\bullet)$ to $K/F^r K$ with the filtration induced by $F^\bullet$;
- $\mathrm{Gr}^r\colon \mathrm{Fil}^f(\mathcal{A}) \to \mathcal{A}$, $(K, \mathrm{Fil}^\bullet) \mapsto \mathrm{Fil}^r K/\mathrm{Fil}^{r-1} K$;
- $\mathrm{Gr} = \bigoplus_{r\in\mathbb{Z}} \mathrm{Gr}^r\colon \mathrm{Fil}^f(\mathcal{A}) \to \mathcal{A}$;
- $\omega\colon \mathrm{Fil}(\mathcal{A}) \to \mathcal{A}$, $(K, F^\bullet) \mapsto K$;
- $i\colon \mathcal{A} \to \mathrm{Fil}^f(\mathcal{A})$ sending $K \in \mathrm{Ob}(\mathcal{A})$ to K with the trivial filtration $F^\bullet$ ($F^r = K$ for $r \le 0$ and $F^r = 0$ for $r \ge 1$).

These induce functors on the categories of complexes, and, as in [29, Lemma 05S3], we get triangulated functors

$$s, \sigma_{\le r}, \sigma_{\ge r+1}\colon \mathrm{D}\left(\mathrm{Fil}^f(\mathcal{A})\right) \to \mathrm{D}\left(\mathrm{Fil}^f(\mathcal{A})\right),$$
$$\mathrm{Gr}^r, \mathrm{Gr}\colon \mathrm{D}\left(\mathrm{Fil}^f(\mathcal{A})\right) \to \mathrm{D}(\mathcal{A}),$$
$$\omega\colon \mathrm{D}\left(\mathrm{Fil}^f(\mathcal{A})\right) \to \mathrm{D}(\mathcal{A})$$

and

$$i\, \mathrm{D}(\mathcal{A}) \to \mathrm{D}\left(\mathrm{Fil}^f(\mathcal{A})\right).$$

For $? \in \{+,-,b\}$, these functors send to $\mathrm{D}^?(\mathrm{Fil}^f(\mathcal{A}))$ (resp. $\mathrm{D}^?(\mathcal{A})$) to $\mathrm{D}^?(\mathrm{Fil}^f(\mathcal{A}))$ (resp. $\mathrm{D}^?(\mathcal{A})$), and they also respect the various filtered derived categories (when it makes sense, for example s sends $\mathrm{DF}^?(\mathcal{A})$ to $\mathrm{DF}?(\mathcal{A})$, and i sends $\mathrm{D}^?(\mathcal{A})$ to $\mathrm{DF}^?(\mathcal{A})$). For every $r \in \mathbb{Z}$, we have a natural transformation $\mathrm{Gr}^r \to \mathrm{Gr}^{r+1}[1]$ (coming from the triangle $\mathrm{Gr}^{r+1} K \to F^r K/F^{r+2}K \to \mathrm{Gr}^r K$, if $(K, F^\bullet)$ is an object of $\mathrm{D}(\mathrm{Fil}^f(\mathcal{A}))$). We also have a natural transformation $\alpha\colon \mathrm{id}_{\mathrm{DF}(\mathcal{A})} \to s$ coming from the inclusions $F^r K \subset F^{r-1}K$, for $(K, F^\bullet) \in \mathrm{Ob}\,\mathrm{Fil}^f(\mathcal{A})$.

For every $n \in \mathbb{Z}$, let

$$\mathrm{DF}(\leq n) = \left\{K \in \mathrm{Ob}\,\mathrm{DF}(\mathcal{A}) \mid \forall r \geq n+1, \mathrm{Gr}^r K = 0\right\}$$

and

$$\mathrm{DF}(\geq n) = \left\{K \in \mathrm{Ob}\,\mathrm{DF}(\mathcal{A}) \mid \forall r \leq n-1, \mathrm{Gr}^r K = 0\right\};$$

these are full subtriangulated categories of $\mathrm{DF}(\mathcal{A})$, stable by isomorphisms. Thanks to our convention on filtered derived categories, we have

$$\mathrm{DF}(\mathcal{A}) = \bigcup_{n\in\mathbb{Z}} \mathrm{DF}(\leq n) = \bigcup_{n\in\mathbb{Z}} \mathrm{DF}(\geq n).$$

Similarly, if $?\in\{+,-,b\}$ and $n\in\mathbb{Z}$, we set $\mathrm{DF}^?(\leq n) = \mathrm{DF}(\leq n) \cap \mathrm{DF}^?(\mathcal{A})$ and $\mathrm{DF}^?(\geq n) = \mathrm{DF}(\geq n) \cap \mathrm{DF}^?(\mathcal{A})$. For $? \in \{\varnothing, +, -, b\}$, the functor i induces an equivalence of categories $\mathrm{D}^?(\mathcal{A}) \to \mathrm{DF}^?(\leq 0) \cap \mathrm{DF}^?(\geq 0)$.

f-categories. In [4, Appendix A], Beilinson introduced f-categories over triangulated categories, that have all the abstract properties of filtered derived categories, and generalized the properties of filtered derived categories to this more general setting. We review his definition and results. Note that Section 6 of Schnürer's paper [26] gives more detailed proofs of many of the results of [4, Appendix A].

The following is [4, Definition A.1].

Definition A.1.1. We introduce the following objects:

(1) A *filtered triangulated category*, or for short a *f-category*, is the data of:

- a triangulated category DF;
- two full triangulated subcategories $\mathrm{DF}(\leq 0)$, $\mathrm{DF}(\geq 0)$ of DF that are stable by isomorphisms;
- a triangulated self-equivalence $s\colon \mathrm{DF} \to \mathrm{DF}$ (called *shift of filtration*);
- a morphism of functors $\alpha\colon \mathrm{id}_{\mathrm{DF}} \to s$;

satisfying the following conditions, where, for every $n \in \mathbb{Z}$, we set

$$\mathrm{DF}(\leq n) = s^n\,\mathrm{DF}(\leq 0) \text{ and } \mathrm{DF}(\geq n) = s^n\,\mathrm{DF}(\geq 0):$$

(i) We have $\mathrm{DF}(\geq 1) \subset \mathrm{DF}(\geq 0)$, $\mathrm{DF}(\leq 1) \supset \mathrm{DF}(\leq 0)$ and

$$\mathrm{DF} = \bigcup_{n\in\mathbb{Z}} \mathrm{DF}(\leq n) = \bigcup_{n\in\mathbb{Z}} \mathrm{DF}(\geq n).$$

(ii) For any $X \in \operatorname{Ob} \mathrm{DF}$, we have $\alpha_X = s(\alpha_{s^{-1}(X)})$.

(iii) For any $X \in \operatorname{Ob} \mathrm{DF}(\geq 1)$ and $Y \in \operatorname{Ob} \mathrm{DF}(\leq 0)$, we have $\operatorname{Hom}(X, Y) = 0$, and the maps

$$\operatorname{Hom}(s(Y), X) \to \operatorname{Hom}(Y, X) \to \operatorname{Hom}\left(Y, s^{-1}(X)\right)$$

induced by α_Y and $\alpha_{s^{-1}(X)}$ are bijective.

(iv) For every $X \in \operatorname{Ob} \mathrm{DF}$, there exists a distinguished triangle $A \to X \to B \xrightarrow{+1}$ with A in $\mathrm{DF}(\geq 1)$ and B in $\mathrm{DF}(\leq 0)$.

(2) If DF and DF′ are f-categories, an *f-functor* from DF to DF′ is the data of a triangulated functor $T : \mathrm{DF} \to \mathrm{DF}'$ and a natural isomorphism $s' \circ T \xrightarrow{\sim} T \circ s$ such that $T(\mathrm{DF}(\leq 0)) \subset \mathrm{DF}'(\leq 0)$, $T(\mathrm{DF}(\geq 0)) \subset \mathrm{DF}(\geq 0)$ and that, for every $X \in \operatorname{Ob} \mathrm{DF}$, the following triangle commutes:

$$\begin{array}{ccc} T(X) & \xrightarrow{\alpha'_{T(X)}} & s'\left(T(X)\right) \\ & {\scriptstyle T(\alpha(X))} \searrow & \downarrow \wr \\ & & T\left(s(X)\right) \end{array}$$

(3) Let $\mathcal{D}$ be a triangulated category. An *f-category over* $\mathcal{D}$ is an f-category DF together with an equivalence $i : \mathcal{D} \to \mathrm{DF}(\leq 0) \cap \mathrm{DF}(\geq 0)$. If $\mathcal{D}'$ is another triangulated category, DF′ is an f-category over $\mathcal{D}'$ and $T : \mathcal{D} \to \mathcal{D}'$ is a triangulated functor, an *f-lifting* of T is an f-functor $FT : \mathrm{DF} \to \mathrm{DF}'$ and a natural isomorphism $i' \circ T \simeq TF \circ i$.

Example A.1.2. Let $\mathcal{A}$ be an abelian category. For $? \in \{\varnothing, +, -, b\}$, the category $\mathrm{DF}^?(\mathcal{A})$ with the subcategories $\mathrm{DF}^?(\leq 0)$ and $\mathrm{DF}^?(\geq 0)$, the functors s and i and the natural transformation α, is an f-category over the triangulated category $\mathrm{D}^?(\mathcal{A})$. Note that this is not be true for $\mathrm{D}^?(\mathrm{Fil}^f(\mathcal{A}))$ instead of $\mathrm{DF}^?(\mathcal{A})$ (except of course if $? = b$), because the third statement of condition (i) of Definition A.1.1 does not hold.

Proposition A.1.3 (Proposition A.3 of [4]). *Let* DF *be an f-category.*

(i) *For every $n \in \mathbb{Z}$, the inclusion $\mathrm{DF}(\leq n) \subset \mathrm{DF}$ admits a left adjoint $\sigma_{\leq n}$, and the inclusion $\mathrm{DF}(\geq n) \subset \mathrm{DF}$ admits a right adjoint $\sigma_{\geq n}$. The functors $\sigma_{\leq n}$, $\sigma_{\geq n}$ are triangulated and preserve the subcategories $\mathrm{DF}(\leq m)$, $\mathrm{DF}(\geq m)$ for every $m \in \mathbb{Z}$.*

(ii) *For $a, b \in \mathbb{Z}$, there exists a unique isomorphism of functors $\sigma_{\leq a}\sigma_{\geq b} \simeq \sigma_{\geq b}\sigma_{\leq a}$ that makes the following diagram commute:*

$$\begin{array}{ccccc} \sigma_{\geq b} & \longrightarrow & \mathrm{id}_{\mathrm{DF}} & \longrightarrow & \sigma_{\leq a} \\ & \searrow & & \nearrow & \\ & \sigma_{\leq a}\sigma_{\geq b} & \xrightarrow{\sim} & \sigma_{\geq b}\sigma_{\leq a} & \end{array}$$

(iii) *Let* $X \in \operatorname{Ob}\mathrm{DF}$. *Then there exists a unique morphism* $\delta\colon \sigma_{\leq 0}X \to \sigma_{\geq 1}[1]$ *making the triangle* $\sigma_{\leq 1}X \to X \to \sigma_{\leq 0}X \xrightarrow{\delta} \sigma_{\geq 1}X[1]$ *distinguished. Any distinguished triangle* $A \to X \to B \xrightarrow{+1}$ *with* $A \in \operatorname{Ob}\mathrm{DF}(\geq 1)$ *and* $B \in \operatorname{Ob}\mathrm{DF}(\leq 0)$ *admits a unique isomorphism to the triangle of the previous sentence.*

(iv) *We have canonical isomorphisms* $\sigma_{\leq n} \circ s = s \circ \sigma_{\leq n-1}$ *and* $\sigma_{\geq n} \circ s = s \circ \sigma_{\geq n-1}$.

Point (iv) is not stated in [4, Proposition A.3] but follows immediately from the fact that $s(\mathrm{DF}(\leq n-1)) = \mathrm{DF}(\leq n)$ (resp. $s(D(\geq n-1)) = D(\geq n)$) and the uniqueness of adjoints.

Definition A.1.4. Let $\mathcal{D}$ be a triangulated category and DF be an f-category over $\mathcal{D}$. For every $n \in \mathbb{Z}$, we define a functor $\mathrm{Gr}^n\colon \mathrm{DF} \to \mathcal{D}$ by $\mathrm{Gr}^n = i^{-1} \circ s^{-n} \circ \sigma_{\leq n}\sigma_{\geq n}$.

Proposition A.1.5. *Let* $\mathcal{D}$ *be a triangulated category and* DF *be an f-category over* $\mathcal{D}$.

(i) *For every* $r \in \mathbb{Z}$, *we have a natural isomorphism* $\mathrm{Gr}^r \circ s = \mathrm{Gr}^{r-1}$.

(ii) *Let* $r \in \mathbb{Z}$. *Then* $\mathrm{Gr}^r \circ i = 0$ *if* $r \neq 0$ *and* $\mathrm{Gr}^r \circ i \simeq \mathrm{id}_{\mathcal{D}}$ *if* $r = 0$.

(iii) *Let* $r, n \in \mathbb{Z}$. *We have*

$$\mathrm{Gr}^r \circ \sigma_{\leq n} = \begin{cases} \mathrm{Gr}^r & \text{if } r \leq n \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathrm{Gr}^r \circ \sigma_{\geq n} = \begin{cases} \mathrm{Gr}^r & \text{if } r \geq n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Point (i) follows from Proposition A.1.3(iv), point (ii) from the fact that the image of i is contained in $\mathrm{DF}(\leq 0) \cap \mathrm{DF}(\geq 0)$, and point (iii) from the definition of Gr^r. ∎

Proposition A.1.6 (Proposition A.3 of [4]). *Let* $\mathcal{D}$ *be a triangulated category and* DF *be an f-category over* $\mathcal{D}$. *Then there exists a triangulated functor* $\omega\colon \mathrm{DF} \to \mathcal{D}$ *such that:*[6]

(a) $\omega_{|\mathrm{DF}(\leq 0)}\colon \mathrm{DF}(\leq 0) \to \mathcal{D}$ *is left adjoint to* $\mathcal{D} \xrightarrow{i} \mathrm{DF}(\leq 0) \cap \mathrm{DF}(\geq 0) \subset \mathrm{DF}(\leq 0)$;

(b) $\omega_{|\mathrm{DF}(\geq 0)}\colon \mathrm{DF}(\geq 0) \to \mathcal{D}$ *is right adjoint to* $\mathcal{D} \xrightarrow{i} \mathrm{DF}(\leq 0) \cap \mathrm{DF}(\geq 0) \subset \mathrm{DF}(\geq 0)$;

(c) *for any* $X \in \operatorname{Ob}\mathrm{DF}$, *the map* $\omega(\alpha_X)\colon \omega(X) \to \omega(s(X))$ *is an isomorphism;*

(d) *if* $A \in \operatorname{Ob}\mathrm{DF}(\leq 0)$ *and* $B \in \mathrm{DF}(\geq 0)$, *then* $\omega\colon \operatorname{Hom}(A, B) \to \operatorname{Hom}(\omega(A), \omega(B))$ *is bijective.*

Moreover, ω *is determined up to unique isomorphism by properties (a) and (c) (resp. (b) and (c)).*

Remark A.1.7. If $\mathcal{A}$ is an abelian category, $? \in \{\varnothing, +, -, n\}$, $\mathrm{DF} = \mathrm{DF}^?(\mathcal{A})$ and $\mathcal{D} = \mathrm{D}^?(\mathcal{A})$, then the functors $\sigma_{\leq n}$, $\sigma_{\geq n}$, Gr^n and ω are isomorphic to the ones defined in the first part of this section.

The following proposition follows easily from the definitions.

[6] Note that there is a typo in [4, Proposition A.3]: the left and right adjoints are switched; see the correction in [26, Proposition 6.6].

Proposition A.1.8. *Let $\mathcal{D}$, $\mathcal{D}'$ be triangulated categories, and let* DF *(resp.* DF'*) be an f-category over $\mathcal{D}$ (resp. $\mathcal{D}'$). Let $T\colon\mathcal{D}\to\mathcal{D}'$ be a triangulated functor, and let $FT\colon\mathrm{DF}\to\mathrm{DF}'$ be an f-lifting of T. Then the following squares commute up to natural isomorphism:*

$$\begin{array}{ccc} \mathrm{DF} & \xrightarrow{FT} & \mathrm{DF}' \\ \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\omega} \\ \mathcal{D} & \xrightarrow[T]{} & \mathcal{D}' \end{array} \qquad \begin{array}{ccc} \mathrm{DF} & \xrightarrow{FT} & \mathrm{DF}' \\ \downarrow{\scriptstyle\mathrm{Gr}^n} & & \downarrow{\scriptstyle\mathrm{Gr}^n} \\ \mathcal{D} & \xrightarrow[T]{} & \mathcal{D}' \end{array} \qquad \begin{array}{ccc} \mathrm{DF} & \xrightarrow{FT} & \mathrm{DF}' \\ \downarrow{\scriptstyle\sigma_{\leq n}} & & \downarrow{\scriptstyle\sigma_{\leq n}} \\ \mathrm{DF} & \xrightarrow[FT]{} & \mathrm{DF}' \end{array}$$

Construction of f-categories and f-liftings.

Proposition A.1.9. *Let $\mathcal{D}$ be a triangulated category,* DF *be an f-category over $\mathcal{D}$, and $\mathcal{D}'$ be a full triangulated subcategory of $\mathcal{D}$ that is stable by isomorphisms. We define a full subcategory* DF' *of* DF *by*

$$\mathrm{Ob}\,\mathrm{DF}' = \left\{K \in \mathrm{Ob}\,\mathcal{D} \mid \forall r \in \mathbb{Z}, \mathrm{Gr}^r\, K \in \mathrm{Ob}\,\mathcal{D}'\right\}.$$

Then DF' *is a triangulated subcategory of* DF*, it is stable by isomorphisms, we have*

$$s(\mathrm{DF}') \subset \mathrm{DF}' \quad \text{and} \quad i(\mathcal{D}') \subset \mathrm{DF}'.$$

The data of DF', $\mathrm{DF}' \cap \mathrm{DF}(\leq 0)$, $\mathrm{DF}' \cap \mathrm{DF}(\geq 0)$, *$s_{|\mathrm{DF}'}\colon \mathrm{DF}' \to \mathrm{DF}'$, α and $i_{|\mathcal{D}'}\colon \mathcal{D}' \to \mathrm{DF}'$ defines an f-category over $\mathcal{D}'$.*

Proof. As the functors Gr^r are triangulated, DF' is a triangulated subcategory of DF; it is clearly stable by isomorphisms, and it stable by s thanks to the isomorphisms $\mathrm{Gr}^r \circ s = \mathrm{Gr}^{r-1}$ (Proposition A.1.5 (i)). If $X \in \mathrm{Ob}\,\mathcal{D}'$, then $\mathrm{Gr}^r(i(X)) = 0$ if $r \neq 0$ and $\mathrm{Gr}^0(i(X)) \simeq X$ (Proposition A.1.5 (ii)), so $i(X) \in \mathrm{Ob}\,\mathrm{DF}'$. To prove the last assertion, we check the conditions of Definition A.1.1. Conditions (i)–(iii) are clear. To check condition (iv), it suffices by Proposition A.1.3 (iii) to prove that the functors $\sigma_{\leq n}$, $\sigma_{\geq n}$ preserve DF'; but this follows immediately from Proposition A.1.5 (iii). ∎

The next proposition, which follows easily from the definitions, is used to construct an f-category over the triangulated category of horizontal constructible complexes in Section 2.4.

Proposition A.1.10. *Let $\mathcal{U}$ be a small category, let $(\mathrm{DF}_x)_{x\in\mathrm{Ob}\,\mathcal{U}}$ be an inductive system of f-categories where all transition functors are f-functors, let $(\mathcal{D}_x)_{x\in\mathrm{Ob}\,\mathcal{U}}$ be an inductive system of triangulated categories where all transition functors are triangulated functors. Suppose that, for every $x \in \mathrm{Ob}\,\mathcal{U}$, we have a functor $i_x\colon \mathcal{D}_x \to \mathrm{DF}_x$ making DF_x an f-lifting of $\mathcal{D}_x$ and that, for every morphism $x \to y$ of $\mathcal{U}$, the diagram*

$$\begin{array}{ccc} \mathrm{DF}_x & \longrightarrow & \mathrm{DF}_y \\ \uparrow{\scriptstyle i_x} & & \uparrow{\scriptstyle i_y} \\ \mathcal{D}_x & \longrightarrow & \mathcal{D}_y \end{array}$$

commutes up to isomorphism; we also suppose that these isomorphisms are compatible with the composition of morphisms of $\mathcal{U}$.

Then $\mathrm{DF} := 2-\varinjlim_{x\in \mathrm{Ob}\,\mathcal{U}} \mathrm{DF}_x$ *is an f-category over* $\mathcal{D} := 2-\varinjlim_{x\in\mathcal{U}} \mathcal{D}_x$ *and, for every* $x \in \mathrm{Ob}\,\mathcal{U}$, *the canonical functor* $\mathrm{DF}_x \to \mathrm{DF}$ *is an f-lifting of the canonical functor* $\mathcal{D}_x \to \mathcal{D}$.

Our main source of f-liftings will be filtered derived functors, so we recall how they are constructed. Let $\mathcal{A}$, $\mathcal{A}'$ be abelian categories, and let $T\colon \mathcal{A} \to \mathcal{A}'$ be a left exact functor. If $(A, F^\bullet)$ is an object of $\mathrm{Fil}^f(\mathcal{A})$, we define a filtration $F^\bullet$ on $T(A)$ by setting $F^i(T(A)) = \Im(T(F^i A) \to T(A))$ for every $i \in \mathbb{Z}$; in particular, we have canonical morphisms $T(F^i A) \to F^i(T(A))$, hence also $T(\mathrm{Gr}^i(A)) \to \mathrm{Gr}^i(T(A))$ (as T is left exact, we have $T(\mathrm{Gr}^i(A)) \simeq T(F^i(A))/T(F^{i+1}(A))$). This defines an additive functor $\mathrm{Fil}^f(\mathcal{A}) \to \mathrm{Fil}^f(\mathcal{A}')$ (the action on morphisms is the restriction of that of T), that we denote by $\mathrm{Fil}^f(T)$.

If $\mathrm{Fil}^f(T)$ admits a right derived functor $R\mathrm{Fil}^f(T)$ in the sense of Definition 1.3.1 of Schneiders's book [25], we see that $R\mathrm{Fil}^f(T)$ is the *filtered right derived functor* of F.[7]

Proposition A.1.11. *Let* $T\colon \mathcal{A} \to \mathcal{A}'$ *be a left exact functor between abelian categories. If* $\mathcal{A}$ *has a* T*-injective subcategory in the sense of* [18, Definition 13.3.4], *then*

$$\mathrm{Fil}^f(T)\colon \mathrm{Fil}^f(\mathcal{A}) \to \mathrm{Fil}^f(\mathcal{A}')$$

has a right derived functor, and the squares

$$\begin{array}{ccc} \mathrm{D}^+\big(\mathrm{Fil}^f(\mathcal{A})\big) & \xrightarrow{R\mathrm{Fil}^f(T)} & \mathrm{D}^+\big(\mathrm{Fil}^f(\mathcal{A}')\big) \\ \omega\downarrow & & \downarrow\omega \\ \mathrm{D}^+(\mathcal{A}) & \xrightarrow[RF]{} & \mathrm{D}^+(\mathcal{A}') \end{array} \qquad \begin{array}{ccc} \mathrm{D}^+\big(\mathrm{Fil}^f(\mathcal{A})\big) & \xrightarrow{R\mathrm{Fil}^f(T)} & \mathrm{D}^+\big(\mathrm{Fil}^f(\mathcal{A}')\big) \\ \mathrm{Gr}\downarrow & & \downarrow\mathrm{Gr} \\ \mathrm{D}^+(\mathcal{A}) & \xrightarrow[RT]{} & \mathrm{D}^+(\mathcal{A}') \end{array}$$

$$\begin{array}{ccc} \mathrm{D}^+\big(\mathrm{Fil}^f(\mathcal{A})\big) & \xrightarrow{R\mathrm{Fil}^f(T)} & \mathrm{D}^+\big(\mathrm{Fil}^f(\mathcal{A}')\big) \\ \sigma_{\le i}\downarrow & & \downarrow\sigma_{\le i} \\ \mathrm{DF}^+(\mathcal{A}) & \xrightarrow[RT]{} & \mathrm{DF}^+(\mathcal{A}') \end{array}$$

commute up to natural isomorphism.

In particular, if RT sends $\mathrm{D}^b(\mathcal{A})$ to $\mathrm{D}^b(\mathcal{A}')$, then $R\mathrm{Fil}^f(T)$ sends $\mathrm{DF}^b(\mathcal{A})$ to $\mathrm{DF}^b(\mathcal{A}')$, because $\mathrm{DF}^b(\mathcal{A}')$ is the full subcategory of $\mathrm{D}^+(\mathrm{Fil}^f(\mathcal{A}'))$ whose objects are the K such that $\mathrm{Gr}(K)$ is in $\mathrm{D}^b(\mathcal{A}')$. In that case, $R\mathrm{Fil}^f(T)$ will be an f-lifting of RT.

Before we prove the proposition, we introduce some definitions. We assume that $\mathcal{A}$ has a T-injective subcategory $\mathcal{I}$. By [18, Proposition 13.3.5], this implies that T has a

[7] The definition of a right derived functor in this setting is analogous to the usual one: a right derived functor of $\mathrm{Fil}^f(T)$ is a left Kan extension of $\mathrm{K}^+(\mathrm{Fil}^f(\mathcal{A})) \xrightarrow{\mathrm{K}^+(\mathrm{Fil}^f(T))} \mathrm{K}^+(\mathrm{Fil}^f(\mathcal{A}')) \to \mathrm{D}^+(\mathrm{Fil}^f(\mathcal{A}'))$ along the canonical functor $\mathrm{K}^+(\mathrm{Fil}^f(\mathcal{A})) \to \mathrm{D}^+(\mathrm{Fil}(\mathcal{A}))$.

right derived functor $RT:\mathrm{D}^+(\mathscr{A}) \to \mathrm{D}^+(\mathscr{A}')$; we say that an object A of $\mathscr{A}$ is *T-acyclic* if $R^r T(A) = 0$ for every $r \geq 1$. Then every object of $\mathcal{I}$ is T-acyclic [18, Remark 13.3.6 (ii)], so we may assume that $\mathcal{I}$ is the full subcategory of T-acyclic objects. The hypothesis on $\mathscr{A}$ implies in particular that every object of $\mathscr{A}$ injects into a T-acyclic object.

We say that an object X of $\mathrm{Fil}^f(\mathscr{A})$ is *filtered T-acyclic* if $\mathrm{Gr}^i(X)$ is T-acyclic for every $i \in \mathbb{Z}$; we denote the full subcategory of filtered T-acyclic objects by $\mathrm{Fil}^f(\mathcal{I})$. Let $X \in \mathrm{Fil}^f(\mathscr{A})$ be filtered T-acyclic; as T-acyclic objects are stable by extensions and by taking cokernels of monomorphisms, and as the filtration on X is finite, the object $\mathrm{Fil}^i(X)/\mathrm{Fil}^j(X)$ is T-acyclic for all $i \leq j$ in $\mathbb{Z}$, and in particular all $\mathrm{Fil}^i(X)$ and X itself are T-acyclic.

Lemma A.1.12. *The following hold:*

(i) *For every $X \in \mathrm{Ob}\,\mathrm{Fil}^f(\mathscr{A})$, there exists a strict monomorphism from X to a filtered T-acyclic object.*

(ii) *Let $0 \to X \to Y \to Z \to 0$ be a strictly exact sequence in $\mathrm{Fil}^f(\mathscr{A})$. If X and Y are filtered T-acyclic, then so is Z.*

(iii) *Let $X \in \mathrm{Ob}(\mathrm{Fil}^f(\mathscr{A}))$ be filtered T-acyclic. Then, for every $i \in \mathbb{Z}$, the canonical morphisms $T(\mathrm{Fil}^i(X)) \to \mathrm{Fil}^i(T(X))$ and $T(\mathrm{Gr}^i(X)) \to \mathrm{Gr}^i(T(X))$ are isomorphisms.*

(iv) *Let $0 \to X \to Y \to Z \to 0$ be a strictly exact sequence in $\mathrm{Fil}^f(\mathscr{A})$. If X, Y and Z are filtered T-acyclic, then the sequence $0 \to \mathrm{Fil}^f(T)(X) \to \mathrm{Fil}^f(T)(Y) \to \mathrm{Fil}^f(T)(Z) \to 0$ is strictly exact.*

Proof. Point (i) is proved exactly like [29, Lemma 05TS].

Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathrm{Fil}^f(\mathscr{A})$. Then it is strictly exact if and only if the sequence $0 \to \mathrm{Gr}^i(X) \to \mathrm{Gr}^i(Y) \to \mathrm{Gr}^i(Z) \to 0$ is exact for every $i \in \mathbb{Z}$; as T-acyclic objects are stable by taking cokernels of monomorphisms, this gives (ii). Point (iv) also follows from this observation and from point (iii), as T sends short exact sequences of T-acyclic objects to short exact sequences.

It remains to prove (iii). So suppose that $X \in \mathrm{Ob}(\mathrm{Fil}^f(\mathscr{A}))$ is filtered T-acyclic. By the paragraph before the statement of the lemmas, this implies that all the $\mathrm{Fil}^i(X)$, $X/\mathrm{Fil}^i(X)$ and $\mathrm{Gr}^i(X)$ (and in particular X itself) are T-acyclic. In particular, if we apply T to the short exact sequence of T-acyclic objects

$$0 \to \mathrm{Fil}^i(X) \to X \to X/\mathrm{Fil}^i(X) \to 0,$$

we get a short exact sequence; this implies that $T(F^i(X)) \to T(X)$ is injective, whence the first assertion of (iii). Now we apply T to the short exact sequence of T-acyclic objects

$$0 \to \mathrm{Fil}^{i+1}(X) \to \mathrm{Fil}^i(X) \to \mathrm{Gr}^i(X) \to 0;$$

this gives again a short exact sequence, so we get that

$$T\big(\mathrm{Gr}^i(X)\big) = \mathrm{Coker}\,\big(T\big(\mathrm{Fil}^{i+1}(X)\big) \to T\big(\mathrm{Fil}^i(X)\big)\big).$$

This, together with the first assertion of (iii), gives the second assertion of (iii). ∎

Proof of Proposition A.1.11. In order to prove that $\mathrm{Fil}^f(T)$ has a right derived functor, it suffices by [25, Proposition 1.3.4] to show that $\mathrm{Fil}^f(\mathcal{I})$ is a $\mathrm{Fil}^f(T)$-injective subcategory of $\mathrm{Fil}^f(\mathcal{A})$ in the sense of [25, Definition 1.3.2]; but this is exactly points (i), (ii) of (iv) of Lemma A.1.12. This also gives a way to compute the right derived functor: for every $K \in \mathrm{D}^+(\mathrm{Fil}^f(\mathcal{A}))$, there exists be a filtered quasi-isomorphism (i.e., a morphism α of complexes of $\mathrm{Fil}^f(\mathcal{A})$ such that $\mathrm{Gr}(\alpha)$ is a quasi-isomorphism) $\alpha\colon K \to I$ with I a complex of filtered T-acyclic objects, and we have $R\mathrm{Fil}^f(T)(K) = \mathrm{Fil}^f(T)(I)$. As the complexes $\omega(I)$, $\mathrm{Gr}(I)$ and $\sigma_{\leq i}(I) = \mathrm{Fil}^i(I)$ are bounded below complexes of T-acyclic objects of $\mathcal{A}$, and as $\omega(\alpha)$, $\mathrm{Gr}(\alpha)$ and $\sigma_{\leq i}(\alpha)$ are quasi-isomorphisms by [29, Lemma 05S3], we also have $RF(\omega(I)) = F(\omega(I)) = \omega(F(I))$, $RF(\mathrm{Gr}(I)) = F(\mathrm{Gr}(I)) \simeq \mathrm{Gr}(F(I))$ and $RF(\sigma_{\leq i}(I)) = F(\sigma_{\leq i}(I)) \simeq \sigma_{\leq i}(F(I))$ (the last isomorphisms are given by point (iii) of Lemma A.1.12). This gives the commutativity of the three squares. ■

A.2. t-structures and the realization functor

We review the construction of the realization functor from [6, Section 3.1] and [4, Appendix A].

Definition A.2.1. Let $\mathcal{D}$ be a triangulated category and DF be an f-category over $\mathcal{D}$. Suppose that we are given a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ and a t-structure $(\mathrm{DF}^{\leq 0}, \mathrm{DF}^{\geq 0})$ on DF. We say that these t-structures are *compatible* if $i\colon \mathcal{D} \to \mathrm{DF}$ is t-exact and if $s(\mathrm{DF}^{\leq 0}) = \mathrm{DF}^{\leq -1}$.

Proposition A.2.2 (Proposition A.5 of [4]). *Let $\mathcal{D}$ be a triangulated category and* DF *be an f-category over $\mathcal{D}$. Suppose that we are given a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$. Then there exists a unique t-structure on* DF *compatible with $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, and it is given by*

$$\mathrm{Ob}\,\mathrm{DF}^{\leq 0} = \{X \in \mathrm{Ob}\,\mathrm{DF} \mid \forall i \in \mathbb{Z}, \mathrm{Gr}^i X[i] \in \mathrm{Ob}\,\mathcal{D}^{\leq 0}\},$$
$$\mathrm{Ob}\,\mathrm{DF}^{\geq 0} = \{X \in \mathrm{Ob}\,\mathrm{DF} \mid \forall i \in \mathbb{Z}, \mathrm{Gr}^i X[i] \in \mathrm{Ob}\,\mathcal{D}^{\geq 0}\}.$$

Theorem A.2.3 (Proposition A.5 and A.6 of [4]). *Let $\mathcal{D}$ be a triangulated category and* DF *be an f-category over $\mathcal{D}$. Suppose that we are given compatible t-structures $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ and $(\mathrm{DF}^{\leq 0}, \mathrm{DF}^{\geq 0})$ on* DF. *Denote by $H\colon \mathcal{D} \to \mathcal{C} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ the cohomology functor. We define a functor $H_F\colon \mathrm{DF} \to \mathrm{C}^b(\mathcal{C})$ in the following way: if $X \in \mathrm{Ob}\,\mathrm{DF}$, we set $H_F(X)^i = H^i\,\mathrm{Gr}^i(X)$, and we take as differential $H_F(X)^i \to H_F(X)^{i+1}$ the map induced from the connection morphism in the distinguished triangle*

$$\omega\big(\sigma_{\leq i+1}\sigma_{\geq i+1}(X)\big) \to \omega\big(\sigma_{\leq i+1}\sigma_{\geq i}(X)\big) \to \omega\big(\sigma_{\leq i}\sigma_{\geq i}(X)\big) \xrightarrow{+1} .$$

(i) *The functor H_F is well-defined, its restriction to the heart $\mathcal{C}_F$ of $(\mathrm{DF}^{\leq 0}, \mathrm{DF}^{\geq 0})$ is an equivalence of categories $G\colon \mathcal{C}_F \xrightarrow{\sim} \mathrm{C}^b(\mathcal{C})$, and $G^{-1} \circ H_F\colon \mathrm{DF} \to \mathcal{C}_F$ is the cohomology functor of the t-structure $(\mathrm{DF}^{\leq 0}, \mathrm{DF}^{\geq 0})$.*

(ii) *The functor $\omega \circ G^{-1}\colon \mathrm{C}^b(\mathcal{C}) \to \mathcal{D}$ factors through $\mathrm{D}^b(\mathcal{C})$.*

Definition A.2.4. In the situation of Theorem A.2.3, we call the functor $\mathrm{D}^b(\mathcal{C}) \to \mathcal{D}$ induced by $\omega \circ G^{-1}$ the *realization functor* and denote it by real.

Example A.2.5. Let $\mathcal{A}$ be an abelian category, let $\mathcal{D}$ be a full triangulated subcategory of $\mathrm{D}^b(\mathcal{A})$, let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure on $\mathcal{D}$, and denote its heart by $\mathcal{C}$. Let DF be the full subcategory of K in $\mathrm{DF}^b(\mathcal{A})$ such that $\mathrm{Gr}^i\, K \in \mathrm{Ob}\,\mathcal{D}$ for every $i \in \mathbb{Z}$. This is an f-category over $\mathcal{D}$ by Proposition A.1.9. By Proposition A.2.2, the t-structure of $\mathcal{D}$ also lifts to a compatible t-structure $(\mathrm{DF}^{\leq 0}, \mathrm{DF}^{\geq 0})$ on DF. The heart of this t-structure is the abelian category with objects

$$\{K \in \mathrm{Ob}\,\mathrm{DF}^b(\mathcal{A}) \mid \forall i \in \mathbb{Z}, \mathrm{Gr}^i\, K[i] \in \mathrm{Ob}\,\mathcal{C}\}.$$

It is the category called "$\mathcal{D}F_{\text{bête}}$" in [6, Section 3.1.7]. If $(K, F^\bullet)$ is an object of this category, then the sequence

$$\cdots \to \mathrm{Gr}^i\, K[i] \to \mathrm{Gr}^{i+1}\, K[i+1] \to \mathrm{Gr}^{i+2}\, K[i+2] \to \cdots$$

is a bounded complex of objects of $\mathcal{C}$, which is the image of $(K, F^\bullet)$ by the functor G of Theorem A.2.3.

A.3. The realization functor and f-liftings

Now we come to the second goal of this subsection. We start with some preliminaries. Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories and $T : \mathcal{D} \to \mathcal{D}'$ be a triangulated functor. Suppose that we are given a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ (resp. $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$) on $\mathcal{D}$ (resp. $\mathcal{D}'$), and denote the cohomology functors of this t-structure by H^i and its heart by $\mathcal{C}$ (resp. $\mathcal{C}'$). We say that an object X of $\mathcal{C}$ is *T-acyclic* if $T(X) \in \mathrm{Ob}\,\mathcal{C}'$. If X is T-acyclic, then we have $\mathrm{H}^n T(X) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$; the converse if true if the t-structure on $\mathcal{D}'$ is nondegenerate.

Lemma A.3.1. *The following hold:*

(i) *The full subcategory of T-acyclic objects of $\mathcal{C}$ is stable by extensions.*

(ii) *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathcal{C}$. If X, Y, Z are T-acyclic, then $0 \to T(X) \to T(Y) \to T(Z) \to 0$ is an exact sequence in $\mathcal{C}'$.*

(iii) *Let $(X^\bullet, d^\bullet)$ be a complex of objects of $\mathcal{C}$ and let $k \in \mathbb{Z}$. If X^k is T-acyclic and $\mathrm{H}^{k+1}(X^\bullet, d^\bullet) = 0$, then we have $\mathrm{H}^r T(\mathrm{Ker}\, d^{k+1}) \simeq \mathrm{H}^{r+1} T(\mathrm{Ker}\, d^k)$ for every $r \in \mathbb{Z} \setminus \{-1, 0\}$.*

(iv) *Suppose that the t-structure on $\mathcal{D}'$ is non-degenerate. Let $(X^\bullet, d^\bullet)$ be an exact complex of T-acyclic objects. Suppose that at least one of the following conditions hold:*

   (a) *The complex $(X^\bullet, d^\bullet)$ is bounded.*

   (b) *There exists $N \in \mathbb{N}$ such that $T(X) \in \mathcal{D}'^{[-N,N]}$ for every $X \in \mathrm{Ob}\,\mathcal{C}$.*

   Then the complex $T(X^\bullet)$ of objects of $\mathcal{C}'$ is exact.

(v) *Suppose that the t-structure on $\mathcal{D}'$ is non-degenerate and that there exists $N \in \mathbb{N}$ such that $T(X) \in \mathcal{D}'^{[-N,N]}$ for every $X \in \mathrm{Ob}\,\mathcal{C}$. Let $(X^\bullet, d^\bullet)$ be a complex of T-acyclic objects of $\mathcal{C}$. If $X^\bullet$ is quasi-isomorphic to a bounded complex, then, for $n \in \mathbb{N}$ big enough, the complexes $\tau_{\leq n}X^\bullet$, $\tau_{\geq -n}X^\bullet$ and $\tau_{\leq n}\tau_{\geq -n}X^\bullet$ are complexes of T-acyclic objects, and all the maps in the square*

$$\begin{array}{ccc} \tau_{\leq n}X^\bullet & \longrightarrow & \tau_{\leq n}\tau_{\geq -n}X^\bullet \\ \downarrow & & \downarrow \\ X^\bullet & \longrightarrow & \tau_{\geq -n}X^\bullet \end{array}$$

are quasi-isomorphisms. In particular, $X^\bullet$ is quasi-isomorphic to a bounded complex of T-acyclic objects.

Proof. We repeatedly use the fact that a complex $X \to Y \to Z$ in $\mathcal{C}$ is a short exact sequence if and only if can completed to a distinguished triangle of $\mathcal{D}$ (see [6, Theorem 1.3.6]).

Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathcal{C}$. If X, Z are T-acyclic, then we have an exact triangle $T(X) \to T(Y) \to T(Z) \xrightarrow{+1}$ with $T(X), T(Z)$ in $\mathcal{C}'$, so $T(Y)$ is in $\mathcal{C}'$ and the sequence $0 \to T(X) \to T(Y) \to T(Z) \to 0$ is exact in $\mathcal{C}'$. This proves (i) and (ii).

In the situation of (iii), we have an exact sequence

$$0 \to \mathrm{Ker}\, d^k \to X^k \xrightarrow{d^k} \Im(d^k) = \mathrm{Ker}(d^{k+1}) \to 0,$$

hence a distinguished triangle $T(\mathrm{Ker}\, d^k) \to T(X^k) \to T(\mathrm{Ker}\, d^{k+1}) \xrightarrow{+1}$. As $\mathrm{H}^r T(X^k) = 0$ for $r \neq 0$, the conclusion of (iii) follows from the long exact cohomology sequence of this triangle.

Suppose that we are in the situation of (iv). Let $k \in \mathbb{Z}$, and let r be a positive integer. By (iii), we have isomorphisms $\mathrm{H}^r T(\mathrm{Ker}\, d^k) \simeq \mathrm{H}^{r+l} T(\mathrm{Ker}\, d^{k-l})$ and $\mathrm{H}^{-r} T(\mathrm{Ker}\, d^k) \simeq \mathrm{H}^{-r-l} T(\mathrm{Ker}\, d^{k+l})$ for every $l \in \mathbb{N}$. Also, for l big enough, we have $\mathrm{H}^{r+l} T(\mathrm{Ker}\, d^{k-l}) = 0$ and $\mathrm{H}^{-r-l} T(\mathrm{Ker}\, d^{k+l}) = 0$; indeed, if (a) holds, this is true because $X^{k+l} = 0$ and $X^{k-l} = 0$ for l big enough, and if (b) holds, this is true as soon as $l \geq N$. We deduce that $\mathrm{H}^r T(\mathrm{Ker}\, d^k) = 0$ and $\mathrm{H}^{-r}(\mathrm{Ker}\, d^k) = 0$ for every $k \in \mathbb{Z}$ and every positive integer r, hence that all $\mathrm{Ker}\, d^k$ are T-acyclic. The conclusion of (iv) then follows by applying (ii) to the short exact sequences $0 \to \mathrm{Ker}\, d^k \to X^k \to \mathrm{Ker}\, d^{k+1} \to 0$.

Finally, suppose that we are in the situation of (v). As $X^\bullet$ is quasi-isomorphic to a bounded complex, there exists $M \in \mathbb{N}$ such that $\mathrm{H}^r(X^\bullet) = 0$ for $r \notin [-M, M]$. Let $k \in \mathbb{N}$. If $k \geq M$ and r is a positive integer, then we have by (iii):

$$\mathrm{H}^{-r} T(\mathrm{Ker}\, d^k) \simeq \mathrm{H}^{-r-N} T(\mathrm{Ker}\, d^{k+N}) = 0$$

and

$$H^r T(\mathrm{Ker}\, d^{-k}) \simeq \mathrm{H}^{r+N}\, \mathrm{Ker}(d^{-k-N}) = 0.$$

Similarly, if $k \geq N + M$ and r is a positive integer, then we have by (iii):

$$\mathrm{H}^r T(\operatorname{Ker} d^k) \simeq \mathrm{H}^{r+N} T(\operatorname{Ker} d^{k-N}) = 0$$

and

$$H^{-r} T(\operatorname{Ker} d^{-k}) \simeq \mathrm{H}^{-r-N} \operatorname{Ker}(d^{-k+N}) = 0.$$

We conclude that $\operatorname{Ker}(d^k)$ is T-acyclic for $k \geq N + M$ or $k \leq -N - M$. Also, if $n \leq -N - 2$, then $\mathrm{H}^n(X^\bullet) = 0$ and $\mathrm{H}^{n+1}(X^\bullet) = 0$, hence $\operatorname{Coker}(d^{n-1}) \simeq \operatorname{Ker}(d^{n+1})$. So the two statements of (v) hold for $n \geq N + M + 2$. ■

The following proposition is essentially proved in [4, Section A.7].

Proposition A.3.2. *Let $\mathcal{D}$, $\mathcal{D}'$ be triangulated categories, and let* DF *(resp.* DF'*) be an f-category over $\mathcal{D}$ (resp. $\mathcal{D}'$). Suppose that we are given compatible t-structures* $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ *and* $(\mathrm{DF}^{\leq 0}, \mathrm{DF}^{\geq 0})$ *(resp.* $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$ *and* $(\mathrm{DF}'^{\leq 0}, \mathrm{DF}'^{\geq 0})$*) on $\mathcal{D}$ and* DF *(resp. $\mathcal{D}'$ and* DF'*), and denote the hearts of this t-structures by $\mathcal{C}$ and $\mathcal{C}_F$ (resp. $\mathcal{C}'$ and $\mathcal{C}'_F$). Suppose also that the t-structure on $\mathcal{D}'$ is non-degenerate.*

Let $T\colon \mathcal{D} \to \mathcal{D}'$ be a triangulated functor. Suppose that the following conditions are satisfied:

(a) *The functor T admits an f-lifting* $FT\colon \mathrm{DF} \to \mathrm{DF}'$.

(b) *Let $\mathcal{I} := \{X \in \mathcal{C} \mid T(X) \in \mathcal{C}'\}$ be the full subcategory of T-acyclic objects of $\mathcal{C}$. Then the functor* $\mathrm{K}^b(\mathcal{I})/\mathrm{N}^b(\mathcal{I}) \to \mathrm{D}^b(\mathcal{C})$ *is an equivalence, where* $\mathrm{K}^b(\mathcal{I})$ *is the category of bounded complexes of objects of $\mathcal{I}$ up to homotopy and* $\mathrm{N}^b(\mathcal{I})$ *is its full subcategory of exact complexes.*

Then the functor $\mathrm{K}^b(\mathcal{I}) \xrightarrow{\mathrm{K}^b(T)} \mathrm{K}^b(\mathcal{C}') \to \mathrm{D}^b(\mathcal{C}')$ *sends* $\mathrm{N}^b(\mathcal{I})$ *to* 0, *hence induces a functor* $DT\colon \mathrm{D}^b(\mathcal{C}) \to \mathrm{D}^b(\mathcal{C}')$*, and the following diagram commutes up to natural isomorphism:*

$$\begin{array}{ccc} \mathrm{D}^b(\mathcal{C}) & \xrightarrow{DT} & \mathrm{D}^b(\mathcal{C}') \\ {\scriptstyle \mathrm{real}}\downarrow & & \downarrow{\scriptstyle \mathrm{real}} \\ \mathcal{D} & \xrightarrow[T]{} & \mathcal{D}' \end{array}$$

Proof. The first statement follows from point (iv) of Lemma A.3.1.

We prove the second statement. In Theorem A.2.3, we defined equivalences $G\colon \mathcal{C}_F \to \mathrm{C}^b(\mathcal{C})$ and $G'\colon \mathcal{C}'_F \to \mathrm{C}^b(\mathcal{C}')$. By point (ii) of the same theorem, the functor

$$\omega \circ G^{-1}\colon \mathrm{C}^b(\mathcal{C}) \to \mathcal{D} \quad \big(\text{resp. } \omega \circ G'^{-1}\colon \mathrm{C}^b(\mathcal{D}') \to \mathcal{D}'\big)$$

sends exact complexes to 0, hence induces a functor $\mathrm{D}^b(\mathcal{C}) \to \mathcal{D}$ (resp. $\mathrm{D}^b(\mathcal{C}') \to \mathcal{D}'$), which is the realization functor real. Now let $\mathcal{I}_F$ be the full subcategory of $\mathcal{C}_F$ whose objects are the X such that $\mathrm{Gr}^i X[i] \in \operatorname{Ob} \mathcal{I}$ for every $i \in \mathbb{Z}$, i.e., such that $G(X)$ is in $\mathrm{C}^b(\mathcal{I})$. Proposition A.1.8 implies that FT sends $\mathcal{I}_F$ to $\mathcal{C}'_F$, and that the restrictions of

$G' \circ FT$ and $\mathrm{C}^b(T) \circ G$ to $\mathcal{I}_F$ are isomorphic. So we get an isomorphism of functors on $\mathrm{C}^b(\mathcal{I})$:

$$T \circ \omega \circ G^{-1} \simeq \omega \circ FT \circ G^{-1} \simeq \omega \circ G'^{-1} \circ \mathrm{C}^b(T).$$

This gives the isomorphism $T \circ \mathrm{real} \simeq \mathrm{real} \circ DT$. ∎

Remark A.3.3. Suppose that we are in the situation of Proposition A.3.2. If moreover $T: \mathcal{D} \to \mathcal{D}'$ is left t-exact and if $\mathcal{I}$ is cogenerating in $\mathcal{C}$ (i.e., every object of $\mathcal{C}$ has a monomorphism into an object of $\mathcal{I}$), then the functor $\mathrm{H}^0(T): \mathcal{C} \to \mathcal{C}'$ admits a right derived functor $RT: \mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+(\mathcal{C}')$ by [18, Proposition 13.3.5], and the construction of RT in that proposition shows that RT sends that $\mathrm{D}^b(\mathcal{C})$ to $\mathrm{D}^b(\mathcal{C}')$ and that DT is the restriction of RT to $\mathrm{D}^b(\mathcal{C})$. We have a similar statement if T is right t-exact and $\mathcal{I}$ is generating in $\mathcal{C}$.

Remark A.3.4. By [18, Proposition 10.2.7], to check assumption (b) in the statement of Proposition A.3.2, it suffices to find triangulated subcategories $\mathcal{D}_0 = \mathrm{K}^b(\mathcal{C}) \supset \mathcal{D}_1 \supset \cdots \supset \mathcal{D}_r = \mathrm{K}^b(\mathcal{I})$ of $\mathrm{K}^b(\mathcal{C})$ such that, for every $i \in \{1, \dots, r-1\}$, one of the following conditions holds:

- For every $X \in \mathrm{Ob}\, \mathcal{D}_i$, there exists a quasi-isomorphism $X \to Y$ with $Y \in \mathrm{Ob}\, \mathcal{D}_{i+1}$.
- For every $X \in \mathrm{Ob}\, \mathcal{D}_i$, there exists a quasi-isomorphism $Y \to X$ with $Y \in \mathrm{Ob}\, \mathcal{D}_{i+1}$.

A.4. Application to horizontal perverse sheaves

In this section, we explain how to construct the f-categories underlying the triangulated categories of the main text, as well as f-liftings of the triangulated functors between these categories.

ℓ-adic complexes. Let X be a scheme and E be an algebraic extension of $\mathbb{Q}_\ell$. We use the notation of Section 2.1.

By Example A.1.2 applied to $\mathcal{A} = \mathrm{Sh}(X_{\text{proét}}, E)$ and Proposition A.1.9 applied to the bounded filtered category of $\mathcal{A}$ and the full subcategory $\mathrm{D}^b_c(X, E)$ of $\mathrm{D}^+(\mathcal{A})$, we get an f-category $\mathrm{DF}^b_c(X, E)$ over the triangulated category $\mathrm{D}^b_c(X, E)$.

Let $f: X \to Y$ be a morphism of finite type. We have triangulated functors f_*, $\otimes$ and $\underline{\mathrm{Hom}}_X$ on $\mathrm{D}^+(X_{\text{proét}}, E)$, $\mathrm{D}^-(X_{\text{proét}}, E) \times \mathrm{D}(X_{\text{proét}}, E)$ and $\mathrm{D}(X_{\text{proét}}, E)^\circ \times \mathrm{D}^+(X_{\text{proét}}, E)$, and they are all derived functors, so, using Proposition A.1.11, we can extend them to triangulated functors on the filtered derived categories $\mathrm{DF}^+(X_{\text{proét}}, E)$, $\mathrm{DF}^-(X_{\text{proét}}, E) \times \mathrm{DF}(X_{\text{proét}}, E)$ and $\mathrm{DF}(X_{\text{proét}}, E)^\circ \times \mathrm{DF}^+(X_{\text{proét}}, E)$.[8] Next, if X has a dimension function, using the fact that $\mathrm{D}^+(X_{\text{proét}}, E)$ is equivalent to the full subcategory of $\mathrm{DF}^+(X_{\text{proét}}, E)$ with objects the K such that $\mathrm{Gr}^i K = 0$ for $i \neq 0$, we can see the dualizing complex $\widehat{K}_X$ as an object of $\mathrm{DF}^+(X_{\text{proét}}, E)$, and so we can define D_X on $\mathrm{DF}(X_{\text{proét}}, E)$ by $D_X(K) = \underline{\mathrm{Hom}}_X(K, K_X)$.

[8] For $\underline{\mathrm{Hom}}_X$ and $\otimes$, we could also use V.2 of [15].

Finally, we extend the inverse image functor. The functor f_* from $\mathrm{D}^+(X_{\text{proét}}, E)$ to $\mathrm{D}^+(Y_{\text{proét}}, E)$ has a left adjoint f^*, given by $f^*K = f^*_{\text{naive}}K \otimes_{f^*_{\text{naive}}E_X} E_Y$, where f^*_{naive} is the regular pullback functor (see [7, Remark 6.8.15]). The functor f^*_{naive} is exact and so extends to $\mathrm{DF}^+(Y_{\text{proét}}, E)$ by Proposition A.1.11, and we can see $f^*_{\text{naive}}E_Y$ and E_X as objects of $\mathrm{DF}^+(X_{\text{proét}}, E)$, so f^* also extends. Restricting all these functors to the subcategories DF^b_c, we get f-liftings of the functors f_*, f^*, $\otimes$, $\underline{\mathrm{Hom}}_X$ and D_X (when X has a dimension function for the last one) on the categories D^b_c.

Perverse t-structure. Fix a scheme X as in Section 2.2. Applying Proposition A.2.2 to the f-lifting $\mathrm{DF}_c(X, E)$ of the triangulated category $\mathrm{D}^b_c(X, E)$ and to the perverse t-structure on $\mathrm{D}^b_c(X, E)$, we get a compatible t-structure on $\mathrm{DF}^b_c(X, E)$. Then Theorem A.2.3 gives a triangulated realization functor real: $\mathrm{D}^b\,\mathrm{Perv}(X, E) \to \mathrm{D}^b_c(X, E)$ extending the inclusion $\mathrm{Perv}(X, E) \subset \mathrm{D}^b_c(X, E)$.

We can apply Proposition A.3.2 and Remarks A.3.3, A.3.4 to any functor between categories $\mathrm{D}^b_c(X, E)$ that is t-exact for the perverse t-structures and constructed from the 6 operations $f_*, f^*, f_!, f^!, \otimes, \underline{\mathrm{Hom}}_X$.

For example, if Y is a scheme satisfying the same conditions as X, and if $T = f_*$ or $T = f_!$ for $f\colon X \to Y$ quasi-finite affine (resp. $T = f^*[d]$ for $f\colon Y \to X$ smooth of relative dimension d), then we get the commutative diagrams of point (i) (resp. (ii)) of Proposition 2.3.1. Similarly, taking for $T = D_X\colon \mathrm{D}^b_c(X, E)^{\mathrm{op}} \to \mathrm{D}^b_c(X, E)$ the duality functor, we get point (iii) of the same Proposition, and taking T to be an appropriate shift of the restriction to the generic fiber functor, we get Proposition 2.3.2.

Horizontal constructible complexes. We use the notation of Section 2.4. In particular, k is a field of finite type over its prime field, X is a separated finite type k-scheme and E is an algebraic extension of $\mathbb{Q}_\ell$.

We define an f-category $\mathrm{DF}^b_h(X, E)$ over $\mathrm{D}^b_h(X, E)$ using Proposition A.1.10. First, for every $(A, \mathcal{X}) \in \mathrm{Ob}\,\mathcal{U}X$, we get an f-category $\mathrm{DF}^b_c(\mathcal{X}, E)$ over $\mathrm{D}^b_c(\mathcal{X}, E)$ by applying Proposition A.1.9 to the triangulated subcategory $\mathrm{D}^b_c(\mathcal{X}, E)$ of the bounded derived category of proétale sheaves of E-modules on $\mathcal{X}$. Next, if $(A, \mathcal{X}) \to (A', \mathcal{X}')$ is a morphism in $\mathcal{U}X$, then the functor $\mathrm{D}^b_c(\mathcal{X}, E) \to \mathrm{D}^b_c(\mathcal{X}', E)$ is the restriction of the trivial derived functor of a functor

$$\mathrm{Sh}(\mathcal{X}_{\text{proét}}, E) \to \mathrm{Sh}(X_{\text{proét}}, E),$$

so, by Proposition A.3.2, it admits an f-lifting $\mathrm{DF}^b_c(\mathcal{X}, E) \to \mathrm{DF}^b_c(\mathcal{X}', E)$. We set

$$\mathrm{DF}^b_h(X, E) = 2-\varinjlim_{(A,\mathcal{X})\in \mathrm{Ob}\,\mathcal{U}X} \mathrm{DF}^b_c(\mathcal{X}, E);$$

this is an f-category over $\mathrm{D}^b_h(X, E)$.

Moreover, if $\eta^*\colon \mathrm{D}^b_h(X, E) \to \mathrm{D}^b_c(X, E)$ is the exact functor induced by the restriction functors

$$\mathrm{D}^b_c(\mathcal{X}, E) \to \mathrm{D}^b_c(\mathcal{X} \otimes_A k, E) \xrightarrow{u^*} \mathrm{D}^b_c(X, E),$$

for $(A, \mathcal{X}, u) \in \mathrm{Ob}\,\mathcal{U}X$, then η^* admits an f-lifting, by Proposition A.3.2.

By Proposition A.2.2, the perverse t-structure on $\mathrm{D}^b_h(X, E)$ of Section 2.5 lifts to a compatible t-structure on $\mathrm{DF}^b_h(X, E)$, so by Theorem A.2.3, we get a realization functor

$$\mathrm{real}\colon \mathrm{D}^b\,\mathrm{Perv}_h(X, E) \to \mathrm{D}^b_h(X, E).$$

Mixed perverse sheaves. We use the notation of Section 2.6, so X and E are as before.

Applying Proposition A.1.9 to the f-category $\mathrm{DF}^b_h(X, E)$ lifting $\mathrm{D}^b_h(X, E)$, we get an f-category $\mathrm{DF}^b_m(X, E)$ lifting $\mathrm{D}^b_m(X, E)$. The realization functor real: $D^b\,\mathrm{Perv}_h(X, E) \to \mathrm{D}^b_h(X, E)$ restricts to a functor

$$\mathrm{real}\colon \mathrm{D}^b\,\mathrm{Perv}_m(X, E) \to \mathrm{D}^b_h(X, E),$$

whose essential image is contained in $\mathrm{D}^b_m(X, E)$ by definition of $\mathrm{D}^b_m(X, E)$; this is also the realization functor that we would get by applying Proposition A.2.2 and Theorem A.2.3 to the f-category $\mathrm{DF}^b_m(X, E)$.

Acknowledgments. The author would like to thank the École Normale Supérieure de Lyon and the Université Lyon 1 for their hospitality and support during the academic year 2017–2018. Most of this papers was written while she was a professor at Princeton University. Also, she has greatly benefited from conversations with Florian Ivorra about the formalism of crossed functors and Beilinson's construction of nearby and vanishing cycles. Last but not least, she would like to thank an anonymous referee for numerous insightful comments and corrections.

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Communicated by Takeshi Saito

Received 25 November 2023; revised 16 September 2024.

Sophie Morel
Unité de mathématiques pures et appliquées, ENS de Lyon site Monod, 46 allée d'Italie,
69364 Cedex 07 Lyon, France; sophie.morel@ens-lyon.fr