Log canonical thresholds and coregularity

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Abstract. We prove the ascending chain condition for log canonical thresholds of bounded coregularity.

1. Introduction

The main approach to study algebraic singularities is to introduce invariants that allow us to measure how singular a point in an algebraic variety is. Once the invariant is introduced, the next step is to understand what values it can take, whether it detects smoothness, how the numerical data reflects on the singularity and vice-versa. The best-known invariant of singularities is the multiplicity. The multiplicity played a fundamental role in the resolution of singularities [25]. However, it is often too coarse of an invariant to obtain information in other contexts. This has led algebraic geometers to search for new invariants of singularities that could guide further developments in the understanding of algebraic singularities.

The log canonical threshold, formerly known as complex singularity exponent, was introduced by Atiyah in the study of the division of distributions using resolution of singularities [2]. The first properties of log canonical thresholds were proved by Varchenko in connection with the asymptotic expansion of integrals and Hodge structures [54]. Shokurov realized that the log canonical threshold could be defined purely in terms of the singularities of the minimal model program [51]. Shokurov conjectured that log canonical thresholds of the same dimension satisfy the ascending chain condition (ACC), i.e., there is no infinite increasing sequence. This conjecture follows a common philosophy in algebraic geometry: in a fixed dimension, we cannot find an infinite sequence of milder and milder singularities. The previous principle is not mathematically precise, but it has led to many important conjectures on the behavior of algebraic singularities (see, e.g., [7, 53]).

Although its modern definition is purely in terms in terms of log discrepancies, the log canonical threshold has deep connections with other topics in mathematics: the growth of the number of polynomial solutions in $\mathbb{Z}/p^n$ [24], dimensions of jet schemes [47], the theory of D-modules [26], the α-invariant and algebraic K-stability [55], vanishing theorems [35], non-rationality of Fano varieties [11], and positive characteristic methods [3].

It is also worth mentioning that the log canonical threshold is deeply related to the termination of flips [4, 18, 45].

We briefly recall some of the developments toward the understanding of log canonical thresholds. In [29–31], Kuwata computed log canonical thresholds of hypersurface singularities, surfaces in $\mathbb{C}^3$, and reducible plane curves. In [49, 50], Prokhorov started the study of accumulation points of log canonical thresholds, conjecturing that they must come from log canonical thresholds in lower dimensions. de Fernex, Ein, and Mustaţă drew a connection between the log canonical threshold and multiplicity in [12]. In [10], Cheltsov studied the log canonical threshold of Fano threefold hypersurfaces and its relation to birational rigidity. In [13], de Fernex, Ein, and Mustaţă proved the ascending chain condition for log canonical thresholds in smooth varieties. In that paper, the authors used an inversion of adjunction for complete intersection varieties [16]. Then, they achieved the ACC for log canonical thresholds on varieties with bounded singularities in [14]. This achievement used the theory of ultrafilters. In [19], Hacon, McKernan, and Xu, proved the ACC for log canonical thresholds. Using ideas from the minimal model program, the authors reduced this statement to a problem about projective varieties of log Calabi–Yau type. Then, they improved the Hacon–McKernan developments on birational boundedness of varieties of log general type to prove this global result. In [8], Birkar and Zhang generalized the results for log canonical thresholds to the context of generalized pairs. This improvement was vital for the effectivity of Iitaka fibrations and boundedness of Fano varieties [5, 6, 8]. In [39], McLean proved that the log canonical threshold can be expressed in terms of Floer cohomology.

Now, we turn to explain the main theorem of this article. When computing log canonical thresholds, it is natural to start with orbifold and toric examples. If $(T; t)$ is the germ of a toric singularity and Γ is a reduced torus invariant divisor, then the pair $(T, \Gamma; t)$ is always log canonical. This means that, regardless of the dimension of the germ, the log canonical threshold is only one. This leads to a natural question: are log canonical thresholds controlled by the dimension or can they be controlled by a weaker invariant? Some computations show that the closer to the toric setting we are, the fewer values for log canonical thresholds we can produce (see Example 5.1). In order to make this notion precise, we define the *coregularity* of a log canonical singularity $(X, B; x)$ to be:

$$\operatorname{coreg}(X, B; x) := \dim X - \dim \mathcal{D}(X, B; x) - 1. \tag{1.1}$$

Here, $\mathcal{D}(X, B; x)$ is the *dual complex* of the singularity, whose homotopy type captures the combinatorial complexity of the exceptional locus of a log resolution (see, e.g., [15, 28]). The coregularity of a log canonical singularity is an integer between zero and its dimension minus one (see Definition 2.22). Toric singularities with the reduced toric boundary have coregularity zero (see Example 5.1). The set of log canonical thresholds of coregularity at most c, with respect to some coefficient sets I and J, is defined to be (see Definition 2.8):

$\operatorname{LCT}_c(I, J)$

$:= \big\{ t \mid t = \operatorname{lct}(X, B; \Gamma), \ \operatorname{coeff}(B) \in I, \ \operatorname{coeff}(\Gamma) \in J, \ \text{and } \operatorname{coreg}(X, B + t\Gamma) \leq c \big\}.$

The main theorem of this article is the following.

Theorem 1. *Let $c \geq 1$ be a positive integer. Let I and J be two sets of nonnegative real numbers satisfying the descending chain condition. Then, the set $\mathrm{LCT}_c(I, J)$ satisfies the ascending chain condition.*

Note that we do not fix the dimension of our germs. Hence, the previous theorem extends to a great extent the main result of [19]. Indeed, the set $\mathrm{LCT}_c(I, J)$ contains all the germs of dimension at most $c + 1$ but it also contain germs of unbounded dimension (see Example 5.4). In Example 5.2, we show that both the conditions on the coefficients and the coregularity are necessary to obtain the ascending chain condition.

Our definition of coregularity in (1.1) is only for the local case; we call it the *local coregularity*. In Definition 2.22, we give the definition of the coregularity for a normal quasi-projective log canonical pair. In a few words, a pair has coregularity c if locally around its minimal log canonical centers it has coregularity at most c. In the log Calabi–Yau setting, this condition can be checked at any point of the dual complex. Indeed, the dual complex of a log Calabi–Yau pair is equi-dimensional [27].

In the minimal model program, it is usual to study singularities by constructing birational modifications and reducing the problem about the singularity to a problem about the exceptional locus of this modification. This sort of argument is known as *global-to-local principle* in birational geometry. Thus, we reduce the proof of the ACC for lct's with bounded coregularity Theorem 1 to the following statement about log Calabi–Yau varieties.

Theorem 2. *Let c be a positive integer. Let I be a set satisfying the descending chain condition (DCC). There exists a finite subset $I_0 \subseteq I$ satisfying the following. Let (X, B) be a projective log canonical pair so that:*

(1) *the equivalence $K_X + B \equiv 0$ holds;*

(2) *the set $\mathrm{coeff}(B)$ is contained in I; and*

(3) *the coregularity of (X, B) is at most c.*

Then, we have that $\mathrm{coeff}(B) \subseteq I_0$.

Statements such as the previous one are known as global ascending chain conditions (or global ACC, for short). We call the previous theorem the *global ACC with bounded coregularity*. As we will show throughout the proof, the advantage of working with log canonical thresholds of small coregularity is that we can often reduce these computations to low-dimensional computations. Following this philosophy, we can give a precise description of the log canonical thresholds of coregularity zero and one (see Theorem 4.4).

Theorem 3. *Let I and J be two sets of nonnegative real numbers. Then, we have that*

$$\mathrm{LCT}_0(I, J) := \left\{ \frac{1 - \sum_{k=1}^{m} n_k i_k}{\sum_{k=1}^{n} m_k j_k} \geq 0 \mid m, n \in \mathbb{Z}_{\geq 0}, \ n_k, m_k \in \mathbb{Z}_{>0}, \ i_k \in I, \ \text{and} \ j_k \in J \right\}.$$

Note that in the previous theorem the set of log canonical thresholds is described with no assumptions on I or J. However, it is clear from the description that it is a set of real numbers satisfying the ascending chain condition if and only if I and J satisfy the descending chain condition.

Finally, even if the set of log canonical thresholds with bounded coregularity satisfies the ascending chain condition, it is natural to study its accumulation points, meaning, the set of infinite decreasing sequences in the set of log canonical thresholds. In [19], the authors prove that accumulation points of n-dimensional log canonical thresholds come from $(n-1)$-dimensional log canonical thresholds (up to increasing the coefficients set). In this direction, we prove that the analogous statement holds for log canonical thresholds with bounded coregularity.

Theorem 4. *Let $I \subset [0, 1]$ be a set satisfying the DCC. Assume that $1 \in I$ is the only accumulation point and $I = I^+$. Then, we have that the accumulation points of $\mathrm{LCT}_c(I)$ are contained in $\mathrm{LCT}_{c-1}(I)$.*

In the previous statement, $\mathrm{LCT}(I)$ stands for $\mathrm{LCT}(I, \mathbb{Z}_{>0})$, the set $D(I)$ is the derived set of I, and I^+ is the set of positive integral combinations of the elements of I (see Definition 2.11). In particular, $D(I)$ and I also satisfy the DCC (see Lemma 2.12). The set $D(I)$ emerges from the set I naturally when performing adjunction to a divisor (see Lemma 2.17).

We prove many of the theorems in this paper in the context of generalized pairs as in [8]. We presented them in the setting of pairs in the introduction to simplify the exposition.

As a counterpart of the coregularity, one can define the *regularity* of a klt singularity (see Definition 2.22). This definition is motivated by Shokurov and Prokhorov's work in the theory of complements for surfaces [48,52]. The regularity of klt singularities has connections with minimal log discrepancies [42–44], boundedness of klt singularities [23,40], and local fundamental groups [9,41,46]. Indeed, the regularity bounds the rank of finite abelian groups acting on klt germs. We expect further developments in the understanding of klt and lc singularities from the perspective of regularity and coregularity.

On the techniques of the article

The idea of using projective geometry to study log canonical thresholds dates back to McKernan and Prokhorov [38]. In [19], this approach, adjunction theory, and the minimal model program are used to settle the ascending condition for log canonical thresholds of bounded dimension. In [8], Birkar and Zhang generalized this result to the setting of generalized pairs. Our article relies on the ideas of both papers [8, 19]. We aim to reduce Theorem 2 to the usual global ACC for generalized pairs [8, Theorem 1.6]. To do so, we will apply the strategy of [19] inductively on the coregularity. For this aim, we use the minimal model program and study how the coregularity behaves under adjunction (Lemma 2.28) and by passing to the general fiber of a fibration (Lemma 2.30). In the following subsection, we expand on the details of the proofs.

Sketch of the proof

In this section, we give a sketch of the proof of our main results (Theorems 1 and 2).

As explained above, Theorem 2 implies Theorem 1 by means of a global-to-local reduction. However, this requires an argument slightly different than the usual one. For simplicity, we consider the local setting of a log canonical singularity $(X, B + t\Gamma; x)$ which is strictly log canonical at x, the coefficients of B and Γ belong to a set I satisfying the DCC and the coregularity of the pair $(X, B + t\Gamma; x)$ is at most c. Then, we consider a dlt modification $(Y, B_Y + t\Gamma_Y + \sum_{i=1}^{r} S_i)$ of the previous pair. Here B_Y (resp. Γ_Y) is the strict transform of B (rep. Γ) on Y. Furthermore, the divisors $S_1, \ldots, S_r$ are the divisors extracted by the dlt modification. We denote by $\phi: Y \to X$ the projective birational morphism. In order to apply the global result, we need to choose a component S_i which intersects Γ_Y and perform adjunction to it. This way, we can use the global ACC in order to control the coefficient t. By Lemma 2.28, we may control the coregularity of the pair induced on S_i. However, to obtain a projective pair, we need to pass to the general fiber of $S_i \to \phi(S_i)$ which may have higher coregularity (see Example 5.3). In order to fix this issue, we will consider a minimal log canonical center Z of $(X, B + t\Gamma; x)$ which contains x. We may choose Z so that the support of $t\Gamma$ contains Z. By possibly blowing up our dlt modification further, we may assume that S_1 maps onto Z. By running a suitable MMP on Y over X, we may assume that $-S_1$ is nef over the base. At this step, we may lose the dlt condition of the log pair on Y, but it will remain log canonical. Under the assumption on S_1, the set-theoretic preimage of Z on Y equals S_1. In particular, $t\Gamma_Y$ must intersect S_1 and the intersection $t\Gamma_Y \cap S_1$ dominates Z. We may take a further dlt modification, keeping the previous conditions (after possibly replacing S_1 with some other component of coefficient one). Thus, we are in the situation that $t\Gamma_Y \cap S_1$ dominates Z. Since Z is a minimal log canonical center, Lemma 2.30 implies that the coregularity of the log pair induced on the general fiber of $S_1 \to Z$ is bounded by c. Thus, we can use the global ACC to control the coefficient t multiplying the divisor Γ.

Now, we proceed to sketch the proof of the global ACC. Consider a sequence of log Calabi–Yau pairs (X_i, B_i) as in the statement of Theorem 2. However, for each one there is a prime component $D_i \subset X_i$, so that $\mathrm{coeff}_{D_i}(B_i)$ violates the ascending chain condition, i.e., it is strictly increasing. If the sequence of varieties X_i has bounded dimension, then the finiteness statement follows from the global ACC proved in [19]. The aim is to replace (X_i, B_i) with a dlt modification and perform adjunction to a component of $\lfloor B_i \rfloor$ that intersects D_i. By Lemma 2.27, as far as $\dim X_i > c$, the divisor $\lfloor B_i \rfloor$ will be non-empty. A priori, it may happen that $\lfloor B_i \rfloor \cap D_i = \emptyset$. However, we can always run a minimal model program for the pair $(X_i, B_i - \varepsilon_i S_i)$ for some prime component $S_i \subset \lfloor B_i \rfloor$ and ε_i small enough. We argue that, after finitely many steps of this MMP, we are in one of the following scenarios:

- the strict transform of D_i will intersect the strict transform of S_i, or
- we obtain a $\mathbb{P}^1$-linking structure and both the strict transform of D_i and S_i are multi-sections for this bundle.

In the former case, we will perform adjunction to S_i and replace (X_i, B_i) with the obtained pair. In the latter case, we will perform adjunction to a general fiber of the $\mathbb{P}^1$-linking structure. When performing adjunction, we may replace X_i with a variety of one dimension lower. By performing this process for each pair (X_i, B_i) in the sequence, we obtain a new sequence (Z_i, B_{Z_i}) of bounded dimension with coefficients on a DCC set $D(I)$. In Lemma 3.2, we will prove that the sequence $\mathrm{coeff}_{D_i}(B_i)$ violating the ACC induces a sequence of prime divisors $D_{Z_i} \subset Z_i$ so that $\mathrm{coeff}_{D_{Z_i}}(B_{Z_i})$ also violates the ascending chain condition. This leads to a contradiction of the usual global ascending chain condition [19].

2. Preliminaries

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. Unless otherwise stated, our varieties are normal and quasi-projective over the base field $\mathbb{K}$. In this section, we introduce some preliminary results regarding generalized pairs, singularities, adjunction, coregularity, and the global ascending chain condition.

2.1. Generalized pairs and singularities

In this subsection, we recall the basics about birational nef divisors, generalized pairs and their singularities.

Definition 2.1. For a normal quasi-projective variety X, consider the set of all birational projective morphisms $\phi : X_\phi \to X$, with X_ϕ normal. A *b-divisor* on X is an element of the inverse limit:
$$\mathrm{Div}_b(X) = \varprojlim_{\phi} \mathrm{Div}(X_\phi).$$

Therefore a b-divisor $\mathbf{M}$ can be written as a countable sum $\sum_i b_i V_i$, where each V_i is a divisorial valuation of $\mathbb{K}(X)$ such that any normal variety Y birational to X has only finitely many V_i's realized as divisors in Y.

The *trace* of $\mathbf{M}$ on a variety Y birational to X is defined as:
$$\mathbf{M}_Y := \sum_i b_i D_i,$$

where i runs over all valuations V_i with $c_Y(V_i) = D_i$ and $\mathrm{codim}_Y D_i = 1$.

A b-divisor on X is *b-$\mathbb{R}$-Cartier* if there exists a birational model X' of X and an $\mathbb{R}$-Cartier divisor $D_{X'}$, such that for any birational projective morphism $\phi : Y \to X'$, we have that $\mathbf{M}_Y = \phi^* D_{X'}$. The b-divisor $\mathbf{M}$ is a *b-nef* divisor if $D_{X'}$ is nef. In the previous context, we say that X' is a model where $\mathbf{M}$ descends and we write $\mathbf{M}'$ for the trace of $\mathbf{M}$ on X'. Note that, a model where $\mathbf{M}$ descend can always be replaced by higher models by blowing-up.

Let $\psi \colon X' \to X$ be a model where $\mathbf{M}$ descends. Assume that $\mathbf{M}$ is $\mathbb{R}$-Cartier on X. Assume that $\mathbf{M}'$ is a nef $\mathbb{R}$-Cartier divisor. By the negativity lemma, we can write $\psi^*\mathbf{M} = \mathbf{M}' + E$, where E is an effective exceptional $\mathbb{R}$-divisor. We say that $X \setminus \psi(E)$ is the locus on X where $\mathbf{M}$ *descends* and we say that $\psi(E)$ is the locus of X where $\mathbf{M}$ *does not descend*.

Definition 2.2. A *generalized pair* $(X, B, \mathbf{M})$ is a triple, consisting of:

- a normal quasi-projective variety X;

- an effective divisor B; and

- a b-nef divisor $\mathbf{M}$;

such that $K_X + B + \mathbf{M}_X$ is $\mathbb{R}$-Cartier. We call B and $\mathbf{M}$ the boundary and moduli parts, respectively.

For a generalized pair $(X, B, \mathbf{M})$ and a projective morphism $\pi : Y \to X$, we can write:

$$\pi^*(K_X + B + \mathbf{M}_X) = K_Y + B_Y + \mathbf{M}_Y.$$

We will call $(Y, B_Y, \mathbf{M}_Y)$ the *log pull-back* of $(X, B, \mathbf{M})$ to Y. Note that B_Y is uniquely determined by the previous equality.

Definition 2.3. Let $(X, B, \mathbf{M})$ be a generalized pair and $\pi : Y \to X$ a birational projective morphism. As above, we can write

$$K_Y + B_Y + \mathbf{M}_Y = \pi^*(K_X + B + \mathbf{M}).$$

For a prime divisor E on Y, we define the *generalized log discrepancy* of $(X, B, \mathbf{M})$ at E to be:

$$a_E(X, B, \mathbf{M}) = 1 - \mathrm{coeff}_E(B_Y).$$

We say that $(X, B, \mathbf{M})$ has *generalized log canonical singularities* (resp. *generalized Kawamata log terminal singularities*) if $a_E(X, B\mathbf{M}) \geq 0$ (resp. $a_E(X, B, \mathbf{M}) > 0$) for any prime divisor E over X. We write glc (resp. gklt) for short.

Definition 2.4. A germ $(X; x)$ is said to be of *generalized klt type* (resp. *generalized lc type*) if there exists a boundary B and a b-nef divisor $\mathbf{M}$ on X, such that $(X, B, \mathbf{M})$ is generalized klt (resp. generalized lc).

Definition 2.5. A *generalized log canonical place* (resp. *generalized non-klt place*) of a generalized pair $(X, B, \mathbf{M})$ is a prime divisor E over X, for which $a_E(X, B, \mathbf{M}) = 0$ (resp. $a_E(X, B, \mathbf{M}) \leq 0$).

A *generalized log canonical center* (resp. *generalized non-klt center*) of a generalized pair $(X, B, \mathbf{M})$ is the center on X of a generalized log canonical place (resp. generalized non-klt place). We abbreviate the set of generalized lc centers of $(X, B, \mathbf{M})$ as $\mathrm{lcc}(X, B, \mathbf{M})$.

A generalized log canonical center that is minimal with respect to inclusion among generalized log canonical centers is said to be a *minimal generalized log canonical center*.

Definition 2.6. A generalized log canonical pair $(X, B, \mathbf{M})$ is *generalized divisorially log terminal*, or *gdlt* for short, if there exists an open set $U \subset X$, such that:

- The coefficients of B are less or equal than one;
- the open set U is smooth and $\lfloor B \rfloor|_U$ has simple normal crossing support;
- all generalized non-klt centers of $(X, B, \mathbf{M})$ intersect U and consist of strata of $\lfloor B \rfloor$; and
- the locus on X where $\mathbf{M}$ descends contains the generic points of each strata of $\lfloor B \rfloor$.

Definition 2.7. Let $(X, B, \mathbf{M})$ be a generalized pair and $\phi : Y \to X$ be a log resolution of (X, B). Assume that $\mathbf{M}$ descends on Y. Let S be the normalization of a component of $\lfloor B \rfloor$ and S_Y its birational transform in Y. Let $(Y, B_Y, \mathbf{M}_Y)$ be the log pull-back of $(X, B, \mathbf{M})$ and let:

$$K_{S_Y} + B_{S_Y} + \mathbf{M}_{S_Y} = (K_Y + B_Y + \mathbf{M}_Y)|_{S_Y},$$

with $B_{S_Y} = (B_Y - S_Y)|_{S_Y}$ and $\mathbf{M}_{S_Y} = \mathbf{M}_Y|_{S_Y}$.

Let f be the induced projective birational morphism from S_Y to S, $B_S = f_* B_{S_Y}$, and $\mathbf{M}_S = f_* \mathbf{M}_{S_Y}$. Then the following equality is called *generalized adjunction*:

$$K_S + B_S + \mathbf{M}_S = (K_X + B + \mathbf{M}_X)|_S.$$

Definition 2.8. Let $(X, B, \mathbf{M})$ be a generalized log canonical pair, let $\Gamma \geq 0$ be an effective divisor, and let $\mathbf{N}$ be a b-nef divisor, such that $\Gamma + \mathbf{N}_X$ is $\mathbb{R}$-Cartier. Then, the *generalized log canonical threshold* of $(\Gamma, \mathbf{N})$ with respect to $(X, B, \mathbf{M})$ is:

$$\mathrm{lct}(X, B, \mathbf{M}; \Gamma, \mathbf{N}) = \sup \left\{ t \in \mathbb{R} \mid (X, B + t\Gamma, \mathbf{M} + t\mathbf{N}) \text{ is generalized log canonical} \right\}.$$

In the previous generalized pair, $B + t\Gamma$ is the boundary divisor, and $\mathbf{M} + t\mathbf{N}$ is the moduli divisor. We define $\mathrm{LCT}_c(I, J)$ to be the set of all $\mathrm{lct}(X, B, \mathbf{M}; \Gamma, \mathbf{N})$, such that the following conditions are satisfied:

(1) the coefficients of B are in I;
(2) the coefficients of Γ are in J;
(3) we can write $\mathbf{M}' = \sum_j \lambda_j M_j'$ where $\lambda_j \in I$ and each M_j' is nef Cartier;
(4) we can write $\mathbf{N}' = \sum_j \mu_j N_j'$ where $\mu_j \in J$ and each N_j' is nef Cartier; and
(5) the coregularity (see Definition 2.22) of $(X, B + t\Gamma, \mathbf{M} + t\mathbf{N})$ is at most c.

The previous set is called the set of log canonical thresholds with indices in I and J and coregularity bounded above by c.

2.2. Generalized dlt modifications

In this subsection, we recall the existence of generalized dlt modifications and results regarding the coefficients obtained after adjunction. Some of the results in this subsection are analogous to those in [8, 19]. In some cases, we give a short proof for the sake of completeness.

Definition 2.9. Let $(X, B, \mathbf{M})$ be a generalized pair, then a generalized pair $(Y, B_Y, \mathbf{M}_Y)$ is a *generalized dlt modification* of $(X, B, \mathbf{M})$ if $(Y, B_Y, \mathbf{M}_Y)$ is a $\mathbb{Q}$-factorial gdlt pair and there exists a projective birational morphism $\phi : Y \to X$ satisfying the following conditions:

(1) the exceptional locus $\mathrm{Ex}(\phi)$ is purely divisorial;

(2) all the divisors E on Y exceptional over X satisfy $a_E(X, B, \mathbf{M}) = 0$; and

(3) we have that $K_Y + B_Y + \mathbf{M}_Y = \phi^*(K_X + B + \mathbf{M}_X)$.

The following lemma is well known to the experts (see, e.g. [17, Theorem 2.9]).

Lemma 2.10. *Let $(X, B, \mathbf{M})$ be a generalized log canonical pair. Then there exists a generalized dlt modification $(Y, B_Y, \mathbf{M}_Y)$ of $(X, B, \mathbf{M})$.*

Definition 2.11. For a set $I \subset \mathbb{R}$, we define:

$$I^+ = \{0\} \cup \left\{ i \in [0, 1] \mid i = \sum_{k=1}^{\ell} j_k, \text{ for some } j_1, \ldots, j_\ell \in I \right\},$$

and

$$D(I) = \left\{ a \leq 1 \mid a = \frac{m - 1 + f}{m}, \; m \in \mathbb{Z}_{>0}, \; f \in I^+ \right\}.$$

The following lemmas are straightforward from the previous definitions, see for example [38, Lemma 4.4].

Lemma 2.12. *Let I be a set of nonnegative real numbers satisfying the DCC, then the sets $D(I)$ and I^+ satisfy the DCC.*

Lemma 2.13. *Let I be a set of nonnegative real numbers, then we have that:*

$$D\big(D(I)\big) = D(I) \cup \{1\}.$$

Definition 2.14. Let I be a set of nonnegative real numbers. Let d be a real number in the interval $[0, 1]$. We define:

$$D_d(I) = \left\{ a \leq 1 \mid a = \frac{m - 1 + f + kd}{m}, \; k, m \in \mathbb{Z}_{>0}, \; f \in I^+ \right\}.$$

Lemma 2.15. *Let I be a set of nonnegative real numbers. Let $d_1 \in D_d(I)$, then we have that $D_{d_1}(I) \subseteq D_d(I)$.*

Proof. Any element in $D_{d_1}(I)$ is of the form:

$$a = \frac{m - 1 + f + k_1 d_1}{m},$$

where $m, k_1 \in \mathbb{Z}_{>0}$ and $f \in I^+$. Similarly d_1 is of the form:

$$d = \frac{n - 1 + g + kd}{n},$$

where $n, k \in \mathbb{Z}_{>0}$ and $g \in I^+$. Thus, we can write

$$
\begin{aligned}
a &= \frac{m - 1 + f + k_1(\frac{n-1+g+kd}{n})}{m} \\
&= \frac{mn - n + fn + k_1 n - k_1 + k_1 g + k_1 kd}{mn} \\
&= \frac{mn - 1 + (k_1 - 1)(n - 1) + fn + k_1 g + k_1 kd}{mn},
\end{aligned}
$$

which is in $D_d(I)$. ∎

The following lemma is [19, Lemma 5.2].

Lemma 2.16. *Let J be a finite set and I a set of nonnegative real numbers satisfying the DCC. Then, the set*

$$
I_1 = \left\{ i \in I \mid \frac{m - 1 + ki + f}{m} \in J \cap [0, 1] \text{ for some } m, k \in \mathbb{Z}_{>0}, \ f \in I^+ \right\}
$$

is also finite.

Proof. We can assume $i \neq 0$. We have that $ki \leq 1$, so no element of $i \in I_1$ can be greater than one, hence we can assume $I \subset [0, 1]$. As I satisfies the DCC, we have that i is bounded below, therefore as $ik \leq 1$, we have that k is bounded above, so it can only take finitely many values.

Call $\ell = \frac{m-1+ki+f}{m}$. If $\ell = 1$, then we can assume $m = 1$, otherwise we have that ℓ is bounded above by a number smaller than one, since J is finite. Therefore $1 - \frac{1}{m}$ is bounded above, and m can only take finitely many values.

For any fixed m, ℓ, k, the set:

$$
I_{m,\ell,k} := \left\{ i \mid i = \frac{m\ell - m + 1 - f}{k}, \ f \in I^+ \right\}
$$

is a subset of I. Hence $I_{m,\ell,k}$ satisfies the DCC. The set $I_{m,\ell,k}$ also satisfies the ACC as the only variable $f \in I$ satisfies the DCC. Therefore, $I_{m,\ell,k}$ is a finite set. As m, ℓ, k can only take finitely many possible values, the set I_1 is also finite. ∎

The following lemma follows from the proof of [8, Proposition 4.9]. It allows us to control the coefficients of the boundary part and the moduli part under adjunction to a divisorial log canonical center.

Lemma 2.17. *Let I be a set of nonnegative real numbers satisfying the DCC. Let $(X, B, \mathbf{M})$ be a generalized dlt pair, with coefficients in I. Let d be a positive real number for which either:*

- *there exists a prime component D on X with $\operatorname{coeff}_D(B) = d$, or*
- *we can write $\mathbf{M}' = \sum_j \lambda_j M_j'$ with λ_j nonnegative real numbers, each $M_j' \not\equiv 0$ is nef Cartier, and $\lambda_j = d$ for some j.*

Let S be the normalization of a component of $\lfloor B \rfloor$ with either $S \cap D \neq \emptyset$ or $S \cap M_j \neq \emptyset$ with $\lambda_j = d$. Let B_S and $\mathbf{M}_S$ be the boundary and moduli part defined by the generalized adjunction. Then, we have that either:

(i)   *the coefficients of B_S belong to $D(I)$,*

(ii)   *for any prime component Q of $D \cap S$ we have that $\mathrm{coeff}_Q(B_S) \in D_d(I)$, or*

(iii)   *we can write $\mathbf{M}'_S = \sum_j \eta_j M'_{S,j}$ with η_j nonnegative real numbers, each $M'_{S,j} \not\equiv 0$ is nef Cartier, and $\eta_j = d$ for some j.*

Proof. Let $(S, B_S, \mathbf{M}_S)$ be the pair obtained by generalized adjunction of $(X, B, \mathbf{M})$ to S (see, e.g., [8, Definition 4.7]). Condition (iii) follows from the definition of generalized adjunction. Hence, we need to prove the first and second statement. Since this statement only depends on codimension one points of S, we may assume that X has dimension 2 and it is $\mathbb{Q}$-factorial. By [51, Corollary 3.10], we can write

$$K_S + B'_S = (K_X + B)|_S,$$

where the coefficients of B'_S belong to $D(I)$. More precisely, if Q is a prime divisor of S, we can write

$$\mathrm{coeff}_Q(B'_S) = 1 - \frac{1}{m} + \frac{\sum_i m_i \lambda_i}{m}, \tag{2.1}$$

where the m is the orbifold index of X at Q, the λ_i's belong to I, and m_i's are positive integers. In the previous formula, λ_i is the coefficient in B of a prime divisor P on X and m_i is the multiplicity of P at Q. Let M_j be the push-forward of M'_j to X. Then, we have that $m\mathbf{M}_X = \sum_j \lambda_j m M_j$. Each mM_j is Cartier on a neighborhood of Q. Let $\pi \colon X' \to X$ be the projective birational morphism. By the negativity lemma, we can write $\pi^*(m\mathbf{M}_X) = \sum_j \lambda_j m M'_j + \lambda_j E_j$ where each E_j is Cartier. Hence, we have that

$$\mathrm{coeff}_Q(B_S) = 1 - \frac{1}{m} + \frac{\sum_i m_i \lambda_i + \sum_j \lambda_j \mathrm{mult}_Q(E_j)}{m}.$$

This finishes the first statement.

Finally, if there is a prime component D of X for which $\mathrm{coeff}_D(B) = d$, then for any prime Q of $D \cap S$ we have $\mathrm{coeff}_Q(B'_S) \in D_d(I)$ by (2.1). Thus, we have that $\mathrm{coeff}_Q(B_S) \in D_d(I)$ as well. This finishes the proof of the second statement. ∎

The following lemma allows to keep track of a prime divisor D when passing to a dlt modification.

Lemma 2.18. *Let $(X, B, \mathbf{M})$ be a generalized log canonical pair. Let D be a prime divisor on X which satisfies:*

$$\mathrm{Supp}(D) \cap \mathrm{lcc}(X, B, \mathbf{M}) \neq \emptyset.$$

Let $(Y, B_Y, \mathbf{M}_Y)$ be a generalized dlt modification of $(X, B, \mathbf{M})$. Let D_Y be the strict transform of D. Then, we have that

$$D_Y \cap \lfloor B_Y \rfloor \neq \emptyset.$$

In particular, there exists a prime component $S \subset \lfloor B_Y \rfloor$ with $D_Y \cap S \neq \emptyset$.

Proof. Let $\phi : Y \to X$ be the dlt modification. Let $p \in D \cap \mathrm{lcc}(X, B, \mathbf{M})$. By definition we have that $D_Y \cap \phi^{-1}(p) \neq \emptyset$. If $\dim(\phi^{-1}(p)) \geq 1$, we have that $\phi^{-1}(p)$ is contained in an exceptional divisor, therefore $\phi^{-1}(p) \subset \lfloor B_Y \rfloor$, as ϕ is a dlt modification. Hence $\emptyset \neq D_Y \cap \phi^{-1}(p) \subset D_Y \cap \lfloor B_Y \rfloor$. Otherwise $\phi^{-1}(p)$ is a point, then $\phi^{-1}(p) \in \mathrm{lcc}(Y, B_Y, \mathbf{M}_Y) = \lfloor B_Y \rfloor$, hence $\phi^{-1}(p) \in D_Y \cap \lfloor B_Y \rfloor$. ∎

2.3. Regularity and coregularity

In this subsection, we introduce the regularity and coregularity. We prove some lemmas regarding the coregularity.

Definition 2.19. Let $(X, B, \mathbf{M})$ be a generalized log canonical pair. A *minimal dlt center* of $(X, B, \mathbf{M})$ is a minimal log canonical center of any generalized dlt modification of $(X, B, \mathbf{M})$.

Definition 2.20. Let $(X, B, \mathbf{M})$ be a generalized pair. Let $\pi : Y \to X$ be a generalized dlt modification such that

$$K_Y + B_Y + \mathbf{M}_Y = \pi^*(K_X + B + \mathbf{M}_X),$$

where $\mathbf{M}_Y$ is the pushforward of $\mathbf{M}'$ on Y. Write

$$(B_Y)^{=1} = E_1 + E_2 + \cdots + E_r,$$

which is a simple normal crossing divisor on Y.

The *dual complex* $\mathcal{D}(Y, B_Y, \mathbf{M}_Y)$ of the dlt modification is a simplicial complex constructed as follows:

- Every divisor E_i corresponds to a vertex v_i in $\mathcal{D}(Y, B_Y, \mathbf{M}_Y)$. For every subset $I \subseteq \{1, 2, \ldots, r\}$ and every irreducible component Z of $\bigcap_{i \in I} E_i$, Z corresponds to a simplex v_Z of dimension $\#I - 1$.

- For every subset $I \subseteq \{1, 2, \ldots, r\}$ and every $j \in I$, we have the following gluing maps. Let $Z \subseteq \bigcap_{i \in I} E_i$ be any irreducible component, then Z lies in a unique component W of $\bigcap_{i \in I \setminus \{j\}} E_i$. We have a gluing map which is the inclusion of v_W into v_Z, as the face of v_Z which do not contain vertex v_i.

The dimension of $\mathcal{D}(Y, B_Y, \mathbf{M}_Y)$ is defined to be the smallest dimension of the maximal simplex, with respect to the inclusion, of $\mathcal{D}(Y, B_Y, \mathbf{M}_Y)$.

We define the *dual complex* $\mathcal{D}(X, B, \mathbf{M})$ associated to $(Y, B_Y, \mathbf{M}_Y)$ as $\mathcal{D}(Y, B_Y, \mathbf{M}_Y)$ and we define

$$\dim \mathcal{D}(X, B, \mathbf{M}) = \dim \mathcal{D}(Y, B_Y, \mathbf{M}_Y).$$

If $Z \subset X$ is an irreducible subvariety with generic point η_Z, we define $\mathcal{D}(X, B, \mathbf{M}; \eta_Z)$ to be $\mathcal{D}(U, B|_U, \mathbf{M}|_U)$ for a sufficiently small neighborhood U of η_Z.

Proposition 2.21. *Let $(X, B, \mathbf{M})$ be a generalized log canonical pair. Then $\dim \mathcal{D}(X, B, \mathbf{M})$ does not depend on the choice of generalized dlt modifications $(Y, B_Y, \mathbf{M}_Y)$ of $(X, B, \mathbf{M})$.*

Proof. Let $\pi_i : (Y_i, B_{Y_i}, \mathbf{M}_{Y_i}) \to (X, B, \mathbf{M})$ be generalized dlt modifications, for $i = 1, 2$. Suppose that $\dim \mathcal{D}(Y_1, B_{Y_1}, \mathbf{M}_{Y_1}) = r$. Then there exists $(r + 1)$ π_1-exceptional divisors whose intersection is non-empty. Let W be an irreducible component of this intersection and $Z = \pi_1(W)$. Then Z is a minimal generalized log canonical center of $(X, B, \mathbf{M})$. So $\dim \mathcal{D}(X, B, \mathbf{M}; \eta_Z) = r$. By [17, Theorems 1.1 and 1.6], the dimension of $\mathcal{D}(X, B, \mathbf{M}; \eta_Z)$ is independent of the choice of the dlt modification. Hence, $\dim \mathcal{D}(Y_2, B_{Y_2}, \mathbf{M}_{Y_2}) \leq r$. Similarly, we can prove that $\dim \mathcal{D}(Y_2, B_{Y_2}, \mathbf{M}_{Y_2}) \geq \dim \mathcal{D}(Y_1, B_{Y_1}, \mathbf{M}_{Y_1})$ and hence the equality holds. ∎

Definition 2.22. Let $(X, B, \mathbf{M})$ be a quasi-projective normal generalized pair. We define the *regularity* of $(X, B, \mathbf{M})$ to be

$$\operatorname{reg}(X, B, \mathbf{M}) := \dim \mathcal{D}(X, B, \mathbf{M}).$$

On the other hand, we define the *coregularity* of $(X, B, \mathbf{M})$ to be

$$\operatorname{coreg}(X, B, \mathbf{M}) := \dim X - \dim \mathcal{D}(X, B, \mathbf{M}) - 1.$$

Hence, by definition, we have that

$$\operatorname{reg}(X, B, \mathbf{M}) + \operatorname{coreg}(X, B, \mathbf{M}) = \dim X - 1.$$

In a few words, a generalized log canonical pair $(X, B, \mathbf{M})$ has coregularity c if for every minimal log canonical center of $(X, B, \mathbf{M})$, we may find a minimal dlt center of dimension at most c mapping onto it.

Analogously, for $Z \subset X$ an irreducible subvariety, we define

$$\operatorname{reg}(X, B, \mathbf{M}; \eta_Z) = \operatorname{reg}\big(U, B|_U, \mathbf{M}|_U\big)$$

for a sufficiently small neighborhood U of η_Z.

Now, we turn to define the absolute regularity and absolute coregularity of klt singularities and Fano type varieties. This definition is not crucial for this paper, as in the log canonical and log Calabi–Yau case they agree with the usual regularity and coregularity (see Remark 2.26). However, it is worth mentioning as it will be considered in forthcoming research.

Definition 2.23. Let $(X, B, \mathbf{M})$ be a projective generalized pair. We say that $(X, B, \mathbf{M})$ is *generalized log Calabi–Yau* if the following conditions are satisfied:

- $(X, B, \mathbf{M})$ has generalized log canonical singularities, and
- $K_X + B + \mathbf{M}_X \sim_{\mathbb{Q}} 0$.

We say that a generalized pair $(X, B, \mathbf{M})$ is of *log Calabi–Yau type* if there exists $\Gamma \geq 0$ for which $(X, B + \Gamma, \mathbf{M})$ is a generalized log Calabi–Yau pair. In this case, to abbreviate, we may say that $(X, B, \mathbf{M})$ is CY type. For a generalized log Calabi–Yau projective variety $(X, B, \mathbf{M})$, two minimal dlt center are birationally equivalent.

Definition 2.24. Let $(X, B, \mathbf{M})$ be a projective generalized pair. We define the *absolute regularity* of $(X, B, \mathbf{M})$ to be

$$\widehat{\mathrm{reg}}(X, B, \mathbf{M})$$
$$:= \max \left\{ \dim \mathcal{D}(X, B + \Gamma, \mathbf{M}) \mid (X, B + \Gamma, \mathbf{M}) \text{ is generalized log Calabi–Yau} \right\}.$$

If the dual complex $\mathcal{D}(X, B + \Gamma, \mathbf{M})$ is empty, then we set $\dim \mathcal{D}(X, B + \Gamma, \mathbf{M}) = -1$. If the previous set is empty, then we set $\mathrm{reg}(X, B, \mathbf{M}) = -\infty$. Hence, a generalized pair has finite absolute regularity if and only if it is of generalized log Calabi–Yau type. A generalized pair has absolute regularity -1 if and only if every log Calabi–Yau structure on it has klt singularities, i.e., absolute regularity -1 is equivalent to being exceptional. The absolute regularity is nonnegative if and only if we can find a log Calabi–Yau structure with a log canonical center. If X is n-dimensional, we have that

$$\widehat{\mathrm{reg}}(X, B, \mathbf{M}) \in \{-\infty, -1, 0, \dots, n-1\}.$$

We define the *absolute coregularity* of $(X, B, \mathbf{M})$ to be

$$\widehat{\mathrm{coreg}}(X, B, \mathbf{M}) := \dim X - \widehat{\mathrm{reg}}(X, B, \mathbf{M}) - 1.$$

Analogously, a generalized pair has finite absolute coregularity if and only if it is of generalized log Calabi–Yau type. A generalized pair $(X, B, \mathbf{M})$ has absolute coregularity $\dim X$ if and only if every log Calabi–Yau $(X, B + \Gamma, \mathbf{M})$ structure has klt singularities, i.e., the generalized pair $(X, B, \mathbf{M})$ is exceptional. The absolute coregularity c is in the interval $\{0, \dots, n-1\}$ if and only if there exists a log Calabi–Yau structure $(X, B + \Gamma, \mathbf{M})$ which admits a log canonical center. In summary, for a projective variety X of dimension n and a generalized pair structure $(X, B, \mathbf{M})$, we have that

$$\widehat{\mathrm{coreg}}(X, B, \mathbf{M}) \in \{0, 1, 2, \dots, n, \infty\}.$$

If the absolute coregularity of $(X, B, \mathbf{M})$ is in the set $\{0, 1, 2, \dots, n\}$ then the absolute coregularity equals the dimension of the smallest minimal dlt center on a log Calabi–Yau structure of $(X, B, \mathbf{M})$. If the log Calabi–Yau structure $(X, B, \mathbf{M})$ is itself klt, then we can set X formally as a log canonical center.

Definition 2.25. Let $(X, B, \mathbf{M}; x)$ be a generalized log canonical singularity. We define the *absolute regularity* of $(X, B, \mathbf{M})$ at x to be

$$\widehat{\mathrm{reg}}(X, B, \mathbf{M}; x)$$
$$:= \max \left\{ \dim \mathcal{D}(X, B + \Gamma, \mathbf{M}; x) \mid (X, B + \Gamma, \mathbf{M}; x) \text{ is generalized log canonical} \right\}.$$

We define the absolute coregularity of the klt to be

$$\widehat{\mathrm{coreg}}(X, B, \mathbf{M}; x) := \dim X - \widehat{\mathrm{reg}}(X, B, \mathbf{M}; x) - 1.$$

The following remark says that in our setting, the coregularity and absolute coregularity agree.

Remark 2.26. If $(X, B, \mathbf{M})$ is a generalized log Calabi–Yau pair, then

$$\operatorname{reg}(X, B, \mathbf{M}) = \widehat{\operatorname{reg}}(X, B, \mathbf{M}) \quad \text{and} \quad \operatorname{coreg}(X, B, \mathbf{M}) = \widehat{\operatorname{coreg}}(X, B, \mathbf{M}).$$

Analogously, if $(X, B, \mathbf{M}; x)$ is a generalized log canonical singularity for which x is a log canonical center, then

$$\operatorname{reg}(X, B, \mathbf{M}; x) = \widehat{\operatorname{reg}}(X, B, \mathbf{M}; x) \quad \text{and} \quad \operatorname{coreg}(X, B, \mathbf{M}; x) = \widehat{\operatorname{coreg}}(X, B, \mathbf{M}; x).$$

The following lemma, which is straightforward from the definition, allows us to perform adjunction whenever the dimension is larger than the coregularity.

Lemma 2.27. *Let* $(X, B, \mathbf{M})$ *be a generalized log canonical pair. If* $\dim X > \operatorname{coreg}(X, B, \mathbf{M})$, *then* $(X, B, \mathbf{M})$ *has a non-trivial generalized log canonical center. In particular, if* $(Y, B_Y, \mathbf{M}_Y)$ *is a generalized dlt modification of* $(X, B, \mathbf{M})$, *then* $\lfloor B_Y \rfloor \neq \emptyset$.

Now, we turn to prove some preliminaries regarding how the coregularity behaves under adjunction, perturbing the boundary, and taking fibrations. The proofs are well known to the experts. In some cases, we give a short argument. The following lemma follows from [17, Theorem 1.6].

Lemma 2.28. *Let* $(X, B, \mathbf{M})$ *be a generalized log Calabi–Yau pair. Let* S *be the normalization of a component of* $\lfloor B \rfloor$. *Let* B_S *and* $\mathbf{M}_S$ *be the boundary and moduli part defined by generalized adjunction, so that*

$$(K_X + B + \mathbf{M}_X)|_S = K_S + B_S + \mathbf{M}_S.$$

Then, we have that $\operatorname{coreg}(S, B_S, \mathbf{M}_S) = \operatorname{coreg}(X, B, \mathbf{M})$.

Lemma 2.29. *Let* $(X, B, \mathbf{M})$ *be a generalized log canonical pair and* V *be a minimal generalized log canonical center of* $(X, B, \mathbf{M})$. *Let* $B_0 \leq B$ *be the sum of all the components of* B *containing* V. *Write* $\mathbf{M}' = \sum_j M_j'$ *where each* M_j' *is a nef Cartier divisor. Let* $\mathbf{M}_0'$ *be the sum of the* M_j' *that do not descend over the generic point of* V *and let* $\mathbf{M}_0$ *the push-forward of* $b\mathbf{M}0.'$ *to* X. *Then, we have that*

$$\operatorname{coreg}(X, B, \mathbf{M}; \eta_V) = \operatorname{coreg}(X, B_0, \mathbf{M}_0; \eta_V).$$

Proof. Since the statement is about coregularity at the generic point of V, we may just localize at the generic point of V and assume that $B_0 = B$ and $\mathbf{M}_0 = \mathbf{M}$. ∎

Lemma 2.30. *Let* $(X, B, \mathbf{M})$ *be a generalized log canonical pair and* $X \to Z$ *be a fibration. Assume that every generalized non-klt center of* $(X, B, \mathbf{M})$ *dominates* Z. *Let* F *be a general fiber of* $X \to Z$ *and* $(F, B_F, \mathbf{M}_F)$ *be the generalized pair induced by adjunction. Then, we have that*

$$\operatorname{coreg}(F, B_F, \mathbf{M}_F) \leq \operatorname{coreg}(X, B, \mathbf{M}) - \dim Z.$$

Proof. We may assume that $(X, B, \mathbf{M})$ is a generalized dlt pair. Let d be the dimension of X. Since $(X, B, \mathbf{M})$ has coregularity c, we can find $d - c$ components $S_1, \ldots, S_{d-c}$ of $\lfloor B \rfloor$ so that

$$S_1 \cap \cdots \cap S_{d-c} \neq \emptyset.$$

Note that $S_{F_i} := S_i \cap F$ is a component with coefficient one in B_F. Then, we conclude that

$$S_{F,1} \cap \cdots \cap S_{F,d-c} \neq \emptyset,$$

as divisors on F. Thus, we have that

$$\mathrm{coreg}(F, B_F, \mathbf{M}_F) \leq \mathrm{coreg}(X, B, \mathbf{M}) - \dim Z$$

as claimed. ∎

We say that a birational map $\psi : X \dashrightarrow Y$ is a birational contraction if ψ is proper and ψ^{-1} does not contract any divisors.

The following lemma states that the coregularity of a generalized log Calabi–Yau pair does not change when we run a minimal model program.

Lemma 2.31. *Let $(X, B, \mathbf{M})$ be a generalized log Calabi–Yau pair. Let $X \dashrightarrow X'$ be a birational contraction. Denote by B' the push-forward of B on X' and by $\mathbf{M}'$ the trace on X' of the b-nef divisor associated to $\mathbf{M}$. Then, we have that*

$$\mathrm{coreg}(X, B, \mathbf{M}) = \mathrm{coreg}(X', B', \mathbf{M}').$$

Proof. Since $(X, B, \mathbf{M})$ is generalized log Calabi–Yau, we can find a common log resolution $p \colon Y \to X$ and $q \colon Y \to X'$ for which

$$K_Y + B_Y + \mathbf{M}_Y = p^*(K_X + B + \mathbf{M}_X) = q^*(K_{X'} + B' + \mathbf{M}'_X).$$

By definition, we have that $\mathcal{D}(X, B, \mathbf{M}) = \mathcal{D}(Y, B_Y^{=1})$, where $B_Y^{=1}$ denotes the sum of the components of B_Y which appear with coefficient one in B_Y. The same applies to $(X', B', \mathbf{M}')$. Then, we conclude that $\dim \mathcal{D}(X', B', \mathbf{M}') = \dim \mathcal{D}(X, B, \mathbf{M})$. This concludes the proof. ∎

2.4. Global ascending chain condition

In this subsection, we recall the ascending chain condition for generalized pairs. We will use the global ascending chain condition for varieties of fixed dimension as the base for an inductive process in the following section. The following result is [8, Theorem 1.6].

Proposition 2.32. *Let n be a positive integer. Let I be a set of nonnegative real numbers satisfying the descending chain condition. Then there is a finite subset $I_0 \subseteq I$ such that if $(X, B, \mathbf{M})$ satisfies the following conditions:*

(1) *the variety X is a projective variety of dimension n;*

(2) *the generalized pair $(X, B, \mathbf{M})$ is generalized log canonical;*

(3) *the coefficients of B belong to I;*

(4) *we can write* $\mathbf{M}' = \sum_j \mu_j M_j'$ *where* $\mu_j \in I$ *and each* M_j' *is a nef Cartier divisor;*

(5) $\mu_j = 0$ *whenever* $M_j' \equiv 0$*; and*

(6) $K_X + B + \mathbf{M}_X \equiv 0$*;*

then the coefficients of B and the μ_j *belong to* I_0*.*

We note that condition (4) is written to avoid trivial counter-examples where we add trivial summands to $\mathbf{M}'$ with arbitrary coefficients in I. Throughout the rest of the paper, we may assume that no M_j' is numerically trivial, unless otherwise stated.

3. Global ascending chain condition

In this section, we prove the main projective result of this paper. We prove the global ascending chain condition for log Calabi–Yau pairs of bounded coregularity.

Theorem 3.1. *Fix a positive integer c and a set I of nonnegative real numbers satisfying the DCC. Then there exists a finite subset* $I_0 \subseteq I$ *such that if* $(X, B, \mathbf{M})$ *satisfies the following conditions:*

(1) *the generalized pair* $(X, B, \mathbf{M})$ *is generalized log canonical;*

(2) *the coefficients of B belong to I;*

(3) *we can write* $\mathbf{M}' = \sum_j \mu_j M_j'$ *where* $\mu_j \in I$ *and each* M_j' *is a nef Cartier divisor;*

(4) *the numerical equivalence* $K_X + B + \mathbf{M}_X \equiv 0$ *holds; and*

(5) *the variety X is projective with an upper bound* $\mathrm{coreg}(X, B, \mathbf{M}) \leq c$*;*

then $\mathrm{coeff}(\Delta) \subseteq I_0$ *and the* μ_j *belong to* I_0*.*

The following lemma allows to replace a log Calabi–Yau pair $(X, B, \mathbf{M})$ by one whose dimension is at most the coregularity of $(X, B, \mathbf{M})$. Furthermore, we are able to control the coefficients of the boundary and moduli part of the new log Calabi–Yau pair.

Lemma 3.2. *Given a generalized log Calabi–Yau pair* $(X, B, \mathbf{M})$ *satisfying the following conditions:*

(1) *The coefficients of B belong to I;*

(2) *we can write* $\mathbf{M}' = \sum_j \mu_j M_j'$ *where* $\mu_j \in I$ *and each* M_j' *is a nef Cartier divisor; and*

(3) *the variety X is projective with an upper bound* $\mathrm{coreg}(X, B, \mathbf{M}) \leq c$*.*

Let d be either one of the μ_j *or in* $\mathrm{coeff}(B)$*. Then, we can construct a generalized log Calabi–Yau pair* $(Z, B_Z, \mathbf{M}_Z)$ *of coregularity at most c satisfying the following conditions:*

(i)   *the variety Z has dimension at most* $c + 1$*;*

(ii) *the coefficients of* B_Z *are in* $D(I)$*; and*

(iii)   *we can write* $\mathbf{M}'_Z = \sum_j \eta_j M'_{Z,j}$ *where* $\eta_j \in I$ *and each* $M'_{Z,j}$ *is a nef Cartier divisor.*

Furthermore, we have that either some $d' \in \mathrm{coeff}(B_Z)$ *satisfies* $d' \in D_d(I)$ *with* $d' < 1$ *or* $\eta_j = d$ *for some* j. *If* $d \in \mathrm{coeff}(B)$, *then the former case holds.*

Proof. By Lemma 2.27, if $(X, B, \mathbf{M})$ is generalized klt, then the dimension of X is at most c, so we are done. Therefore, we can assume that $(X, B, \mathbf{M})$ is generalized lc but not generalized klt. By Lemma 2.15, we only have to prove that there exists a pair of dimension strictly smaller than $\dim X$ satisfying the conditions (i)–(iii). Inductively, by Lemma 2.15 we get the desired pair.

We start by taking a generalized dlt modification $(Y, B_Y, \mathbf{M}_Y)$ of $(X, B, \mathbf{M})$. Let S_Y be a prime component of $\lfloor B_Y \rfloor$. Without loss of generality, we assume that $d \in \mathrm{coeff}(B_Y)$. The case in which d is one of the μ_j proceeds analogously. We call D_Y a prime component of the divisor B_Y that has coefficient d. Pick $\varepsilon > 0$ to be small enough, we run a $(K_Y + B_Y - \varepsilon S_Y + \mathbf{M}_Y)$-MMP. As $K_Y + B_Y + \mathbf{M}_Y \equiv 0$, all the steps of the minimal model program are S_Y-positive. By Lemma 2.31 the coregularity of $(Y, B_Y, \mathbf{M}_Y)$ remains constant.

Claim. The divisor S_Y does not get contracted by any step of the minimal model program.

Proof. Suppose S_Y gets contracted. Without loss of generality, as all our conditions are preserved by the MMP, we may assume it is contracted by the very first step of the MMP. Then, we have a birational contraction $\phi : Y \to Y'$. As $K_Y + B_Y + \mathbf{M}_Y \equiv 0$, we have that

$$K_Y + B_Y + \mathbf{M}_Y = \phi^* \phi_*(K_Y + B_Y + \mathbf{M}_Y) = \phi^*(K_{Y'} + B_{Y'} + \mathbf{M}_{Y'}).$$

Here, as usual, $B_{Y'}$ (resp. $\mathbf{M}_{Y'}$) is the push-forward of B_Y (resp $\mathbf{M}_Y$) to Y'. Therefore $a_{S_Y}(Y', B_{Y'}, \mathbf{M}_{Y'}) = 0$. However, we have

$$\varepsilon = a_{S_Y}(Y, B_Y - \varepsilon S_Y, \mathbf{M}_Y) \leq a_{S_Y}(Y', B_{Y'}, \mathbf{M}_{Y'}),$$

as the log discrepancies only increase after a step of the MMP. A contradiction. ∎

As $K_Y + B_Y + \mathbf{M}_Y \equiv 0$ and Y is projective, we have that $K_Y + B_Y - \varepsilon S_Y + \mathbf{M}_Y$ is not pseudo-effective. By [8, Lemma 4.4], we know that this minimal model program must terminate with a Mori fiber space. We will split into different cases, depending on the steps and outcome of the minimal model program.

Case 1. The strict transform of the divisor D_Y gets contracted by a step of the MMP.

Let $\pi_1 : Y_1 \to Y_2$ be the step where the strict transform of D_Y is contracted. We denote by B_1 (resp. $\mathbf{M}_1$) the push-forward of B_Y (resp. $\mathbf{M}_Y$) to the model Y_1. Analogously, we denote by S_1 (resp. D_1) the push-forward of S_Y (resp. D_Y) on Y_1. As π_1 contracts $(K_{Y_1} + B_1 - \varepsilon S_1 + \mathbf{M}_1)$-negative curves, we have that π_1 contracts S_1-positive curves. As some contracted curve C must be contained in D_1, we have that

$$D_1 \cap S_1 \supseteq C \cap S_1 \neq \emptyset.$$

By Lemma 2.18, may replace $(Y_1, B_1, \mathbf{M}_1)$ with a dlt modification and assume that D_1 intersects a component S of $\lfloor B_1 \rfloor$. Let $(S, B_S, \mathbf{M}_S)$ be the generalized pair obtained by adjunction of $K_{Y_1} + B_1 + \mathbf{M}_1$ to S. By Lemma 2.28, $(S, B_S, \mathbf{M}_S)$ has coregularity at most c.

The following conditions are satisfied:

    (i)   the generalized pair $(S, B_S, \mathbf{M}_S)$ is generalized log Calabi–Yau,

    (ii)   the coefficients of B_S belong to $D(I)$, and

    (iii)   we can write $\mathbf{M}_{S'} = \sum_j \eta_j M_{S',j}$ where $\eta_j \in I$ and each $M_{S',j}$ is a nef Cartier divisor.

Observe that $D_1 \cap S \neq \emptyset$ and $\mathrm{coeff}_{D_1}(B_1) = d$. By Lemma 2.17, we conclude that any prime component Q of $D_1 \cap S$ will satisfy that $d' = \mathrm{coeff}_Q(B_S) \in D_d(I)$. Since we are performing generalized dlt adjunction, we have that $d' < 1$. This finishes the proof in the case that the minimal model program eventually contracts D.

Case 2. The divisor D is never contracted by the MMP and the MMP terminates with a Mori fiber space $\phi : Y_1 \to Z$ with general fiber of dimension at least 2.

Here, we have that $\rho(Y_1/Z) = 1$. Hence S_1 is ample over Z, where S_1 is the push-forward of S_Y to Y_1. We denote by B_1 (resp $\mathbf{M}_1$) the push-forward of B_Y (resp $\mathbf{M}_Y$) to Y_1. Analogously, we denote by D_1 the push-forward of D_Y to Y_1. As D_1 is effective it can either be vertical over the base or ample over the base. We argue in each of these cases.

Case 2(a). The divisor D_1 is numerically trivial over the base Z.

The divisor D_1 contains a curve C that gets contracted to a point in Z. Indeed, we can take a general point $P \in \phi(D_1)$ and a curve $C \subset \phi^{-1}(P)$. Therefore $C \cap D_1 \neq \emptyset$, but since D_1 is vertical over the base, we have that $C \cdot D_1 = 0$, hence $C \subset D_1$ is the desired curve. As S_1 is ample over the base $C \cdot S_1 > 0$, so $C \cap S_1 \neq \emptyset$, therefore $D_1 \cap S_1 \neq \emptyset$. Then, we can replace $(Y_1, B_1, \mathbf{M}_1)$ with a dlt modification, perform adjunction to a component S of $\lfloor B_1 \rfloor$, and proceed as in the first case.

Case 2(b). The divisor D_1 is ample over the base Z.

Let F be the general fiber. The divisors $D_1|_F$ and $S_1|_F$ are ample in F. As $\dim(D)|_F \geq 1$, we can find a curve $C \subset D_1|_F$. Since $S_1|_F$ is ample, we have that $C \cdot S_1|_F > 0$. Therefore $C \cap S_1 \neq \emptyset$, so $S_1 \cap D_1 \neq \emptyset$. Then, we can replace $(Y_1, B_1, \mathbf{M}_1)$ with a dlt modification, perform adjunction to a component S of $\lfloor B_1 \rfloor$, and proceed as in the first case.

Case 3. The divisor D is never contracted by the MMP and the MMP terminates with a Mori fiber space $Y_1 \to Z$ with general fiber of dimension one.

If D_1 is vertical over the base, we can proceed as in Case 2(a). Otherwise, we restrict $(Y_1, B_1, \mathbf{M}_1)$ to the general fiber and we obtain a one-dimensional generalized log CY pair $(\mathbb{P}^1, B_{\mathbb{P}^1}, \mathbf{M}_{\mathbb{P}^1})$ so that $d \in \mathrm{coeff}(B_{\mathbb{P}^1})$ is contained in $D_d(I)$. ∎

Remark 3.3. In the proof of Lemma 3.2, if $c = 0$, then we can always take $Z \simeq \mathbb{P}^1$. On the other hand, if $c \geq 1$, then the projective variety Z can be chosen to have dimension c or $Z \simeq \mathbb{P}^1$.

Proof of Theorem 3.1. Suppose there is no finite set I_0. Then there exists an infinite set $I' \subseteq I$ such that for each $i \in I'$, we have that

- the coefficient i is in $\mathrm{coeff}(B)$ for some generalized pair $(X, B, \mathbf{M})$ satisfying conditions (1)–(5) of the statement, or

- the coefficient $i = \mu_k$ for some generalized pair $(X, B, \mathbf{M})$ satisfying conditions (1)–(5) of the statement.

We take an infinite sequence $(d_j)_{j \geq 0}$ of values $d_j \in I'$ and generalized pairs $(X_j, B_j, \mathbf{M}_j)$ satisfying the conditions (1)–(5) of the theorem and either $d_j \in \mathrm{coeff}(B_j)$ or $d_j = \mu_{k,j}$ for some k, j. Since I satisfies the DCC, we can take a subsequence of $(d_j)_{j \geq 0}$ that is strictly increasing. By abuse of notation we may also call this sequence $(d_j)_{j \geq 0}$.

Applying Lemma 3.2 to each generalized pair $(X_j, B_j, \mathbf{M}_j)$, with either $d_j \in \mathrm{coeff}(B_j)$ or $d_j = \mu_{k,j}$, we obtain a sequence of generalized pairs $(Z_j, B_{Z_j}, \mathbf{M}_{Z_j})$ satisfying the following conditions:

- the projective variety Z_j has dimension at most $c + 1$;

- the generalized pair $(Z_j, B_{Z_j}, \mathbf{M}_{Z_j})$ is generalized log Calabi–Yau;

- the coefficients of B_{Z_j} are in $D(I)$;

- we can write $\mathbf{M}'_Z = \sum_j \eta_{k,j} M'_{Z,j}$ where $\eta_{k,j} \in I$ and each $M'_{Z,j}$ is a nef Cartier divisor; and

- either $d'_j \in \mathrm{coeff}(B_{Z_j})$ with $d'_j \in D_{d_j}(I)$ and $d'_j < 1$ or $d'_j = d_j = \eta_{k,j}$.

By Lemma 2.12, we know that $D(I)$ satisfies the descending chain condition. If we have that $d'_j = d_j = \eta_{k,j}$ for infinitely many j, we get a contradiction of Proposition 2.32. Otherwise, we assume that each d'_j is of the following form:

$$d'_j = \frac{N_j - 1 + m_{j,1} d_j + \sum_{k \geq 2} m_{j,k} d_{j,k}}{N_j}$$

for some positive integers $N_j, m_{j,1}$ and $m_{j,k}$, and that $d'_j < 1$. By Proposition 2.32, the coefficients d'_j belong to a finite set I_1, so we may assume they are all equal to a fixed number $d' \in (0, 1)$. As $d' \geq 1 - \frac{1}{N_j}$, the integer N_j can take only finitely many values. By abuse of notation, we take a subsequence with constant $N_j = N$. As $m_{j,1} d_j < 1$, and d_i is an increasing sequence, we have that

$$m_{j,1} < \frac{1}{d_j} < \frac{1}{d_1}.$$

Therefore, $m_{j,1}$ can also take finitely many values and we can assume it to be a constant m. Then, we can write:

$$\sum_{k \geq 2} m_{j,k} d_{j,k} = Nd' - N + 1 - md_j.$$

The left-hand side is in $D(I)$, hence it also satisfies the DCC. However, the right-hand side is strictly decreasing, as the only variable is d_j, which we chose to be strictly increasing. We have a contradiction, so there exists a finite set I_0 with the desired property.   ∎

4. Proof of the theorems

In this section, we prove the theorems presented in the introduction. In Section 4.1, we prove the local ascending chain condition for log canonical singularities with bounded coregularity. In Section 4.2, we prove the ascending chain condition for log canonical thresholds with bounded coregularity. In Section 4.3, we study the log canonical thresholds of coregularity zero and one. Finally, in Section 4.4, we study the accumulation points of log canonical thresholds with bounded coregularity.

4.1. Local ascending chain condition

In this subsection, we prove a local version of the ascending chain condition for generalized log canonical singularities with bounded coregularity.

Theorem 4.1. *Let c be a positive integer. Let I be a set of nonnegative real numbers satisfying the descending chain condition. Then, there exists a finite subset $I_0 \subseteq I$ such that if $(X, B, \mathbf{M})$ satisfies the following conditions:*

(1) *the variety X is normal and quasi-projective;*

(2) *the generalized pair $(X, B, \mathbf{M})$ is generalized log canonical;*

(3) *the coefficient of any component of B lies in I;*

(4) *on some model X' where $\mathbf{M}$ descends, we can write the trace of $\mathbf{M}$ on X' as $\mathbf{M}' = \sum_j \lambda_j M'_j$, where each $\lambda_j \in I$ and each M'_j is a nef Cartier divisor;*

(5) *the coregularity of $(X, B, \mathbf{M})$ is at most c; and*

(6) *there exists a minimal generalized non-klt center $Z \subset X$ of $(X, B, \mathbf{M})$ which is contained in every component of B and no M'_j descends over the generic point of Z;*

then the coefficient of any component of B lie in I_0 and each λ_j is contained in I_0.

Proof. Assume $1 \in I$. Let $J \subseteq I$ be the finite set we obtain in Theorem 3.1. Let

$$I_1 = \left\{ i \in I \mid \frac{m - 1 + ki + f}{m} \in J \cap [0, 1] \text{ for some } m, k \in \mathbb{Z}_{>0}, \ f \in I \right\},$$

which is finite by Lemma 2.16. Let $I_0 = I_1 \cup \{1\}$. Without loss of generality, we may assume that Z is a minimal generalized non-klt center. Let $i \in I$ be the coefficient of a component D of B. We show that i must belong to the finite subset I_0. By condition (6), Z is contained in the support of D. *Furthermore, after pulled back to a certain log resolution, every component of $\mathbf{M}$ contributes to the coefficient of some exceptional divisor with center Z.* Thus, an analogous argument would also show that λ_j belongs to I_0.

If Z is a divisor on X, then $B = Z$ and hence $i = 1$. Suppose that Z is not a divisor on X. Let $\pi : (Y, B_Y, \mathbf{M}_Y) \to (X, B, \mathbf{M})$ be a generalized dlt modification, where B_Y is the strict transform of B. We denote by D_Y the strict transform of D on Y. We may choose π so that the center of some component S of $\lfloor B_Y \rfloor$ on X is Z.

Case 1. The divisor D_Y intersects S over the generic point of Z.

Adjunction formula gives us:

$$(K_Y + B_Y + \mathbf{M}_Y)|_S \sim_{\mathbb{R}} K_S + B_S + \mathbf{M}_S,$$

where $(S, B_S, \mathbf{M}_S)$ is generalized dlt and the coefficients of B_S belong to $D(I)$. *Furthermore, some component of B_S, which comes from irreducible components of $D_Y \cap S$, has a coefficient in $D_i(I)$ and dominates Z by assumption.* Since Z is a minimal non-klt center of $(X, B, \mathbf{M})$, every non-klt center of $(Y, B_Y, \mathbf{M}_Y)$ dominates Z. By adjunction, every non-klt center of $(S, B_S, \mathbf{M}_S)$ also dominates Z. Let $(S_z, B_{S_z}, \mathbf{M}_{S_z})$ be the generalized pair induced on a general fiber S_z of the morphism $S \to Z$. Then $(S_z, B_{S_z}, \mathbf{M}_{S_z})$ is gdlt, $K_{S_z} + B_{S_z} + \mathbf{M}_{S_z} \equiv 0$, and some component of B_{S_z} has a coefficient in $D_i(I)$. Furthermore, by Lemma 2.30, the coregularity of $(S_z, B_{S_z}, \mathbf{M}_{S_z})$ is at most c. By Theorem 3.1, the coefficient of B_{S_z} lies in J and hence $i \in I_0$.

Case 2. The divisor D_Y does not intersect S over the generic point of Z.

Pick $\varepsilon > 0$ to be sufficiently small and run a $(K_Y + B_Y - \varepsilon S + \mathbf{M}_Y)$-MMP on Y over X, with scaling of an ample divisor. Since,

$$K_Y + B_Y - \varepsilon S + \mathbf{M}_Y = \pi^*(K_X + B + \mathbf{M}) - \varepsilon S \equiv_X -\varepsilon S,$$

every step of this MMP is S-positive. Furthermore, this MMP terminates with a minimal model Y_0 over X by [8, Lemma 4.4]. The divisor S is not contracted by this MMP (see Claim of the proof of Lemma 3.2). No component of the divisor B_Y is contracted by this MMP since their center on X are divisors.

Let S_{Y_0} be the push-forward of S on Y_0. Analogously, we denote by B_{Y_0} (resp. $\mathbf{M}_{Y_0}$ and D_{Y_0}) the push-forward of B_Y (resp. $\mathbf{M}_Y$ and D_Y) to Y_0. We claim that S_{Y_0} is the set-theoretic preimage of Z with respect to the structure morphism $\pi_0 : Y_0 \to X$. From our construction, $\pi_0(S_{Y_0}) = \pi(S) = Z$, so it suffices to prove $\pi_0^{-1}(Z) \subseteq S_{Y_0}$. Suppose there is a component of $\pi_0^{-1}(Z)$ which is not contained in S_{Y_0}. Then, we can choose a curve $C \not\subseteq S_{Y_0}$ such that $\pi_0(C)$ is a point in Z and $C \cap S_{Y_0} \neq \emptyset$. However, $-S_{Y_0}$ is nef over X because $(Y_0, B_{Y_0} - \varepsilon S_{Y_0}, \mathbf{M}_{Y_0})$ is a minimal model and $K_{Y_0} + B_{Y_0} + \mathbf{M}_{Y_0}$ is numerically trivial over X. This is a contradiction, so the claim follows.

We note that the image of D_{Y_0} in X dominates Z, so $D_{Y_0} \cap S_{Y_0}$ also dominates Z by the above claim. By Lemma 2.31, the coregularity of $(Y_0, B_{Y_0}, \mathbf{M}_{Y_0})$ is equal to the coregularity of $(Y, B_Y, \mathbf{M}_Y)$. Thus, we may replace $(Y, B_Y, \mathbf{M}_Y)$ by $(Y_0, B_{Y_0}, \mathbf{M}_{Y_0})$ to assume that D_Y intersects S over the generic point of Z, $-S$ is nef over X, and $\pi^{-1}(Z) = S$ holds set-theoretically. However, we may lose the property that $(Y, B_Y, \mathbf{M}_Y)$ is generalized dlt.

Let $\pi_1 \colon (Y_1, B_{Y_1}, \mathbf{M}_{Y_1}) \to (Y, B_Y, \mathbf{M}_Y)$ be a generalized dlt modification. Let $F = T_1 + \cdots + T_n$ be the sum of all the π_1-exceptional divisors. Let S_{Y_1} be the strict transform of S on Y_1. Write

$$\pi_1^* S = S_{Y_1} + \sum_{j=1}^n \lambda_j T_j,$$

for some $\lambda_j \geq 0$. Let $\psi \colon Y_1 \to X$ be the induced projective birational morphism. We show that

$$\psi^{-1}(Z) = S_{Y_1} \cup \bigcup_{\lambda_j > 0} T_j,$$

holds set-theoretically. We apply a similar argument as before. It suffices to show that

$$\psi^{-1}(Z) \subseteq S_{Y_1} \cup \bigcup_{\lambda_j > 0} T_j.$$

Suppose that the inclusion does not hold. Then, we can find a curve C such that

- $C \not\subseteq \operatorname{supp}(S_{Y_1} \cup \bigcup_{\lambda_j > 0} T_j)$,
- $\psi(C)$ is a point in Z,
- C intersects non-trivially with either S_{Y_1} or some T_j for some $\lambda_j > 0$.

This implies that

$$\pi_1^* S \cdot C = \left(S_{Y_1} + \sum_{j=1}^n \lambda_j T_j \right) \cdot C > 0,$$

which contradicts the fact that $-S$ is nef over X. Hence, there is a prime divisor S_1 in $\{S_{Y_1}\} \cup \{T_j : \lambda_j > 0\}$ for which $D_{Y_1} \cap S_1$ dominates Z. This reduces to the first case. Hence, we conclude that $i \in I_0$. ∎

4.2. Log canonical thresholds with bounded coregularity

In this section, we state and we prove the ascending chain condition for generalized log canonical thresholds with bounded coregularity.

Theorem 4.2. *Let c be a positive integer. Let I and J be two sets of nonnegative real numbers satisfying the descending chain condition. Suppose $\mathbf{M}$ and $\mathbf{N}$ are b-divisors that descend on some birational model X' over X with traces $\mathbf{M}'$ and $\mathbf{N}'$. Let $\operatorname{LCT}_c(I, J)$ be the set of thresholds $t = \operatorname{lct}(X, B, \mathbf{M}; \Gamma, \mathbf{N})$ where the following conditions are satisfied:*

- *$(X, B, \mathbf{M})$ is a generalized log canonical pair;*
- *the coefficients of B lie in I and the coefficients of Γ lie in J;*
- *we can write $\mathbf{M}' = \sum_j \lambda_j M_j'$ where $\lambda_j \in I$ and each M_j' is a nef Cartier divisor;*
- *we can write $\mathbf{N}' = \sum_j \mu_j N_j'$ where $\mu_j \in J$ and each N_j' is a nef Cartier divisor; and*
- *the coregularity of $(X, B + t\Gamma, \mathbf{M} + t\mathbf{N})$ is at most c.*

Then, the set $\operatorname{LCT}_c(I, J)$ satisfies the ACC.

Proof. Let $t_k := \mathrm{lct}(X_k, B_k, \mathbf{M}_k; \Gamma_k, \mathbf{N}_k)$ be a strictly increasing sequence in $\mathrm{LCT}_c(I, J)$. For each k, let V_k be a minimal generalized log canonical center of $(X_k, B_k + t_k \Gamma_k + \mathbf{M}_k + \mathbf{N}_k)$ which is either contained in the support of Γ_k or in the locus where $\mathbf{N}_k$ does not descend. By Lemma 2.29, we may assume that every component of Γ_k and B_k contains V_k. Furthermore, we may assume that neither $N'_{k,j}$ nor $M'_{k,j}$ descend at the generic point of V_k. Define

$$L = \{i + t_k j \mid i \in I \cup \{0\}, \ j \in J \cup \{0\}, \ k \in \mathbb{Z}_{>0}\}$$

so that the coefficients of $B_k + t_k \Gamma_k$ lie in L and the coefficients of $\mathbf{M}_k + t_k \mathbf{N}_k$, in the models where they descend, lie in L. Then, L satisfies the descending chain condition. By Theorem 4.1, there exists a finite subset $L_0 \subseteq L$ such that the coefficients of $B_k + t_k \Gamma_k$ lie in L_0 and the coefficients of $\mathbf{M}_k + t_k \mathbf{N}_k$, in the models where they descend, lie in L_0. Passing to a subsequence on t_k, we may find $i_k \in I$ and $j_k \in J \setminus \{0\}$ with $i_k + t_k j_k = \ell_0$ for some fixed $\ell_0 \in L_0$. Thus, t_k belongs to the set

$$\left\{\frac{\ell_0 - i}{j} : i \in I \cup \{0\}, \ j \in J\right\},$$

which satisfies ACC because both I and J satisfy DCC. This contradicts with $\{t_k\}_{k \geq 0}$ being a strictly increasing sequence. We conclude that the set $\mathrm{LCT}_c(I, J)$ satisfies the ascending chain condition. ∎

4.3. Log canonical thresholds of coregularity zero and one

In this subsection, we study the set of log canonical thresholds of coregularity zero and coregularity one.

Theorem 4.3. *Let I and J be two sets of nonnegative real numbers. We have that*

$$\mathrm{LCT}_0(I, J) = \left\{\frac{1 - i}{j} \mid i \in I^+ \cap [0, 1], \ j \in J^+\right\}.$$

Proof. Denote the set on the right-hand side to be $\mathcal{I}$. Let $t = \mathrm{lct}(X, B, \mathbf{M}; \Gamma, \mathbf{N})$ be a threshold as in the statement. Let V be a minimal generalized lc center of $(X, B + t\Gamma, \mathbf{M} + t\mathbf{N})$ which is either contained in the support of Γ or the locus where $\mathbf{N}$ does not descend. By Lemma 2.29, we may assume that every component of Γ contains V and no N'_j descend over the generic point of V. Denote the set

$$L = \{i + tj \mid i \in I \cup \{0\}, \ j \in J \cup \{0\}\} \tag{4.1}$$

be the set of all possible coefficient of $B + t\Gamma$ and coefficients of $\mathbf{M} + t\mathbf{N}$, where it descends. Let $\gamma = i + tj$ be the coefficient of a component of $B + t\Gamma$ or $\mathbf{M} + t\mathbf{N}$ with $j \neq 0$. By the proof of Theorem 4.1, we may find a projective generalized pair $(S, B_S, \mathbf{M}_S)$ such that

- $(S, B_S, \mathbf{M}_S)$ is generalized log canonical,

- $K_S + B_S + \mathbf{M}_S \equiv 0,$

- the coregularity of $(S, B_S, \mathbf{M}_S)$ is zero,

- every component of B_S has coefficient in $D(L)$,

- we can write $\mathbf{M}'_S = \sum_j \lambda_j M'_{S,j}$ where $\lambda_j \in L$ and each $M'_{S,j}$ is nef Cartier, and

- either component of B_S has coefficient in $D_\gamma(L)$ or $\gamma = \lambda_j$ for some j.

By Lemma 3.2 and Remark 3.3, we can obtain a generalized pair $(\mathbb{P}^1, B_{\mathbb{P}^1}, \mathbf{M}_{\mathbb{P}^1})$ such that

(1) the generalized pair $(\mathbb{P}^1, B_{\mathbb{P}^1}, \mathbf{M}_{\mathbb{P}^1})$ is generalized log canonical,

(2) $K_{\mathbb{P}^1} + B_{\mathbb{P}^1} + \mathbf{M}_{\mathbb{P}^1} \equiv 0,$

(3) the generalized pair $(\mathbb{P}^1, B_{\mathbb{P}^1}, \mathbf{M}_{\mathbb{P}^1})$ has coregularity zero,

(4) every coefficient of $B_{\mathbb{P}^1}$ lies in $D(L)$,

(5) we can write $\mathbf{M}_{\mathbb{P}^1} = \sum_s \lambda_s M_{\mathbb{P}^1,s}$ where $\lambda_s \in L$ and each $M'_{\mathbb{P}^1,s}$ is nef and integral, and

(6) either some component of $B_{\mathbb{P}^1}$ has coefficient in $D_\gamma(L)$ or $\gamma = \lambda_j$ for some j.

Since $(\mathbb{P}^1, B_{\mathbb{P}^1}, \mathbf{M}_{\mathbb{P}^1})$ has coregularity zero, then some coefficient of $B_{\mathbb{P}^1}$ equals one. Now, we have that

$$\deg(K_{\mathbb{P}^1} + B_{\mathbb{P}^1} + \mathbf{M}_{\mathbb{P}^1}) = 0.$$

Expanding this expression, we get an equation of the form

$$\sum_{k=1}^{m} \frac{N_k - 1 + d_k}{N_k} = 1,$$

where $d_k \in L^+$ and $N_k \in \mathbb{Z}_{>0}$ and the term one comes from the coefficient-one component of $B_{\mathbb{P}^1}$. Notice that $1 - N_k^{-1} \geq 1/2$ if $N_k \geq 2$ and property (6) implies that $d_k > 0$ for some k. Hence we have at most one k for which $N_k \neq 1$. Suppose $N_k = 1$ for all $k \geq 2$. Then

$$d_1 + N_1(d_2 + \cdots + d_m) = 1.$$

Write $d'_k = N_1 d_k$ if $k \geq 2$ and $d'_1 = d_1$. Then $d'_k \in L^+$ for all k. We further write $d'_k = i_k + t j_k$ for some $i_k \in I^+$ and $j_k \in J^+$. We have

$$\sum_{k=1}^{m} (i_k + t j_k) = 1.$$

Property (6) ensures that some $j_k \neq 0$ and hence we can solve

$$t = \frac{1 - \sum_{k=1}^{m} i_k}{\sum_{k=1}^{m} j_k} \in \mathcal{I},$$

as desired. This shows the inclusion $\mathrm{LCT}_c(I, J) \subseteq \mathcal{I}$ holds.

Now, let $t = (1-i)j^{-1}$ with $i \in I^+ \cap [0,1]$ and $j \in J^+$. We can write $i = \sum_{k=1}^{a} m_k i_k$ for $m_k \in \mathbb{Z}_{>0}$ and $i_k \in I$. We can write $j = \sum_{k=1}^{b} n_k j_k$ for $n_k \in \mathbb{Z}_{>0}$ and $j_k \in J$. Let $X = \mathrm{Bl}_q(\mathrm{Bl}_p \mathbb{P}^2)$ be a sequence of two blow-ups of $\mathbb{P}^2$, where p is a point on $\mathbb{P}^2$ and q is a point on the exceptional divisor of $\mathrm{Bl}_p \mathbb{P}^2 \to \mathbb{P}^2$. Denote the blow-up morphism $X \to \mathbb{P}^2$ as π. Let E_1 and E_2 be the two exceptional divisors of π, such that the center of E_1 is a divisor on $\mathrm{Bl}_p \mathbb{P}^2$. Let $p_1, \ldots, p_a, p_1', \ldots, p_b' \in E_1 \setminus E_2$ be distinct points. Let $q_1, \ldots, q_a, q_1', \ldots, q_b' \in E_2 \setminus E_1$ be distinct points.

For $1 \le k \le a$, let $C_{k,s}$ and $D_{k,s}$ be pairwise different curves on X such that:

- $C_{k,s} \cap E_1 = \{p_k\}$ with intersection multiplicity one and $C_{k,s} \cap E_2 = \emptyset$; and

- $D_{k,s} \cap E_2 = \{q_k\}$ with intersection multiplicity one and $D_{k,s} \cap E_1 = \emptyset$.

For $1 \le k \le b$ and $1 \le s \le n_k$, let $C_{k,s}'$ and $D_{k,s}'$ be pairwise different curves on X such that:

- $C_{k,s}' \cap E_1 = \{p_k'\}$ with intersection multiplicity one and $C_{k,s}' \cap E_2 = \emptyset$; and

- $D_{k,s}' \cap E_2 = \{q_k'\}$ with intersection multiplicity one and $D_{k,s}' \cap E_1 = \emptyset$.

Consider the triple

$$(X, B_X; \Gamma_X) = \left(X, \sum_{k=1}^{a} \sum_{s=1}^{m_k} i_k (C_{k,s} + D_{k,s}); \sum_{k=1}^{b} \sum_{s=1}^{n_k} j_k (C_{k,s}' + D_{k,s}') \right).$$

Denote $B = \pi_* B_X$ and $\Gamma = \pi_* \Gamma_X$. Then the coefficients of B lie in I and the coefficients of Γ lie in J. We can compute

$$\pi^*(K_{\mathbb{P}^2} + B + t\Gamma)$$

$$= K_X - E_1 - 2E_2 + B_X + \sum_{k=1}^{a} i_k m_k (2E_1 + 3E_2) + t\Gamma_X + t \sum_{k=1}^{b} j_k n_k (2E_1 + 3E_2)$$

$$= K_X + B_X + t\Gamma_X + E_1 + E_2.$$

As a result, $\pi : X \to \mathbb{P}^2$ is a dlt modification of $(X, B_X + t\Gamma_X)$ and $\mathrm{coreg}(X, B_X + t\Gamma_X) = 0$. Then the triple $(\mathbb{P}^2, B; \Gamma)$ gives an example with coregularity zero and log canonical threshold t. We conclude that $\mathcal{I} \subseteq \mathrm{LCT}_0(I, J)$. This finishes the proof of the statement. ∎

In order to state the theorem for the coregularity one case, we need to introduce some notation. Let I be a set of nonnegative real numbers. Let p, q, r be three positive integers. We set

$$I_{(p,q,r)}^+ := \{qri_1 + pri_2 + pqi_3 \mid i_1, i_2, i_3 \in I^+\}.$$

We define the set

$$\mathrm{LCT}_{1,(p,q,r)}(I, J) := \left\{ \frac{qr + pr + pq - pqr - i}{j} \mid i \in I_{(p,q,r)}^+ \text{ and } j \in J_{(p,q,r)}^+ \right\}.$$

We call this set, the set of *weighted log canonical thresholds* of coregularity one.

Theorem 4.4. *Let I and J be two sets of nonnegative real numbers. We have that*

$$\mathrm{LCT}_1(I, J) := \bigcup_{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1} \mathrm{LCT}_{1,(p,q,r)}(I, J).$$

Proof. Let $t \in \mathrm{LCT}_1(I, J)$. Proceeding as in the proof of Theorem 4.3, we can produce a generalized log Calabi–Yau pair $(\mathbb{P}^1, B_{\mathbb{P}^1}, \mathbf{M}_{\mathbb{P}^1})$ satisfying the following conditions:

(1) the coefficients of $B_{\mathbb{P}^1}$ belong to $D(L)$,

(2) we can write $\mathbf{M}_{\mathbb{P}^1} = \sum_s \lambda_s M_{\mathbb{P}^1,s}$ where $\lambda_s \in L$ and each $M_{\mathbb{P}^1,s}$ is a nef integral divisor, and

(3) either some component of $B_{\mathbb{P}^1}$ has coefficient in $D_\gamma(L)$ or $\gamma = \lambda_s$ for some s.

Above, L is defined as in (4.1) and γ is an element of the form $i + tj$ with $j \neq 0$. Expanding the equality

$$\deg(K_{\mathbb{P}^1} + B_{\mathbb{P}^1} + \mathbf{M}_{\mathbb{P}^1}) = 0,$$

we obtain an equation of the form

$$\sum_{k=1}^{m} \frac{N_k - 1 + d_k}{N_k} = 2,$$

where $d_k \in L^+$ and $N_k \in \mathbb{Z}_{>0}$ for each k. Note that $1 - N_k^{-1} \geq 1/2$ if $N_k \geq 2$ and property (3) implies that $d_k > 0$ for some k. Thus, there are at most three k's for which $N_k \geq 2$. Assume that $N_k = 1$ for all $k \geq 4$ and $m \geq 3$ (add zero terms if $m \leq 2$). Denote $N_1 = p$, $N_2 = q$, and $N_3 = r$. Then, the previous equation becomes

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 + \frac{d_1}{p} + \frac{d_2}{q} + \frac{d_3}{r} + \sum_{k=4}^{m} d_k.$$

In particular, $1/p + 1/q + 1/r > 1$. For each k, we write $d_k = i_k + tj_k$ for some $i_k \in I^+$ and $j_k \in J^+$. Then we have

$$qr + pr + pq - pqr = qr(i_1 + tj_1) + pr(i_2 + tj_2) + pq(i_3 + tj_3) + \sum_{k=4}^{m} pqr(i_k + tj_k).$$

Property (3) ensures that some $j_k \neq 0$, so we can solve

$$t = \frac{qr + pr + pq - pqr - qri_1 - pri_2 - pqi_3 - \sum_{k=4}^{m} pqri_k}{qrj_1 + prj_2 + pqj_3 + \sum_{k=4}^{m} pqrj_k} \in I^+_{(p,q,r)}(I, J),$$

as desired. Therefore, we conclude that every element $t \in \mathrm{LCT}_1(I, J)$ is contained in a set $\mathrm{LCT}_{1,(p,q,r)}(I, J)$ for some $p, q, r \in \mathbb{Z}_{>0}$ with $p^{-1} + q^{-1} + r^{-1} > 1$.

Now, let $t \in I^+_{(p,q,r)}(I, J)$ for some $p, q, r \in \mathbb{Z}_{>0}$ such that $1/p + 1/q + 1/r > 1$. We can write

$$t = \frac{qr + pr + pq - pqr - (qri_1 + pri_2 + pqi_3)}{qrj_1 + prj_2 + pqj_3}$$

for some $i_1, i_2, i_3 \in I^+$ and $j_1, j_2, j_3 \in J^+$. Write

$$i_k = \sum_\ell m_{k,\ell} i_{k,\ell} \quad \text{and} \quad j_k = \sum_\ell n_{k,\ell} j_{k,\ell}$$

for some $i_{k,\ell} \in I$, $j_{k,\ell} \in J$, and $m_{k,\ell}, n_{k,\ell} \in \mathbb{Z}_{>0}$.

Let p_1, p_2 and p_3 be three distinct points on $\mathbb{P}^1$. Let $(X; x)$ be the orbifold cone over $\mathbb{P}^1$ with respect to the $\mathbb{Q}$-divisor $\frac{1}{p}\{p_1\} + \frac{1}{q}\{p_2\} + \frac{1}{r}\{p_3\}$. Let $\widetilde{X} \to X$ be the blow-up of $x \in X$. Let $E \subset \widetilde{X}$ be the exceptional divisor, isomorphic to $\mathbb{P}^1$. The surface $\widetilde{X}$ has three singularities $x_1, x_2, x_3 \in E$. These singularities are of type A_p, A_q and A_r, respectively. For each $k = 1, 2, 3$ and ℓ, define:

- $\widetilde{B}_{k,\ell}$ be an irreducible curve on $\widetilde{X}$ such that $\widetilde{B}_{k,\ell}|_E = m_{k,\ell} x_k$, and
- $\widetilde{\Gamma}_{k,\ell}$ be an irreducible divisor on $\widetilde{X}$ such that $\widetilde{\Gamma}_{k,\ell}|_E = n_{k,\ell} x_k$.

Let

$$\widetilde{B} = \sum_{k,\ell} i_{k,\ell} \widetilde{B}_{k,\ell} \quad \text{and} \quad \widetilde{\Gamma} = \sum_{k,\ell} j_{k,\ell} \widetilde{\Gamma}_{k,\ell}.$$

Denote $B = \phi_* \widetilde{B}$ and $\Gamma = \phi_* \widetilde{\Gamma}$. Then the coefficients of B lie in I and the coefficients of Γ lie in J. Note that we have

$$(K_{\widetilde{X}} + \widetilde{B} + t\widetilde{\Gamma} + E)|_E$$

$$\sim_\mathbb{Q} K_E + \left(\frac{p-1+i_1+tj_1}{p}\right)x_1 + \left(\frac{q-1+i_2+tj_2}{q}\right)x_2 + \left(\frac{r-1+i_3+tj_3}{r}\right)x_3.$$

Thus, the divisor $K_{\widetilde{X}} + \widetilde{B} + t\widetilde{\Gamma} + E$ intersects E trivially. Hence, we have that

$$\phi^*(K_X + B + t\Gamma) = K_{\widetilde{X}} + \widetilde{B} + t\widetilde{\Gamma} + E.$$

Then, the pair $(X, B + t\Gamma; x)$ is log canonical but not klt at the vertex. Furthermore, the pair has coregularity one as E is the only log canonical place. We conclude that $t \in \mathrm{LCT}_1(I, J)$. This finishes the proof. ∎

4.4. Proof of the theorems

In this subsection, we prove the theorems of the introduction. The first three theorems presented in the introduction are already proved in the previous sections in a broader context. For the statement about accumulation points, we need a short argument.

Proof of Theorem 1. This follows from Theorem 4.2 for generalized pairs. ∎

Proof of Theorem 2. This follows from Theorem 3.1 for generalized pairs. ∎

Proof of Theorem 3. This follows from Theorem 4.3 for generalized pairs. ∎

In order to prove the last theorem, we need to use the following notation.

Notation 4.5. *We denote by $\mathcal{N}_c(I, t)$ the set of log Calabi–Yau log canonical pairs $(X, B + \Gamma)$ so that the coefficients of B belong to $D(I)$, the coefficients of $\Gamma \neq 0$ belong to $D_t(I)$, and $\mathrm{coreg}(X, B + \Gamma) \leq c$. We set*

$$N_c(I) := \{t \mid \mathcal{N}_c(I, t) \text{ is non-empty}\}.$$

Analogously, we denote by $\mathcal{N}_{\dim \leq c}(I, t)$ the set of log Calabi–Yau log canonical pairs $(X, B + \Gamma)$ so that the coefficients of B belong to $D(I)$, the coefficients of $\Gamma \neq 0$ belong to $D_t(I)$, and the dimension of X is at most c. We set

$$N_{\dim \leq c}(I) := \{t \mid \mathcal{N}_{\dim \leq c}(I, t) \text{ is non-empty}\}.$$

Note that $N_{\dim \leq c}(I) \subset N_c(I)$ as every log Calabi–Yau pair of dimension at most c has coregularity at most c.

Proof of Theorem 4. Let $t \in \mathrm{LCT}_c(I)$, by the proof of Theorem 4.2, we deduce that $t \in N_c(I)$. By the proof of Theorem 3.1, it follows that $t \in N_{\dim \leq c}(I)$. Thus, we conclude that we have an inclusion

$$\mathrm{LCT}_c(I) \subseteq N_{\dim \leq c}(I).$$

As in the proof of Theorem 4.4, taking the orbifold cone over a log Calabi–Yau pair in $\mathcal{N}_{\dim \leq c}(I, t)$ gives us the opposite inclusion

$$N_{\dim \leq c}(I) \subseteq \mathrm{LCT}_c(I).$$

Thus, we conclude that

$$\mathrm{LCT}_c(I) = N_{\dim \leq c}(I). \tag{4.2}$$

Now, we take accumulation points at both sides of the previous equality and get

$$\mathrm{Acc}(\mathrm{LCT}_c(I)) = \mathrm{Acc}(N_{\dim \leq c}(I)),$$

where $\mathrm{Acc}(S)$ stands for the accumulation points of the set S. By [19, Proposition 11.7], we have that

$$\mathrm{Acc}(N_{\dim \leq c}(I)) \subseteq N_{\dim \leq c-1}(I),$$

we conclude that

$$\mathrm{Acc}(\mathrm{LCT}_c(I)) = \mathrm{Acc}(N_{\dim \leq c}(I)) \subseteq N_{\dim \leq c-1}(I) = \mathrm{LCT}_{c-1}(I),$$

where we used equality (4.2) in the first and last equalities. This finishes the proof of the theorem. ∎

5. Examples and questions

In this section, we give a couple of examples related to the coregularity of fibrations and singularities. We also propose some questions for further research. First, we show

an example of log canonical thresholds on singularities that are closely related to toric singularities, the so-called complexity one $\mathbb{T}$-singularities (see, e.g., [32–34]).

Example 5.1. Let $(X, B; x)$ be a toric singularity with reduced boundary B. Let Γ be a reduced torus invariant divisor through x. Then, we have that

$$\operatorname{lct}(X, B; \Gamma) \in \{0, 1\},$$

holds independently of the dimension of the germ $(X; x)$.

Let $(X, B; x)$ be a complexity one rational log canonical $\mathbb{T}$-singularity with B reduced, i.e., X is n-dimensional and admits the effective action of a $(n-1)$-dimensional torus $\mathbb{T}$ and B is $\mathbb{T}$-invariant. Let Γ be an effective reduced $\mathbb{T}$-invariant divisor. By [1, Theorem 3.1], we can find a $\mathbb{T}$-equivariant projective birational morphism $\phi: Y \to X$ so that $\phi^{-1}(x) \simeq \mathbb{P}^1 =: C$. Furthermore, the variety Y admits a good $\mathbb{T}$-quotient to $Z \simeq \mathbb{P}^1$ and the log pair

$$\phi^*(K_X + B + t\Gamma) = K_Y + B_Y + t\Gamma_Y$$

has toroidal singularities. Furthermore, as we are assuming $(X, B + t\Gamma)$ is strictly log canonical at x, we may assume that some log canonical center of $(Y, B_Y + t\Gamma_Y)$ is contained in C. Hence, we conclude that all the $\mathbb{T}$-invariant divisors of Y which dominate Z must appear with coefficient one in $B_Y + t\Gamma_Y$. In particular, performing adjunction to C, we obtain a one-dimensional dlt pair on $C \simeq \mathbb{P}^1$. Taking the degree of this dlt pair, we either get a relation of the form

$$\left(1 - \frac{1}{p}\right) + \left(1 - \frac{1}{q}\right) + \left(1 - \frac{1}{r}\right) + t = 2,$$

or

$$\left(1 - \frac{1}{p} + \frac{t}{p}\right) + \left(1 - \frac{1}{q}\right) + \left(1 - \frac{1}{r}\right) = 2,$$

for certain positive integers p, q, r. We conclude that t must belong to the set

$$\left\{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 \mid p, q, r \in \mathbb{Z}_{>0}\right\} \cap [0, 1] \cup \left\{p\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1\right) \mid p, q, r \in \mathbb{Z}_{>0}\right\} \cap [0, 1].$$

Note that the previous set of log canonical thresholds t is independent of the dimension of the germ. Indeed, the coregularity of the previous pairs is at most one independently of their dimension.

The following example shows that the ascending chain condition does not hold if we do not bound the coregularity or control the coefficients of the involved divisors.

Example 5.2. First, we have $\operatorname{lct}(\mathbb{A}^1, i\{0\}; j\{0\}) = (1 - i)j^{-1}$. Then, if either I or J does not satisfy the descending chain condition, then the set $\operatorname{LCT}_1(I, J)$ does not satisfy the ascending chain condition. Thus, the DCC condition for I and J in the statement of Theorem 1 is necessary.

Now, we turn to show that the condition on the coregularity in Theorem 1 is also a necessary one. Let H_n be a smooth hypersurface of degree $n + 2$ in $\mathbb{P}^n$. Then, we have that the pair

$$\left(\mathbb{P}^n, \frac{n+1}{n+2} H_n\right)$$

is log Calabi–Yau. Let Γ_n be the cone over H_n in $\mathbb{A}^{n+1}$. Then, Γ_n is an irreducible divisor on $\mathbb{A}^{n+1}$ and for every t, the log pair $(\mathbb{A}^{n+1}, t\Gamma_n)$ has an isolated singularity at the origin. We have that

$$t_n := \mathrm{lct}(\mathbb{A}^{n+1}, \Gamma_n) = 1 - \frac{1}{n+2}.$$

The log pair $(\mathbb{A}^{n+1}, t_n \Gamma_n)$ has coregularity n. Indeed, it has a unique log canonical center which corresponds to the blow-up of the maximal ideal of $\mathbb{A}^{n+1}$ at the origin. Hence, we have that

$$\mathrm{coreg}(\mathbb{A}^{n+1}, t_n \Gamma_n) = n + 1 - \dim \mathcal{D}(\mathbb{A}^{n+1}, t_n \Gamma_n) - 1 = n.$$

Note that the sequence of t_n violates the ascending chain condition. However, the coregularity of the sequence $(\mathbb{A}^{n+1}, t_n \Gamma_n)$ is also unbounded.

In the following example, we show that in a log Calabi–Yau fibration the coregularity of the general fiber can be arbitrary, even if the coregularity of the log pair in the domain is zero.

Example 5.3. Consider a maximal Fano degeneration $\pi \colon \mathcal{X} \to \mathbb{A}^1$ of $\mathbb{P}^n$, i.e.,

$$\pi^{-1}\left(\mathbb{A}^1 - \{0\}\right) \simeq \mathbb{P}^n \times \left(\mathbb{A}^1 - \{0\}\right)$$

and $\mathcal{X}_0 := \pi^{-1}(\{0\})$ is the union the $n + 1$ projective toric varieties intersecting along toric subvarieties and the dual complex of $\mathcal{X}_0$ is a n-simplex (see, e.g., [37, Theorem 1.2]). We consider a general element $H \in |-2(K_{\mathcal{X}} + \mathcal{X}_0)|$, then the pair

$$(\mathcal{X}, \mathcal{X}_0 + H/2) \tag{5.1}$$

is log Calabi–Yau over the base and log canonical. Note that all the log canonical centers of (5.1) map to zero. The general fiber of this log pair is

$$(\mathbb{P}^n, H|_{\mathbb{P}^n}/2)$$

which is a log Calabi–Yau klt pair. Thus, the coregularity of the general fiber is n. However, the log pair (5.1) has coregularity zero as it is dlt and it has a zero-dimensional log canonical center contained in $\mathcal{X}_0$. Note that this example is over an affine base, however, it can be turned into a projective example by compactifying at infinity as a product.

In the following example, we show examples of germs with arbitrary dimension and bounded coregularity.

Example 5.4. For examples of log Calabi–Yau pairs with coregularity zero and arbitrary dimension, we can simply consider a log Calabi–Yau toric pair (T, Γ_T). In the local case, we can consider toric singularities with the reduced torus invariant boundary. In these examples, the number of components of the boundary tends to be high. Indeed, we have exactly $\dim T + \operatorname{rank} \operatorname{Cl}_{\mathbb{Q}}(T)$ prime components in the boundary.

We show that, in arbitrary dimension, there are examples of log canonical singularities of coregularity zero with a unique prime component in the boundary and coefficient $\frac{1}{2}$. Let $(\mathcal{X}, \mathcal{X}_0 + H_1/2)$ and $(\mathcal{X}, \mathcal{X}_0 + H_2/2)$ be two log Calabi–Yaus constructed as in Example 5.3. Here, H_1 and H_2 are two different general elements of the linear system $|-2(K_{\mathcal{X}} + \mathcal{X}_0)|$. We let $\pi\colon \mathcal{Y} \to \mathcal{X}$ be the blow-up of the intersection of H_1 with a component of $\mathcal{X}_0$. We let E be the exceptional divisor of this projective birational morphism. We denote by F_1 (resp. F_2) the strict transform of H_1 (resp. H_2) on $\mathcal{Y}$, and by $\mathcal{Y}_0$ the strict transform of $\mathcal{X}_0$ on $\mathcal{Y}$. Note that we have

$$\pi^*(K_{\mathcal{X}} + \mathcal{X}_0 + H_1/2) = K_{\mathcal{Y}} + \mathcal{Y}_0 + F_1/2 + E/2, \tag{5.2}$$

$$\pi^*(K_{\mathcal{X}} + \mathcal{X}_0 + H_2/2) = K_{\mathcal{Y}} + \mathcal{Y}_0 + F_2/2. \tag{5.3}$$

Due to (5.2), we know that the morphism $\mathcal{Y} \to \mathbb{A}^1$ is of Fano type. Observe that

$$K_{\mathcal{Y}} + \mathcal{Y}_0 + F_1/2 + (1 - \varepsilon)E/2, \tag{5.4}$$

is dlt and pseudo-effective over the base. We run a minimal model program for (5.4) over the base $\mathbb{A}^1$. Since $\mathcal{Y} \to \mathbb{A}^1$ is of Fano type, this minimal model program terminates with a good minimal model over the base. We call this model $\mathcal{Z} \to \mathbb{A}^1$. Let $E_{\mathcal{Z}}$ be the push-forward of E on $\mathcal{Z}$. Analogously, let $F_{\mathcal{Z},i}$ be the push-forward of F_i on $\mathcal{Z}$. Then, we conclude that the log pair

$$(\mathcal{Z}, F_{\mathcal{Z},2}/2)$$

is log crepant to the pair (5.3). Thus, it has coregularity zero. Furthermore, this log pair has a unique irreducible component of coefficient $\frac{1}{2}$. Let X be the cone of $-K_{\mathcal{Z}}$ over $\mathbb{A}^1$ and Γ be the cone over $F_{\mathcal{Z},1}$. We denote by $x \in X$ the point corresponding to $\{0\}$ in this cone with one-dimensional fixed point locus. Then, we have that $(X; x)$ is an n-dimensional $\mathbb{Q}$-factorial klt singularity. On the other hand, the pair $(X, \Gamma/2; x)$ is an n-dimensional $\mathbb{Q}$-factorial pair, Γ is prime, and the coregularity of $(X, \Gamma/2; x)$ is zero.

A polytopal variant of the log canonical threshold, the so-called log canonical threshold polytopes, were introduced by Libgober and Mustață in [36]. Given a projective variety X and a sequence of $\mathbb{Q}$-Cartier effective divisors $D_1, \ldots, D_s$, we can define the *log canonical threshold polytope*, to be

$$\mathrm{LCT}(X; D_1, \ldots, D_r) := \big\{ (t_1, \ldots, t_s) \in \mathbb{R}^s \mid (X, t_1 D_1 + \cdots + t_s D_s) \text{ is log canonical} \big\}.$$

The ACC for log canonical threshold polytopes of the same dimension and number of divisors was proved in [22]. This means that there is no infinite sequence of these polytopes so each of them strictly contains the previous one. Instead, we can consider log

canonical threshold polytopes on which the coregularity is bounded when the thresholds are computed. Analogously to the main theorem of this article, we can ask.

Question 5.5. *Does the ascending chain condition for log canonical threshold polytopes with bounded coregularity hold?*

The study of numerical thresholds naturally leads to the study of pseudo-effective thresholds, i.e., questions related to the Fujita's spectrum conjecture (see, e.g., [20, 21]). Given a log canonical pair (X, Δ) and an effective divisor Γ on X, we define the *pseudo-effective threshold* of (X, Δ) with respect to Γ, denoted by $p(X, \Delta; \Gamma)$ to be the smallest positive real number p for which $K_X + \Delta + p\Gamma$ is pseudo-effective. We set $p = \infty$ if the previous divisor is never pseudo-effective and zero if $K_X + \Delta$ is already pseudo-effective. We can define:

$$\mathcal{P}_c(I, J) := \{ p \mid p = p(X, \Delta; \Gamma), \ \mathrm{coeff}(\Delta) \in I, \ \mathrm{coeff}(\Gamma) \in J, \ \text{and} \ \mathrm{coreg}(X, \Delta + p\Gamma) \leq c \}.$$

Then, it is natural to ask.

Question 5.6. *Let c be a positive integer. Let I and J be two sets of nonnegative real numbers satisfying the descending chain condition. Does $\mathcal{P}_c(I, J)$ satisfy the ascending chain condition?*

We expect both previous questions to have positive answers. However, some new ideas must be introduced in order to prove them.

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