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Frobenius rigidity in $\mathbb{A}^1$-homotopy theory

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Abstract. We study the homotopy fixed points under the Frobenius endomorphism on the stable $\mathbb{A}^1$-homotopy category of schemes in characteristic $p > 0$ and prove a rigidity result for cellular objects in these categories after inverting p. As a consequence we determine the analogous fixed points on the K-theory of algebraically closed fields in positive characteristic. We also prove a rigidity result for the homotopy fixed points of the partial Frobenius pullback on motivic cohomology groups in weights at most 1.

1. Introduction

A functor $\kappa\colon \mathrm{CAlg}_k \to \mathrm{Sp}$ from the category of commutative algebras over a field k to the ∞-category of spectra (or the derived category of abelian groups, or groups, or sets) is called *rigid* if for any extension $F \subset E$ of algebraically closed overfields of k, the induced map

$$\kappa(F) \to \kappa(E)$$

is an isomorphism. For example, for some proper k-scheme X and some prime number ℓ, the functor $R \mapsto \mathrm{H}^n_{\text{ét}}(X \times_k \operatorname{Spec} R, \mathbb{Z}/\ell)$ given by taking étale cohomology is rigid for all $n \in \mathbb{Z}$. Similarly, Suslin's celebrated rigidity result [43] states that mod-ℓ K-theory $R \mapsto K_n(R)/\ell$ is rigid, provided ℓ is prime to the characteristic of k. In particular, when F is of characteristic $p > 0$ prime to ℓ, Quillen's computation of $K_n(\overline{\mathbb{F}}_p)$ gives explicit results for the mod-ℓ K-groups of F. Suslin's argument is robust enough to allow for various extensions, including the rigidity result of Röndigs–Østvær [39] asserting the full faithfulness of the pullback functor $\mathrm{SH}(F)/\ell \to \mathrm{SH}(E)/\ell$ between the mod-ℓ stable $\mathbb{A}^1$-homotopy categories, for two algebraically closed fields $F \subset E$ of characteristic prime to ℓ.

Of course, the full stable $\mathbb{A}^1$-homotopy category fails to be rigid, as is visible already for the first K-group $K_1(F) = F^\times$. For $k = \mathbb{F}_p$, the present paper studies the idea of rigidifying various functors by applying (homotopy) fixed points under the Frobenius endomorphism, as opposed to considering classes modulo ℓ. As a first indication note that for an algebraically closed field F of characteristic $p > 0$ the complex

$$F^\times \xrightarrow{x \mapsto x/x^p} F^\times \tag{1.1}$$

is quasi-isomorphic to $\mathbb{F}_p^\times$ by Kummer theory.

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For an $\mathbb{F}_p$-scheme S, let $\mathrm{Frob}_S\colon S \to S$ be the Frobenius endomorphism given by $f \mapsto f^p$ on functions. If S is understood, we abbreviate Frob_S simply by Frob.

Definition 1.1. For an $\mathbb{F}_p$-scheme S, the *Frobenius stable* $\mathbb{A}^1$*-homotopy category* is the fixed point category

$$\mathrm{SH}(S/\mathrm{Frob}) := \lim\big(\mathrm{SH}(S) \underset{\mathrm{id}}{\overset{\mathrm{Frob}^*}{\rightrightarrows}} \mathrm{SH}(S)\big)$$

i.e., the homotopy fixed points of Frobenius acting on the stable $\mathbb{A}^1$-homotopy category.

Objects in this category are pairs of objects $M \in \mathrm{SH}(S)$ together with an isomorphism $M \cong \mathrm{Frob}^* M$. By construction, the canonical pullback functor $\mathrm{SH}(\mathbb{F}_p) \to \mathrm{SH}(S)$ factors through a functor

$$\mathrm{can}_S\colon \mathrm{SH}(\mathbb{F}_p) \to \mathrm{SH}(S/\mathrm{Frob}).$$

The idea of rigidity after taking Frobenius fixed points leads to the following question.

Question 1.2. *Is the functor*

$$\mathrm{CAlg}_{\mathbb{F}_p} \to \mathrm{Sp}, \quad R \mapsto \mathrm{Map}_{\mathrm{SH}(R/\mathrm{Frob})[p^{-1}]}(\mathrm{can}_R M, \mathrm{can}_R N) \tag{1.2}$$

rigid for all $M, N \in \mathrm{SH}(\mathbb{F}_p)[p^{-1}]$*?*

The main results of this paper exhibit two situations in which we can answer special cases of this question affirmatively. To state the first, recall that the subcategory

$$\mathrm{SH}(\mathbb{F}_p)_{\mathrm{cell}} \subset \mathrm{SH}(\mathbb{F}_p)$$

of *cellular objects* is the full presentable subcategory generated by the motivic spheres $\mathbb{S}^{m,n}$ for all $m, n \in \mathbb{Z}$.

Theorem 1.3 (Theorem 5.5). *The functor in* (1.2) *is rigid for all* $M, N \in \mathrm{SH}(\mathbb{F}_p)_{\mathrm{cell}}[p^{-1}]$.

Applying the theorem to cellular spectra (see also Section 5.4 for more examples) such as the homotopy invariant K-theory spectrum implies the following result, where amusingly p^{-1}-localization is not necessary.

Corollary 1.4 (Corollary 5.11). *For any algebraically closed field* F *of characteristic* $p > 0$*, one has*

$$\pi_n\big(K(F/\mathrm{Frob})\big) = \begin{cases} \mathbb{Z} & n = -1, 0, \\ \mathbb{F}_{p^i}^\times & n = 2i - 1 > 0, \\ 0 & \textit{else}, \end{cases}$$

where $K(F/\mathrm{Frob})$ *denotes the homotopy fixed points of the Frobenius endomorphism on the* K*-theory spectrum* $K(F)$.

In the formulation of the next result, we denote by $\mathrm{SH}^{\mathrm{eff}}(\mathbb{F}_p)$ the stable, full subcategory of $\mathrm{SH}(\mathbb{F}_p)$ generated under colimits by motives of smooth $\mathbb{F}_p$-schemes, but not allowing negative Tate twists.

Theorem 1.5 (Corollary 5.26). *The functor in* (1.2) *is rigid for all* $M \in \mathrm{SH}^{\mathrm{eff}}(\mathbb{F}_p)[p^{-1}]$ *and* $N = \mathbb{S}^{n,n}[p^{-1}]$ *(or,* $N = \mathbb{Z}[p^{-1}](n)$*) for all* $n \leq 1$.

Let us unwind the meaning of this assertion in terms of Bloch's cycle complex. For a smooth $\mathbb{F}_p$-scheme X of finite type, we define the *Frobenius–Bloch cycle complex* $\mathrm{R}\Gamma(X \times F/\operatorname{Frob}_F, \mathbb{Z}(n))$ to be the total complex associated to the two-term double complex

$$\mathrm{R}\Gamma\big(X \times F, \mathbb{Z}(n)\big) \xrightarrow{\mathrm{id}-(\mathrm{id}_X \times \operatorname{Frob}_F)^*} \mathrm{R}\Gamma\big(X \times F, \mathbb{Z}(n)\big), \tag{1.3}$$

where $\mathrm{R}\Gamma(X \times F, \mathbb{Z}(n))$ denotes Bloch's cycle complex. Equivalently, this is the homotopy fixed point of the action of the partial Frobenius pullback $(\mathrm{id}_X \times \operatorname{Frob}_F)^*$ on $\mathrm{R}\Gamma(X \times F, \mathbb{Z}(n))$. If we take $F = \overline{\mathbb{F}}_p$, and consider étale motivic cohomology $\mathrm{R}\Gamma_{\text{ét}}(-, \mathbb{Z}(n))$, this recovers Weil-étale cohomology of schemes in characteristic $p > 0$ as introduced by Lichtenbaum [28] and studied in particular by Geisser [15]. Still for $F = \overline{\mathbb{F}}_p$, the cohomologies of the complex in (1.3) were studied under the name of *Frobenius cohomology* by Geisser [16]. Geisser conjectured that these groups are finitely generated, which relates to Lichtenbaum's conjectures on finiteness of Weil-étale cohomology. We (also) refer to the above concept as *Frobenius motivic cohomology* (as opposed to Weil motivic cohomology) in order to emphasize that fixed points under partial Frobenius are considered even for transcendental extensions F over $\mathbb{F}_p$. For M being the motive of X and $N = \mathbb{Z}(n)$, the rigidity asserted above amounts to the claim that the complex $\mathrm{R}\Gamma(X \times F/\operatorname{Frob}_F, \mathbb{Z}(n))$ is rigid after inverting p, i.e., is independent, up to quasi-isomorphism, of the choice of an algebraically closed field F of characteristic $p > 0$.

The proof for $n = 1$ is based on the following observation. Resolution of singularities (by alterations) allows to reduce to the case of X being smooth and proper over $\mathbb{F}_p$. The maps

$$\mathbb{G}_{\mathrm{m}}(X) \times \mathbb{G}_{\mathrm{m}}(Y) \xrightarrow{\boxtimes} \mathbb{G}_{\mathrm{m}}(X \times_F Y),$$
$$\operatorname{Pic}(X) \times \operatorname{Pic}(Y) \xrightarrow{\boxtimes} \operatorname{Pic}(X \times_F Y)$$

fail to be isomorphisms for algebraically closed fields F in general. However, the "error terms" are under control, cf. (5.9) and (5.10), and the homotopy fixed points of the action by a partial Frobenius on these error terms do vanish. From this perspective, Theorem 1.5 shares a kinship with a statement known as *Drinfeld's lemma* [9, Proposition 1.1], which rectifies the failure of étale fundamental groups of $\mathbb{F}_p$-schemes to satisfy a Künneth formula, and is a key point in the Langlands program over fields such as $\mathbb{F}_p(t)$ or $\mathbb{Q}_p$ [14,27].

One may ask whether Frobenius motivic cohomology is rigid for $n \geq 2$ as well. In that direction, we recall the following result, which is also in the vicinity of Drinfeld's lemma [20, Lemma 4.7]: for a finite type $\mathbb{F}_p$-scheme X, and any extension of algebraically closed fields $F \subset E$ in characteristic $p > 0$, the base change

$$\{\text{constructible subsets of } X \times F\} \to \{\text{constructible subsets of } X \times E\}$$

induces a bijection after restricting to those subsets that are set-theoretically stable under $\mathrm{id}_X \times \operatorname{Frob}_E$, resp. $\mathrm{id}_X \times \operatorname{Frob}_F$. In fact, these are precisely the subsets descending to X.

This result gives control over the degree-wise kernel of $(\mathrm{id}_X \times \mathrm{Frob}_F)^*$ on Bloch's cycle complex $\mathrm{R}\Gamma(X \times F, \mathbb{Z}(n))$. The obstacle towards an analogue of Theorem 1.5 for $n \geq 2$ is a similar control of the cokernel. For $F = \overline{\mathbb{F}}_p$, this cokernel was studied by Geisser [16], who pointed out the relation of these groups to the Parshin conjecture.

We conclude this paper with a short appendix on the homotopy fixed points of the partial Frobenius on topological Hochschild homology. Frobenius THH is again rigid (cf. Proposition A.1), with the Artin–Schreier sequence playing the rôle of the Kummer sequence in the context of Frobenius K-theory.

2. Rigid functors

Let AffSch_k be the category of affine schemes over a field k. We identify its opposite category $\mathrm{AffSch}_k^{\mathrm{op}}$ with the category of commutative k-algebras CAlg_k whenever convenient. Let Ani be the ∞-category of anima (also called spaces or ∞-groupoids).

Definition 2.1. A functor $\kappa\colon \mathrm{AffSch}_k^{\mathrm{op}} \to \mathrm{Ani}$ is called *rigid* if, for any extension $F \subset E$ of algebraically closed fields over k, the map $\kappa(F) \to \kappa(E)$ is an equivalence.

Since equivalences in Ani are detected on homotopy groups, a functor κ is rigid if and only if $\pi_n(\kappa(F), \star) \to \pi_n(\kappa(E), \star)$ is a bijection for all base points $\star \to \kappa(F)$. Recall that a functor is called *finitary* if it preserves filtered colimits.

Lemma 2.2 (Suslin). *Let* $\kappa\colon \mathrm{AffSch}_k^{\mathrm{op}} \to \mathrm{Ani}$ *be a finitary functor. Then, the following are equivalent:*

(1) *The functor* κ *is rigid.*

(2) *For any algebraically closed field* F *over* k*, any base point* $\star \to \kappa(F)$*, any connected, smooth, affine* F*-curve* C*, any* $n \in \mathbb{Z}_{\geq 0}$ *and any* $\alpha \in \pi_n(\kappa(C), \star)$*, there exists a non-empty open (automatically affine) subset* $U_\alpha \subset C$ *such that the map* $U_\alpha(E) \to \pi_n(\kappa(E), \star)$*,* $c \mapsto c^*\alpha$ *is constant for any algebraically closed field extension* E *over* F*.*

Proof. Since the formation of homotopy groups commutes with filtered colimits, we may and do assume that κ takes values in the category of sets.

Let F be an algebraically closed field over k. Then, the map $\kappa(F) \to \kappa(E)$ is injective for any F-algebra E: by finitariness of κ, and expressing E as the filtered colimit of the finitely generated F-subalgebras, we may assume E is a finitely generated F-algebra. Since F is algebraically closed, Hilbert's Nullstellensatz supplies a section of the structural map $F \to E$ implying the injectivity of $\kappa(F) \to \kappa(E)$. This uses neither (1) nor (2).

Now assume (1) holds. Let $C \to \operatorname{Spec} F$ and $\alpha \in \kappa(C)$ be as in (2). Let K be an algebraic closure of the function field of C. Then, $K = \mathrm{colim}_{\widetilde{C} \to C} \Gamma(\widetilde{C}, \mathcal{O})$ is a filtered colimit where $\widetilde{C}$ ranges over the connected, smooth, affine F-curves equipped with a flat (necessarily generically finite) map to C. Using that κ is finitary, we get maps of sets

$$\kappa(F) \to \kappa(C) \to \kappa(K) = \operatorname*{colim}_{\widetilde{C} \to C} \kappa(\widetilde{C}),$$

whose composition is bijective by (1). Thus, there exists some $\widetilde{C} \to C$ such that the pullback $\alpha|_{\widetilde{C}}$ lies in the image of $\kappa(F) \to \kappa(\widetilde{C})$. In particular, the map

$$\widetilde{C}(E) \to \kappa(E), \quad \tilde{c} \mapsto \tilde{c}^*(\alpha|_{\widetilde{C}})$$

is constant for any F-algebra E where $\widetilde{C}(E)$ denotes the set of F-maps $\operatorname{Spec} E \to \widetilde{C}$. Let U_α be the (necessarily open by flatness) image of $\tilde{C} \to C$. Then, $\tilde{c}^*(\alpha|_{\widetilde{C}}) = c^*\alpha$ for

$$\widetilde{C}(E) \to U_\alpha(E), \quad \tilde{c} \mapsto c.$$

So, (2) follows from the surjectivity of $\widetilde{C}(E) \twoheadrightarrow U_\alpha(E)$ for algebraically closed fields E.

Conversely, assume that (2) holds. Let $F \subset E$ be an algebraically closed field extension. It remains to show that the injection $\kappa(F) \hookrightarrow \kappa(E)$ is surjective. By finitariness of κ, we reduce to fields E of finite transcendence degree over F, then to transcendence degree 1 by induction. Again, by finitariness of κ, any element $\alpha \in \kappa(E)$ arises by pullback from some $\alpha_C \in \kappa(C)$ for a connected, smooth, affine F-curve C whose algebraically closed function field identifies with E. Denote by $\eta \in C(E)$ the canonical map. Using (2) and replacing C by U_α if necessary, we may and do assume that the map $C(E) \to \kappa(E)$, $c \mapsto c^*\alpha_C$ is constant. Applying this to $c = \eta$ gives $\eta^*\alpha_C = \alpha$ by construction. Hence, choosing any section $\operatorname{Spec} F \to C$ and looking at the composition

$$\operatorname{Spec} E \to \operatorname{Spec} F \to C$$

implies (1). ∎

In practice the following corollary is useful.

Corollary 2.3. *Let $\kappa\colon \mathrm{AffSch}_k^{\mathrm{op}} \to \mathrm{Ani}$ be a finitary functor such that for any algebraically closed field F over k, any connected, smooth, affine F-curve C and any points $c_0, c_1 \in C(F)$, the maps*

$$\pi_n\big(\kappa(c_i)\big)\colon \pi_n\big(\kappa(C), \star\big) \to \pi_n\big(\kappa(F), \star\big), \quad i = 0, 1$$

agree for all $n \in \mathbb{Z}_{\geq 0}$ and all base points $\star \to \kappa(F)$. Then, κ is rigid.

Proof. Lemma 2.2 (2) is satisfied with $U_\alpha = C$ for all $\alpha \in \pi_n(\kappa(C))$, noting that F-maps $\operatorname{Spec} E \to C$ are the same as sections of the base change $C \times_F E \to \operatorname{Spec} E$. ∎

Remark 2.4. Definition 2.1, Lemma 2.2 and Corollary 2.3 admit obvious analogues for finitary functors $\kappa\colon \mathrm{AffSch}_k^{\mathrm{op}} \to \mathrm{Sp}$ with values in the ∞-category of spectra Sp, i.e., the stabilization of Ani. Indeed, in the proof of Lemma 2.2 one uses the non-degeneracy of the t-structure on Sp and the commutation of π_n with filtered colimits to reduce to functors valued in the category of abelian groups Ab. The rest of the argument is the same. Likewise, for finitary functors $\kappa\colon \mathrm{AffSch}_k^{\mathrm{op}} \to \mathrm{D}(\mathbb{Z})$ valued in the derived category of abelian groups.

Example 2.5. The following (non-)examples are of interest throughout:

(1) Let $M, N \in \mathrm{SH}(k)$ be motivic spectra, see Section 4 for the definition of SH. Then, for any $n \in \mathbb{Z}$ prime to the characteristic of k, the functor $\mathrm{CAlg}_k \to \mathrm{Sp}$ given for any k-algebra R by the cofiber

$$\mathrm{cofib}\left(\mathrm{Map}_{\mathrm{SH}(R)}(M_R, N_R) \xrightarrow{n\cdot} \mathrm{Map}_{\mathrm{SH}(R)}(M_R, N_R)\right)$$

is rigid [39]. The functor is finitary if M is compact.

(2) K-theory defines a finitary functor $K\colon \mathrm{CAlg}_k \to \mathrm{Sp}$ that is not rigid: $K_1(F) = F^\times$ highly depends on F. In particular, the functor

$$\mathbb{G}_{\mathrm{m}}\colon \mathrm{CAlg}_k \to \mathrm{Ab}, \quad R \mapsto R^\times$$

is not rigid, evidently. This plays well with the fact that for $C = \mathbb{A}^1_F - \{0\}$ and a point $c \in C(F) = F^\times$, the map

$$\mathbb{G}_{\mathrm{m}}(C) = \left(F[t^\pm]\right)^\times = F^\times \times t^{\mathbb{Z}} \xrightarrow{t \mapsto c} F^\times$$

depends on the choice of c. In the subsequent sections, we show that the homotopy fixed points under the (partial) Frobenius endomorphism are rigid.

3. Fixed point categories

Definition 3.1. For an endofunctor of an ∞-category $\varphi\colon \mathcal{C} \to \mathcal{C}$ the *fixed point category of φ on $\mathcal{C}$* is the ∞-category

$$\mathcal{C}^\varphi := \lim\left(\mathcal{C} \underset{\mathrm{id}}{\overset{\varphi}{\rightrightarrows}} \mathcal{C}\right) = \mathcal{C} \times_{\varphi \times \mathrm{id}, \mathcal{C} \times \mathcal{C}, \Delta} \mathcal{C}.$$

Remark 3.2. Objects in $\mathcal{C}^\varphi$ are triples $(c_1 \in \mathcal{C}, c_2 \in \mathcal{C}, (c_1, c_1) \cong (\varphi c_2, c_2))$. Any such object is isomorphic to one of the form $(c, c, (c \cong \varphi c, \mathrm{id}_c))$, i.e., one can think of objects as pairs $(c, c \cong \varphi c)$. The anima (or space) of maps between two such objects $(c, \lambda\colon c \cong \varphi c)$ and $(c', \lambda'\colon c' \cong \varphi c')$ is the equalizer in Ani of the following two maps

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(c, c') \arrow[r, "\varphi"'] \arrow[rr, bend left, "\lambda'_*"] & \mathrm{Map}_{\mathcal{C}}(\varphi c, \varphi c') \arrow[r, "\lambda^*"'] & \mathrm{Map}_{\mathcal{C}}(c, \varphi c'). \end{tikzcd} \tag{3.1}$$

Thus, maps in $\mathcal{C}^\varphi$ are maps $f\colon c \to c'$ in $\mathcal{C}$ together with a commutative diagram:

$$\begin{tikzcd} c \arrow[r, "\lambda", "\cong"'] \arrow[d, "f"] & \varphi c \arrow[d, "\varphi f"] \\ c' \arrow[r, "\lambda'", "\cong"'] & \varphi c'. \end{tikzcd} \tag{3.2}$$

Example 3.3. For $\varphi = \mathrm{id}_{\mathcal{C}}$, one has

$$\mathrm{Map}_{\mathcal{C}^{\mathrm{id}}}(\mathrm{triv}\, c, \mathrm{triv}\, c') = \mathrm{Map}_{\mathcal{C}}(c, c') \oplus \mathrm{Map}_{\mathcal{C}}(c, c')[-1],$$

since the two maps $\lambda^* \circ \varphi$ and λ'_* in (3.1) agree. In addition, there is a functor

$$\mathrm{triv}\colon \mathcal{C} \to \mathcal{C}^{\mathrm{id}}, \quad c \mapsto (c, c \xrightarrow{\mathrm{id}} c),$$

which is not fully faithful due to the shifted copy of the mapping spectrum.

Remark 3.4. The fixed point category $\mathcal{C}^{\varphi}$ can also be regarded as the limit of the functor $\Phi_{\mathcal{C},\varphi}\colon B\mathbb{N} \to \mathrm{Cat}_\infty$ sending $* \mapsto \mathcal{C}$ and $\mathbb{N} \ni 1 \mapsto \varphi$. In particular, an equivalence of functors $\beta\colon \varphi \to \varphi'$ gives rise to an isomorphism of diagrams $\Phi_{\mathcal{C},\varphi} \cong \Phi_{\mathcal{C},\varphi'}$, and therefore an equivalence

$$\mathcal{C}^{\varphi} \underset{\cong}{\xrightarrow{\beta}} \mathcal{C}^{\varphi'}, \quad \left(c, c \underset{\cong}{\xrightarrow{\lambda}} \varphi c\right) \mapsto \left(c, c \underset{\cong}{\xrightarrow{\lambda}} \varphi c \underset{\cong}{\xrightarrow{\beta(c)}} \varphi' c\right).$$

So, given an equivalence $\beta\colon \mathrm{id} \xrightarrow{\cong} \varphi$, there is a "twisting" functor

$$\mathrm{tw} := \mathrm{tw}_\beta\colon \mathcal{C} \xrightarrow{\mathrm{triv}} \mathcal{C}^{\mathrm{id}} \xrightarrow{\cong} \mathcal{C}^{\varphi}.$$

Remark 3.5. If $\mathcal{C}$ is a presentably symmetric monoidal (i.e., it is presentable, symmetric monoidal and the $\otimes$-product commutes with colimits in each variable), stable ∞-category and φ a symmetric monoidal endofunctor, then so is $\mathcal{C}^{\varphi}$. Indeed, the forgetful functors

$$\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{St}}) \to \mathrm{Pr}^{\mathrm{St}} \to \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Cat}_\infty$$

preserve limits. See [30, Proposition 3.2.2.1] for the first arrow, [30, Proposition 4.8.2.18] for the second one, and [29, Proposition 5.5.3.13] for the last one. In addition, if $\mathcal{C}$ is compactly generated and φ preserves compact objects, then $\mathcal{C}^{\varphi}$ is compactly generated [20, Proof of Lemma 2.5]. In the situation of Example 3.3 (or Remark 3.4, where β is supposed to be an equivalence of symmetric monoidal colimit-preserving functors) the functors triv (resp. tw) will again be functors in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{St}})$.

4. Frobenius stable homotopy category

For a scheme S, we denote by $\mathrm{SH}(S)$ the *stable* $\mathbb{A}^1$*-homotopy category*, i.e., the presentably symmetric monoidal ∞-category given by the $\mathbb{P}^1$-stabilization of $\mathbb{A}^1$-invariant Nisnevich ∞-sheaves of spectra on the category Sm_S of smooth S-schemes, cf. the discussion around [23, equation (C.11)], and also [2, Appendix A] for the definition of the Nisnevich topology in full generality. The construction of SH gives a functor

$$M\colon \mathrm{Sm}_S \to \mathrm{SH}(S), \tag{4.1}$$

which associates to a smooth S-scheme X its *motive* $M(X)$. If S is quasi-compact and quasi-separated (qcqs), then $\mathrm{SH}(S)$ is compactly generated [23, Proposition C.12], up to desuspensions, by the motives $M(X)$ of *finitely presented*, smooth S-schemes X. If S is qcqs of finite Krull dimension, then every Nisnevich sheaf is a hypersheaf [5, Theorem 1.7], so the above definition of $\mathrm{SH}(S)$ agrees with more classical definitions using model categorical language [38, Section 2.4.1]. The construction of the stable $\mathbb{A}^1$-homotopy category can be upgraded to a functor

$$\mathrm{SH}\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{St}})$$

using $*$-pullback functoriality and further to a six functors formalism [24, 25].

We use the following standard notation for the *motivic spheres*: let $\mathbb{S}^{1,1} \in \mathrm{SH}(S)$ be the object represented by $\mathbb{G}_{\mathrm{m},S}$, and denote by $\mathbb{S}^{1,0} \in \mathrm{SH}(S)$ the suspension of the monoidal unit. By definition of $\mathrm{SH}(S)$, both objects are dualizable. So, the definition

$$\mathbb{S}^{n+r,n} := (\mathbb{S}^{1,1})^{\otimes n} \otimes (\mathbb{S}^{1,0})^{\otimes r}$$

makes sense for all $n, r \in \mathbb{Z}$. Note that $\mathbb{S}^{0,0} = 1$ is the monoidal unit in $\mathrm{SH}(S)$.

4.1. The Frobenius stable homotopy category

Fix a prime number p. For an $\mathbb{F}_p$-scheme S, we denote by $\mathrm{Frob}_S\colon S \to S$ the Frobenius endomorphism given by $f \mapsto f^p$ on functions. If S is understood, we abbreviate Frob_S simply by Frob. The pullback Frob^* induces a symmetric monoidal endofunctor of $\mathrm{SH}(S)$, so the setting of Section 3, in particular Remark 3.5, applies.

Definition 4.1. The *Frobenius stable $\mathbb{A}^1$-homotopy category of S* is the fixed point category under the pullback along the Frobenius map:

$$\mathrm{SH}(S/\mathrm{Frob}) := \mathrm{SH}(S)^{\mathrm{Frob}^*}$$

Remark 4.2. We use an analogous notation also for other ∞-categories:

(1) If P is a set of prime numbers, then we denote by $\mathrm{SH}(S)[P^{-1}]$ the full subcategory in $\mathrm{SH}(S)$ of P^{-1}-localized objects, i.e., $M \in \mathrm{SH}(S)$ with $M \otimes 1/\ell = 0$ for all $\ell \in P$. The inclusion is right adjoint to the localization functor

$$\mathrm{SH}(S) \to \mathrm{SH}(S)[P^{-1}],$$

see e.g. [31, Section 3.2] for a general discussion. Further, P^{-1}-localization commutes with taking Frobenius fixed points, and we denote by $\mathrm{SH}(S/\mathrm{Frob})[P^{-1}]$ the resulting full subcategory of $\mathrm{SH}(S/\mathrm{Frob})$. We apply this to the cases where $P = \{p\}$ and where P is the set of all primes. The resulting categories of p^{-1}-localized and rational objects are denoted by $\mathrm{SH}(S/\mathrm{Frob})[p^{-1}]$ and $\mathrm{SH}(S/\mathrm{Frob})_{\mathbb{Q}}$ respectively.

(2) Similarly, we consider the category $\mathrm{DM}(S/\mathrm{Frob})$, where DM denotes the category of Beilinson motives with rational coefficients [4].

Remark 4.3. The formation of $\mathrm{SH}(S/\mathrm{Frob})$ is functorial since the Frobenius endomorphism is so. That is, for a map $s\colon S' \to S$, the $*$-pullback induces a symmetric monoidal functor

$$\begin{aligned} s^*\colon \mathrm{SH}(S/\mathrm{Frob}) &\to \mathrm{SH}(S'/\mathrm{Frob}),\\ \big(M, M \xrightarrow[\cong]{\lambda} \mathrm{Frob}^* M\big) &\mapsto \big(s^*M, s^*M \xrightarrow[\cong]{s^*\lambda} s^*\mathrm{Frob}^* M = \mathrm{Frob}^* s^* M\big). \end{aligned}$$

Construction 4.4. Remark 4.3 applies to the structural map $s\colon S \to \operatorname{Spec}\mathbb{F}_p$ and gives the symmetric monoidal functor

$$\begin{aligned} \mathrm{can}_S\colon \mathrm{SH}(\mathbb{F}_p) \xrightarrow{\mathrm{triv}} \mathrm{SH}(\mathbb{F}_p/\mathrm{id}) &\xrightarrow{s^*} \mathrm{SH}(S/\mathrm{Frob}),\\ M &\mapsto (s^*M, s^*M \xrightarrow{\mathrm{id}} s^*M = \mathrm{Frob}^* s^*M). \end{aligned}$$

using $\mathrm{Frob} = \mathrm{id}$ over $\mathbb{F}_p$ and so $s \circ \mathrm{Frob} = s$. We also use the same notation for the variants in Remark 4.2.

4.2. Twisted Frobenius objects

If $t\colon T \to S$ is a morphism of $\mathbb{F}_p$-schemes, we consider throughout the usual diagram involving the relative Frobenius $\mathrm{Frob}_{T/S}$ where the square is cartesian:

$$\begin{tikzcd} T \arrow[r, "\mathrm{Frob}_{T/S}"] \arrow[dr, "t"'] & T' \arrow[r] \arrow[d, "t'"] & T \arrow[d, "t"] \\ & S \arrow[r, "\mathrm{Frob}_S"] & S. \end{tikzcd} \tag{4.2}$$

Example 4.5. The relative Frobenius $\mathrm{Frob}_{\mathbb{G}_{\mathrm{m},S}/S} = \mathrm{Frob}_{\mathbb{G}_{\mathrm{m},\mathbb{F}_p}} \times \mathrm{id}_S$ agrees with the p-multiplication of the S-group scheme $\mathbb{G}_{\mathrm{m},S}$. The induced map on $\mathbb{S}^{1,1}$ is the multiplication by

$$p_\varepsilon := \sum_{i=0}^{p-1} \big\langle (-1)^i \big\rangle \in K_0^{MW}(\mathbb{F}_p), \tag{4.3}$$

[1, Corollaire C.9 and proof of Proposition C.5]. After p^{-1}-localization, this element is invertible, see [1, Section C] and, e.g., [8, Example 2.1.5, Proposition 2.3.1]. Note that $\langle -1 \rangle = \langle 1 \rangle = 1$ if -1 is a square in $\mathbb{F}_p^\times$, i.e., if $p = 2$ or $p \equiv 1 \bmod 4$. In this case, one has $p_\varepsilon = p$.

Proposition 4.6. *Let S be an $\mathbb{F}_p$-scheme. Then, there is a canonical isomorphism of symmetric monoidal endofunctors on* $\mathrm{SH}(S)[p^{-1}]$*,*

$$\beta\colon \mathrm{id} \xrightarrow{\cong} \mathrm{Frob}^*$$

given on motives of smooth S-schemes T by the relative Frobenius maps:

$$\beta\big(\mathrm{M}(T)\big)\colon \mathrm{M}(T) \xrightarrow{\mathrm{Frob}_{T/S}} \mathrm{M}(T \times_{S,\mathrm{Frob}} S) = \mathrm{Frob}^* \mathrm{M}(T).$$

Proof. In order to construct β we use the universal property of SH, see [38]. Using the notation in (4.2) gives a functor

$$\mathrm{tw}\colon \mathrm{Sm}_S \to \mathrm{Fun}(\Delta^1, \mathrm{Sm}_S), \quad T \mapsto (T \to T')$$

whose evaluations at the two endpoints of Δ^1 are the identity, respectively Frob^*. This functor has a symmetric monoidal structure with respect to the pointwise monoidal structure on the target category. On the unstable $\mathbb{A}^1$-homotopy category $\mathrm{H}(S)$ (i.e., on $\mathbb{A}^1$-invariant Nisnevich sheaves of spectra), one has a symmetric monoidal functor $\mathrm{H}(S) \to \mathrm{Fun}(\Delta^1, \mathrm{H}(S))$ without inverting p. Its evaluation at the object represented by $\mathbb{G}_{\mathrm{m},S}$ is the multiplication with p_ε on $\mathbb{S}^{1,1} \in \mathrm{H}(S)$ (Example 4.5), which becomes invertible upon passing to p^{-1}-localizations. Thus its image in $\mathrm{Fun}(\Delta^1, \mathrm{SH}(S)[p^{-1}])$ is $\otimes$-invertible.

By the universal property of SH (Remark 4.7) it then descends to a symmetric monoidal functor

$$\mathrm{SH}(S)[p^{-1}] \to \mathrm{Fun}\big(\Delta^1, \mathrm{SH}(S)[p^{-1}]\big)$$

whose evaluations at the two endpoints of Δ^1 are again id and Frob^*. Therefore, we obtain a natural transformation of symmetric monoidal functors β as stated.

It remains to show that $\beta(M)$ is an equivalence for all $M \in \mathrm{SH}(S)$. It suffices to do this for $M = \mathrm{M}_S(T) = t_! t^! 1_S$ for some smooth $t\colon T \to S$ as above. The map $\mathrm{Frob}_{T/S}$ is a universal homeomorphism [45, Tag 0CCB]. So, the functor $\mathrm{Frob}^*_{T/S}\colon \mathrm{SH}(T')[p^{-1}] \to \mathrm{SH}(T)[p^{-1}]$ is an equivalence [12] with inverse $\mathrm{Frob}_{T/S,*} = \mathrm{Frob}_{T/S,!}$, hence $\mathrm{Frob}^*_{T/S} = \mathrm{Frob}^!_{T/S}$ as well. We have $t = t' \circ \mathrm{Frob}_{T/S}$ with notation as in (4.2). So, the counit map

$$\mathrm{Frob}_{T/S,!}\,\mathrm{Frob}^!_{T/S} \to \mathrm{id},$$

which is an isomorphism, induces the isomorphism

$$\mathrm{M}_S(T) = t_! t^! 1_S = t'_!\,\mathrm{Frob}_{T/S,!}\,\mathrm{Frob}^!_{T/S}\, t'^! 1_S \xrightarrow{\cong} t'_! t'^! 1_S = \mathrm{M}_S(T') = \mathrm{Frob}^*\,\mathrm{M}_S(T).$$

It agrees with the map induced by $\mathrm{Frob}_{T/S}\colon T \to T'$ under the functor (4.1) on motives. ∎

Remark 4.7. We thank the anonymous referee for pointing out the following observation and minor correction to [38, Corollary 2.39] (and similarly [2, Lemma 4.1]): the universal property of the symmetric monoidal localization $C[X^{-1}]$ of a stable presentably symmetric monoidal ∞-category C at a symmetric object $X \in C$ is that the composition with $C \to C[X^{-1}]$ induces an equivalence, for any symmetric monoidal ∞-category D,

$$\mathrm{Fun}^{\otimes}\big(C[X^{-1}], D\big) \xrightarrow{\cong} \mathrm{Fun}^{\otimes}(C, D) \times_{\mathrm{ev}_X, D} \mathrm{Pic}(D),$$

where $\mathrm{Pic}(D)$ denotes the ∞-groupoid of invertible objects in D.

Indeed, any natural transformation of symmetric monoidal functors $F \to G$ of functors $C[X^{-1}] \to D$ is necessarily an *isomorphism* when evaluated on an invertible (or, more generally, dualizable) object in $C[X^{-1}]$. In particular, this is true for the evaluation at X itself.

Applying Remark 3.4 to Proposition 4.6, we get the following result.

Corollary 4.8. *For any $\mathbb{F}_p$-scheme S, there is the symmetric monoidal* "twisting" *functor*

$$\mathrm{tw}\colon \mathrm{SH}(S)[p^{-1}] \xrightarrow{\mathrm{triv}} \mathrm{SH}(S/\mathrm{id})[p^{-1}] \xrightarrow[\cong]{\beta} \mathrm{SH}(S/\,\mathrm{Frob})[p^{-1}].$$

For a smooth $\mathbb{F}_p$-scheme X, one has

$$\mathrm{tw}\,\big(\mathrm{M}(X)\big) = \big(\mathrm{M}(X \times S), \mathrm{M}(X \times S) \xrightarrow{\mathrm{Frob}_X \times \mathrm{id}_S} \mathrm{M}(X \times S)\big).$$

Remark 4.9. The two functors

$$\mathrm{SH}(\mathbb{F}_p)[p^{-1}] \underset{\mathrm{tw}}{\overset{\mathrm{triv}}{\rightrightarrows}} \mathrm{SH}(\mathbb{F}_p/\mathrm{id})[p^{-1}]$$

do not agree. Indeed, $\mathrm{tw}(\mathbb{S}^{r+n,n}) = (\mathbb{S}^{r+n,n}, p_\varepsilon^n \cdot \mathrm{id})$, by Example 4.5.

Remark 4.10. For a map $s\colon S' \to S$, the pullback functor from Remark 4.3 together with the twisting functors give a diagram

$$\begin{tikzcd}
\mathrm{SH}(S)[p^{-1}] \arrow[r, "\mathrm{tw}"] \arrow[d, "s^*"] & \mathrm{SH}(S/\,\mathrm{Frob})[p^{-1}] \arrow[d, "s^*"] \\
\mathrm{SH}(S')[p^{-1}] \arrow[r, "\mathrm{tw}"] & \mathrm{SH}(S'/\,\mathrm{Frob})[p^{-1}],
\end{tikzcd}$$

which commutes since forming relative Frobenii is functorial.

5. Frobenius rigidity

Recall the functor $\mathrm{can}_S\colon \mathrm{SH}(\mathbb{F}_p)[p^{-1}] \to \mathrm{SH}(S/\,\mathrm{Frob})[p^{-1}]$ from Construction 4.4.

Definition 5.1. An ordered pair of objects $M, N \in \mathrm{SH}(\mathbb{F}_p)[p^{-1}]$ is *Frobenius rigid* if the functor $\mathrm{AffSch}_{\mathbb{F}_p}^{\mathrm{op}} \to \mathrm{Sp}$ given by

$$S \mapsto \mathrm{Map}_{\mathrm{SH}(S/\,\mathrm{Frob})[p^{-1}]}(\mathrm{can}_S\, M, \mathrm{can}_S\, N)$$

is rigid. That is, if for any extension of algebraically closed fields $f\colon \mathrm{Spec}\, E \to \mathrm{Spec}\, F$ in characteristic $p > 0$ the induced map

$$\mathrm{Map}_{\mathrm{SH}(F/\,\mathrm{Frob})[p^{-1}]}(\mathrm{can}_F\, M, \mathrm{can}_F\, N) \xrightarrow{f^*} \mathrm{Map}_{\mathrm{SH}(E/\,\mathrm{Frob})[p^{-1}]}(\mathrm{can}_E\, M, \mathrm{can}_E\, N) \quad (5.1)$$

is an equivalence.

Remark 5.2. This definition suggests the question to what extent the functor

$$f^*\colon \mathrm{SH}(F/\,\mathrm{Frob})[p^{-1}] \to \mathrm{SH}(E/\,\mathrm{Frob})[p^{-1}]$$

is fully faithful. On the whole of $\mathrm{SH}(F/\mathrm{Frob})[p^{-1}]$, f^* is *not* fully faithful, however. Indeed, using the twisting functor tw, both categories are equivalent to $\mathrm{SH}(-)[p^{-1}]^{\mathrm{id}}$. By Example 3.3 (and given that the functor f^* then identifies with the usual f^*, by Remark 4.10), f^* is not fully faithful, compare also Example 2.5 (2).

Remark 5.3. If the pair M, N is Frobenius rigid, then the invariance of $\mathrm{SH}(-)[p^{-1}]$ under perfection [12] implies a similar rigidity property for any extension of separably (as opposed to algebraically) closed fields.

Recall the variants of the Frobenius fixed point categories from Remark 4.2.

Lemma 5.4. *Let $M, N \in \mathrm{SH}(\mathbb{F}_p)[p^{-1}]$. The following are equivalent:*

(1) *The pair M, N is Frobenius rigid.*

(2) *Their rationalizations $M_{\mathbb{Q}}, N_{\mathbb{Q}}$ satisfy the property of* (5.1) *in* $\mathrm{SH}(-/\mathrm{Frob})_{\mathbb{Q}}$.

(3) *The Beilinson motives associated with $M_{\mathbb{Q}}, N_{\mathbb{Q}}$ satisfy the property of* (5.1) *in* $\mathrm{DM}(-/\mathrm{Frob})$.

Proof. Let A be the fiber of the map in (5.1). By definition, p-multiplication is invertible. The arithmetic fracture square [31, (3.17)] implies that $A = 0$ if and only if both its rationalization $A_{\mathbb{Q}} = 0$ and $A/n := \mathrm{cofib}(A \xrightarrow{n\cdot} A) = 0$ for all n prime to p. Röndigs–Østvær's version of Suslin rigidity for SH, i.e., the full faithfulness of $\mathrm{SH}(F)/n \to \mathrm{SH}(E)/n$ [39], ensures that the latter holds for any M, N as above. This proves (1)⇔(2).

For any field containing a square root of -1, in particular for $K = E$ and $K = F$, $\mathrm{SH}(K)_{\mathbb{Q}} = \mathrm{DM}(K)$ [4, Corollary 16.2.14]. Again passing to homotopy fixed points under Frobenius pullback shows the equivalence of (2) and (3). ■

5.1. Frobenius rigidity for cellular objects

Recall, e.g. from [10, Section 2.8] that the subcategory

$$\mathrm{SH}(S)_{\mathrm{cell}} \subset \mathrm{SH}(S)$$

of *cellular objects* is the stable full subcategory generated under colimits by the spheres $\mathbb{S}^{r+n,n}$, which lie in the essential image of $\mathrm{SH}(\mathbb{F}_p) \to \mathrm{SH}(S)$, for all $r, n \in \mathbb{Z}$. These objects are dualizable and, if S is qcqs, also compact.

Theorem 5.5. *Any pair of p^{-1}-localized cellular objects $M, N \in \mathrm{SH}(\mathbb{F}_p)_{\mathrm{cell}}[p^{-1}]$ is Frobenius rigid.*

Proof. As $\mathrm{SH}(\mathbb{F}_p)_{\mathrm{cell}}$ is compactly generated by dualizable objects, we may assume $M = 1_{\mathbb{F}_p}$. Again using compactness, we may then also assume that $N = \mathbb{S}^{r+n,n}$ is a compact generator of $\mathrm{SH}(\mathbb{F}_p)_{\mathrm{cell}}$. Let $F \subset E$ be an extension of algebraically closed fields in characteristic $p > 0$. We have to prove that the map between the p^{-1}-localized mapping spectra

$$\mathrm{Map}_{\mathrm{SH}(F/\mathrm{Frob})}(1, \mathrm{can}_F \mathbb{S}^{r+n,n})[p^{-1}] \to \mathrm{Map}_{\mathrm{SH}(E/\mathrm{Frob})}(1, \mathrm{can}_E \mathbb{S}^{r+n,n})[p^{-1}] \quad (5.2)$$

is an isomorphism for all $r, n \in \mathbb{Z}$. Since $\mathbb{S}^{1,0}$ is the object associated with the constant presheaf with values the circle, we may assume $r = 0$.

Letting S denote either Spec F or Spec E, we will show that these mapping spectra are insensitive to the choice of Spec F or Spec E. By definition, $\mathrm{can}_S 1_{\mathbb{F}_p} = (1_S, \mathrm{can}_{1_S}\colon 1_S \cong \mathrm{Frob}_S^* 1_S)$, see Construction 4.4. Abbreviating $\mathbb{S} := s^*\mathbb{S}^{n,n}$, the same description holds for $\mathrm{can}_S \mathbb{S}^{n,n} = (\mathbb{S}, \mathrm{can}_{\mathbb{S}}\colon \mathbb{S} \cong \mathrm{Frob}_S^* \mathbb{S})$. We have the following canonical identifications, where all mapping spectra appearing at the right are in $\mathrm{SH}(S)[p^{-1}]$:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{SH}(S/\mathrm{Frob})[p^{-1}]}(1, \mathrm{can}_S \mathbb{S}^{n,n}) &= \lim \Big(\mathrm{Map}(1, \mathbb{S}) \overset{\mathrm{Frob}_S^*(-)\circ \mathrm{can}_{1_S}}{\underset{\mathrm{can}_{\mathbb{S}}\circ -}{\rightrightarrows}} \mathrm{Map}(1, \mathrm{Frob}_S^* \mathbb{S}) \Big) \\
&= \lim \Big(\mathrm{Map}(1, \mathbb{S}) \overset{\mathrm{can}_{\mathbb{S}}^{-1}\circ \mathrm{Frob}_S^*(-)\circ \mathrm{can}_{1_S}}{\underset{\mathrm{id}}{\rightrightarrows}} \mathrm{Map}(1, \mathbb{S}) \Big) \\
&= \lim \Big(\mathrm{Map}(1, \mathbb{S}) \overset{\mathrm{can}_{\mathbb{S}}^{-1}\circ \beta(\mathbb{S})\circ -}{\underset{\mathrm{id}}{\rightrightarrows}} \mathrm{Map}(1, \mathbb{S}) \Big) \\
&= \lim \Big(\mathrm{Map}(1, \mathbb{S}) \overset{p_\varepsilon^n \cdot -}{\underset{\mathrm{id}}{\rightrightarrows}} \mathrm{Map}(1, \mathbb{S}) \Big).
\end{aligned}$$

The first equality follows from (3.1), the second by postcomposing with $\mathrm{can}_{\mathbb{S}}^{-1}$, the equality $\mathrm{Frob}_S^*(-) \circ \mathrm{can}_{1_S} = \beta(\mathbb{S})$ from $\mathrm{can}_{1_S} = \beta(1_S)$ and the functoriality of β (Proposition 4.6) and the last from Example 4.5, according to which the composite $\mathrm{can}_{\mathbb{S}}^{-1}\beta(\mathbb{S})$ equals $p_\varepsilon^n \cdot \mathrm{id}$. Thus, it remains to show that the fiber of the multiplication with $1 - p_\varepsilon^n$ on $\mathrm{Map}_{\mathrm{SH}(S)[p^{-1}]}(1, \mathbb{S}) = \mathrm{Map}_{\mathrm{SH}(S)}(1, \mathbb{S})[p^{-1}]$ is insensitive to replacing $S = \mathrm{Spec}\, F$ by $S = \mathrm{Spec}\, E$.

By Suslin rigidity (Lemma 5.4), we may consider the category of Beilinson motives $\mathrm{DM}(-)$ instead of $\mathrm{SH}(-)[p^{-1}]$. Then, each homotopy group of the associated mapping spectra is a $\mathbb{Q}$-vector space, so multiplication by $1 - p_\varepsilon^n \neq 0$ is an isomorphism for all $n \neq 0$. Therefore $\mathrm{Map}_{\mathrm{DM}(S/\mathrm{Frob})}(1, \mathrm{can}_S \mathbb{Q}(n)[n]) = 0$ in this case. For $n = 0$, already the mapping spectra $\mathrm{Map}_{\mathrm{DM}(S)}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}$ are independent of the chosen S. Passing to homotopy fixed points under the trivial Frobenius actions preserves that independence. ■

5.2. Frobenius stable homotopy groups

Recall that the *stable* $\mathbb{A}^1$*-homotopy groups* of a field F are defined as

$$\pi_{r,n}(F) := \mathrm{Hom}_{\mathrm{SH}(F)}(\mathbb{S}^{r+n,n}, 1),$$

where 1 denotes the monoidal unit.

Morel showed that these groups vanish for $r < 0$ [33, Theorem 4.9]. For $r = 0$ they are isomorphic to *Milnor–Witt* K*-groups* $K_{-n}^{MW}(F)$, which for algebraically closed fields reduce to the Milnor K-groups $K_{-n}^M(F)$. For odd primes p and any p-power q, the p^{-1}-localized groups $\pi_{1,n}(\mathbb{F}_q)[p^{-1}]$ have been computed in [36, Section 8.10]. We also have $K_n^M(\mathbb{F}_q) = 0$ for $n \geq 2$ [32, Example 1.5]. These computations, and the continuity of SH, which allows to pass to $\overline{\mathbb{F}}_p = \operatorname{colim} \mathbb{F}_q$, give the following results for $\pi_{r,n}(\overline{\mathbb{F}}_p)[p^{-1}]$ and odd primes p:

n	≤ -2	-1	0	1	2	≥ 2
$r = 1$	0	0	$(\mathbb{Z}/2)^{\oplus 2}$	$\mathbb{Z}/2$	$\mathbb{Z}/24[p^{-1}]$	0
$r = 0$	0	$\overline{\mathbb{F}}_p^\times$	$\mathbb{Z}[p^{-1}]$	0	0	0

Definition 5.6. Let F be a field of characteristic $p > 0$. The *p^{-1}-localized Frobenius stable $\mathbb{A}^1$-homotopy groups* are the groups

$$\pi_{r,n}(F/\operatorname{Frob})[p^{-1}] := \operatorname{Hom}_{\mathrm{SH}(F/\operatorname{Frob})[p^{-1}]}(\operatorname{can}_F \mathbb{S}^{r+n,n}, 1).$$

These groups appear in a long exact sequence:

$$\dots \pi_{r,n}(F/\operatorname{Frob})[p^{-1}] \to \pi_{r,n}(F)[p^{-1}] \xrightarrow{\mathrm{id}-\operatorname{Frob}} \pi_{r,n}(F)[p^{-1}] \to \pi_{r-1,n}(F/\operatorname{Frob})[p^{-1}] \dots$$

The Frobenius rigidity of cellular spectra (Theorem 5.5) implies the following computation.

Corollary 5.7. *The groups $\pi_{r,n}(F/\operatorname{Frob})[p^{-1}]$ are independent of the choice of an algebraically closed field F of characteristic $p > 0$. For small values of r, and odd primes p, the groups are given by*

n	≤ -2	-1	0	1	2	≥ 2
$r = 0$	0	$\mathbb{F}_p^\times$	$(\mathbb{Z}/2)^{\oplus 2} \oplus \mathbb{Z}[p^{-1}]$	$\mathbb{Z}/2$	$\mathbb{Z}/24[p^{-1}]$	0
$r = -1$	0	0	$\mathbb{Z}[p^{-1}]$	0	0	0

5.3. Frobenius K-theory

Definition 5.8. The *Frobenius K-theory spectrum* of S, with respect to an $\mathbb{F}_p$-scheme X, is defined as the equalizer in the ∞-category of spectra

$$K(X \times S/\operatorname{Frob}_S) := \lim \left(K(X \times S) \underset{\mathrm{id}}{\overset{(\mathrm{id}_X \times \operatorname{Frob}_S)^*}{\rightrightarrows}} K(X \times S) \right),$$

i.e., the homotopy fixed points of the pullback along the partial Frobenius $\mathrm{id}_X \times \operatorname{Frob}_S$. The homotopy groups of this spectrum, denoted by $K_n(X \times S/\operatorname{Frob}_S)$, appear in a long exact sequence

$$\dots K_n(X \times S/\operatorname{Frob}_S) \to K_n(X \times S) \xrightarrow{\mathrm{id}-\operatorname{Frob}_S^*} K_n(X \times S) \to K_{n-1}(X \times S/\operatorname{Frob}_S) \dots . \tag{5.3}$$

In order to relate these groups to the Frobenius fixed points on SH, we place the spectrum $\mathrm{KGL} \in \mathrm{SH}(\mathbb{F}_p)$ representing homotopy K-theory inside $\mathrm{SH}(S/\operatorname{Frob})$ as follows:

Definition 5.9. For a scheme S of characteristic $p > 0$, let $\mathrm{KGL}_S/\operatorname{Frob} := \operatorname{can}_S \mathrm{KGL} \in \mathrm{SH}(S/\operatorname{Frob})$.

By Construction 4.4, $\mathrm{KGL}_S/\,\mathrm{Frob}$ consists of the spectrum KGL_S together with the map

$$\mathrm{can}_{\mathrm{KGL}_S}\colon \mathrm{KGL}_S = s^*\mathrm{KGL} \xrightarrow{\mathrm{id}} s^*\mathrm{KGL} = \mathrm{Frob}_S^*\, s^*\mathrm{KGL},$$

where $s\colon S \to \operatorname{Spec}\mathbb{F}_p$ denotes the structure map. Being the image of a commutative algebra object in $\mathrm{SH}(\mathbb{F}_p)$, the object $\mathrm{KGL}_S/\,\mathrm{Frob}$ again has the structure of a commutative algebra object in $\mathrm{SH}(S/\,\mathrm{Frob})$. This object represents Frobenius K-theory as follows.

Lemma 5.10. *Let S be regular Noetherian, and let X be smooth of finite type over $\mathbb{F}_p$. Then, there is an isomorphism of spectra*

$$K(X \times S/\,\mathrm{Frob}_S) = \mathrm{Map}_{\mathrm{SH}(S/\,\mathrm{Frob})}\big(\mathrm{can}_S \mathrm{M}(X), \mathrm{KGL}_S/\,\mathrm{Frob}\big).$$

Proof. We have $s^*\mathrm{M}(X) = \mathrm{M}(X \times S)$. By the assumptions, $X \times S$ is regular, so that there is an identification of mapping spectra

$$\mathrm{Map}_{\mathrm{SH}(S)}(s^*\mathrm{M}(X), \mathrm{KGL}_S) = \mathrm{Map}_{\mathrm{SH}(X\times S)}(1, \mathrm{KGL}_{X\times S}) = K(X \times S).$$

By construction [4, Section 13.1], for an endomorphism φ of $X \times S$, such as $\varphi = \mathrm{id}_X \times \mathrm{Frob}_S$, the map

$$\mathrm{Map}_{\mathrm{SH}(X\times S)}(1, \mathrm{KGL}_{X\times S}) \xrightarrow{\varphi^*} \mathrm{Map}(\varphi^*1, \varphi^*\mathrm{KGL}_{X\times S}) \xrightarrow[\cong]{\mathrm{can}_{\mathrm{KGL}}^{-1}\circ(-)\circ\mathrm{can}_1} \mathrm{Map}(1, \mathrm{KGL}_{X\times S})$$

identifies with the pullback φ^* on the K-theory spectrum. (Here at the right $\mathrm{can}_\bullet$ denotes again the canonical isomorphisms coming from functoriality of $*$-pullback, see Construction 4.4). The following computation is analogous to the proof of Theorem 5.5, where $\mathrm{Map}_- := \mathrm{Map}_{\mathrm{SH}(-)}$:

$$\begin{aligned}
&\mathrm{Map}_{\mathrm{SH}(S/\,\mathrm{Frob})}\big(\mathrm{can}_S\mathrm{M}(X), \mathrm{can}_S\mathrm{KGL}\big)\\
&= \lim\Big(\mathrm{Map}_S\big(\mathrm{M}(X\times S), \mathrm{KGL}_S\big) \underset{\mathrm{can}_{\mathrm{KGL}}\circ -}{\overset{\mathrm{Frob}_S^*(-)\circ\mathrm{can}_{\mathrm{M}(X\times S)}}{\rightrightarrows}} \mathrm{Map}_S\big(\mathrm{M}(X\times S), \mathrm{Frob}_S^*\,\mathrm{KGL}\big)\Big)\\
&= \lim\Big(\mathrm{Map}_S\big(\mathrm{M}(X\times S), \mathrm{KGL}\big) \underset{\mathrm{id}}{\overset{\mathrm{can}_{\mathrm{KGL}}^{-1}\circ\mathrm{Frob}_S^*(-)\circ\mathrm{can}_{\mathrm{M}(X\times S)}}{\rightrightarrows}} \mathrm{Map}_S\big(\mathrm{M}(X\times S), \mathrm{KGL}\big)\Big)\\
&= \lim\Big(\mathrm{Map}_{X\times S}(1, \mathrm{KGL}_{X\times S}) \underset{\mathrm{id}}{\overset{\mathrm{can}_{\mathrm{KGL}}^{-1}\circ(\mathrm{id}_X\times\mathrm{Frob}_S)^*(-)\circ\mathrm{can}_{\mathrm{M}(X\times S)}}{\rightrightarrows}} \mathrm{Map}_{X\times S}(1, \mathrm{KGL}_{X\times S})\Big)\\
&= \lim\Big(K(X\times S) \underset{\mathrm{id}}{\overset{(\mathrm{id}_X\times\mathrm{Frob}_S)^*}{\rightrightarrows}} K(X\times S)\Big) =: K(X\times S/\,\mathrm{Frob}_S).
\end{aligned}$$

■

The following result asserts that the Frobenius acts so richly on the K-theory of (large enough) fields that hardly anything is fixed under Frobenius pullback.

Corollary 5.11. *Frobenius K-theory is rigid. That is, for an extension $F \subset E$ of algebraically closed fields in characteristic $p > 0$, the pullback map*

$$K(F/\,\mathrm{Frob}) \to K(E/\,\mathrm{Frob})$$

is an equivalence of spectra. The individual Frobenius K-groups are given by

$$K_n(F/\operatorname{Frob}) = \begin{cases} \mathbb{Z} & n = -1, 0, \\ \mathbb{F}_{p^i}^{\times} & n = 2i - 1 > 0, \\ 0 & \textit{else}. \end{cases} \tag{5.4}$$

Proof. The statement is clear for $n \leq 0$, cf. the discussion around (1.1). By [22, Theorem 5.4], the groups $K_n(F)$ are uniquely p-divisible for $n > 0$. Thus $K_n(F/\operatorname{Frob}) = K_n(F/\operatorname{Frob})[p^{-1}]$ for $n > 0$. The first statement now follows from the cellularity of KGL [10, Theorem 6.2], Lemma 5.10 for $X = \operatorname{Spec} \mathbb{F}_p$ and Theorem 5.5.

To see the concrete values in (5.4), we may assume $F = \overline{\mathbb{F}}_p$ and use Quillen's computation of $K(\overline{\mathbb{F}}_p)$ and its Frobenius action, as reported e.g., in [46, Section VI.1, p. 465]: as an abelian group $K_{2i-1}(\overline{\mathbb{F}}_p)$ is isomorphic to $\overline{\mathbb{F}}_p^{\times}$, with Frob^* acting by raising to the p^i-th power. Then, our statement follows from the Kummer sequence. ∎

Remark 5.12. As communicated to us by Georg Tamme, Corollary 5.11 can be proven directly by using that for $n > 0$, the map Frob^* on $K_n(F)$ agrees with the p-th Adams operation, which acts by multiplication with p^k on the k-th Adams eigenspace inside $K_n(F)_{\mathbb{Q}}$. Such an argument seems not applicable to cellular objects other than KGL.

Corollary 5.13. *The rationalized Frobenius K-groups of* any *field F of characteristic $p > 0$ are given by*

$$K_n(F/\operatorname{Frob})_{\mathbb{Q}} = \begin{cases} \mathbb{Q} & n = -1, 0, \\ 0 & \textit{else}. \end{cases} \tag{5.5}$$

In particular, the Beilinson–Soulé vanishing *holds for Frobenius K-theory of fields:*

$$\mathrm{H}^p\big(F/\operatorname{Frob}, \mathbb{Q}(q)\big) = K_{2q-p}(F/\operatorname{Frob})_{\mathbb{Q}}^{(q)} = 0$$

for $q > 0$ and $p \leq 0$.

Proof. The cases $n = -1, 0$ are clear. Suppose now $n \neq -1, 0$. To show the claimed vanishing, we may assume F is perfect, since p^{-1}-localized K-theory is insensitive to perfection. Let $\overline{F}$ be an algebraic closure of F. Combining Corollary 5.11 with finitaryness of Frobenius K-theory we have

$$0 = K_n(\overline{F}/\operatorname{Frob})_{\mathbb{Q}} = \operatorname*{colim}_{F \subset L \subset \overline{F}} K_n(L/\operatorname{Frob})_{\mathbb{Q}}, \tag{5.6}$$

where the colimit runs over the finite, separable extensions $F \subset L \subset \overline{F}$.

It suffices to show that the transition maps $f^*\colon K(F/\operatorname{Frob})_{\mathbb{Q}} \to K(L/\operatorname{Frob})_{\mathbb{Q}}$ in (5.6) are injective. By Lemma 5.14 below, the usual $f_* f^*$ on K-theory extends to Frobenius K-theory. The maps $f_* f^*\colon K_n(F) \to K_n(F)$ are equal to $[L : F] \cdot \mathrm{id}$ [37, Section 7, Proposition 4.8], and are therefore isomorphisms after passing to rationalizations. So, $f^*\colon K(F/\operatorname{Frob})_{\mathbb{Q}} \to K(L/\operatorname{Frob})_{\mathbb{Q}}$ is injective. ∎

Lemma 5.14. *If $f\colon S' \to S$ is a finite étale map, there is a natural pushforward map $f_*\colon K(X \times S'/\operatorname{Frob}_{S'}) \to K(X \times S/\operatorname{Frob}_S)$, compatible with the usual pushforward on K-theory. The same holds for pullback along arbitrary maps f.*

Proof. The map f^* always exists since $f^* \operatorname{Frob}_S^* = \operatorname{Frob}_{S'}^* f^*\colon K(X \times S) \to K(X \times S')$. For étale maps f, the natural map $S' \to S' \times_{f,S,\operatorname{Frob}_S} S$ is an isomorphism [45, Tag 0EBS], so that the base-change formula

$$\operatorname{Frob}_S^* f_* = f_* \operatorname{Frob}_{S'}^*\colon K(X \times S') \to K(X \times S)$$

implies the existence of the pushforward on Frobenius K-theory. ∎

Recall a conjecture of Beilinson: for all fields $F/\mathbb{F}_p$ the canonical map

$$K_*^M(F)_{\mathbb{Q}} \to K_*(F)_{\mathbb{Q}}$$

is an isomorphism. This conjecture is implied by the Bass conjecture or, alternatively, also by the Tate conjecture, see [17, Introduction] for references. The next result confirms this conjecture for the Frobenius variants of these two theories. For an abelian group A, we write $A_{(p)} := A \otimes \mathbb{Z}_{(p)}$ for the localization at the prime ideal (p).

Corollary 5.15. *For any field F of characteristic $p > 0$, and any $n \in \mathbb{Z}$, the map*

$$\left[K_n^M(F)_{(p)} \xrightarrow{\mathrm{id}-\mathrm{Frob}} K_n^M(F)_{(p)}\right] \to \left[K_n(F)_{(p)} \xrightarrow{\mathrm{id}-\mathrm{Frob}} K_n(F)_{(p)}\right]$$

is a quasi-isomorphism.

Proof. By [17, Proof of Theorem 8.1], the natural map $K_n^M(F) \to K_n(F)$ is injective and its cokernel C is a $\mathbb{Z}[p^{-1}]$-module. We therefore have $C_{(p)} = C_{\mathbb{Q}}$. Thus, we may replace the localization at (p) in the statement by the rationalization.

On $F^\times$ (resp. $K_n^M(F)$), the Frobenius is the multiplication by p (resp. by p^n). Therefore, $\mathrm{id} - \mathrm{Frob}$ is an isomorphism on $K_n^M(F) \otimes \mathbb{Q}$ for $n \geq 1$. Thus, the claim for algebraically closed fields follows immediately from Corollary 5.7 and Corollary 5.11.

For arbitrary F, the argument from the proof of Corollary 5.13 carries over: Milnor K-theory is continuous, and for any finite field extension $E \subset F$, the composite $K_*^M(E) \to K_*^M(F) \to K_*^M(E)$ is $[F : E] \cdot \mathrm{id}$. (This is one of the joint properties of Milnor K-theory and K-theory, cf. also [41, Axiom R2d].) ∎

Remark 5.16. The localization at the prime ideal (p) is necessary in the statement above: for $n > 0$, the group $K_n^M(\mathbb{F}_2)$ vanishes, but $K_{2i-1}(\mathbb{F}_2) = \mathbb{F}_{2^i}^\times \neq 0$ [46, Corollary IV.1.13].

5.4. Further cellular spectra

In addition to 1 and KGL, further cellular spectra include the cobordism spectrum MGL [10, Theorem 6.4] as well as, for $p \neq 2$, hermitian K-theory and Witt theory [40, Theorem 1.1]. Thus, Corollary 5.11 admits analogues for Frobenius cobordism groups, Frobenius hermitian K-groups and Witt groups.

5.5. Motivic cohomology of small weight

An interesting case of Definition 5.1 not covered by the results thus far is the case $M = \mathrm{M}(X)[p^{-1}]$ and $N = \mathbb{S}^{n,n}[p^{-1}]$ (or $N = \mathbb{Z}[p^{-1}](n)$ or $\mathbb{Q}(n)$, which makes no difference in view of Lemma 5.4). For an $\mathbb{F}_p$-scheme X of finite type and any field F of characteristic $p > 0$, we write $\mathrm{R}\Gamma(X \times F, \mathbb{Z}(n))$ for Bloch's complex of codimension n-cycles on $X \times F$. Its m-th homology identifies with the higher Chow group $\mathrm{CH}^n(X \times F, m)$. As before, we define the Frobenius variant of this by taking homotopy fixed points under the partial Frobenius pullback:

$$\mathrm{R}\Gamma\big(X \times F/\operatorname{Frob}_F, \mathbb{Z}(n)\big) := \lim \Big(\mathrm{R}\Gamma\big(X \times F, \mathbb{Z}(n)\big) \overset{(\mathrm{id}_X \times \operatorname{Frob}_F)^*}{\underset{\mathrm{id}}{\rightrightarrows}} \mathrm{R}\Gamma\big(X \times F, \mathbb{Z}(n)\big)\Big)$$

A concrete representative for this complex is the total complex of a two-step double complex, as in (1.3). The cohomology groups of this complex, denoted by

$$\mathrm{H}^*\big(X \times \operatorname{Spec} F/\operatorname{Frob}_F, \mathbb{Z}(n)\big),$$

again sit in a long exact sequence similar to (5.3).

Theorem 5.17. *Let X be smooth, proper $\mathbb{F}_p$-scheme. Then, Frobenius rigidity holds for the pair $\mathrm{M}(X)[p^{-1}]$ and $\mathbb{S}^{n,n}[p^{-1}]$ for all $n \leq 1$. In particular, the p^{-1}-localized Frobenius motivic cohomology groups*

$$\mathrm{H}^*\big(X \times \operatorname{Spec} F/\operatorname{Frob}_F, \mathbb{Z}(n)\big)[p^{-1}] \tag{5.7}$$

are independent of the choice of an algebraically closed field F of characteristic $p > 0$, for all $n \leq 1$.

After some preparation, the proof will be given at the end of this subsection.

Example 5.18. For $F = \overline{\mathbb{F}}_p$, the étale version of (5.7) is studied in [15,28] and in [19,20] for constructible ℓ-adic sheaves.

5.5.1. Rigidity of Frobenius units. Starting with the non-rigid presheaf $\mathbb{G}_{\mathrm{m}}$ (Example 2.5), we do get a rigid functor once we apply homotopy fixed points under the partial Frobenius:

Lemma 5.19. *For a geometrically connected and geometrically reduced scheme X, the following functor is rigid:*

$$\mathbb{G}_{\mathrm{m}}(X \times -/\operatorname{Frob}_-)\colon \mathrm{CAlg}_{\mathbb{F}_p} \to \mathrm{D}(\mathbb{Z}), \quad R \mapsto \big[\mathbb{G}_{\mathrm{m}}(X \times R) \xrightarrow{\mathrm{id}-\operatorname{Frob}_R^*} \mathbb{G}_{\mathrm{m}}(X \times R)\big]. \tag{5.8}$$

Proof. Since the functor is finitary it suffices to check the criterion in Corollary 2.3. Let $C = \operatorname{Spec} R$ be a connected, smooth, affine curve over an algebraically closed field F. We first consider the case when X is geometrically integral. We can then apply the unit theorem due to Sweedler [44] and Rosenlicht [7] to X_F and C over F and obtain the following short exact sequences:

$$\begin{tikzcd} 1 \arrow[r] & F^\times \arrow[r] \arrow[d, "\mathrm{id}-\mathrm{Frob}_F^*"] & \mathbb{G}_\mathrm{m}(X_F) \times \mathbb{G}_\mathrm{m}(C) \arrow[r] \arrow[d, "\mathrm{id}-(\mathrm{Frob}_F^*, \mathrm{Frob}_C^*)"] & \mathbb{G}_\mathrm{m}(X \times C) \arrow[r] \arrow[d, "\mathrm{id}-\mathrm{Frob}_C^*"] & 1 \\ 1 \arrow[r] & F^\times \arrow[r] & \mathbb{G}_\mathrm{m}(X_F) \times \mathbb{G}_\mathrm{m}(C) \arrow[r] & \mathbb{G}_\mathrm{m}(X \times C) \arrow[r] & 1, \end{tikzcd} \tag{5.9}$$

which are compatible with the displayed vertical maps.

Below, we write (co)ker for the (co)kernel of the vertical maps. Using the snake lemma along with the Kummer sequence, we obtain a short exact sequence and an isomorphism:

$$1 \to \mathbb{F}_p^\times \to \ker\big|_{\mathbb{G}_\mathrm{m}(X_F)\oplus\mathbb{G}_\mathrm{m}(C)} \to \ker\big|_{\mathbb{G}_\mathrm{m}(X\times C)} \to 1,$$
$$\operatorname{coker}\big|_{\mathbb{G}_\mathrm{m}(X_F)\oplus\mathbb{G}_\mathrm{m}(C)} \xrightarrow{\cong} \operatorname{coker}\big|_{\mathbb{G}_\mathrm{m}(X\times C)}.$$

For $f \in \mathbb{G}_\mathrm{m}(X \times C)$ and $c \in C(F)$, the pullback $c^* f \in \mathbb{G}_\mathrm{m}(X_F)$ is clearly independent of c if f factors over the projection $X \times C \to X_F$. Thus, the rigidity of $R \mapsto \text{(co)ker}|_{\mathbb{G}_\mathrm{m}(X\times R)}$ follows from the one of $R \mapsto \text{co)ker}|_{\mathbb{G}_\mathrm{m}(R)}$, i.e., we may and do assume $X = \operatorname{Spec} \mathbb{F}_p$. Then, the pullback map

$$c^*\colon \text{(co)ker}|_{\mathbb{G}_\mathrm{m}(C)} \to \text{(co)ker}|_{\mathbb{G}_\mathrm{m}(F)}$$

is independent of c.

For the kernel, the left-hand group is $\mathbb{F}_p^\times$ since C is integral, and the value of constant functions is clearly independent of c. For the cokernel, it is independent since the target group $\operatorname{coker}|_{\mathbb{G}_\mathrm{m}(F)}$ is trivial because F is algebraically closed. ∎

Corollary 5.20. *For a geometrically reduced $\mathbb{F}_p$-scheme X with finitely many geometric connected components, the rationalization*

$$\mathbb{G}_\mathrm{m}(X \times -/\operatorname{Frob}_-)_\mathbb{Q} \colon \mathrm{CAlg}_{\mathbb{F}_p} \to \mathrm{D}(\mathbb{Q})$$

of (5.8) is rigid.

Proof. First off, we have $\pi_0(X_{\overline{\mathbb{F}}_p}) = \pi_0(X_F)$ for any algebraically closed field F of characteristic p, see [45, Tag 0363]. Clearly, the rows in diagram (5.9) remain exact when replacing $F^\times$ at the left by $G := \mathbb{Z}[\pi_0(X_F)] \otimes_\mathbb{Z} F^\times$. On this group, Frob_F^* acts as usual on $F^\times$ and by permutation on the set $\pi_0(X_F)$. We claim that, after rationalization, the map $\mathrm{id} - \mathrm{Frob}_F^*$ is invertible on $G_\mathbb{Q}$. In particular, its cokernel vanishes and the proof of Lemma 5.19 carries over. To show the claim, let $\mathbb{Z}$ act on $G_\mathbb{Q}$ through $1 \mapsto \mathrm{Frob}_F^*$. We have to show that $\mathrm{H}^i(\mathbb{Z}, G_\mathbb{Q}) = 0$ for $i = 0$ (injectivity) and $i = 1$ (surjectivity). Observe that $(\mathrm{Frob}_F^*)^n$ acts as the identity on the finite set $\pi_0(X_F)$ for some suitable $n \in \mathbb{Z}_{\geq 1}$. Thus, the Kummer sequence shows $\mathrm{H}^i(n\mathbb{Z}, G_\mathbb{Q}) = 0$ for $i = 0, 1$. This obviously implies $\mathrm{H}^0(\mathbb{Z}, G_\mathbb{Q}) = 0$. The vanishing of H^1 now follows from the inflation-restriction exact sequence

$$0 \to \mathrm{H}^1\big(\mathbb{Z}/n\mathbb{Z}, \mathrm{H}^0(n\mathbb{Z}, G_\mathbb{Q})\big) \to \mathrm{H}^1(\mathbb{Z}, G_\mathbb{Q}) \to \mathrm{H}^1(n\mathbb{Z}, G_\mathbb{Q}).$$ ∎

5.5.2. Verschiebung. To show rigidity of a Frobenius version of the Picard group, we use some generalities about the *Verschiebung* of abelian varieties, see, e.g., [11, Section 5.2]. Recall that for an abelian variety A over $\mathbb{F}_p$, the Verschiebung is an isogeny

$$V_A\colon A \to A$$

of degree $p^{\dim A}$. It commutes with any morphism of abelian varieties $A \to A'$.

Fix a (geometrically) normal, proper $\mathbb{F}_p$-scheme X. We consider the Verschiebung of the Picard variety $A := \mathrm{Pic}^0_{X/\mathbb{F}_p,\mathrm{red}}$, which is an abelian variety over $\mathbb{F}_p$. Indeed, $\mathrm{Pic}^0_{X/\mathbb{F}_p}$ is a geometrically irreducible, proper $\mathbb{F}_p$-group scheme [13, Lemma 9.5.1, Theorem 9.5.4, Remark 9.5.6]. Its reduction $\mathrm{Pic}^0_{X/\mathbb{F}_p,\mathrm{red}}$ is geometrically reduced and still an $\mathbb{F}_p$-group scheme (since $\mathbb{F}_p$ is perfect both properties are clear, but also hold over general fields by [6, Discussion above Theorem 5.1.1]), hence an abelian variety [45, Tag 03RO].

Lemma 5.21. *The Verschiebung of* $A = \mathrm{Pic}^0_{X/\mathbb{F}_p,\mathrm{red}}$ *and the map induced by pulling back line bundles along the Frobenius* Frob_X *agree:*

$$V_A = \mathrm{Frob}_X^* \,.$$

Proof. It suffices to see

$$\mathrm{Frob}_X^* \circ \mathrm{Frob}_A = V_A \circ \mathrm{Frob}_A,$$

because Frob_A is an epimorphism (since A is reduced). By construction of the Verschiebung, the composition $V_A \circ \mathrm{Frob}_A$ is multiplication by p.

The simple, but crucial observation (e.g., [42, Lemme 1.4]) is that the map Frob_A sends a T-point $a\colon T \to A$ to $\mathrm{Frob}_A \circ a = a \circ \mathrm{Frob}_T$. Interpreting a as a line bundle $\mathcal{L}$ on $X \times_{\mathbb{F}_p} T$, this means that $\mathrm{Frob}_A(\mathcal{L}) = (\mathrm{id}_X \times \mathrm{Frob}_T)^*\mathcal{L}$. Composing this with Frob_X^*, we see that it gets sent to $(\mathrm{Frob}_X \times \mathrm{Frob}_T)^*\mathcal{L} = \mathrm{Frob}_{X\times T}^*\mathcal{L}$. Generally, pulling back line bundles along the total Frobenius on a scheme, such as $X \times T$, sends $\mathcal{L}$ to $\mathcal{L}^{\otimes p}$, as can be seen by regarding the transition functions, which are raised to their p-th power. Hence, $\mathrm{Frob}_X^* \circ \mathrm{Frob}_A$ is also the p-multiplication. ∎

Proposition 5.22. *For any abelian variety A over $\mathbb{F}_p$, and any $\lambda \in \mathbb{Q}$, the element*

$$\mathrm{id} + \lambda V_A$$

is an isogeny, i.e., an invertible element in $\mathrm{End}(A)_{\mathbb{Q}}$.

Proof. Using that the Verschiebung is compatible with any morphism of abelian varieties, we may replace A by any isogeneous abelian variety A' to check this claim (since then $\mathrm{End}(A)_{\mathbb{Q}} = \mathrm{End}(A')_{\mathbb{Q}}$). Therefore we may assume $A = \prod A_i$ is a product of simple abelian varieties $A_i/\mathbb{F}_p$. The morphism $\mathrm{id} + \lambda V_A$ respects this product decomposition, so we may assume A is simple. Then $\mathrm{End}(A)_{\mathbb{Q}}$ is a skew field, so it suffices to show that $\mathrm{id} + \lambda V_A$ is a non-zero element in $\mathrm{End}(A)_{\mathbb{Q}}$. The case $\lambda = 0$ being trivial, we now consider $\lambda = \frac{r}{s} \in \mathbb{Q}$ with $r, s \in \mathbb{Z} \setminus \{0\}$. If $s\mathrm{id}_A = rV_A$, then taking degrees ($\deg V_A = p^{\dim A}$, [11, Proposition 5.20]), we get

$$s^{2\dim A} = r^{2\dim A} p^{\dim A},$$

which is a contradiction. ∎

Remark 5.23. If q is an *odd* p-power, then the analogue of Proposition 5.22 holds for abelian varieties $A/\mathbb{F}_q$ equipped with their Verschiebung $V_{A/\mathbb{F}_q}$. Indeed, since $\deg V_{A/\mathbb{F}_q} = q^{\dim A}$ the same arguments lead to the equation $s^{2\dim A} = r^{2\dim A} q^{\dim A}$ which contradicts the assumption that $\log_p(q)$ is odd.

5.5.3. Rigidity for Frobenius–Picard groups.

Proposition 5.24. *Let X be a smooth, proper $\mathbb{F}_p$-scheme. Then, the following functor is rigid:*

$$\mathrm{Pic}(X \times -/\,\mathrm{Frob}_-)_{\mathbb{Q}} \colon \mathrm{AffSch}^{\mathrm{op}}_{\mathbb{F}_p} \to \mathrm{D}(\mathbb{Q}),$$
$$S \mapsto \left[\mathrm{Pic}(X \times S)_{\mathbb{Q}} \xrightarrow{\mathrm{id}-(\mathrm{id}_X \times \mathrm{Frob}_S)^*} \mathrm{Pic}(X \times S)_{\mathbb{Q}}\right].$$

Proof. We have $p = \mathrm{Frob}^*_{X\times S} = \mathrm{Frob}^*_X \circ \mathrm{Frob}^*_S$ on $\mathrm{Pic}(X \times S)$. In particular, Frob^*_X is invertible on the rationalization $\mathrm{Pic}(X \times S)_{\mathbb{Q}}$. The above complex is therefore quasi-isomorphic to

$$\left[\mathrm{Pic}(X \times S)_{\mathbb{Q}} \xrightarrow{p-\mathrm{Frob}^*_X} \mathrm{Pic}(X \times S)_{\mathbb{Q}}\right].$$

Let F be an algebraically closed field of characteristic p, C a smooth, affine, connected F-curve, and let $c_0, c_1 \in C(F)$ be points. Write $X_F = X \times F$. In order to show rigidity, let $\bar{C}$ be the smooth compactification of C, and let $D = \bar{C} \setminus C$ be the boundary points. There is an exact sequence

$$\begin{aligned} 0 \to \mathbb{G}_{\mathrm{m}}(X_F \times_F \bar{C}) \to \mathbb{G}_{\mathrm{m}}(X_F \times_F C) \to \mathbb{Z}^{\pi_0(X\times D)} \\ \to \mathrm{Pic}(X_F \times_F \bar{C}) \to \mathrm{Pic}(X_F \times_F C) \to 0 \end{aligned}$$

using that X_F is smooth and proper. The sequence is functorial under Frob^*_X. On the rationalization of the middle term, $\mathbb{Q}^{\pi_0(X\times D)}$, the map $p - \mathrm{Frob}^*_X$ is easily seen to be invertible: Frob^*_X acts through permutation on the finite set $\pi_0(X \times D)$, so its eigenvalues on $\mathbb{Q}^{\pi_0(X\times D)}$ are roots of unity.

Therefore, we may assume C is projective in the sequel.

We compute the Picard group of $X \times C = X_F \times_F C$ using the short exact sequence [6, (5.31)]

$$0 \to \mathrm{Pic}(X_F) \oplus \mathrm{Pic}(C) \to \mathrm{Pic}(X \times C) \to \mathrm{Hom}_{\mathrm{AbVar}_F}(B^\vee, A) \to 0, \tag{5.10}$$

where $B = \mathrm{Pic}^0_{C/F}$ is the Picard variety of the smooth, projective, connected curve C, $B^\vee$ its dual abelian variety, and $A = \mathrm{Pic}^0_{X_F/F,\mathrm{red}} = \mathrm{Pic}^0_{X/\mathbb{F}_p,\mathrm{red}} \times F$ the Picard variety of X_F.

This sequence is compatible with $(\mathrm{Frob}_X \times \mathrm{id}_F)^* \oplus \mathrm{id}$, resp. $(\mathrm{Frob}_X \times \mathrm{id}_C)^*$, resp. the map $A \to A$ induced by pullback along Frob_X (and id_F). By Lemma 5.21, the map induced by pullback along Frob_X on the reduced Picard scheme $\mathrm{Pic}^0_{X/\mathbb{F}_p,\mathrm{red}}$ is the Verschiebung; here we use the assumptions on X. In $\mathrm{End}(A)_{\mathbb{Q}}$, the element $[p]_A - \mathrm{Frob}_X^* = [p]_A - V_A$ is invertible by Proposition 5.22, if $\dim A > 0$. This implies that postcomposing with $[p]_A - \mathrm{Frob}_X^*$ is an isomorphism on $\mathrm{Hom}(B^\vee, A)_{\mathbb{Q}}$ (if $\dim A = 0$, this Hom-group is trivial). Thus, the "error term" $\mathrm{Hom}(B^\vee, A)_{\mathbb{Q}}$ vanishes after passing to homotopy fixed points under $p - \mathrm{Frob}_X^*$. The restriction of c_i^* on the subgroup $\mathrm{Pic}(X_F) \oplus \mathrm{Pic}(C)$ is clearly independent of the point $c_i \in C(F)$, since it is the identity on $\mathrm{Pic}(X_F)$ and $0\colon \mathrm{Pic}(C) \to \mathrm{Pic}(F) = 0$. ∎

Proof of Theorem 5.17. The case $n \leq 0$ is trivial since $\mathrm{R\Gamma}(X \times \mathrm{Spec}\, F, \mathbb{Z}(n)) = 0$ for $n < 0$ and is quasi-isomorphic to $\mathbb{Z}[0]$ for $n = 0$.

We now turn to $n = 1$, using that $\mathbb{Z}(1) = \mathbb{G}_{\mathrm{m}}[-1]$. The only non-zero groups

$$\mathrm{H}^r(X \times \mathrm{Spec}\, F, \mathbb{G}_{\mathrm{m}})$$

are for $r = 0$ and $r = 1$, so it suffices to show that the two-term complex

$$\left[\mathrm{H}^r(X \times \mathrm{Spec}\, F, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\mathrm{id}-\mathrm{Frob}_F^*} \mathrm{H}^r(X \times \mathrm{Spec}\, F, \mathbb{G}_{\mathrm{m}})\right]$$

is insensitive (up to quasi-isomorphism) to the choice of an algebraically closed field, at least after p^{-1}-localization. By Suslin rigidity (Lemma 5.4), it suffices to consider the rationalization of these two-term complexes.

The formation of this complex is finitary in F. Our claim then follows for $r = 0$ by Corollary 5.20 and for $r = 1$ by Proposition 5.24. ∎

Remark 5.25. It would be interesting to apply the above ideas towards *Gabber rigidity* for Frobenius motivic cohomology, along the lines of [18, Section 4]. More precisely, one can ask whether for a Henselian local ring A of a smooth variety over an algebraically closed field in characteristic p, with residue field k, the map

$$\mathrm{H}^n\big(X \times A/\mathrm{Frob}_A, \mathbb{Z}(1)[p^{-1}]\big) \to \mathrm{H}^n\big(X \times k/\mathrm{Frob}_k, \mathbb{Z}(1)[p^{-1}]\big)$$

is an isomorphism.

To round off the discussion concerning Frobenius motivic cohomology of small weight, we consider the stable, full subcategory $\mathrm{SH}^{\mathrm{eff}}(\mathbb{F}_p)$ in $\mathrm{SH}(\mathbb{F}_p)$ generated under colimits by motives of smooth $\mathbb{F}_p$-schemes X.

Corollary 5.26. *Suppose $M \in \mathrm{SH}^{\mathrm{eff}}(\mathbb{F}_p)[p^{-1}]$ and $N = \mathbb{S}^{n,n}[p^{-1}]$ (or $N = \mathbb{Z}[p^{-1}](n)$) with $n \leq 1$. Then, the pair M, N is Frobenius rigid.*

Proof. By resolution of singularities (via alterations), it is known that $\mathrm{SH}^{\mathrm{eff}}(\mathbb{F}_p)[p^{-1}]$ is the stable, full subcategory generated under colimits by $\mathrm{M}(X)(e)[e]$, with $X/\mathbb{F}_p$ being smooth and proper, and $e \geq 0$ [3, Theorem 2.4.3]. Thus, the corollary follows from Theorem 5.17. ∎

Appendix: Frobenius topological Hochschild homology

In this aside, we consider homotopy fixed points under Frobenius pullbacks for topological Hochschild homology (THH). Since THH is not representable in SH, the following result is not strictly an example of Frobenius rigidity as in Definition 5.1, but may still be illustrational.

We fix an $\mathbb{F}_p$-scheme X. Recall, e.g., from [35] the topological Hochschild homology functor

$$\mathrm{THH}(X \times -)\colon \mathrm{Sch}_{\mathbb{F}_p}^{\mathrm{op}} \to \mathrm{Sp}.$$

We let Frobenius THH be again the homotopy fixed points of partial Frobenius:

$$\mathrm{THH}(X \times S/\,\mathrm{Frob}_S) := \lim\big(\mathrm{THH}(X \times S) \underset{\mathrm{id}}{\overset{(\mathrm{id}_X \times \mathrm{Frob}_S)^*}{\rightrightarrows}} \mathrm{THH}(X \times S)\big).$$

Proposition A.1. *Let X be an affine $\mathbb{F}_p$-scheme. Then, Frobenius THH with respect to X is rigid. More precisely, for any algebraically closed field F of characteristic p, the following natural map is an equivalence:*

$$\mathrm{THH}(X) \xrightarrow{\cong} \mathrm{THH}(X \times \operatorname{Spec} F/\,\mathrm{Frob}_F).$$

Proof. We give two proofs for this, the first of which was suggested to us by Markus Land and Zhouhang Mao.

Using the symmetric monoidality of THH, i.e., the equality

$$\mathrm{THH}(X \times_{\operatorname{Spec}\mathbb{F}_p} \operatorname{Spec} F) = \mathrm{THH}(X) \otimes_{\mathrm{THH}(\mathbb{F}_p)} \mathrm{THH}(F),$$

we may assume $X = \operatorname{Spec}\mathbb{F}_p$. We then use Bökstedt periodicity, i.e., the fact that the natural map $F \otimes_{\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p) \to \mathrm{THH}(F)$ is an isomorphism (in the ∞-category of spectra) for any perfect field F [26, Proposition 2.1]. Finally, we conclude using the Artin–Schreier sequence

$$0 \to \mathbb{F}_p \to F \xrightarrow{x \mapsto x^p - x} F \to 0.$$

The second proof works, as is, for smooth $X/\mathbb{F}_p$: for such X, the Hochschild–Kostant–Rosenberg theorem for THH due to Hesselholt [21, Theorem B],[1] gives an isomorphism

$$\mathrm{THH}_n(X \times \operatorname{Spec} F) = \bigoplus_{i \geq 0} \Omega^{n-2i}_{X \times F/F} = \bigoplus_{i \geq 0} \Omega^{n-2i}_{X/\mathbb{F}_p} \otimes_{\mathbb{F}_p} F.$$

Again, the Artin–Schreier sequence shows that the homotopy fixed points of $(\mathrm{id}_X \times \mathrm{Frob}_F)^*$ acting on this agree with $\mathrm{THH}_n(X)$. ∎

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[1] Or see [34, Proposition 5.6] for a recent exposition.

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