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Convergence of the environment seen from geodesics in exponential last-passage percolation

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Abstract. A well-known question in planar first-passage percolation concerns the convergence of the empirical distribution of weights as seen along geodesics. We demonstrate this convergence for an explicit model, directed last-passage percolation on $\mathbb{Z}^2$ with i.i.d. exponential weights, and provide explicit formulae for the limiting distributions, which depend on the asymptotic direction. For example, for geodesics in the direction of the diagonal, the limiting weight distribution has density $(1/4 + x/2 + x^2/8)e^{-x}$, and so is a mixture of Gamma(1, 1), Gamma(2, 1), and Gamma(3, 1) distributions with weights $1/4$, $1/2$, and $1/4$ respectively. More generally, we study the local environment as seen from vertices along geodesics (including information about the shape of the path and about the weights on and off the path in a local neighborhood). We consider finite geodesics from $(0, 0)$ to $n\boldsymbol{\rho}$ for some vector $\boldsymbol{\rho}$ in the first quadrant, in the limit as $n \to \infty$, as well as semi-infinite geodesics in direction $\boldsymbol{\rho}$. We show almost sure convergence of the empirical distributions of the environments along these geodesics, as well as convergence of the distributions of the environment around a typical point in these geodesics, to the same limiting distribution, for which we give an explicit description.

We make extensive use of a correspondence with TASEP as seen from an isolated second-class particle for which we prove new results concerning ergodicity and convergence to equilibrium. Our analysis relies on geometric arguments involving estimates for last-passage times, available from the integrable probability literature.

Keywords: last passage percolation, KPZ universality class, exclusion process, empirical measure, competition interface, geodesic.

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1. Introduction

In this article we study exactly solvable models of planar directed last-passage percolation (LPP), an instance of the more general Kardar–Parisi–Zhang (KPZ) universality class, which dates back to the seminal work of [42]. The KPZ universality class has been a major topic of interest both in statistical physics and in probability theory in recent decades. In [42], the authors predicted universal scaling behavior for a large number of

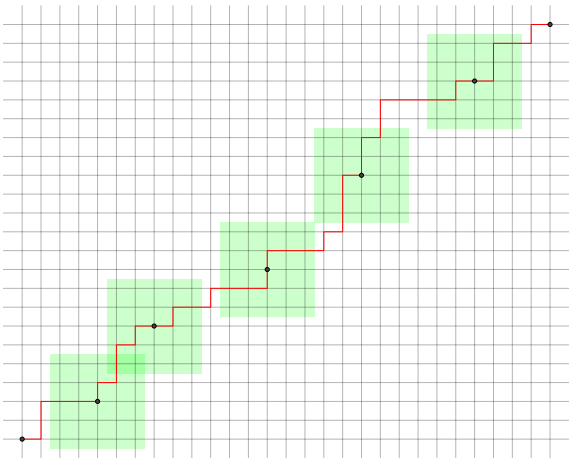


Fig. 1. An illustration of local environments along a finite geodesic.

planar random growth processes, including first-passage percolation (FPP) and corner growth processes; in particular, it is predicted that these models have length fluctuation exponent $1/3$ and transversal fluctuation exponent $2/3$. Since then, rigorous progress has been made only in a handful of cases. The first breakthrough was made by Baik, Deift and Johansson [4] who established $n^{1/3}$ fluctuations of the length of the longest up-right path from $(0,0)$ to (n,n) in a homogeneous Poissonian field on $\mathbb{R}^2$, and also established the GUE Tracy–Widom scaling limit. Then Johansson proved a transversal fluctuation exponent of $2/3$ for that model, and also $n^{1/3}$ fluctuations and a Tracy–Widom scaling limit for LPP on $\mathbb{Z}^2$ with i.i.d. geometric or exponential weights [40,41]. For these models such results could be obtained due to their exact solvability, using exact distributional formulae from algebraic combinatorics, random matrix theory, or queueing theory in some cases. Since then there have been tremendous developments in achieving a detailed understanding of these exactly solvable models, with notable progress concerning scaling limits (see e.g. the recent works of [24,45]). For surveys in this direction, see e.g. [21,48,56].

In another related direction, there has been great interest in studying FPP with general weights. In the 2D setting, such models are also conjectured to be in the KPZ universality class, but much less is known due to the lack of exact formulae. The geometry of the set of geodesics has been an important tool in the study of these models; see e.g. [3,46]. When trying to understand the behavior of large finite or infinite geodesics, a well-known open question is whether the empirical distributions of weights as seen along geodesics converge; see e.g. [39] where it was proposed by Hoffman during a 2015 American Institute of Mathematics workshop. Recently, Bates gave an affirmative answer to this question for various abstract dense families of weight distributions [15]. The proof uses a variational formula, and does not rely on any exactly solvable structure.

In this paper we study the limiting local behavior for LPP in the exactly solvable case. We focus on LPP on $\mathbb{Z}^2$ with i.i.d. exponential weights. Rather than the weights along geodesics, we consider the more general 'empirical environments' around vertices, along finite or semi-infinite geodesics, and we show that they converge to a deterministic measure. By the environment around a vertex, we mean the weights of nearby vertices, and the path of the geodesic through them. In particular, this positively answers the question of Hoffman for a first explicit model. Our approach is different from [15] and relies on information provided by the exactly solvable structure. In addition to proving convergence results, we also give an explicit description of the limiting distribution, which depends on the direction of the geodesics considered. Using this description one can compute any limiting local statistics along the geodesics, and we give some first examples in this paper.

A particular exactly solvable input that we use is the connection between LPP and the totally asymmetric exclusion process (TASEP), dating back to [49]. We use the correspondence between LPP semi-infinite geodesics and the trajectory of a second-class particle in TASEP, as developed in a series of works [35, 36, 47]. Then in order to understand local environments along LPP geodesics, we study stationary distributions of TASEP as seen from an isolated second-class particle. Models involving second-class particles have been proved powerful in understanding the evolution of TASEP

[7, 8, 27, 30, 33, 50, 51], and stationary distributions for multi-type systems have been widely studied [1, 2, 27, 29, 32, 34, 54]. See also [31] for a recent survey of related ideas.

Before formally stating our results, we remark that (besides this paper and [15]) there are several other recent works on environments along geodesics in random planar geometry. In [25], the authors study geodesics in the directed landscape, the joint scaling limit of exponential LPP (see [24, 26]). They proved that when zooming in around a point in the geodesic, the local environment converges to an object termed 'the directed landscape with Brownian–Bessel boundary conditions' [25, Theorem 1.1]. Back to the non-exactly-solvable model of general weights FPP, tail estimates for the averaged empirical distribution of weights along geodesics have been obtained in [23]. In [9], convergence of the empirical distribution of environments along geodesics has been obtained in the Liouville Quantum Gravity setting.

1.1. Model definition and main results

We study the exponential weight planar directed last-passage percolation (LPP) model, which is defined as follows. To each vertex $v \in \mathbb{Z}^2$ we associate an independent weight $\xi(v)$ with Exp(1) distribution. For two vertices $u, v \in \mathbb{Z}^2$, we say $u \leq v$ if u is coordinate-wise less than or equal to v. For such u, v and any up-right path γ from u to v, we define the *passage time of the path* to be

$$T(\gamma) := \sum_{w \in \gamma} \xi(w).$$

Then almost surely there is a unique up-right path from u to v that has the largest passage time. We call this path the *geodesic* $\Gamma_{u,v}$, and call $T_{u,v} := T(\Gamma_{u,v})$ the (*last-*)*passage time from u to v*. In this paper we always work under the event that there is a unique geodesic between any $u \leq v$.

For any fixed $\rho \in (0, 1)$, it is known that almost surely the following statements hold (see [22, 36]). For each $u \in \mathbb{Z}^2$ there is a unique infinite up-right path from u (called the *semi-infinite geodesic* and denoted by Γ_u^ρ) asymptotically going to the $\boldsymbol{\rho} := ((1-\rho)^2, \rho^2)$ direction, such that for any $v \leq w$ contained in Γ_u^ρ, the part of Γ_u^ρ between v and w is the geodesic $\Gamma_{v,w}$. For any $u, v \in \mathbb{Z}^2$, the semi-infinite geodesics Γ_u^ρ and Γ_v^ρ coalesce, i.e. both $\Gamma_u^\rho \setminus \Gamma_v^\rho$ and $\Gamma_v^\rho \setminus \Gamma_u^\rho$ are finite. Below and whenever we consider a specific ρ, we always work under the almost sure event where these statements hold.

Our main results concern the local behavior around vertices along geodesics. For each $v \in \mathbb{Z}^2$, we denote $\xi\{v\} := \{\xi(v+u)\}_{u \in \mathbb{Z}^2}$. For any (finite or semi-infinite) up-right path γ we let $\gamma[i]$ be the i-th vertex in γ.

For any $u \leq v$ in $\mathbb{Z}^2$, and each $w \in \Gamma_{u,v}$, we regard $(\xi\{w\}, \Gamma_{u,v} - w)$ as a point in $\mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2}$ (equipped with the product topology and the cylinder σ-algebra), and we define the *empirical environment along* $\Gamma_{u,v}$ as

$$\mu_{u,v} := \frac{1}{|\Gamma_{u,v}|} \sum_{w \in \Gamma_{u,v}} \delta_{(\xi\{w\}, \Gamma_{u,v} - w)},$$

where $\delta_{(\xi\{w\},\Gamma_{u,v}-w)}$ is the Dirac measure concentrated on $(\xi\{w\}, \Gamma_{u,v} - w)$. Similarly, we define the *empirical environment along the semi-infinite geodesic* Γ_v^ρ as

$$\mu_{v;r}^\rho := \frac{1}{2r+1} \sum_{i=1}^{2r+1} \delta_{(\xi\{\Gamma_v^\rho[i]\},\Gamma_v^\rho - \Gamma_v^\rho[i])}$$

for any $v \in \mathbb{Z}^2$, $\rho \in (0,1)$, and $r \in \mathbb{Z}_{\geq 0}$. We will show that these empirical environments converge. For each ρ, there is a limiting measure ν^ρ on $\mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2}$, which is explicit and will be defined in Section 4.

For any $n \in \mathbb{Z}$ we denote

$$\mathbf{n}^\rho := \left(\left\lfloor \frac{2(1-\rho)^2 n}{\rho^2 + (1-\rho)^2} \right\rfloor, \left\lceil \frac{2\rho^2 n}{\rho^2 + (1-\rho)^2} \right\rceil \right).$$

We also denote $\mathbf{0} := (0,0)$. For the following results we fix any $\rho \in (0,1)$.

Theorem 1.1. *Almost surely the measures $\mu_{\mathbf{0},\mathbf{n}^\rho}$ converge to ν^ρ weakly as $n \to \infty$.*

Theorem 1.2. *Almost surely the measures $\mu_{\mathbf{0};r}^\rho$ converge to ν^ρ weakly as $r \to \infty$.*

In other words, for any bounded continuous function $f : \mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2} \to \mathbb{R}$ we have $\mu_{\mathbf{0},\mathbf{n}^\rho}(f) \to \nu^\rho(f)$ as $n \to \infty$ almost surely, and $\mu_{\mathbf{0};r}^\rho(f) \to \nu^\rho(f)$ as $r \to \infty$ almost surely.

We also have convergence of distributions.

Theorem 1.3. *The laws of $(\xi\{\Gamma_{\mathbf{0}}^\rho[i]\}, \Gamma_{\mathbf{0}}^\rho - \Gamma_{\mathbf{0}}^\rho[i])$ converge to ν^ρ weakly as $i \to \infty$.*

Theorem 1.4. *For each $0 < \alpha < 2$, the laws of $(\xi\{\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n \rfloor]\}, \Gamma_{\mathbf{0},\mathbf{n}^\rho} - \Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n \rfloor])$ converge to ν^ρ weakly as $n \to \infty$.*

These results in particular imply that the marginal distribution of ν^ρ on $\mathbb{R}^{\mathbb{Z}^2}$ is singular to the i.i.d. Exp(1) distribution. Specifically, these convergence results imply that for $(\xi, \gamma) \sim \nu^\rho$, the path γ is a *bigeodesic* for ξ, i.e. γ is a bi-infinite up-right path such that for any $u \leq v$ contained in γ, the part of γ between u and v is the geodesic from u to v, under the weights ξ. However, for ξ being i.i.d. Exp(1), almost surely there is no bigeodesic, as proved in [5, 12].

For the limiting measure ν^ρ to be defined in Section 4, its construction is explicit, and from it one can compute any finite-dimensional distributions of ν^ρ, thus any limiting local statistics along exponential LPP geodesics. Here we give a first example, which is the distribution function of $\xi(\mathbf{0})$ under ν^ρ.

Proposition 1.5. *For $(\xi, \gamma) \sim \nu^\rho$, we have*

$$\mathbb{P}[\xi(\mathbf{0}) > h] = \left(1 + \frac{\rho(1-\rho)h}{(1-\rho)^2 + \rho^2}\right)(1 + \rho(1-\rho)h)e^{-h}.$$

The distribution of $\xi(\mathbf{0})$ given in Proposition 1.5 is a mixture of Gamma(1, 1), Gamma(2, 1), and Gamma(3, 1) distributions. In the case $\rho = 1/2$, for example, the

weights of this mixture are $1/4$, $1/2$, and $1/4$ respectively, and the distribution of $\xi(\mathbf{0})$ can be interpreted as that of $2\min(E_1 + E_2, E_3 + BE_4)$ with $B \sim \text{Bernoulli}(1/2)$ and $(E_i)_{1\le 1\le 4}$ i.i.d. $\sim \text{Exp}(1)$ independently of B. Related but slightly less simple representations can be given for general ρ. See the discussion after the proof of Proposition 1.6 in Section 4.

One interesting question in exponential LPP is to derive descriptions for geodesics. They are known to be different from simple random walks, as their scaling limits are known be Hölder-$2/3^-$ regular [24, 25], and for $\Gamma_{\mathbf{0},\mathbf{n}^\rho}$ its transversal fluctuation is on the order of $n^{2/3}$ [7]. Exact formulae for the geodesic one-point distribution have also been obtained recently [44]. Our next result implies that $\Gamma_{\mathbf{0},\mathbf{n}^\rho}$ is not like a simple random walk even at a small scale, by showing that one step is more likely to follow the same direction as the previous step than to make a 'turning'. This follows from the convergence results, and our explicit construction of the limiting measure ν^ρ.

Proposition 1.6. *Denote by $N_{n,\rho}$ the number of 'corners' along $\Gamma_{\mathbf{0},\mathbf{n}^\rho}$, that is, the number of $v \in \mathbb{Z}^2$ such that $\{v-(1,0), v, v+(0,1)\} \subset \Gamma_{\mathbf{0},\mathbf{n}^\rho}$ or $\{v-(0,1), v, v+(1,0)\} \subset \Gamma_{\mathbf{0},\mathbf{n}^\rho}$. Then almost surely we have*

$$\frac{N_{n,\rho}}{2n} \to \frac{2\rho^2(1-\rho)^2(1+2\rho-2\rho^2)}{(1-\rho)^2+\rho^2} \quad \text{as } n \to \infty.$$

For example, for $\rho = 1/2$, the proportion of steps which are 'corners' converges to $3/8$. For $(\xi, \gamma) \sim \nu^\rho$, the limiting path γ can also be described as the 'competition interface' in a growth process with some explicit random initial configurations. See the discussion at the end of Section 4.

In our proofs of the above results we will use the connection between LPP and the totally asymmetric exclusion process (TASEP), which can be described as a Markov process $(\eta_t)_{t\in\mathbb{R}}$ on the space $\{0,1\}^{\mathbb{Z}}$ (also equipped with the product topology and the cylinder σ-algebra), where $\eta_t(x) = 1$ means that there is a particle at site x at time t, whereas $\eta_t(x) = 0$ means that there is a hole at site x at time t. If there is a particle at site x and a hole at site $x+1$, they switch at rate 1, independently for all such x. We shall consider TASEP with a single 'second-class particle', which is denoted by $*$ and can switch with a hole to the right of it, or with a (normal) particle to the left of it. We prove a corresponding result for TASEP with a single second-class particle as well, which may be of independent interest.

Theorem 1.7. $\lim_{t\to\infty} \Phi_t^\rho = \Psi^\rho$ *weakly.*

Here Φ_t^ρ and Ψ^ρ are measures on $\{0,1,*\}^{\mathbb{Z}}$ (with the product topology) to be defined in Section 2, and we describe them here. Consider TASEP with a single second-class particle, where initially the second-class particle is at the origin, and any other site has a (normal) particle with probability ρ independently. Then Φ_t^ρ is the law of such TASEP at time t, as seen from the only second-class particle. The measure Ψ^ρ is the stationary distribution of TASEP as seen from an isolated second-class particle, with particle density ρ. In proving this theorem, we will also show that the corresponding stationary

process (of TASEP as seen from an isolated second-class particle) is ergodic in time (Proposition 2.2).

1.2. A roadmap of our arguments

There are two main ingredients in our proofs of the above results: geometry of geodesics in exponential LPP, and TASEP as seen from an isolated second-class particle.

For each $\rho \in (0, 1)$ there is a (density ρ) stationary distribution for TASEP, where for each site there is a particle with probability ρ and a hole with probability $1 - \rho$ independently (i.e. i.i.d. Bernoulli(ρ)). Such i.i.d. Bernoulli TASEP corresponds to a growth process in $\mathbb{Z}^2$, which (when rotated by $\pi/4$) is a random walk at any time. Dividing the interface into two competing clusters, this gives a competition interface which corresponds to a semi-infinite geodesic in LPP; see e.g. [35, 47]. On the other hand, such a competition interface corresponds to a second-class particle in TASEP. Thus, the environment seen from a semi-infinite geodesic corresponds to TASEP as seen from an isolated second-class particle. Connections between TASEP and LPP will be discussed in detail in Section 3.

We will construct the limiting measure ν^ρ in Section 4, using the density ρ stationary measure of TASEP as seen from an isolated second-class particle, as described in [32] and to be studied in Section 2; we then prove Propositions 1.5 and 1.6 in Section 4 assuming the convergence results.

For the convergence results we take the following approach. For Theorem 1.7, in Section 2.2 we first prove a weak version of convergence in the averaged sense (Proposition 2.7), using a coupling argument of interacting particle systems. In Section 6 we upgrade Proposition 2.7 to Theorem 1.7 using LPP and geometric arguments. In Section 7 we prove weak versions of Theorems 1.1 and 1.2, involving convergence in probability. Convergence in probability along semi-infinite geodesics (Theorem 7.1) is deduced from the TASEP convergence result of Theorem 1.7 (or the averaged version, Proposition 2.7) and ergodicity of the TASEP stationary process, which we have proved as Proposition 2.2 in Section 2.1. From then on we work completely in the LPP setting. In Section 7.2 we prove the convergence in probability version of Theorem 1.1 (Theorem 7.3), by using Theorem 7.1 and covering a finite geodesic with an infinite one.

The next several sections rely on a generalization of Theorem 7.3, which is Proposition 8.1, the main result of Theorem 8. It says that for geodesics whose endpoints vary along two anti-diagonal segments, their empirical environments converge jointly (in probability). The proof is via taking a finite (i.e. size not growing) dense family of geodesics, and showing that each geodesic connecting the two anti-diagonal segments can be mostly covered by one geodesic in the family. Using this result, in Section 9 we prove Theorems 1.3 and 1.4, by showing that environments of nearby vertices (along geodesics) are close to each other in distribution. In Section 10, by covering a long or semi-infinite geodesic by short ones, we prove that its empirical environment concentrates exponentially fast, and thus upgrade Theorem 7.1 to Theorem 1.2 and Theorem 7.3 to Theorem 1.1.

At the end of this roadmap, we comment on how much our arguments rely on exact solvability. As mentioned above, while we do not work directly on formulae, we rely on the structure of the exponential LPP model considered. The construction of ν^ρ in Section 4 uses the exact equivalence between exponential LPP and TASEP (as stated in Section 3); and Section 2 contains purely interacting particle system arguments. Most other proofs in this paper are via LPP geometric arguments, using basic estimates on passage times and geometric properties that have appeared in the literature (and are stated in Section 5). For Section 6, while we prove Theorem 1.7 which is about TASEP, the arguments are mainly via the connection with LPP and its geometry. Starting from Section 7 all the proofs use only geometric arguments, except for the short Section 7.1 (where the convergence in probability of empirical environments along semi-infinite geodesics is quickly deduced using TASEP results). We point out that the LPP geometric arguments throughout this paper are robust, with the only inputs from exact solvability being the passage time distribution tail estimates (Theorem 5.2 below), and that the so-called Busemann function (to be defined in Section 3.1) in an anti-diagonal is a random walk.

1.3. Further applications and questions

With the limiting measure ν^ρ one can get any local information along geodesics in LPP. Before closing the introduction we discuss some questions, which can potentially be answered using our explicit description of ν^ρ, either as direct applications or requiring some further analysis.

The first question has been communicated to us by Alan Hammond. Given that a vertex on a geodesic has a large weight, how would the local environment behave? For a vertex with a large weight, it would force the geodesic to go through it. Thus we expect that conditioned on this, weights of nearby vertices are distributed like i.i.d. Exp(1) random variables. From the TASEP aspect, a large weight corresponds to a long waiting time between two jumps of the second-class particle, and this is mostly due to a 'jam' in TASEP, i.e. a consecutive sequence of particles to the right of the second-class particle, and a sequence of holes to the left. This resembles a 'reversed' step initial condition.

A related question is about vertices near but off a geodesic. For such vertices we have the following result.

Lemma 1.8. *For $(\xi, \gamma) \sim \nu^\rho$, and any vertex $v \neq \mathbf{0}$, the random variable $\xi(v)$ conditioned on $v \notin \gamma$ is stochastically dominated by* Exp(1).

Proof. For any vertices $u \leq v$, any up-right path Γ from u to v, any vertex $w \notin \Gamma$ with $u \leq w \leq v$, and any $x > 0$, the events $\Gamma_{u,v} = \Gamma$ and $\xi(w) > x$ are negatively correlated, by the FKG inequality. Thus the law of $\xi(w)$ is stochastically dominated by Exp(1), conditioned on $\Gamma_{u,v} = \Gamma$. This implies that for any vertex $v \neq \mathbf{0}$, $n \in \mathbb{N}$, the random variable $\xi(\Gamma_{\mathbf{0},\mathbf{n}^\rho}[n] + v)$ conditioned on $v \notin \Gamma_{\mathbf{0},\mathbf{n}^\rho} - \Gamma_{\mathbf{0},\mathbf{n}^\rho}[n]$ is stochastically dominated by Exp(1). By Theorem 1.4 and sending $n \to \infty$ we get the conclusion. ∎

It is then interesting to see if the distribution converges to Exp(1) as the distance of v to the geodesic increases to infinity.

The next question is about a slightly different setting, that of LPP with i.i.d. geometric weights. The main difference is that since the weights are discrete, geodesics are not necessarily unique in this case. However, one could still consider 'rightmost' geodesics. Geometric LPP corresponds to discrete-time TASEP, and one can similarly construct stationary measures for such TASEP as seen from an isolated second-class particle. For a correspondence with rightmost geodesics, in discrete-time TASEP one takes a second-class particle which is prioritized to jump to the right rather than to the left. One can similarly construct limiting measures, and thus get local information about the environment along rightmost geodesics. One question that would be interesting to study is the proportion of 'unique geodesic vertices'. For fixed endpoints (or for one fixed endpoint and a fixed direction), take the intersection of all the geodesics, and call those vertices in that intersection 'unique geodesic vertices'. Do these unique geodesic vertices asymptotically make up a positive proportion of the geodesics? Furthermore, does the proportion converge in probability, and can we compute the limit explicitly? We think such questions are related to the constructed limiting measures of the environment along rightmost geodesics, because we expect that a vertex in the geodesics is unlikely to be 'locally unique' without being a unique geodesic vertex in the sense mentioned above. Anomalous 'locally but not globally unique' vertices should make up a vanishing proportion of the geodesics in the limit.

Another direction concerns the scaling limit of the measure ν^ρ. As mentioned above, in [25] the authors constructed the small scaling limit of the local environment around a vertex in the geodesic, in the directed landscape setting. It is reasonable to expect that when zooming out, the measure ν^ρ would converge to the local environment constructed there. Also, once this is established, we would like to see if our explicit description of ν^ρ could be used to get some explicit information about the local environment and geodesics in the directed landscape (see e.g. [25, Problem 4]). In fact, for the geodesic under ν^ρ, one can possibly obtain various information on its large scale behavior using the description as a competition interface (see the end of Section 4).

We expect that the LPP geometric arguments in this paper can be extended to get more information on environments along geodesics. For example, it can be shown that, for any $0 < \alpha < \beta < 2$, the two environments $(\xi\{\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n \rfloor]\}, \Gamma_{\mathbf{0},\mathbf{n}^\rho} - \Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n \rfloor])$ and $(\xi\{\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \beta n \rfloor]\}, \Gamma_{\mathbf{0},\mathbf{n}^\rho} - \Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \beta n \rfloor])$ are asymptotically independent as $n \to \infty$. A possible route to prove this statement is as follows. Consider the point-to-line profiles from $\mathbf{0}$ to $\{(a,b) : a + b = \lfloor (\alpha - \varepsilon) n \rfloor\}$ and from $\{(a,b) : a + b = \lfloor (\alpha + \varepsilon) n \rfloor\}$ to $\mathbf{n}^\rho$, i.e. consider the passage times $T_{\mathbf{0},u}$ and $T_{v,\mathbf{n}^\rho}$, for u, v varying in these two lines respectively. Here $\varepsilon > 0$ is a small number. These two point-to-line profiles are independent, each converges (after rescaling) to the so-called Airy_2 process [17, 18], which is locally like a Brownian motion. Then it can be shown that in small neighborhoods of the intersections of the geodesic $\Gamma_{\mathbf{0},\mathbf{n}^\rho}$ with these two lines, the point-to-line profiles (after rescaling) are similar to two independent Brownian motions around the maximum of their sum, or equivalently $R - B$, $R + B$, where R is a Bessel_3 process and B is a Brownian motion. (In [25],

such behavior is observed for geodesics in the directed landscape.) Using coalescence of geodesics, such picture can be established even conditioned on the environment around $\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \beta n \rfloor]$. One can also show that the part of the geodesic $\Gamma_{\mathbf{0},\mathbf{n}^\rho}$ between these lines is stable with respect to small perturbations of the point-to-line profiles. This implies that, no matter how the environment around $\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \beta n \rfloor]$ behaves, conditioned on it the distribution of the environment around $\Gamma_{\mathbf{0},\mathbf{n}^\rho}[\lfloor \alpha n \rfloor]$ remains roughly the same as $n \to \infty$. In fact, such asymptotic independence can be used to give an alternative proof of the convergence of $\mu_{\mathbf{0},\mathbf{n}^\rho}$, without using any TASEP arguments or identifying the limit as ν^ρ. Indeed, it implies that for any bounded continuous f the variance of $\mu_{\mathbf{0},\mathbf{n}^\rho}(f)$ decays to zero. To upgrade such decay of variance to convergence, one needs to cover long geodesics with short ones, using arguments similar to those in Sections 7–10.

Notations. Throughout the rest of this paper the following notations will be used. For any $x, y \in \mathbb{R} \cup \{-\infty, \infty\}$ we denote $x \vee y = \max(x, y)$, and $x \wedge y = \min(x, y)$, and $[\![x, y]\!]$ is the set $[x, y] \cap \mathbb{Z}$. For each $u = (a, b) \in \mathbb{Z}^2$, we denote $d(u) = a + b$ and $ad(u) = a - b$. For $n \in \mathbb{Z}$ we denote $\mathbb{L}_n = \{u \in \mathbb{Z}^2 : d(u) = 2n\}$. Unless otherwise stated (mainly in Section 5), for the rest of this paper we always fix $\rho \in (0, 1)$, and the choice of all other parameters and constants can depend on ρ. We denote $\boldsymbol{\rho} = ((1-\rho)^2, \rho^2)$. We also drop ρ from some notations. Specifically, we write Γ_u for Γ_u^ρ, $\mu_{v;r}$ for $\mu_{v;r}^\rho$, $\mathbf{n}$ for $\mathbf{n}^\rho$, and ν, Φ_t, Ψ for ν^ρ, Φ_t^ρ, Ψ^ρ.

2. Stationary distribution of TASEP with a second-class particle

We start with the totally asymmetric simple exclusion process (TASEP), which is a classical interacting particle system. For TASEP with second-class particles, we represent it as a Markov process on $\{1, *, 0\}^{\mathbb{Z}}$, where the symbols 1, $*$, and 0 represent particles, second-class particles, and holes respectively. As in ordinary TASEP, any (normal) particle can switch with a hole to its right. In addition, any second-class particle can switch with a hole to its right, and can switch with a (normal) particle to its left. We consider TASEP as seen from an isolated second-class particle, which is related to LPP semi-infinite geodesics, as will be explained later in Section 3.3. Namely, suppose that $(\eta_t^*)_{t\in I}$ for some interval $I \subset \mathbb{R}$ is TASEP containing a single second-class particle, then the process $(\eta_t^*(l_t + \cdot))_{t\in I}$ is the corresponding TASEP as seen from an isolated second-class particle, where l_t is the location of the second-class particle at time t. There is a family of stationary distributions of TASEP as seen from an isolated second-class particle, constructed in [32]. In this section we study a particular one $\Psi = \Psi^\rho$, under which the configuration has the same asymptotic density ρ of particles in both directions.

We first construct Ψ following [32]. We start by constructing a stationary distribution for TASEP with infinitely many second-class particles.

Let $Y_1(x), x \geq 1$, and $Y_2(x), x \geq 1$, be independent collections of i.i.d. Bernoulli(ρ) random variables. Let $R_1(x) = \sum_{y=1}^x Y_1(y)$ and $R_2(x) = \sum_{y=1}^x Y_2(y)$. Then we can

define a symmetric random walk W by

$$W(x) = R_2(x) - R_1(x) \tag{2.1}$$

for $x \geq 0$. We also define

$$M(x) = \sup_{0 \leq y \leq x} W(y), \tag{2.2}$$

and $\mathcal{E} = \{x \geq 1 : M(x) > M(x-1)\}$. Then $M(x) = |\mathcal{E} \cap [\![1, x]\!]|$.

Then we can see $M(x) - W(x)$ as a symmetric random walk with steps in $\{-1, 0, 1\}$ and forced to stay non-negative: if at one step this walk 'tries' to go from 0 to -1, it will be altered and stay at 0. The points of $\mathcal{E}$, i.e. the points of increase of M, are those steps where such alternation occurs. More precisely, $x \in \mathcal{E}$ if and only if $M(x-1) = W(x-1)$ and $Y_2(x) = 1$, $Y_1(x) = 0$. By well-known properties of symmetric random walks, we can show that as $x \to \infty$, $\mathbb{P}[x \in \mathcal{E}]$ decays like $x^{-1/2}$, while $M(x)/x^{1/2} = |\mathcal{E} \cap [\![1, x]\!]|/x^{1/2}$ converges in distribution to a random variable supported on $(0, \infty)$.

Now we define a configuration σ on $\mathbb{Z}_{\geq 0}$, by copying Y_1 except at points of $\mathcal{E}$. We set $\sigma(0) = *$ and, for $x \geq 1$,

$$\sigma(x) = \begin{cases} 1 & \text{if } Y_1(x) = 1, \\ 0 & \text{if } Y_1(x) = 0 \text{ and } x \notin \mathcal{E}, \\ * & \text{if } Y_1(x) = 0 \text{ and } x \in \mathcal{E}. \end{cases} \tag{2.3}$$

(There is a natural interpretation involving the departure process of a discrete-time $M/M/1$ queue – see [34].) We wish to extend this to give a configuration $\sigma(x)$ on the whole line $\mathbb{Z}$. We can do it in two equivalent ways:

(1) Note that $\sigma(x), x \geq 0$, is a renewal process with renewals at points x where $\sigma(x) = *$, i.e. where $x \in \mathcal{E}$. Between successive renewal points, we see an i.i.d. sequence of finite strings in $\bigcup_{n \geq 0} \{0, 1\}^n$ (but the length of each string has an infinite expectation). We can extend σ to a renewal process on the whole line by extending this sequence of i.i.d. strings, separated by stars, leftward from the origin also.

(2) Alternatively, we can exploit the symmetry of TASEP under exchanging holes/particles and left/right. Write π_ρ for the distribution defined above on $\sigma(x)$, $x \geq 0$. Now generate another configuration $\tilde{\sigma}(x), x \geq 0$, from $\pi_{1-\rho}$, independently of σ, and for $x \geq 1$ set

$$\sigma(-x) = \begin{cases} 1 & \text{if } \tilde{\sigma}(x) = 0, \\ 0 & \text{if } \tilde{\sigma}(x) = 1, \\ * & \text{if } \tilde{\sigma}(x) = *. \end{cases}$$

The equivalence of these two definitions follows from the random walk construction above. If we look at the configuration between 0 and the first $*$ to the right of 0, we obtain a finite string of holes and particles whose distribution is invariant under exchanging both left/right and hole/particle; this invariance comes from the invariance under reflection of

the random walk path beginning and ending at level 0 which is used to construct the configuration.

We also extend the definition of $\mathcal{E}$ to the whole line, by saying $x \in \mathcal{E}$ whenever $\sigma(x) = *$.

Now we have defined the distribution of $\{\sigma(x)\}_{x\in\mathbb{Z}}$. From the construction we note immediately that $\{\sigma(x)\}_{x\in\mathbb{Z}_+}$ is independent of $\{\sigma(x)\}_{x\in\mathbb{Z}_-}$. Also if we consider the interval $[\![-x, x]\!]$, as $x \to \infty$ the density of $*$ in this interval converges to 0 (since $\mathbb{P}[x \in \mathcal{E}] \to 0$ as $x \to \infty$), and the densities of 1 and 0 converge to ρ and $1-\rho$ respectively.

This distribution is stationary for TASEP with second-class particles, as seen from one of the second-class particles.

Proposition 2.1 ([32, Theorem 1]). *Let $(\sigma_t)_{t\geq 0}$ be TASEP with second-class particles, started from $\sigma_0 = \sigma$. Suppose that at time $t \geq 0$, the second-class particle starting from the origin is at site l_t. Then $\sigma_t(l_t + \cdot)$ has the same distribution as σ.*

Given σ, there are two related projections of it which involve setting all the $*$ symbols except for the one at the origin to be either 1s or 0s.

(1) The simpler one consists of setting all $*$ symbols on positive sites (i.e. $\mathbb{Z}_+$) to be 0, and all $*$ symbols on negative sites (i.e. $\mathbb{Z}_-$) to be 1. This gives a configuration where the non-zero sites are i.i.d. Bernoulli(ρ).

(2) Alternatively, we can follow the opposite rule of setting all $*$ symbols on positive sites to be 1 and all $*$ symbols on negative sites to be 0. Specifically, define a configuration ζ^* by $\zeta^*(0) = *$ and for $x \neq 0$,

$$\zeta^*(x) = \begin{cases} 0 & \text{if } \sigma(x) = 0, \text{ or if } \sigma(x) = * \text{ and } x < 0, \\ 1 & \text{if } \sigma(x) = 1, \text{ or if } \sigma(x) = * \text{ and } x > 0. \end{cases}$$

This gives a configuration which, compared to the product measure of Bernoulli(ρ), has a bias towards particles on positive sites and towards holes on negative sites. This bias decays as one gets further away from the origin.

We define Ψ to be the distribution of this ζ^*. Theorem 2 of [32] says that it is stationary for TASEP as seen from an isolated second-class particle. The bias above reflects the tendency created by the dynamics of the process for the second-class particle to get stuck behind particles and to get stuck in front of holes.

The combination of the two projections above gives a coupling between the configuration ζ^* and the i.i.d. Bernoulli(ρ) configuration in which the discrepancies are precisely the non-zero members of $\mathcal{E}$. The fact that $|\mathcal{E} \cap [\![1, x]\!]|$ grows on the order of $\sqrt{x}$ implies that the product measure of Bernoulli(ρ) and the stationary distribution of TASEP as seen from an isolated second-class particle are mutually singular.

For later calculation, it will be useful to look at the position of the first hole to the right of the origin in $\zeta^* \sim \Psi$ (and similarly the first particle to the left).

Let $X_+ = \min\{x \geq 1 : \zeta^*(x) = 0\}$, which is also $\min\{x \geq 1 : \sigma(x) = 0\}$. From the random walk construction of $\sigma(x), x > 0$, one gets

$$X_+ = \min\{x \geq 1 : Y_1(x) = 0, \text{ and for some } y \in [\![1, x]\!],\ Y_2(y) = 0\}.$$

That is, to find X_+ we look for the first 0 in the process Y_2, and then we look for the first 0 in the process Y_1 from then on. Since all the variables $Y_1(x)$ and $Y_2(x)$ are i.i.d. Bernoulli(ρ), this implies that $X_+ + 1$ is the sum of two Geometric($1-\rho$) random variables, and so

$$\mathbb{P}[X_+ = k] = k(1-\rho)^2\rho^{k-1} \tag{2.4}$$

for $k \geq 1$. Similarly if X_- is the location of the first particle to the left of the origin, then

$$\mathbb{P}[X_- = -k] = k\rho^2(1-\rho)^{k-1}. \tag{2.5}$$

In the next two subsections, we prove two properties of Ψ, respectively: (1) the corresponding stationary process of TASEP as seen from an isolated second-class particle is ergodic in time, and (2) convergence to Ψ starting from the i.i.d. Bernoulli(ρ) configuration, in the averaged sense (in other words, a weak version of Theorem 1.7). These two properties will be key inputs to the rest of this paper.

2.1. Ergodicity

This subsection is devoted to proving the following ergodicity statement. We let $(\zeta_t^*)_{t\in\mathbb{R}}$ denote the process of TASEP as seen from an isolated second-class particle, such that $\zeta_t^* \sim \Psi$ for each t.

Proposition 2.2. *The process $(\zeta_t^*)_{t\in\mathbb{R}}$ is ergodic in time.*

The key step is the following coupling between Ψ and itself.

Lemma 2.3. *For any $L \in \mathbb{N}$ and $\epsilon > 0$, there exist an integer $M > L$, and a coupling between Ψ and itself such that the following is true. Let $\zeta^{(1)}$ and $\zeta^{(2)}$ be sampled from this coupling. Then*

(1) *restricted to $[\![-L, L]\!]$, $\zeta^{(1)}$ and $\zeta^{(2)}$ are independent,*

(2) *with probability $> 1-\epsilon$, $\zeta^{(1)}$ and $\zeta^{(2)}$ have the same number of particles in $[\![-M, -1]\!]$ and in $[\![1, M]\!]$, and $\zeta^{(1)}$ and $\zeta^{(2)}$ are identical on $\mathbb{Z} \setminus [\![-M, M]\!]$.*

To construct this coupling, we revisit the construction of Ψ. For $\zeta^* \sim \Psi$, recall that we defined it on $\mathbb{Z}_+$ using two independent collections of i.i.d. Bernoulli(ρ) random variables $Y_1(x), x \geq 1$, and $Y_2(x), x \geq 1$; and $R_1(x) = \sum_{y=1}^{x} Y_1(y)$, $R_2(x) = \sum_{y=1}^{x} Y_2(y)$. For $x \geq 1$, let

$$\overline{Y}_1(x) = \zeta^*(x) = \begin{cases} 1, & Y_1(x) = 1 \text{ or } x \in \mathcal{E}, \\ 0, & Y_1(x) = 0 \text{ and } x \notin \mathcal{E}, \end{cases}$$

$$\overline{Y}_2(x) = \begin{cases} 0, & Y_2(x) = 0 \text{ or } x \in \mathcal{E}, \\ 1, & Y_2(x) = 1 \text{ and } x \notin \mathcal{E}. \end{cases}$$

Namely, $\bar{Y}_1$ is just ζ^* on $\mathbb{Z}_+$, and $\bar{Y}_2$ is 'paired with' $\bar{Y}_1$ such that $\bar{Y}_1 + \bar{Y}_2 = Y_1 + Y_2$. To see why we define $\bar{Y}_2$, consider

$$\bar{R}_1(x) = \sum_{y=1}^{x} \bar{Y}_1(x) = R_1(x) + M(x),$$
$$\bar{R}_2(x) = \sum_{y=1}^{x} \bar{Y}_2(x) = R_2(x) - M(x).$$

We have $\bar{R}_1(x) - \bar{R}_2(x) = 2M(x) - W(x)$, where W and M are defined in (2.1) and (2.2). In particular, $\bar{R}_1(x) \geq \bar{R}_2(x)$ for all x. Note that $\bar{R}_1(x)$ is the number of particles of ζ^* in $[\![1, x]\!]$. The process $\bar{R}_1$ is certainly not Markovian; however, the process $(\bar{R}_1(x), \bar{R}_2(x)), x \geq 0$, is a Markov chain, and we will exploit this fact.

Consider the transition function $\mathbf{T} : \mathbb{Z}^2_{\geq 0} \times \mathbb{Z}^2_{\geq 0} \to [0, 1]$ defined by

$$\begin{aligned}
&\mathbf{T}((a, b), (a + 1, b + 1)) = \rho^2,\\
&\mathbf{T}((a, b), (a + 1, b)) = \rho(1 - \rho)\frac{a - b + 2}{a - b + 1},\\
&\mathbf{T}((a, b), (a, b + 1)) = \rho(1 - \rho)\frac{a - b}{a - b + 1},\\
&\mathbf{T}((a, b), (a, b)) = (1 - \rho)^2.
\end{aligned}$$

Lemma 2.4. *The process $(\bar{R}_1, \bar{R}_2)$ is a Markov chain in $\mathbb{Z}^2_{\geq 0}$ with transition probability* $\mathbf{T}$.

Proof. For any $x \geq 0$, we show that

$$\begin{aligned}
\mathbb{P}[\{\bar{R}_1(y)\}_{y=0}^x = \{r_1(y)\}_{y=0}^x, \{\bar{R}_2(y)\}_{y=0}^x = \{r_2(y)\}_{y=0}^x, M(x) = h]\\
= \rho^{r_1(x)+r_2(x)}(1 - \rho)^{2x-r_1(x)-r_2(x)}
\end{aligned} \tag{2.6}$$

for any integers $\{r_1(y)\}_{y=0}^x$, $\{r_2(y)\}_{y=0}^x$ and h such that

(1) $r_1(0) = r_2(0) = 0$,

(2) $r_1(y) - r_1(y - 1), r_2(y) - r_2(y - 1) \in \{0, 1\}$, and $r_1(y) \geq r_2(y)$ for any $1 \leq y \leq x$,

(3) $0 \leq h \leq r_1(x) - r_2(x)$.

We prove this by induction on x. The base case (of $x = 0$) is trivial, and now we assume that it is true for x, and consider $x + 1$.

Note that $x + 1 \in \mathcal{E}$ if the following three conditions all hold: (i) $M(x) = W(x)$ (i.e. $\bar{R}_1(x) - \bar{R}_2(x) = M(x)$); (ii) $Y_1(x + 1) = 0$; (iii) $Y_2(x + 1) = 1$. In that case, $\bar{R}_1(x + 1) = \bar{R}_1(x) + 1$, $\bar{R}_2(x + 1) = \bar{R}_2(x)$, and $M(x + 1) = M(x) + 1$. In any other case, $\bar{R}_1(x + 1) = \bar{R}_1(x) + Y_1(x + 1)$, $\bar{R}_2(x + 1) = \bar{R}_2(x) + Y_2(x + 1)$, and $M(x + 1) = M(x)$.

Denote $y_1(x+1) = r_1(x+1) - r_1(x)$ and $y_2(x+1) = r_2(x+1) - r_2(x)$. From the above transition we deduce that when $h \le r_1(x) - r_2(x)$,

$$\begin{aligned}
&\mathbb{P}[\{\bar{R}_1(y)\}_{y=0}^{x+1} = \{r_1(y)\}_{y=0}^{x+1},\ \{\bar{R}_2(y)\}_{y=0}^{x+1} = \{r_2(y)\}_{y=0}^{x+1},\ M(x+1) = h] \\
&\quad = \mathbb{P}[\{\bar{R}_1(y)\}_{y=0}^{x} = \{r_1(y)\}_{y=0}^{x},\ \{\bar{R}_2(y)\}_{y=0}^{x} = \{r_2(y)\}_{y=0}^{x},\ M(x) = h] \\
&\qquad \times \mathbb{P}[Y_1(x+1) = y_1(x+1),\ Y_2(x+1) = y_2(x+1)],
\end{aligned}$$

where the second probability on the right-hand side equals

$$\rho^{y_1(x+1)+y_2(x+1)}(1-\rho)^{2-y_1(x+1)-y_2(x+1)}.$$

When $h > r_1(x) - r_2(x)$, we must have $h = r_1(x) - r_2(x) + 1$ and $y_1(x+1) = 1$, $y_2(x+1) = 0$, and

$$\begin{aligned}
&\mathbb{P}[\{\bar{R}_1(y)\}_{y=0}^{x+1} = \{r_1(y)\}_{y=0}^{x+1},\ \{\bar{R}_2(y)\}_{y=0}^{x+1} = \{r_2(y)\}_{y=0}^{x+1},\ M(x+1) = h] \\
&\quad = \mathbb{P}[\{\bar{R}_1(y)\}_{y=0}^{x} = \{r_1(y)\}_{y=0}^{x},\ \{\bar{R}_2(y)\}_{y=0}^{x} = \{r_2(y)\}_{y=0}^{x},\ M(x) = h-1] \\
&\qquad \times \mathbb{P}[Y_1(x+1) = 0,\ Y_2(x+1) = 1],
\end{aligned}$$

where the second probability on the right-hand side equals $\rho(1-\rho)$, which also equals $\rho^{y_1(x+1)+y_2(x+1)}(1-\rho)^{2-y_1(x+1)-y_2(x+1)}$. Thus by the induction hypothesis ((2.6) for x), we get (2.6) for $x+1$.

Finally, by summing over all h, we conclude that

$$\begin{aligned}
&\mathbb{P}[\{\bar{R}_1(y)\}_{y=0}^{x} = \{r_1(y)\}_{y=0}^{x},\ \{\bar{R}_2(y)\}_{y=0}^{x} = \{r_2(y)\}_{y=0}^{x}] \\
&\qquad = (r_1(x) - r_2(x) + 1)\rho^{r_1(x)+r_2(x)}(1-\rho)^{2x-r_1(x)-r_2(x)}.
\end{aligned}$$

Using this we conclude that

$$\begin{aligned}
&\mathbb{P}[\bar{R}_1(x+1) = r_1(x+1),\ \bar{R}_2(x+1) = r_2(x+1) \mid \\
&\qquad \{\bar{R}_1(y)\}_{y=0}^{x} = \{r_1(y)\}_{y=0}^{x}, \{\bar{R}_2(y)\}_{y=0}^{x} = \{r_2(y)\}_{y=0}^{x}] \\
&\qquad = \frac{r_1(x+1) - r_2(x+1) + 1}{r_1(x) - r_2(x) + 1}\rho^{y_1(x)+y_2(x)}(1-\rho)^{2-y_1(x)-y_2(x)},
\end{aligned}$$

which implies the conclusion. ∎

We have the following mixing property of this Markov chain.

Lemma 2.5. *For any $u, v \in \mathbb{Z}^2_{\ge 0}$, we have* $\lim_{n\to\infty} \|\mathbf{T}^n(u,\cdot) - \mathbf{T}^n(v,\cdot)\|_1 = 0$.

Proof. Our strategy is to construct a coupling between two Markov chains, each with transition probability $\mathbf{T}$, starting from u and v respectively.

To construct the coupling, we recursively define a random process

$$(A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)}) : \mathbb{Z}_{\ge 0} \to \mathbb{Z}^2 \times \mathbb{Z}^2_{\ge 0}.$$

For $x \in \mathbb{Z}_{\geq 0}$, given $A^{(1)}(y)$, $B^{(1)}(y)$, $A^{(2)}(y)$, $B^{(2)}(y)$ for each $y \in [\![0, x]\!]$, we define $A^{(1)}(x+1)$, $B^{(1)}(x+1)$, $A^{(2)}(x+1)$, $B^{(2)}(x+1)$ as follows. First, we let U_0 be a Bernoulli$(2\rho(1-\rho))$ random variable.

(1) If $U_0 = 0$ we do the following. Let $B^{(1)}(x+1) = B^{(1)}(x)$ and $B^{(2)}(x+1) = B^{(2)}(x)$. If $A^{(1)}(x) = A^{(2)}(x)$, we let

$$A^{(1)}(x+1) = A^{(2)}(x+1) = A^{(1)}(x) + 2U_1 - 1 = A^{(2)}(x) + 2U_1 - 1;$$

otherwise we let

$$A^{(1)}(x+1) = A^{(1)}(x) + 2U_1 - 1 \quad \text{and} \quad A^{(2)}(x+1) = A^{(2)}(x) + 2U_2 - 1.$$

Here U_1 and U_2 are independent Bernoulli$(\frac{\rho^2}{\rho^2+(1-\rho)^2})$ random variables, and are independent of U_0.

(2) If $U_0 = 1$ we do the following. Let $A^{(1)}(x+1) = A^{(1)}(x)$ and $A^{(2)}(x+1) = A^{(2)}(x)$.

- If $B^{(1)}(x) = B^{(2)}(x)$, we let $B^{(1)}(x+1) = B^{(2)}(x+1) = B^{(1)}(x) + 2U_3 - 1 = B^{(2)}(x) + 2U_3 - 1$.
- If $B^{(1)}(x) \neq B^{(2)}(x)$ and $\max_{0 \leq y \leq x} B^{(1)}(y) \geq N$, we let $B^{(1)}(x+1) = B^{(1)}(x) + 2U_3 - 1$ and $B^{(2)}(x+1) = B^{(2)}(x) + 2U_4 - 1$.
- If $B^{(1)}(x) < B^{(2)}(x)$ and $\max_{0 \leq y \leq x} B^{(1)}(y) < N$, we let $B^{(1)}(x+1) = B^{(1)}(x) + 2U_3 - 1$ and $B^{(2)}(x+1) = B^{(2)}(x) + 2U_3U_5 - 1$.
- If $B^{(1)}(x) > B^{(2)}(x)$ and $\max_{0 \leq y \leq x} B^{(1)}(y) < N$, we let $B^{(1)}(x+1) = B^{(1)}(x) + 2U_4U_6 - 1$ and $B^{(2)}(x+1) = B^{(2)}(x) + 2U_4 - 1$.

Here $N > 0$ is a number to be determined; and U_3, U_4, U_5, U_6 are independent with distribution being

$$\begin{aligned}
&\text{Bernoulli}\left(\frac{B^{(1)}(x)+2}{2B^{(1)}(x)+2}\right),\\
&\text{Bernoulli}\left(\frac{B^{(2)}(x)+2}{2B^{(2)}(x)+2}\right),\\
&\text{Bernoulli}\left(\frac{B^{(2)}(x)+2}{2B^{(2)}(x)+2}\cdot\frac{2B^{(1)}(x)+2}{B^{(1)}(x)+2}\right),\\
&\text{Bernoulli}\left(\frac{2B^{(2)}(x)+2}{B^{(2)}(x)+2}\cdot\frac{B^{(1)}(x)+2}{2B^{(1)}(x)+2}\right),
\end{aligned}$$

respectively; and they are independent of U_0, U_1, U_2.

The reason behind the construction of $(A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)})$ is that, if we set the initial condition to be $A^{(1)}(0) = d(u)$, $B^{(1)}(0) = ad(u)$, and $A^{(2)}(0) = d(v)$, $B^{(2)}(0) = ad(v)$, for $u, v \in \mathbb{Z}^2_{\geq 0}$, then the processes $x \mapsto (\frac{A^{(1)}(x)+B^{(1)}(x)+x}{2}, \frac{A^{(1)}(x)-B^{(1)}(x)+x}{2})$ and $x \mapsto (\frac{A^{(2)}(x)+B^{(2)}(x)+x}{2}, \frac{A^{(2)}(x)-B^{(2)}(x)+x}{2})$ are Markov chains with the same transition probability $\mathbf{T}$, starting from u and v respectively. Indeed, from this construction, it is easy to check that for each $i = 1, 2$, $(A^{(i)}, B^{(i)})$ is a Markov chain, with transition probability

given by

$$\begin{aligned}
&\mathbb{P}[A^{(i)}(x+1) = A^{(i)}(x)+1,\ B^{(i)}(x+1) = B^{(i)}(x) \mid A^{(i)}(x), B^{(i)}(x)] = \rho^2,\\
&\mathbb{P}[A^{(i)}(x+1) = A^{(i)}(x)-1,\ B^{(i)}(x+1) = B^{(i)}(x) \mid A^{(i)}(x), B^{(i)}(x)] = (1-\rho)^2,\\
&\mathbb{P}[A^{(i)}(x+1) = A^{(i)}(x),\ B^{(i)}(x+1) = B^{(i)}(x)+1 \mid A^{(i)}(x), B^{(i)}(x)]\\
&\qquad\qquad = \rho(1-\rho)\frac{B^{(i)}(x)+2}{B^{(i)}(x)+1},\\
&\mathbb{P}[A^{(i)}(x+1) = A^{(i)}(x),\ B^{(i)}(x+1) = B^{(i)}(x)-1 \mid A^{(i)}(x), B^{(i)}(x)]\\
&\qquad\qquad = \rho(1-\rho)\frac{B^{(i)}(x)}{B^{(i)}(x)+1}.
\end{aligned} \tag{2.7}$$

From the construction above there are several other key properties to note:

(1) If $B^{(1)}(x) \neq B^{(2)}(x)$ and $\max_{0\le y\le x} B^{(1)}(y) < N$, then always

$$|B^{(1)}(x+1) - B^{(2)}(x+1)| \le |B^{(1)}(x) - B^{(2)}(x)|.$$

(2) If $A^{(1)}(x) = A^{(2)}(x)$ (resp. $B^{(1)}(x) = B^{(2)}(x)$), then for any $y \ge x$ we must have $A^{(1)}(y) = A^{(2)}(y)$ (resp. $B^{(1)}(y) = B^{(2)}(y)$).

(3) The processes $A^{(1)}$ and $A^{(2)}$ are independent random walks until they are equal; starting from the first time when $B^{(1)}$ reaches N, the processes $B^{(1)}$ and $B^{(2)}$ are independent until they are equal.

To show that $\lim_{n\to\infty} \|\mathbf{T}^n(u,\cdot) - \mathbf{T}^n(v,\cdot)\|_1 = 0$, it now suffices to show that

$$\liminf_{x\to\infty} \mathbb{P}[A^{(1)}(x) = A^{(2)}(x),\ B^{(1)}(x) = B^{(2)}(x)] > 1-\epsilon \tag{2.8}$$

for any $\epsilon > 0$ and some choice of N. First, we have $A^{(1)}(x) = A^{(2)}(x)$ for all large enough x, by the third property above.

We next show that when N is large enough depending on u, v, ϵ, with probability at least $1-\epsilon$ we have $B^{(1)}(x) = B^{(2)}(x)$ for some large enough x (thus for all large x, by (2) above). Let $x_0 = \min\{x \in \mathbb{Z}_{\ge 0} : B^{(1)}(x) = N\}$. We have $x_0 < \infty$ almost surely, since $B^{(1)}$ dominates a simple random walk.

As stated in (3), given $B^{(1)}(x_0)$ and $B^{(2)}(x_0)$, the processes $B^{(1)}(x_0+x)$ and $B^{(2)}(x_0+x)$ for $x \ge 0$ are independent (until they are equal); and we further note that when N is taken large they should be very close to two independent random walks. To make this more precise, we define proxies of $B^{(1)}$ and $B^{(2)}$. For $i = 1, 2$, let $V^{(i)} : \mathbb{Z}_{\ge 0} \to \mathbb{Z}$ be a random walk satisfying $V^{(i)}(0) = B^{(i)}(x_0)$, and

$$\begin{aligned}
\mathbb{P}[V^{(i)}(x+1) = V^{(i)}(x) \mid V^{(i)}(x)] &= \rho^2 + (1-\rho)^2,\\
\mathbb{P}[V^{(i)}(x+1) = V^{(i)}(x)+1 \mid V^{(i)}(x)] &= \rho(1-\rho),\\
\mathbb{P}[V^{(i)}(x+1) = V^{(i)}(x)-1 \mid V^{(i)}(x)] &= \rho(1-\rho).
\end{aligned} \tag{2.9}$$

Also we let $V^{(1)}$ and $V^{(2)}$ be independent, until $V^{(1)}(x_1) = V^{(2)}(x_1)$ for some $x_1 > 0$, and let $V^{(1)}(x) = V^{(2)}(x)$ for all $x > x_1$. For some N_1 large enough (depending on u, v, ϵ) we have $\mathbb{P}[x_1 < N_1] > 1 - \epsilon/2$, thus

$$\mathbb{P}[V^{(1)}(N_1) = V^{(2)}(N_1)] > 1 - \epsilon/2. \tag{2.10}$$

By comparing the transition probabilities (2.7) and (2.9), we can couple $B^{(1)}, B^{(2)}$ with $V^{(1)}, V^{(2)}$ in such a way that for any $x \geq 0$, given that $B^{(1)}(x_0 + x) = V^{(1)}(x)$ and $B^{(2)}(x_0 + x) = V^{(2)}(x)$, we have $B^{(1)}(x_0 + x + 1) = V^{(1)}(x + 1)$ and $B^{(2)}(x_0 + x + 1) = V^{(2)}(x + 1)$ with probability at least

$$1 - \rho(1-\rho)\left(\frac{1}{B^{(1)}(x_0 + x) + 1} + \frac{1}{B^{(2)}(x_0 + x) + 1}\right) > 1 - \frac{2\rho(1-\rho)}{N - x - \|u - v\|_1}.$$

Here the inequality is due to that $B^{(1)}(x_0 + x) \geq B^{(1)}(x_0) - x = N - x$, and $B^{(2)}(x_0 + x) \geq B^{(1)}(x_0 + x) - |B^{(1)}(0) - B^{(2)}(0)| \geq N - x - \|u - v\|_1$, using (1) above. Under this coupling, by taking a union bound over x we find that $V^{(1)}(x) = B^{(1)}(x_0 + x)$ and $V^{(2)}(x) = B^{(2)}(x_0 + x)$ for any $0 \leq x \leq N_1$ with probability at least $1 - \frac{2\rho(1-\rho)N_1}{N - N_1 - \|u-v\|_1}$. By taking N large enough (depending on N_1, ϵ, u, v) we can make this probability $> 1 - \epsilon/2$. From this and (2.10), we have $\mathbb{P}[B^{(1)}(x_0 + N_1) = B^{(2)}(x_0 + N_1)] > 1 - \epsilon$. This implies (2.8), and the conclusion follows. ∎

We let $\mathcal{S}$ denote the law of a Markov chain starting from $(0, 0)$ with transition probability $\mathbf{T}$, i.e. the law of $(\bar{R}_1, \bar{R}_2)$. From the above lemma we can construct a coupling between $\mathcal{S}$ and itself, as follows.

Lemma 2.6. *For any $L \in \mathbb{N}$ and $\epsilon > 0$, there exist an integer $M > L$ and a coupling between $\mathcal{S}$ and itself such that the following is true. Let $(\bar{R}_1^{(1)}, \bar{R}_2^{(1)})$ and $(\bar{R}_1^{(2)}, \bar{R}_2^{(2)})$ be sampled from this coupling. Then*

(1) *restricted to $[\![0, L]\!]$, $(\bar{R}_1^{(1)}, \bar{R}_2^{(1)})$ and $(\bar{R}_1^{(2)}, \bar{R}_2^{(2)})$ are independent,*

(2) $\mathbb{P}[\bar{R}_1^{(1)}(M) = \bar{R}_1^{(2)}(M), \bar{R}_2^{(1)}(M) = \bar{R}_2^{(2)}(M)] > 1 - \epsilon$.

Proof. We construct the coupling by first allowing $(\bar{R}_1^{(1)}, \bar{R}_2^{(1)})$ and $(\bar{R}_1^{(2)}, \bar{R}_2^{(2)})$ to evolve independently for the first L steps. Then conditioned on $(\bar{R}_1^{(1)}(L), \bar{R}_2^{(1)}(L))$ and on $(\bar{R}_1^{(2)}(L), \bar{R}_2^{(2)}(L))$, we couple $(\bar{R}_1^{(1)}(M), \bar{R}_2^{(1)}(M))$ and $(\bar{R}_1^{(2)}(M), \bar{R}_2^{(2)}(M))$ to maximize the probability that they coincide. The conclusion follows from Lemma 2.5 by taking M large enough, since there are only finitely many possible values of $(\bar{R}_1^{(1)}(L), \bar{R}_2^{(1)}(L))$ and $(\bar{R}_1^{(2)}(L), \bar{R}_2^{(2)}(L))$. ∎

Proof of Lemma 2.3. From the coupling of two copies of $\bar{R}_1$ given by Lemma 2.6, we get a coupling between two copies of $\bar{Y}_1$, thus two copies of $\zeta^* \sim \Psi$ on $\mathbb{Z}_+$. We can similarly couple two copies of $\zeta^* \sim \Psi$ on $\mathbb{Z}_-$. As Ψ on $\mathbb{Z}_+$ and Ψ on $\mathbb{Z}_-$ are independent, we get the desired coupling satisfying the statement of this lemma. ∎

We can now prove ergodicity of the stationary process of TASEP as seen from an isolated second-class particle, using the coupling given by Lemma 2.3.

Proof of Proposition 2.2. We assume the contrary. Then there is a measurable set $B \subset \{0, 1, *\}^{\mathbb{Z}}$ invariant under the Markov process (of TASEP as seen from an isolated second-class particle) with $0 < \Psi(B) < 1$. Let $\zeta^* \sim \Psi$. For any $L \in \mathbb{N}$ we consider the random variable $\chi_L(\zeta^*) = \mathbb{P}[\zeta^* \in B \mid \{\zeta^*(x)\}_{x \in [\![-L,L]\!]}]$. Note that this is a martingale in L, and almost surely converges to $\mathbb{1}[\zeta^* \in B]$. Thus for any $\epsilon > 0$, we can take L large enough such that $\mathbb{P}[|\chi_L(\zeta^*) - \mathbb{1}[\zeta^* \in B]| > \epsilon] < \epsilon$.

For the above L and ϵ, by Lemma 2.3 we can find $M > L$ and a coupling between Ψ and itself. Suppose that $\zeta^{(1)}, \zeta^{(2)}$ are sampled from this coupling. By the first property of the coupling, and since χ_L only depends on the configuration in $[\![-L, L]\!]$, we see that $\chi_L(\zeta^{(1)})$ and $\chi_L(\zeta^{(2)})$ are independent. Thus

$$\mathbb{P}[\chi_L(\zeta^{(1)}) > 1 - \epsilon,\ \chi_L(\zeta^{(2)}) < \epsilon] = \mathbb{P}[\chi_L(\zeta^{(1)}) > 1 - \epsilon]\mathbb{P}[\chi_L(\zeta^{(2)}) < \epsilon].$$

Note $\zeta^{(1)} \in B$ and $|\chi_L(\zeta^{(1)}) - \mathbb{1}[\zeta^{(1)} \in B]| < \epsilon$ imply that $\chi_L(\zeta^{(1)}) > 1 - \epsilon$, so

$$\begin{aligned}\mathbb{P}[\chi_L(\zeta^{(1)}) > 1 - \epsilon] &\geq \mathbb{P}[\zeta^{(1)} \in B] - \mathbb{P}\big[|\chi_L(\zeta^{(1)}) - \mathbb{1}[\zeta^{(1)} \in B]| > \epsilon\big] \\ &> \Psi(B) - \epsilon,\end{aligned}$$

and similarly

$$\begin{aligned}\mathbb{P}[\chi_L(\zeta^{(2)}) < \epsilon] &\geq \mathbb{P}[\zeta^{(2)} \notin B] - \mathbb{P}\big[|\chi_L(\zeta^{(2)}) - \mathbb{1}[\zeta^{(2)} \in B]| > \epsilon\big] \\ &> 1 - \Psi(B) - \epsilon.\end{aligned}$$

Combining the above three inequalities, we have

$$\mathbb{P}[\chi_L(\zeta^{(1)}) > 1 - \epsilon,\ \chi_L(\zeta^{(2)}) < \epsilon] > \Psi(B)(1 - \Psi(B)) - \epsilon.$$

Using $\mathbb{P}[|\chi_L(\zeta^{(1)}) - \mathbb{1}[\zeta^{(1)} \in B]| > \epsilon] < \epsilon$ and $\mathbb{P}[|\chi_L(\zeta^{(2)}) - \mathbb{1}[\zeta^{(2)} \in B]| > \epsilon] < \epsilon$ again, we have

$$\mathbb{P}[\zeta^{(1)} \in B,\ \zeta^{(2)} \notin B] > \Psi(B)(1 - \Psi(B)) - 3\epsilon.$$

Using the second property of the coupling (from Lemma 2.3), and by taking ϵ small enough, we conclude that with probability $> \Psi(B)(1 - \Psi(B)) - 4\epsilon > 0$, all of the following conditions are satisfied: $\zeta^{(1)} \in B$ and $\zeta^{(2)} \notin B$, and $\zeta^{(1)}$ and $\zeta^{(2)}$ are identical on $\mathbb{Z} \setminus [\![-M, M]\!]$, and they have the same number of particles in $[\![-M, -1]\!]$ and in $[\![1, M]\!]$.

Assuming that $\zeta^{(1)}$ and $\zeta^{(2)}$ satisfy the above conditions, we next couple two TASEPs starting from $\zeta^{(1)}$ and $\zeta^{(2)}$ at time 0, such that switches happen between neighboring sites with the same Poisson clocks. With positive probability the following happens: from time 0 to time 1, no switch happens between sites x and $x + 1$, for $x \in \{-M - 1, -1, 0, M\}$; and switches happen between sites x and $x + 1$, sequentially for $x = -M, \ldots, -2$ and for $x = 1, \ldots, M - 1$, and repeat this for M times. Then at time 1 the two processes starting from $\zeta^{(1)}$ and $\zeta^{(2)}$ would be identical. However, as B and $\{0, 1, *\}^{\mathbb{Z}} \setminus B$ are assumed to be invariant under the evolution of TASEP as seen from an isolated second-class particle, we get a subset of $\{0, 1, *\}^{\mathbb{Z}}$ with positive Ψ measure, and contained (up to a zero measure set) in both B and $\{0, 1, *\}^{\mathbb{Z}} \setminus B$. This is a contradiction. ∎

2.2. *Convergence in the averaged sense*

As indicated in the introduction, we consider the process $(\eta_t^*)_{t\geq 0}$, which is TASEP with a single second-class particle such that $\eta_0^*(x)$ are i.i.d Bernoulli(ρ) for $x \in \mathbb{Z} \setminus \{0\}$ and $\eta_0^*(0) = *$. We define Φ_t to be the law of $\eta_t^*(l_t + \cdot)$, where l_t is the location of the second-class particle at time t. In this subsection we prove a weak version of Theorem 1.7, i.e. the convergence of Φ_t to Ψ in the averaged sense.

Proposition 2.7. *We have $T^{-1}\int_0^T \Phi_t\, dt \to \Psi$ weakly as $T \to \infty$.*

Our strategy to prove this is to construct a coupling between Φ_t and Ψ using σ, the stationary configuration of TASEP with infinitely many second-class particles constructed in (2.3).

Recall that we have the following two projections of σ: first, if we set all $*$ symbols on positive sites to be 0, and all $*$ symbols on negative sites to be 1, we get i.i.d. Bernoulli(ρ) on all non-zero sites; second, if we set all $*$ symbols on positive sites to be 1, and all $*$ symbols on negative sites to be 0, we get a distribution which is stationary for TASEP as seen from an isolated second-class particle (see the discussion after Proposition 2.1).

Now let $(\sigma_t)_{t\geq 0}$ be TASEP with (infinitely many) second-class particles, and starting from $\sigma_0 = \sigma$. At time 0, we label all the second-class particles with $\mathbb{Z}$ from right to left, so that the one at the origin is labeled 0. We consider two ways in which the labels evolve:

- Rule (a): for all second-class particles, the labels never change.
- Rule (b): for two second-class particles labeled $i > j$, if they are at sites x and $x+1$, then with rate 1 they exchange their labels.

We note that when forgetting the labels, the dynamic is unchanged. For each $i \in \mathbb{Z}$ and $t \geq 0$, we denote by $l_t^{a,i}$ the location of the second-class particle labeled by i at time t, under Rule (a). Then for each $i \in \mathbb{Z}$ we have $l_t^{a,i} > l_t^{a,i+1}$, and there is no other second-class particle between sites $l_t^{a,i}$ and $l_t^{a,i+1}$. We also denote by $l_t^{b,i}$ the location of the second-class particle labeled by i at time t, under Rule (b). Define $\sigma_t^{a,i}, \sigma_t^{b,i} : \mathbb{Z} \to \{0, 1, *\}$ as $\sigma_t^{a,i}(x) = \sigma_t(x + l_t^{a,i})$ and $\sigma_t^{b,i}(x) = \sigma_t(x + l_t^{b,i})$, which is σ_t as seen from the second-class particle labeled by i, under each rule.

Our strategy to construct the coupling between Ψ and Φ_t is to project $\sigma_t^{b,0}$ in two different ways, to get these two measures respectively (see Figure 2). For the first way, we just look at the law of $\sigma_t^{b,0}$ without considering the labels. As σ is a renewal process, and σ is stationary (Proposition 2.1), we know that $\sigma_t^{a,i}$ has the same distribution as σ. We next show that the same is true for $\sigma_t^{b,i}$.

Lemma 2.8. *For each $i \in \mathbb{Z}$ and $t \geq 0$, $\sigma_t^{b,i}$ has the same distribution as σ.*

Proof. Take any measurable set $B \subset \{0, 1, *\}^{\mathbb{Z}}$; it suffices to show that $\mathbb{P}[\sigma_t^{b,i} \in B] = \mathbb{P}[\sigma \in B]$.

We fix $t \geq 0$. As each second-class particle jumps with rate at most 1, for any $\epsilon > 0$ we can find $M > 0$ such that $\mathbb{P}[|l_t^{a,i} - l_t^{b,i}| > M] < \epsilon$ for any $i \in \mathbb{Z}$. Take a large $N \in \mathbb{N}$.

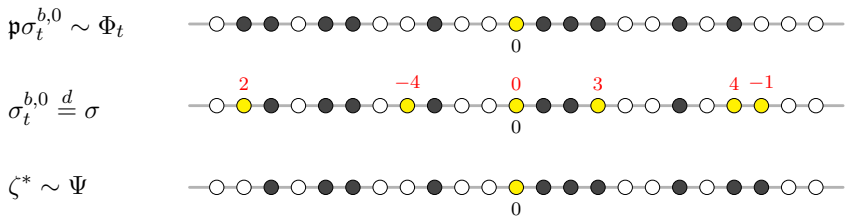


Fig. 2. A coupling between Ψ and Φ_t via $\sigma_t^{b,0}$. The red numbers are labels of second-class particles. Here ζ^* and $\mathfrak{p}\sigma_t^{b,0}$ are the same on $[\![-9, 9]\!]$.

For each i with $|i| \le N - M$, if $|l_t^{b,i} - l_t^{a,i}| \le M$, we must have $l_t^{b,i} \in \{l_t^{a,j} : i - M \le j \le i + M\} \subset \{l_t^{a,j} : -N \le j \le N\}$, since the set $\{l_t^{a,j} : i - M \le j \le i + M\}$ contains all locations of second-class particles in $[\![l_t^{a,i} - M, l_t^{a,i} + M]\!]$. We then have

$$\begin{aligned}
&\mathbb{E}[|\{l_t^{b,i} : -N \le i \le N\} \setminus \{l_t^{a,i} : -N \le i \le N\}|] \\
&\qquad = \sum_{|i| \le N} \mathbb{P}[l_t^{a,i} \notin \{l_t^{a,j} : -N \le j \le N\}] \\
&\qquad \le 2M + \sum_{|i| \le N-M} \mathbb{P}[l_t^{a,i} \notin \{l_t^{a,j} : -N \le j \le N\}] \\
&\qquad \le 2M + \sum_{|i| \le N-M} \mathbb{P}[|l_t^{b,i} - l_t^{a,i}| > M] \\
&\qquad \le 2M + 2N\epsilon. \qquad (2.11)
\end{aligned}$$

Since both $|\{l_t^{b,i} : -N \le i \le N\}|$ and $|\{l_t^{a,i} : -N \le i \le N\}|$ equal $2N + 1$, we have

$$\begin{aligned}
&|\{l_t^{b,i} : -N \le i \le N\} \setminus \{l_t^{a,i} : -N \le i \le N\}| \\
&\qquad = |\{l_t^{a,i} : -N \le i \le N\} \setminus \{l_t^{b,i} : -N \le i \le N\}|,
\end{aligned}$$

so

$$\mathbb{E}[|\{l_t^{a,i} : -N \le i \le N\} \setminus \{l_t^{b,i} : -N \le i \le N\}|] \le 2M + 2N\epsilon.$$

Thus since ϵ is arbitrarily taken, we have

$$\lim_{N\to\infty} \frac{1}{2N+1} \big(\mathbb{E}[|\{-N \le i \le N : \sigma_t^{a,i} \in B\}|] - \mathbb{E}[|\{-N \le i \le N : \sigma_t^{b,i} \in B\}|]\big) = 0.$$

Since for each $i \in \mathbb{Z}$, $\sigma_t^{a,i}$ has the same distribution as σ, we have

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{2N+1} \mathbb{E}[|\{-N \le i \le N : \sigma_t^{a,i} \in B\}|] &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} \mathbb{P}[\sigma_t^{a,i} \in B] \\
&= \mathbb{P}[\sigma \in B].
\end{aligned}$$

By combining the above two equalities, we have

$$\lim_{N\to\infty} \frac{1}{2N+1}\mathbb{E}[|\{-N \le i \le N : \sigma_t^{b,i} \in B\}|] = \mathbb{P}[\sigma \in B].$$

Now that σ is a renewal process, $\sigma_0^{b,i}$, and thus $\sigma_t^{b,i}$, has the same distribution for all i. Thus the left-hand side in the previous equation equals $\mathbb{P}[\sigma_t^{b,i} \in B]$ for each $i \in \mathbb{Z}$, and the conclusion follows. ∎

Now since $\sigma_t^{b,0}$ has the same distribution as σ, we can just identify all $*$ with 1 in $\mathbb{Z}_+$, and identify all $*$ with 0 in $\mathbb{Z}_-$, and get $\zeta^* \sim \Psi$ (by Lemma 2.8). For the other projection we need to look at the labels. We define $\mathfrak{p}\sigma_t^{b,0} : \mathbb{Z} \to \{0, 1, *\}$ from $\sigma_t^{b,0}$, by identifying all second-class particles whose labels are < 0 with holes, and all second-class particles whose labels are > 0 with particles. Formally, we let $\mathfrak{p}\sigma_t^{b,0}(0) = *$, and $\mathfrak{p}\sigma_t^{b,0}(x) = 1$ for any x such that $\sigma_t(x + l_t^{b,0}) = 1$ or $x = l_t^{b,i} - l_t^{b,0}$ for some $i > 0$; and $\mathfrak{p}\sigma_t^{b,0}(x) = 0$ for any x such that $\sigma_t(x + l_t^{b,0}) = 0$ or $x = l_t^{b,i} - l_t^{b,0}$ for some $i < 0$. See Figure 2 for an illustration of $\mathfrak{p}\sigma_t^{b,0}$.

Lemma 2.9. *For each $t \ge 0$, we have $\mathfrak{p}\sigma_t^{b,0} \sim \Phi_t$.*

Proof. We just need to check that $(\mathfrak{p}\sigma_t^{b,0})_{t\ge 0}$ is TASEP as seen from an isolated second-class particle, and $\mathfrak{p}\sigma_0^{b,0}$ is i.i.d. Bernoulli(ρ) on all non-zero sites.

We first consider the initial configuration $\mathfrak{p}\sigma_0^{b,0}$. It is obtained from $\sigma_0 = \sigma$ by setting all $*$ symbols in $\mathbb{Z}_+$ to be 0 and all $*$ symbols in $\mathbb{Z}_-$ to be 1. This is because at $t = 0$, the second-class particles in $\mathbb{Z}_+$ have negative labels, and the second-class particles in $\mathbb{Z}_-$ have positive labels. Recall (from the discussion after Proposition 2.1) that this implies that $\mathfrak{p}\sigma_0^{b,0}$ is i.i.d. Bernoulli(ρ) on all non-zero sites.

We next consider the evolution of $(\mathfrak{p}\sigma_t^{b,0})_{t\ge 0}$. We now define $(\mathfrak{p}\sigma_t)_{t\ge 0}$ from σ_t, by identifying all second-class particles whose labels are < 0 with holes and all second-class particles whose labels are > 0 with particles. Then $\mathfrak{p}\sigma_t(l_t^{b,0}) = *$, and $\mathfrak{p}\sigma_t(x) = 1$ for any x such that $\sigma_t(x) = 1$ or $x = l_t^{b,i}$ for some $i > 0$; and $\mathfrak{p}\sigma_t(x) = 0$ for any x such that $\sigma_t(x) = 0$ or $x = l_t^{b,i}$ for some $i < 0$. Then $\mathfrak{p}\sigma_t$ is precisely $\mathfrak{p}\sigma_t^{b,0}$ shifted by $l_t^{b,i}$, and it suffices to check that the evolution of $(\mathfrak{p}\sigma_t)_{t\ge 0}$ is given by TASEP with a single second-class particle. For $(\sigma_t)_{t\ge 0}$ and the labels evolving under Rule (b), recall that it is driven by the following generators, independently for all $x \in \mathbb{Z}$:

(1) If $\sigma_t(x) = 1$ and $\sigma_t(x + 1) = 0$, with rate 1 we switch $\sigma_t(x)$ and $\sigma_t(x + 1)$.

(2) If $\sigma_t(x) = 1$ and $\sigma_t(x + 1) = *$ with $l_t^{b,i} = x + 1$ for some $i \in \mathbb{Z}$, with rate 1 we switch $\sigma_t(x)$ and $\sigma_t(x + 1)$ and set $l_t^{b,i} = x$.

(3) If $\sigma_t(x) = *$ with $l_t^{b,i} = x + 1$ for some $i \in \mathbb{Z}$, and $\sigma_t(x + 1) = 0$, with rate 1 we switch $\sigma_t(x)$ and $\sigma_t(x + 1)$ and set $l_t^{b,i} = x + 1$.

(4) If $\sigma_t(x) = \sigma_t(x + 1) = *$ with $l_t^{b,i} = x$ and $l_t^{b,j} = x + 1$ for some $i > j$, with rate 1 we set $l_t^{b,i} = x + 1$ and $l_t^{b,j} = x$.

From the definition of $(\mathfrak{p}\sigma_t)_{t\ge 0}$, these generators degenerate in the sense that for each $x \in \mathbb{Z}$ we switch $\mathfrak{p}\sigma_t(x)$ and $\mathfrak{p}\sigma_t(x + 1)$ with rate 1, if one of the following happens:

(a) $\mathfrak{p}\sigma_t(x) = 1$ and $\mathfrak{p}\sigma_t(x+1) = 0$.

(b) $\mathfrak{p}\sigma_t(x) = 1$ and $\mathfrak{p}\sigma_t(x+1) = *$.

(c) $\mathfrak{p}\sigma_t(x) = *$ and $\mathfrak{p}\sigma_t(x+1) = 0$.

More precisely: (1) degenerates into (a); (2) degenerates into no change or (b) or (a), depending on whether $i > 0$, $i = 0$, or $i < 0$; (3) degenerates into (a) or (c) or no change, depending on whether $i > 0$, $i = 0$, or $i < 0$; (4) degenerates into (c) or (b) or (a) or no change, depending on whether $i = 0$, $j = 0$, $ij < 0$, or $ij > 0$. These verify that $(\mathfrak{p}\sigma_t)_{t\geq 0}$ has the same generators as TASEP with a single second-class particle, so the conclusion follows. ∎

Now we finish the proof of Proposition 2.7, by using the two projections of $\sigma_t^{b,0}$.

Proof of Proposition 2.7. It suffices to take any cylinder set $B = B' \times \{0,1\}^{\mathbb{Z}\setminus[\![-L,L]\!]} \subset \{0,1\}^{\mathbb{Z}}$ for some $L \in \mathbb{N}$ and $B' \subset \{0,1\}^{[\![-L,L]\!]}$, and show that $T^{-1}\int_0^T \Phi_t(B)\,dt \to \Psi(B)$.

By Lemma 2.8, from $\sigma_t^{b,0}$, by identifying all $*$ with 1 in $\mathbb{Z}_+$ and all $*$ with 0 in $\mathbb{Z}_-$, we get $\zeta^* \sim \Psi$; and by Lemma 2.9, from $\sigma_t^{b,0}$ we get $\mathfrak{p}\sigma_t^{b,0} \sim \Phi_t$, by identifying all negatively labeled $*$ with 0, and identifying all positively labeled $*$ with 1 (see Figure 2). Then we have

$$\begin{aligned}|\Phi_t(B) - \Psi(B)| &\leq \mathbb{P}[\zeta^* \in B,\ \mathfrak{p}\sigma_t^{b,0} \notin B] + \mathbb{P}[\zeta^* \notin B,\ \mathfrak{p}\sigma_t^{b,0} \in B] \\ &\leq \mathbb{P}[\zeta^*|_{[\![-L,L]\!]} \neq \mathfrak{p}\sigma_t^{b,0}|_{[\![-L,L]\!]}].\end{aligned}$$

The event on the right-hand side is equivalent to the event that, in $\sigma^{b,0}$, each $*$ in $[\![1,L]\!]$ has a positive label and each $*$ in $[\![-L,-1]\!]$ has a negative label. In other words, for any $i \in \mathbb{Z}$ with $l_t^{b,i} - l_t^{b,0} \in [\![-L,0]\!]$, we must have $i \leq 0$; and for any $i \in \mathbb{Z}$ with $l_t^{b,i} - l_t^{b,0} \in [\![0,L]\!]$, we must have $i \geq 0$. So we have

$$\begin{aligned}&|\Phi_t(B) - \Psi(B)| \\ &\quad\leq 1 - \mathbb{P}\big[\{l_t^{b,i} - l_t^{b,0} : i > 0\} \cap [\![-L,0]\!] = \{l_t^{b,i} - l_t^{b,0} : i < 0\} \cap [\![0,L]\!] = \emptyset\big] \\ &\quad\leq \mathbb{E}\big[|\{l_t^{b,i} - l_t^{b,0} : i > 0\} \cap [\![-L,0]\!]|\big] + \mathbb{E}\big[|\{l_t^{b,i} - l_t^{b,0} : i < 0\} \cap [\![0,L]\!]|\big].\end{aligned}$$

By integrating over t we have

$$\begin{aligned}&\int_0^T |\Phi_t(B) - \Psi(B)|\,dt \\ &\leq \sum_{i\in\mathbb{Z}_+} \int_0^T \mathbb{P}\big[l_t^{b,i} - l_t^{b,0} \in [\![-L,0]\!]\big]\,dt + \sum_{i\in\mathbb{Z}_-} \int_0^T \mathbb{P}\big[l_t^{b,i} - l_t^{b,0} \in [\![0,L]\!]\big]\,dt. \qquad (2.12)\end{aligned}$$

We first bound the first term on the right-hand side. For each $i \in \mathbb{Z}_+$ we recursively define a sequence of stopping times: let $T_{i,1} = \inf\{t \geq 0 : l_t^{b,i} - l_t^{b,0} \in [\![-L,0]\!]\} \cup \{\infty\}$; and given $T_{i,n} < \infty$, let $T_{i,n+1} = \inf\{t \geq T_{i,n} + 1 : l_t^{b,i} - l_t^{b,0} \in [\![-L,0]\!]\} \cup \{\infty\}$.

It is not difficult to see that there exists $\delta > 0$, depending only on L, such that for any $t \geq 0$ and $n \in \mathbb{N}$ we have $\mathbb{P}[l_{t+1}^{b,i} > l_{t+1}^{b,0} \mid T_{i,n} = t] > \delta$. Note that since $i > 0$, if $l_{t_0}^{b,i} > l_{t_0}^{b,0}$

for some $t_0 \geq 0$, we must have $l_t^{b,i} > l_t^{b,0}$ for any $t > t_0$. Thus the event $l_{T_{i,n}+1}^{b,i} > l_{T_{i,n}+1}^{b,0}$ implies that $T_{i,n+1} = \infty$. So for any $t \geq 0$ and $n \in \mathbb{N}$, we have

$$\mathbb{P}[T_{i,n+1} < \infty \mid T_{i,n} = t] < 1 - \delta.$$

Then

$$\mathbb{P}[T_{i,n} < T] = \mathbb{P}[T_{i,1} < T_{i,n} < T] \leq \mathbb{P}[T_{i,1} < T \text{ and } T_{i,n} < \infty] \leq (1-\delta)^{n-1}\mathbb{P}[T_{i,1} < T].$$

Also note that $\int_0^T \mathbb{1}[l_t^{b,i} - l_t^{b,0} \in [\![-L, 0]\!]]\, dt \leq \sum_{n=1}^{\infty} \mathbb{1}[T_{i,n} < T]$. So we have

$$\begin{aligned}\int_0^T \mathbb{P}\big[l_t^{b,i} - l_t^{b,0} \in [\![-L, 0]\!]\big]\, dt &\leq \sum_{n=1}^{\infty} \mathbb{P}[T_{i,n} < T] \leq \sum_{n=1}^{\infty} (1-\delta)^{n-1}\mathbb{P}[T_{i,1} < T] \\ &= \delta^{-1}\mathbb{P}[T_{i,1} < T]. \end{aligned} \tag{2.13}$$

Next we bound $\sum_{i\in\mathbb{Z}_+} \mathbb{P}[T_{i,1} < T]$. Take any $\epsilon > 0$. From the renewal construction of σ, we know that $l_0^{b,0} - l_0^{b,i}$ is the sum of i i.i.d. positive random variables, each with infinite expectation. Thus we have

$$\lim_{T\to\infty} \mathbb{P}[l_0^{b,0} - l_0^{b,\lceil \epsilon T\rceil} < 3T] = 0. \tag{2.14}$$

Given $\{l_0^{b,i}\}_{i\in\mathbb{Z}}$ satisfying $l_0^{b,0} - l_0^{b,\lceil \epsilon T\rceil} \geq 3T$, for each $j \in \mathbb{Z}_{\geq 0}$, $\mathbb{P}[T_{\lceil \epsilon T\rceil + j,1} < T \mid \{l_0^{b,i}\}_{i\in\mathbb{Z}}]$ is bounded by the probability of the following event: there are two particles starting from 0 and $-\lceil 3T\rceil - j$ respectively, jumping left and right respectively with rate 1 independently, and the first time when they are within distance L of each other is $< T$. This is just the probability that the sum of $\lceil 3T\rceil + j - L$ independent Exp(2) random variables is less than T (since for the distance to decrease by 1, the waiting time is the minimum of two independent Exp(1) random variables). Summing such probabilities for all j and using (2.14), we get

$$\lim_{T\to\infty} \sum_{i\geq \epsilon T} \mathbb{P}[T_{i,1} < T] = 0.$$

Plugging this into (2.13) and summing over $i \in \mathbb{Z}_+$ there, we get

$$\limsup_{T\to\infty} \sum_{i\in\mathbb{Z}_+} \int_0^T \mathbb{P}\big[l_t^{b,i} - l_t^{b,0} \in [\![-L, 0]\!]\big]\, dt - \delta^{-1}\epsilon T \leq 0.$$

Similarly,

$$\limsup_{T\to\infty} \sum_{i\in\mathbb{Z}_-} \int_0^T \mathbb{P}\big[l_t^{b,i} - l_t^{b,0} \in [\![0, L]\!]\big]\, dt - \delta^{-1}\epsilon T \leq 0.$$

Adding them up and using (2.12), we get

$$\limsup_{T\to\infty} T^{-1} \int_0^T |\Phi_t(B) - \Psi(B)|\, dt \leq 2\delta^{-1}\epsilon.$$

Since $\epsilon > 0$ is arbitrary, the conclusion follows. ∎

3. Coupling between TASEP and LPP

In this section we connect TASEP and LPP, and other objects such as an up-right growth process to be defined shortly. These results are mostly from the literature, and will motivate the construction of the LPP limiting environment in Section 4.

3.1. Semi-infinite geodesics and the Busemann function

We start by introducing a useful tool in studying LPP, namely, the Busemann function, and its beautiful duality property.

For any $u, v \in \mathbb{Z}^2$, we denote $\mathbf{B}(u, v) := T_{u,\mathbf{c}} - T_{v,\mathbf{c}}$, where $\mathbf{c} \in \mathbb{Z}^2$ is the coalescing point of Γ_u and Γ_v, i.e. $\mathbf{c}$ is the vertex in $\Gamma_u \cap \Gamma_v$ with the smallest $d(\mathbf{c})$. Such $\mathbf{B}$ is called the *Busemann function* (*in direction* $\boldsymbol{\rho}$). We also write $\mathbf{G}(u) := \mathbf{B}(\mathbf{0}, u)$. The Busemann function satisfies the following properties:

(1) For any $u, v, w \in \mathbb{Z}^2$, we have $\mathbf{B}(u, v) + \mathbf{B}(v, w) = \mathbf{B}(u, w)$. In particular, $\mathbf{B}(u, v) = \mathbf{G}(v) - \mathbf{G}(u)$.

(2) For each $u \in \mathbb{Z}^2$, $\mathbf{G}(u) = \mathbf{G}(u + (1, 0)) \wedge \mathbf{G}(u + (0, 1)) - \xi(u)$.

(3) For each $u \in \mathbb{Z}^2$, define the *dual weight*

$$\xi^{\vee}(u) := \mathbf{G}(u) - \mathbf{G}(u - (1, 0)) \vee \mathbf{G}(u - (0, 1)).$$

Then its distribution is Exp(1).

(4) For each $u \in \mathbb{Z}^2$, the distribution of $\mathbf{B}(u, u + (0, 1))$ is $\mathrm{Exp}(\rho)$, and the distribution of $\mathbf{B}(u, u + (1, 0))$ is $\mathrm{Exp}(1 - \rho)$.

(5) For any down-right path $\mathcal{U} = \{u_k\}_{k\in\mathbb{Z}}$, let $\mathcal{U}_- = \{u_k - (a, a) : k \in \mathbb{Z},\ a \in \mathbb{N}\}$ and $\mathcal{U}_+ = \{u_k + (a, a) : k \in \mathbb{Z},\ a \in \mathbb{N}\}$. Then the following random variables are independent: $\mathbf{B}(u_k, u_{k-1})$ for each $k \in \mathbb{Z}$, $\xi(u)$ for each $u \in \mathcal{U}_-$, and $\xi^{\vee}(u)$ for each $u \in \mathcal{U}_+$.

The first two properties are by definition. The third property comes from [35, Lemma 4.2] (see also [7]). For the last two properties, a proof can be found in [52].

All the semi-infinite geodesics (in direction $\boldsymbol{\rho}$) can be characterized by the Busemann function $\mathbf{G}$ and passage times.

Lemma 3.1. *For any $u \le v$ we have $\mathbf{B}(u, v) = -\mathbf{G}(u) + \mathbf{G}(v) \ge T_{u,v} - \xi(v)$, and equality holds if and only if $v \in \Gamma_u$.*

Proof. Let $\mathbf{c}$ be the coalescing point of Γ_u and Γ_v. Then $\mathbf{B}(u, v) = -\mathbf{G}(u) + \mathbf{G}(v) = T_{u,\mathbf{c}} - T_{v,\mathbf{c}}$. From the definition of passage times, we know that $T_{u,\mathbf{c}} \ge T_{u,v} + T_{v,\mathbf{c}} - \xi(v)$, and equality holds if and only if $v \in \Gamma_{u,\mathbf{c}}$. ∎

In particular, by taking $v = u + (0, 1)$ and $v = u + (1, 0)$ in Lemma 3.1, we must have $\mathbf{G}(u + (1, 0)) \ne \mathbf{G}(u + (0, 1))$ for any $u \in \mathbb{Z}^2$. This is true as we have assumed the existence and uniqueness of all the finite geodesics, and the existence, uniqueness, and

coalescence of all the semi-infinite geodesics in direction $\boldsymbol{\rho}$. These properties are used in defining the Busemann function and in the proof of Lemma 3.1.

The Busemann function $\mathbf{G}$ actually contains all the information to reconstruct all the semi-infinite geodesics in direction $\boldsymbol{\rho}$.

Lemma 3.2. *The semi-infinite geodesic Γ_u for any $u \in \mathbb{Z}^2$ can be reconstructed recursively using $\mathbf{G}$ as follows: first let $\Gamma_u[1] = u$, and then let*

$$\Gamma_u[i+1] = \operatorname{argmin}_{v \in \{\Gamma_u[i]+(1,0), \Gamma_u[i]+(0,1)\}} \mathbf{G}(v) \quad \textit{for each } i \in \mathbb{N}.$$

This is proved by repeatedly using Lemma 3.1, and we omit the details.

Using the dual weights $\xi^\vee$, which are also i.i.d. Exp(1) (by the third and last properties of the Busemann function), we define 'downward semi-infinite geodesics'. For any $u \in \mathbb{Z}^2$, we let $\Gamma_u^\vee$ be the semi-infinite geodesic from u in direction $-\boldsymbol{\rho} = (-(1-\rho)^2, -\rho^2)$, under the weights $\xi^\vee$. Below we work under the almost sure event that such $\Gamma_u^\vee$ exists and is unique, and $\Gamma_u^\vee$ and $\Gamma_v^\vee$ coalesce for any $u, v \in \mathbb{Z}^2$. Such downward semi-infinite geodesics can also be constructed recursively using $\mathbf{G}$. More precisely, we let $\Gamma_u^\vee[1] = u$, and for each $i \in \mathbb{N}$ we let

$$\Gamma_u^\vee[i+1] = \operatorname{argmax}_{v \in \{\Gamma_u^\vee[i]-(1,0), \Gamma_u^\vee[i]-(0,1)\}} \mathbf{G}(v). \tag{3.1}$$

By the definition of $\xi^\vee$ and using induction, it is straightforward to check that each finite segment of the path $\Gamma_u^\vee$ constructed from (3.1) is a geodesic under $\xi^\vee$. Also the path $\Gamma_u^\vee$ constructed from (3.1) has the same law as $-\Gamma_{-u}$ (since $\mathbf{G}$ and $v \mapsto -\mathbf{G}(-v)$ have the same law), so it has the desired asymptotic direction.

A quick observation is the following 'non-crossing' property between semi-infinite geodesics and downward semi-infinite geodesics.

Lemma 3.3. *For any Γ_u and $\Gamma_v^\vee$ we cannot find $w \in \mathbb{Z}^2$ with $w, w-(1,0) \in \Gamma_u$ and $w, w+(0,1) \in \Gamma_v^\vee$ simultaneously, or $w, w-(0,1) \in \Gamma_u$ and $w, w+(1,0) \in \Gamma_v^\vee$ simultaneously. This implies that the path $\Gamma_u + (1/2, 1/2)$ divides $u + (\mathbb{Z}^2 \setminus \mathbb{Z}^2_{\leq 0})$ into two parts, which are $\bigcup_{w \in \Gamma_u}(w + \mathbb{Z}_+ \times \mathbb{Z}_{\leq 0})$ and $\bigcup_{w \in \Gamma_u}(w + \mathbb{Z}_{\leq 0} \times \mathbb{Z}_+)$, so that $\Gamma_v^\vee$ cannot intersect both. Equivalently, the path $\Gamma_v^\vee - (1/2, 1/2)$ divides $v + (\mathbb{Z}^2 \setminus \mathbb{Z}^2_{\geq 0})$ into two parts, which are $\bigcup_{w \in \Gamma_v^\vee}(w + \mathbb{Z}_- \times \mathbb{Z}_{\geq 0})$ and $\bigcup_{w \in \Gamma_v^\vee}(w + \mathbb{Z}_{\geq 0} \times \mathbb{Z}_-)$, so that Γ_u cannot intersect both.*

Proof. From the recursive constructions of Γ_u and $\Gamma_v^\vee$, the event $w, w-(1,0) \in \Gamma_u$ implies $\mathbf{G}(w) < \mathbf{G}(w+(-1,1))$, while $w, w+(0,1) \in \Gamma_v^\vee$ implies $\mathbf{G}(w) > \mathbf{G}(w+(-1,1))$. Thus the first statement holds. The second statement follows similarly. ∎

3.2. Growth process

The function $\mathbf{G}$ can also be used to describe an up-right growth process. For each $t \in \mathbb{R}$, we let $I_t := \{u \in \mathbb{Z}^2 : \mathbf{G}(u) \leq t\}$ be the set of vertices occupied by time t. Then for any $u \in \mathbb{Z}^2$, the waiting time for it to be occupied (since the first time when both $u-(1,0)$

and $u-(0,1)$ are occupied) is $\xi^{\vee}(u)$, which is i.i.d. Exp(1) for all u. Thus $(I_t)_{t\in\mathbb{R}}$ is a Markov process such that given I_t, each vertex $u \notin I_t$ with $u-(0,1), u-(1,0) \in I_t$ becomes occupied with rate 1 independently.

We next define several objects that will be useful in proofs in later sections. For any $t \in \mathbb{R}$ and $u \in \mathbb{Z}^2$, we denote

$$\xi^{\vee,t}(u) := \mathbf{G}(u) \vee t - \mathbf{G}(u-(1,0)) \vee \mathbf{G}(u-(0,1)) \vee t.$$

This can be understood as the waiting time for u to be occupied, starting from I_t. Note that for any u such that $\{u-(1,0), u-(0,1)\} \not\subset I_t$, we have $\xi^{\vee}(u) = \xi^{\vee,t}(u)$. A key property for $\xi^{\vee,t}$ is that it is still i.i.d. Exp(1) outside I_t.

Lemma 3.4. *For any $t \geq 0$, conditioned on I_t and $\{\mathbf{G}(u)\}_{u\in I_t}$, the random variables $\xi^{\vee,t}(u)$ are i.i.d.* Exp(1) *for all $u \notin I_t$.*

Proof. Take any down-right path $\mathcal{U} = \{u_k\}_{k\in\mathbb{Z}}$, and denote $\mathcal{U}_- = \{u_k - (a,a) : k \in \mathbb{Z}, a \in \mathbb{N}\}$, $\mathcal{U}_+ = \{u_k + (a,a) : k \in \mathbb{Z}, a \in \mathbb{N}\}$. Let $\mathcal{U}_c$ contain all $u \in \mathcal{U}_+$ such that $\{u-(1,0), u-(0,1)\} \not\subset \mathcal{U}_+$. Assume that $\mathbf{0} \in \mathcal{U} \cup \mathcal{U}_-$.

We now consider the event $I_t = \mathcal{U} \cup \mathcal{U}_-$. It is equivalent to $\mathbf{G}(u_k) \leq t$ for each $k \in \mathbb{Z}$, and $\mathbf{G}(u) > t$ (or equivalently $\xi^{\vee}(u) > t - \mathbf{G}(u-(1,0)) \vee \mathbf{G}(u-(0,1))$) for any $u \in \mathcal{U}_c$. We next study the distribution of $\{\xi^{\vee}(u)\}_{u\in\mathcal{U}_+}$, conditioned on this event.

By (1)–(2) in Section 3.1, we know that $\{\mathbf{G}(u)\}_{u\in\mathcal{U}\cup\mathcal{U}_-}$ determines $\{\mathbf{B}(u_k, u_{k-1})\}_{k\in\mathbb{Z}}$ and $\{\xi(u)\}_{u\in\mathcal{U}_-}$. We will next show that $\{\mathbf{G}(u)\}_{u\in\mathcal{U}\cup\mathcal{U}_-}$ is also determined by $\{\mathbf{B}(u_k, u_{k-1})\}_{k\in\mathbb{Z}}$ and $\{\xi(u)\}_{u\in\mathcal{U}_-}$. Indeed, by (1), for any $k \in \mathbb{Z}$ the value $\mathbf{G}(u_k) - \mathbf{G}(u_0) = \mathbf{B}(u_0, u_k)$ is determined by $\{\mathbf{B}(u_k, u_{k-1})\}_{k\in\mathbb{Z}}$. Then using (2), and the fact that $\mathbf{0} \in \mathcal{U} \cup \mathcal{U}_-$, we find that for any $u \in \mathcal{U} \cup \mathcal{U}_-$, $\mathbf{G}(u) - \mathbf{G}(u_0)$ is determined by $\{\mathbf{B}(u_k, u_{k-1})\}_{k\in\mathbb{Z}}$ and $\{\xi(u)\}_{u\in\mathcal{U}_-}$, in particular for $u = \mathbf{0}$. Since $\mathbf{G}(\mathbf{0}) = 0$, we conclude that $\mathbf{G}(u_0)$, thus $\mathbf{G}(u)$ for any $u \in \mathcal{U} \cup \mathcal{U}_-$, is determined by $\{\mathbf{B}(u_k, u_{k-1})\}_{k\in\mathbb{Z}}$ and $\{\xi(u)\}_{u\in\mathcal{U}_-}$.

By (3)–(5) in Section 3.1, $\{\mathbf{B}(u_k, u_{k-1})\}_{k\in\mathbb{Z}}$, $\{\xi(u)\}_{u\in\mathcal{U}_-}$, and $\{\xi^{\vee}(u)\}_{u\in\mathcal{U}_+}$ are independent exponential random variables. Thus conditioned on $\{\mathbf{G}(u)\}_{u\in\mathcal{U}\cup\mathcal{U}_-}$ and the event $I_t = \mathcal{U} \cup \mathcal{U}_-$, we deduce that

- $\{\xi^{\vee}(u)\}_{u\in\mathcal{U}_+}$ are independent random variables,
- $\xi^{\vee}(u) \sim$ Exp(1) for each $u \in \mathcal{U}_+ \setminus \mathcal{U}_c$,
- for each $u \in \mathcal{U}_c$, $\xi^{\vee}(u)$ has the distribution of Exp(1) conditioned on $> t - \mathbf{G}(u-(1,0)) \vee \mathbf{G}(u-(0,1))$.

Since an Exp(1) random variable conditioned on $> x$ for any $x \geq 0$ is just $x +$ Exp(1), we have

$$\xi^{\vee}(u) - (t - \mathbf{G}(u-(1,0)) \vee \mathbf{G}(u-(0,1))) \sim \text{Exp}(1)$$

for each $u \in \mathcal{U}_c$. We note that (still conditioned on $\{\mathbf{G}(u)\}_{u\in\mathcal{U}\cup\mathcal{U}_-}$ and the event $I_t = \mathcal{U} \cup \mathcal{U}_-$) we have $\xi^{\vee,t}(u) = \xi^{\vee}(u)$ for any $u \in \mathcal{U}_+ \setminus \mathcal{U}_c$ and $\xi^{\vee,t}(u) = \xi^{\vee}(u) - (t - \mathbf{G}(u-(1,0)) \vee \mathbf{G}(u-(0,1)))$, so $\{\xi^{\vee,t}(u)\}_{u\in\mathcal{U}_+}$ are i.i.d. Exp(1) random variables. Thus the conclusion follows. ∎

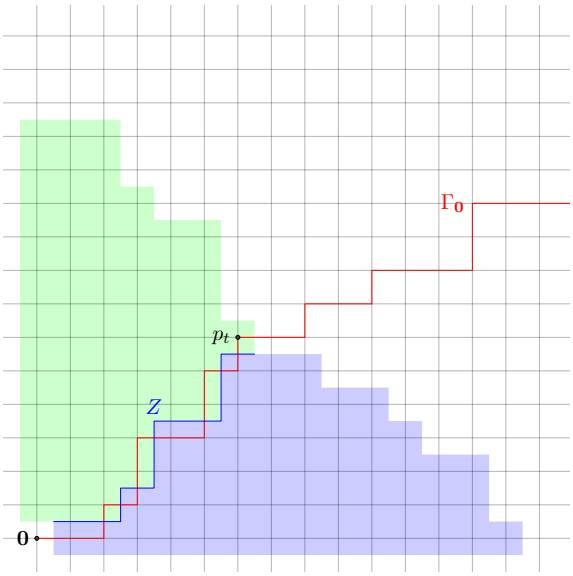


Fig. 3. An illustration of the growth process from LPP: the blue and green areas are the two clusters $C_1 \cap I_t$ and $C_2 \cap I_t$ respectively, and the red curve is the semi-infinite geodesic $\Gamma_{\mathbf{0}}$.

For any $t \in \mathbb{R}$ and $u \notin I_t$, the path $\Gamma_u^\vee \setminus I_t$ can be constructed as the geodesic with boundary I_t, under the weights $\xi^{\vee,t}$. For any $u \leq v$, $u, v \notin I_t$, let $T_{u,v}^{\vee,t}$ and $\Gamma_{u,v}^{\vee,t}$ denote the passage time and geodesic from u to v under the weights $\xi^{\vee,t}$.

Lemma 3.5. *For any $v \notin I_t$ we have*

$$\mathbf{G}(v) - t = \max_{u \leq v,\, u \notin I_t} T_{u,v}^{\vee,t} \quad \textit{and} \quad \Gamma_v^\vee \setminus I_t = \Gamma_{u_*,v}^{\vee,t} \quad \textit{for } u_* = \operatorname{argmax}_{u \leq v,\, u \notin I_t} T_{u,v}^{\vee,t}.$$

The proof of this lemma is by straightforward induction on u in the up-right direction; we omit the details.

3.3. The coupling and the competition interface

We now describe the coupling between LPP and TASEP (denoted as a Markov process on $\{0,1\}^{\mathbb{Z}}$). In this subsection we let $(\eta_t)_{t \geq 0}$ denote TASEP with the following initial condition: $\eta_0(0) = 0$ and $\eta_0(1) = 1$, and let $\eta_0(x)$ be i.i.d. Bernoulli(ρ) for all other x. We label the holes by $\mathbb{Z}$ from left to right, with the one at site 0 labeled 0; and label the particles by $\mathbb{Z}$ from right to left, with the one at site 1 labeled 0. For any $(a,b) \in \mathbb{Z}^2$, if at time 0 the particle labeled b is to the right of the hole labeled a, we set $L(a,b) = 0$; otherwise, we define $L(a,b) > 0$ to be the time when the particle switches with the hole. Then $\{L(a,b)\}_{(a,b)\in\mathbb{Z}^2}$ has the same distribution as $\{\mathbf{G}(a,b) \vee 0\}_{(a,b)\in\mathbb{Z}^2}$. Indeed, using (5) in Section 3.1, we can deduce that I_0 and $\{(a,b) : L(a,b) = 0\}$ have the same distribution;

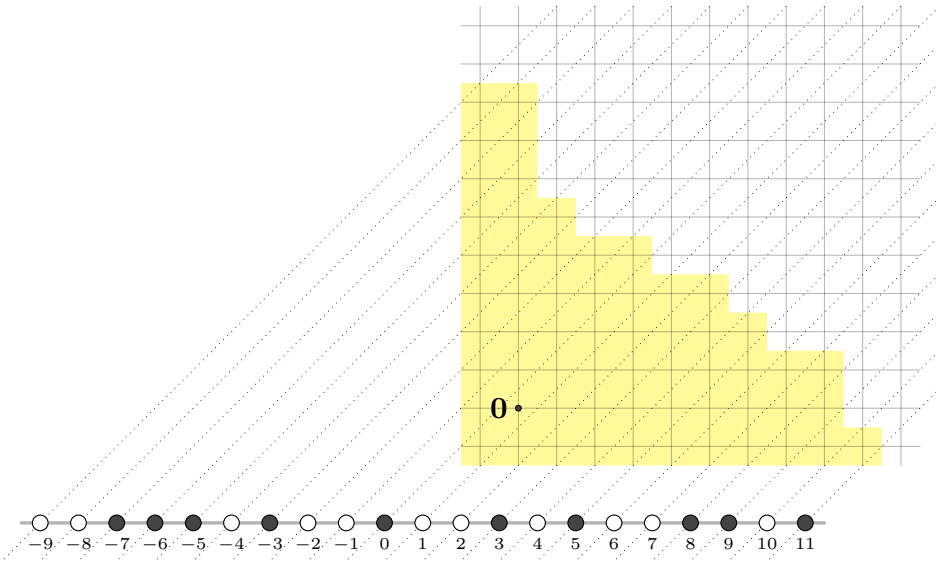


Fig. 4. An illustration of the correspondence between TASEP and the growth process.

and given η_0, the random variables

$$L(a,b) - L(a-1,b) \vee L(a,b-1)$$

for all (a,b) with $L(a,b) > 0$ are i.i.d. Exp(1), because this is the waiting time for the particle labeled b and the hole labeled a to switch since the time they are next to each other. Thus they have the same distribution as $\{\xi^{\vee,0}(u)\}_{u \notin I_0}$ conditioned on I_0, according to Lemma 3.4. See e.g. [35, Section 4.2] for more details on the equality in distribution between L and $\mathbf{G} \vee 0$. We couple $(\eta_t)_{t \geq 0}$ with LPP so that $L = \mathbf{G} \vee 0$ almost surely, and in the rest of this section we work under the event that this equality holds. Then the TASEP configuration η_t can be directly read off from I_t (see Figure 4).

Lemma 3.6. *For any $t \geq 0$ and $x \in \mathbb{Z}$, $\eta_t(x) = 0$ if and only if there is some $y \in \mathbb{Z}$ such that $(x+y, y) \in I_t$ and $(x+y, y+1) \notin I_t$, and the hole at site x has label $x+y$; and $\eta_t(x) = 1$ if and only if there is some $y \in \mathbb{Z}$ such that $(x+y-1, y) \in I_t$ and $(x+y, y) \notin I_t$, and the particle at site x has label y. Equivalently, if we let $f_t : \mathbb{Z} \to \mathbb{Z}$ be the function such that $f_t(x)$ is the largest integer with $(f_t(x)+x, f_t(x)) \in I_t$, then $f_t(x-1) - f_t(x) = \eta_t(x)$.*

Proof. For simplicity of notation we denote by $\mathcal{E}_1$ the event where there is $y \in \mathbb{Z}$ such that $(x+y, y) \in I_t$ and $(x+y, y+1) \notin I_t$, and by $\mathcal{E}_2$ the event where there is $y \in \mathbb{Z}$ such that $(x+y-1, y) \in I_t$ and $(x+y, y) \notin I_t$. Note that exactly one of $\mathcal{E}_1$ and $\mathcal{E}_2$ happens, so it suffices to show that $\mathcal{E}_1$ implies $\eta_t(x) = 0$, since by symmetry we would know that $\mathcal{E}_2$ implies $\eta_t(x) = 1$, and then the conclusion follows.

If $(x+y, y) \in I_t$ and $(x+y, y+1) \notin I_t$, then $L(x+y, y) \le t$ and $L(x+y, y+1) > t$ under the coupling. This means that at time t, the hole labeled $x+y$ is to the left of the particle labeled y, but to the right of the particle labeled $y+1$. Suppose that at time 0, the hole labeled $x+y$ is at site z. Since the hole at site 0 is labeled 0, if $x+y>0$ we must have $z>1$, and there are $z-(x+y)$ particles between sites 0 and z; and if $x+y<0$ we must have $z<0$, and there are $(x+y)-z$ particles between sites z and 0. In either case, the nearest particle to the left of the hole labeled $x+y$ (at time 0) must be labeled $x+y-z+1$ (since the particle at site 1 is labeled 0). This means that at time t, the hole labeled $x+y$ has already swapped with $(y+1)-(x+y-z+1) = z-x$ particles. So at time t it is at site x, and $\eta_t(x) = 0$. ∎

We next consider the semi-infinite geodesic $\Gamma_{\mathbf{0}}$ under this coupling. It actually corresponds to the competition interface starting from $\mathbf{0}$, which we describe now (see e.g. [35,36]). We define two clusters C_1 and C_2 for the growth process $(I_t)_{t\ge 0}$. Let $\mathbb{Z}_+ \times \{0\} \subset C_1$ and $\{0\} \times \mathbb{Z}_+ \subset C_2$. For any $(a,b) \in \mathbb{Z}_+^2$ let its 'parent' be either $(a-1,b)$ or $(a,b-1)$, whichever is occupied later; then (a,b) is in the same cluster as its parent. Starting from any u and by taking parent repeatedly, we can actually get $\Gamma_u^\vee \setminus I_0$, by (3.1); thus we can equivalently define C_1 and C_2 such that for any $u \in \mathbb{Z}_{\ge 0}^2$, $u \ne \mathbf{0}$, we have $u \in C_1$ if $\Gamma_u^\vee$ intersects $\mathbb{Z}_+ \times \{0\}$, and $u \in C_2$ if $\Gamma_u^\vee$ intersects $\{0\} \times \mathbb{Z}_+$. By Lemma 3.5, such clusters are determined by I_0 and $\{\xi^{\vee,0}(u)\}_{u \notin I_0}$, which are i.i.d. Exp(1) conditioned on I_0. The competition interface Z is defined to be the boundary of these clusters C_1 and C_2. Namely, we let $Z \subset (1/2,1/2) + \mathbb{Z}_{\ge 0}^2$ be such that for any $v \in Z$, every vertex in $\mathbb{Z}_{\ge 0}^2$ to the upper-left of v is in C_2, and every vertex in $\mathbb{Z}_{\ge 0}^2$ to the lower-right of v is in C_1. By Lemma 3.3, $Z = \Gamma_{\mathbf{0}} + (1/2,1/2)$. In words, the competition interface Z defined from I_0 and $\{\xi^{\vee,0}(u)\}_{u \notin I_0}$ is equivalent to the semi-infinite geodesic $\Gamma_{\mathbf{0}}$ defined from $\{\xi(u)\}_{u \in \mathbb{Z}^2}$. We also define the process $(p_t)_{t\ge 0}$ such that p_t is the last vertex in $\Gamma_{\mathbf{0}} \cap I_t$ (see Figure 3).

In the TASEP side, in $(\eta_t)_{t\ge 0}$ we keep track of a 'hole-particle pair', which is a hole with a particle next to the right. At $t=0$ it is the hole at site 0 and particle at site 1. Whenever the particle is switched with a hole to the right, we move this pair to the right; and whenever the hole is switched with a particle to the left, we move this pair to the left (see Figure 5 for an illustration). We have the following lemma from [36], which says that the trajectory of this 'hole-particle pair' can be mapped to the competition interface.

Lemma 3.7. *Under the above coupling between LPP and TASEP, for the hole-particle pair at time t, let b_t be the label of the particle and a_t be the label of the hole. Then $p_t = (a_t, b_t)$.*

We note that this hole-particle pair can also be replaced by a second-class particle. For this, note that a_t is also the number of times the pair moved to the right up to time t, and b_t is the number of times the pair moved to the left up to time t. Thus at time t the hole-particle pair is at sites $a_t - b_t$ and $a_t - b_t + 1$. If we take $\eta_t^*(x) = \eta_t(x)$ for $x < a_t - b_t$, $\eta_t^*(x) = \eta_t(x+1)$ for $x > a_t - b_t$, and $\eta_t^*(a_t - b_t) = *$, then $(\eta_t^*)_{t\ge 0}$ is TASEP with a second-class particle, starting from i.i.d. Bernoulli(ρ) on $\mathbb{Z} \setminus \{0\}$. So far we have seen

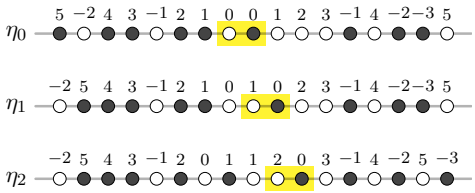


Fig. 5. An illustration of the evolution of a hole-particle pair in $(\eta_t)_{\geq 0}$: the numbers above the particles/holes are the labels, which increase from left to right for holes, and decrease from left to right for particles. The yellow boxes indicate the tracked hole-particle pairs.

that the process $(\eta_t^*)_{t\geq 0}$ contains the same information as $\mathbf{G} \vee 0$ (by Lemma 3.6), thus the same information as I_0 and $\{\xi^{\vee,0}(u)\}_{u\notin I_0}$, and the trajectory of the second-class particle gives the semi-infinite geodesic $\Gamma_{\mathbf{0}}$ (Lemma 3.7). Recall from Section 2 that Φ_t is the law of $\eta_t^*(a_t - b_t + \cdot)$, and Ψ is the stationary distribution of TASEP as seen from an isolated second-class particle. In light of the convergence of Φ_t to Ψ as $t \to \infty$, stated in Theorem 1.7 or Proposition 2.7, the LPP limiting environment measure ν should be constructed from Ψ. We give such construction in the next section.

4. The LPP limiting environment

We are now ready to define ν. As before, we use $(\zeta_t^*)_{t\in\mathbb{R}}$ to denote the process of TASEP as seen from an isolated second-class particle, such that $\zeta_t^* \sim \Psi$ for each t, where Ψ is the stationary distribution defined in Section 2. The idea is to construct a growth process from $(\zeta_t^*)_{t\in\mathbb{R}}$, then take the environment around the origin. This would give the limiting environment along the geodesic $\Gamma_{\mathbf{0}}$, as seen at a uniform time, i.e. the environment as seen from p_t for a uniform t, where p_t is the last vertex in $\Gamma_{\mathbf{0}} \cap I_t$. To get the environment ν, which is as seen from a uniformly chosen vertex, we would do an extra reweighting.

We first replace the second-class particle in $(\zeta_t^*)_{t\in\mathbb{R}}$ by a hole-particle pair. Namely, we let $(\zeta_t)_{t\in\mathbb{R}}$ be the process such that $\zeta_t(x) = \zeta_t^*(x)$ for $x < 0$, $\zeta_t(x) = \zeta_t^*(x-1)$ for $x > 1$, and $\zeta_t(0) = 0$, $\zeta_t(1) = 1$. The process $(\zeta_t)_{t\in\mathbb{R}}$ is then the stationary process of TASEP as seen from a hole-particle pair. We use $\tilde{\mathbf{\Psi}}$ to denote the law of this process.

We next describe the procedure of obtaining the environment from $(\zeta_t)_{t\in\mathbb{R}}$. We give the growth process in terms of the occupation time function, which we also denote by L as a slight abuse of notation. Similar to the i.i.d. Bernoulli initial setting in Section 3.3, we label the particles from right to left, and the holes from left to right, so that at time 0 the particle at site 1 and the hole at site 0 are both labeled 0. Let $L(a,b)$ be the time when the particle labeled b is switched with the hole labeled a. Unlike the i.i.d. Bernoulli initial setting, here $(\zeta_t)_{t\in\mathbb{R}}$ is a stationary process and evolves from time $-\infty$ to ∞, so $L(a,b)$ may be negative and is well-defined for all $(a,b) \in \mathbb{Z}^2$. We then use L to define the limiting weights and path, which we denote by ξ and γ by slightly abusing these notations within this section. We define ξ via $\xi(a,b) = L(a+1,b) \wedge L(a,b+1) - L(a,b)$, and define $\gamma \subset \mathbb{Z}^2$ as the collection of all (a,b) such that there is a time t when the particle

labeled b is at site 1 and the hole labeled a is at site 0 in η_t. We let $\tilde{\nu}$ be the measure given by the law of such (ξ, γ) constructed from $(\zeta_t)_{t\in\mathbb{R}} \sim \tilde{\mathbf{\Psi}}$.

We next do the reweighting. We let $\mathbf{\Psi}$ be the measure $\tilde{\mathbf{\Psi}}$ conditioned on $L(\mathbf{0}) = 0$, i.e. we let

$$d\mathbf{\Psi} = \lim_{\epsilon\to 0_+} \frac{\mathbb{1}[L(\mathbf{0}) > -\epsilon] d\tilde{\mathbf{\Psi}}}{\mathbb{P}_{\tilde{\mathbf{\Psi}}}[L(\mathbf{0}) > -\epsilon]}.$$

As $(\zeta_t)_{t\in\mathbb{R}}$ under $\tilde{\mathbf{\Psi}}$ is a Markov process, the limit could be computed as $\tilde{\mathbf{\Psi}}$ conditioned on there being a jump of the hole-particle pair at time 0; i.e. we first reweight $\tilde{\mathbf{\Psi}}$ by $\mathbb{1}[\zeta_{0-}(2) = 0] + \mathbb{1}[\zeta_{0-}(-1) = 1]$, the events where a jump is allowed, then let the jump happen at time 0. More precisely, we can describe $\mathbf{\Psi}$ as follows. We have

$$\mathbf{\Psi} = \frac{\mathbb{P}_{\tilde{\mathbf{\Psi}}}[\zeta_{0-}(2) = 0]\mathbf{\Psi}^{(1)} + \mathbb{P}_{\tilde{\mathbf{\Psi}}}[\zeta_{0-}(-1) = 1]\mathbf{\Psi}^{(2)}}{\mathbb{P}_{\tilde{\mathbf{\Psi}}}[\zeta_{0-}(2) = 0] + \mathbb{P}_{\tilde{\mathbf{\Psi}}}[\zeta_{0-}(-1) = 1]} = \frac{(1-\rho)^2\mathbf{\Psi}^{(1)} + \rho^2\mathbf{\Psi}^{(2)}}{(1-\rho)^2 + \rho^2},$$

where $\mathbf{\Psi}^{(1)}$ (resp. $\mathbf{\Psi}^{(2)}$) is $\tilde{\mathbf{\Psi}}$ conditioned on a jump of the hole-particle pair to the right (resp. to the left) happening at time 0. More precisely, we define these measures as follows. Let $(\zeta_t)_{t\in\mathbb{R}} \sim \mathbf{\Psi}^{(1)}$. Then the negative time part $(\zeta_t)_{t<0}$ has distribution given by

$$\frac{\mathbb{1}[\zeta_{0-}(2) = 0] d\tilde{\mathbf{\Psi}}}{\mathbb{P}_{\tilde{\mathbf{\Psi}}}[\zeta_{0-}(2) = 0]};$$

and given ζ_{0-} we let ζ_0 be such that $\zeta_0(-1) = \zeta_0(0) = 0$, $\zeta_0(1) = 1$, and $\zeta_0(x) = \zeta_{0-}(x+1)$ for any $x \notin \{-1, 0, 1\}$; and let $(\zeta_t)_{t\geq 0}$ be the Markov process of TASEP as seen from a hole-particle pair starting from ζ_0. Similarly, for $(\zeta_t)_{t\in\mathbb{R}} \sim \mathbf{\Psi}^{(2)}$, the negative part $(\zeta_t)_{t<0}$ has distribution given by

$$\frac{\mathbb{1}[\zeta_{0-}(-1) = 1] d\tilde{\mathbf{\Psi}}}{\mathbb{P}_{\tilde{\mathbf{\Psi}}}[\zeta_{0-}(-1) = 1]};$$

and given ζ_{0-}, we have $\zeta_0(0) = 0$, $\zeta_0(1) = \zeta_0(2) = 1$, and $\zeta_0(x) = \zeta_{0-}(x-1)$ for any $x \notin \{0, 1, 2\}$; and $(\zeta_t)_{t\geq 0}$ is the Markov process of TASEP as seen from a hole-particle pair starting from ζ_0.

From this construction, the laws of ζ_0 under $\mathbf{\Psi}^{(1)}$ and $\mathbf{\Psi}^{(2)}$ can also be described as follows. Let Ψ_+ be the law of $\{\zeta^*(x)\}_{x\in\mathbb{N}}$ and Ψ_- be the law of $\{\zeta^*(-x)\}_{x\in\mathbb{N}}$, for $\zeta^* \sim \Psi$. Under $\mathbf{\Psi}^{(1)}$, we have $\zeta_0(-1) = \zeta_0(0) = 0$, $\zeta_0(1) = 1$, and $\{\zeta_0(x+1)\}_{x\in\mathbb{N}} \sim \Psi_+$ and $\{\zeta_0(-x-1)\}_{x\in\mathbb{N}} \sim \Psi_-$, and they are independent. Under $\mathbf{\Psi}^{(2)}$, we have $\zeta_0(0) = 0$, $\zeta_0(1) = \zeta_0(2) = 1$, and $\{\zeta_0(x+2)\}_{x\in\mathbb{N}} \sim \Psi_+$, $\{\zeta_0(-x)\}_{x\in\mathbb{N}} \sim \Psi_-$, and they are independent.

We define ν as the measure given by the law of (ξ, γ), obtained using the procedure above from $(\zeta_t)_{t\in\mathbb{R}} \sim \mathbf{\Psi}$. By Lemma 4.2 below we can see that $\xi(\mathbf{0})$ has exponential tail under ν, so $\mathbb{E}_\nu[\xi(\mathbf{0})] < \infty$. We then show that $\tilde{\nu}$ is ν reweighted by $\xi(\mathbf{0})$.

Lemma 4.1. *We have*

$$d\tilde{\nu} = \frac{\xi(\mathbf{0}) d\nu}{\mathbb{E}_\nu[\xi(\mathbf{0})]}.$$

Let us explain why such a relation is expected. Consider the sequence $\{L(u)\}_{u\in\gamma}$ of times when the hole-particle pair jumps. Under $\tilde{\mathbf{\Psi}}$ this is a stationary point process in $\mathbb{R}$. Then ν corresponds to the environment as seen from the hole-particle at a typical jump time. On the other hand, $\tilde{\nu}$ corresponds to the environment as seen from the hole-particle at time 0. Because of the 'inspection effect', this is biased by the length of the interval in the point process containing time 0, which is $\xi(\mathbf{0})$.

Proof of Lemma 4.1. For each $s > 0$, we let $\mathbf{\Psi}_{-s}$ be the measure of $\tilde{\mathbf{\Psi}}$ conditioned on $L(\mathbf{0}) = -s$, i.e.

$$d\mathbf{\Psi}_{-s} = \lim_{\epsilon\to 0_+} \frac{\mathbb{1}[-s-\epsilon < L(\mathbf{0}) < -s]d\tilde{\mathbf{\Psi}}}{\mathbb{P}_{\tilde{\mathbf{\Psi}}}[-s-\epsilon < L(\mathbf{0}) < -s]}.$$

Note that under $\tilde{\mathbf{\Psi}}$, almost surely $L(0,1), L(1,0) > 0$ and $L(\mathbf{0}) < 0$, since at time 0 the following objects are ordered from left to right: the particle labeled 1, the hole labeled 0, the particle labeled 0, and the hole labeled 1. So $\mathbb{1}[-s-\epsilon < L(\mathbf{0}) < -s] = \mathbb{1}[-s-\epsilon < L(\mathbf{0}) < -s]\mathbb{1}[\xi(\mathbf{0}) > s]$. Then since $\tilde{\mathbf{\Psi}}$ is stationary, we have

$$\mathbb{1}[-s-\epsilon < L(\mathbf{0}) < -s]d\tilde{\mathbf{\Psi}} = (\mathbb{1}[L(\mathbf{0}) > -\epsilon]\mathbb{1}[\xi(\mathbf{0}) > s]d\tilde{\mathbf{\Psi}}) \circ \mathcal{T}_{-s},$$

where $\mathcal{T}_{-s}$ is the time translation operator: for any process $P = (P_w)_{w\in\mathbb{R}}$, we denote by $\mathcal{T}_{-s}P$ the process $(P_{-s+w})_{w\in\mathbb{R}}$. By multiplying by ϵ^{-1} and sending $\epsilon \to 0_+$, we have

$$\mathbb{P}_{\tilde{\mathbf{\Psi}}}[L(\mathbf{0}) = -s]d\mathbf{\Psi}_{-s} = \mathbb{P}_{\tilde{\mathbf{\Psi}}}[L(\mathbf{0}) = 0](\mathbb{1}[\xi(\mathbf{0}) > s]d\mathbf{\Psi}) \circ \mathcal{T}_{-s}, \tag{4.1}$$

where

$$\mathbb{P}_{\tilde{\mathbf{\Psi}}}[L(\mathbf{0}) = -s] = \lim_{\epsilon\to 0_+} \epsilon^{-1}\mathbb{P}_{\tilde{\mathbf{\Psi}}}[-s-\epsilon < L(\mathbf{0}) < -s],$$
$$\mathbb{P}_{\tilde{\mathbf{\Psi}}}[L(\mathbf{0}) = 0] = \lim_{\epsilon\to 0_+} \epsilon^{-1}\mathbb{P}_{\tilde{\mathbf{\Psi}}}[L(\mathbf{0}) > -\epsilon]$$

are the probability densities. By integrating the left-hand side of (4.1) over $s > 0$ we get $d\tilde{\mathbf{\Psi}}$, under which the law of (ξ, γ) is $\tilde{\nu}$. For the right-hand side of (4.1), we note that the laws of (ξ, γ) are the same under $(\mathbb{1}[\xi(\mathbf{0}) > s]d\mathbf{\Psi}) \circ \mathcal{T}_{-s}$ or $\mathbb{1}[\xi(\mathbf{0}) > s]d\mathbf{\Psi}$. So by integrating over $s > 0$ and taking the law of (ξ, γ), we get $\mathbb{P}_{\tilde{\mathbf{\Psi}}}[L(\mathbf{0}) = 0]\xi(\mathbf{0})d\nu$. Thus we conclude that $d\tilde{\nu} = \mathbb{P}_{\tilde{\mathbf{\Psi}}}[L(\mathbf{0}) = 0]\xi(\mathbf{0})d\nu$. Since $\tilde{\nu}$ and ν are probability measures, by integrating both sides we get

$$\mathbb{P}_{\tilde{\mathbf{\Psi}}}[L(\mathbf{0}) = 0]\mathbb{E}_{\nu}[\xi(\mathbf{0})] = 1,$$

so the conclusion follows. ∎

The above construction allows us to explicitly compute finite-dimensional distributions of ν and thus local geodesic statistics (assuming the main results of this paper). For this rest of this section we illustrate such computations, and prove Propositions 1.5 and 1.6.

We start with the following computations on the next jump times.

Lemma 4.2. *For any $h \geq 0$ we have*

$$\mathbb{P}_{\mathbf{\Psi}^{(1)}}[L(1,0) > h] = (1 + \rho(1-\rho)h)e^{-(1-\rho)h},$$
$$\mathbb{P}_{\mathbf{\Psi}^{(1)}}[L(0,1) > h] = (1 + \rho h)e^{-\rho h},$$
$$\mathbb{P}_{\mathbf{\Psi}^{(2)}}[L(1,0) > h] = (1 + (1-\rho)h)e^{-(1-\rho)h},$$
$$\mathbb{P}_{\mathbf{\Psi}^{(2)}}[L(0,1) > h] = (1 + \rho(1-\rho)h)e^{-\rho h}.$$

Proof. Let $D_+ = \min\{x \geq 1 : \zeta_0(x+1) = 0\}$, the number of particles between the origin and the leftmost hole at a positive site. Similarly, let $D_- = \min\{x \geq 1 : \zeta_0(-x) = 1\}$, the number of holes to the right of the rightmost particle at a negative site, up to and including the origin.

The distribution of D_+ under $\mathbf{\Psi}^{(1)}$ is that of X_+ given by (2.4), while the distribution of D_+ under $\mathbf{\Psi}^{(2)}$ is that of $X_+ + 1$ (which is the distribution of the sum of two independent Geometric$(1-\rho)$ random variables).

Similarly the distribution of D_- under $\mathbf{\Psi}^{(2)}$ is that of X_- at (2.5), while the distribution of D_- under $\mathbf{\Psi}^{(1)}$ is that of $X_- + 1$.

In order for the particle which is at site 1 at time 0 to jump, the hole starting at site $D_+ + 1$ must switch with each of the D_+ particles starting in $[\![1, D_+]\!]$. So given D_+, the distribution of $L(1,0)$ is the sum of D_+ independent Exp(1) random variables, that is, a Gamma$(D_+, 1)$ distribution. A random variable V with Gamma$(k,1)$ distribution has $\mathbb{E}[e^{-sV}] = (1+s)^{-k}$, and from this we obtain, for any $s > -1 + \rho$,

$$\mathbb{E}_{\mathbf{\Psi}^{(1)}}[e^{-s(L(1,0))}] = \sum_{k=1}^{\infty} k(1-\rho)^2\rho^{k-1}(1+s)^{-k} = \frac{(1+s)(1-\rho)^2}{(1+s-\rho)^2},$$

which can be shown to match the expression for $\mathbb{P}_{\mathbf{\Psi}^{(1)}}[L(1,0) > h]$ given in the statement.

Similarly, in order for the hole which is at site 0 at time 0 to jump, the particle starting at site $-D_-$ must switch with each of the D_- holes starting in $[\![-D_- + 1, 0]\!]$. One obtains

$$\mathbb{E}_{\mathbf{\Psi}^{(1)}}[e^{-s(L(0,1))}] = \sum_{k=1}^{\infty} k\rho^2(1-\rho)^{k-1}(1+s)^{-(k+1)} = \frac{\rho^2}{(\rho+s)^2},$$

which matches the desired expression for $\mathbb{P}_{\mathbf{\Psi}^{(1)}}[L(0,1) > h]$.

Analogous calculations give the probabilities under $\mathbf{\Psi}^{(2)}$. ∎

Now we compute the law of the weights on geodesics.

Proof of Proposition 1.5. It suffices to compute the law of $L(1,0) \wedge L(0,1)$, under the measure $\mathbf{\Psi} = \frac{(1-\rho)^2\mathbf{\Psi}^{(1)} + \rho^2\mathbf{\Psi}^{(2)}}{(1-\rho)^2+\rho^2}$. Note that under either $\mathbf{\Psi}^{(1)}$ or $\mathbf{\Psi}^{(2)}$, the random variables $L(1,0)$ and $L(0,1)$ are independent. Thus by Lemma 4.2 we get

$$\mathbb{P}_{\mathbf{\Psi}^{(1)}}[L(1,0) \wedge L(0,1) > h] = (1+\rho h)(1+\rho(1-\rho)h)e^{-h},$$
$$\mathbb{P}_{\mathbf{\Psi}^{(2)}}[L(1,0) \wedge L(0,1) > h] = (1+(1-\rho)h)(1+\rho(1-\rho)h)e^{-h},$$

and the conclusion follows. ∎

Assuming Theorem 1.1, we can also compute the proportion of 'corners' in geodesics.

Proof of Proposition 1.6. Assuming Theorem 1.1, we have

$$\frac{N_{n,\rho}}{2n} \to \mathbb{P}_\nu[\{(0,0),(0,1),(-1,0)\} \subset \gamma] + \mathbb{P}_\nu[\{(0,0),(0,-1),(1,0)\} \subset \gamma]$$

almost surely as $n \to \infty$. From the construction of ν, this equals

$$\frac{(1-\rho)^2 \mathbb{P}_{\mathbf{\Psi}^{(1)}}[L(1,0) > L(0,1)] + \rho^2 \mathbb{P}_{\mathbf{\Psi}^{(2)}}[L(1,0) < L(0,1)]}{(1-\rho)^2 + \rho^2}.$$

Since $L(1,0) - L(0,0)$ and $L(0,1) - L(0,0)$ are independent under either $\mathbf{\Psi}^{(1)}$ or $\mathbf{\Psi}^{(2)}$, by Lemma 4.2 we have

$$\mathbb{P}_{\mathbf{\Psi}^{(1)}}[L(1,0) > L(0,1)] = \rho^2(1 + 2\rho - 2\rho^2),$$
$$\mathbb{P}_{\mathbf{\Psi}^{(2)}}[L(1,0) < L(0,1)] = (1-\rho)^2(1 + 2\rho - 2\rho^2),$$

and the conclusion follows. ■

An alternative representation of the weights on geodesics. We also give an outline of alternative derivation of the formulae in Propositions 1.5 and 1.6, which also leads to representations of the type mentioned after the statement of Proposition 1.5.

Note that under $\mathbf{\Psi}^{(2)}$, D_+ takes values in $\{2, 3, \dots\}$ and has the distribution of the sum of two independent geometric random variables with parameter $1-\rho$. Given D_+, the random variable $L(1,0)$ is the sum of D_+ independent Exp(1) random variables. From this, $L(1,0)$ has the same distribution as the sum of two Exp$(1-\rho)$ random variables, or equivalently of $\frac{1}{1-\rho}(E_1 + E_2)$ for E_1, E_2 i.i.d. $\sim$ Exp(1).

Meanwhile under $\mathbf{\Psi}^{(2)}$, D_- takes values in $\{1, 2, \dots\}$ and has the distribution of the sum of two independent Geometric(ρ) random variables minus 1. Note that if $X \sim$ Geometric(ρ), then $X - 1 \overset{d}{=} BX$ where $B \sim$ Bernoulli(ρ) independently of X. We infer that $L(0,1)$ has the distribution of $\frac{1}{\rho}(E_3 + BE_4)$, for $B \sim$ Bernoulli(ρ) and E_3, E_4 i.i.d. $\sim$ Exp(1) independently of B.

Note $L(0,1)$ and $L(1,0)$ are independent under $\mathbf{\Psi}^{(2)}$. So we can combine the previous two paragraphs to deduce that the distribution of $\xi(\mathbf{0}) = L(0,1) \wedge L(1,0)$ under $\mathbf{\Psi}^{(2)}$ is that of

$$\frac{1}{1-\rho}(E_1 + E_2) \wedge \frac{1}{\rho}(E_3 + BE_4)$$

for $B \sim$ Bernoulli(ρ) and $(E_i)_{1 \le i \le 4}$ i.i.d. $\sim$ Exp(1) independently of B.

We continue in the particular case $\rho = 1/2$. Then the distribution of $\xi(\mathbf{0})$ is the same under $\mathbf{\Psi}^{(1)}$ as under $\mathbf{\Psi}^{(2)}$, and so its distribution under $\mathbf{\Psi}$ is again the same, that of $2((E_1 + E_2) \wedge (E_3 + BE_4))$ for $B \sim$ Bernoulli$(1/2)$ and $(E_i)_{1 \le i \le 4}$ i.i.d. $\sim$ Exp(1) independently of B.

By elementary arguments involving the memoryless property of exponentials, this distribution can be seen to be a $(1/4, 1/2, 1/4)$ mixture of Gamma$(1,1)$, Gamma$(2,1)$, and Gamma$(3,1)$ distributions.

A similar but slightly more involved argument can be made for the case of general ρ, to give that the distribution of $\xi(\mathbf{0})$ is again a mixture of Gamma(1, 1), Gamma(2, 1), and Gamma(3, 1) distributions, now with weights

$$\left(\frac{\rho^4 + (1-\rho)^4}{\rho^2 + (1-\rho)^2}, 2\rho(1-\rho), \frac{2\rho^2(1-\rho)^2}{\rho^2 + (1-\rho)^2}\right).$$

As a function of $\rho \in (0, 1)$, this distribution is stochastically increasing on $(0, 1/2]$, and symmetric around $1/2$.

The path as a competition interface. As $\Gamma_{\mathbf{0}}$ in the i.i.d. Exp(1) random field, the path γ under ν can also be described as a competition interface. For L under $\mathbf{\Psi}$, by slightly abusing the notations we let $I_0 := \{u \in \mathbb{Z}^2 : L(u) \le 0\}$ and

$$\xi^{\vee,0}(u) := L(u) \vee 0 - L(u - (1,0)) \vee L(u - (0,1)) \vee 0$$

for each $u \in \mathbb{Z}^2$. Then like Lemma 3.6, we can show that I_0 contains the same information as ζ_0 (whose law under $\mathbf{\Psi}$ is explicitly described using Ψ_+ and Ψ_- above). Namely, we have $(0, 0) \in I_0$ and $(0, 1), (1, 0) \notin I_0$; and for any $x \in \mathbb{Z}$, $\zeta_0(x) = 0$ if and only if there is some $y \in \mathbb{Z}$ such that $(x + y, y) \in I_0$ and $(x + y, y + 1) \notin I_0$, and $\zeta_0(x) = 1$ if and only if there is some $y \in \mathbb{Z}$ such that $(x + y - 1, y) \in I_0$ and $(x + y, y) \notin I_0$.

Under $\mathbf{\Psi}$ and conditioned on I_0, the weights $\{\xi^{\vee,0}(u)\}_{u \notin I_0}$ are i.i.d. Exp(1). This is because, for any $(a, b) \in \mathbb{Z}^2$, $\xi^{\vee,0}(a, b)$ is the waiting time for the particle labeled b and the hole labeled a to switch, since they are next to each other; and that is i.i.d. Exp(1) for all $(a, b) \notin I_0$, given ζ_0.

From I_0 and $\xi^{\vee,0}$ under $\mathbf{\Psi}$, we define a competition interface, similar to how the competition interface is defined in Section 3.3. Specifically, for any $u \le v$, $u, v \notin I_0$, let $T^{\vee,0}_{u,v}$ and $\Gamma^{\vee,0}_{u,v}$ be the passage time and geodesic from u to v under the weights $\xi^{\vee,0}$. For any $v \in \mathbb{Z}^2_{\ge 0} \setminus \{\mathbf{0}\}$, we consider the vertex $u_* \notin I_0$ with the maximum $T^{\vee,0}_{u_*,v}$. If $\Gamma^{\vee,0}_{u_*,v}$ intersects $\mathbb{Z}_+ \times \{0\}$ we let $v \in C_1$; otherwise $\Gamma^{\vee,0}_{u_*,v}$ intersects $\{0\} \times \mathbb{Z}_+$ and we let $v \in C_2$. Then $(\gamma \cap \mathbb{Z}^2_{\ge 0}) + (1/2, 1/2)$ is the boundary between C_1 and C_2, by analogues of Lemmas 3.3 and 3.5.

Using this formulation of γ and the explicit description of ζ_0 under $\mathbf{\Psi}$, and passage time estimates (e.g. Theorem 5.2 below) or the convergence of the passage time point-to-line profile to the so-called Airy_2 process (see Theorem 8.7 below), one can possibly show that $\gamma[i]$ has transversal fluctuation on the order of $i^{2/3}$ for large i (here $\gamma[i]$ denotes the i-th vertex in $\gamma \cap \mathbb{Z}^2_{\ge 0}$), and even obtain exact formulae for the distribution of its scaling limit. We leave these for future explorations.

5. Geometric estimates for LPP

While so far most arguments are on TASEP and use interacting particle system techniques, for the rest of this paper we will mainly use various LPP geometric arguments. In this

section we make some preparations, by providing some useful tools. Most results in this section have appeared in the literature.

In this section, we do not fix $\rho \in (0,1)$, and all constants are assumed not to depend on ρ, unless otherwise stated. For $a, b \in \mathbb{Z}$ and $\rho \in (0,1)$, we denote

$$\langle a, b\rangle_\rho := \left(\left\lfloor \frac{2(1-\rho)^2 a}{\rho^2 + (1-\rho)^2}\right\rfloor + b, \left\lceil \frac{2\rho^2 a}{\rho^2 + (1-\rho)^2}\right\rceil - b\right).$$

Then recall that $\mathbf{n}^\rho = \langle n, 0\rangle_\rho$ for any $n \in \mathbb{Z}$. We also write $\langle 0, b\rangle := (b, -b)$ for any $b \in \mathbb{Z}$.

We start with a basic geometric property called *ordering of geodesics*. Note that for any $\mathbb{Z}^2$ vertices $u \le v$, if $u' \le v'$ and $u', v' \in \Gamma_{u,v}$, then $\Gamma_{u',v'}$ is the part of $\Gamma_{u,v}$ between u' and v' (including u', v'). This immediately leads to the following result.

Lemma 5.1. *For any (a_1, b_1) and (a_2, b_2), write $(a_1, b_1) \preceq (a_2, b_2)$ if $a_1 \le a_2$ and $b_1 \ge b_2$. For any u_1, u_2 and v_1, v_2 such that $u_1 \le v_1$, $u_2 \le v_2$ and $u_1 \preceq u_2$, $v_1 \preceq v_2$, and any $w_1 \in \Gamma_{u_1,v_1}$, $w_2 \in \Gamma_{u_2,v_2}$, we cannot have $w_2 \preceq w_1$ unless $w_1 = w_2$.*

We next give estimates on passage times. We know that $T_{\mathbf{0},(m,n)}$ has the same law as the largest eigenvalue of X^*X where X is an $(m+1)\times(n+1)$ matrix of i.i.d. standard complex Gaussian entries (see [40, Proposition 1.4]). Hence we get the following one-point estimates from [43, Theorem 2].

Theorem 5.2. *There exist constants $c, C > 0$ such that for any $m \ge n \ge 1$ and $x > 0$,*

$$\mathbb{P}[T_{\mathbf{0},(m,n)} - (\sqrt{m} + \sqrt{n})^2 \ge x m^{1/2} n^{-1/6}] \le C e^{-cx}. \tag{5.1}$$

In addition, for each $\psi > 1$, there exist $C', c' > 0$ depending on ψ such that if $m/n < \psi$, then

$$\begin{aligned} &\mathbb{P}[T_{\mathbf{0},(m,n)} - (\sqrt{m} + \sqrt{n})^2 \ge x n^{1/3}] \le C' e^{-c'(x^{3/2} \wedge x n^{1/3})}, \\ &\mathbb{P}[T_{\mathbf{0},(m,n)} - (\sqrt{m} + \sqrt{n})^2 \le -x n^{1/3}] \le C' e^{-c' x^3}, \end{aligned} \tag{5.2}$$

and as a consequence

$$|\mathbb{E}[T_{\mathbf{0},(m,n)}] - (\sqrt{m} + \sqrt{n})^2| \le C' n^{1/3}. \tag{5.3}$$

Below we will frequently use parallelograms in $\mathbb{R}^2$. For simplicity of notation, in the rest of this section, for any vertices $u \le v$ and $x > 0$, we let $U^x_{u,v}$ denote the parallelogram in which one pair of opposite sides have length $2x$, parallel to the anti-diagonal, and have midpoints u and v respectively. Formally, we let

$$U^x_{u,v} = \{u + \alpha(v-u) + (y, -y) : \alpha \in [0,1],\ y \in [-x, x]\}.$$

We next state the following parallelogram estimate.

Proposition 5.3 ([11, Theorem 4.2]). *Let U_1, U_2 be the part of $U^{n^{2/3}}_{(-m,m),(n,n)}$ below $\mathbb{L}_{\lfloor n/3\rfloor}$ and above $\mathbb{L}_{\lceil 2n/3\rceil}$ respectively. For each $\psi < 1$, there exist constants $c, C > 0$, depending only on ψ, such that when $|m| < \psi n$,*

$$\mathbb{P}\Big[\sup_{u\in U_1,\, v\in U_2} |T_{u,v} - \mathbb{E}[T_{u,v}]| \ge x n^{1/3}\Big] \le C e^{-c(x^{3/2} \wedge x n^{1/3})}.$$

Such a result was first proved as [14, Propositions 10.1, 10.5], in the setting of Poissonian LPP. In the setting of exponential LPP a proof was given in [11, Appendix C], following the ideas in [14].

We will also need the following estimate on the coalescing probability of two geodesics, for finite and semi-infinite geodesics respectively.

Proposition 5.4 ([55]). *For each $\psi \in (0,1)$, there exists $C > 0$ such that*

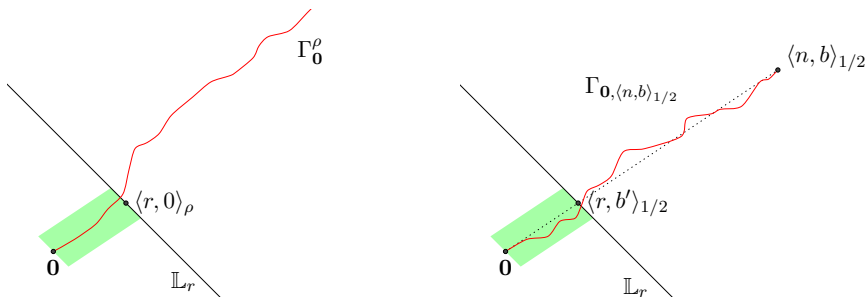
$$\mathbb{P}[\Gamma_{\mathbf{0},\langle n,b_1\rangle_{1/2}} \cap \mathbb{L}_{n-m} = \Gamma_{\mathbf{0},\langle n,b_2\rangle_{1/2}} \cap \mathbb{L}_{n-m}] > 1 - Cm^{-2/3}|b_1 - b_2|$$

for any $n, m \in \mathbb{N}$ and $b_1, b_2 \in \mathbb{Z}$ such that $m < n/3$ and $|b_1|, |b_2| < \psi n$.

Proposition 5.5 ([13, Theorem 2]). *For any $\rho \in (0,1)$, there is a constant $C > 0$ such that for any $r \in \mathbb{N}$ and $k > 1$, we have $\mathbb{P}[\Gamma^\rho_{\mathbf{0}} \cap \mathbb{L}_{\lfloor r^{3/2}k\rfloor} \neq \Gamma^\rho_{\langle 0,r\rangle} \cap \mathbb{L}_{\lfloor r^{3/2}k\rfloor}] < Ck^{-2/3}$.*

Similar coalescence estimates have also been obtained in various other papers, such as [6, 53].

We next give some estimates on transversal fluctuations of geodesics (see Figure 6). Such geodesic fluctuation estimates have been proved using various methods in the literature [10, 11, 13, 14, 16, 19, 28, 38, 55]. We start with an estimate for semi-infinite geodesics, which is illustrated by Figure 6 (a).



(a) For the semi-infinite geodesic $\Gamma^\rho_{\mathbf{0}}$: Lemma 5.6 states that with probability $> 1 - Ce^{-cx^3}$, its intersection with $\mathbb{L}_r$ is within distance $xr^{2/3}$ of $\langle r, 0\rangle_\rho$; Corollary 5.8 states that with probability $> 1 - Ce^{-cx^3}$, below $\mathbb{L}_r$ it is contained in $U^{xr^{2/3}}_{\mathbf{0},\langle r,0\rangle_\rho}$.

(b) For the infinite geodesic $\Gamma_{\mathbf{0},\langle n,b\rangle_{1/2}}$: Corollary 5.9 states that with probability $> 1 - Ce^{-cx^3}$, below $\mathbb{L}_r$ it is contained in $U^{xr^{2/3}}_{\mathbf{0},\langle r,b'\rangle_{1/2}}$.

Fig. 6. Illustrations of the transversal fluctuation estimates.

Lemma 5.6. *For any $\psi \in (0,1)$, there exist $c, C > 0$ such that the following is true. Let $\rho \in (\psi, 1-\psi)$ and $r, b_r \in \mathbb{Z}$ be such that $\Gamma^\rho_{\mathbf{0}}[2r+1] = \langle r, b_r\rangle_\rho$. Then $\mathbb{P}[|b_r| > xr^{2/3}] < Ce^{-cx^3}$ for any $x > 0$.*

This bound can be quickly deduced from [28, Theorem 3.1] or [16, Theorem 2.4]. Here we give a proof using the above passage time estimates, and properties of the Busemann function.

Proof of Lemma 5.6. In this proof we let $c, C > 0$ denote small and large constants that depend on ψ, and whose values can change from line to line. We assume that r and x are large enough (depending on ψ), since otherwise the conclusion is obvious.

For simplicity of notation we denote $T^{\bullet}_{u,v} = T_{u,v} - \xi(v)$ for any vertices $u \le v$. Let $\mathbf{B}$ be the Busemann function in direction $\boldsymbol{\rho}$, as defined in Section 3.1. By Lemma 3.1, the event $|b_r| > xr^{2/3}$ implies that there exists $b \in \mathbb{Z}$ such that $|b| > xr^{2/3}$, and $T^{\bullet}_{\mathbf{0},\langle r,b\rangle_\rho} + \mathbf{B}(\langle r,b\rangle_\rho, \mathbf{r}^\rho) > T^{\bullet}_{\mathbf{0},\mathbf{r}^\rho}$. To bound this event, we denote

$$L_j := \{\langle r,b\rangle_\rho : |b - 2j\lfloor r^{2/3}\rfloor| \le r^{2/3}\} \quad \text{for } j \in \mathbb{Z}.$$

For each j such that L_j intersects $\mathbb{Z}^2_{\ge 0}$, we have

$$\begin{aligned}
&\mathbb{P}\Big[\max_{u\in L_j\cap\mathbb{Z}^2_{\ge 0}} (T^{\bullet}_{\mathbf{0},u} + \mathbf{B}(u,\mathbf{r}^\rho)) > T^{\bullet}_{\mathbf{0},\mathbf{r}^\rho}\Big] \\
&\quad < \mathbb{P}\Big[\max_{u\in L_j\cap\mathbb{Z}^2_{\ge 0}} (T^{\bullet}_{\mathbf{0},u} - \mathbb{E}[T^{\bullet}_{\mathbf{0},u}]) > c_0 j^2 r^{1/3}\Big] + \mathbb{P}\big[T^{\bullet}_{\mathbf{0},\mathbf{r}^\rho} - \mathbb{E}[T^{\bullet}_{\mathbf{0},\mathbf{r}^\rho}] < -c_0 j^2 r^{1/3}\big] \\
&\qquad + \mathbb{P}\Big[\max_{u\in L_j\cap\mathbb{Z}^2_{\ge 0}} (\mathbf{B}(u,\mathbf{r}^\rho) + \mathbb{E}[T^{\bullet}_{\mathbf{0},u}]) - \mathbb{E}[T^{\bullet}_{\mathbf{0},\mathbf{r}^\rho}] > -2c_0 j^2 r^{1/3}\Big],
\end{aligned} \tag{5.4}$$

where $c_0 > 0$ is a small constant depending only on ψ, and satisfies several conditions to be specified below. We next show that each term on the right-hand side of (5.4) is bounded by $Ce^{-c|j|^3}$; then by summing over $j \in \mathbb{Z}$ such that $2|j| + 1 > x$ and L_j intersects $\mathbb{Z}^2_{\ge 0}$ (note that the latter implies that $|j| \le 2r(2\lfloor r^{2/3}\rfloor)^{-1} < 2r^{1/3}$) we get the conclusion.

- For the first term, we let $\psi' > 0$ be a small number (depending on c_0 and ψ and to be determined). When L_j is contained in $\{\langle r,b\rangle_{1/2} : |b| < (1-\psi')r\}$, we consider the parallelogram $U^{r^{2/3}}_{\mathbf{0},\langle r,2j\lfloor r^{2/3}\rfloor\rangle_\rho}$. Using Proposition 5.3 with this parallelogram we get the desired bound. When L_j is not contained in $\{\langle r,b\rangle_{1/2} : |b| < (1-\psi')r\}$ we cannot directly apply Proposition 5.3, since the above parallelogram may not satisfy the slope condition there. Instead, we take some small $\alpha > 0$ (depending on c_0 and ψ and to be determined), and consider the parallelogram $U^{r^{2/3}}_{-\langle\lfloor\alpha r\rfloor,0\rangle_{1/2},\langle r,2j\lfloor r^{2/3}\rfloor\rangle_\rho}$. Using Proposition 5.3 with this parallelogram we get

$$\mathbb{P}\Big[\max_{u\in L_j\cap\mathbb{Z}^2_{\ge 0}} (T^{\bullet}_{-\langle\lfloor\alpha r\rfloor,0\rangle_{1/2},u} - \mathbb{E}[T^{\bullet}_{-\langle\lfloor\alpha r\rfloor,0\rangle_{1/2},u}]) > 2^{-1}c_0 j^2 r^{1/3}\Big] < Ce^{-c|j|^3}. \tag{5.5}$$

For any $u \in L_j \cap \mathbb{Z}^2_{\ge 0}$ we have $T^{\bullet}_{\mathbf{0},u} \le T^{\bullet}_{\langle -\lfloor\alpha r\rfloor,0\rangle_{1/2},u}$, and

$$\mathbb{E}[T^{\bullet}_{\mathbf{0},u}] > \mathbb{E}[T^{\bullet}_{\langle -\lfloor\alpha r\rfloor,0\rangle_{1/2},u}] - 200^{-1}c_0\psi^2 r > \mathbb{E}[T^{\bullet}_{\langle -\lfloor\alpha r\rfloor,0\rangle_{1/2},u}] - 2^{-1}c_0 j^2 r^{1/3},$$

where the two inequalities are due to the following reasons. The first inequality is by $\mathbb{E}[T^{\bullet}_{\mathbf{0},u}] \ge 2r$ and $\mathbb{E}[T^{\bullet}_{\langle -\lfloor\alpha r\rfloor,0\rangle_{1/2},u}] < 2r + 200^{-1}c_0\psi^2 r$, which is due to (5.3) and the fact that L_j is not contained in $\{\langle r,b\rangle_{1/2} : |b| < (1-\psi')r\}$, and taking ψ' and α small enough (depending on ψ and c_0). The second inequality follows from $|j| > 0.1\psi r^{1/3}$,

which is implied by the fact that L_j is not contained in $\{\langle r,b\rangle_{1/2} : |b| < (1-\psi')r\}$. Thus we have

$$\max_{u\in L_j\cap\mathbb{Z}^2_{\geq 0}} (T^{\bullet}_{-\langle\lfloor\alpha r\rfloor,0\rangle_{1/2},u} - \mathbb{E}[T^{\bullet}_{-\langle\lfloor\alpha r\rfloor,0\rangle_{1/2},u}]) > \max_{u\in L_j\cap\mathbb{Z}^2_{\geq 0}} (T^{\bullet}_{\mathbf{0},u} - \mathbb{E}[T^{\bullet}_{\mathbf{0},u}]) - 2^{-1}c_0 j^2 r^{1/3},$$

so the first term on the right-hand side of (5.4) is bounded as desired by (5.5).

- For the second term we apply Theorem 5.2.
- For the last term, by (5.1) and (5.3) from Theorem 5.2, we have $\mathbb{E}[T^{\bullet}_{\mathbf{0},\langle r,b\rangle_\rho}] - \mathbb{E}[T^{\bullet}_{\mathbf{0},\mathbf{r}^\rho}] \leq Cr^{1/3} - b(\rho^{-1}-(1-\rho)^{-1}) - c_1 b^2 r^{-1}$ for any $\langle r,b\rangle_\rho \in \mathbb{Z}^2_{\geq 0}$, where $c_1 > 0$ is determined by ψ. By taking $c_0 < c_1$, and assuming that both r and $|j|$ are large enough, we have $c_1 b^2 r^{-1} - Cr^{1/3} - 2c_0 j^2 r^{1/3} > c_0 j^2 r^{1/3}$ when $|b - 2j\lfloor r^{2/3}\rfloor| \leq r^{2/3}$. Then the last term on the right-hand side of (5.4) is bounded by

$$\mathbb{P}\Big[\max_{|b-2j\lfloor r^{2/3}\rfloor|\leq r^{2/3}} \big(\mathbf{B}(\langle r,b\rangle_\rho, \mathbf{r}^\rho) - b(\rho^{-1}-(1-\rho)^{-1})\big) > c_0 j^2 r^{1/3}\Big].$$

Note that $b \mapsto \mathbf{B}(\langle r,b\rangle_\rho, \mathbf{r}^\rho) - b(\rho^{-1}-(1-\rho)^{-1})$ is a (two-sided) centered random walk, where each step has exponential tail determined by ρ (see Section 3.1). We can apply concentration inequalities to get the desired bound.

(For example, we can do a Chernoff type estimate as follows. Take any $c_2 > 0$, small enough depending on ψ. Without loss of generality we assume $j > 0$. Write the random walk as $b \mapsto \sum_{i=1}^b X_i$ for $b > 0$, where each X_i has exponential tail determined by ρ. Consider $e^{c_2 j r^{-1/3}\sum_{i=1}^b X_i}$, which is a supermartingale in b. Let τ be the first time after $(2j-1)\lfloor r^{2/3}\rfloor$ when this supermartingale is at least $e^{c_2^{3/2} j^3}$, or $(2j+1)\lfloor r^{2/3}\rfloor + 1$, whichever is smaller. Then we have

$$\begin{aligned}
&\mathbb{P}\Big[\max_{|b-2j\lfloor r^{2/3}\rfloor|\leq r^{2/3}} \sum_{i=1}^b X_i > c_2^{1/2} j^2 r^{1/3}\Big] \\
&\leq \mathbb{P}\Big[\sum_{i=1}^\tau X_i > c_2^{1/2} j^2 r^{1/3}\Big] = \mathbb{P}[e^{c_2 j r^{-1/3}\sum_{i=1}^\tau X_i} > e^{c_2^{3/2} j^3}] \\
&\leq e^{-c_2^{3/2} j^3}\mathbb{E}[e^{c_2 j r^{-1/3}\sum_{i=1}^\tau X_i}] \\
&\leq e^{-c_2^{3/2} j^3}\mathbb{E}[e^{c_2 j r^{-1/3}\sum_{i=1}^{(2j+1)\lfloor r^{2/3}\rfloor+1} X_i}] = e^{-c_2^{3/2} j^3}\mathbb{E}[e^{c_2 j r^{-1/3} X_1}]^{(2j+1)\lfloor r^{2/3}\rfloor+1},
\end{aligned}$$

and this is bounded by $e^{-c_2^{3/2} j^3/2}$ when c_2 is small enough, since $\mathbb{E}[e^{c_2 j r^{-1/3} X_1}] < e^{Cc_2^2 j^2 r^{-2/3}}$.)

From these bounds the conclusion follows. ∎

In addition to the above one-point bound, we also quote the following uniform bound on transversal fluctuations of geodesics.

Lemma 5.7 ([11, Proposition C.9]). *For each $\psi \in (0,1)$ there exist constants $c, C > 0$ such that the following is true. For $x > 0$, $n \in \mathbb{N}$, and $|b| < \psi n$, the probability that the geodesic $\Gamma_{\mathbf{0},\langle n,b\rangle_{1/2}}$ exits $U^{xn^{2/3}}_{\mathbf{0},\langle n,b\rangle_{1/2}}$ is at most Ce^{-cx^3}.*

The following result for semi-infinite geodesics follows immediately by combining Lemmas 5.6 and 5.7. See also Figure 6 (a).

Corollary 5.8. *For each $\psi \in (0,1)$, there exist constants $c, C > 0$ such that the following is true. Take any $r \in \mathbb{N}$ large enough, any $x > 0$, and any $\rho \in (\psi, 1-\psi)$. Then with probability at least $1 - Ce^{-cx^3}$, the part of the geodesic $\Gamma^\rho_{\mathbf{0}}$ below $\mathbb{L}_r$ is contained in $U^{xr^{2/3}}_{\mathbf{0},\langle r,0\rangle_\rho}$.*

We also have such a near-end transversal fluctuation estimate for finite geodesics (see Figure 6 (b)).

Corollary 5.9. *For each $\psi \in (0,1)$, there exist constants $c, C > 0$ such that the following is true. Take any integers $0 < r < n$ that are large enough, any $x > 0$, and any integer b with $|b| < \psi n$. Let $\langle r, b'\rangle_{1/2}$ be the vertex in $\mathbb{L}_r$ that is closest to the straight line connecting $\mathbf{0}$ and $\langle n, b\rangle_{1/2}$. Then with probability at least $1 - Ce^{-cx^3}$, the geodesic $\Gamma_{\mathbf{0},\langle n,b\rangle_{1/2}}$ below $\mathbb{L}_r$ is contained in $U^{xr^{2/3}}_{\mathbf{0},\langle r,b'\rangle_{1/2}}$.*

Proof. Let $c, C > 0$ denote small and large constants depending only on ψ, and whose values can change from line to line. When $x > 2r^{1/3}$ the conclusion follows obviously, so we can assume that $x \le 2r^{1/3}$. We now take $\rho_-, \rho_+ \in (0,1)$ such that $\mathbf{n}^{\rho_-} = \langle n, b - \lfloor cxn^{1/3}\rfloor\rangle_{1/2}$ and $\mathbf{n}^{\rho_+} = \langle n, b + \lfloor cxn^{1/3}\rfloor\rangle_{1/2}$. Take $b_-, b_+ \in \mathbb{Z}$ such that $\langle n, b_-\rangle_{1/2} \in \Gamma^{\rho_-}_{\mathbf{0}}$ and $\langle n, b_+\rangle_{1/2} \in \Gamma^{\rho_+}_{\mathbf{0}}$. By Lemma 5.6, with probability at least $1 - Ce^{-cx^3}$ we have $b_- < b < b_+$, thus $\Gamma_{\mathbf{0},\langle n,b\rangle_{1/2}}$ is sandwiched between $\Gamma^{\rho_-}_{\mathbf{0}}$ and $\Gamma^{\rho_+}_{\mathbf{0}}$ below $\mathbb{L}_r$ by ordering of geodesics (Lemma 5.1). By Corollary 5.8, with probability at least $1 - Ce^{-cx^3}$, $\Gamma^{\rho_-}_{\mathbf{0}}$ and $\Gamma^{\rho_+}_{\mathbf{0}}$ below $\mathbb{L}_r$ are both contained in $U^{xr^{2/3}}_{\mathbf{0},\langle r,b'\rangle_{1/2}}$, so the conclusion follows. ∎

Finally, we have the following estimate on the passage time along a semi-infinite geodesic. For simplicity of notation, below we denote (recall that $\boldsymbol{\rho} = ((1-\rho)^2, \rho^2)$)

$$\mathbb{H}^\rho_x := \{x\boldsymbol{\rho} + y((1-\rho), -\rho) : y \in \mathbb{R}\}$$

for any $\rho \in (0,1)$ and $x \in \mathbb{R}$. Note that $\mathbb{H}_x$ intersects the axes at $(0, x\rho)$ and $(x(1-\rho), 0)$.

Lemma 5.10. *For each $\psi \in (0,1)$, there exist constants $c, C > 0$ such that the following is true. Take any $\rho \in (\psi, 1-\psi)$ and $l > 0$. Let u_* be the first vertex in $\Gamma^\rho_{\mathbf{0}}$ above the line $\mathbb{H}^\rho_l$. Then $\mathbb{P}[|T_{\mathbf{0},u_*} - l| > xl^{1/3}] < Ce^{-cx}$ for any $0 < x < cl^{2/3}$.*

Proof. Let $c, C > 0$ denote small and large constants depending only on ψ, and whose values can change from line to line. Let $U = U^{cx^{1/2}l^{2/3}}_{\langle\lfloor cl\rfloor,0\rangle_\rho,\langle\lfloor Cl\rfloor,0\rangle_\rho}$. Let V be the set of all $v \in U$ that are within distance 1 of the line $\mathbb{H}^\rho_l$. By Corollary 5.8, with probability at least $1 - Ce^{-cx^{3/2}}$, the geodesic $\Gamma^\rho_{\mathbf{0}}$ between $\mathbb{L}_{\lfloor cl\rfloor}$ and $\mathbb{L}_{\lfloor Cl\rfloor}$ is contained in U, thus

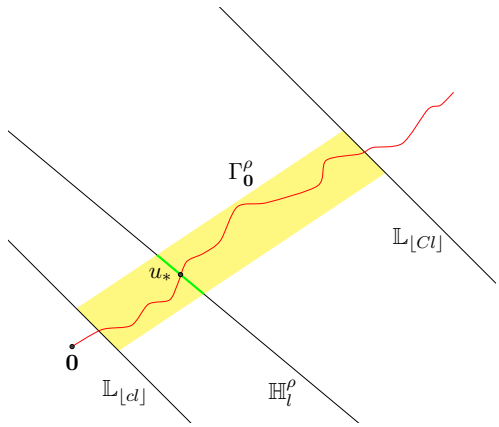


Fig. 7. An illustration of Lemma 5.10 and its proof: the yellow region is the parallelogram $U = U^{cx^{1/2}l^{2/3}}_{\langle \lfloor cl \rfloor, 0\rangle_\rho, \langle \lfloor Cl \rfloor, 0\rangle_\rho}$, and V is the set of vertices within distance 1 of the green segment. When c is small and C is large (depending on ρ), if $\Gamma_{\mathbf{0}}^{\rho}$ between $\mathbb{L}_{\lfloor cl \rfloor}$ and $\mathbb{L}_{\lfloor Cl \rfloor}$ is contained in U, it must intersect the line $\mathbb{H}_l^{\rho}$ inside U.

$u_* \in U$ and $u_* \in V$, since (when c is small and C is large) $\mathbb{L}_{\lfloor cl \rfloor} \cap \mathbb{Z}^2_{\geq 0}$ is below $\mathbb{H}_l^{\rho}$ and $\mathbb{L}_{\lfloor Cl \rfloor} \cap \mathbb{Z}^2_{\geq 0}$ is above $\mathbb{H}_l^{\rho}$ (see Figure 7).

By (5.3) in Theorem 5.2, we have $|\mathbb{E}[T_{\mathbf{0},v}] - l| < cxl^{1/3}$ for any $v \in V$. It remains to show that

$$\mathbb{P}\Big[\max_{v \in V} |T_{\mathbf{0},v} - \mathbb{E}[T_{\mathbf{0},v}]| > cxl^{1/3}\Big] < Ce^{-cx}. \tag{5.6}$$

For this, we split V into $\lceil x^{1/2} \rceil$ sets $V_1, \dots, V_{\lceil x^{1/2} \rceil}$, each with diameter $< cl^{2/3}$. Since $cx^{1/2}l^{2/3} < cl$, the slope of any line passing through $\mathbf{0}$ and intersecting V is bounded away from 0 and ∞. So we can apply Proposition 5.3 to each V_i to conclude that

$$\mathbb{P}\Big[\max_{v \in V_i} |T_{\mathbf{0},v} - \mathbb{E}[T_{\mathbf{0},v}]| > cxl^{1/3}\Big] < Ce^{-c(x^{3/2} \wedge xl^{2/3})}.$$

By a union bound over i we get (5.6), and the conclusion follows. ∎

Combining Corollary 5.8 and Lemma 5.10 we get the following (see Figure 8).

Corollary 5.11. *For each $\psi \in (0,1)$, there exist constants $c, C > 0$ such that the following is true. Take any $\rho \in (\psi, 1-\psi)$ and $l > 0$, and let u_* be the last vertex in $\Gamma_{\mathbf{0}}^{\rho}$ with $T_{\mathbf{0},u_*} \leq l$. Then for any $0 < x < cl^{2/9}$, with probability $> 1 - Ce^{-cx^3}$ the vertex u_* is between the lines $\mathbb{H}^{\rho}_{l-x^3l^{1/3}}$ and $\mathbb{H}^{\rho}_{l+x^3l^{1/3}}$, and $\Gamma_{\mathbf{0},u_*} \subset U^{xl^{2/3}}_{\mathbf{0},\langle l,0\rangle_\rho}$.*

6. Convergence of TASEP as seen from an isolated second-class particle

Starting from this section, we again always fix $\rho \in (0,1)$, and the choice of all other parameters and constants can depend on ρ unless otherwise stated.

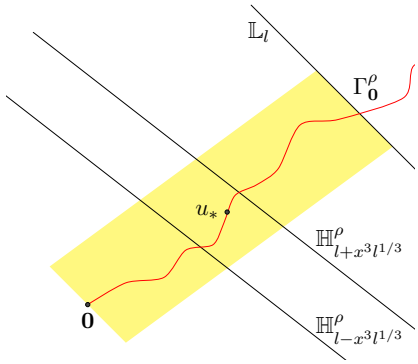


Fig. 8. An illustration of Corollary 5.11: the yellow region is the parallelogram $U^{xl^{2/3}}_{\mathbf{0},\langle l,0\rangle_\rho}$. When $x < cl^{2/9}$, the intersections between $U^{xl^{2/3}}_{\mathbf{0},\langle l,0\rangle_\rho}$ and the lines $\mathbb{H}^{\rho}_{l-x^3l^{1/3}}$ and $\mathbb{H}^{\rho}_{l+x^3l^{1/3}}$ are strictly below $\mathbb{L}_l$. Thus if $\Gamma^{\rho}_{\mathbf{0}}$ below $\mathbb{L}_l$ is contained in $U^{xl^{2/3}}_{\mathbf{0},\langle l,0\rangle_\rho}$, the part of $\Gamma^{\rho}_{\mathbf{0}}$ between $\mathbb{H}^{\rho}_{l-x^3l^{1/3}}$ and $\mathbb{H}^{\rho}_{l+x^3l^{1/3}}$ must also be contained in $U^{xl^{2/3}}_{\mathbf{0},\langle l,0\rangle_\rho}$; and if in addition u_* is between $\mathbb{H}^{\rho}_{l-x^3l^{1/3}}$ and $\mathbb{H}^{\rho}_{l+x^3l^{1/3}}$, we must have $\Gamma_{\mathbf{0},u_*} \subset U^{xl^{2/3}}_{\mathbf{0},\langle l,0\rangle_\rho}$.

Using geometric arguments and estimates from Section 5, in this section we upgrade Proposition 2.7 to Theorem 1.7. The general idea is to show that Φ_t and Φ_{t+s} are close when s is much smaller than t.

Proposition 6.1. *For any $N \in \mathbb{N}$, there is a constant $C > 0$ such that the following is true. For any $s, t > C$ with $t < s^{1.01}$, and any continuous function $f : \{0, 1, *\}^{[\![-N,N]\!]} \to [0, 1]$, regarded as a function on $\{0, 1, *\}^{\mathbb{Z}}$, we have $|\Phi_t(f) - \Phi_{t+s}(f)| < C(s/t)^{1/30}$.*

Using this we can deduce Theorem 1.7.

Proof of Theorem 1.7. Take any $N \in \mathbb{N}$ and $f : \{0, 1, *\}^{[\![-N,N]\!]} \to [0, 1]$, regarded as a function on $\{0, 1, *\}^{\mathbb{Z}}$; it suffices to show that

$$\lim_{t\to\infty} \Phi_t(f) = \Psi(f). \tag{6.1}$$

Take any $\delta > 0$. By Proposition 2.7, $(\delta t)^{-1} \int_0^{\delta t} \Phi_{t+s}(f)\, ds \to \Psi(f)$ as $t \to \infty$. By Proposition 6.1, for any $t > C$,

$$\begin{aligned}&\left|\Phi_t(f) - (\delta t)^{-1} \int_0^{\delta t} \Phi_{t+s}(f)\, ds\right| \\ &\quad \le (\delta t)^{-1} \int_{t^{1/1.01}}^{\delta t} |\Phi_t(f) - \Phi_{t+s}(f)|\, ds + (\delta t)^{-1} t^{1/1.01} < C\delta^{1/30} + (\delta t)^{-1} t^{1/1.01},\end{aligned}$$

where C is a constant depending on N. Thus $\limsup_{t\to\infty} |\Phi_t(f) - \Psi(f)| \le C\delta^{1/30}$, and by sending $\delta \to 0$ we get (6.1). ∎

To prove Proposition 6.1, we construct a coupling between Φ_t and Φ_{t+s}. For this we recall the setup of TASEP as seen from a hole-particle pair (or equivalently a second-class particle).

Let $(\eta_t^-)_{t\geq 0}$ and $(\eta_t^+)_{t\geq 0}$ be two copies of TASEP with $\eta_0^-(0) = \eta_0^+(0) = 0$ and $\eta_0^-(1) = \eta_0^+(1) = 1$; and all $\eta_0^-(x), \eta_0^+(x)$ for $x \in \mathbb{Z} \setminus \{0, 1\}$ are i.i.d. Bernoulli(ρ). In both copies, we label the holes by $\mathbb{Z}$ from left to right, with the hole at site 0 at time 0 labeled 0; and label the particles by $\mathbb{Z}$ from right to left, with the particle at site 1 at time 0 labeled 0. Keeping track of the hole-particle pair starting from sites 0 and 1, as described in Section 3.3, we let $p_t^- = (a_t^-, b_t^-)$ and $p_t^+ = (a_t^+, b_t^+)$ be the labels of the tracked hole and particle at time t (in $(\eta_t^-)_{t\geq 0}$ and $(\eta_t^+)_{t\geq 0}$ respectively).

For notational convenience, we also denote

$$\hat{\eta}_t^- = \eta_t^-(x + a_t^- - b_t^- + \cdot), \quad \hat{\eta}_t^+ = \eta_t^+(x + a_t^+ - b_t^+ + \cdot)$$

for any $t \geq 0$. Then $(\hat{\eta}_t^-)_{t\geq 0}$ and $(\hat{\eta}_t^+)_{t\geq 0}$ are TASEPs as seen from a hole-particle pair, and by replacing the hole-particle pair in $\hat{\eta}_t^-$ or $\hat{\eta}_t^+$ by a second-class particle we get the distribution Φ_t (defined in Section 2.2).

Below we fix $s > 0$. Our general strategy is to couple the processes $(\hat{\eta}_t^-)_{t\geq 0}$ and $(\hat{\eta}_{t+s}^+)_{t\geq 0}$ so that they evolve with the same set of waiting times (to be explained shortly). We implement this via coupling TASEP and LPP as described at the beginning of Section 3.3, and coupling the corresponding LPP models. For this, let us set up some useful notations.

- For any $a, b \in \mathbb{Z}$, if in η_0^- (resp. η_0^+) the particle with label b is to the left of the hole with label a, we let $L^-(a, b)$ (resp. $L^+(a, b)$) be the time when they switch; otherwise we set $L^-(a, b) = 0$ (resp. $L^+(a, b) = 0$). Let $\{\xi^-(u)\}_{u\in\mathbb{Z}^2}$ (resp. $\{\xi^+(u)\}_{u\in\mathbb{Z}^2}$) be i.i.d. Exp(1) weights; below we work under the almost sure event that there is a unique geodesic between any $\mathbb{Z}^2$ vertices $u \leq v$ under these weights, and from any $u \in \mathbb{Z}^2$ there is a unique semi-infinite geodesic in direction $\boldsymbol{\rho}$ under these random weights, and all these semi-infinite geodesics coalesce. We let $\mathbf{G}^-$ (resp. $\mathbf{G}^+$) be the LPP Busemann function in direction $\boldsymbol{\rho}$ under these random weights (defined like $\mathbf{G}$ in Section 3.1). We couple $(\eta_t^-)_{t\geq 0}$ (resp. $(\eta_t^+)_{t\geq 0}$) with $\{\xi^-(u)\}_{u\in\mathbb{Z}^2}$ (resp. $\{\xi^+(u)\}_{u\in\mathbb{Z}^2}$) so that $\mathbf{G}^- \vee 0 = L^-$ (resp. $\mathbf{G}^+ \vee 0 = L^+$) almost surely, and below we work under the event that this equality holds.

- We use $I_t^-, \xi^{-,\vee}, \xi^{-,\vee,t}, T_{u,v}^-, \Gamma_{u,v}^-, \Gamma_u^-, \Gamma_u^{-,\vee}$ (resp. $I_t^+, \xi^{+,\vee}, \xi^{+,\vee,t}, T_{u,v}^+, \Gamma_{u,v}^+, \Gamma_u^+, \Gamma_u^{+,\vee}$) to denote the objects I_t (growth process), $\xi^\vee$ (dual weights), $\xi^{\vee,t}$ (dual weights from I_t), $T_{u,v}$ (passage time), $\Gamma_{u,v}$ (finite geodesic), Γ_u (semi-infinite geodesic), $\Gamma_u^\vee$ (downward semi-infinite geodesic) defined in the introduction, and Sections 3.1 and 3.2, under the weights ξ^- (resp. ξ^+). We shall also work under the almost sure event that these downward semi-infinite geodesics (under these weights) exist and enjoy uniqueness and coalescence. In addition, for any $t \in \mathbb{R}$ we let

$$\partial I_t^+ := \{u \in I_t^+ : \mathbf{G}^+(u + (1, 0)) \vee \mathbf{G}^+(u + (0, 1)) > t\},$$
$$\partial I_t^- := \{u \in I_t^- : \mathbf{G}^-(u + (1, 0)) \vee \mathbf{G}^-(u + (0, 1)) > t\}.$$

• For $t \geq 0$, $p_t^- = (a_t^-, b_t^-)$ (resp. $p_t^+ = (a_t^+, b_t^+)$) is the last vertex in $\Gamma_{\mathbf{0}}^- \cap I_t^-$ (resp. $\Gamma_{\mathbf{0}}^+ \cap I_t^+$), by Lemma 3.7.

We first describe the coupling between $\hat{\eta}_0^-$ and $\hat{\eta}_s^+$. Take $r \in \mathbb{N}$. For any coupling between $\hat{\eta}_0^-$ and $\hat{\eta}_s^+$, we denote by $\mathcal{A}$ the event where

$$\hat{\eta}_0^-(x) = \hat{\eta}_s^+(x), \forall x \in \mathbb{Z}, |x| > r; \quad \sum_{x=-r}^{r} \hat{\eta}_0^-(x) = \sum_{x=-r}^{r} \hat{\eta}_s^+(x).$$

By Lemma 3.6, under $\mathcal{A}$ we can find a (unique) $p^* \in \mathbb{Z}^2$ such that $I_0^- \cap \{u \in \mathbb{Z}^2 : |ad(u)| > r\} = (I_s^+ - p^*) \cap \{u \in \mathbb{Z}^2 : |ad(u)| > r\}$, and $ad(p^*) = ad(p_s^+)$. In particular, this implies that

$$I_0^- \setminus \{u \in \mathbb{Z}^2 : |ad(u)|, |d(u)| \leq r\} = (I_s^+ - p^*) \setminus \{u \in \mathbb{Z}^2 : |ad(u)|, |d(u)| \leq r\}. \tag{6.2}$$

Lemma 6.2. *There is a coupling of $\hat{\eta}_0^-$ and $\hat{\eta}_s^+$ such that $\mathbb{P}[\mathcal{A}] > 1 - C(rs^{-2/3})^{-1/10}$ when $Cs^{2/3} < r < s^{2/3+0.01}$ and $s > C$, where $C > 0$ is a constant.*

We leave the construction of this coupling to the next subsection, and proceed to couple $(\hat{\eta}_t^-)_{t\geq 0}$ and $(\hat{\eta}_{s+t}^+)_{t\geq 0}$. The idea is actually straightforward: we just couple the exponential waiting times. Namely, we note that for any $(a, b) \notin I_0^-$, $\xi^{-,\vee,0}(a, b)$ is precisely the waiting time for the hole labeled a to switch with the particle labeled b since they are next to each other; and conditioned on I_0^-, $\{\xi^{-,\vee,0}(u)\}_{u\in\mathbb{Z}^2\setminus I_0^-}$ are i.i.d. Exp(1) (see also Lemma 3.4). The same is true for $\{\xi^{+,\vee,s}(u)\}_{u\in\mathbb{Z}^2\setminus I_s^+}$ conditioned on I_s^+. So we just couple these two sets of waiting times (as much as possible), up to a translation by p^*.

We note that, for $(\hat{\eta}_t^-)_{t\geq 0}$ and $(\hat{\eta}_{s+t}^+)_{t\geq 0}$, we need to couple them conditioned on $\hat{\eta}_0^-$ and $\hat{\eta}_s^+$. We next show that, under $\mathcal{A}$, $\hat{\eta}_0^-$ and $\hat{\eta}_s^+$ and p^* contain precisely the same information as I_0^- and I_s^+. (Then conditioned on $\hat{\eta}_0^-$ and $\hat{\eta}_s^+$ and p^*, the waiting times $\{\xi^{-,\vee,0}(u)\}_{u\in\mathbb{Z}^2\setminus I_0^-}$ and $\{\xi^{+,\vee,s}(u)\}_{u\in\mathbb{Z}^2\setminus I_s^+}$ are i.i.d. Exp(1).) Indeed, $\hat{\eta}_0^-$ is just η_0^-, which determines I_0^- according to Lemma 3.6. Using Lemma 3.6 we also see that $\hat{\eta}_s^+$ determines I_s^+, up to a translation of $\mathbb{Z}^2$; and the translation can be uniquely determined using p^* and the fact that I_0^- and $I_s^+ - p^*$ are the same outside a compact set. In the other direction, given I_s^+ and I_0^- we can uniquely find p^*, and (by Lemma 3.6) $\eta_0^- = \hat{\eta}_0^-$ is determined by I_0^-, and η_s^+ is determined by I_s^+. To get $\hat{\eta}_s^+$ we just shift η_s^+ by $ad(p^*)$.

We now couple $(\hat{\eta}_t^-)_{t\geq 0}$ and $(\hat{\eta}_{s+t}^+)_{t\geq 0}$. We let $\hat{\eta}_0^-$ and $\hat{\eta}_s^+$ be coupled using the coupling from Lemma 6.2. If $\mathcal{A}$ does not hold, we just couple $(\hat{\eta}_t^-)_{t\geq 0}$ and $(\hat{\eta}_{s+t}^+)_{t\geq 0}$ (conditioned on $\hat{\eta}_0^-$ and $\hat{\eta}_s^+$) arbitrarily. If $\mathcal{A}$ holds, we couple $(\hat{\eta}_t^-)_{t\geq 0}$ and $(\hat{\eta}_{s+t}^+)_{t\geq 0}$ (conditioned on $\hat{\eta}_0^-$ and $\hat{\eta}_s^+$ and p^*) so that the event

$$\xi^{-,\vee,0}(u) = \xi^{+,\vee,s}(u + p^*), \quad \forall u \in \mathbb{Z}^2 \setminus (I_0^- \cup (I_s^+ - p^*)), \tag{6.3}$$

holds with probability 1. Below we work under this event whenever $\mathcal{A}$ holds.

We bound the total variation distance between the $\{0, 1, *\}^{[\![-N,N]\!]}$ marginals of Φ_t and Φ_{t+s} (and thus prove Proposition 6.1) by bounding the probability that $\hat{\eta}_t^-$ and $\hat{\eta}_{t+s}^+$ are different in $[\![-N, N+1]\!]$ under this coupling.

In the LPP setting, this is to show that with high probability, for any u around p_t^- we have $L^-(u) = L^+(u + p^*) - s$. By Lemma 3.5, this is implied by the fact that for such u the paths $\Gamma_u^{-,\vee} \setminus I_0^-$ and $\Gamma_{u+p^*}^{+,\vee} \setminus I_s^- - p^*$ are the same path, contained in the area $\mathbb{Z}^2 \setminus (I_0^- \cup (I_s^+ - p^*))$ where the weights are coupled. Using the non-crossing property (Lemma 3.3), this is ensured by coalescence of (upward) semi-finite geodesics. More precisely, we consider the following events (which are also illustrated in Figure 9).

Take $m, r \in \mathbb{N}$ with $r < m$.

- Let $\mathcal{B}_-$ be the event where

$$\exists u_{-,1}, u_{-,2} \in \partial I_0^- : \quad ad(u_{-,1}) < -r,\ ad(u_{-,2}) > r,\ d(u_{-,1}), d(u_{-,2}) < 2m,$$
$$\Gamma_{u_{-,1}}^- \cap \mathbb{L}_m = \Gamma_{u_{-,2}}^- \cap \mathbb{L}_m.$$

- Let $\mathcal{B}_+$ be the event where $\mathcal{A}$ happens (with the same r), and in addition

$$\begin{aligned}\exists u_{+,1}, u_{+,2} \in \partial I_s^+ : \quad & ad(u_{+,1} - p^*) < -r,\ ad(u_{+,2} - p^*) > r,\\ & d(u_{+,1} - p*), d(u_{+,2} - p*) < 2m,\\ & (\Gamma_{u_{+,1}}^- - p^*) \cap \mathbb{L}_m = (\Gamma_{u_{+,2}}^- - p^*) \cap \mathbb{L}_m.\end{aligned}$$

- For any $t > 0$ we let $\mathcal{F}_t$ be the event where $d(p_t^-) > 2m + 2N + 2$.

Lemma 6.3. *For any $t > 0$, under $\mathcal{B}_- \cap \mathcal{B}_+ \cap \mathcal{F}_t$ the process $\hat{\eta}_t^-$ equals $\hat{\eta}_{t+s}^+$ in $[\![-N, N+1]\!]$.*

To prove this, we mainly need to establish the following result.

Lemma 6.4. *Under $\mathcal{B}_- \cap \mathcal{B}_+$, we have*

(1) $L^-(u) = L^+(u + p^*) - s$ *for any* $u \in \mathbb{Z}^2$ *with* $d(u) > 2m$, $u \notin I_0^-$, $u + p^* \notin I_s^+$,

(2) $p_t^- = p_{t+s}^+ - p^*$ *for any* $t > 0$ *with* $d(p_t^-) \geq 2m$.

Proof. Since $r < m$, under $\mathcal{A}$ we have $\{u \in \mathbb{Z}^2 : d(u) > 2m\} \cap I_0^- = \{u \in \mathbb{Z}^2 : d(u) > 2m\} \cap (I_s^+ - p^*)$ by (6.2). Denote $U = \mathbb{Z}^2 \setminus (I_0^- \cup (I_s^+ - p^*))$.

We first show that, under $\mathcal{A} \cap \mathcal{B}_-$, we must have $\Gamma_u^{-,\vee} \setminus I_0^- \subset U$ for any $u \in U$ with $d(u) > 2m$. Indeed, by the non-crossing property (Lemma 3.3), the path $\Gamma_{u_{-,1}}^- + (1/2, 1/2)$ divides $u_{-,1} + (\mathbb{Z}^2 \setminus \mathbb{Z}_{\leq 0}^2)$ into two parts $P_{1,\downarrow}$ (the lower-right part) and $P_{1,\uparrow}$ (the upper-left part) such that $\Gamma_u^{-,\vee}$ intersects at most one of them, and the path $\Gamma_{u_{-,2}}^- + (1/2, 1/2)$ divides $u_{-,2} + (\mathbb{Z}^2 \setminus \mathbb{Z}_{\leq 0}^2)$ into two parts $P_{2,\downarrow}$ (the lower-right part) and $P_{2,\uparrow}$ (the upper-left part) such that $\Gamma_u^{-,\vee}$ intersects at most one of them. Since $\Gamma_{u_{-,1}}^- \cap \mathbb{L}_m = \Gamma_{u_{-,2}}^- \cap \mathbb{L}_m$, we must have

$$\{u : d(u) > 2m\} \cap P_{1,\downarrow} = \{u : d(u) > 2m\} \cap P_{2,\downarrow},$$
$$\{u : d(u) > 2m\} \cap P_{1,\uparrow} = \{u : d(u) > 2m\} \cap P_{2,\uparrow}.$$

For any $u \in U$ with $d(u) > 2m$, depending on whether $u \in P_{1,\uparrow}, P_{2,\uparrow}$ or $u \in P_{1,\downarrow}, P_{2,\downarrow}$, we must have $\Gamma_u^{-,\vee} \setminus I_0^- \subset P_{1,\uparrow}$ or $\Gamma_u^{-,\vee} \setminus I_0^- \subset P_{2,\downarrow}$. Denote the lower endpoint

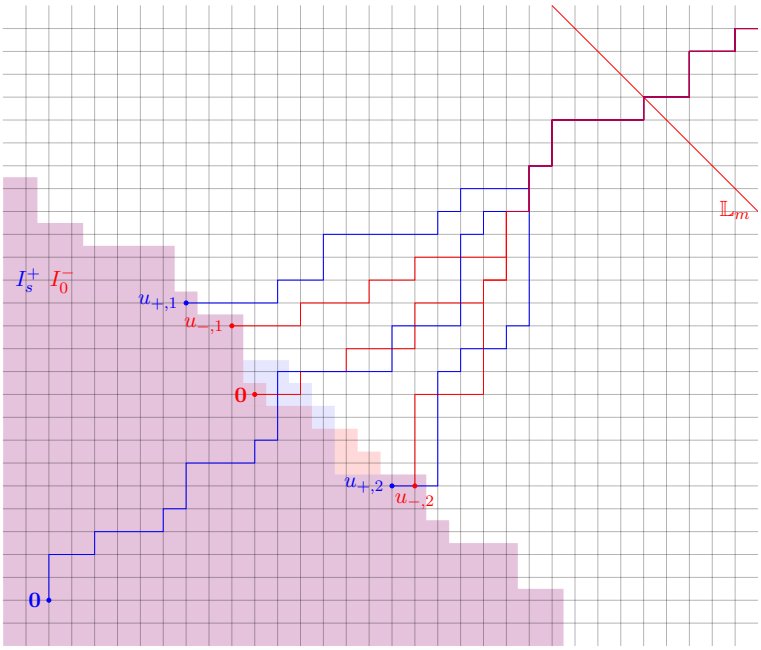


Fig. 9. An illustration of the events $\mathcal{B}_-$ and $\mathcal{B}_+$, translated and superposed together. The red objects are for $\mathcal{B}_-$, and are constructed from $(\eta_t^-)_{t\geq 0}$; and the blue objects are for $\mathcal{B}_+$ and are constructed from $(\eta_t^+)_{t\geq 0}$. The difference between the red $\mathbf{0}$ and blue $\mathbf{0}$ is p^*.

of $\Gamma_u^{-,\vee} \setminus I_0^-$ by v_0. By Lemma 3.5, we have $v_0 - (0,1), v_0 - (1,0) \in I_0^-$, so $v_0 \in u_{-,1} + \mathbb{Z}_{\leq 0} \times \mathbb{Z}_+$ (if $v_0 \in P_{1,\uparrow}$) or $v_0 \in u_{-,2} + \mathbb{Z}_+ \times \mathbb{Z}_{\leq 0}$ (if $v_0 \in P_{2,\downarrow}$). Thus $v_0 \notin I_s^+ - p^*$, by (6.2) and the fact that $ad(u_{-,1}) < -r$, $ad(u_{-,2}) > r$. So $v_0 \in U$, which implies that $\Gamma_u^{-,\vee} \setminus I_0^- \subset U$.

Similarly, under $\mathcal{B}_+$ we have $\Gamma_{u+p^*}^{+,\vee} \setminus I_s^+ - p^* \subset U$ for any $u \in U$ with $d(u) > 2m$.

According to (6.3) we have $\xi^{-,\vee,0}(u) = \xi^{+,\vee,s}(u + p^*)$ for any $u \in U$. Then by Lemma 3.5, for any $u \in U$ with $d(u) > 2m$, since we have shown that $\Gamma_u^{-,\vee} \setminus I_0^- \subset U$ and $\Gamma_{u+p^*}^{+,\vee} \setminus I_s^+ - p^* \subset U$, we must have

$$\Gamma_u^{-,\vee} \setminus I_0^- = \Gamma_{u+p^*}^{+,\vee} \setminus I_s^+ - p^*, \tag{6.4}$$

and $L^-(u) = L^+(u + p^*) - s$. Thus statement (1) holds.

We next prove (2). Below we always assume $\mathcal{B}_- \cap \mathcal{B}_+$. Using the fact that p_t^- is the last vertex in $\Gamma_{\mathbf{0}}^- \cap I_t^-$ and p_{t+s}^+ is the last vertex in $\Gamma_{\mathbf{0}}^+ \cap I_{t+s}^+$, and statement (1), it suffices to show that

$$\Gamma_{\mathbf{0}}^- \cap \{u \in \mathbb{Z}^2 : d(u) > 2m\} + p^* = \Gamma_{\mathbf{0}}^+ \cap \{u \in \mathbb{Z}^2 : d(u) > 2m + d(p^*)\}. \tag{6.5}$$

By the non-crossing property (Lemma 3.3), $\Gamma_{\mathbf{0}}^- \cap \{u \in \mathbb{Z}^2 : d(u) > 2m\}$ is determined by $\Gamma_u^{-,\vee}$ for all $u \in U$ with $d(u) \geq 2m$. More precisely, we divide the set of $u \in U$ with

$d(u) > 2m$ into two parts, depending on whether the lower endpoint of $\Gamma_u^{-,\vee} \setminus I_0^-$ is in $\mathbb{Z}_{\le 0} \times \mathbb{Z}_+$ or $\mathbb{Z}_+ \times \mathbb{Z}_{\le 0}$, and $\Gamma_{\mathbf{0}}^- \cap \{u \in \mathbb{Z}^2 : d(u) \ge 2m\} + (1/2, 1/2)$ is the boundary of these two parts. Similarly, we can also divide the set of $u \in U$ with $d(u) > 2m$ into two parts, depending on whether the lower endpoint of $\Gamma_{u+p^*}^{+,\vee} \setminus I_s^+$ is in $p_s^+ + \mathbb{Z}_{\le 0} \times \mathbb{Z}_+$ or $p_s^+ + \mathbb{Z}_+ \times \mathbb{Z}_{\le 0}$, and $(\Gamma_{\mathbf{0}}^+ - p^*) \cap \{u \in \mathbb{Z}^2 : d(u) \ge 2m\} + (1/2, 1/2)$ is the boundary of these two parts.

To prove (6.5), it then remains to show that, for any $u \in U$ with $d(u) > 2m$,

$$\begin{aligned} &\text{the lower endpoint of } \Gamma_u^{-,\vee} \setminus I_0^- \text{ is in } \mathbb{Z}_{\le 0} \times \mathbb{Z}_+ \\ &\qquad \iff \text{the lower endpoint of } \Gamma_{u+p^*}^{+,\vee} \setminus I_s^+ \text{ is in } p_s^+ + \mathbb{Z}_{\le 0} \times \mathbb{Z}_+. \end{aligned} \tag{6.6}$$

For this, we denote the lower endpoint of $\Gamma_u^{-,\vee} \setminus I_0^-$ as v_0. By (6.4), $v_0 + p^*$ is the lower endpoint of $\Gamma_{u+p^*}^{+,\vee} \setminus I_s^+$. Recall that $ad(p_s^+ - p^*) = 0$, so v_0 is either in both $\mathbb{Z}_{\le 0} \times \mathbb{Z}_+$ and $p_s^+ - p^* + \mathbb{Z}_{\le 0} \times \mathbb{Z}_+$, or in both $\mathbb{Z}_+ \times \mathbb{Z}_{\le 0}$ and $p_s^+ - p^* + \mathbb{Z}_+ \times \mathbb{Z}_{\le 0}$. Thus we get (6.6), which implies (6.5), and the conclusion follows. ∎

Proof of Lemma 6.3. The event that $\hat{\eta}_t^-$ equals $\hat{\eta}_{t+s}^+$ in $[\![-N, N+1]\!]$ can be written as

$$\{\eta_t^-(x + ad(p_t^-))\}_{x \in [\![-N,N+1]\!]} = \{\eta_{t+s}^+(x + ad(p_{t+s}^+))\}_{x \in [\![-N,N+1]\!]}.$$

By Lemma 3.6, this is implied by

$$(I_t^- - p_t^-) \cap [\![-N-1, N+1]\!]^2 = (I_{t+s}^+ - p_{t+s}^+) \cap [\![-N-1, N+1]\!]^2. \tag{6.7}$$

Below we assume $\mathcal{B}_- \cap \mathcal{B}_+ \cap \mathcal{F}_t$. For any $u \in [\![-N-1, N+1]\!]^2 + p_t^-$, we have $d(u) > 2m$ by $\mathcal{F}_t$. Since under $\mathcal{A}$ the sets I_0^- and $I_s^+ - p^*$ are the same outside $\{u \in \mathbb{Z}^2 : |ad(u)|, |d(u)| \le r\}$ (as stated by (6.2)), and $m > r$, we have either $u \in I_0^- \cap (I_s^+ - p^*)$ or $u \notin I_0^- \cup (I_s^+ - p^*)$. In the latter case we have $L^-(u) = L^+(u + p^*) - s$, by Lemma 6.4 (1). Thus we always have either $u \in I_t^- \cap (I_{t+s}^+ - p^*)$, or $u \notin I_t^- \cup (I_{t+s}^+ - p^*)$. Consequently,

$$([\![-N-1, N+1]\!]^2 + p_t^-) \cap I_t^- = ([\![-N-1, N+1]\!]^2 + p_t^-) \cap (I_{t+s}^+ - p^*).$$

By Lemma 6.4 (2) we have $p_t^- = p_{t+s}^+ - p^*$, so we get (6.7), and the conclusion follows. ∎

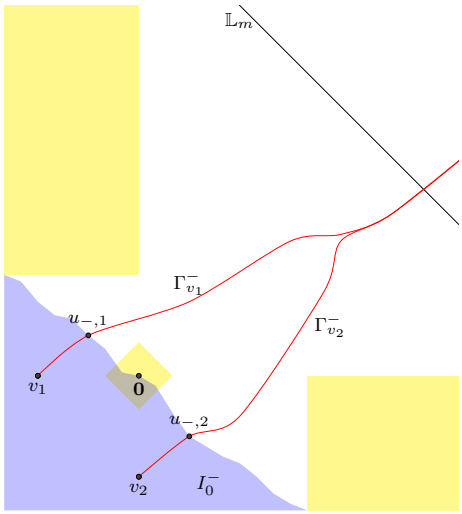
To prove Proposition 6.1, it remains to lower bound $\mathbb{P}[\mathcal{B}_-]$, $\mathbb{P}[\mathcal{B}_+]$, and $\mathbb{P}[\mathcal{F}_t]$. For this we set up some additional notations. Recall that $\boldsymbol{\rho} = ((1-\rho)^2, \rho^2)$. As in Section 5 (but omitting ρ from the notation), we set

$$\mathbb{H}_x := \{x\boldsymbol{\rho} + y((1-\rho), -\rho) : y \in \mathbb{R}\}$$

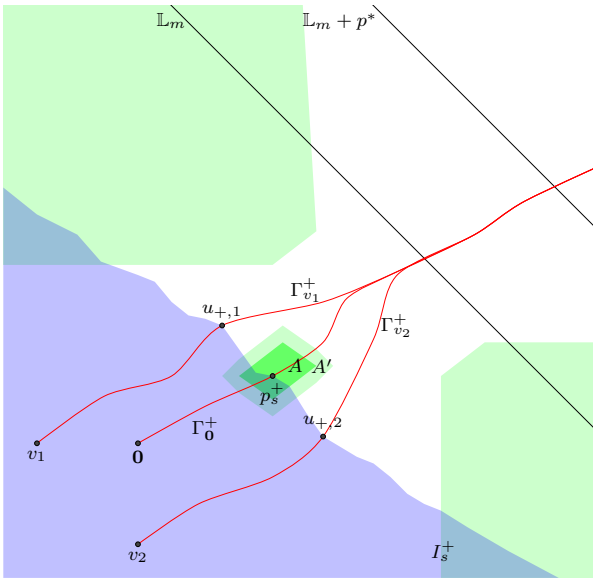
for any $x \in \mathbb{R}$. We also denote

$$\mathbb{V}_x := \{(x, -x) + y\boldsymbol{\rho} : y \in \mathbb{R}\},$$

and for any set $A \subset \mathbb{R}$ we define $\mathbb{V}_A := \bigcup_{x \in A} \mathbb{V}_x$ and $\mathbb{H}_A := \bigcup_{x \in A} \mathbb{H}_x$.



(a) Estimating the probability of $\mathcal{B}_-$: by Corollary 5.8, with probability $> 1 - Ce^{-cr}$, the geodesics $\Gamma^-_{v_1}$ and $\Gamma^-_{v_2}$ are disjoint from the sets $\{u \in \mathbb{Z}^2 : |d(u)|, |ad(u)| \leq r\}$ and $\mathbb{Z}_{\leq 0} \times [\![2r(1-\rho)^{-4}, \infty]\!]$ and $[\![2r\rho^{-4}, \infty]\!] \times \mathbb{Z}_{\leq 0}$.



(b) Estimating the probability of $\mathcal{A} \setminus \mathcal{B}_+$: by Corollaries 5.8 and 5.11, with probability $> 1 - Ce^{-cr^3 s^{-2}}$ we have $p^+_s \in A$, and the geodesics $\Gamma^+_{v_1}$ and $\Gamma^+_{v_2}$ are disjoint from the sets A' and $A + \mathbb{Z}_{\leq 0} \times [\![2r(1-\rho)^{-4}, \infty]\!]$ and $A + [\![2r\rho^{-4}, \infty]\!] \times \mathbb{Z}_{\leq 0}$.

Fig. 10. Illustrations of the proof of Proposition 6.1. The probabilities of the coalescence events are estimated using Proposition 5.5.

Proof of Proposition 6.1. In this proof we let $c, C > 0$ be small and large constants which depend on N, and their values can change from line to line.

We will show that $\hat{\eta}_t^-$ equals $\hat{\eta}_{t+s}^+$ in $[\![-N, N+1]\!]$ with probability $> 1 - C(s/t)^{1/30}$, assuming that t, s are large enough depending on N. We could also assume that t/s is large enough depending on N, since otherwise we would have $1 - C(s/t)^{1/30} < 0$. For the parameters in the definition of the events $\mathcal{A}$, $\mathcal{B}_-$, $\mathcal{B}_+$, and $\mathcal{F}_t$, we take $m = \lfloor t/10 \rfloor$ and $r = \lfloor s^{1/3} t^{1/3} \rfloor$. Denote $v_1 = (-\lfloor r\rho^{-2}(1-\rho)^{-2} \rfloor, 0)$ and $v_2 = (0, -\lfloor r\rho^{-2}(1-\rho)^{-2} \rfloor)$.

We first lower bound $\mathbb{P}[\mathcal{B}_-]$. We take $u_{-,1}$ to be the last vertex in $\Gamma_{v_1}^- \cap I_0^-$, and $u_{-,2}$ to be the last vertex in $\Gamma_{v_2}^- \cap I_0^-$. Then $u_{-,1} \in \mathbb{Z}_{\le 0} \times \mathbb{Z}_{\ge 0}$ and $u_{-,2} \in \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\le 0}$. By Corollary 5.8, we have

$$\mathbb{P}[ad(u_{-,1}) < -r], \mathbb{P}[ad(u_{-,2}) > r] > 1 - Ce^{-cr},$$

and

$$\mathbb{P}\big[u_{-,1} \in \mathbb{Z}_{\le 0} \times [\![0, 2r(1-\rho)^{-4}]\!]\big], \mathbb{P}\big[u_{-,2} \in [\![0, 2r\rho^{-4}]\!] \times \mathbb{Z}_{\le 0}\big] > 1 - Ce^{-cr}.$$

See Figure 10 (a) for an illustration. Since $2r(1-\rho)^{-4}, 2r\rho^{-4} < 2m$ (as $t, s, t/s$ are large enough), we have $\mathbb{P}[d(u_{-,1}), d(u_{-,2}) < 2m] > 1 - Ce^{-cr}$. By Proposition 5.5 we have $\mathbb{P}[\Gamma_{v_1}^- \cap \mathbb{L}_m = \Gamma_{v_2}^- \cap \mathbb{L}_m] > 1 - Crm^{-2/3}$. Thus we conclude that $\mathbb{P}[\mathcal{B}_-] > 1 - Crm^{-2/3} - Ce^{-cr}$.

We next bound $\mathbb{P}[\mathcal{A} \setminus \mathcal{B}_+]$ (see Figure 10 (b) for several sets in $\mathbb{Z}^2$ to be defined). We take $u_{+,1}$ to be the last vertex in $\Gamma_{v_1}^+ \cap I_s^+$, and $u_{+,2}$ to be the last vertex in $\Gamma_{v_2}^+ \cap I_s^+$. Then by ordering of geodesics (Lemma 5.1), and since p_s^+, $u_{+,1}$, and $u_{+,2}$ are all in ∂I_s^+, we must have $u_{+,1} \in p_s^+ + \mathbb{Z}_{\le 0} \times \mathbb{Z}_{\ge 0}$ and $u_{+,2} \in p_s^+ + \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\le 0}$.

Let $A = \mathbb{V}_{(-r,r)} \cap \mathbb{H}_{(s-(r^3s^{-2})s^{1/3}, s+(r^3s^{-2})s^{1/3})}$. Note that p_s^+ is the last vertex in $\{u \in \Gamma_{\mathbf{0}}^+ : \mathbf{G}^+(u) \le s\}$, so by Lemma 3.1, $p_s^+ - (1,0)$ or $p_s^+ - (0,1)$ is the last vertex in $\{u \in \Gamma_{\mathbf{0}}^+ : T_{\mathbf{0},u}^+ \le s\}$. Then by Corollary 5.11 we have $\mathbb{P}[p_s^+ \in A] > 1 - Ce^{-cr^3s^{-2}}$.

- When $p_s^+ \in A$, we must have $ad(u_{+,1}) < ad(p_s^+) - r$ and $ad(u_{+,2}) > ad(p_s^+) + r$, unless $u_{+,1} \in A'$ or $u_{+,2} \in A'$, where

$$A' := A + \{u \in \mathbb{R}^2 : |d(u)|, |ad(u)| \le r\} \subset \mathbb{V}_{(-2r,2r)} \cap \mathbb{H}_{(s-Cr, s+Cr)}.$$

By Corollary 5.8, we have $\mathbb{P}[u_{+,1} \in A'], \mathbb{P}[u_{+,2} \in A'] < Ce^{-cr^3s^{-2}}$. Recalling that under $\mathcal{A}$ we have $ad(p_s^+) = ad(p^*)$, we now conclude that

$$\mathbb{P}[\{ad(u_{+,1}) > ad(p^*) - r\} \cap \mathcal{A}], \mathbb{P}[\{ad(u_{+,2}) < ad(p^*) + r\} \cap \mathcal{A}] < Ce^{-cr^3s^{-2}}.$$

- When $p_s^+ \in A$, we must also have $d(u_{+,1}) < d(p_s^+) + 2r(1-\rho)^{-4}$ unless $u_{+,1} \in A + \mathbb{Z}_{\le 0} \times [\![2r(1-\rho)^{-4}, \infty]\!]$. By Corollary 5.8 the latter happens with probability $< Ce^{-cr^3s^{-2}}$, so we have

$$\mathbb{P}[d(u_{+,1} - p_s^+) \ge 2r(1-\rho)^{-4}] < Ce^{-cr^3s^{-2}}.$$

When $\mathcal{A}$ holds, we can find some $w_1, w_2 \in \mathbb{Z}^2$ with $|ad(w_1)|, |ad(w_2)| \le r+1$ such that $w_1 \in (\mathbb{Z}_- \times \mathbb{Z}_+) \cap (p_s^+ - p^* + \mathbb{Z}_- \times \mathbb{Z}_+)$, and $w_2 \in (\mathbb{Z}_+ \times \mathbb{Z}_-) \cap (p_s^+ - p^* + \mathbb{Z}_+ \times \mathbb{Z}_-)$. This implies that $|d(p_s^+ - p^*)| \le 2r$. Thus we conclude that

$$\mathbb{P}[\{d(u_{+,1} - p^*) \ge 2m\} \cap \mathcal{A}] \le \mathbb{P}[d(u_{+,1} - p_s^+) \ge 2r(1-\rho)^{-4}] < Ce^{-cr^3s^{-2}}.$$

Here the first inequality is by $2m - 2r \ge 2r(1-\rho)^{-4}$ (due to taking $s, t, t/s$ large). Similarly, $\mathbb{P}[\{d(u_{+,2} - p^*) \ge 2m\} \cap \mathcal{A}] < Ce^{-cr^3s^{-2}}$.

• We have shown that $\mathcal{A}$ implies $|d(p_s^+ - p^*)| \le 2r$. If in addition $p_s^+ \in A$, we must have $d(p^*) \ge d(p_s^+) - 2r > 0$ (since $s, t, t/s$ are taken to be large). So using Proposition 5.5 we have

$$\begin{aligned}&\mathbb{P}[\{(\Gamma_{v_1}^+ - p^*) \cap \mathbb{L}_m \ne (\Gamma_{v_2}^+ - p^*) \cap \mathbb{L}_m\} \cap \mathcal{A}]\\ &\qquad \le \mathbb{P}[\Gamma_{v_1}^+ \cap \mathbb{L}_m \ne \Gamma_{v_2}^+ \cap \mathbb{L}_m] + \mathbb{P}[p_s^+ \notin A] < Crm^{-2/3} + Ce^{-cr^3s^{-2}}.\end{aligned}$$

Thus we conclude that $\mathbb{P}[\mathcal{A} \setminus \mathcal{B}_+] < Crm^{-2/3} + Ce^{-cr^3s^{-2}}$.

Finally, we consider $\mathbb{P}[\mathcal{F}_t]$. Since p_t^- is the last vertex in $\{u \in \Gamma_{\mathbf{0}}^- : \mathbf{G}^-(u) \le t\}$, Lemma 3.1 implies that $p_t^- - (1,0)$ or $p_t^- - (0,1)$ is the last vertex in $\{u \in \Gamma_{\mathbf{0}}^- : T_{\mathbf{0},u}^- \le t\}$. Then by Corollary 5.11, with probability $> 1 - Ce^{-ct^{2/3}}$ we have $p_t^- \in \mathbb{H}_{(t/2,2t)} \cap \mathbb{V}_{(-t^{8/9},t^{8/9})}$, thus $d(p_t^-) \ge ((1-\rho)^2 + \rho^2)t/2 - t^{8/9} \ge t/4 - t^{8/9}$. Note that since $m = \lfloor t/10 \rfloor$ and t is taken large enough depending on N, we have $t/4 - t^{8/9} > 2m + 2N + 2$. So $\mathbb{P}[\mathcal{F}_t] > 1 - Ce^{-ct^{2/3}}$.

By Lemma 6.3, $\hat{\eta}_t^-$ equals $\hat{\eta}_{t+s}^+$ in $[\![-N, N+1]\!]$ with probability at least

$$\begin{aligned}&\mathbb{P}[\mathcal{B}_-] + \mathbb{P}[\mathcal{A}] - \mathbb{P}[\mathcal{A} \setminus \mathcal{B}_+] + \mathbb{P}[\mathcal{F}_t] - 2\\ &> 1 - (Crm^{-2/3} + Ce^{-cr}) - C(rs^{-2/3})^{-1/10} - (Crm^{-2/3} + Ce^{-cr^3s^{-2}}) - Ce^{-ct^{2/3}}\\ &\qquad > 1 - C(s/t)^{1/30},\end{aligned}$$

where the first inequality uses the estimates of $\mathbb{P}[\mathcal{B}_-]$ and $\mathbb{P}[\mathcal{A} \setminus \mathcal{B}_+]$ above, and the estimates on $\mathbb{P}[\mathcal{A}]$ from Lemma 6.2. Thus the conclusion follows. ■

6.1. The initial step coupling

This subsection is devoted to proving Lemma 6.2.

We define $(\tau_t)_{t\in\mathbb{R}}$ as the process of stationary TASEP with density ρ, i.e. for any $t \in \mathbb{R}$, $\tau_t(x)$ is Bernoulli(ρ) for each $x \in \mathbb{Z}$ independently. Our strategy is to construct a coupling between the processes $(\eta_t^+)_{t\ge0}$ and $(\tau_t)_{t\ge0}$, where (with high probability) $\hat{\eta}_s^+$ and τ_s are identical outside $[\![-r, r]\!]$, and have the same number of particles in $[\![-r, r]\!]$. It would be straightforward to couple $\hat{\eta}_0^-$ and τ_s since both are Bernoulli(ρ) on $\mathbb{Z} \setminus \{0,1\}$.

We denote $\alpha = (rs^{-2/3})^{1/5}$ and $r_i = \alpha^i s^{2/3}$ for $i = 1,2,3,4$. Below we assume that α and s are large enough, and also $\alpha < r^{0.01}$. Recall the notations L^-, $\mathbf{G}^-$, I_t^-, ∂I_t^-,

$\xi^{-,\vee}$, $\xi^{-,\vee,t}$, $T^-_{u,v}$, $\Gamma^-_{u,v}$, Γ^-_u, $\Gamma^{-,\vee}_u$, p^-_t (resp. L^+, $\mathbf{G}^+$, I^+_t, ∂I^+_t, $\xi^{+,\vee}$, $\xi^{+,\vee,t}$, $T^+_{u,v}$, $\Gamma^+_{u,v}$, Γ^+_u, $\Gamma^{+,\vee}_u$, p^+_t) for LPP with weights ξ^- (resp. ξ^+). Also recall the notations $\mathbb{V}_x$, $\mathbb{H}_x$ and $\mathbb{V}_A$, $\mathbb{H}_A$ for $x \in \mathbb{R}$ and $A \subset \mathbb{R}$.

We now explain the coupling between $(\eta^+_t)_{t\geq 0}$ and $(\tau_t)_{t\geq 0}$. One straightforward way is to couple η^+_0 and τ_0 so that they are the same outside a compact interval, and let them evolve using the same exponential waiting times (just like $(\eta^-_t)_{t\geq 0}$ and $(\eta^+_{s+t})_{t\geq 0}$ are coupled). One can show that under this coupling, with high probability η^+_s and τ_s are the same, as in the proof of Proposition 6.1 assuming Lemma 6.2. However, we need to compare $\hat{\eta}^+_s$ and τ_s instead. For this, we first shift η^+_0 by $ad(p^+_s)$, and then couple it with τ_0, so that they are the same outside a compact interval, and then let them evolve using the same exponential waiting times. One problem is that the number $ad(p^+_s)$ depends on the evolution of $(\eta^+_t)_{t\geq 0}$. To solve this, we exploit the fact that $ad(p^+_s)$ mostly depends on the evolution around the hole-particle pair. Specifically, we take the following approach: we first sample the evolution of $(\eta^+_t)_{t\geq 0}$ around the pair (which corresponds to sampling the waiting times $\xi^{+,\vee,0}$ in a tube $\mathbb{V}_{(-r_1,r_1)}$) to get a proxy of p^+_s which equals p^+_s with high probability. Using that, we could shift τ_0 and couple the rest of the waiting times $\xi^{+,\vee,0}$ with the waiting times of $(\tau_t)_{t\geq 0}$.

We start by defining the proxy of p^+_s. Denote $P = \mathbb{V}_{(-r_1,r_1)} \cap \mathbb{Z}^2$. First we define L^P by letting $L^P(u) = 0$ for $u \in I^+_0 \cup (\mathbb{Z}^2 \setminus P)$, and setting $L^P(u) = L^P(u-(1,0)) \vee L^P(u-(0,1)) + \xi^{+,\vee,0}(u)$ recursively for each $u \in P \setminus I^+_0$. As in Lemma 3.5, $L^P(u)$ can also be defined as the maximum passage time in $\mathbb{Z}^2 \setminus I^+_0$ to u under the weights $\{\xi^{+,\vee,0}(v)\mathbb{1}[v \in P]\}_{v\in\mathbb{Z}^2\setminus I^+_0}$. Then with probability 1, $L^P(u)$ for $u \in P \setminus I^+_0$ are mutually different, and below we work under this event. Analogously to the inductive construction of $\Gamma^+_{\mathbf{0}}$ (see Lemma 3.2), we define $\Gamma^P_{\mathbf{0}}$ by letting $\Gamma^P_{\mathbf{0}}[1] = \mathbf{0}$ and

$$\Gamma^P_{\mathbf{0}}[i+1] = \operatorname{argmin}_{v\in\{\Gamma^P_{\mathbf{0}}[i]+(1,0),\Gamma^P_{\mathbf{0}}[i]+(0,1)\}\cap P} L^P(v) \tag{6.8}$$

for each $i \in \mathbb{N}$. Recall that p^+_s is the last vertex in $\Gamma^+_{\mathbf{0}} \cap I^+_s$. We let p^P be the last vertex in $\{u \in \Gamma^P_{\mathbf{0}} : L^P(u) \leq s\}$. Denote $M = ad(p^P)$. Then M is determined by $I^+_0 \cap P$ and $\{\xi^{+,\vee,0}(u)\}_{u\in P\setminus I^+_0}$.

We next show that this proxy p^P equals p^+_s with high probability.

Lemma 6.5. $\mathbb{P}[p^P = p^+_s] > 1 - Ce^{-c\alpha^3}$ *for some constants* $c, C > 0$.

Proof. According to Lemma 3.5 and as stated above, $L^+(u)$ and $L^P(u)$ are the maximum passage times from a vertex in $\mathbb{Z}^2 \setminus I^+_0$ to u, under the weights $\{\xi^{+,\vee,0}(v)\}_{v\in\mathbb{Z}^2\setminus I^+_0}$ and $\{\xi^{+,\vee,0}(v)\mathbb{1}[v \in P]\}_{v\in\mathbb{Z}^2\setminus I^+_0}$ respectively. Also Lemma 3.5 states that the path with the maximum passage time to u under the weights $\{\xi^{+,\vee,0}(v)\}_{v\in\mathbb{Z}^2\setminus I^+_0}$ is precisely $\Gamma^{+,\vee}_u \setminus I^+_0$. Then $L^+(u) = L^P(u)$ for any u with $\Gamma^{+,\vee}_u \setminus I^+_0 \subset P$.

We take

$$u_1 = \left(\left\lfloor -\frac{3((1-\rho)^2+\rho^2)r_1}{4\rho^2}\right\rfloor, 0\right), \quad u_2 = \left(0, \left\lfloor -\frac{3((1-\rho)^2+\rho^2)r_1}{4(1-\rho)^2}\right\rfloor\right).$$

We let $\mathcal{D}_1$ be the event where

$$\Gamma^+_{u_1} \cap \mathbb{H}_{(-\infty,2s)} \subset \mathbb{V}_{(-r_1,-r_1/2)}, \quad \Gamma^+_{u_2} \cap \mathbb{H}_{(-\infty,2s)} \subset \mathbb{V}_{(r_1/2,r_1)}.$$

Assuming that $\mathcal{D}_1$ holds, we denote

$$S = \Big(\bigcup_{v\in\Gamma^+_{u_1}} (v + \mathbb{Z}_+ \times \mathbb{Z}_{\le 0}) \Big) \cap \Big(\bigcup_{v\in\Gamma^+_{u_2}} (v + \mathbb{Z}_{\le 0} \times \mathbb{Z}_+) \Big) \cap (\mathbb{H}_{(-\infty,2s)} \setminus I^+_0).$$

In other words, S is the set consisting of the vertices in $\mathbb{H}_{(-\infty,2s)} \setminus I^+_0$ between $\Gamma^+_{u_1}$ and $\Gamma^+_{u_2}$. Take any $u \in S$. By the non-crossing property (Lemma 3.3), $\Gamma^{+,\vee}_u$ is disjoint from $\bigcup_{v\in\Gamma^+_{u_1}} (v + \mathbb{Z}_{\le 0} \times \mathbb{Z}_+)$. As $u_1 + \mathbb{Z}^2_{\le 0}$, $\bigcup_{v\in\Gamma^+_{u_1}} (v + \mathbb{Z}_{\le 0} \times \mathbb{Z}_+)$, $\bigcup_{v\in\Gamma^+_{u_1}} (v + \mathbb{Z}_+ \times \mathbb{Z}_{\le 0})$ is a disjoint partition of $\mathbb{Z}^2$, and $u_1 + \mathbb{Z}^2_{\le 0} \subset I^+_0$, we must have $\Gamma^{+,\vee}_u \setminus I^+_0 \subset \bigcup_{v\in\Gamma^+_{u_1}} (v + \mathbb{Z}_+ \times \mathbb{Z}_{\le 0})$. Since $u \in \mathbb{H}_{(-\infty,2s)}$, we further have $\Gamma^{+,\vee}_u \setminus I^+_0 \subset \bigcup_{v\in\Gamma^+_{u_1}\cap\mathbb{H}_{(-\infty,2s)}} (v + \mathbb{Z}_+ \times \mathbb{Z}_{\le 0})$. Similarly $\Gamma^{+,\vee}_u \setminus I^+_0 \subset \bigcup_{v\in\Gamma^+_{u_2}\cap\mathbb{H}_{(-\infty,2s)}} (v + \mathbb{Z}_{\le 0} \times \mathbb{Z}_+)$. Then by $\mathcal{D}_1$ we must have $\Gamma^{+,\vee}_u \setminus I^+_0 \subset P$, so $L^+(u) = L^P(u)$.

Let $\mathcal{D}_2$ be the event where $\Gamma^+_{\mathbf{0}} \cap I^+_s \subset \mathbb{V}_{(-r_1/3,r_1/3)} \cap \mathbb{H}_{(-\infty,3s/2)}$. Under $\mathcal{D}_1$ we have $\mathbb{V}_{(-r_1/3-1,r_1/3+1)} \cap \mathbb{H}_{(-\infty,2s)} \setminus I^+_0 \subset S$, so under $\mathcal{D}_1 \cap \mathcal{D}_2$ we have $(\Gamma^+_{\mathbf{0}} \cap I^+_s) + \{(1,0),(0,1)\} \subset S$. Then by the inductive constructions of $\Gamma^+_{\mathbf{0}}$ and $\Gamma^P_{\mathbf{0}}$ (Lemma 3.2 and (6.8)), one can inductively show that $\Gamma^+_{\mathbf{0}}[i] = \Gamma^P_{\mathbf{0}}[i]$ and $L^+(\Gamma^+_{\mathbf{0}}[i] + (1,0)) = L^P(\Gamma^P_{\mathbf{0}}[i] + (1,0))$, $L^+(\Gamma^+_{\mathbf{0}}[i] + (0,1)) = L^P(\Gamma^P_{\mathbf{0}}[i] + (0,1))$ for all $i \in \mathbb{N}$ such that $L^+(\Gamma^+_{\mathbf{0}}[i]) \le s$. By considering the largest such i we conclude that $p^P = p^+_s$.

Now $\mathbb{P}[p^P = p^+_s] \ge \mathbb{P}[\mathcal{D}_1 \cap \mathcal{D}_2]$, and it remains to lower bound $\mathbb{P}[\mathcal{D}_1]$ and $\mathbb{P}[\mathcal{D}_2]$. By Corollary 5.8 we have $\mathbb{P}[\mathcal{D}_1] > 1 - Ce^{-cr_1^3 s^{-2}}$. For $\mathcal{D}_2$, as p^+_s is the last vertex in $\{u \in \Gamma^+_{\mathbf{0}} : \mathbf{G}^+(u) \le s\}$, by Lemma 3.1 we know that $p^+_s - (1,0)$ or $p^+_s - (0,1)$ is the last vertex in $\{u \in \Gamma^+_{\mathbf{0}} : T^+_{\mathbf{0},u} \le s\}$. Then Corollary 5.11 implies that $\mathbb{P}[\mathcal{D}_2] > 1 - Ce^{-cr_1^3 s^{-2}}$ (noting that $3s/2 > s + C(r_1^3 s^{-2})s^{1/3}$ by our choice of the parameters). Thus the conclusion follows. ∎

We next couple τ_0 and $(\eta^+_t)_{t\ge 0}$. We state the coupling by constructing τ_0 conditioned on $(\eta^+_t)_{t\ge 0}$, using the following steps.

(1) Let $\tau_0(x - M)$ be i.i.d. Bernoulli(ρ) for each $x \in [\![-r_2, r_2]\!]$.

(2) For each $x = \lfloor -r_2 \rfloor, \lfloor -r_2 \rfloor - 1, \dots,$ we take $\tau_0(x - M)$ to be i.i.d. Bernoulli(ρ), until the first $x_* \in \mathbb{Z}$ with $\sum_{x=x_*}^{0} \eta^+_0(x) - \tau_0(x - M) = 0$. Then for each $x < x_*$ we take $\tau_0(x - M) = \eta^+_0(x)$.

(3) For each $x = \lceil r_2 \rceil, \lceil r_2 \rceil + 1, \dots,$ we take $\tau_0(x - M)$ to be i.i.d. Bernoulli(ρ), until the first $x^* \in \mathbb{Z}$ with $\sum_{x=1}^{x^*} \eta^+_0(x) - \tau_0(x - M) = 0$. Then for each $x > x^*$ we take $\tau_0(x - M) = \eta^+_0(x)$.

As $\alpha = r_2/r_1$ is large enough, the set $P \cap (\mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\le 0} \cup \mathbb{Z}_{\le 0} \times \mathbb{Z}_{\ge 0})$ is contained in $\{u \in \mathbb{Z}^2 : |ad(u)| < r_2\}$. Also note that $I^+_0 \cap P$ is determined by $I^+_0 \cap P \cap$

$(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\leq 0} \cup \mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\geq 0})$, as $P \cap \mathbb{Z}^2_- \subset I_0^+$ and $P \cap \mathbb{Z}^2_+$ is disjoint from I_0^+. Then $I_0^+ \cap P$ is determined by $I_0^+ \cap \{u \in \mathbb{Z}^2 : |ad(u)| < r_2\}$, which (by Lemma 3.6) is determined by $\{\eta_0^+(x)\}_{|x|<r_2}$, thus is independent of $\{\eta_0^+(x)\}_{|x|\geq r_2}$. Since M is determined by $I_0^+ \cap P$ and $\{\xi^{+,\vee,0}(u)\}_{u\in P\setminus I_0^+}$, we see that M is also independent of $\{\eta_0^+(x)\}_{|x|\geq r_2}$. Then from the construction of τ_0, and since $\{\eta_0^+(x)\}_{|x|\geq r_2}$ are i.i.d. Bernoulli(ρ), it follows that $\{\tau_0(x-M)\}_{x\in\mathbb{Z}}$ are i.i.d. Bernoulli(ρ) conditioned on M. Thus $\{\tau_0(x)\}_{x\in\mathbb{Z}}$ are i.i.d. Bernoulli(ρ) and independent of M. Conditioned on I_0^+, both τ_0 and M are independent of $\{\xi^{+,\vee,0}(u)\}_{u\in\mathbb{Z}^2\setminus(P\cup I_0^+)}$.

We wish to let $(\tau_t)_{t\geq 0}$ evolve using the waiting times $\{\xi^{+,\vee,0}(u)\}_{u\in\mathbb{Z}^2\setminus(P\cup I_0^+)}$. For this, in τ_0 we label the holes by $\mathbb{Z}$ from left to right, and the particles by $\mathbb{Z}$ from right to left, in such a way that for any $x \in \mathbb{Z} \setminus [\![x_*, x^*]\!]$, the particle (or hole) at site $x - M$ has the same label as the particle (or hole) at site x in η_0^+. This can be achieved since τ_0 and η_0^+ are the same outside $[\![x_*, x^*]\!]$, and they have the same number of particles in $[\![x_*, 0]\!]$ and $[\![1, x^*]\!]$, respectively.

Let $L^\tau(a,b)$ be the time when the particle labeled b switches with the hole labeled a if in τ_0 this particle is to the left of this hole, and let $L^\tau(a,b) = 0$ otherwise. Note that unlike L^+, this function L^τ does not have the same distribution as the Busemann function in LPP. However, we can still define a growth process from it. For each $t \geq 0$ denote $I_t^\tau := \{u \in \mathbb{Z}^2 : L^\tau(u) \leq t\}$ and $\partial I_t^\tau := \{u \in I_t^\tau : L^\tau(u+(1,0)) \vee L^\tau(u+(0,1)) > t\}$. Then I_0^τ is the same as I_0^+ outside a compact set.

Lemma 6.6. *For any $u \in (I_0^\tau \setminus I_0^+) \cup (I_0^+ \setminus I_0^\tau)$ we have $ad(u) \in [\![x_*, x^*]\!]$*

Proof. Write $u = (a,b)$. If $u \notin I_0^+$, there is some $x \leq 0$ such that $(a-x, b-x) \notin I_0^+$, $(a-x-1, b-x) \in I_0^+$, or $(a-x-1, b-x) \notin I_0^+$, $(a-x-1, b-x-1) \in I_0^+$. Then by Lemma 3.6, either $\eta_0^+(a-b) = 1$ and the particle at site $a-b$ (in η_0^+) has label $b-x$, or $\eta_0^+(a-b) = 0$ and the hole at site $a-b$ (in η_0^+) has label $a-x-1$. If $u \in I_0^\tau$, we can similarly deduce that there is some $y \geq 0$ such that $(a+y+1, b+y+1) \notin I_0^\tau$, $(a+y, b+y+1) \in I_0^\tau$, or $(a+y, b+y+1) \notin I_0^\tau$, $(a+y, b+y) \in I_0^\tau$. Then by (an analogue of) Lemma 3.6, either $\tau_0(a-b-M) = 1$ and the particle at site $a-b-M$ (in τ_0) has label $b+y+1$, or $\tau_0(a-b-M) = 0$ and the hole at site $a-b-M$ (in τ_0) has label $a+y$.

Thus if $u \in I_0^\tau \setminus I_0^+$, we must have $\eta_0^+(a-b) \neq \tau_0(a-b-M)$ or the particles/holes do not have the same label. So from the coupling between η_0^+ and τ_0 we infer that $a-b \in [\![x_*, x^*]\!]$, and the conclusion follows. The case where $u \in I_0^+ \setminus I_0^\tau$ follows from similar arguments. ∎

We can also define the waiting times by letting

$$\xi^{\tau,\vee}(u) = L^\tau(u) - L^\tau(u-(1,0)) \vee L^\tau(u-(0,1))$$

for any $u \in \mathbb{Z}^2 \setminus I_0^\tau$. Given I_0^τ (equivalently, τ_0 and the labels), we find that $\{\xi^{\tau,\vee}(u)\}_{u\notin I_0^\tau}$ are i.i.d. Exp(1), since they are precisely the waiting times for certain particles and holes

to switch. Then almost surely $L^\tau(u)$ are mutually different for all $u \in \mathbb{Z}^2 \setminus I_0^\tau$, and below we assume this event.

We now couple $\{\xi^{\tau,\vee}(u)\}_{u \notin I_0^\tau}$ with $(\eta_t^+)_{\geq 0}$ in such a way that conditioned on τ_0 and $(\eta_t^+)_{t\geq 0}$, we have $\xi^{\tau,\vee}(u) = \xi^{+,\vee,0}(u)$ for any $u \in \mathbb{Z}^2 \setminus (I_0^+ \cup I_0^\tau \cup P)$, and $\xi^{\tau,\vee}(u)$ for $u \in (P \cup I_0^+) \setminus I_0^\tau$ are i.i.d. Exp(1). Under this coupling, we denote by $\mathcal{E}_1$ the event where for any $x < -r$, $\tau_s(x) = \hat{\eta}_s^+(x) = \eta_s^+(x + ad(p_s^+))$, and the particle or hole at site x has the same label for τ_s and $\hat{\eta}_s^+$; denote by $\mathcal{E}_2$ the event where the same is true for any $x > r$.

Lemma 6.7. *We have* $\mathbb{P}[\mathcal{E}_1], \mathbb{P}[\mathcal{E}_2] > 1 - C\alpha^{-1/2}$ *when* $Cs^{2/3} < r < s^{2/3+0.01}$ *and* $s > C$, *where* $C > 0$ *is a constant.*

We can now prove Lemma 6.2 assuming Lemma 6.7.

Proof of Lemma 6.2. Under $\mathcal{E}_1 \cap \mathcal{E}_2$, we have $\tau_s(x) = \hat{\eta}_s^+(x)$ for any $x \in \mathbb{Z}$ with $|x| > r$. Also, note $1 + \sum_{x=-r}^r \hat{\eta}_s^+(x)$ and $1 + \sum_{x=-r}^r \tau_s(x)$ are precisely the differences between the label of the first particle to the left of $-r$ and the label of the first particle to the right of r in $\hat{\eta}_s^+$ and τ_s respectively, so $\sum_{x=-r}^r \hat{\eta}_s^+(x) = \sum_{x=-r}^r \tau_s(x)$ under $\mathcal{E}_1 \cap \mathcal{E}_2$.

We can couple $\hat{\eta}_0^-$ with τ_s as follows. Conditioned on $\hat{\eta}_0^-$, we let $\tau_s(x)$ for $x = 1, 2, \ldots$ be i.i.d. Bernoulli(ρ) until some $y^* \in \mathbb{Z}$ such that $\sum_{x=1}^{y^*} \tau_s(x) - \hat{\eta}_0^-(x) = 0$, and for any $x > y^*$ we let $\tau_s(x) = \hat{\eta}_0^-(x)$; we also let $\tau_s(x)$ for $x = 0, -1, \ldots$ be i.i.d. Bernoulli(ρ) until some $y_* \in \mathbb{Z}$ such that $\sum_{x=y_*}^0 \tau_s(x) - \hat{\eta}_0^-(x) = 0$, and for any $x < y_*$ we let $\tau_s(x) = \hat{\eta}_0^-(x)$. Let $\mathcal{E}_*$ be the event $|y_*|, |y^*| \leq r$. Then $\mathbb{P}[\mathcal{E}_*] > 1 - Cr^{-1/2}$ for some constant $C > 0$, since $\hat{\eta}_0^-$ is Bernoulli(ρ) on $\mathbb{Z} \setminus \{0, 1\}$. On the other hand, under $\mathcal{E}_*$ we have $\hat{\eta}_0^-(x) = \tau_s(x)$ for any $x \in \mathbb{Z}$ with $|x| > r$ and $\sum_{x=-r}^r \hat{\eta}_0^-(x) = \sum_{x=-r}^r \tau_s(x)$. Thus $\mathcal{E}_* \cap \mathcal{E}_1 \cap \mathcal{E}_2$ implies $\mathcal{A}$, and $\mathbb{P}[\mathcal{A}] > \mathbb{P}[\mathcal{E}_1] + \mathbb{P}[\mathcal{E}_2] + \mathbb{P}[\mathcal{E}_*] - 2$. Using $\mathbb{P}[\mathcal{E}_*] > 1 - Cr^{-1/2}$ and Lemma 6.7, the conclusion follows. ∎

In the rest of this section we prove Lemma 6.7.

For any $u \in (\mathbb{Z}^2 \setminus I_0^\tau) \cup \partial I_0^\tau$, we also define the 'semi-infinite geodesic' Γ_u^τ recursively, by letting $\Gamma_u^\tau[1] = u$, and $\Gamma_u^\tau[i+1] = \operatorname{argmin}_{v \in \{\Gamma_u^\tau[i]+(1,0), \Gamma_u^\tau[i]+(0,1)\}} L^\tau(v)$ for each $i \in \mathbb{N}$. Note that since L^τ is not coupled with the LPP Busemann function, these Γ_u^τ are not actual geodesics.

We consider the following events (see Figure 11 for an illustration of the geometric objects).

$\mathcal{E}_3$: there exists a vertex $u_+ \in \partial I_0^+$ such that $ad(u_+) < x_*$ and $\Gamma_{u_+}^+ \cap I_s^+ \subset \mathbb{V}_{(-6r_3,-r_1)}$, and $a'_+ > (1-\rho)^2 s - r_4$ for $u'_+ = (a'_+, b'_+)$ being the last vertex in $\Gamma_{u_+}^+ \cap I_s^+$.

$\mathcal{E}_3^\tau$: there exists a vertex $u_\tau \in \partial I_0^\tau$ such that $ad(u_\tau) < x_*$ and $\Gamma_{u_\tau}^\tau \cap I_s^\tau \subset \mathbb{V}_{(-6r_3,-r_1)}$, and $a'_\tau > (1-\rho)^2 s - r_4$ for $u'_\tau = (a'_\tau, b'_\tau)$ being the last vertex in $\Gamma_{u_\tau}^+ \cap I_s^\tau$.

$\mathcal{E}_4$: for each $u = (a, b) \in \partial I_s^+$ with $ad(u) \leq M - r$, we have $a < (1-\rho)^2 s - r_4 - 1$, and $u + (1, 0) \in \mathbb{V}_{(-\infty,-6r_3)}$.

The purpose of these events is as follows. $\mathcal{E}_3$ and $\mathcal{E}_3^\tau$ ensure that for u in a certain region (around $\{u \in \mathbb{Z}^2 : ad(u) \leq M - r\} \cap \partial I_s^+$), the downward geodesics $\Gamma_u^{+,\vee}$ and

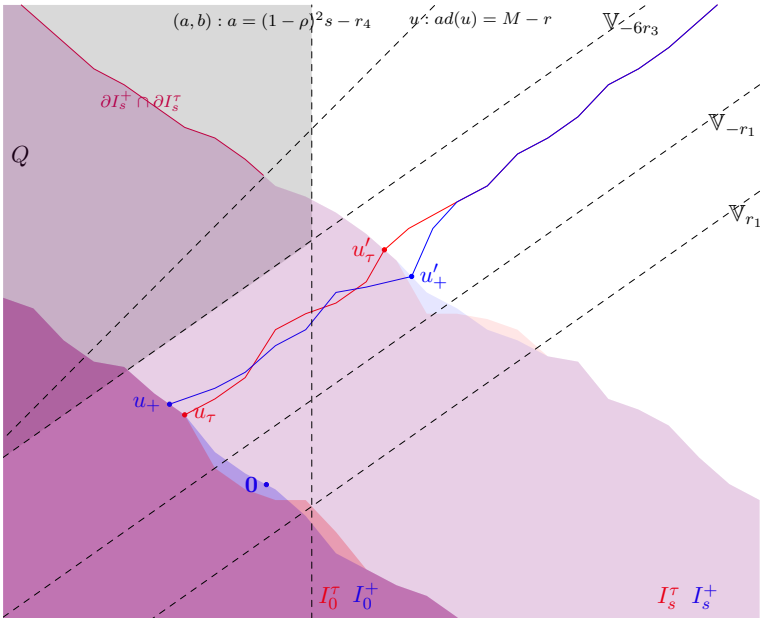


Fig. 11. The events $\mathcal{E}_3, \mathcal{E}_3^\tau, \mathcal{E}_4$, assuming $p^P = p_s^+$. The shaded region is Q in the proof of Lemma 6.8.

$\Gamma_u^{\tau,\vee}$ are disjoint from P. Thus using Lemma 6.6 and the coupling between $\xi^{\tau,\vee}$ and $\xi^{+,\vee,0}$ we have $L^+(u) = L^\tau(u)$ for these u. Then we can deduce that $\{u \in \mathbb{Z}^2 : ad(u) \le M - r\} \cap \partial I_s^+$ is the same as $\{u \in \mathbb{Z}^2 : ad(u) \le M - r\} \cap \partial I_s^\tau$. Then using $p^P = p_s^+$ and Lemma 3.6, we find that $\mathcal{E}_1$ holds. The event $\mathcal{E}_4$ is to define this 'certain region'. In summary, we have the following statement.

Lemma 6.8. $\{p^P = p_s^+\} \cap \mathcal{E}_3 \cap \mathcal{E}_3^\tau \cap \mathcal{E}_4 \subset \mathcal{E}_1$.

Proof. Below we assume that $\mathcal{E}_3 \cap \mathcal{E}_3^\tau \cap \mathcal{E}_4$ holds and $p^P = p_s^+$. Denote

$$Q = \{(a,b) \in \mathbb{Z}^2 : a < (1-\rho)^2 s - r_4,\ (a,b) \in \mathbb{V}_{(-\infty,-6r_3)}\}.$$

See Figure 11. Then $Q \setminus I_0^+ = Q \setminus I_0^\tau$. Indeed, otherwise we can find some $u \in Q$ with $u \in \partial I_0^\tau$ and $u \notin I_0^+$, or $u \in \partial I_0^+$ and $u \notin I_0^\tau$. In the first case, $u \in u_\tau + \mathbb{Z}_{\le 0} \times \mathbb{Z}_+$ since $u \in \mathbb{V}_{(-\infty,-6r_3)}$, $u_\tau \in \mathbb{V}_{(-6r_3,-r_1)}$, and both $u, u_\tau \in \partial I_0^\tau$. So we must have $ad(u) < ad(u_\tau) < x_*$. But $ad(u) \in [\![x_*, x^*]\!]$ by Lemma 6.6, so we get a contradiction. A similar contradiction can be obtained in the second case.

Now we take any $u \in Q \setminus I_0^+ = Q \setminus I_0^\tau$, and we show that $L^+(u) \le L^\tau(u)$. By Lemma 3.5 we know that $L^+(u) = \sum_{v \in \Gamma_u^{+,\vee} \setminus I_0^+} \xi^{+,\vee,0}(v)$, and this is the maximum passage time to u from a vertex in $\mathbb{Z}^2 \setminus I_0^+$ under the weights $\xi^{+,\vee,0}$. Analogously, $L^\tau(u)$ equals the maximum passage time to u from a vertex in $\mathbb{Z}^2 \setminus I_0^\tau$ under the weights $\xi^{\tau,\vee}$. It then suffices to show that $\Gamma_u^{+,\vee} \setminus I_0^+$ is disjoint from P and I_0^τ, since then $\Gamma_u^{+,\vee} \setminus I_0^+$

is an up-right path from a vertex in $\mathbb{Z}^2 \setminus I_0^\tau$ to u, and for any $v \in \Gamma_u^{+,\vee} \setminus I_0^+$ we have $\xi^{\tau,\vee}(v) = \xi^{+,\vee,0}(v)$, thus

$$L^+(u) = \sum_{v \in \Gamma_u^{+,\vee} \setminus I_0^+} \xi^{+,\vee,0}(v) = \sum_{v \in \Gamma_u^{+,\vee} \setminus I_0^+} \xi^{\tau,\vee}(v) \le L^\tau(u).$$

We show that $(\Gamma_u^{+,\vee} \setminus I_0^+) \cap (P \cup I_0^\tau) = \emptyset$, using the following steps.

Step 1: By Lemma 3.3, the path $\Gamma_{u_+}^+ + (1/2, 1/2)$ divides $u_+ + (\mathbb{Z}^2 \setminus \mathbb{Z}^2_{\le 0})$ into two parts such that $\Gamma_u^{+,\vee}$ intersects at most one of them. By $\mathcal{E}_3$ and $u \in Q \setminus I_0^+$, we know that u must be in the upper-left part, so $\Gamma_u^{+,\vee}$ cannot intersect the lower-right part. In particular, $\Gamma_u^{+,\vee}$ is disjoint from $u_+ + \mathbb{Z}_+ \times \mathbb{Z}_{\le 0}$. Also $u_+ + \mathbb{Z}^2_{\le 0} \subset I_0^+$ since $u_+ \in I_0^+$, so $\Gamma_u^{+,\vee} \setminus I_0^+ \subset u_+ + \mathbb{Z} \times \mathbb{Z}_+$.

Step 2: If $\Gamma_u^{+,\vee} \setminus I_0^+$ is not disjoint from I_0^τ, take any $v \in (\Gamma_u^{+,\vee} \setminus I_0^+) \cap I_0^\tau$. Then $v \in u_+ + \mathbb{Z} \times \mathbb{Z}_+$ according to the previous step. By Lemma 6.6 we have $ad(v) \ge x_*$, and $\mathcal{E}_3$ states that $ad(u_+) < x_*$. So $ad(v) > ad(u_+)$, and thus $v - u_+ \in \mathbb{Z}^2_+$. But this implies that $u_+ + (1,1) \in I_0^\tau \setminus I_0^+$, which contradicts Lemma 6.6 since $ad(u_+ + (1,1)) = ad(u_+) < x_*$.

Step 3: Since $u_+ \subset \mathbb{V}_{(-6r_3, -r_1)}$ (by $\mathcal{E}_3$), we have $P \cap (u_+ + \mathbb{Z}_{\le 0} \times \mathbb{Z}_{\ge 0}) = \emptyset$. We also have $u_+ + \mathbb{Z}^2_{\le 0} \subset I_0^+$ since $u_+ \in I_0^+$. Thus $P \setminus I_0^+ \subset u_+ + \mathbb{Z}_+ \times \mathbb{Z}$. Take any $(a,b) \in P \setminus I_0^+$. If $a \ge (1-\rho)^2 s - r_4$, we cannot have $(a,b) \in \Gamma_u^{+,\vee}$ since $u \in Q$. If $a < (1-\rho)^2 s - r_4$, by $\mathcal{E}_3$ the point (a,b) is in the lower-right part from Step 1, so still $(a,b) \notin \Gamma_u^{+,\vee}$. Hence $P \setminus I_0^+$ is disjoint from $\Gamma_u^{+,\vee}$, and equivalently $\Gamma_u^{+,\vee} \setminus I_0^+$ is disjoint from P.

So far we have shown $L^+(u) \le L^\tau(u)$. We can also show $L^+(u) \le L^\tau(u)$ with essentially verbatim arguments, using $\mathcal{E}_3^\tau$ instead of $\mathcal{E}_3$. We then conclude that $L^+(u) = L^\tau(u)$ for any $u \in Q \setminus I_0^+ = Q \setminus I_0^\tau$.

We then show that $\mathcal{E}_1$ holds, using Lemma 3.6. Specifically, take any $x \in \mathbb{Z}$ with $x < -r$. We next show that $\hat{\eta}_s^+(x) = \tau(x)$, and the particles (or holes) have the same labels.

We first assume that $\hat{\eta}_s^+(x) = 1$. Then $\eta_s^+(x + ad(p_s^+)) = 1$. By $p^P = p_s^+$ we have $M = ad(p_s^+)$, so $\eta_s^+(x + M) = 1$. By Lemma 3.6, there is some $y \in \mathbb{Z}$ such that $(M + x + y - 1, y) \in I_s^+$ and $(M + x + y, y) \notin I_s^+$, and the particle at x in $\hat{\eta}_s^+$ has label y. Since $ad(M + x + y - 1, y) = M + x - 1 \le M - r$ and $(M + x + y - 1, y) \in \partial I_s^+$, by $\mathcal{E}_4$ we have $M + x + y < (1-\rho)^2 s - r_4$ and $(M + x + y, y) \in \mathbb{V}_{(-\infty, -6r_3)}$. Thus $(M + x + y, y), (M + x + y - 1, y) \in Q$, and

$$\begin{aligned} L^\tau(M + x + y, y) &= L^+(M + x + y, y) \\ &> s \ge L^+(M + x + y - 1, y) = L^\tau(M + x + y - 1, y). \end{aligned}$$

This implies that $(M + x + y - 1, y) \in I_s^\tau$ and $(M + x + y, y) \notin I_s^\tau$. Then by (an analogue of) Lemma 3.6 we have $\tau_s(x) = 1$, and the particle at x in τ_s has label y.

Similarly, if we assume that $\hat{\eta}_s^+(x) = 0$, we can deduce that $\tau_s(x) = 0$, and the holes have the same label. By taking x over all integers $< -r$, we conclude that $\mathcal{E}_1$ holds assuming $p^P = p_s^+$ and $\mathcal{E}_3 \cap \mathcal{E}_4 \cap \mathcal{E}_3^\tau$. ∎

It now suffices to lower bound the probabilities of the events $\mathcal{E}_3$, $\mathcal{E}_3^\tau$, $\mathcal{E}_4$.

Lemma 6.9. $\mathbb{P}[\mathcal{E}_3], \mathbb{P}[\mathcal{E}_3^\tau] > 1 - C\alpha^{-1/2}$ *for constants* $c, C > 0$.

Lemma 6.10. $\mathbb{P}[\mathcal{E}_4] > 1 - Ce^{-c\alpha}$ *for constants* $c, C > 0$.

Using Lemmas 6.8–6.10, we get a lower bound for $\mathbb{P}[\mathcal{E}_1]$. We can lower bound $\mathbb{P}[\mathcal{E}_2]$ similarly. Thus Lemma 6.7 follows.

To prove these estimates we introduce some other setups. For the convenience of notation, we extend $\xi^{+,\vee,0}$ from $\mathbb{Z}^2 \setminus I_0^+$ to $\mathbb{Z}^2$, so that conditioned on I_0^+, $\{\xi^{+,\vee,0}(v)\}_{v \in I_0^+}$ are i.i.d. Exp(1) and are independent of everything else. For any $u \le v$, we let $T_{u,v}^{+,\vee,0}$ and $\Gamma_{u,v}^{+,\vee,0}$ be the passage time and geodesic from u to v under the weights $\xi^{+,\vee,0}$. For any $v \notin I_0^+$ we denote $\Gamma_{I,v}^{+,\vee,0} = \Gamma_{u_*,v}^{+,\vee,0}$ and $T_{I,v}^{+,\vee,0} = T_{u_*,v}^{+,\vee,0}$, where $u_* = \operatorname{argmax}_{u \le v, u \notin I_0^+} T_{u,v}^{+,\vee,0}$. In words, $\Gamma_{I,v}^{+,\vee,0}$ and $T_{I,v}^{+,\vee,0}$ are the geodesic and passage time from boundary I_0^+ to v, under the weights $\xi^{+,\vee,0}$. By Lemma 3.5 we have $\Gamma_{I,v}^{+,\vee,0} = \Gamma_v^{+,\vee} \setminus I_0^+$ and $T_{I,v}^{+,\vee,0} = L^+(v)$.

Proof of Lemma 6.9. We shall write out the proof for $\mathbb{P}[\mathcal{E}_3]$ only, as the approach we take applies to $\mathbb{P}[\mathcal{E}_3^\tau]$ essentially verbatim. We will use $c, C > 0$ to denote small and large enough constants whose values can change from line to line.

We consider the following events (see Figure 12):

$\mathcal{E}_5$: $x_* > -r_3$.

$\mathcal{E}_6$: $\mathbb{V}_{(-jr_4, jr_4)} \cap \partial I_0^+ \subset \mathbb{H}_{(-j\alpha r_4^{1/2}, j\alpha r_4^{1/2})}$ for each $j \in \mathbb{N}$.

$\mathcal{E}_7$: $\mathbb{V}_{(-6r_3, -r_3)} \cap \partial I_s^+ \subset \mathbb{H}_{(s-2\alpha r_4^{1/2}, s+2\alpha r_4^{1/2})}$.

$\mathcal{E}_8$: Let u_1 be the intersection of $\mathbb{H}_{2s}$ with $\mathbb{V}_{-5r_3}$ and u_2 be the intersection of $\mathbb{H}_{2s}$ with $\mathbb{V}_{-2r_3}$ (rounded to the nearest lattice vertex). Then $\Gamma_{I,u_1}^{+,\vee,0} \subset \mathbb{V}_{(-6r_3,-4r_3)}$ and $\Gamma_{I,u_2}^{+,\vee,0} \subset \mathbb{V}_{(-3r_3,-r_3)}$.

The events $\mathcal{E}_6$ and $\mathcal{E}_7$ just say that ∂I_0^+ and ∂I_s^+ behave 'typically' in certain regions. The event $\mathcal{E}_8$ is to bound the transversal fluctuation of $\Gamma_{u_+}^+$ (for some $u_+ \in \partial I_0^+$), using the non-crossing property of downward and upward semi-infinite geodesics (Lemma 3.3).

We next show that $\mathcal{E}_5 \cap \mathcal{E}_6 \cap \mathcal{E}_7 \cap \mathcal{E}_8 \subset \mathcal{E}_3$. For this, we take any $u_+ \in \partial I_0^+ \cap \mathbb{V}_{(-4r_3,-3r_3)}$, and let $u'_+ = (a'_+, b'_+)$ be the last vertex in $\Gamma_{u_+}^+ \cap I_s^+$. Then we need to show that $ad(u_+) < x_*$, $\Gamma_{u_+}^+ \cap I_s^+ \subset \mathbb{V}_{(-6r_3,-r_3)}$, and $a'_+ > (1-\rho)^2 s - r_4$, assuming $\mathcal{E}_5 \cap \mathcal{E}_6 \cap \mathcal{E}_7 \cap \mathcal{E}_8$.

- By $\mathcal{E}_6$, and since $r_3 > C\alpha r_4^{1/2}$ by our choice of the parameters, we have $ad(u_+) < -r_3$. So under $\mathcal{E}_5 \cap \mathcal{E}_6$ we have $ad(u_+) < x_*$.

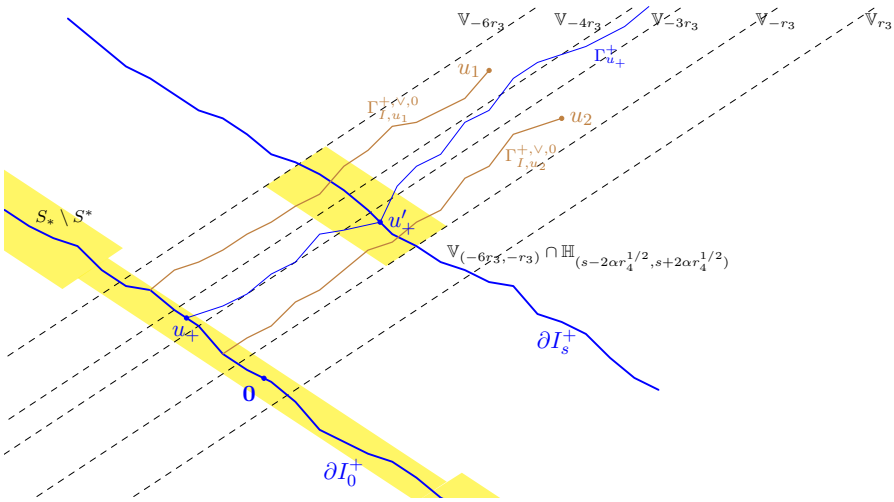


Fig. 12. The events to lower bound $\mathbb{P}[\mathcal{E}_3]$.

- Under $\mathcal{E}_8$, the path $\Gamma^{+,\vee,0}_{I,u_1} - (1/2,1/2)$ divides $(u_1 + (\mathbb{Z}^2 \setminus \mathbb{Z}^2_{\geq 0})) \setminus I^+_s$ into two parts: $(\bigcup_{v \in \Gamma^{+,\vee,0}_{I,u_1}} v + \mathbb{Z}_- \times \mathbb{Z}_{\geq 0}) \setminus I^+_s$ and $(\bigcup_{v \in \Gamma^{+,\vee,0}_{I,u_1}} v + \mathbb{Z}_{\geq 0} \times \mathbb{Z}_-) \setminus I^+_s$. Then u_+ must be in the second part, so by Lemma 3.3, $\Gamma^+_{u_+}$ must be disjoint from the first part, thus $\Gamma^+_{u_+} \cap \mathbb{V}_{(-\infty,-6r_3]} \subset u_1 + \mathbb{Z}^2_{\geq 0}$. If $\Gamma^+_{u_+} \cap \mathbb{V}_{(-\infty,-6r_3]} \cap I^+_s$ is not empty, we must have $u_1 \in I^+_s$, which contradicts $\mathcal{E}_7$. So under $\mathcal{E}_7 \cap \mathcal{E}_8$, $\Gamma^+_{u_+} \cap I^+_s$ is disjoint from $\mathbb{V}_{(-\infty,-6r_3]}$, and similarly also from $\mathbb{V}_{[-r_3,\infty)}$. This means that $\Gamma^+_{u_+} \cap I^+_s \subset \mathbb{V}_{(-6r_3,-r_3)}$.
- $\{\Gamma^+_{u_+} \cap I^+_s \subset \mathbb{V}_{(-6r_3,-r_3)}\} \cap \mathcal{E}_7$ implies that $u'_+ \in \mathbb{H}_{(s-2\alpha r_4^{1/2}, s+2\alpha r_4^{1/2})} \cap \mathbb{V}_{(-6r_3,-r_3)}$. Thus we get $a'_+ > (1-\rho)^2 s - r_4$ since $r_4 > Cr_3, C\alpha r_4^{1/2}$ by our choice of the parameters.

It remains to estimate the probabilities of these events and take a union bound.

Bounding $\mathbb{P}[\mathcal{E}_5]$. By the coupling between τ_0 and $(\eta^+_t)_{t\geq 0}$ (stated after the proof of Lemma 6.5), the number $-x_*$ is just the time of a symmetric random walk hitting 0 after r_2. Thus $\mathbb{P}[\mathcal{E}_5] \geq 1 - Cr_2^{1/2} r_3^{-1/2} = 1 - C\alpha^{-1/2}$.

Bounding $\mathbb{P}[\mathcal{E}_6]$. The event $\mathcal{E}_6$ is again on the hitting probability of a random walk. Indeed, by Lemma 3.6, if we let $f(x)$ be the largest integer with $(f(x)+x, f(x)) \in I^+_0$, we must have $f(0)=0$, $f(x) = \sum_{y=1}^{x} -\eta^+_0(x)$ for any $x \geq 1$, and $f(x) = \sum_{y=x+1}^{0} \eta^+_0(x)$ for any $x \leq -1$; and $\{\eta^+_0(x)\}_{x\in\mathbb{Z}\setminus\{0,1\}}$ are i.i.d. Bernoulli(ρ). Thus for each $j \in \mathbb{N}$ we have $\mathbb{P}[\mathbb{V}_{(-jr_4,jr_4)} \cap \partial I^+_0 \subset \mathbb{H}_{(-j\alpha r_4^{1/2}, j\alpha r_4^{1/2})}] > 1 - Ce^{-cj\alpha^2}$, so when $\alpha > C$ we have $\mathbb{P}[\mathcal{E}_6] \geq 1 - Ce^{-c\alpha^2}$.

For $\mathbb{P}[\mathcal{E}_7]$ and $\mathbb{P}[\mathcal{E}_8]$, we reduce to estimates on last-passage times and geodesic transversal fluctuations under the weights $\xi^{+,\vee,0}$, and use results from Section 5.

Bounding $\mathbb{P}[\mathcal{E}_7]$. We note that $\mathcal{E}_7$ is implied by the following two events:

- $T_{I,v}^{+,\vee,0} = L^+(v) > s$ whenever $v \in \mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{H}_{[s+2\alpha r_4^{1/2},\infty)} \cap \mathbb{Z}^2$,
- $T_{I,v}^{+,\vee,0} = L^+(v) \le s$ whenever $v - (1,1) \in \mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{H}_{(-\infty,s-2\alpha r_4^{1/2}]} \cap \mathbb{Z}^2$.

These two events imply that $\mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{H}_{(-\infty,s-2\alpha r_4^{1/2}]\cup[s+2\alpha r_4^{1/2},\infty)}$ is disjoint from ∂I_s^+, so $\mathcal{E}_7$ holds.

To estimate the probabilities of these events, we need to bound the passage times under $\xi^{+,\vee,0}$. For this, we set up the following notations. For each $j \in \mathbb{N}$ we let

$$S_j = \mathbb{V}_{(-jr_4,-(j-1)r_4]\cup[(j-1)r_4,jr_4)} \cap \mathbb{H}_{(-j\alpha r_4^{1/2},\infty)},$$
$$S^j = \mathbb{V}_{(-jr_4,-(j-1)r_4]\cup[(j-1)r_4,jr_4)} \cap \mathbb{H}_{[j\alpha r_4^{1/2},\infty)}.$$

Let $S_* = \bigcup_{j\in\mathbb{N}} S_j$ and $S^* = \bigcup_{j\in\mathbb{N}} S^j$. Then the event $\mathcal{E}_6$ precisely says that $\partial I_0^+ \subset S_* \setminus S^*$ (see Figure 12), and implies that $S^* \subset \mathbb{Z}^2 \setminus I_0^+ \subset S_*$.

We consider the following events:

$\mathcal{E}_7'$: $T_{u,v}^{+,\vee,0} \le s$ for any vertices $u \in S_*$ and $v \in \mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{H}_{(-\infty,s-2\alpha r_4^{1/2}]}$ with $u \le v$.

$\mathcal{E}_7''$: For any $v \in \mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{H}_{[s+2\alpha r_4^{1/2},\infty)} \cap \mathbb{Z}^2$, there exists $u \in \mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{H}_{(\alpha r_4^{1/2},\infty)} \cap \mathbb{Z}^2$ such that $T_{u,v}^{+,\vee,0} > s$.

Then under $\mathcal{E}_7' \cap \mathcal{E}_7'' \cap \mathcal{E}_6$, the two events above hold, and thus $\mathcal{E}_7$ holds.

We next lower bound the probabilities $\mathbb{P}[\mathcal{E}_7']$ and $\mathbb{P}[\mathcal{E}_7'']$. These bounds are deduced from the estimates of Theorem 5.2 and Proposition 5.3.

We first consider $\mathbb{P}[\mathcal{E}_7']$. For each $j \in \mathbb{N}$, we let P_j be the collection of all vertices in $\mathbb{Z}^2$ that are within distance 1 of $\mathbb{V}_{(-jr_4,-(j-1)r_4]\cup[(j-1)r_4,jr_4)} \cap \mathbb{H}_{-j\alpha r_4^{1/2}}$, and let P_* be the collection of all vertices in $\mathbb{Z}^2$ that are within distance 1 of $\mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{H}_{s-2\alpha r_4^{1/2}}$. To lower bound $\mathbb{P}[\mathcal{E}_7']$, we just need to consider $T_{u,v}^{+,\vee,0}$ for all $u \in \bigcup_{j\in\mathbb{N}} P_j$ and $v \in P_*$ (see Figure 13 (a)). We note that for any $j \in \mathbb{N}$ and any $u \in P_j$ and $v \in P_*$ with $u \le v$, if we write $(a,b) = v - u$ we have

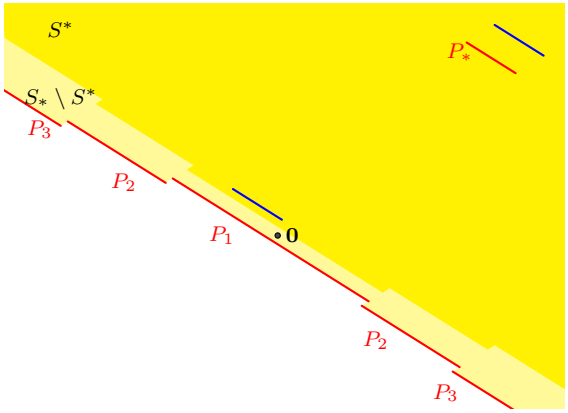
$$(\sqrt{a} + \sqrt{b})^2 < s - c\alpha r_4^{1/2} - c(j-1)^2 r_4^2 s^{-1}. \tag{6.9}$$

For $j > csr_4^{-1} + 1$, we apply (5.1) in Theorem 5.2 to each $u \in P_j$ and $v \in P_*$ and take a union bound to conclude that

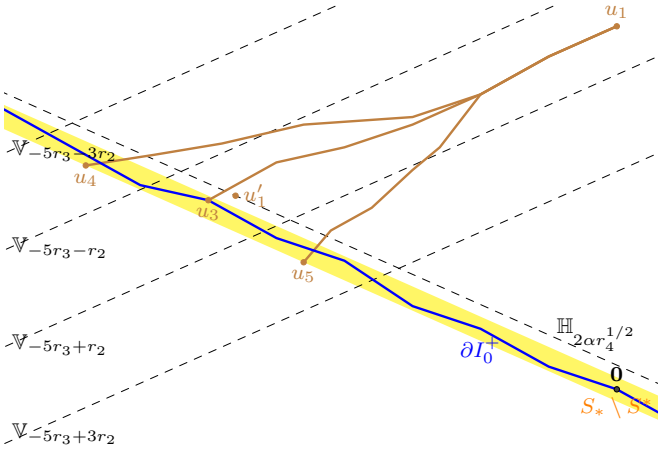
$$\mathbb{P}[T_{u,v}^{+,\vee,0} \le s,\ \forall u \in P_j,\ v \in P_*,\ u \le v,\ j > csr_4^{-1} + 1] > 1 - Csr_3 e^{-c\sqrt{s}}.$$

For any $j \le csr_4^{-1} + 1$, the slope of $v - u$ for any $u \in P_j$ and $v \in P_*$ is bounded away from 0 and ∞. Thus we can split P_j and P_* into $Cr_4 s^{-2/3}$ and $Cr_3 s^{-2/3}$ segments of length $< Cs^{2/3}$, and apply Proposition 5.3. Note that using (5.3) from Theorem 5.2 and (6.9), we find that $\mathbb{E}[T_{u,v}^{+,\vee,0}] < s - cj^2\alpha^3 s^{1/3}$ for any $u \in P_j$ and $v \in P_*$. We then conclude that

$$\mathbb{P}[T_{u,v}^{+,\vee,0} \le s,\ \forall u \in P_j,\ v \in P_*] > 1 - C(r_3 s^{-2/3})(r_4 s^{-2/3}) e^{-cj^2\alpha^3}.$$



(a) $\mathbb{P}(\mathcal{E}_7)$: assuming $\mathcal{E}_6$, the event $\mathcal{E}_7$ is implied by $\mathcal{E}_7' \cap \mathcal{E}_7''$, about passage times under the weights $\xi^{+,\vee,\mathbf{0}}$. For $\mathbb{P}(\mathcal{E}_7')$, we need to upper bound the passage times from $\bigcup_{j\in\mathbb{N}} P_j$ to P_*; for $\mathbb{P}(\mathcal{E}_7'')$, we need to lower bound the passage times from around $\mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{H}_{3\alpha r_4^{1/2}/2}$ to $\mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{H}_{s+2\alpha r_4^{1/2}}$ (the blue segments).



(b) $\mathbb{P}(\mathcal{E}_8)$: under $\mathcal{E}_8' \cap \mathcal{E}_6$, u_3 (the lower endpoint of $\Gamma^{+,\vee,\mathbf{0}}_{I,u_1}$) is in $\mathbb{V}_{(-5r_3-r_2,-5r_3+r_2)}$. Then if $\Gamma^{+,\vee,\mathbf{0}}_{u_4,u_1}$ and $\Gamma^{+,\vee,\mathbf{0}}_{u_5,u_1}$ below $\mathbb{H}_{2\alpha r_4^{1/2}}$ are contained in $\mathbb{V}_{(-5r_3-3r_2,-5r_3-r_2)}$ and $\mathbb{V}_{(-5r_3+r_2,-5r_3+3r_2)}$, respectively, $\Gamma^{+,\vee,\mathbf{0}}_{I,u_1}$ is sandwiched between them, and the transversal fluctuation of $\Gamma^{+,\vee,\mathbf{0}}_{I,u_1}$ is controlled by $\Gamma^{+,\vee,\mathbf{0}}_{u_4,u_1}$ and $\Gamma^{+,\vee,\mathbf{0}}_{u_5,u_1}$.

Fig. 13. Illustrations of bounding $\mathbb{P}[\mathcal{E}_7]$ and $\mathbb{P}[\mathcal{E}_8]$ in the proof of Lemma 6.9.

Thus

$$\mathbb{P}[\mathcal{E}_7'] > 1 - Csr_3e^{-c\sqrt{s}} - Cr_3r_4s^{-4/3}\sum_{j\in\mathbb{N}} e^{-cj^2\alpha^3}$$
$$= 1 - Cs^{5/3}\alpha^3e^{-c\sqrt{s}} - C\alpha^7\sum_{j\in\mathbb{N}} e^{-cj^2\alpha^3}.$$

For $\mathbb{P}[\mathcal{E}_7'']$, we need to consider $T^{+,\vee,0}_{u,v}$ for all $v \in \mathbb{Z}^2 \cap \mathbb{V}_{(-6r_3,-r_3)}$ within distance 1 of $\mathbb{H}_{s+2\alpha r_4^{1/2}}$, and $u \in \mathbb{V}_{(-6r_3,-r_3)} \cap \mathbb{H}_{(\alpha r_4^{1/2},\infty)} \cap \mathbb{Z}^2$ within distance 1 of $v - (s + \alpha r_4^{1/2}/2)\boldsymbol{\rho}$ (see Figure 13 (a)). For such u and v, the slope of $v - u$ is bounded away from 0 and ∞. By (5.3) we have $\mathbb{E}[T^{+,\vee,0}_{u,v}] > s + c\alpha^3 s^{1/3}$. We then apply Proposition 5.3 by covering all such u, v with $Cr_3s^{-2/3}$ parallelograms of size $Cs \times Cs^{2/3}$, and we conclude that $\mathbb{P}[\mathcal{E}_7''] > 1 - Cr_3s^{-2/3}e^{-c\alpha^3} = 1 - C\alpha^3e^{-c\alpha^3}$.

In summary and using the fact that $Cs^{2/3} < r < s^{2/3+0.01}$ from the statement of Lemma 6.2 (thus $\alpha > C$ and $\alpha < s^{0.002}$), we have $\mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_7] < Ce^{-c\alpha^3}$.

Bounding $\mathbb{P}[\mathcal{E}_8]$. Let u_3 denote the lower endpoint of $\Gamma^{+,\vee,0}_{I,u_1}$. Consider the event $\mathcal{E}_8'$ where

- for any $u \in (S_* \setminus \mathbb{V}_{(-5r_3-r_2,-5r_3+r_2)}) \cap \mathbb{Z}^2$, we have $T^{+,\vee,0}_{u,u_1} < 2s - 4\alpha r_4^{1/2}$,
- $T^{+,\vee,0}_{u_1',u_1} > 2s - 4\alpha r_4^{1/2}$, where u_1' is the intersection of $\mathbb{H}_{2\alpha r_4^{1/2}}$ with $\mathbb{V}_{-5r_3}$ (rounded to the nearest lattice vertex). Note that $u_1' \in S^* \cap \mathbb{V}_{(-5r_3-r_2,-5r_3+r_2)}$.

Under $\mathcal{E}_8' \cap \mathcal{E}_6$ we must have $u_3 \in \mathbb{V}_{(-5r_3-r_2,-5r_3+r_2)}$, since $T^{+,\vee,0}_{u_3,u_1}$ is the maximum passage time from I_0^+ to u_1 (see Figure 13 (b)). We can deduce that $\mathbb{P}[\mathcal{E}_8'] > 1 - Ce^{-c\alpha^3}$ similar to how $\mathbb{P}[\mathcal{E}_7']$ and $\mathbb{P}[\mathcal{E}_7'']$ are bounded above using Theorem 5.2 and Proposition 5.3; we omit the details.

Now let u_4, u_5 be the intersection points of $\mathbb{H}_{-\alpha r_4^{1/2}}$ with $\mathbb{V}_{-5r_3-2r_2}$ and $\mathbb{V}_{-5r_3+2r_2}$, respectively (rounded to the nearest lattice vertex, see Figure 13 (b)). Consider $\Gamma^{+,\vee,0}_{u_4,u_1}$ and $\Gamma^{+,\vee,0}_{u_5,u_1}$. By Corollary 5.9 we have

$$\mathbb{P}[\Gamma^{+,\vee,0}_{u_4,u_1} \cap \mathbb{H}_{(-\alpha r_4^{1/2},2\alpha r_4^{1/2})} \subset \mathbb{V}_{(-5r_3-3r_2,-5r_3-r_2)}] > 1 - Ce^{-cr_2^3\alpha^{-2}r_4^{-1}}, \tag{6.10}$$

$$\mathbb{P}[\Gamma^{+,\vee,0}_{u_5,u_1} \cap \mathbb{H}_{(-\alpha r_4^{1/2},2\alpha r_4^{1/2})} \subset \mathbb{V}_{(-5r_3+r_2,-5r_3+3r_2)}] > 1 - Ce^{-cr_2^3\alpha^{-2}r_4^{-1}}, \tag{6.11}$$

and by Lemma 5.7 we have

$$\mathbb{P}[\Gamma^{+,\vee,0}_{u_4,u_1} \subset \mathbb{V}_{(-6r_3,-4r_3)}], \mathbb{P}[\Gamma^{+,\vee,0}_{u_5,u_1} \subset \mathbb{V}_{(-6r_3,-4r_3)}] > 1 - Ce^{-cr_3^3s^{-2}}. \tag{6.12}$$

When $\mathcal{E}_8' \cap \mathcal{E}_6$ happens, we have $u_3 \in \mathbb{V}_{(-5r_3-r_2,-5r_3+r_2)} \cap \mathbb{H}_{(-\alpha r_4^{1/2},2\alpha r_4^{1/2})}$. If the events on the left-hand side of (6.10) and (6.11) also happen, then $\Gamma^{+,\vee,0}_{I,u_1} = \Gamma^{+,\vee,0}_{u_3,u_1}$ is between $\Gamma^{+,\vee,0}_{u_4,u_1}$ and $\Gamma^{+,\vee,0}_{u_5,u_1}$, by ordering of geodesics (Lemma 5.1). If in addition the event on the left-hand side of (6.12) happens, we have $\Gamma^{+,\vee,0}_{I,u_1} = \Gamma^{+,\vee,0}_{u_3,u_1} \subset \mathbb{V}_{(-6r_3,-4r_3)}$.

We can use similar arguments to study the event $\Gamma^{+,\vee,0}_{I,u_2} \subset \mathbb{V}_{(-3r_3,-r_3)}$. Then from $\mathbb{P}[\mathcal{E}'_8] > 1 - Ce^{-c\alpha^3}$ we conclude that

$$\mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_8] < Ce^{-c\alpha^3} + Ce^{-cr_2^3\alpha^{-2}r_4^{-1}} + Ce^{-cr_3^3 s^{-2}} = Ce^{-c\alpha^3} + Ce^{-cs^{4/3}} + Ce^{-c\alpha^9}.$$

Since $Cs^{2/3} < r < s^{2/3+0.01}$ (from the statement of Lemma 6.2), this is bounded by $Ce^{-c\alpha^3}$.

Putting together the bounds for $\mathbb{P}[\mathcal{E}_5], \mathbb{P}[\mathcal{E}_6], \mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_7], \mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_8]$ we conclude that $\mathbb{P}[\mathcal{E}_3] > 1 - C\alpha^{-1/2}$. ∎

Proof of Lemma 6.10. We again use $c, C > 0$ to denote small and large enough constants, whose values can change from line to line. We consider three events.

$\mathcal{E}_9$: $|M - (1-2\rho)s| < r_2$.

$\mathcal{E}_{10}$: for any $u = (a,b)$ with $ad(u) = a - b < (1-2\rho)s + r_2 - r$ and $a \geq (1-\rho)^2 s - r_4 - 1$, we have $u \notin I_s^+$.

$\mathcal{E}_{11}$: for any $u = (a,b)$ with $a < (1-\rho)^2 s - r_4$ and $u - (0,1) \in \mathbb{V}_{[-6r_3,\infty)}$, we have $u \in I_s^+$.

Note that $\mathcal{E}_9 \cap \mathcal{E}_{10} \cap \mathcal{E}_{11} \subset \mathcal{E}_4$. Indeed, $\mathcal{E}_9 \cap \mathcal{E}_{10}$ implies that ∂I_s^+ is disjoint from

$$\{(a,b) \in \mathbb{Z}^2 : a - b \leq M - r,\ a \geq (1-\rho)^2 s - r_4 - 1\};$$

and $\mathcal{E}_{11}$ implies that ∂I_s^+ is disjoint from

$$\{(a,b) \in \mathbb{Z}^2 : (a+1,b) \in \mathbb{V}_{[-6r_3,\infty)},\ a < (1-\rho)^2 s - r_4 - 1\},$$

since this set shifted by $(1,1)$ is contained in I_s^+ by $\mathcal{E}_{11}$. See Figure 11 for an illustration of these regions. Thus under $\mathcal{E}_9 \cap \mathcal{E}_{10} \cap \mathcal{E}_{11}$, for any $u = (a,b) \in \partial I_s^+$ with $ad(u) \leq M - r$ we must have $a < (1-\rho)^2 s - r_4 - 1$ and $u + (1,0) \in \mathbb{V}_{(-\infty,-6r_3)}$. So we conclude that $\mathcal{E}_9 \cap \mathcal{E}_{10} \cap \mathcal{E}_{11} \subset \mathcal{E}_4$, and it remains to lower bound $\mathbb{P}[\mathcal{E}_9]$, $\mathbb{P}[\mathcal{E}_{10}]$, and $\mathbb{P}[\mathcal{E}_{11}]$.

Bounding $\mathbb{P}[\mathcal{E}_9]$. By Lemma 3.1, $p_s^+ - (1,0)$ or $p_s^+ - (0,1)$ is the last vertex in $\{u \in \Gamma_{\mathbf{0}}^+ : T^+_{\mathbf{0},u} \leq s\}$ since p_s^+ is the last vertex in $\{u \in \Gamma_{\mathbf{0}}^+ : \mathbf{G}^+(u) \leq s\}$. So by Corollary 5.11 we have $\mathbb{P}[|ad(p_s^+) - (1-2\rho)s| < r_2] > 1 - Ce^{-cr_2^3 s^{-2}}$. Then Lemma 6.5 implies that $\mathbb{P}[\mathcal{E}_9] > 1 - Ce^{-cr_2^3 s^{-2}} - Ce^{-c\alpha^3} = 1 - Ce^{-c\alpha^6} - Ce^{-c\alpha^3}$.

To bound $\mathbb{P}[\mathcal{E}_{10}]$ and $\mathbb{P}[\mathcal{E}_{11}]$, we just need to bound the function L^+ at certain vertices. For this, we recall the event $\mathcal{E}_6$ and the sets S_*, S^* from the proof of Lemma 6.9.

Bounding $\mathbb{P}[\mathcal{E}_{10}]$. We take $u^* = (a^*, b^*)$ where $a^* = \lceil (1-\rho)^2 s - r_4 - 1 \rceil$ and $b^* = a^* - \lceil (1-2\rho)s + r_2 - r \rceil$. Then $\mathcal{E}_{10}$ is equivalent to $L^+(u^*) = T^{+,\vee,0}_{I,u^*} > s$. Denote $u^*_- = (\lfloor r_4 \rfloor, \lfloor r_4 \rfloor)$. As $u^*_- \in S^*$, under $\mathcal{E}_6$ we have $u^*_- \notin I_0^+$. Thus under $\mathcal{E}_6 \setminus \mathcal{E}_{10}$ we have $T^{+,\vee,0}_{u^*_-,u^*} \leq s$. Then $\mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_{10}] \leq \mathbb{P}[T^{+,\vee,0}_{u^*_-,u^*} \leq s] < Ce^{-cr^3/s}$, where the last inequality is by the fact that $(\sqrt{a_* - \lfloor r_4 \rfloor} + \sqrt{b_* - \lfloor r_4 \rfloor})^2 > s + cr$ and (5.2) in Theorem 5.2.

Bounding $\mathbb{P}[\mathcal{E}_{11}]$. Let $u_* = (a_*, b_*)$ where $a_* = \lceil (1-\rho)^2 s - r_4 - 1 \rceil$, and b_* is the largest integer such that $u_* - (0,1) \in \mathbb{V}_{[-6r_3,\infty)}$. Then $\mathcal{E}_{11}$ is equivalent to $L^+(u) = T^{+,v,0}_{I,u_*} \le s$. Under $\mathcal{E}_6 \setminus \mathcal{E}_{11}$ there is some $u \in S_*$ with $u \le u_*$ such that $T^{+,v,0}_{u,u_*} > s$. Note that for any $u \in S_*$ with $u \le u_*$, if we let $(a,b) = u_* - u$, then $(\sqrt{a} + \sqrt{b})^2 < s - cr_4$. Then by (5.1) in Theorem 5.2 and a union bound (over all $u \in S_*$ such that $u \in u_* + \mathbb{Z}^2_{\le 0}$ and $u - (1,1) \notin S_*$), we have $\mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_{11}] < Cse^{-cr_4 s^{-1/2}}$.

Putting together the bounds for $\mathbb{P}[\mathcal{E}_9]$, $\mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_{10}]$, $\mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_{11}]$ and the bound for $\mathbb{P}[\mathcal{E}_6]$ in the proof of Lemma 6.9, and using the fact that $Cs^{2/3} < r < s^{2/3+0.01}$ from the statement of Lemma 6.2 (thus $\alpha > C$ and $Ce^{-cr^3/s}, Cse^{-cr_4 s^{-1/2}} < Ce^{-c\alpha}$), we conclude that $\mathbb{P}[\mathcal{E}_4] > 1 - Ce^{-c\alpha}$. ∎

7. Convergence in probability of empirical environments

In this section we prove convergence in probability versions of the main results, Theorems 1.1 and 1.2. The semi-infinite geodesic case (Theorem 7.1 below) follows quickly from the convergence of TASEP as seen from an isolated second-class particle (Proposition 2.7 or Theorem 1.7), and ergodicity of the stationary process (Proposition 2.2). The finite geodesic one (Theorem 7.3) is via geometric arguments, specifically, covering finite geodesics by semi-infinite geodesics.

7.1. Semi-infinite geodesics

We start with convergence along semi-infinite geodesics and giving a weak version of Theorem 1.2.

Theorem 7.1. *For any bounded continuous function* $f : \mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2} \to \mathbb{R}$, *we have* $\mu_{\mathbf{0};r}(f) \to \nu(f)$ *in probability as* $r \to \infty$.

We let $(\eta^*_t)_{t \ge 0}$ be the process of TASEP starting from i.i.d. Bernoulli(ρ) on $\mathbb{Z} \setminus \{0\}$, and $\eta^*_0(0) = *$. Then recall (from Section 2.2) that $\eta^*_t(l_t + \cdot) \sim \Phi_t$, with l_t being the location of the second-class particle at time t. We also let $\boldsymbol{\zeta}^* = (\zeta^*_t)_{t \in \mathbb{R}}$ be the stationary process of TASEP as seen from an isolated second-class, i.e. for each t we have $\zeta^*_t \sim \Psi$ (defined in Section 2).

For any process $P = (P_w)_{w \in \mathbb{R}}$ and $t \in \mathbb{R}$, we let $\mathcal{T}_t P$ denote the process $(P_{t+w})_{w \in \mathbb{R}}$. By Lemmas 3.7 and 4.1, we can deduce Theorem 7.1 from the following result. To make things well-defined, we let $\boldsymbol{\eta}^* = (\eta^*_t(l_t + \cdot))_{t \in \mathbb{R}}$ with $\eta^*_t = \eta^*_0$ and $l_t = 0$ for each $t < 0$. Let $\{0,1,*\}^{\mathbb{Z} \times \mathbb{R}}$ be equipped with the product topology.

Proposition 7.2. *For any bounded continuous function* $f : \{0,1,*\}^{\mathbb{Z} \times \mathbb{R}} \to \mathbb{R}$, *we have*

$$T^{-1} \int_0^T f(\mathcal{T}_t \boldsymbol{\eta}^*)\, dt \to \mathbb{E}[f(\boldsymbol{\zeta}^*)] \quad \textit{in probability as } T \to \infty.$$

By Birkhoff's Ergodic Theorem, this proposition follows from Proposition 2.7 or Theorem 1.7, and Proposition 2.2.

Proof of Proposition 7.2. Without loss of generality we assume that $0 \le f \le 1$, and for some $s > 0$ the function is measurable with respect to the σ-algebra generated by $A \times \{0,1\}^{\mathbb{Z}\times(-\infty,-s)\cup(s,\infty)}$ for all measurable $A \subset \{0,1\}^{\mathbb{Z}\times[-s,s]}$. Take any $\delta > 0$. Then by Birkhoff's Ergodic Theorem and Proposition 2.2, we can find r large enough such that

$$\mathbb{P}\left[\left| r^{-1}\int_0^r f(\mathcal{T}_t\boldsymbol{\zeta}^*)\,dt - \mathbb{E}[f(\boldsymbol{\zeta}^*)]\right| > \delta\right] < \delta.$$

For each $t \ge 0$, denote

$$\chi_t = \mathbb{1}\left[\left| r^{-1}\int_{t+s}^{t+s+r} f(\mathcal{T}_w\boldsymbol{\eta}^*)\,dw - \mathbb{E}[f(\boldsymbol{\zeta}^*)]\right| \ge \delta\right].$$

Let $F : \eta \mapsto \mathbb{E}[\chi_t \mid \eta_t^* = \eta]$. Then F is the same for all $t \ge 0$, and is an upper semi-continuous function on $\{\eta : \eta(0) = *, \eta(x) \in \{0,1\}, \forall x \ne 0\} \subset \{0,1,*\}^{\mathbb{Z}}$ since f is continuous. Then by Theorem 1.7 we have

$$\begin{aligned}\limsup_{N\to\infty} N^{-1}\sum_{i=0}^{N-1}\mathbb{E}[\chi_{ir}] &= \limsup_{N\to\infty} N^{-1}\sum_{i=0}^{N-1}\mathbb{E}[F(\eta_{ir}^*)]\\ &\le \mathbb{E}[F(\zeta_0^*)] = \mathbb{P}\left[\left| r^{-1}\int_0^r f(\mathcal{T}_t\boldsymbol{\zeta}^*)\,dt - \mathbb{E}[f(\boldsymbol{\zeta}^*)]\right| \ge \delta\right] < \delta.\end{aligned}$$

This implies that for any N large enough, we have $\mathbb{P}[\sum_{i=0}^{N-1}\chi_{ir} > \sqrt{\delta}N] < \sqrt{\delta}$, thus

$$\mathbb{P}\left[\left| (Nr)^{-1}\int_s^{Nr+s} f(\mathcal{T}_t\boldsymbol{\eta}^*)\,dt - \mathbb{E}[f(\boldsymbol{\zeta}^*)]\right| > \sqrt{\delta}+\delta\right] < \sqrt{\delta},$$

which implies our conclusion since $\delta > 0$ is arbitrary. ∎

7.2. *From semi-infinite geodesics to point-to-point geodesics*

From convergence in probability along semi-infinite geodesics (Theorem 7.1), we now deduce convergence in probability along finite geodesics. It can be viewed as a weak version of Theorem 1.1.

Recall (from Section 5) that we let $\langle a,b\rangle_\rho = (\lfloor \frac{2(1-\rho)^2 a}{\rho^2+(1-\rho)^2}\rfloor + b, \lceil \frac{2\rho^2 a}{\rho^2+(1-\rho)^2}\rceil - b)$. Since ρ is fixed, for the rest of this paper we also write $\langle a,b\rangle = \langle a,b\rangle_\rho$.

Theorem 7.3. *Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of integers with $\limsup_{n\to\infty} n^{-2/3}|b_n| < \infty$. Then for any bounded continuous function $f : \mathbb{R}^{\mathbb{Z}^2}\times\{0,1\}^{\mathbb{Z}^2} \to \mathbb{R}$, we have*

$$\mu_{\mathbf{0},\langle n,b_n\rangle}(f) \to \nu(f) \quad \textit{in probability as } n\to\infty.$$

We explain the strategy of proving this theorem. The general idea is to cover the finite geodesic $\Gamma_{\mathbf{0},\langle n,b_n\rangle}$ with a semi-infinite geodesic. More precisely, for any $\epsilon > 0$, we con-

struct an event that depends only on the i.i.d. random weights ξ or on the above $\mathbb{L}_n$, such that (1) this event happens with positive probability (lower bounded uniformly in n) and (2) assuming this event, with high probability a $1-\epsilon$ portion of $\Gamma_{\mathbf{0},\langle n,b_n\rangle}$ is contained in $\Gamma_{\mathbf{0}}$. Then by Theorem 7.1, conditioned on this event the empirical environment $\mu_{\mathbf{0},\langle n,b_n\rangle}$ would be 'ϵ-close' to ν with high probability for n large enough. On the other hand, since $\mu_{\mathbf{0},\langle n,b_n\rangle}$ depends mainly on the random i.i.d. weights ξ below $\mathbb{L}_n$, it is roughly 'independent' of the event constructed, so it would always be close to ν with high probability for n large enough.

We start by describing the event. Recall $\mathbf{B}$ (and also $\mathbf{G}$), the Busemann function in direction $\boldsymbol{\rho}$. The event basically says that the Busemann function $\mathbf{B}(\langle n, b_n+b\rangle, \langle n, b_n\rangle)$ decays fast when b is slightly away from 0. By Lemma 3.1 this can force $\Gamma_{\mathbf{0}}$ to intersect $\mathbb{L}_n$ near $\langle n, b_n+b\rangle$, and that $\Gamma_{\mathbf{0},\langle n,b_n\rangle}$ overlaps with $\Gamma_{\mathbf{0}}$ can be deduced using coalescence and ordering of geodesics (Proposition 5.4 and Lemma 5.1).

We now formally define this event and study its probability. For simplicity of notation, we shift this event by $-\langle n, b_n\rangle$ and look at the Busemann function on $\mathbb{L}_0$. Let $\mathcal{E}_{h,n}$ denote the following event: for any $b\in\mathbb{Z}$ with $h^{-1}n^{2/3}<|b|<hn^{2/3}$, we have $\mathbf{G}(\langle 0,b\rangle)+b(\rho^{-1}-(1-\rho)^{-1})>hn^{1/3}$; and for $b\in\mathbb{Z}$ with $|b|\geq hn^{2/3}$, we have $\mathbf{G}(\langle 0,b\rangle)+b(\rho^{-1}-(1-\rho)^{-1})>-|b|n^{-1/3}$. We show that its probability is lower bounded uniformly in n.

Lemma 7.4. *For any $h>1$, there is $\delta>0$ such that $\mathbb{P}[\mathcal{E}_{h,n}]>\delta$ for all n large enough.*

Proof. Denote $F(b)=-\mathbf{G}(\langle 0,b\rangle)-b(\rho^{-1}-(1-\rho)^{-1})$. Then F is a (two-sided) random walk, where each step is centered with exponential tail. By independence of all the steps, we have

$$\begin{aligned}\mathbb{P}[\mathcal{E}_{h,n}] \geq\ & \mathbb{P}\Big[\max_{h^{-1}n^{2/3}<|b|<hn^{2/3}} F(b)<-hn^{1/3}\Big]\\ &\times\mathbb{P}\Big[\max_{b\geq hn^{2/3}}\big(F(b)-F(\lfloor hn^{2/3}\rfloor)-bn^{-1/3}\big)<hn^{1/3}\Big]\\ &\times\mathbb{P}\Big[\max_{b\leq -hn^{2/3}}\big(F(b)-F(-\lfloor hn^{2/3}\rfloor)+bn^{-1/3}\big)<hn^{1/3}\Big].\end{aligned}$$

As the process F converges to a (two-sided) Brownian motion (weakly in the uniform topology) on compact sets, the first factor on the right-hand side is lower bounded by a positive constant. We next lower bound the factor in the second line, and the third line could be lower bounded in a similar way. The second line is at least

$$\begin{aligned}\mathbb{P}\Big[\max_{b\in\mathbb{N}}(F(b)-bn^{-1/3})<hn^{1/3}\Big] \geq\ & \mathbb{P}\Big[\max_{b\in[\![0,In^{2/3}]\!]}(F(b)-bn^{-1/3})<hn^{1/3}\Big]\\ &-\sum_{i=I}^{\infty}\mathbb{P}\Big[\max_{b\in[\![in^{2/3},(i+1)n^{2/3}]\!]}F(b)\geq(i+h)n^{1/3}\Big],\end{aligned}$$

where I is a large integer. As $n\to\infty$, the first term on the right-hand side converges to the probability that a Brownian motion is bounded below a (sloped) line in $[0,I]$, and that

probability is lower bounded uniformly in I. For the sum in the second line, the i-th term is upper bounded by

$$\mathbb{P}[F(\lceil in^{2/3}\rceil) \geq (i+h)n^{1/3}/2] + \mathbb{P}\Big[\max_{b\in[\![0,n^{2/3}]\!]} F(b) \geq (i+h)n^{1/3}/2\Big]$$
$$\leq \mathbb{P}[F(\lceil in^{2/3}\rceil) \geq (i+h)n^{1/3}/2] + 2\mathbb{P}[F(\lfloor n^{2/3}\rfloor) \geq (i+h)n^{1/3}/2],$$

where the inequality is by the reflection principle. By a Bernstein type estimate for the sum of independent random variables with exponential tails, this can be bounded by Ce^{-ci} for some $c, C > 0$, independent of n. Thus by taking I large enough the conclusion follows. ■

Proof of Theorem 7.3. It suffices to show that, for any $s \in \mathbb{N}$ and any continuous $f : \mathbb{R}^{[\![-s,s]\!]^2} \times \{0,1\}^{[\![-s,s]\!]^2} \to [0,1]$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2}$, we have $\mu_{\mathbf{0},\langle n,b_n\rangle}(f) \to \nu(f)$ in probability.

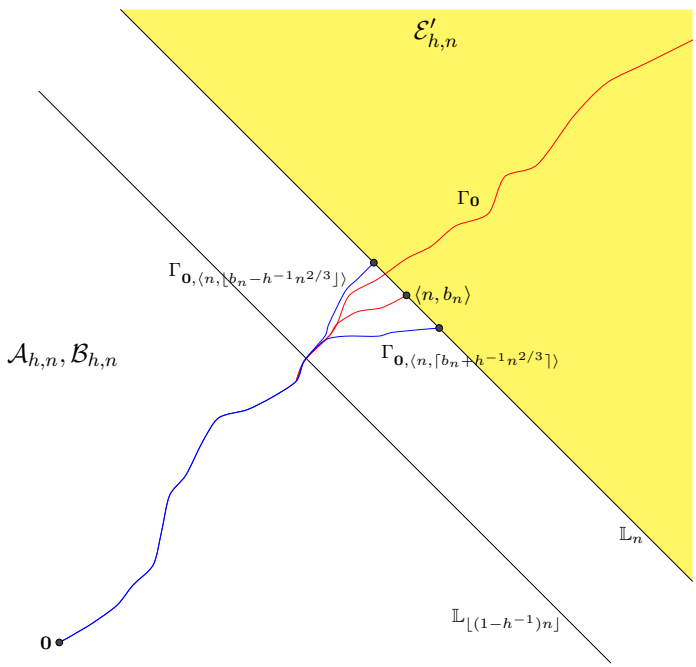


Fig. 14. An illustration of the proof of Theorem 7.3. The event $\mathcal{E}'_{h,n}$ is on the spiky behavior of the Busemann function, the event $\mathcal{A}_{h,n}$ is on passage times from $\mathbf{0}$ to $\mathbb{L}_n$, and the event $\mathcal{B}_{h,n}$ is on coalescence of geodesics. Under their intersection, most of $\Gamma_{\mathbf{0},\langle n,b_n\rangle}$ is also in $\Gamma_{\mathbf{0}}$. The events $\mathcal{A}_{h,n}$ and $\mathcal{B}_{h,n}$ happen with high probability, and $\mathcal{E}'_{h,n}$ happens with positive probability lower bounded uniformly in n. The event $\mathcal{E}'_{h,n}$ depends only on ξ in the yellow region, while $\mathcal{A}_{h,n}$ and $\mathcal{B}_{h,n}$ depend only on ξ in the remaining region (and roughly so does $\mu_{\mathbf{0},\langle n,b_n\rangle}$).

In this proof we use $c, C > 0$ to denote small and large enough constants, whose values may change from line to line. Then $|b_n| < Cn^{2/3}$ for any $n \in \mathbb{N}$. For simplicity of notation we denote $T^{\bullet}_{u,v} = T_{u,v} - \xi(v)$ for any vertices $u \le v$.

We let $\mathcal{E}'_{h,n}$ be $\mathcal{E}_{h,n}$ translated by $\langle n, b_n\rangle$, i.e. $\mathcal{E}'_{h,n}$ is the event where

$$\mathbf{B}(\langle n,b\rangle, \langle n,b_n\rangle) < \begin{cases} (b-b_n)(\rho^{-1}-(1-\rho)^{-1}) - hn^{1/3} \\ \qquad\qquad\qquad\qquad \text{for any } h^{-1}n^{2/3} < |b-b_n| < hn^{2/3}, \\ (b-b_n)(\rho^{-1}-(1-\rho)^{-1}) + |b-b_n|n^{-1/3} \\ \qquad\qquad\qquad\qquad \text{for any } |b-b_n| \ge hn^{2/3}. \end{cases}$$

Denote by $\langle n, b'_n\rangle$ the intersection of $\Gamma_{\mathbf{0}}$ with $\mathbb{L}_n$. Take any $\epsilon > 0$. By Theorem 7.1, for any n large enough (depending on ϵ, f), we have

$$\mathbb{P}[|\mu_{\mathbf{0},\langle n,b'_n\rangle}(f) - \nu(f)| < \epsilon] > 1-\epsilon.$$

By Lemma 7.4, when ϵ is taken small enough depending on h, we have

$$\mathbb{P}\big[|\mu_{\mathbf{0},\langle n,b'_n\rangle}(f) - \nu(f)| < \epsilon \;\big|\; \mathcal{E}'_{h,n}\big] > 1-\sqrt{\epsilon} \tag{7.1}$$

for any n large enough (depending on h, ϵ, f).

We next study the overlap between $\Gamma_{\mathbf{0}}$ and $\Gamma_{\mathbf{0},\langle n,b_n\rangle}$ under the event $\mathcal{E}'_{h,n}$. We denote by $\mathcal{A}_{h,n}$ the following event: for any $b \in \mathbb{Z}$, we have

$$T^{\bullet}_{\mathbf{0},\langle n,b\rangle} + b(\rho^{-1}-(1-\rho)^{-1}) \begin{cases} > \mathbb{E}[T_{\mathbf{0},\langle n,0\rangle}] - hn^{1/3}/2 & \text{if } |b-b_n| \le h^{-1}n^{2/3}, \\ < \mathbb{E}[T_{\mathbf{0},\langle n,0\rangle}] + hn^{1/3}/2 & \text{if } h^{-1}n^{2/3} < |b-b_n| < hn^{2/3}, \\ > \mathbb{E}[T_{\mathbf{0},\langle n,0\rangle}] - hn^{1/3}/2 - |b-b_n|n^{-1/3} & \text{if } |b-b_n| \ge hn^{2/3}. \end{cases}$$

We have $\mathbb{P}[\mathcal{A}_{h,n}] > 1 - e^{-ch}$ for n and h large enough. This can be deduced by applying (5.1) in Theorem 5.2 to $T_{\mathbf{0},\langle n,b\rangle}$ for each $b \in [\![-n,n]\!]$ with $|b| > (\rho^2 \wedge (1-\rho)^2)n$, and splitting $\{\langle n,b\rangle : b \in [\![-(\rho^2 \wedge (1-\rho)^2)n, (\rho^2 \wedge (1-\rho)^2)n]\!]\}$ into segments of length $n^{2/3}$ and using Proposition 5.3 for each of them.

We also denote by $\mathcal{B}_{h,n}$ the following event:

$$\Gamma_{\mathbf{0},\langle n,\lfloor b_n - h^{-1}n^{2/3}\rfloor\rangle} \cap \mathbb{L}_{\lfloor(1-h^{-1})n\rfloor} = \Gamma_{\mathbf{0},\langle n,\lceil b_n + h^{-1}n^{2/3}\rceil\rangle} \cap \mathbb{L}_{\lfloor(1-h^{-1})n\rfloor}.$$

By Proposition 5.4, we have $\mathbb{P}[\mathcal{B}_{h,n}] > 1 - Ch^{-1/3}$ for $h < cn^{2/3}$ and for h large enough.

Note that $\mathcal{A}_{h,n}$ and $\mathcal{B}_{h,n}$ only depend on the i.i.d. random weights ξ below $\mathbb{L}_n$, and $\mathcal{E}'_{h,n}$ only depends on ξ on or above $\mathbb{L}_n$, so the events $\mathcal{A}_{h,n}, \mathcal{B}_{h,n}$ are independent of $\mathcal{E}'_{h,n}$ (see Figure 14). Using $\mathbb{P}[\mathcal{A}_{h,n}] > 1 - e^{-ch}$, $\mathbb{P}[\mathcal{B}_{h,n}] > 1 - Ch^{-1/3}$, and (7.1), for n large enough (depending on h, ϵ, f) we have

$$\mathbb{P}\big[\mathcal{A}_{h,n}, \mathcal{B}_{h,n}, |\mu_{\mathbf{0},\langle n,b'_n\rangle}(f) - \nu(f)| < \epsilon \;\big|\; \mathcal{E}'_{h,n}\big] > 1 - \sqrt{\epsilon} - e^{-ch} - Ch^{-1/3}.$$

Under $\mathcal{A}_{h,n} \cap \mathcal{E}'_{h,n}$, we have

$$T^{\bullet}_{\mathbf{0},\langle n,b\rangle} + \mathbf{B}(\langle n,b\rangle, \langle n,b_n\rangle) < T^{\bullet}_{\mathbf{0},\langle n,b_n\rangle}$$

for any $b \in \mathbb{Z}$ with $|b - b_n| > h^{-1}n^{2/3}$. Thus $|b_n - b'_n| \le h^{-1}n^{2/3}$ by Lemma 3.1. Then under $\mathcal{A}_{h,n} \cap \mathcal{B}_{h,n} \cap \mathcal{E}'_{h,n}$, we must have $\Gamma_{\mathbf{0},\langle n,b_n\rangle} \cap \mathbb{L}_{\lfloor(1-h^{-1})n\rfloor} = \Gamma_{\mathbf{0},\langle n,b'_n\rangle} \cap \mathbb{L}_{\lfloor(1-h^{-1})n\rfloor}$ by ordering of geodesics (Lemma 5.1), and $|\mu_{\mathbf{0},\langle n,b'_n\rangle}(f) - \nu(f)| < \epsilon$ implies that

$$|\mu_{\mathbf{0},\langle n,b_n\rangle}(f) - \nu(f)| < \epsilon + h^{-1}.$$

So we have

$$\mathbb{P}\big[|\mu_{\mathbf{0},\langle n,b_n\rangle}(f) - \nu(f)| < \epsilon + h^{-1} \;\big|\; \mathcal{E}'_{h,n}\big] > 1 - \sqrt{\epsilon} - e^{-ch} - Ch^{-1/3}.$$

Note that $\Gamma_{\mathbf{0},\langle n,b_n\rangle}$ is determined by the weights ξ below $\mathbb{L}_n$, so it is independent of $\mathcal{E}'_{h,n}$. For each $v \in \Gamma_{\mathbf{0},\langle n,b_n\rangle}$ with $d(v) < 2n - 2s$, $f(v)$ is determined by the weights ξ in $v + [\![-s,s]\!]^2$, so it is also independent of $\mathcal{E}'_{h,n}$. Thus we conclude that

$$\mathbb{P}[|\mu_{\mathbf{0},\langle n,b_n\rangle}(f) - \nu(f)| < \epsilon + h^{-1} + s/n] > 1 - \sqrt{\epsilon} - e^{-ch} - Ch^{-1/3}$$

for any n large enough (depending on h, ϵ, f). Since h can be taken arbitrarily large and ϵ is any number small enough depending on h, we conclude that $\mu_{\mathbf{0},\langle n,b_n\rangle}(f) \to \nu(f)$ in probability. ∎

8. Parallelogram uniform covering

The goal of this section is to prove the following upgraded version of Theorem 7.3. It will be the key input for the next two sections.

Proposition 8.1. *For any $h > 0$, $s \in \mathbb{N}$, and any bounded continuous function $f : \mathbb{R}^{[\![-s,s]\!]^2} \times \{0,1\}^{[\![-s,s]\!]^2} \to \mathbb{R}$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2}$, we have*

$$\max_{a,b\in\mathbb{Z},\,|a|,|b|<hn^{2/3}} \mu_{\langle 0,a\rangle,\langle n,b\rangle}(f), \quad \min_{a,b\in\mathbb{Z},\,|a|,|b|<hn^{2/3}} \mu_{\langle 0,a\rangle,\langle n,b\rangle}(f) \to \nu(f)$$

in probability.

For simplicity of notation, below we write out the proof for $\rho = 1/2$, while the general ρ case follows essentially verbatim.

We now explain our strategy. We will take two families of vertices, $\mathfrak{P}_1$ and $\mathfrak{P}_2$, around the segment connecting $\langle 0, -hn^{2/3}\rangle$ and $\langle 0, hn^{2/3}\rangle$ and the segment connecting $\langle n, -hn^{2/3}\rangle$ and $\langle n, hn^{2/3}\rangle$, respectively. Both $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are finite, in the sense that their sizes do not increase as $n \to \infty$. Then by Theorem 7.3, when n is large enough, with high probability, for any $u \in \mathfrak{P}_1$ and $v \in \mathfrak{P}_2$, $\mu_{u,v}(f)$ is close to $\nu(f)$. We will show that with high probability, for any $|a|, |b| < hn^{2/3}$, the geodesic $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ is mostly covered by some $\Gamma_{u,v}$ with $u \in \mathfrak{P}_1$ and $v \in \mathfrak{P}_2$, thus $\mu_{\langle 0,a\rangle,\langle n,b\rangle}(f)$ is also close to $\nu(f)$.

The main task is to establish the covering statement. To motivate our arguments, we start with the following attempt. For $a^- < a^+$ and $b^- < b^+$, if the geodesics $\Gamma_{\langle 0,a^-\rangle,\langle n,b^-\rangle}$ and $\Gamma_{\langle 0,a^+\rangle,\langle n,b^+\rangle}$ coalesce near both ends, then they must mostly stay together; and by ordering of geodesics (Lemma 5.1), for any $a^- < a < a^+$ and $b^- < b < b^+$, the geodesic $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ must be covered by $\Gamma_{\langle 0,a^-\rangle,\langle n,b^-\rangle}$, except for a small portion. By estimates on coalescence of geodesics (e.g. Proposition 5.4), if we let $b^+ - b^- = a^+ - a^-$ be on the order of $\delta_0 n^{2/3}$ (for some small $\delta_0 > 0$), the probability for $\Gamma_{\langle 0,a^-\rangle,\langle n,b^-\rangle}$ and $\Gamma_{\langle 0,a^+\rangle,\langle n,b^+\rangle}$ to stay disjoint within order n distance of their endpoints is on the order of δ_0. Now we take $\mathfrak{P}_1$ and $\mathfrak{P}_2$ to be contained in the segment connecting $\langle 0, -hn^{2/3}\rangle$ and $\langle 0, hn^{2/3}\rangle$ and the segment connecting $\langle n, -hn^{2/3}\rangle$ and $\langle n, hn^{2/3}\rangle$, respectively. Let these vertices split these two segments into $h\delta_0^{-1}$ small segments, each of length $\delta_0 n^{-2/3}$. By taking a union bound over all pairs of such small segments, we conclude that the probability of there existing some $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ not being mostly covered (by one geodesic with two endpoints in $\mathfrak{P}_1$ and $\mathfrak{P}_2$) is upper bounded by $(\delta_0^{-1})^2\delta_0$, which is too large.

To resolve this issue, we need to get a better bound on the probability of the following event: there exist $a^- < a < a^+$ and $b^- < b < b^+$ such that the geodesic $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ is not mostly covered by any geodesic with endpoints in $\mathfrak{P}_1$ and $\mathfrak{P}_2$. If this probability could be upper bounded by $\delta_0^{2+\epsilon}$ for some $\epsilon > 0$ (rather than δ_0), then by a union bound and sending $\delta_0 \to 0$, the conclusion follows. Towards this, we need to take $\mathfrak{P}_1$ and $\mathfrak{P}_2$ larger (but still finite). Instead of having them contained in $\mathbb{L}_0$ and $\mathbb{L}_n$, we let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ have $h\delta_0^{-1} \times \delta_0^{-1}$ vertices in the rectangles $\{u : 0 \le d(u) \le 2n/3, -2hn^{2/3} \le ad(u) \le 2hn^{2/3}\}$ and $\{u : 4n/3 \le d(u) \le 2n, -2hn^{2/3} \le ad(u) \le 2hn^{2/3}\}$, respectively. Fix some small $\kappa > 0$. Using ordering of geodesics (Lemma 5.1) and a union bound, the above task can roughly be reduced to proving the following statement. For given a^-, a^+ and b^-, b_+ that are contained in $[-hn^{2/3}, hn^{2/3}]$ with $b^+ - b^- = a^+ - a^-$ on the order of $\delta_0 n^{2/3}$, the following event happens with probability on the order of at most $\delta_0^{2+\epsilon}$ for some $\epsilon > 0$: there exist $a^- < a < a^+$ and $b^- < b < b^+$ such that for any $u \in \mathfrak{P}_1$ and $v \in \mathfrak{P}_2$ in the same side of $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$, $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle} \cap \Gamma_{u,v}$ contains no vertex below $\mathbb{L}_{2\kappa n}$.

Now let us consider the scenario where the above event happens. Take any $v \in \mathfrak{P}_2$ that is within distance $\delta_0 n^{2/3}$ of $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$. We find vertices u_1, u_2, u_3, u_4, u_5 in $\mathfrak{P}_1$ such that (1) they are between $\mathbb{L}_{\kappa n}$ and $\mathbb{L}_{2\kappa n}$; (2) these vertices are on the same side of $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ as v; (3) each is within distance $\delta_0 n^{2/3}$ of $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$. Consider the geodesics from each of these vertices to v; these geodesics are disjoint from $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ below $\mathbb{L}_{2\kappa n}$, by the above event. We can show that (with high probability), any two geodesics cannot stay close to each other while being disjoint for a long distance. By choosing the vertices u_5, u_4, u_3, u_2, u_1 sequentially and in a multi-scale way (see Figure 17 below for an illustration), we can actually find α_0 with $\kappa < \alpha_0 \in 2\kappa$ such that for $\Gamma_{u_i,v}$ with $i = 1, 2, 3, 4, 5$ and $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$, their intersections with $\mathbb{L}_{\alpha_0 n}$ are far from each other (with distances on the order of at least $\delta_0^{1/150} n^{2/3}$).

However, using $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ and each $\Gamma_{u_i,v}$, (with high probability) one can construct a path from $\langle 0, a^-\rangle$ to $\langle n, b^-\rangle$, and the difference between its passage time and $T_{\langle 0,a^-\rangle,\langle n,b^-\rangle}$ is at most on the order of $\delta_0^{1/2} n^{1/3}$. Indeed, one can just mainly use the path of $\Gamma_{u_i,v}$, and

switch to $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ only near u_i and v, and switch to $\langle 0,a^-\rangle$ and $\langle n,b^-\rangle$ near the ends. One can also just mainly use the path $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ and switch to $\langle 0,a^-\rangle$ and $\langle n,b^-\rangle$ near the ends. This way we get in total six paths from $\langle 0,a^-\rangle$ to $\langle n,b^-\rangle$, each with total passage time at least $T_{\langle 0,a^-\rangle,\langle n,b^-\rangle} - \delta_0^{1/2} n^{1/3}$; and they intersect $\mathbb{L}_{\alpha_0 n}$ at vertices far away from each other. Now consider the optimal passage time from $\langle 0,a^-\rangle$ to $\langle n,b^-\rangle$ passing through $\langle \alpha_0 n, b'\rangle$ as a function of b'. This is roughly the sum of two independent point-to-line last-passage profiles (see Section 8.1 below). Its scaling limit is known to behave like a Brownian motion, and the event that there are six paths with near optimal passage times is reduced to that, for a Brownian motion in a compact interval one can find six points such that their distances are at least on the order of $\delta_0^{1/150}$, and the Brownian motion values at these points are at least the maximum (of the Brownian motion) minus $\delta_0^{1/2}$. This event has probability on the order of at most $(\delta_0^{1/2-1/300})^5 = \delta_0^{5/2-1/60}$, which is smaller than $\delta_0^{2+\epsilon}$ as needed (and this is also why we need to find five alternative paths).

We now explain the organization of the rest of this section. We will first list some useful ingredients that will be useful in carrying out the above plan; the proofs of some of these ingredients are delayed to Sections 8.1 and 8.2. Then we will define several events, each with a small probability. The main arguments are contained in the proof of Lemma 8.6 below, where we show that under the intersection of the complements of these events, every $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ is mostly covered by one geodesic in a finite family. Finally, we deduce Proposition 8.1 using Lemma 8.6.

Ingredients: The first one concerns continuity of the function $(a,b) \mapsto T_{\langle 0,a\rangle,\langle n,b\rangle}$.

Lemma 8.2. *There exist constants $c, C > 0$ such that the following is true. For $h > 0$, $0 < \theta < 1$, and $t > 1$, we have*

$$\mathbb{P}\Big[\max_{\substack{|a|,|a'|,|b|,|b'|<hn^{2/3}\\ |a-a'|,|b-b'|<\theta n^{2/3}}} |T_{\langle 0,a\rangle,\langle n,b\rangle} - T_{\langle 0,a'\rangle,\langle n,b'\rangle}| > t\theta^{1/2-0.01} n^{1/3} + Ch\theta n^{1/3}\Big] < Che^{-ct}$$

when n is large enough (depending on h, θ, t).

The proof of this lemma will be given in Section 8.1.

We next state a bound on transversal fluctuations of geodesics. It actually immediately follows from the results in Section 5, and we state it here mainly for the convenience of the proof of Proposition 8.1. For vertices $u \le v$, and $0 \le l \le d(v) - d(u)$, $t > 1$, let $\mathcal{T}_{l,t}^{u,v}$ be the event where $\Gamma_{u,v}$ below $u + \mathbb{L}_l$ is not contained in a rectangle of width $2tl^{2/3}$, or $\Gamma_{u,v}$ above $v - \mathbb{L}_l$ is not contained in a rectangle of width $2tl^{2/3}$ (see Figure 15). Formally, we let $\Gamma_{u,v}$ be the event where there exists $w \in \Gamma_{u,v}$ with $d(u) \le d(w) \le d(u) + 2l$ and $|ad(w) - ad(u)| \ge 2tl^{2/3}$, or with $d(v) - 2l \le d(w) \le d(v)$ and $|ad(w) - ad(v)| \ge 2tl^{2/3}$.

Lemma 8.3. *For $h > 0$, there exist constants $c, C > 0$ such that the following is true. For any $0 \le l \le n$ large enough, and $|b| < hn^{2/3}$, $t > 1$, we have $\mathbb{P}[\mathcal{T}_{l,t}^{\mathbf{0},\langle n,b\rangle}] < Ce^{-ct^3}$.*

This lemma can be obtained by applying Corollary 5.9 twice, and we omit its proof.

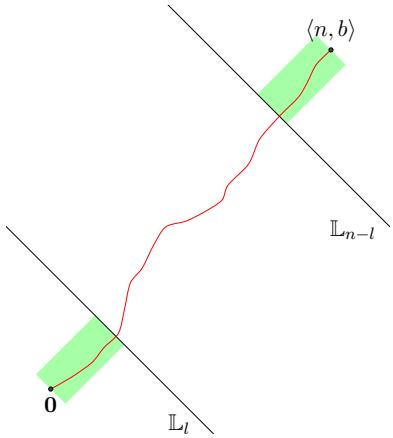


Fig. 15. The complement of the event $\mathcal{T}_{l,t}^{\mathbf{0},\langle n,b\rangle}$: the geodesic $\Gamma_{\mathbf{0},\langle n,b\rangle}$ is restricted within the green boxes with width $tl^{2/3}$, below $\mathbb{L}_l$ or above $\mathbb{L}_{n-l}$.

Our next lemma establishes that, for a geodesic and a path with a 'near-optimal' passage time, it is unlikely for them to stay together for a while but remain disjoint.

For any vertices $u \le v$, and $M, l \in \mathbb{N}, m \in \mathbb{Z}$ with $d(u) \le 2m < 2m + 2Ml \le d(v)$, and a small enough parameter $c_0 > 0$, we let $\mathcal{D}_{M,l,m}^{u,v}$ be the following event (see Figure 16): there exists an up-right path γ from $\mathbb{L}_m$ to $\mathbb{L}_{m+Ml}$ such that

- γ is disjoint from $\Gamma_{u,v}$,
- the passage time of γ (i.e. $T(\gamma)$) is at least $4Ml - c_0 Ml^{1/3}$,
- for each $i = 0, 1, \dots, M$, $|ad(\Gamma_{u,v} \cap \mathbb{L}_{m+il}) - ad(\gamma \cap \mathbb{L}_{m+il})| < 2c_0 l^{2/3}$.

Lemma 8.4. *There exist universal constants $c, C > 0$ such that the following is true. For any $M, l, n \in \mathbb{N}$ and $m, b \in \mathbb{Z}$ with $l > C$, $c_0 < c$, $|b| \le n$, and $0 \le m < m + Ml \le n$, we have $\mathbb{P}[\mathcal{D}_{M,l,m}^{\mathbf{0},\langle n,b\rangle}] < Ce^{-cM}$.*

The last ingredient we need is to bound the probability of multiple peaks in the sum of two independent point-to-line profiles.

As in previous sections, we denote $T_{u,v}^{\bullet} = T_{u,v} - \xi(v)$ for any vertices $u \le v$ (i.e. remove the weight of the last vertex). For any vertices $u \le v$, and $m \in \mathbb{Z}$ with $d(u) \le 2m \le d(v)$, and $\lambda, t > 0$, we denote by $\mathcal{M}_{\lambda,t,m,g}^{u,v}$ the following event: there exist $-g \le b_1 < b_2 < b_3 < b_4 < b_5 < b_6 \le g$ with $b_2 - b_1, b_3 - b_2, b_4 - b_3, b_5 - b_4, b_6 - b_5 \ge \lambda$ such that $T_{u,v} = T_{u,\langle m,b_1\rangle}^{\bullet} + T_{\langle m,b_1\rangle,v}$ and

$$T_{u,\langle m,b_i\rangle}^{\bullet} + T_{\langle m,b_i\rangle,v} > T_{u,v} - t\lambda^{1/2}, \quad \forall i \in \{2,3,4,5,6\}.$$

Lemma 8.5. *For $h > 0$ and $0 < \kappa < 1/2$, there exists a constant $C > 0$ such that the following is true. For any $\theta > 0$, $0 < t < 1$, $\kappa < \alpha < 1 - \kappa$, $|\beta| < h$, we have*

$$\mathbb{P}[\mathcal{M}_{\theta n^{2/3},t,\lfloor \alpha n\rfloor,hn^{2/3}}^{\mathbf{0},\langle n,\lfloor \beta n^{2/3}\rfloor\rangle}] < Ct^{5-0.01}$$

for n large enough depending on $h, \theta, t, \alpha, \beta$.

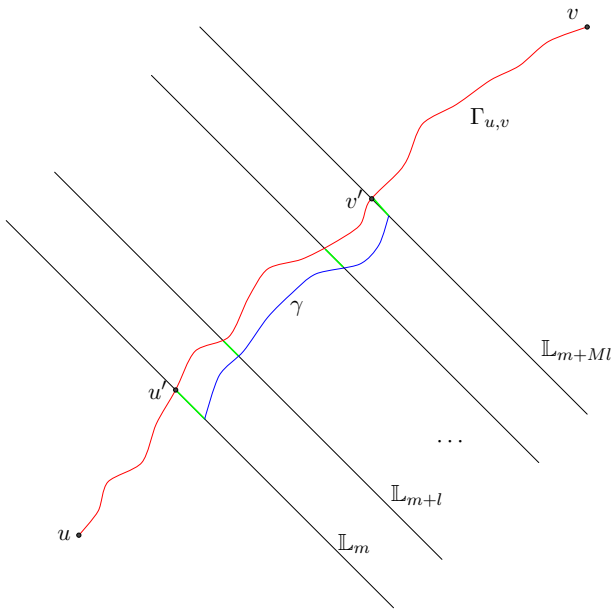


Fig. 16. The event $\mathcal{D}^{u,v}_{M,l,m}$: each green segment has length $< c_0 l^{2/3}$, and $T(\gamma) \geq 4Ml - c_0 M l^{1/3}$.

Lemma 8.4 will be proved in Section 8.2, and Lemma 8.5 will be proved in Section 8.1.

Assuming all the lemmas above, we now prove Proposition 8.1. We set up the events to be used in the proof of Proposition 8.1, for which we first define the parameters.

Parameters: From now on we fix h in the statement of Proposition 8.1. As indicated above, we will choose vertices u_1, u_2, u_3, u_4, u_5 in $\mathfrak{P}_1$ in a multi-scale way. Thus we define the scales as follows. We take a small number $\delta > 0$, and let $\delta_i = \delta^{100^{6-i}}$ for $i = 0, 1, 2, 3, 4, 5$. So we have $0 < \delta_0 < \delta_1 < \delta_2 < \delta_3 < \delta_4 < \delta_5 < \delta$. We also take small $\kappa > 0$ and large $\hat{h}$, and we can assume that δ is small enough depending on κ and $\hat{h}$, and $\hat{h}$ is large enough depending on h. The values of the parameters $\delta, \kappa, \hat{h}$ are to be determined, but we always ensure that $\delta^{-1}, \kappa^{-1}, \hat{h}$ are integers. Then there exists some integer N such that if we denote $\mathcal{N} = \{Nk^3 : k \in \mathbb{N}\}$, then for any $n \in \mathcal{N}$ the numbers $\delta_0 n, \delta_0 n^{2/3}, \hat{h} n^{2/3}, \kappa n, \delta^{-1}$, and all $\delta_i^{-1}\delta_{i+1}$ for $i \in \{0, 1, 2, 3, 4\}$ are integers. From now on we assume that $n \in \mathcal{N}$ is large enough depending on all these parameters. Only inside the proof of Proposition 8.1 will we treat general large n.

Below we use $c, C > 0$ to denote small and large enough constants, which can only depend on $\hat{h}$ and κ, and whose values may change from line to line.

Events: We take the two families of vertices as $\mathfrak{P}_1 = \{\langle i\delta_0 n, j\delta_0 n^{2/3}\rangle : i, j \in \mathbb{Z}, 0 \leq i \leq \delta_0^{-1}/3, |j| < 4\hat{h}\delta_0^{-1}\}$ and $\mathfrak{P}_2 = \mathbf{n} - \mathfrak{P}_1$. Note that here we take $\mathfrak{P}_1$ and $\mathfrak{P}_2$ to be in rectangles with width on the order of $\hat{h}n^{2/3}$ rather than $hn^{2/3}$, because the geodesics (that

we will study) can potentially have large transversal fluctuations. Consider the following events:

- Let $\mathcal{T}$ be the union of $\mathcal{T}^{u,v}_{l,\delta^{-1}}$ for all $u \in \mathfrak{P}_1$, $v \in \mathfrak{P}_2$, and $l \in \delta_0 n\mathbb{Z}$ with $0 \le l < d(v) - d(u)$. By Lemma 8.3 we have $\mathbb{P}[\mathcal{T}] < C\delta_0^{-5}e^{-c\delta^{-3}}$.
- Let
$$\mathcal{T}_* = \mathcal{T}^{\langle 0,\hat{h}n^{2/3}\rangle,\langle n,\hat{h}n^{2/3}\rangle}_{n,\hat{h}} \cup \mathcal{T}^{\langle 0,-\hat{h}n^{2/3}\rangle,\langle n,-\hat{h}n^{2/3}\rangle}_{n,\hat{h}} \cup \mathcal{T}^{\langle 0,3\hat{h}n^{2/3}\rangle,\langle n,3\hat{h}n^{2/3}\rangle}_{n,\hat{h}} \cup \mathcal{T}^{\langle 0,-3\hat{h}n^{2/3}\rangle,\langle n,-3\hat{h}n^{2/3}\rangle}_{n,\hat{h}}.$$
In other words, $\mathcal{T}_*^c$ is just the event where for each $j \in \{-3,-1,1,3\}$ the geodesic $\Gamma_{\langle 0,j\hat{h}n^{2/3}\rangle,\langle n,j\hat{h}n^{2/3}\rangle}$ is contained in $\{u \in \mathbb{Z}^2 : 0 \le d(u) \le 2n, |ad(u) - 2j\hat{h}n^{2/3}| \le 2\hat{h}n^{2/3}\}$. By Lemma 8.3, $\mathbb{P}[\mathcal{T}_*] \to 0$ as $\hat{h} \to \infty$, uniformly in n.
- Let $\mathcal{F}$ be the event where
$$|T^{\bullet}_{\langle i\delta_1 n,a\rangle,\langle j\delta_1 n,b\rangle} - T^{\bullet}_{\langle i\delta_1 n,a'\rangle,\langle j\delta_1 n,b'\rangle}| > \delta_0^{1/2-0.02}n^{1/3}$$
for some integers $0 \le i < j \le \delta_1^{-1}$, and $|a|,|a'|,|b|,|b'| \le 4\hat{h}n^{2/3}$ with $|a-a'|,|b-b'| \le \delta_0 n^{2/3}$. By applying Lemma 8.2 to this event with each fixed i,j and taking a union bound, we have $\mathbb{P}[\mathcal{F}] < C\delta_1^{-2-2/3}e^{-c\delta_0^{-0.01}}$.
- Let $\mathcal{D}$ be the union of $\mathcal{D}^{u,v}_{\delta^{-7},l,m}$ for all $u \in \mathfrak{P}_1$, $v \in \mathfrak{P}_2$, $l \in \{\delta_i n : i = 1,2,3,4,5\}$, $m \in \delta_0 n\mathbb{Z}$ such that $d(u) \le 2m < 2m + 2\delta^{-7}l \le d(v)$. Here we take c_0 to be small enough as required by Lemma 8.4. Then by applying Lemma 8.4 to each $\mathcal{D}^{u,v}_{\delta^{-7},l,m}$ and taking a union bound, we have $\mathbb{P}[\mathcal{D}] < C\delta_0^{-5}e^{-c\delta^{-7}}$.
- Let $\mathcal{H}$ denote the event where there exist some $m \in \delta_0 n\mathbb{Z}$, $0 \le m \le n$, and $l \in \{\delta_i n : i = 1,2,3,4,5\}$, $|a|,|b| < 4\hat{h}n^{2/3}$, $|a-b| < \delta^{-6}l^{2/3}$, such that
$$T_{\langle m,a\rangle,\langle m+\delta^{-7}l,b\rangle} < 4\delta^{-7}l - c_0\delta^{-6}l^{1/3},$$
where c_0 is as in the event $\mathcal{D}$. By applying Proposition 5.3 via splitting the lines $\mathbb{L}_m$ and $\mathbb{L}_{m+\delta^{-7}l}$ into segments of length $\delta_0 n^{2/3}$, we have $\mathbb{P}[\mathcal{H}] < C\delta_0^{-3}e^{-c\delta^{-11/2}}$.
- Let $\mathcal{M}$ be the union of $\mathcal{M}^{u,v}_{c_0(\delta_1 n)^{2/3},\delta_0^{1/2-0.03}\delta_1^{-1/3},\alpha n,4\hat{h}n^{2/3}}$ for all $u \in \mathfrak{P}_1 \cap \mathbb{L}_0$, $v \in \mathfrak{P}_2 \cap \mathbb{L}_n$, and $\alpha \in \delta_1\mathbb{Z}$ with $\kappa < \alpha < 1-\kappa$, and c_0 be as in the event $\mathcal{D}$. By Lemma 8.5, we have $\mathbb{P}[\mathcal{M}] < C\delta_0^{-2}\delta_1^{-1}(\delta_0^{1/2-0.03}\delta_1^{-1/3})^{5-0.01} < C\delta_0^{0.3}\delta_1^{-8/3}$.

We denote $\mathcal{E} = \mathcal{T}^c \cap \mathcal{T}_*^c \cap \mathcal{F}^c \cap \mathcal{D}^c \cap \mathcal{H}^c \cap \mathcal{M}^c$. These events are designed so that $\mathcal{E}$ happens with high probability, and under $\mathcal{E}$ we have covering of geodesics.

Lemma 8.6. *Under $\mathcal{E}$, for any $|a|,|b| < \hat{h}n^{2/3}$, there exist $u \in \mathfrak{P}_1$ and $v \in \mathfrak{P}_2$ with $d(u) < 4\kappa n$ and $d(v) > (1-4\kappa)n$ such that $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ is the same as $\Gamma_{u,v}$ between $\mathbb{L}_{2\kappa n}$ and $\mathbb{L}_{(1-2\kappa)n}$.*

Proof. Assume $\mathcal{E}$ holds, and fix a,b such that $|a|,|b| < \hat{h}n^{2/3}$. By ordering of geodesics (Lemma 5.1), $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ is between $\Gamma_{\langle 0,-\hat{h}n^{2/3}\rangle,\langle n,-\hat{h}n^{2/3}\rangle}$ and $\Gamma_{\langle 0,\hat{h}n^{2/3}\rangle,\langle n,\hat{h}n^{2/3}\rangle}$. Then

by $\mathcal{T}_*^c$, we have

$$\Gamma_{\langle 0,a\rangle,\langle n,b\rangle} \subset \{u \in \mathbb{Z}^2 : 0 \le d(u) \le 2n,\ |ad(u)| \le 4\hat{h}n^{2/3}\}. \tag{8.1}$$

Let b^+ be the smallest number with $b^+ \in \delta_0 n^{2/3}\mathbb{Z}$ and $b^+ \ge b$. As indicated above, we now show that we can find $u^* \in \mathfrak{P}_1$ with $d(u^*) < 4\kappa n$ such that there exists $u \in \Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ with $d(u) = d(u^*)$ and $ad(u) \le ad(u^*) \le ad(u) + 2\delta_0 n^{2/3}$, and $\Gamma_{u^*,\langle n,b^+\rangle}$ intersects $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ before $\mathbb{L}_{2\kappa n}$.

Indeed, assume that no such u^* exists. We will sequentially find the vertices u_5, u_4, u_3, u_2, u_1 (as illustrated in Figure 17) and then use them to find some multiple peaks, and thus get a contradiction with $\mathcal{M}^c$. The idea is to take each u_i as the vertex in $\mathfrak{P}_1 \cap \mathbb{L}_{\alpha_i n}$ that is to the right of and closest to $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$. Here α_i are numbers to be chosen sequentially: given u_{i+1}, we find α_i such that the intersections of $\mathbb{L}_{\alpha_i}$ with $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ and $\Gamma_{u_{i+1},\langle n,b^+\rangle}$ are $c_0(\delta_{i+1}n)^{2/3}$ apart, using $\mathcal{D}^c$. Finally, we consider the intersections of each $\Gamma_{u_i,\langle n,b^+\rangle}$ with $\mathbb{L}_{\alpha_0 n}$; we can ensure that they are still $c_0(\delta_1 n)^{2/3}$ apart, using transversal fluctuation bounds (from the event $\mathcal{T}^c$) and the fact that $\alpha_0 - \alpha_i$ is chosen to be on the order of δ_i.

Sequential construction. Let us start by choosing u_5. We take α_5 as the smallest number such that $\alpha_5 \in \delta_5\mathbb{Z}$ and $\alpha_5 > \kappa$, and take $u_5 \in \mathfrak{P}_1 \cap \mathbb{L}_{\alpha_5 n}$ being the first one on or to the right of $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$. In other words, we have $0 \le ad(u_5) - ad(\Gamma_{\langle 0,a\rangle,\langle n,b\rangle} \cap \mathbb{L}_{\alpha_5 n}) < 2\delta_0 n^{2/3}$. Then by (8.1), we have $|ad(u_5)| \le 4\hat{h}n^{2/3}$. Consider the path $\Gamma_{u_5,\langle n,b^+\rangle}$. Again by $\mathcal{T}_*^c$ and ordering of geodesics (Lemma 5.1), it is between $\Gamma_{\langle 0,-3\hat{h}n^{2/3}\rangle,\langle n,-3\hat{h}n^{2/3}\rangle}$ and $\Gamma_{\langle 0,3\hat{h}n^{2/3}\rangle,\langle n,3\hat{h}n^{2/3}\rangle}$ and

$$\Gamma_{u_5,\langle n,b^+\rangle} \subset \{u \in \mathbb{Z}^2 : 0 \le d(u) \le 2n,\ |ad(u)| \le 8\hat{h}n^{2/3}\}. \tag{8.2}$$

For each $j \in [\![0, \delta^{-7}]\!]$, we have

$$ad(\mathbb{L}_{(\alpha_5+j\delta_5)n} \cap \Gamma_{u_5,\langle n,b^+\rangle}) - ad(\mathbb{L}_{(\alpha_5+j\delta_5)n} \cap \Gamma_{\langle 0,a\rangle,\langle n,b\rangle}) \ge 0,$$

by ordering of geodesics (Lemma 5.1). We claim that there must exist $j_5 \in [\![0, \delta^{-7}]\!]$ such that the left-hand side above for $j = j_5$ is at least $2c_0(\delta_5 n)^{2/3}$. Indeed, otherwise we can show that the event $\mathcal{D}^{\langle 0,a\rangle,\langle n,b\rangle}_{\delta^{-7},\delta_5 n,\alpha_5 n}$ holds with the path being Γ_{u_5,w_5}, where $w_5 = \Gamma_{u_5,\langle n,b^+\rangle} \cap \mathbb{L}_{(\alpha_5+\delta^{-7}\delta_5)n}$ (see Figure 17). For this we just verify several things:

- By the assumption above (that no such u^* exists), $\Gamma_{u_5,\langle n,b^+\rangle}$ is disjoint from $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ before $\mathbb{L}_{2\kappa n}$, thus Γ_{u_5,w_5} is disjoint from $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ since by taking δ small enough depending on κ we have $\alpha_5 + \delta^{-7}\delta_5 < 2\kappa$.
- We have $|ad(w_5)| \le 8\hat{h}n^{2/3}$ by (8.2). By $\mathcal{T}^c$ we have

$$|ad(u_5) - ad(w_5)| < 2\delta^{-1}(\delta^{-7}\delta_5 n)^{2/3} = 2\delta^{-17/3}(\delta_5 n)^{2/3}.$$

  Then $T_{u_5,w_5} \ge 4\delta^{-7}\delta_5 n - c_0\delta^{-6}(\delta_5 n)^{1/3}$ by $\mathcal{H}^c$.

Thus the event $\mathcal{D}^{\langle 0,a\rangle,\langle n,b\rangle}_{\delta^{-7},\delta_5 n,\alpha_5 n}$ holds, contradicting $\mathcal{D}^c$. So such j_5 must exist.

We next let $\alpha_4 = \alpha_5 + j_5\delta_5$, and take $u_4 \in \mathfrak{P}_1 \cap \mathbb{L}_{\alpha_4 n}$ being the first one on or to the right of $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$. Using the same arguments we find $0 \le j_4 \le \delta^{-7}$ such that

$$|ad(\mathbb{L}_{(\alpha_4+j_4\delta_4)n} \cap \Gamma_{\langle 0,a\rangle,\langle n,b\rangle}) - ad(\mathbb{L}_{(\alpha_4+j_4\delta_4)n} \cap \Gamma_{u_4,\langle n,b^+\rangle})| \ge 2c_0(\delta_4 n)^{2/3}.$$

Then we let $\alpha_3 = \alpha_4 + j_4\delta_4$. Similarly we find $j_3, j_2, j_1 \in [\![0, \delta^{-7}]\!]$ and $\alpha_2 = \alpha_3 + j_3\delta_3$, $\alpha_1 = \alpha_2 + j_2\delta_2$, $\alpha_0 = \alpha_1 + j_1\delta_1$, and vertices $u_3 \in \mathfrak{P}_1 \cap \mathbb{L}_{\alpha_3 n}$, $u_2 \in \mathfrak{P}_1 \cap \mathbb{L}_{\alpha_2 n}$, $u_1 \in \mathfrak{P}_1 \cap \mathbb{L}_{\alpha_1 n}$, such that for each $i = 1, 2, 3$ we have

$$0 \le ad(u_i) - ad(\mathbb{L}_{\alpha_i n} \cap \Gamma_{\langle 0,a\rangle,\langle n,b\rangle}) < 2\delta_0 n^{2/3} \tag{8.3}$$

and

$$ad(\mathbb{L}_{\alpha_{i-1}n} \cap \Gamma_{u_i,\langle n,b^+\rangle}) - ad(\mathbb{L}_{\alpha_{i-1}n} \cap \Gamma_{\langle 0,a\rangle,\langle n,b\rangle}) \ge 2c_0(\delta_i n)^{2/3}. \tag{8.4}$$

Note that $\mathbb{L}_{(\alpha_i+j_i\delta_i)n} = \mathbb{L}_{\alpha_{i-1}n}$ for each $i = 1, 2, 3, 4, 5$, and (8.3) and (8.4) also hold for $i = 4, 5$ as stated above. See Figure 17 for (some of) these constructed objects.

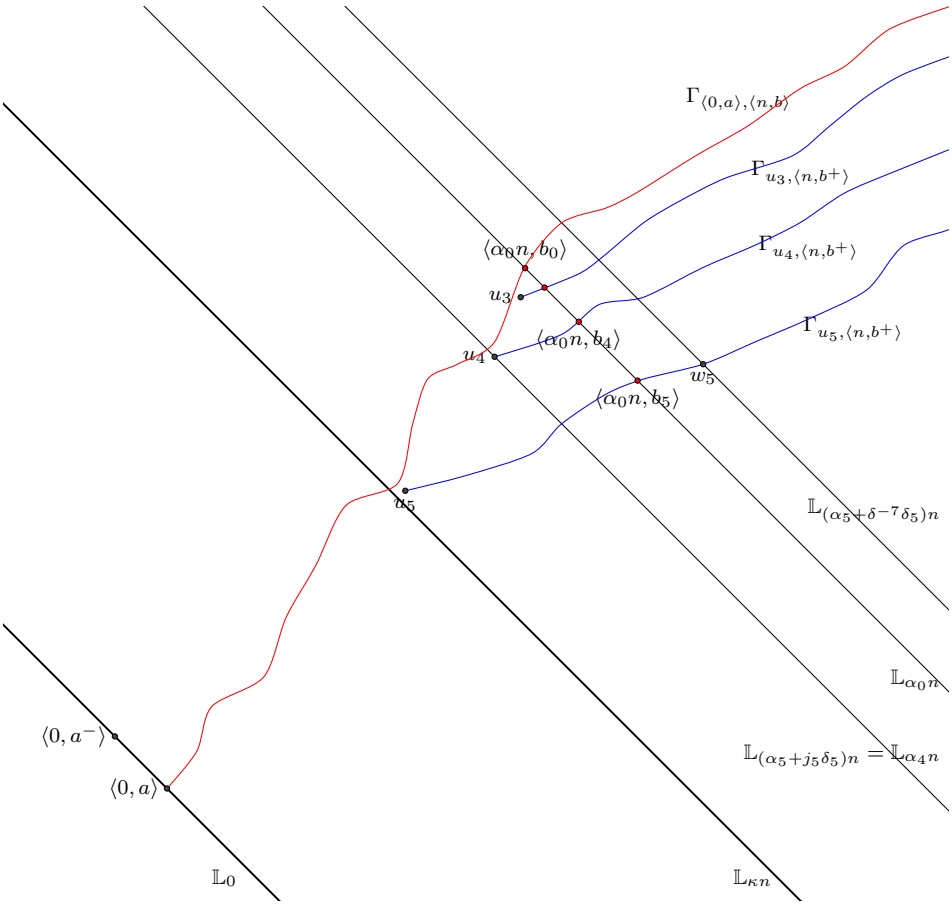


Fig. 17. An illustration of the geodesics $\Gamma_{u_i,\langle n,b^+\rangle}$ for $i = 5, 4, 3$. Their intersections with $\mathbb{L}_{\alpha_0 n}$ are separated by $c_0(\delta_1 n)^{2/3}$.

Multiple peaks event. We denote the intersections of $\mathbb{L}_{\alpha_0 n}$ with $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ and $\Gamma_{u_i,\langle n,b+\rangle}$ by $\langle \alpha_0 n, b_0\rangle$ and $\langle \alpha_0 n, b_i\rangle$, for $i = 1,2,3,4,5$. We next lower bound the differences between these b_i.

From (8.4) we have $b_1 - b_0 \geq c_0(\delta_1 n)^{2/3}$. We next show that $b_i - b_{i-1} \geq c_0(\delta_1 n)^{2/3}$ for each $i = 2,3,4,5$. By $\mathcal{T}^c$ and considering $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ above $\mathbb{L}_{\alpha_i n}$ and $\Gamma_{u_i,\langle n,b+\rangle}$, and using (8.3), we have

$$b_i - b_0 < \delta_0 n^{2/3} + 2\delta^{-1}(\alpha_0 - \alpha_i)^{2/3} n^{2/3} < \delta_0 n^{2/3} + 2\delta^{-1}(2\delta^{-7}\delta_i n)^{2/3}$$

for each $i = 1,2,3,4,5$, where the last inequality is by $\alpha_0 - \alpha_i \leq \delta^{-7}\sum_{i'=1}^{i}\delta_{i'} < 2\delta^{-7}\delta_i$. Similarly, by $\mathcal{T}^c$ and considering $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ and $\Gamma_{u_i,\langle n,b+\rangle}$ above $\mathbb{L}_{\alpha_{i-1}n}$, and using (8.4), we have

$$\begin{aligned} b_i - b_0 &\geq c_0(\delta_i n)^{2/3} - 2\delta_0 n^{2/3} - 2\delta^{-1}(\alpha_0 - \alpha_{i-1})^{2/3} n^{2/3} \\ &> c_0(\delta_i n)^{2/3} - 2\delta_0 n^{2/3} - 2\delta^{-1}(2\delta^{-7}\delta_{i-1} n)^{2/3} \end{aligned}$$

for each $i = 2,3,4,5$. Thus we get

$$b_i - b_{i-1} > c_0(\delta_i n)^{2/3} - 3\delta_0 n^{2/3} - 4\delta^{-1}(2\delta^{-7}\delta_{i-1} n)^{2/3} > c_0(\delta_1 n)^{2/3}$$

for each $i = 2,3,4,5$.

Moreover, since $|b_0| \leq 2\hat{h} n^{2/3}$ (by (8.1)), we have $-2\hat{h} n^{2/3} \leq b_0 < b_5 < (2\hat{h}+1)n^{2/3}$.

To obtain the multiple peaks event at these b_i, the remaining task is to bound the passage times through each $\langle \alpha_0 n, b_i\rangle$, from $\langle 0, a^-\rangle$ to $\langle n, b^-\rangle$, where a^-, b^- are the largest numbers satisfying $a^-, b^- \in \delta_0 n^{2/3}\mathbb{Z}$ and $a^- \leq a$, $b^- \leq b$. Recall that we denote $T^{\bullet}_{u,v} = T_{u,v} - \xi(v)$ for any vertices $u \leq v$. For each $i = 1,2,3,4,5$, denote $u_i' = \Gamma_{\langle 0,a\rangle,\langle n,b\rangle} \cap \mathbb{L}_{\alpha_i n}$; we have $|ad(u_i)|, |ad(u_i')| \leq 4\hat{h} n^{2/3}$ by (8.1) and (8.3). We then have

$$\begin{aligned} &T^{\bullet}_{\langle 0,a^-\rangle,\langle \alpha_0 n, b_i\rangle} + T_{\langle \alpha_0 n, b_i\rangle,\langle n,b^-\rangle} \\ &\qquad \geq T^{\bullet}_{\langle 0,a\rangle,\langle \alpha_0 n, b_i\rangle} + T_{\langle \alpha_0 n, b_i\rangle,\langle n,b^+\rangle} - 2\delta_0^{1/2-0.02} n^{1/3} \\ &\qquad \geq T^{\bullet}_{\langle 0,a\rangle,u_i'} + T^{\bullet}_{u_i',\langle \alpha_0 n, b_i\rangle} + T_{\langle \alpha_0 n, b_i\rangle,\langle n,b^+\rangle} - 2\delta_0^{1/2-0.02} n^{1/3} \\ &\qquad \geq T^{\bullet}_{\langle 0,a\rangle,u_i'} + T^{\bullet}_{u_i,\langle \alpha_0 n, b_i\rangle} + T_{\langle \alpha_0 n, b_i\rangle,\langle n,b^+\rangle} - 3\delta_0^{1/2-0.02} n^{1/3} \\ &\qquad = T^{\bullet}_{\langle 0,a\rangle,u_i'} + T_{u_i,\langle n,b^+\rangle} - 3\delta_0^{1/2-0.02} n^{1/3} \\ &\qquad \geq T^{\bullet}_{\langle 0,a\rangle,u_i'} + T_{u_i',\langle n,b\rangle} - 4\delta_0^{1/2-0.02} n^{1/3} \\ &\qquad = T_{\langle 0,a\rangle,\langle n,b\rangle} - 4\delta_0^{1/2-0.02} n^{1/3} \\ &\qquad \geq T_{\langle 0,a^-\rangle,\langle n,b^-\rangle} - 5\delta_0^{1/2-0.02} n^{1/3}, \end{aligned}$$

where the second inequality is by $T^{\bullet}_{\langle 0,a\rangle,\langle \alpha_0 n, b_i\rangle} \geq T^{\bullet}_{\langle 0,a\rangle,u_i'} + T^{\bullet}_{u_i',\langle \alpha_0 n, b_i\rangle}$ which follows from the definition of passage times, and all the other inequalities are due to $\mathcal{F}^c$. Note that if $\langle \alpha_0 n, b_0^-\rangle$ is the intersection of $\Gamma_{\langle 0,a^-\rangle,\langle n,b^-\rangle}$ with $\mathbb{L}_{\alpha_0 n}$, then $-2\hat{h} n^{2/3} \leq b_0^- \leq b_0$

by $\mathcal{T}_*^c$ and ordering of geodesics (Lemma 5.1). Thus $\mathcal{M}^{\langle 0,a^-\rangle,\langle n,b^-\rangle}_{c_0(\delta_1 n)^{2/3},\delta_0^{1/2-0.03}\delta_1^{-1/3},\alpha_0 n,4\hat{h}n^{2/3}}$ holds with $b_0^- < b_1 < b_2 < b_3 < b_4 < b_5$. Also note that $\alpha_0 \geq \alpha_5 > \kappa$, and

$$\alpha_0 \leq \alpha_5 + \sum_{i=1}^{5} \delta^{-7}\delta_i < \alpha_5 + 2\delta^{-7}\delta_5 \leq \kappa + \delta_5 + 2\delta^{-7}\delta_5 < 1-\kappa.$$

Thus we get a contradiction with $\mathcal{M}^c$. Now we conclude that there exists $u^* \in \mathfrak{P}_1$ with $d(u^*) < 4\kappa n$ such that there is $u \in \Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ with $d(u) = d(u^*)$ and $ad(u) \leq ad(u^*) \leq ad(u) + 2\delta_0 n^{2/3}$, and $\Gamma_{u^*,\langle n,b+\rangle}$ intersects $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ before $\mathbb{L}_{2\kappa n}$.

Final steps. Using the same arguments, we can find $v^* \in \mathfrak{P}_2$ with $d(v^*) > (2-4\kappa)n$ such that there is $v \in \Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ with $d(v) = d(v^*)$ and $ad(v) \leq ad(v^*) \leq ad(v) + 2\delta_0 n^{2/3}$, and Γ_{u^*,v^*} intersects $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ after $\mathbb{L}_{(1-2\kappa)n}$. We now consider the geodesics $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$, $\Gamma_{u^*,\langle n,b+\rangle}$, and Γ_{u^*,v^*} between $\mathbb{L}_{2\kappa n}$ and $\mathbb{L}_{(1-2\kappa)n}$. By ordering of geodesics (Lemma 5.1), either $\Gamma_{u^*,\langle n,b+\rangle}$ is sandwiched between $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ and Γ_{u^*,v^*}, or Γ_{u^*,v^*} is sandwiched between $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ and $\Gamma_{u^*,\langle n,b+\rangle}$. In the former case $\Gamma_{u^*,\langle n,b+\rangle}$ intersects $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ before $\mathbb{L}_{2\kappa n}$ and after $\mathbb{L}_{(1-2\kappa)n}$, so $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ is the same as $\Gamma_{u^*,\langle n,b+\rangle}$ between $\mathbb{L}_{2\kappa n}$ and $\mathbb{L}_{(1-2\kappa)n}$; in the latter case Γ_{u^*,v^*} intersects $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ before $\mathbb{L}_{2\kappa n}$ and after $\mathbb{L}_{(1-2\kappa)n}$, so $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ is the same as Γ_{u^*,v^*} between $\mathbb{L}_{2\kappa n}$ and $\mathbb{L}_{(1-2\kappa)n}$. Thus the conclusion follows. ∎

We can now finish the proof of Proposition 8.1 using Lemma 8.6.

Proof of Proposition 8.1. As stated above, we write the proof for $\rho = 1/2$ for simplicity of notation.

We now consider general n, i.e. not necessarily in $\mathcal{N}$. We let n' be the largest number such that $n' \leq n$ and $n' \in \mathcal{N}$. Then $n' \to \infty$ and $n'/n \to 1$ as $n \to \infty$. We define $\mathcal{E}'$ as $\mathcal{E}$ for n' instead of n, and $\mathfrak{P}_1'$, $\mathfrak{P}_2'$ as $\mathfrak{P}_1$, $\mathfrak{P}_2$ for n' instead of n.

By Theorem 7.3, as $n \to \infty$ we have

$$\max_{u\in\mathfrak{P}_1',\, v\in\mathfrak{P}_2'} |\mu_{u,v}(f) - \nu(f)| \to 0 \quad \text{in probability.}$$

Thus by Lemma 8.6, and since f is bounded on $\mathbb{R}^{[\![-s,s]\!]^2} \times \{0,1\}^{[\![-s,s]\!]^2}$, we have

$$\mathbb{P}\Big[\mathcal{E}', \max_{a,b\in\mathbb{Z},\, |a|,|b|<\hat{h}n'^{2/3}} |\mu_{\langle 0,a\rangle,\langle n',b\rangle}(f) - \nu(f)| > 10\kappa\|f\|_\infty\Big] \to 0.$$

Denote

$$\mathcal{T}' = \mathcal{T}^{\langle 0,\lceil hn^{2/3}\rceil\rangle,\langle n,\lceil hn^{2/3}\rceil\rangle}_{n-n',\hat{h}n'^{2/3}(n-n')^{-2/3}/2} \cup \mathcal{T}^{\langle 0,-\lceil hn^{2/3}\rceil\rangle,\langle n,-\lceil hn^{2/3}\rceil\rangle}_{n-n',\hat{h}n'^{2/3}(n-n')^{-2/3}/2}.$$

By ordering of geodesics (Lemma 5.1), $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ for $|a|, |b| < hn^{2/3}$ is sandwiched between $\Gamma_{\langle 0,-\lceil hn^{2/3}\rceil\rangle,\langle n,-\lceil hn^{2/3}\rceil\rangle}$ and $\Gamma_{\langle 0,\lceil hn^{2/3}\rceil\rangle,\langle n,\lceil hn^{2/3}\rceil\rangle}$; so assuming the complement of $\mathcal{T}'$, every $\Gamma_{\langle 0,a\rangle,\langle n,b\rangle}$ with $|a|, |b| < hn^{2/3}$ intersects $\mathbb{L}_{n'}$ at some vertex $\langle n', b'\rangle$ with

$$|b'| \leq \lceil hn^{2/3}\rceil + \hat{h}n'^{2/3}/2 < \hat{h}n'^{2/3},$$

where the second inequality is by taking $\hat{h}$ much larger than h. Thus

$$\mathbb{P}\Big[\mathcal{E}' \cap \mathcal{T}'^c, \max_{a,b\in\mathbb{Z},\,|a|,|b|<hn^{2/3}/2} |\mu_{\langle 0,a\rangle,\langle n,b\rangle}(f) - \nu(f)| > (10\kappa + 2(n-n')/n)\|f\|_\infty\Big]$$

tends to 0. Then since $(n-n')/n \to 0$ as $n \to \infty$, we have

$$\limsup_{n\to\infty} \mathbb{P}\Big[\max_{a,b\in\mathbb{Z},\,|a|,|b|<hn^{2/3}/2} |\mu_{\langle 0,a\rangle,\langle n,b\rangle}(f) - \nu(f)| > 11\kappa\|f\|_\infty\Big] \le \limsup_{n\to\infty} \mathbb{P}[\mathcal{T}'] + \mathbb{P}[\mathcal{E}'^c]. \tag{8.5}$$

By Lemma 8.3, $\limsup_{n\to\infty} \mathbb{P}[\mathcal{T}'] = 0$. Also, by the discussion of the events $\mathcal{T}$, $\mathcal{T}_*$, $\mathcal{F}$, $\mathcal{D}$, $\mathcal{H}$, $\mathcal{M}$ before Lemma 8.6, $\lim_{\hat{h}\to\infty} \limsup_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}[\mathcal{E}'^c] = 0$. Thus the left-hand side of (8.5) equals 0. Since κ can be arbitrary, the conclusion follows. ∎

In the next two subsections we prove Lemmas 8.2, 8.4, and 8.5.

8.1. Continuity of passage times and multiple peaks

In this subsection we prove Lemmas 8.2 and 8.5. For both we use the convergence of the point-to-line profile to the Airy_2 process, which is a stationary ergodic process minus a parabola. Such convergence in the sense of finite-dimensional distributions is from [17, 18]. Using the so-called slow decorrelation phenomenon, and proving equicontinuity of the point-to-line profile, it also follows that weak convergence holds in the topology of uniform convergence on compact sets [11, 37]. More precisely, let $\mathcal{A}_2$ denote the stationary Airy_2 process on $\mathbb{R}$, and let us define the stochastic process $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{L}(x) := \mathcal{A}_2(x) - x^2.$$

We quote the following result.

Theorem 8.7 ([11, Theorem 3.8]). *Consider the function*

$$\mathcal{L}_n : x \mapsto 2^{-4/3} n^{-1/3} (T_{\mathbf{0},\langle n, x(2n)^{2/3}\rangle} - 4n),$$

where we linearly interpolate between points in $(2n)^{-2/3}\mathbb{Z}$. *As* $n \to \infty$, *we have* $\mathcal{L}_n \to \mathcal{L}$ *weakly in the topology of uniform convergence on compact sets.*

We shall also use the following (quantitative) comparison between the Airy_2 process and a Brownian motion.

For $K \in \mathbb{R}, d > 0$, let $\mathcal{B}^{[K,K+d]}$ denote the law of a Brownian motion with diffusivity 2 on $[K, K+d]$, taking value 0 at K. Let $\mathcal{L}^{[K,K+d]}$ denote the random function on $[K, K+d]$ defined by

$$\mathcal{L}^{[K,K+d]}(x) := \mathcal{L}(x) - \mathcal{L}(K), \quad \forall x \in [K, K+d].$$

Let $C_*([K, K+d], \mathbb{R})$ denote the space of all real-valued continuous functions defined on $[K, K+d]$ which vanish at K, with the topology of uniform convergence. The following result can be obtained from [20].

Theorem 8.8 ([20, Theorem 1.1]). *There exists a universal constant $G > 0$ such that the following holds. For any fixed $M > 0$, there exists $a_0 = a_0(M)$ such that for all intervals $[K, K+d] \subset [-M, M]$ and for all measurable $A \subset C_*([K, K+d], \mathbb{R})$ with $0 < \mathcal{B}^{[K,K+d]}(A) = a \le a_0$,*

$$\mathbb{P}[\mathcal{L}^{[K,K+d]} \in A] \le a e^{GM(\log a^{-1})^{5/6}}.$$

Now we prove Lemma 8.2. We start with the following estimate on deviations when moving one endpoint.

Lemma 8.9. *There are constants $c, C > 0$ such that for any $h \in \mathbb{R}$, $0 < \theta < 1$, and $t > 1$, we have*

$$\mathbb{P}\Big[\max_{hn^{2/3} < b, b' < (h+1)n^{2/3},\, |b-b'| < \theta n^{2/3}} |T_{\mathbf{0},\langle n,b\rangle} - T_{\mathbf{0},\langle n,b'\rangle}| > t\theta^{1/2-0.01} n^{1/3} + C(|h|+1)\theta n^{1/3}\Big] < Ce^{-ct} \tag{8.6}$$

for n large enough (depending on h, θ, t).

Proof. For any continuous function $f : \mathbb{R} \to \mathbb{R}$, we let

$$\mathcal{M}(f) := \max_{2^{-2/3}h \le x, x' \le 2^{-2/3}(h+1),\, |x-x'| \le 2^{-2/3}\theta} |f(x) - f(x')|.$$

It is straightforward to check that $\mathcal{M}$ is a continuous functional on the space of all continuous real-valued functions on $\mathbb{R}$, with the topology of uniform convergence on compact sets.

By Theorem 8.8, $\mathcal{M}(\mathcal{L})$ has continuous distribution since this is the case when $\mathcal{L}$ is replaced by a Brownian motion. Thus by Theorem 8.7, as $n \to \infty$ we have $\mathbb{P}[\mathcal{M}(\mathcal{L}_n) > x] \to \mathbb{P}[\mathcal{M}(\mathcal{L}) > x]$ for any $x > 0$. We note that the left-hand side of (8.6) is bounded by

$$\mathbb{P}[\mathcal{M}(\mathcal{L}_n) > 2^{-4/3} t\theta^{1/2-0.01} + 2^{-4/3} C(|h|+1)\theta].$$

Thus as $n \to \infty$, the lim sup of the left-hand side of (8.6) is bounded by

$$\mathbb{P}[\mathcal{M}(\mathcal{L}) > 2^{-4/3} t\theta^{1/2-0.01} + 2^{-4/3} C(|h|+1)\theta].$$

We next show that this is bounded by Ce^{-ct}. When $C > 2$ it follows that $|x^2 - x'^2| < 2^{-4/3}C(|h|+1)\theta$ for all x, x' with $2^{-2/3}h \le x, x' \le 2^{-2/3}(h+1)$ and $|x - x'| \le 2^{-2/3}\theta$. Then by stationarity of $\mathcal{A}_2$, we can bound this probability by

$$\mathbb{P}\Big[\max_{0 \le x, x' \le 2^{-2/3},\, |x-x'| \le 2^{-2/3}\theta} |\mathcal{L}(x) - \mathcal{L}(x')| > 2^{-4/3} t\theta^{1/2-0.01}\Big],$$

where the event only relies on $\mathcal{L}^{[0,2^{-2/3}]}$. Using modulus of continuity for Brownian motions and Theorem 8.8, we can bound this by Ce^{-ct} as desired. ∎

We can now prove Lemma 8.2 by using Lemma 8.9 repeatedly.

Proof of Lemma 8.2. First, note that we have the following inequality for passage times:

$$T_{\langle 0,a\rangle,\langle n,b\rangle} - T_{\langle 0,a\rangle,\langle n,b'\rangle} \geq T_{\langle 0,a'\rangle,\langle n,b\rangle} - T_{\langle 0,a'\rangle,\langle n,b'\rangle}$$

for any $a \leq a', b \leq b'$. Indeed, the geodesics $\Gamma_{\langle 0,a\rangle,\langle n,b'\rangle}$ and $\Gamma_{\langle 0,a'\rangle,\langle n,b\rangle}$ must intersect. By switching the paths after their first intersection, we get two up-right paths, from $\langle 0,a\rangle$ to $\langle n,b\rangle$ and from $\langle 0,a'\rangle$ to $\langle n,b'\rangle$, and the sum of their passage times equals $T_{\langle 0,a'\rangle,\langle n,b\rangle} + T_{\langle 0,a\rangle,\langle n,b'\rangle}$. Thus we get the above inequality from the definition of last-passage times.

Using this inequality, for any $|a|, |a'|, |b|, |b'| < hn^{2/3}$ we have

$$\begin{aligned}|T_{\langle 0,a\rangle,\langle n,b\rangle} - T_{\langle 0,a\rangle,\langle n,b'\rangle}| \leq\ & |T_{\langle 0,-\lceil hn^{2/3}\rceil\rangle,\langle n,b\rangle} - T_{\langle 0,-\lceil hn^{2/3}\rceil\rangle,\langle n,b'\rangle}| \\ & \vee |T_{\langle 0,\lceil hn^{2/3}\rceil\rangle,\langle n,b\rangle} - T_{\langle 0,\lceil hn^{2/3}\rceil\rangle,\langle n,b'\rangle}|\end{aligned}$$

and

$$\begin{aligned}|T_{\langle 0,a\rangle,\langle n,b'\rangle} - T_{\langle 0,a'\rangle,\langle n,b'\rangle}| \leq\ & |T_{\langle 0,a\rangle,\langle n,-\lceil hn^{2/3}\rceil\rangle} - T_{\langle 0,a'\rangle,\langle n,-\lceil hn^{2/3}\rceil\rangle}| \\ & \vee |T_{\langle 0,a\rangle,\langle n,\lceil hn^{2/3}\rceil\rangle} - T_{\langle 0,a'\rangle,\langle n,\lceil hn^{2/3}\rceil\rangle}|.\end{aligned}$$

By adding up these two inequalities and using the triangle inequality, we have

$$\begin{aligned}|T_{\langle 0,a\rangle,\langle n,b\rangle} - T_{\langle 0,a'\rangle,\langle n,b'\rangle}| \leq\ & |T_{\langle 0,-\lceil hn^{2/3}\rceil\rangle,\langle n,b\rangle} - T_{\langle 0,-\lceil hn^{2/3}\rceil\rangle,\langle n,b'\rangle}| \\ & \vee |T_{\langle 0,\lceil hn^{2/3}\rceil\rangle,\langle n,b\rangle} - T_{\langle 0,\lceil hn^{2/3}\rceil\rangle,\langle n,b'\rangle}| \\ & + |T_{\langle 0,a\rangle,\langle n,-\lceil hn^{2/3}\rceil\rangle} - T_{\langle 0,a'\rangle,\langle n,-\lceil hn^{2/3}\rceil\rangle}| \\ & \vee |T_{\langle 0,a\rangle,\langle n,\lceil hn^{2/3}\rceil\rangle} - T_{\langle 0,a'\rangle,\langle n,\lceil hn^{2/3}\rceil\rangle}|.\end{aligned}$$

By symmetry, it now suffices to bound the probability of

$$\max_{\substack{|b|,|b'|<hn^{2/3}\\ |b-b'|<\theta n^{2/3}}} |T_{\langle 0,-\lceil hn^{2/3}\rceil\rangle,\langle n,b\rangle} - T_{\langle 0,-\lceil hn^{2/3}\rceil\rangle,\langle n,b'\rangle}| > \frac{1}{2}(t\theta^{1/2-0.01}n^{1/3} + Ch\theta n^{1/3}).$$

For this we split $\{\langle 0,b\rangle : |b| < hn^{2/3}\}$ into overlapping segments of length $n^{2/3}$, and apply Lemma 8.9 to each of them to get the desired bound. ∎

We next prove Lemma 8.5. Again, using Theorem 8.7 we reduce the point-to-line profiles to Airy_2 processes, and then by applying Theorem 8.8 we can just prove the result for Brownian motions.

Proof of Lemma 8.5. Denote by $\mathcal{L}_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ the process given by

$$\mathcal{L}_{\alpha,\beta}(x) := \alpha^{1/3}\mathcal{L}(\alpha^{-2/3}x) + (1-\alpha)^{1/3}\mathcal{L}'\big((1-\alpha)^{-2/3}(x - 2^{-2/3}\beta)\big),$$

where $\mathcal{L}'$ is an independent copy of $\mathcal{L}$. Denote

$$\mathcal{L}_{n,\alpha,\beta}(x) := 2^{-4/3}n^{-1/3}(T^{\bullet}_{\mathbf{0},\langle\lfloor\alpha n\rfloor,x(2n)^{2/3}\rangle} + T_{\langle\lfloor\alpha n\rfloor,x(2n)^{2/3}\rangle,\langle n,\lfloor\beta n^{2/3}\rfloor\rangle} - 4n),$$

where we linearly interpolate between points in $(2n)^{-2/3}\mathbb{Z}$. Using Theorem 8.7, we can

deduce that $\mathcal{L}_{n,\alpha,\beta} \to \mathcal{L}_{\alpha,\beta}$ as $n \to \infty$, weakly in the topology of uniform convergence on compact sets.

We let Ω be the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that there exist $-2^{1/3}h \le x_1 < x_2 < x_3 < x_4 < x_5 < x_6 \le 2^{1/3}h$ with $x_2 - x_1, x_3 - x_2, x_4 - x_3, x_5 - x_4, x_6 - x_5 \ge 2^{-2/3}\theta$ and $x_1 = \operatorname{argmax}_{[-2^{4/3}h, 2^{4/3}h]} f$ and

$$f(x_1) \le f(x_i) + 2^{-4/3} t\theta^{1/2}, \quad \forall i = 2, 3, 4, 5, 6.$$

It is straightforward to check that Ω is a closed set in the space of all continuous functions with the topology of uniform convergence on compact sets. It is also straightforward to check that $\mathcal{M}^{\mathbf{0},\langle n,\lfloor \beta n^{2/3}\rfloor\rangle}_{\theta n^{2/3},t,\lfloor \alpha n\rfloor,2hn^{2/3}}$ implies $\mathcal{L}_{n,\alpha,\beta} \in \Omega$. So by Theorem 8.7 we have

$$\limsup_{n\to\infty} \mathbb{P}[\mathcal{M}^{\mathbf{0},\langle n,\lfloor \beta n^{2/3}\rfloor\rangle}_{\theta n^{2/3},t,\lfloor \alpha n\rfloor,2hn^{2/3}}] \le \limsup_{n\to\infty} \mathbb{P}[\mathcal{L}_{n,\alpha,\beta} \in \Omega] \le \mathbb{P}[\mathcal{L}_{\alpha,\beta} \in \Omega].$$

We just need to bound the right-hand side. By Theorem 8.8, we can consider the probability of a (two-sided) Brownian motion (with diffusivity 4) belonging to Ω. By Lemma 8.10 below this probability is bounded by Ct^5 for $C > 0$ being a universal constant, so the conclusion follows. ∎

We finally bound the event on Brownian motions.

Lemma 8.10. *There exists a universal constant $C > 0$ such that for any $t, \theta > 0$, the following event holds with probability at most Ct^5. For $W : [-2, 2] \to \mathbb{R}$ being a two-sided Brownian motion, there are $-1 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < 1$ with $x_2 - x_1$, $x_3 - x_2, x_4 - x_3, x_5 - x_4, x_6 - x_5 > \theta$ such that $x_1 = \operatorname{argmax}_{[-2,2]} W$ and*

$$W(x_1) < W(x_i) + t\theta^{1/2}, \quad \forall i = 2, 3, 4, 5, 6.$$

Proof. Fix $T_1 \in [-1, 1]$, and let $\mathcal{E}$ be the event where $W(T_1) = \max_{[-2,2]} W$. For $i = 2, 3, 4, 5, 6$, let $T_i = \min\{x \ge T_{i-1} + \theta : W(x) \ge W(x_1) - t\theta^{1/2}\}$. It suffices to show that $\mathbb{P}[T_6 \le 1 \mid \mathcal{E}] < Ct^5$ for some universal constant $C > 0$. For $i = 2, 3, 4, 5, 6$, conditioned on $\mathcal{E}$ and the event $T_{i-1} \le 1$, and given the values of T_{i-1} and $W(T_{i-1}) - W(T_1)$, the process $x \mapsto W(T_{i-1} + x) - W(T_1)$ on $[0, 2 - T_{i-1}]$ has the same law as W', which is a Brownian motion on $[0, 2 - T_{i-1}]$ starting from $W'(0) = W(T_{i-1}) - W(T_1)$ and conditioned to stay below zero (for $i = 2$ this degenerates to a Brownian meander). Using the reflection principle we find that $\mathbb{P}[\max_{[\theta, 2-T_{i-1}]} W' \ge -t\theta^{1/2}] < C't$ for some universal constant $C' > 0$, so $\mathbb{P}[T_i \le 1 \mid \mathcal{E}, T_{i-1} \le 1] < C't$. Thus $\mathbb{P}[T_6 \le 1 \mid \mathcal{E}] < (C't)^5$, which implies the conclusion. ∎

8.2. Disjoint paths

In this subsection we prove Lemma 8.4. The idea is to show that for a path restricted to be close to another (deterministic) path for a while, its passage time is unlikely to be small (compared to that of a geodesic with the same endpoints). We then use the FKG inequality to move from a deterministic path to a geodesic.

Lemma 8.11. *For sufficiently small $c_0 > 0$, there is $c_1 > 0$ such that for $l \in \mathbb{N}$ large enough (depending on c_0) and any $r \in \mathbb{Z}$, we have*

$$\mathbb{E}\Big[\max_{a,b\in[\![0,c_0 l^{2/3}]\!]} T_{\langle 0,a\rangle,\langle l,r+b\rangle}\Big] < 4l - c_1 l^{1/3}.$$

Proof. Take $u = \langle -\lfloor c_0^{3/2} l\rfloor, 0\rangle$ and $v = \langle l + \lfloor c_0^{3/2} l\rfloor, r'\rangle$, where r' is the number in $\lfloor c_0 l^{2/3}\rfloor\mathbb{Z}$ with $r \le r' < r + \lfloor c_0 l^{2/3}\rfloor$. Note that

$$\begin{aligned}\mathbb{E}\Big[&\max_{a,b\in[\![0,c_0 l^{2/3}]\!]} T_{\langle 0,a\rangle,\langle l,r+b\rangle}\Big]\\ &\le \mathbb{E}[T_{u,v}] - \mathbb{E}\Big[\min_{a\in[\![0,c_0 l^{2/3}]\!]} T_{u,\langle -1,a\rangle}\Big] - \mathbb{E}\Big[\min_{b\in[\![0,c_0 l^{2/3}]\!]} T_{\langle l+1,r+b\rangle,v}\Big].\end{aligned}$$

By Proposition 5.3, we have

$$\mathbb{E}\Big[\min_{a\in[\![0,c_0 l^{2/3}]\!]} T_{u,\langle -1,a\rangle}\Big], \mathbb{E}\Big[\min_{b\in[\![0,c_0 l^{2/3}]\!]} T_{\langle l+1,r+b\rangle,v}\Big] \ge 4c_0^{3/2} l - Cc_0^{1/2} l^{1/3},$$

where $C > 0$ is a universal constant. We also claim that for l sufficiently large,

$$\mathbb{E}[T_{u,v}] \le 4(l + 2c_0^{3/2} l) - c_2 l^{1/3} \tag{8.7}$$

for some small universal constant $c_2 > 0$. Let $C' > 0$ be a large enough universal constant. When $l^{-2/3}|r| > C'$, (8.7) follows from (5.3). When $l^{-2/3}|r| \le C'$, for each l there are at most $3C'/c_0$ possible values r' can take. For each of them, by Theorem 8.7 the corresponding $T_{u,v}$ after rescaling converges (as $l \to \infty$) to one point of the Airy$_2$ process, whose law is given by the GUE Tracy–Widom distribution. Thus (8.7) (for l large enough) follows since the GUE Tracy–Widom distribution has negative expectation. By choosing c_0 such that $2Cc_0^{1/2} < c_2/2$ and letting $c_1 = c_2/2$, we complete the proof. ∎

For the next lemma, as before we denote $T^{\bullet}_{u,v} = T_{u,v} - \xi(v)$ for any vertices $u \le v$.

Lemma 8.12. *For $l, M \in \mathbb{N}$ and any $r_0, \dots, r_M \in \mathbb{Z}$, we have*

$$\mathbb{P}\Big[\max_{a_0,\dots,a_M\in[\![0,cl^{2/3}]\!]} \sum_{i=0}^{M-1} T^{\bullet}_{\langle il,r_i+a_i\rangle,\langle (i+1)l,r_{i+1}+a_{i+1}\rangle} \ge 4Ml - cMl^{1/3}\Big] < Ce^{-cM}$$

for some universal constants $c, C > 0$ when l is large enough.

Proof. In this proof we let $c, C > 0$ denote small and large enough universal constants, whose values can change from line to line.

Take $c_0, c_1 > 0$ such that Lemma 8.11 holds. For each $0 \le i \le M-1$ we denote

$$S_i = \max_{a_i,a_{i+1}\in[\![0,c_0 l^{2/3}]\!]} T^{\bullet}_{\langle il,r_i+a_i\rangle,\langle (i+1)l,r_{i+1}+a_{i+1}\rangle}.$$

Then (by Lemma 8.11) we have $\mathbb{E}[S_i] < 4l - c_1 l^{1/3}$ for each i when l is large enough.

Next we apply Proposition 5.3. When $|r_i - r_{i+1}| \le 0.9l$ we can apply it directly; and when $|r_i - r_{i+1}| > 0.9l$, the slope condition may not be satisfied, so we use the fact that $T^{\bullet}_{\langle il, r_i+a_i\rangle, \langle (i+1)l, r_{i+1}+a_{i+1}\rangle} < T^{\bullet}_{\langle il, r_i+a_i\rangle, \langle (i+1)l+\lfloor 0.1l\rfloor, r_{i+1}+a_{i+1}\rangle}$, and upper bound the latter using Proposition 5.3. In either case we conclude that $\mathbb{P}[S_i > 4l + xl^{1/3}] < Ce^{-cx}$ for any $x > 0$.

Note that S_i for each i are independent. Thus by a Bernstein type bound on the sum of independent random variables with exponential tails, we have

$$\mathbb{P}\left[\max_{a_0,\dots,a_M \in [\![0, c_0 l^{2/3}]\!]} \sum_{i=0}^{M-1} T^{\bullet}_{\langle il, r_i+a_i\rangle, \langle (i+1)l, r_{i+1}+a_{i+1}\rangle} \ge 4Ml - \frac{c_1}{2} Ml^{1/3}\right] < Ce^{-cM},$$

and the conclusion follows. ∎

Proof of Lemma 8.4. Take any up-right path Γ from $\mathbf{0}$ to $\langle n, b\rangle$. Denote by $\mathcal{D}_\Gamma$ the following event: there exists an up-right path γ from $\mathbb{L}_m$ to $\mathbb{L}_{m+Ml}$ such that

- γ is disjoint from Γ,
- the passage time of γ (i.e. $T(\gamma)$) is at least $4Ml - c_0 Ml^{1/3}$,
- for each $i = 0, 1, \dots, M$, $|ad(\Gamma \cap \mathbb{L}_{m+il}) - ad(\gamma \cap \mathbb{L}_{m+il})| < 2c_0 l^{2/3}$.

Here $c_0 > 0$ is as in the definition of $\mathcal{D}^{\mathbf{0},\langle n,b\rangle}_{M,l,m}$. Now we consider the event $\Gamma_{\mathbf{0},\langle n,b\rangle} = \Gamma$. Under this event, $\mathcal{D}^{\mathbf{0},\langle n,b\rangle}_{M,l,m} = \mathcal{D}_\Gamma$. Also, $\Gamma_{u,v} = \Gamma$ is a negative event of the field on $\mathbb{Z}^2 \setminus \Gamma$, while $\mathcal{D}_\Gamma$ is determined by the field on $\mathbb{Z}^2 \setminus \Gamma$, and is a positive event of the field on $\mathbb{Z}^2 \setminus \Gamma$. By the FKG inequality we have

$$\mathbb{P}[\mathcal{D}^{\mathbf{0},\langle n,b\rangle}_{M,l,m} \mid \Gamma_{u,v} = \Gamma] = \mathbb{P}[\mathcal{D}_\Gamma \mid \Gamma_{u,v} = \Gamma] \le \mathbb{P}[\mathcal{D}_\Gamma].$$

By Lemma 8.12, $\mathbb{P}[\mathcal{D}_\Gamma] < Ce^{-cM}$ when $c_0 < c$ and $l > C$, for $c, C > 0$ being universal constants. By averaging over all Γ we get the conclusion. ∎

9. Convergence of one point distribution

In this section we prove Theorems 1.3 and 1.4. The general idea is to show that the law of the environment around a specific vertex in the geodesic is close to that of nearby vertices along the geodesic; and this is achieved by a coalescing argument. Then we use Proposition 8.1 to argue that a certain time average (of environments along the geodesic) is close to the stationary measure ν.

To prove Theorem 1.3, a key step would be to bound the total variation distance between $(\xi\{\Gamma_{\mathbf{0}}[i]\}, \Gamma_{\mathbf{0}} - \Gamma_{\mathbf{0}}[i])$ and $(\xi\{\Gamma_{\mathbf{0}}[i-r]\}, \Gamma_{\mathbf{0}} - \Gamma_{\mathbf{0}}[i-r])$ in a finite box, for any i large and r much smaller than i. For this, we use translation invariance, and consider the environment around $\Gamma_{v[r]}[i-r]$ instead of $\Gamma_{\mathbf{0}}[i-r]$, where

$$v[r] = \begin{cases} \langle \lfloor r/2\rfloor, 0\rangle, & r \text{ is even,} \\ \langle \lfloor r/2\rfloor, 0\rangle + (1,0), & r \text{ is odd.} \end{cases}$$

We define $v[r]$ this way so that always $d(v[r]) = r$. We show that with high probability $\Gamma_{v[r]}[i-r] = \Gamma_{\mathbf{0}}[i]$, and in a finite box around this vertex the paths $\Gamma_{v[r]}$ and $\Gamma_{\mathbf{0}}$ are the same. Towards this we need the following estimate on coalescence of geodesics, which directly follows from Proposition 5.5 and Lemma 5.6.

Lemma 9.1. *There is a constant $C > 0$ such that for any $r \in \mathbb{N}$ and $k > 2$, we have* $\mathbb{P}[\Gamma_{\mathbf{0}} \cap \mathbb{L}_{\lfloor rk \rfloor} \neq \Gamma_{v[r]} \cap \mathbb{L}_{\lfloor rk \rfloor}] < C \log(k) k^{-2/3}$.

Proof. Denote the intersections of $\Gamma_{\mathbf{0}}$ and $\Gamma_{v[r]}$ with $\mathbb{L}_r$ as $\langle r, b_r \rangle$ and $\langle r, b'_r \rangle$, respectively. By Lemma 5.6 and Proposition 5.5, there is a constant $C_0 > 0$ such that

$$\mathbb{P}[|b_r|, |b'_r| \leq C_0 \log(k) r^{2/3}] > 1 - C_0 k^{-1}$$

and

$$\mathbb{P}[\Gamma_{\langle r, -\lfloor C_0 \log(k) r^{2/3} \rfloor - 1 \rangle} \cap \mathbb{L}_{\lfloor rk \rfloor} \neq \Gamma_{\langle r, \lfloor C_0 \log(k) r^{2/3} \rfloor + 1 \rangle} \cap \mathbb{L}_{\lfloor rk \rfloor}] < C_0^2 \log(k)(k-1)^{-2/3}.$$

Thus the conclusion follows by ordering of geodesics (Lemma 5.1). ∎

Proof of Theorem 1.3. Take any $s \in \mathbb{N}$ and any continuous $f : \mathbb{R}^{[\![-s,s]\!]^2} \times \{0,1\}^{[\![-s,s]\!]^2} \to [0,1]$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2}$. We need to show that

$$\lim_{i \to \infty} \mathbb{E}\big[f(\xi\{\Gamma_{\mathbf{0}}[i]\}, \Gamma_{\mathbf{0}} - \Gamma_{\mathbf{0}}[i])\big] = \nu(f).$$

For $i, r \in \mathbb{N}$ and $k > 2$ with $i - 2s > 2rk$, by Lemma 9.1, with probability at least $1 - C \log(k) k^{-2/3}$ we have $\Gamma_{\mathbf{0}}[j] = \Gamma_{v[r]}[j-r]$ for any $j \geq i - 2s$; thus the pairs $(\xi\{\Gamma_{\mathbf{0}}[i]\}, \Gamma_{\mathbf{0}} - \Gamma_{\mathbf{0}}[i])$ and $(\xi\{\Gamma_{v[r]}[i-r]\}, \Gamma_{v[r]} - \Gamma_{v[r]}[i-r])$ are the same in $[\![-s,s]\!]^2$. Since $(\xi\{\Gamma_{v[r]}[i-r]\}, \Gamma_{v[r]} - \Gamma_{v[r]}[i-r])$ have the same joint distribution as $(\xi\{\Gamma_{\mathbf{0}}[i-r]\}, \Gamma_{\mathbf{0}} - \Gamma_{\mathbf{0}}[i-r])$, we must have

$$\big|\mathbb{E}\big[f(\xi\{\Gamma_{\mathbf{0}}[i]\}, \Gamma_{\mathbf{0}} - \Gamma_{\mathbf{0}}[i])\big] - \mathbb{E}\big[f(\xi\{\Gamma_{\mathbf{0}}[i-r], \Gamma_{\mathbf{0}} - \Gamma_{\mathbf{0}}[i-r]\})\big]\big| \leq C \log(k) k^{-2/3}.$$

By averaging over $r \in [\![0, i/4k]\!]$, we have (when $i > 4s$)

$$\big|\mathbb{E}\big[f(\xi\{\Gamma_{\mathbf{0}}[i]\}, \Gamma_{\mathbf{0}} - \Gamma_{\mathbf{0}}[i])\big] - \mathbb{E}[\mu_{\Gamma_{\mathbf{0}}[i - \lfloor i/4k \rfloor], \Gamma_{\mathbf{0}}[i]}(f)]\big| \leq C \log(k) k^{-2/3}.$$

By Lemma 5.6 and Proposition 8.1, for any fixed $k > 0$, $\mu_{\Gamma_{\mathbf{0}}[i - \lfloor i/4k \rfloor], \Gamma_{\mathbf{0}}[i]}(f) \to \nu(f)$ in probability as $i \to \infty$. Thus

$$\limsup_{i \to \infty} \big|\mathbb{E}\big[f(\xi\{\Gamma_{\mathbf{0}}[i]\}, \Gamma_{\mathbf{0}} - \Gamma_{\mathbf{0}}[i])\big] - \nu(f)\big| \leq C \log(k) k^{-2/3}.$$

Since k can be arbitrarily large, the conclusion follows. ∎

The proof of Theorem 1.4 is similar. Again we need the following estimate on coalescence of geodesics, which follows from Corollary 5.9 and Proposition 5.4. Recall that we denote $\mathbf{n} = \mathbf{n}^{\rho} = \langle n, 0 \rangle = (\lfloor \frac{2(1-\rho)^2 n}{\rho^2 + (1-\rho)^2} \rfloor, \lceil \frac{2\rho^2 n}{\rho^2 + (1-\rho)^2} \rceil)$ for any $n \in \mathbb{Z}$.

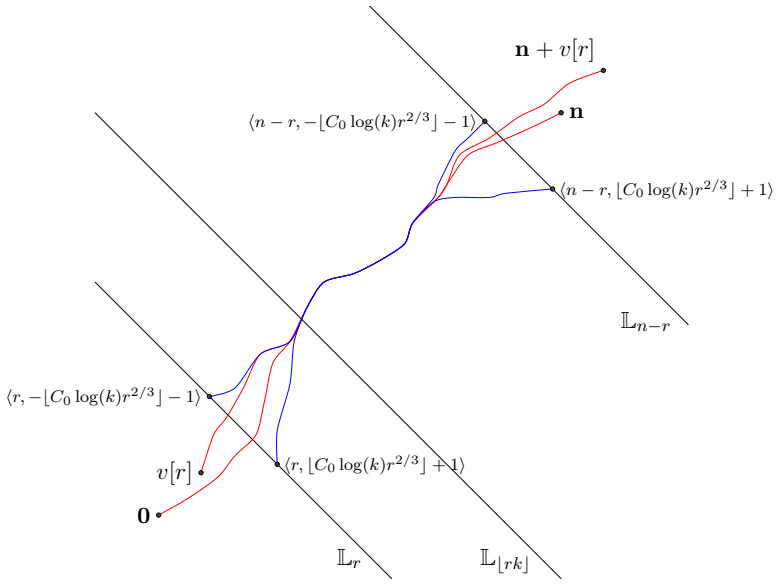


Fig. 18. An illustration of the proof of Lemma 9.2. The geodesics $\Gamma_{\mathbf{0},\mathbf{n}}$ and $\Gamma_{v[r],\mathbf{n}+v[r]}$ are sandwiched between $\Gamma_{\langle r,-\lfloor C_0 \log(k) r^{2/3}\rfloor-1\rangle,\langle n-r,-\lfloor C_0 \log(k) r^{2/3}\rfloor-1\rangle}$ and $\Gamma_{\langle r,\lfloor C_0 \log(k) r^{2/3}\rfloor+1\rangle,\langle n-r,\lfloor C_0 \log(k) r^{2/3}\rfloor+1\rangle}$.

Lemma 9.2. *There is a constant $C > 0$ such that for any $r, n \in \mathbb{N}$ and $k > 2$ with $n \geq 2rk$, we have*

$$\mathbb{P}[\Gamma_{\mathbf{0},\mathbf{n}} \cap \mathbb{L}_{\lfloor rk \rfloor} \neq \Gamma_{v[r],\mathbf{n}+v[r]} \cap \mathbb{L}_{\lfloor rk \rfloor}] < C \log(k) k^{-2/3},$$
$$\mathbb{P}[\Gamma_{\mathbf{0},\mathbf{n}} \cap \mathbb{L}_{n-\lfloor rk \rfloor} \neq \Gamma_{v[r],\mathbf{n}+v[r]} \cap \mathbb{L}_{n-\lfloor rk \rfloor}] < C \log(k) k^{-2/3}.$$

Proof. Since $n \geq 2rk$, we just show the first inequality; the other follows by symmetry.

Denote the intersections of $\Gamma_{\mathbf{0},\mathbf{n}}$ and $\Gamma_{v[r],\mathbf{n}+v[r]}$ with $\mathbb{L}_r$ as $\langle r, b_-\rangle$ and $\langle r, b'_-\rangle$, respectively; and the intersections of $\Gamma_{\mathbf{0},\mathbf{n}}$ and $\Gamma_{v[r],\mathbf{n}+v[r]}$ with $\mathbb{L}_{n-r}$ as $\langle n-r, b_+\rangle$ and $\langle n-r, b'_+\rangle$, respectively. There is a constant $C_0 > 0$ such that

$$\mathbb{P}[|b_-|, |b'_-| \leq C_0 \log(k) r^{2/3}], \mathbb{P}[|b_+|, |b'_+| \leq C_0 \log(k) r^{2/3}] > 1 - C_0 k^{-1}$$

by Corollary 5.9; and

$$\begin{aligned}
&\mathbb{P}[\Gamma_{\langle r,-\lfloor C_0 \log(k) r^{2/3}\rfloor-1\rangle,\langle n-r,-\lfloor C_0 \log(k) r^{2/3}\rfloor-1\rangle} \cap \mathbb{L}_{\lfloor rk \rfloor} \\
&\qquad \neq \Gamma_{\langle r,\lfloor C_0 \log(k) r^{2/3}\rfloor+1\rangle,\langle n-r,\lfloor C_0 \log(k) r^{2/3}\rfloor+1\rangle} \cap \mathbb{L}_{\lfloor rk \rfloor}] \\
&\leq \mathbb{P}\big[\Gamma_{\langle r,-\lfloor C_0 \log(k) r^{2/3}\rfloor-1\rangle,\langle n-r,-\lfloor C_0 \log(k) r^{2/3}\rfloor-1\rangle} \cap \mathbb{L}_{\lfloor rk \rfloor} \\
&\qquad \neq \Gamma_{\langle r,-\lfloor C_0 \log(k) r^{2/3}\rfloor-1\rangle,\langle n-r,\lfloor C_0 \log(k) r^{2/3}\rfloor+1\rangle} \cap \mathbb{L}_{\lfloor rk \rfloor}\big] \\
&\quad + \mathbb{P}\big[\Gamma_{\langle r,-\lfloor C_0 \log(k) r^{2/3}\rfloor-1\rangle,\langle n-r,\lfloor C_0 \log(k) r^{2/3}\rfloor+1\rangle} \cap \mathbb{L}_{\lfloor rk \rfloor} \\
&\qquad \neq \Gamma_{\langle r,\lfloor C_0 \log(k) r^{2/3}\rfloor+1\rangle,\langle n-r,\lfloor C_0 \log(k) r^{2/3}\rfloor+1\rangle} \cap \mathbb{L}_{\lfloor rk \rfloor}\big] \\
&< C_0^2 \log(k)(k-1)^{-2/3},
\end{aligned}$$

where the last inequality is by Proposition 5.4. Then the conclusion follows by ordering of geodesics (Lemma 5.1). See Figure 18 for an illustration. ∎

Proof of Theorem 1.4. Take any $s \in \mathbb{N}$ and any continuous $f : \mathbb{R}^{[\![-s,s]\!]^2} \times \{0,1\}^{[\![-s,s]\!]^2} \to [0,1]$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2}$. We need to show that

$$\lim_{n\to\infty} \mathbb{E}\big[f(\xi\{\Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor]\}, \Gamma_{\mathbf{0},\mathbf{n}} - \Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor])\big] = \nu(f).$$

Without loss of generality we assume that $\alpha \le 1$. For $n, r \in \mathbb{N}$ and $k > 2$ with $\alpha n - 2s > 2rk$ and $\alpha n + 2s < 2n - 2rk$, by Lemma 9.2 we have

$$\mathbb{P}\big[\Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor + j] = \Gamma_{v[r],\mathbf{n}+v[r]}[\lfloor \alpha n\rfloor - r + j]\},\ \forall j \in [\![-2s, 2s]\!]\big] \ge 1 - C\log(k)k^{-2/3}.$$

By translation invariance, $(\xi\{\Gamma_{v[r],\mathbf{n}+v[r]}[\lfloor \alpha n\rfloor - r]\}, \Gamma_{v[r],\mathbf{n}+v[r]} - \Gamma_{v[r],\mathbf{n}+v[r]}[\lfloor \alpha n\rfloor - r])$ have the same joint distribution as $(\xi\{\Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor - r]\}, \Gamma_{\mathbf{0},\mathbf{n}} - \Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor - r])$, so

$$\begin{aligned}\big|\mathbb{E}\big[f(\xi\{\Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor]\}, \Gamma_{\mathbf{0},\mathbf{n}} - \Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor])\big] \\ - \mathbb{E}\big[f(\xi\{\Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor - r]\}, \Gamma_{\mathbf{0},\mathbf{n}} - \Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor - r])\big]\big| \le C\log(k)k^{-2/3}.\end{aligned}$$

By averaging over $r \in [\![0, \alpha n/(4k)]\!]$, we have (when $\alpha n > 4s$)

$$\begin{aligned}\big|\mathbb{E}\big[f(\xi\{\Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor]\}, \Gamma_{\mathbf{0},\mathbf{n}} - \Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor])\big] - \mathbb{E}[\mu_{\Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor - \lfloor \alpha n/(4k)\rfloor], \Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor]}(f)]\big| \\ \le C\log(k)k^{-2/3}.\end{aligned}$$

By Corollary 5.9 and Proposition 8.1, for fixed k we have

$$\mu_{\Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor - \lfloor \alpha n/(4k)\rfloor], \Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor]}(f) \to \nu(f) \quad \text{in probability as } n \to \infty.$$

Thus

$$\limsup_{i\to\infty} \big|\mathbb{E}\big[f(\xi\{\Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor]\}, \Gamma_{\mathbf{0},\mathbf{n}} - \Gamma_{\mathbf{0},\mathbf{n}}[\lfloor \alpha n\rfloor])\big] - \nu(f)\big| \le C\log(k)k^{-2/3}.$$

Then the conclusion follows since k can be arbitrarily large. ∎

10. Exponential concentration via counting argument

Using a covering argument, we can prove the following exponential concentration of the empirical environment, for both finite and semi-infinite geodesics.

Proposition 10.1. *For any $s \in \mathbb{N}$, any bounded continuous $f : \mathbb{R}^{[\![-s,s]\!]^2} \times \{0,1\}^{[\![-s,s]\!]^2} \to \mathbb{R}$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2}$, and any $\epsilon > 0$, we have*

$$\mathbb{P}[|\mu_{\mathbf{0};r}(f) - \nu(f)| > \epsilon] < Ce^{-cr}$$

for r large enough, and $c, C > 0$ depending on s, f, ϵ.

Proposition 10.2. *Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of integers with $\lim_{n\to\infty} n^{-2/3}|b_n| < \infty$. Then for any $s \in \mathbb{N}$, any bounded continuous $f : \mathbb{R}^{[\![-s,s]\!]^2} \times \{0,1\}^{[\![-s,s]\!]^2} \to \mathbb{R}$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2}$, and any $\epsilon > 0$, we have*

$$\mathbb{P}[|\mu_{\mathbf{0},\langle n,b_n\rangle}(f) - \nu(f)| > \epsilon] < Ce^{-cn}$$

for n large enough, and $c, C > 0$ depending on s, f, ϵ.

From Proposition 10.1 we can deduce Theorem 1.2.

Proof of Theorem 1.2. By Proposition 10.1, for any bounded continuous $f : \mathbb{R}^{[\![-s,s]\!]^2} \times \{0,1\}^{[\![-s,s]\!]^2} \to \mathbb{R}$ (regarded as a function on $\mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2}$) and $\epsilon > 0$, we have

$$\sum_{r\in\mathbb{N}} \mathbb{P}[|\mu_{\mathbf{0};r}(f) - \nu(f)| > \epsilon] < \infty.$$

So almost surely, there exists some (random) r_0 such that $|\mu_{\mathbf{0};r}(f) - \nu(f)| \le \epsilon$ for any $r > r_0$. Thus $\mu_{\mathbf{0};r}(f) \to \nu(f)$ almost surely. The conclusion follows by taking all $s \in \mathbb{N}$, and f over a countable dense subset of the space of continuous and compactly supported functions on $\mathbb{R}^{[\![-s,s]\!]^2} \times \{0,1\}^{[\![-s,s]\!]^2}$ with the uniform convergence topology. ∎

Using the same arguments we can deduce Theorem 1.1 from Proposition 10.2. We omit the details.

To prove the exponential concentration bounds (Propositions 10.1 and 10.2), we cover the geodesics with short finite ones, and use Proposition 8.1.

We take $m \in \mathbb{N}$ such that $m^{2/3} \in \mathbb{Z}$. For each $i, j \in \mathbb{Z}$ we denote by $L_{i,j}$ the segment joining $\langle im, (2j-1)m^{2/3}\rangle$ and $\langle im, (2j+1)m^{2/3}\rangle$. For each integer sequence $j_0, j_1, \dots, j_k$, we let $P_{j_0,\dots,j_k}$ be the collection of paths from L_{0,j_0} to L_{k,j_k}, intersecting each L_{i,j_i}, $0 \le i \le k$. For any $k \in \mathbb{N}$ and $D > 0$, we define $P_{k,D}$ to be the union of all $P_{j_0,j_1,\dots,j_k}$ such that $j_0 = 0$ and $\sum_{i=1}^k (j_i - j_{i-1})^2 > Dk$. In words, $P_{k,D}$ contains all paths from $L_{0,0}$ to $\mathbb{L}_{km}$ with 'quadratic variation' $> Dk$. We next upper bound the passage times of these paths.

Lemma 10.3. *There exists $c_0 > 0$ such that when m, k, D are large enough,*

$$\mathbb{P}\Big[\exists\gamma \in P_{k,D}, T(\gamma) > \frac{2km}{(1-\rho)^2+\rho^2} - (b_+ - b_-)(\rho^{-1} - (1-\rho)^{-1}) - c_0 Dk m^{1/3}\Big] < e^{-c_0 k},$$

where $b_-, b_+ \in \mathbb{Z}$ are such that $\langle 0, b_-\rangle, \langle km, b_+\rangle$ are the intersections of γ with $\mathbb{L}_0, \mathbb{L}_{km}$, respectively.

Proof. First, there exist $c_1, C_1 > 0$ such that for m large enough and any $j \in \mathbb{Z}$, $x > 0$,

$$\mathbb{E}\Big[\max_{\substack{\langle 0,b\rangle\in L_{0,0}\\ \langle m,b'\rangle\in L_{1,j}}} \big(T_{\langle 0,b\rangle,\langle m,b'\rangle} + (b'-b)(\rho^{-1} - (1-\rho)^{-1})\big)\Big] < \frac{2m}{(1-\rho)^2+\rho^2} + (C_1 - c_1 j^2)m^{1/3}$$

and

$$\mathbb{P}\Bigg[\max_{\substack{\langle 0,b\rangle\in L_{0,0}\\ \langle m,b'\rangle\in L_{1,j}}}\big(T_{\langle 0,b\rangle,\langle m,b'\rangle}+(b'-b)(\rho^{-1}-(1-\rho)^{-1})\big) < \frac{2m}{(1-\rho)^2+\rho^2}+(x-c_1j^2)m^{1/3}\Bigg] < C_1e^{-c_1x}.$$

When $|j|<(\rho^2\wedge(1-\rho)^2)m^{1/3}$ these inequalities follow from Proposition 5.3 and (5.3), and fundamental computations. When $|j|\geq(\rho^2\wedge(1-\rho)^2)m^{1/3}$ the inequalities can be obtained by applying (5.1) in Theorem 5.2 to each $T_{u,v}$ with $u\in L_{0,0}$ and $v\in L_{1,j}$ and taking a union bound.

Note that

$$\max_{\gamma\in P_{j_0,j_1,\dots,j_k}} T(\gamma)\leq\sum_{i=1}^{k-1}\max_{\substack{u\in L_{i-1,j_{i-1}}\\ v\in L_{i,j_i}}} T^{\bullet}_{u,v}+\max_{\substack{u\in L_{k-1,j_{k-1}}\\ v\in L_{k,j_k}}} T_{u,v}.$$

Here $T^{\bullet}_{u,v}=T_{u,v}-\xi(v)$ for any $u\leq v\in\mathbb{Z}^2$. Then by a Bernstein type estimate for independent random variables with exponential tails, we have

$$\mathbb{P}\Bigg[\max_{\gamma\in P_{j_0,j_1,\dots,j_k}} T(\gamma)+(b_+-b_-)(\rho^{-1}-(1-\rho)^{-1})>\frac{2km}{(1-\rho)^2+\rho^2}-\frac{c_1}{2}Dkm^{1/3}\Bigg] < C_2e^{-c_2\sum_{i=1}^k(j_i-j_{i-1})^2}$$

for any D large (depending on c_1,C_1) and any integer sequence $j_0,\dots,j_k$ with $j_0=0$, $\sum_{i=1}^k(j_i-j_{i-1})^2>Dk$. Here $c_2,C_2>0$ are constants, and $\langle 0,b_-\rangle,\langle km,b_+\rangle$ are the intersections of γ with $\mathbb{L}_0,\mathbb{L}_{km}$. Summing over all such sequences $j_0,j_1,\dots,j_k$, the right-hand side is bounded by

$$C_2e^{-c_2Dk/2}\Big(\sum_{j\in\mathbb{Z}}e^{-c_2j^2/2}\Big)^k.$$

By taking D so large that $e^{c_2D/4}>\sum_{j\in\mathbb{Z}}e^{-c_2j^2/2}$, we get the conclusion. ∎

We next prove Proposition 10.1. The general idea is to upper bound the 'quadratic variation' of the first r steps of $\Gamma_{\mathbf{0}}$, and use Proposition 8.1 to show that the empirical environment between each $\mathbb{L}_{im}$ and $\mathbb{L}_{(i+1)m}$ is close to ν, and use independence to deduce exponential concentration.

Proof of Proposition 10.1. For any vertices $u\leq v$, denote

$$\mu^{\bullet}_{u,v}:=\frac{1}{|\Gamma_{u,v}|-1}\sum_{w\in\Gamma_{u,v},\,w\neq v}\delta_{(\xi\{w\},\Gamma_{u,v}-w)},$$

i.e. it is the empirical environment along $\Gamma_{u,v}$, excluding the last vertex v. Without loss of generality we assume that $0\leq f\leq 1$, and ϵ is small enough (depending on s and f).

We first consider paths with small 'quadratic variation'. Take $D > 0$ and $m \in \mathbb{N}$ such that $m^{2/3} \in \mathbb{Z}$, and let them be large enough as required by Lemma 10.3. We also choose m large enough such that

$$\mathbb{P}\Big[\max_{|a|,|b|<\epsilon^{-2}m^{2/3}} |\mu^{\bullet}_{\langle 0,a\rangle,\langle m,b\rangle}(f) - \nu(f)| > \epsilon^2\Big] < \varepsilon, \tag{10.1}$$

by Proposition 8.1. Here ε is a small number depending on D, ϵ and to be determined. Take any $k \in \mathbb{N}$ (also large enough as required by Lemma 10.3), and a sequence $j_0, \dots, j_k$ such that $j_0 = 0$ and $\sum_{i=1}^{k}(j_i - j_{i-1})^2 \le Dk$. We let $I' \subset \{1, \dots, k\}$ be the collection of indices such that $|j_i - j_{i-1}| < \epsilon^{-2}/2 - 1$ for each $i \in I'$. Then $|I'| > (1 - \epsilon/2)k$ when ϵ is small enough (depending on D). Next we let $I \subset I'$ be such that for each $i \in I$,

$$\max_{u\in L_{i-1,j_{i-1}},\, v\in L_{i,j_i}} |\mu^{\bullet}_{u,v}(f) - \nu(f)| \le \epsilon^2.$$

By (10.1) we have $\mathbb{P}[i \in I] > 1 - \varepsilon$ for each $i \in I'$. Also note that $i_1 \in I$ and $i_2 \in I$ are independent for any $i_1, i_2 \in I'$ with $i_1 - i_2 \ge 2$. Then by a Chernoff bound and taking ε small enough (depending on D, ϵ), we can make $\mathbb{P}[|I'| - |I| > \epsilon^2 k] < (D+1)^{-2k}$.

Let γ be the path consisting of the first $2km + 1$ vertices of $\Gamma_{\mathbf{0}}$, i.e. γ is the part of $\Gamma_{\mathbf{0}}$ on and between $\mathbb{L}_0$ and $\mathbb{L}_{km}$. Given that $\gamma \in P_{j_0,\dots,j_k}$ and $|I'| - |I| \le \epsilon^2 k$, for any $r \in [\![2km, 2(k+1)m]\!]$ we must have $|\mu_{\mathbf{0};r}(f) - \nu(f)| \le \epsilon/2 + 2\epsilon^2 + 1/(k+1)$. So when $k > \epsilon^{-2}$ and ϵ is large enough, we have

$$\mathbb{P}[\gamma \in P_{j_0,\dots,j_k},\ |\mu_{\mathbf{0};r}(f) - \nu(f)| > \epsilon] < (D+1)^{-2k}.$$

Thus by summing over all sequences $j_0, \dots, j_k$ with $j_0 = 0$, $\sum_{i=1}^{k}(j_i - j_{i-1})^2 \le Dk$, we have

$$\mathbb{P}[\gamma \notin P_{k,D},\ |\mu_{\mathbf{0};r}(f) - \nu(f)| > \epsilon] < \binom{\lfloor Dk\rfloor + k - 1}{k-1}(D+1)^{-2k} < e^{-ck}$$

for some $c > 0$ depending on D.

Now it remains to bound $\mathbb{P}[\gamma \in P_{k,D}]$. By Lemma 10.3, we have

$$\begin{aligned}&\mathbb{P}[\gamma \in P_{k,D}]\\ &< e^{-c_0 k} + \mathbb{P}\Big[T(\gamma) \le \frac{2km}{(1-\rho)^2 + \rho^2} - b_+(\rho^{-1} - (1-\rho)^{-1}) - c_0 Dkm^{1/3}\Big],\end{aligned} \tag{10.2}$$

where $\langle km, b_+\rangle$ is the intersection of $\Gamma_{\mathbf{0}}$ with $\mathbb{L}_{km}$, and recall that $c_0 > 0$ is a constant independent of m, k, D. When the event on the right-hand side of (10.2) happens, (at least) one of the following must happen:

- $|b_+| > km^{2/3}$,
- $\max_{|b|\le km^{2/3}}(\mathbf{B}(\langle km, b\rangle, \langle km, 0\rangle) - b(\rho^{-1} - (1-\rho)^{-1})) \ge c_0 Dkm^{1/3}/3$,
- $T^{\bullet}_{\mathbf{0},\langle km,0\rangle} \le \frac{2km}{(1-\rho)^2+\rho^2} - c_0 Dkm^{1/3}/2$,

where $T^{\bullet}_{u,v} = T_{u,v} - \xi(v)$ for any vertices $u \le v$ as before, and recall that $\mathbf{B}$ is the Busemann function (defined in Section 3.1). To see this, we assume the contrary, i.e. none of the above three events happen (while the event on the right-hand side of (10.2) happens). Then we must have $T^{\bullet}_{\mathbf{0},\langle km,0\rangle} - T^{\bullet}_{\mathbf{0},\langle km,0\rangle} > \mathbf{B}(\langle km, b_+\rangle, \langle km, 0\rangle)$, which contradicts Lemma 3.1.

We claim that we can bound the probability of each of the three events by $C'e^{-c'k}$ for some $c', C' > 0$ depending on m, D. For the first event the bound is by Lemma 5.6. For the second event, note that $b \mapsto \mathbf{B}(\langle km, b\rangle, \langle km, 0\rangle) - b(\rho^{-1} - (1-\rho)^{-1})$ is a (two-sided) centered random walk; for the third event, use Theorem 5.2.

Finally, by sequentially choosing $D, \epsilon, \varepsilon, m$, and considering all large enough k and each $r \in [\![2km, 2(k+1)m]\!]$, the conclusion follows. ∎

We prove Proposition 10.2 using a similar strategy.

Proof of Proposition 10.2. The first half of this proof goes the same way as the proof of Proposition 10.1. We omit the details, and conclude that the following is true for any $D > 0$, $\epsilon > 0$, $m \in \mathbb{N}$ with $m^{2/3} \in \mathbb{Z}$, and $k \in \mathbb{N}$, such that D, m are large enough as required by Lemma 10.3, ϵ is small enough depending on D, and m is large enough depending on D, ϵ. Take any $k \in \mathbb{N}$ which is $> \epsilon^{-2}$ and large enough as required by Lemma 10.3, and take any $n \in [\![km, (k+1)m]\!]$. Let γ be the path from $\mathbb{L}_0$ to $\mathbb{L}_{km}$, consisting of the first $2km+1$ vertices of $\Gamma_{\mathbf{0},\langle n,b_n\rangle}$. Then

$$\mathbb{P}[\gamma \notin P_{k,D}, |\mu_{\mathbf{0},\langle n,b_n\rangle}(f) - \nu(f)| > \epsilon] < \binom{\lfloor Dk\rfloor + k - 1}{k-1}(D+1)^{-2k} < e^{-ck}$$

for some $c > 0$ depending on D. It remains to bound $\mathbb{P}[\gamma \in P_{k,D}]$. By Lemma 10.3,

$$\mathbb{P}[\gamma \in P_{k,D}] < e^{-c_0 k} + \mathbb{P}\left[T(\gamma) \le \frac{2km}{(1-\rho)^2+\rho^2} - b_+(\rho^{-1} - (1-\rho)^{-1}) - c_0 Dkm^{1/3}\right], \tag{10.3}$$

where $\langle km, b_+\rangle$ is the intersection of $\Gamma_{\mathbf{0},\langle n,b_n\rangle}$ with $\mathbb{L}_{km}$. When the event on the right-hand side of (10.3) happens, (at least) one of the following must happen:

- $\max_{b\in\mathbb{Z}}(T_{\langle km,b\rangle,\langle n,b_n\rangle} - (b-b_n)(\rho^{-1} - (1-\rho)^{-1})) \ge c_0 Dkm^{1/3}/3$,
- $T_{\mathbf{0},\langle km,b_n\rangle} \le \frac{2km}{(1-\rho)^2+\rho^2} - b_n(\rho^{-1} - (1-\rho)^{-1}) - c_0 Dkm^{1/3}/2$.

To see this, assume that none of the above events happen. Then

$$\begin{aligned} T(\gamma) &> T_{\mathbf{0},\langle n,b_n\rangle} - T_{\langle km,b_+\rangle,\langle n,b_n\rangle} \ge T_{\mathbf{0},\langle km,b_n\rangle} - T_{\langle km,b_+\rangle,\langle n,b_n\rangle} \\ &> \frac{2km}{(1-\rho)^2+\rho^2} - b_+(\rho^{-1} - (1-\rho)^{-1}) - 5c_0 Dkm^{1/3}/6, \end{aligned}$$

which contradicts the event on the right-hand side of (10.3).

We claim that we can bound the probability of each of the two events by $C'e^{-c'k}$ for some $c', C' > 0$ depending on m, D. For the first event, note that $n - km \le m$; then the bound can be obtained by taking a union bound over all up-right paths from $\mathbb{L}_{km}$ to

$\langle n, b_n \rangle$ (there are at most 2^{2m} such paths, and the passage time of each is the sum of at most $2m + 1$ i.i.d. Exp(1) random variables). For the second event, apply Theorem 5.2. Thus the conclusion follows. ∎

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References

[1] Angel, O.: The stationary measure of a 2-type totally asymmetric exclusion process. J. Combin. Theory Ser. A **113**, 625–635 (2006) Zbl 1087.60067 MR 2216456

[2] Arita, C., Ayyer, A., Mallick, K., Prolhac, S.: Recursive structures in the multispecies TASEP. J. Phys. A Math. Theor. **44**, article no. 335004, 26 pp. (2011) Zbl 1251.82036

[3] Auffinger, A., Damron, M., Hanson, J.: 50 years of first-passage percolation. Univ. Lecture Ser. 68, American Mathematical Society, Providence, RI (2017) Zbl 1452.60002 MR 3729447

[4] Baik, J., Deift, P., Johansson, K.: On the distribution of the length of the longest increasing subsequence of random permutations. J. Amer. Math. Soc. **12**, 1119–1178 (1999) Zbl 0932.05001 MR 1682248

[5] Balázs, M., Busani, O., Seppäläinen, T.: Non-existence of bi-infinite geodesics in the exponential corner growth model. Forum Math. Sigma **8**, article no. e46, 34 pp. (2020) Zbl 1456.60248 MR 4176750

[6] Balázs, M., Busani, O., Seppäläinen, T.: Local stationarity in exponential last-passage percolation. Probab. Theory Related Fields **180**, 113–162 (2021) Zbl 1483.60144 MR 4265019

[7] Balázs, M., Cator, E., Seppäläinen, T.: Cube root fluctuations for the corner growth model associated to the exclusion process. Electron. J. Probab. **11**, 1094–1132 (2006) Zbl 1139.60046 MR 2268539

[8] Balázs, M., Seppäläinen, T.: Order of current variance and diffusivity in the asymmetric simple exclusion process. Ann. of Math. (2) **171**, 1237–1265 (2010) Zbl 1200.60083 MR 2630064

[9] Basu, R., Bhatia, M., Ganguly, S.: Environment seen from infinite geodesics in Liouville quantum gravity. Ann. Probab. **52**, 1399–1486 (2024) Zbl 07889241 MR 4766837

[10] Basu, R., Ganguly, S.: Time correlation exponents in last passage percolation. In: In and out of equilibrium 3: Celebrating Vladas Sidoravicius, Progr. Probab. 77, Birkhäuser/Springer, Cham, 101–123 (2021) Zbl 1469.60305 MR 4237265

[11] Basu, R., Ganguly, S., Zhang, L.: Temporal correlation in last passage percolation with flat initial condition via Brownian comparison. Comm. Math. Phys. **383**, 1805–1888 (2021) Zbl 1468.60111 MR 4244262

[12] Basu, R., Hoffman, C., Sly, A.: Nonexistence of bigeodesics in planar exponential last passage percolation. Comm. Math. Phys. **389**, 1–30 (2022) Zbl 1494.60092 MR 4365136

[13] Basu, R., Sarkar, S., Sly, A.: Coalescence of geodesics in exactly solvable models of last passage percolation. J. Math. Phys. **60**, article no. 093301, 22 pp. (2019) Zbl 1480.60284 MR 4002528

[14] Basu, R., Sidoravicius, V., Sly, A.: Last passage percolation with a defect line and the solution of the slow bond problem. arXiv:1408.3464v3 (2016)

[15] Bates, E.: Empirical measures, geodesic lengths, and a variational formula in first-passage percolation. Mem. Amer. Math. Soc. **293**, no. 1460, xi+92 pp. (2024) Zbl 07807562 MR 4684338

[16] Bhatia, M.: Moderate deviation and exit time estimates for stationary last passage percolation. J. Statist. Phys. **181**, 1410–1432 (2020) Zbl 1462.82034 MR 4163506

[17] Borodin, A., Ferrari, P. L.: Large time asymptotics of growth models on space-like paths. I. PushASEP. Electron. J. Probab. **13**, 1380–1418 (2008) Zbl 1187.82084 MR 2438811

[18] Borodin, A., Péché, S.: Airy kernel with two sets of parameters in directed percolation and random matrix theory. J. Statist. Phys. **132**, 275–290 (2008) Zbl 1145.82021 MR 2415103

[19] Busani, O., Ferrari, P. L.: Universality of the geodesic tree in last passage percolation. Ann. Probab. **50**, 90–130 (2022) Zbl 1499.60325 MR 4385124

[20] Calvert, J., Hammond, A., Hegde, M.: Brownian structure in the KPZ fixed point. Astérisque **441** (2023) Zbl 1530.82001 MR 4612529

[21] Corwin, I.: The Kardar–Parisi–Zhang equation and universality class. Random Matrices Theory Appl. **1**, article no. 1130001, 76 pp. (2012) Zbl 1247.82040 MR 2930377

[22] Coupier, D.: Multiple geodesics with the same direction. Electron. Comm. Probab. **16**, 517–527 (2011) Zbl 1244.60093 MR 2836758

[23] Damron, M., Hanson, J., Janjigian, C., Lam, W.-K., Shen, X.: Estimates for the empirical distribution along a geodesic in first-passage percolation. arXiv:2010.08072v2 (2024)

[24] Dauvergne, D., Ortmann, J., Virág, B.: The directed landscape. Acta Math. **229**, 201–285 (2022) Zbl 1522.60075 MR 4554223

[25] Dauvergne, D., Sarkar, S., Virág, B.: Three-halves variation of geodesics in the directed landscape. Ann. Probab. **50**, 1947–1985 (2022) Zbl 1510.60087 MR 4474505

[26] Dauvergne, D., Virág, B.: The scaling limit of the longest increasing subsequence. arXiv:2104.08210 (2021)

[27] Derrida, B., Janowsky, S. A., Lebowitz, J. L., Speer, E. R.: Exact solution of the totally asymmetric simple exclusion process: shock profiles. J. Statist. Phys. **73**, 813–842 (1993) Zbl 1102.60320 MR 1251221

[28] Emrah, E., Janjigian, C., Seppäläinen, T.: Optimal-order exit point bounds in exponential last-passage percolation via the coupling technique. Probab. Math. Phys. **4**, 609–666 (2023) Zbl 1533.60174 MR 4620410

[29] Evans, M. R., Ferrari, P. A., Mallick, K.: Matrix representation of the stationary measure for the multispecies TASEP. J. Statist. Phys. **135**, 217–239 (2009) Zbl 1167.82017 MR 2505734

[30] Ferrari, P. A.: Shock fluctuations in asymmetric simple exclusion. Probab. Theory Related Fields **91**, 81–101 (1992) Zbl 0744.60117 MR 1142763

[31] Ferrari, P. A.: TASEP hydrodynamics using microscopic characteristics. Probab. Surv. **15**, 1–27 (2018) Zbl 1434.60288 MR 3769188

[32] Ferrari, P. A., Fontes, L. R. G., Kohayakawa, Y.: Invariant measures for a two-species asymmetric process. J. Statist. Phys. **76**, 1153–1177 (1994) Zbl 0841.60085 MR 1298099

[33] Ferrari, P. A., Kipnis, C., Saada, E.: Microscopic structure of travelling waves in the asymmetric simple exclusion process. Ann. Probab. **19**, 226–244 (1991) Zbl 0725.60113 MR 1085334

[34] Ferrari, P. A., Martin, J. B.: Stationary distributions of multi-type totally asymmetric exclusion processes. Ann. Probab. **35**, 807–832 (2007) Zbl 1117.60089 MR 2319708

[35] Ferrari, P. A., Martin, J. B., Pimentel, L. P. R.: A phase transition for competition interfaces. Ann. Appl. Probab. **19**, 281–317 (2009) Zbl 1185.60109 MR 2498679

[36] Ferrari, P. A., Pimentel, L. P. R.: Competition interfaces and second class particles. Ann. Probab. **33**, 1235–1254 (2005) Zbl 1078.60083 MR 2150188

[37] Ferrari, P. L., Occelli, A.: Universality of the GOE Tracy–Widom distribution for TASEP with arbitrary particle density. Electron. J. Probab. **23**, article no. 51, 24 pp. (2018) Zbl 1406.82012 MR 3814245

[38] Hammond, A., Sarkar, S.: Modulus of continuity for polymer fluctuations and weight profiles in Poissonian last passage percolation. Electron. J. Probab. **25**, article no. 29, 38 pp. (2020) Zbl 1439.82014 MR 4073690

[39] Hoffman, C.: American Institute of Mathematics problem lists: first passage percolation. http://aimpl.org/firstpercolation/. Aug. 2015. Edited by Daniel Ahlberg (2015); visited on January 3, 2025

[40] Johansson, K.: Shape fluctuations and random matrices. Comm. Math. Phys. **209**, 437–476 (2000) Zbl 0969.15008 MR 1737991

[41] Johansson, K.: Transversal fluctuations for increasing subsequences on the plane. Probab. Theory Related Fields **116**, 445–456 (2000) Zbl 0960.60097 MR 1757595

[42] Kardar, M., Parisi, G., Zhang, Y.-C.: Dynamic scaling of growing interfaces. Phys. Rev. Lett. **56**, 889–892 (1986) Zbl 1101.82329

[43] Ledoux, M., Rider, B.: Small deviations for beta ensembles. Electron. J. Probab. **15**, no. 41, 1319–1343 (2010) Zbl 1228.60015 MR 2678393

[44] Liu, Z.: One-point distribution of the geodesic in directed last passage percolation. Probab. Theory Related Fields **184**, 425–491 (2022) Zbl 1510.60089 MR 4498515

[45] Matetski, K., Quastel, J., Remenik, D.: The KPZ fixed point. Acta Math. **227**, 115–203 (2021) Zbl 1505.82041 MR 4346267

[46] Newman, C. M.: A surface view of first-passage percolation. In: Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1017–1023 (1995) Zbl 0848.60089 MR 1404001

[47] Pimentel, L. P. R.: Duality between coalescence times and exit points in last-passage percolation models. Ann. Probab. **44**, 3187–3206 (2016) Zbl 1361.60095 MR 3551194

[48] Quastel, J., Remenik, D.: Airy processes and variational problems. In: Topics in percolative and disordered systems, Springer Proc. Math. Statist. 69, Springer, New York, 121–171 (2014) Zbl 1329.82059 MR 3229288

[49] Rost, H.: Nonequilibrium behaviour of a many particle process: density profile and local equilibria. Z. Wahrsch. Verw. Gebiete **58**, 41–53 (1981) Zbl 0451.60097 MR 0635270

[50] Schmid, D.: Mixing times for the TASEP in the maximal current phase. Ann. Probab. **51**, 1342–1379 (2023) Zbl 1535.60154 MR 4597321

[51] Schmid, D., Sly, A.: Mixing times for the TASEP on the circle. arXiv:2203.11896v2 (2024)

[52] Seppäläinen, T.: Existence, uniqueness and coalescence of directed planar geodesics: proof via the increment-stationary growth process. Ann. Inst. Henri Poincaré Probab. Statist. **56**, 1775–1791 (2020) Zbl 1454.60146 MR 4116707

[53] Seppäläinen, T., Shen, X.: Coalescence estimates for the corner growth model with exponential weights. Electron. J. Probab. **25**, article no. 85, 31 pp. (2020) Zbl 1459.60208 MR 4125790

[54] Speer, E. R.: The two species totally asymmetric simple exclusion process. In: On three levels: Micro, meso and macroscopic approaches in physics, Springer, 91–102 (1994) Zbl 0872.60084

[55] Zhang, L.: Optimal exponent for coalescence of finite geodesics in exponential last passage percolation. Electron. Comm. Probab. **25**, article no. 74, 14 pp. (2020) Zbl 1469.60352 MR 4167385

[56] Zygouras, N.: Some algebraic structures in KPZ universality. Probab. Surv. **19**, 590–700 (2022) Zbl 1506.05217 MR 4524523