Rev. Mat. Iberoam. (Online first)
DOI 10.4171/RMI/1493

Published by EMS Press

Stability of weighted norm inequalities

Michel Alexis, José Luis Luna-Garcia, Eric Sawyer and Ignacio Uriarte-Tuero

Abstract. We show that while individual Riesz transforms are two-weight norm *stable* under biLipschitz change of variables on A_∞ weights, they are two-weight norm *unstable* under even rotational change of variables on doubling weights. More precisely, we show that individual Riesz transforms are unstable under a set of rotations having full measure, which includes rotations arbitrarily close to the identity. This provides an operator theoretic distinction between A_∞ weights and doubling weights. More generally, all iterated Riesz transforms of odd order are rotationally unstable on pairs of doubling weights, thus demonstrating the need for characterizations of iterated Riesz transform inequalities using testing conditions as appearing in the work of Nazarov, Treil and Volberg, and other works by subsets of the authors Alexis, Lacey, Sawyer, Shen, Uriarte-Tuero and Wick, as opposed to the typically stable 'bump' conditions.

Contents

1. Introduction

We begin by describing two stability theorems for operator norms, given three decades apart, that motivate the main results of this paper.

Mathematics Subject Classification 2020: 42B20.
Keywords: singular integrals, Calderón–Zygmund operators, weighted norm inequalities, Riesz transforms, biLipschitz stability, doubling measures, two weight.

1.1. Previous stability results

Thirty-five years ago, Johnson and Neugebauer (see Theorem 2.10 (a) in [18], see also the preceding Remark 1) characterized the smooth homeomorphisms $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ that preserve Muckenhoupt's $A_p(\mathbb{R}^n)$ condition for a weight w under pushforward by Φ, as precisely those quasiconformal maps Φ having their Jacobian $J = |\det D\Phi|$ in the intersection $\bigcap_{r>1} A_r(\mathbb{R}^n)$ of the A_r classes over $r > 1$. A variant of the one-dimensional case of this beautiful characterization, see Theorem 2.7 of [18] with $\alpha = 1$, can be reformulated in terms of *stability* of the 'Muckenhoupt' one-weight norm inequality for the Hilbert transform under homeomorphisms of the real line.

Theorem 1.1. *Suppose that $w\colon \mathbb{R} \to [0,\infty)$ is a nonnegative weight on the real line $\mathbb{R}$, that $\varphi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with φ and φ^{-1} absolutely continuous, and that H is the Hilbert transform, $Hf(x) = \text{p.v.} \int_{-\infty}^{\infty} f(y)/(y-x)$.*

For $1<p<\infty$, denote by $\mathfrak{N}_{H;p}[w]$ the operator norm of the map $H\colon L^p(w)\to L^p(w)$, i.e., the best constant C in the inequality

$$\int_{\mathbb{R}} |Hf(x)|^p\, w(x)\, dx \le C^p \int_{\mathbb{R}} |f(x)|^p\, w(x)\, dx.$$

Then there is a positive constant C_1, such that

$$\mathfrak{N}_{H;p}[(w\circ\varphi)\varphi'] \le C_1 \mathfrak{N}_{H,p}[w], \quad \textit{for all weights } w,$$

if and only if $\varphi' \in \bigcap_{r>1} A_r(\mathbb{R})$.

More recently, Tolsa (see the abstract of [42]) characterized the 'Ahlfors–David' one-weight inequality for the Cauchy transform, equivalently for the 1-fractional vector Riesz transform $\mathbf{R}^{1,2}$ in the plane $\mathbb{R}^2$ (defined in (1.1) below), in the case $p = 2$, namely,

$$\int_{\mathbb{R}^2} |\mathbf{R}^{1,2}(f\mu)(x)|^2\, d\mu(x) \le \mathfrak{N}^2_{\mathbf{R}^{1,2};2}(\mu) \int_{\mathbb{R}^2} |f(x)|^2\, d\mu(x),$$

in terms of a growth condition and Menger curvature. As a consequence, Tolsa obtained stability of the operator norm $\mathfrak{N}_{\mathbf{R}^{1,2};2}(\mu)$ under biLipschitz pushforwards of the measure μ. Even more recently, in papers by Dąbrowski and Tolsa [9], and Tolsa [43], this result was extended to higher dimensions, and as a consequence they obtained stability of the operator norm $\mathfrak{N}_{\mathbf{R}^{1,n};2}(\mu)$ of the 1-fractional vector Riesz transform $\mathbf{R}^{1,n}$ under biLipschitz pushforwards of the measure μ in $\mathbb{R}^n$, see the comment at the top of page 6 of [9], and see [43] as well. As an important application of norm stability, they obtain the stability of removable sets for Lipschitz harmonic functions under biLipschitz mappings, see Corollary 1.6 of [43] and the discussion surrounding it.

Here we define the α-fractional vector Riesz transform in $\mathbb{R}^n$ by

$$\text{(1.1)} \qquad \mathbf{R}^{\alpha,n} f(x) \equiv c_{\alpha,n}\ \text{p.v.}\int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n+1-\alpha}}\, f(y)\, dy, \quad x\in\mathbb{R}^n,\ 0\le\alpha<n.$$

Let $\mathbf{R}^{\alpha,n}_j = (R^{\alpha,n}_1, \dots, R^{\alpha,n}_2)$, where we refer to the components $R^{\alpha,n}_j$ as individual α-fractional Riesz transforms in $\mathbb{R}^n$. We are primarily concerned with the classical case

$\alpha = 0$ in this paper, so we will usually drop the superscript α and write $\mathbf{R} = (R_1, \dots, R_n)$ when the dimension n is understood, and refer to the components R_j as Riesz transforms.

The main problem we consider in this paper is the extent to which the above theorems hold in the setting of *two-weight* norm inequalities, and to include more general operators in higher dimensions. The complexities inherent in dealing with two-weight norm inequalities – mainly that they are no longer characterized simply by A_p-like conditions or more generally by conditions of 'positive nature', but require testing conditions of 'singular nature' as well – suggests that we should limit ourselves to consideration of biLipschitz maps. Indeed, this much smaller class of maps is much more amenable to current two-weight techniques, and allows for a rich theory where stability holds in certain 'nice' situations, while failing in small perturbations of these 'nice' situations. We also show in Appendix A that any reasonable group of transformations under which the two-weight A_2 condition is stable is contained in the group of biLipschitz transformations.

Our analysis will be mainly restricted to the case $p = 2$ and iterated Riesz transforms of odd order in $\mathbb{R}^n$, where we show that stability of the two-weight norm inequality is sensitive to the distinction between doubling and A_∞ weights, even when the biLipschitz maps are restricted to rotations of $\mathbb{R}^n$.

1.2. Description of results

The two-weight norm inequality for an operator T with a pair (σ, ω) of positive locally finite Borel measures on $\mathbb{R}^n$ and exponents $1 < p \le q < \infty$ is informally

$$\left(\int_{\mathbb{R}^n} |T(f\sigma)|^q \, d\omega \right)^{1/q} \le \mathfrak{N}_T \left(\int_{\mathbb{R}^n} |f|^p \, d\sigma \right)^{1/p}, \quad f \in L^p(\sigma). \tag{1.2}$$

See Definition 5.5 for a formal definition of the two-weight norm inequality. In the case $p = q = 2$, we first establish a distinction between weighted norm inequalities for positive operators T in (1.2), such as the maximal function and fractional integrals, on the one hand; and singular integral operators T in (1.2), such as the individual Riesz transforms and iterated Riesz transforms, on the other hand. Namely, that the former are two-weight norm stable under biLipschitz change of variables for arbitrary locally finite positive Borel measures, while the latter are not in general, even on pairs of doubling measures.

Our main result, Theorem 1.4, shows that while individual Riesz transforms are two-weight norm *stable* under biLipschitz change of variables on pairs of A_∞ weights, they are two-weight norm *unstable* under even a rotational change of variables on doubling weights. This provides an operator theoretic distinction between A_∞ weights and doubling weights.[1]

We also show that all iterated Riesz transforms of odd order are rotationally unstable on pairs of doubling weights, thus demonstrating the need for characterizations of iterated Riesz transform inequalities using unstable conditions, such as the testing conditions in [4,21,24,31,37,38], as opposed to the typically stable 'bump' conditions, see Section A.3.

[1] In 1974, C. Fefferman and B. Muckenhoupt [10] constructed an example of a doubling weight that was not A_∞ using a self similar construction, on which many subsequent results have been based.

1.3. BiLipschitz and rotational stability

In this section, we precisely define stability.

Definition 1.2. Let $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuous and invertible.

(1) Φ is *biLipschitz* if

$$\|\Phi\|_{\mathrm{biLip}} \equiv \sup_{x,y\in\mathbb{R}^n} \frac{|\Phi(x)-\Phi(y)|}{|x-y|} + \sup_{x,y\in\mathbb{R}^n} \frac{|\Phi^{-1}(x)-\Phi^{-1}(y)|}{|x-y|} < \infty.$$

(2) Φ is a rotation if Φ is linear and $\Phi\Phi^* = I$ and $\det \Phi = 1$.

Let $\mathcal{X}$ be a group of continuous invertible maps on $\mathbb{R}^n$, such as the group of biLipschitz or rotation transformations, which we denote by $\mathcal{X}_{\mathrm{biLip}}$ and $\mathcal{X}_{\mathrm{rot}}$, respectively.[2] Denote by $\mathcal{M}$ the space of positive Borel measures on $\mathbb{R}^n$, and by $\Phi_*\mu$ the pushforward of $\mu \in \mathcal{M}$ by a continuous map $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$, i.e., $\Phi_*\mu(B) \equiv \mu(\Phi^{-1}(B))$. We say that a subclass $\mathcal{S} \subset \mathcal{M}$ of positive Borel measures is *$\mathcal{X}$-invariant* if $\Phi_*\mu \in \mathcal{S}$ for all $\mu \in \mathcal{S}$ and $\Phi \in \mathcal{X}$. Of course, $\mathcal{M}$ itself is $\mathcal{X}$-invariant for the group $\mathcal{X}_{\mathrm{cont\,inv}}$ of all continuous invertible maps, but less trivial examples of *biLipschitz* invariant classes include

$$\begin{aligned}
\mathcal{S}_{A_p} &\equiv \{\mu \in \mathcal{M} : d\mu(x) = u(x)\,dx \text{ with } u \in A_p\} \quad \text{for } 1 \le p < \infty,\\
\mathcal{S}_{A_\infty} &\equiv \{\mu \in \mathcal{M} : d\mu(x) = u(x)\,dx \text{ with } u \in A_\infty\},\\
\mathcal{S}_{\mathrm{doub}} &\equiv \{\mu \in \mathcal{M} : \mu \text{ is a doubling measure}\},\\
\mathcal{S}_{\mathrm{ADs}} &\equiv \{\mu \in \mathcal{M} : \mu \text{ is Ahlfors–David regular of degree } s\},\\
\mathcal{S}_{\mathrm{lfpB}} &\equiv \{\mu \in \mathcal{M} : \mu \text{ is a locally finite positive Borel measure}\}.
\end{aligned}$$

To each of the above classes $\mathcal{S}$, we can associate a functional $\|\mu\|_{\mathcal{S}}$ for which $\mathcal{S} \equiv \{\mu \in \mathcal{M} : \|\mu\|_{\mathcal{S}} < \infty\}$. For example, we take

$$\|\mu\|_{\mathcal{S}_{A_\infty}} = [\mu]_{A_\infty} \equiv \sup_Q \Big(\frac{1}{|Q|}\int_Q \mu\Big) \exp\Big(\frac{1}{|Q|}\int_Q \ln\frac{1}{\mu}\Big),$$

and $\|\mu\|_{\mathcal{S}_{\mathrm{doub}}} = C_{\mathrm{doub}}(\mu)$ as in Definition 2.1. In the case that $\mathcal{S} = \mathcal{S}_{\mathrm{lfpB}}$, there is no 'natural' choice of $\|\cdot\|_{\mathcal{S}}$ that measures the 'size' of the measure μ, and so instead we may, for instance, define

$$\|\mu\|_{\mathcal{S}_{\mathrm{lfpB}}} = \begin{cases} 1 & \text{if } \mu \in \mathcal{S}_{\mathrm{lfpB}},\\ \infty & \text{otherwise.}\end{cases}$$

We also define

$$\|\Phi\|_{\mathcal{X}} = \begin{cases} \|\Phi\|_{\mathrm{biLip}} & \text{if } \mathcal{X} = \mathcal{X}_{\mathrm{biLip}},\\ 1 & \text{if } \mathcal{X} = \mathcal{X}_{\mathrm{rot}} \text{ and } \Phi \in \mathcal{X}_{\mathrm{rot}},\\ \infty & \text{if } \mathcal{X} = \mathcal{X}_{\mathrm{rot}} \text{ and } \Phi \notin \mathcal{X}_{\mathrm{rot}}.\end{cases}$$

Here is the main stability definition for a function $\mathcal{F}$ on measure pairs, a group $\mathcal{X} \in \{\mathcal{X}_{\mathrm{biLip}}, \mathcal{X}_{\mathrm{rot}}\}$ and an $\mathcal{X}$-invariant class $\mathcal{S}$ (or to be precise, for $(\mathcal{S}, \|\cdot\|_{\mathcal{S}})$).

[2] See Lemma A.8 in Appendix A for a justification of considering subgroups of biLipschitz transformations.

Definition 1.3. Let $\mathcal{X} \in \{\mathcal{X}_{\text{biLip}}, \mathcal{X}_{\text{rot}}\}$, $\mathcal{S} \subset \mathcal{M}$ be $\mathcal{X}$-invariant, and let $\mathcal{F}: \mathcal{S} \times \mathcal{S} \to [0, \infty]$ be a nonnegative extended real-valued function on the product set $\mathcal{S} \times \mathcal{S}$. We say that the function $\mathcal{F}$ is *$\mathcal{X}$-stable on $\mathcal{S}$* if there is a function $\mathcal{G}: [0, \infty)^4 \to [0, \infty)$, which maps bounded subsets of $[0, \infty)^4$ to bounded subsets of $[0, \infty)$, such that

$$\mathcal{F}(\Phi_* \sigma, \Phi_* \omega) \le \mathcal{G}(\|\Phi\|_{\mathcal{X}}, \mathcal{F}(\sigma, \omega), \|\sigma\|_{\mathcal{S}}, \|\omega\|_{\mathcal{S}}), \tag{1.3}$$

for all $\sigma, \omega \in \mathcal{S}$, such that $\mathcal{F}(\sigma, \omega) < \infty$ and all $\Phi \in \mathcal{X}$.

Note that to check that $\mathcal{G}$ maps bounded sets to bounded sets, it suffices to show for instance, that $\mathcal{G}$ is continuous. Typically, we will take $\mathcal{F}$ to be an operator norm on weighted spaces, in which case we say an operator T is (un)stable on a class of measures $\mathcal{S}$ if its two-weight operator norm is (un)stable on $\mathcal{S}$. One may also take $\mathcal{F}$ to be a common two-weight bump condition.

A simple example of a biLipschitz stable function on the class $\mathcal{S}_{\text{lfpB}}$ is the classical two-weight A_2 characteristic for a pair of measures, namely,

$$\mathcal{F}(\sigma, \omega) = A_2(\sigma, \omega) = \sup_{\text{cubes } Q \text{ in } \mathbb{R}^n} \frac{|Q|_\sigma}{|Q|} \frac{|Q|_\omega}{|Q|}.$$

Indeed,

$$\frac{|Q|_{\Phi_*\sigma}}{|Q|} \frac{|Q|_{\Phi_*\omega}}{|Q|} = \frac{|\Phi^{-1}Q|_\sigma}{|Q|} \frac{|\Phi^{-1}Q|_\omega}{|Q|} \approx \frac{|\Phi^{-1}Q|_\sigma}{|\Phi^{-1}Q|} \frac{|\Phi^{-1}Q|_\omega}{|\Phi^{-1}Q|},$$

since Φ^{-1} is biLipschitz, and observe that there is a cube P such that $P \subset \Phi^{-1}Q \subset \rho P$ for some $\rho > 1$ by quasiconformality of Φ, see Lemma 3.4.5 in [5], where ρ depends only on $\|\Phi\|_{\text{biLip}}$. Thus, we have

$$\frac{|Q|_{\Phi_*\sigma}}{|Q|} \frac{|Q|_{\Phi_*\omega}}{|Q|} \lesssim \frac{|\rho P|_\sigma}{|\rho P|} \frac{|\rho P|_\omega}{|\rho P|} \le A_2(\sigma, \omega),$$

and by taking supremums over cubes gives

$$\begin{aligned} A_2(\Phi_*\sigma, \Phi_*\omega) &\le \mathcal{G}(\|\Phi\|_{\text{biLip}}, A_2(\sigma, \omega), \|\sigma\|_{\mathcal{S}_{\text{lfpB}}}, \|\omega\|_{\mathcal{S}_{\text{lfpB}}}) \\ &= \mathcal{G}(\|\Phi\|_{\text{biLip}}, A_2(\sigma, \omega), 1, 1) \end{aligned} \tag{1.4}$$

for $\mathcal{G}(w, x, y, z) = c w^{4n} x$, where $c > 0$ is independent of Φ, σ and ω.

The reader can also check that all of the usual 'Orlicz bump' conditions

$$\sup_{Q \text{ a ball}} \|u^{1/p}\|_{A,Q} \|v^{-1/p}\|_{B,Q} < \infty,$$

where

$$\|f\|_{A,Q} \equiv \inf\Big\{\lambda > 0 : \frac{1}{|B|} \int_B A\Big(\frac{|f(x)|}{\lambda}\Big)\, dx\Big\},$$

on a pair of absolutely continuous measures $\sigma(x)\, dx$ and $\omega(x)\, dx$ on $\mathbb{R}^n$ as in the conjecture of Cruz-Uribe and Pérez [8] (proved by Lerner, see [26]), are biLipschitz stable on any biLipschitz invariant subclass $\mathcal{S}$, e.g., Neugebauer's bump condition,

$$A_{2,r}(\sigma, \omega) = \sup_{\text{cubes } Q \text{ in } \mathbb{R}^n} \Big(\frac{1}{|Q|} \int_Q \sigma(x)^r\, dx\Big)^{1/r} \Big(\frac{1}{|Q|} \int_Q \omega(x)^r\, dx\Big)^{1/r},$$

where $1 < r < \infty$, see Appendix A.3.

More recently, additional variants of bump condition, such as entropy bumps and separated bumps, have arisen in work of Treil and Volberg [45], Lacey and Spencer [23] to mention just a few. The sufficiency of these bump conditions for two-weight singular integral inequalities all go through the boundedness of sparse operators, see Lerner [26] for a proof of the optimal result to date, and a history of this fascinating subject. In Appendix A.3, we show that no such bump conditions can characterize the two-weight norm inequality for an iterated Riesz transform T of odd order even when the measures are doubling (or for any Calderón–Zygmund operator T that is biLipschitz unstable on doubling measures).

We mention in passing that the following form of the two-weight A_p condition on the real line,

$$\widetilde{A}_p(v,w) \equiv \sup_{I \text{ an interval}} \Big(\frac{1}{|I|}\int_I w\Big)\Big(\frac{1}{|I|}\int_I \frac{1}{v^{p'-1}}\Big)^{p-1},$$

has been proved stable under an increasing homeomorphic change of variable φ (with both φ and φ^{-1} absolutely continuous) if and only if $\varphi' \in A_1(\mathbb{R})$, see Corollary 4.4 in [18], but this condition is no longer equivalent to boundedness of the Hilbert transform for two weights, and moreover, the definitions of stability of $\widetilde{A}_2(v,w)$ and $A_2(\sigma,\omega)$ considered above are a priori different since composition and pushforward do not commute, e.g., when $p = 2$, $\Phi_* v \neq (\Phi_* v^{-1})^{-1}$ in general.

1.3.1. Main results. Our main result below on both *stability* and *instability* involves Riesz transforms and doubling measures, as well as Stein elliptic Calderón–Zygmund operators. Recall that if K is a Calderón–Zygmund kernel, i.e., it satisfies

$$\begin{aligned} |K(x,y)| &\le C_{\mathrm{CZ}}\,|x-y|^{-n}, \\ |\nabla_x K(x,y)| + |\nabla_y K(x,y)| &\lesssim C_{\mathrm{CZ}}\,|x-y|^{-n-1}, \end{aligned} \tag{1.5}$$

and if T is a bounded linear operator on unweighted $L^2(\mathbb{R}^n)$, we say that T is *associated with* the kernel K if

$$Tf(x) = \int K(x,y) f(y)\,dy \quad \text{for all } x \in \mathbb{R}^n \setminus \operatorname{supp} f,$$

and we refer to such operators as *Calderón–Zygmund operators*. Note, in particular, that a Calderón–Zygmund operator T is bounded on unweighted $L^2(\mathbb{R}^n)$ by definition. Following equation (39) on p. 210 of [41], we say that a Calderón–Zygmund operator T is *elliptic in the sense of Stein* if there is a unit vector $\mathbf{u}_0 \in \mathbb{R}^n$ and a constant $c > 0$ such that

$$|K(x, x + t\mathbf{u}_0)| \ge c\,|t|^{-n} \quad \text{for all } t \in \mathbb{R},$$

where $K(x,y)$ is the kernel of T.

Note that a function $\mathcal{F}$ being $\mathcal{X}$-stable means the estimate (1.3) holds across *all* measure pairs and *all* functions in the class $\mathcal{X}$, while to show a function $\mathcal{F}$ is *not* $\mathcal{X}$-stable, it suffices to construct *a sequence* of measure pairs and *a sequence* of functions in $\mathcal{X}$ for which the arguments of $\mathcal{G}$ in (1.3) remain bounded, but $\mathcal{F}$ diverges to ∞, i.e., (1.3) fails for any choice of $\mathcal{G}$. For this last point, in this paper we will always prove instability via this last strategy. In this paper, we consider norms as in (1.2) for $p = q = 2$.

Theorem 1.4. *The two-weight operator norms for individual Riesz transforms R_j, and, more generally, any Stein elliptic Calderón–Zygmund operator, are biLipschitz stable on $\mathcal{S}_{A_\infty}$. The individual Riesz transforms, as well as iterated Riesz transforms of odd order, are not even* rotationally *stable on $\mathcal{S}_{\text{doub}}$, and even when the measures are restricted to have doubling constants C_{doub} arbitrarily close to 2^n.*

In fact, we can prove the following stronger rotational instability for iterated Riesz transforms of odd order, which, in particular, shows that instability can hold for rotations arbitrarily close to the identity.

Theorem 1.5. *Iterated Riesz transforms of odd order are unstable on $\mathcal{S}_{\text{doub}}$ under a set of rotations having full measure.*

In contrast to the instability assertions in these theorems, most positive operators, such as maximal functions and fractional integral operators, are easily seen to be biLipschitz stable on $\mathcal{S}_{A_p}$, $\mathcal{S}_{A_\infty}$, $\mathcal{S}_{\text{doub}}$ and $\mathcal{S}_{\text{lfpB}}$.

For example, if $T = I_\alpha$ is the fractional integral of order $0 < \alpha < n$, and if $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is biLipschitz, then

$$\begin{aligned}
\|T_{\Phi_*\sigma} f\|^2_{L^2(\Phi_*\omega)} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, d\Phi_*\sigma(y) \right|^2 d\Phi_*\omega(x) \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} |\Phi^{-1}x - \Phi^{-1}y|^{\alpha-n} f(\Phi^{-1}y)\, d\sigma(y) \right|^2 d\omega(x) \\
&\approx \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} |x-y|^{\alpha-n} (f\circ\Phi^{-1})(y)\, d\sigma(y) \right|^2 d\omega(x) \\
&= \|T_\sigma(f\circ\Phi^{-1})\|^2_{L^2(\omega)}
\end{aligned}$$

and

$$\|f\|^2_{L^2(\Phi_*\sigma)} = \int_{\mathbb{R}^n} |f(y)|^2\, d\Phi_*\sigma(y) = \int_{\mathbb{R}^n} |f(\Phi^{-1}y)|^2\, d\sigma(y) = \|f\circ\Phi^{-1}\|^2_{L^2(\sigma)}.$$

A similar proof holds for the case when T is a fractional maximal operator of order $0 \le \alpha < n$.

1.4. History of stability

The class of Calderón–Zygmund kernels $K(x,y)$ satisfying (1.5) has long been known to be invariant under biLipschitz change of variable $x = \Phi(u)$. For example, if $K_\Phi(u,v) = K(\Phi(u), \Phi(v))$, then the chain rule gives

$$\begin{aligned}
|\nabla_u K_\Phi(u,v)| &= |D\Phi(u)(\nabla_x K)(u,v)| \\
&\lesssim \|D\Phi\|_\infty C_{\text{CZ}} |u-v|^{-n-1} \le \|\Phi\|_{\text{biLip}} C_{\text{CZ}} |u-v|^{-n-1}.
\end{aligned}$$

It follows that if a Calderón–Zygmund operator T associated with such a kernel K satisfies the two-weight norm inequality (1.2), then the pullback T_Φ with kernel K_Φ is also a Calderón–Zygmund operator (by a simple change of variables using that the Jacobian of Φ is bounded between two positive constants), and satisfies the inequality (1.2) with the pair of measures (σ, ω) replaced by the pair of pushforwards $(\Phi_*\sigma, \Phi_*\omega)$. This raises the question of when T itself satisfies (1.2) with the pair of pushforwards $(\Phi_*\sigma, \Phi_*\omega)$ when Φ is biLipschitz. Roughly speaking, our results show that the answer is *yes* if the measures σ, ω are A_∞ weights, but *no* in general if the measures σ, ω are just doubling.

In [22], it was mentioned that the two-weight norm inequality for the Hilbert transform is "unstable," in the sense that for ω equal to the Cantor measure, and σ an appropriate choice of weighted point masses in each removed middle third, the norm of the operator could go from finite to infinite with just arbitrarily small perturbations of the locations of the point masses, while the $\mathcal{A}_2$ condition remained in force. In Appendix A, we use this example to show that the Hilbert transform is two-weight norm *unstable* under biLipschitz pushforwards of arbitrary measure pairs, and this instability extends to Riesz transforms in higher dimensions in a straightforward way. Thus, the Riesz transforms in higher dimensions are biLipschitz *unstable* on arbitrary weight pairs, something which already shows that the more familiar bump-type conditions, e.g., Theorem 3 in [33], cannot characterize the two-weight problem for Riesz transforms alone.

On the other hand, we show below that Riesz transforms are biLipschitz stable under pairs of A_∞ weights. So on one hand, for pairs of arbitrary measures we have instability, and on the other hand for pairs of A_∞ weights, we have stability. This begs the question, what side-conditions on the weights in our weight pairs will give stability/instability for Riesz transforms? Now it is trivial that A_∞ weights are doubling weights, but it was not until the famous construction of Fefferman and Muckenhoupt in [10] that one knew the two classes were in fact different. Because of this, doubling is often considered to be the next more general condition on a weight than A_∞.

The main result of this paper is that individual Riesz transforms are biLipschitz – and even *rotationally* – unstable for pairs of doubling weights. This provides an operator-theoretic means of distinguishing A_∞ weights from doubling weights, sharpening the result of Fefferman and Muckenhoupt, by showing that stability differentiates the two classes.

1.4.1. Our methods and their history. In 1976, Muckenhoupt and Wheeden showed in [29] that the two-weight norm inequality for the maximal function M implies the one-tailed $\mathcal{A}_2$ condition, and conjectured that it was sufficient. Then in 1982, the third author disproved that conjecture in [34] by starting with a pair of simple radially decreasing weights V, U constructed by Muckenhoupt in [28], that were essentially constant on dyadic intervals $I_k = [2^{-k-1}, 2^{-k}]$ and failed the two-weight inequality for M. Then the weights were *disarranged* into weights v and u, i.e., dilates and translates of the weights restricted to the dyadic intervals I_k were essentially redistributed onto the unit interval $[0, 1]$ according to a self-similar "transplantation" rule. The resulting weights satisfied the one-tailed $\mathcal{A}_2$ condition on $[0, 1]$, but failed the two-weight norm inequality for M.[3] However, such weights were not doubling, as follows from calculations in [34]. This significant obstacle remained until the pioneering work of Nazarov [30], and Nazarov and Volberg [32], to which we now turn.

[3] The reader can easily check that for a discretized version of these weights, the dyadic square function defined in Section 2 also has infinite two-weight norm.

Some years later, Treil and Volberg showed in [44] that the two-weight norm inequality for the Hilbert transform H implies the two-tailed $\mathcal{A}_2$ condition, and Sarason conjectured the two-tailed condition was also sufficient, see Section 7.9 in [14]. Shortly after that, Nazarov disproved the conjecture in [30] (which we were unable to locate till very recently, using the references in [19]), even using *doubling* weights, in a beautiful proof involving the Bellman technique and a brilliant supervisor, or remodeling, argument, see also [32] for the details. This use of doubling weights here turns out to be crucial for our purposes. More specifically, Nazarov's method consisted of first using the Bellman technique in a delicate argument to construct a weight pair (v, u) on $\mathbb{T}$ that failed to satisfy the two-weight inequality for the discrete Hilbert transform, but satisfied both dyadic doubling, with constant arbitrarily close to that of Lebesgue, and dyadic A_2. Then he transplanted highly oscillating functions according to a certain self-similar 'supervisor' rule having roots in [6], that resulted in a pair of weights (v, u) on $\mathbb{T}$ that satisfied the two-tailed $\mathcal{A}_2$ condition, with doubling constant arbitrarily close to that of Lebesgue measure, and for which the testing condition was increasingly unbounded. Nazarov's argument requires the clever use of highly oscillatory functions in order to deal with the singularity of the Hilbert transform, and the use of holomorphic function theory to prove weak convergence results for these increasingly oscillatory functions.

Very recently, it has come to our attention that Kakaroumpas and Treil extended Nazarov's results to $p \neq 2$ using a non-Bellman and 'remodeling' construction [19]. More precisely, Kakaroumpas and Treil first began with a pair of discretized weights with the A_p condition under control, a bilinear form involving the Haar shift having increasingly large norm, but doubling constant just as large. They then apply an iterative *disarrangement* of these weights to then construct new weights for which the A_p condition and the norm of the bilinear form remain essentially unchanged, but the dyadic doubling constant of the weights is much closer to that of Lebesgue measure. This clever disarrangement is one of the innovative ideas which replaces Nazarov's Bellman construction, and provides weights for which one can compute explicit quantities. It is possible that our Riesz transform results can be proved using the Haar shift scheme of Kakaroumpas and Treil in place of the square function scheme of Nazarov, but we have not checked the details.

Note that the rotational stability problem is only significant in dimension two or higher, since in one dimension the only rotation is reflection about the origin, and that preserves the Hilbert transform. Our proof of rotational instability in higher dimensions begins by using the Bellman construction in [32], and is then inspired by Nazarov and Volberg's supervisor argument with highly oscillatory functions. In particular, we extend Nazarov's supervisor/remodeling construction to higher dimensions, which we call "transplantation", and which makes explicit how v and u are constructed by *transplanting averages* of V and U.

We also need to extend Nazarov's weak convergence results to higher dimensions, where holomorphic function theory is no longer available. This requires the new arguments in Section 4, comprising much of the technical difficulty of the present paper. We must also prove that testing conditions hold at all scales for one of the Riesz transforms, something not considered in [32]. Finally, in Appendix A, we provide proofs of those portions of the supervisor argument required for our theorem that not detailed in [32]; one may also consult [19] for additional arguments.

Remark 1.6. In our construction, we show that a given iterated Riesz transform T_0 of order $N = 2m + 1$ fails one of the testing conditions, while all other iterated Riesz transforms T of order $N = 2m + 1$ satisfy both testing conditions. Thus, at this point, we have doubling measures satisfying the A_2 condition with doubling constant arbitrarily close to that of Lebesgue measure and both testing conditions for T. We now need to conclude that T is two-weight bounded. Since the doubling constants can be taken arbitrarily close to that of Lebesgue measure, the A_2 condition implies the classical energy condition [12], and so one can apply the T1 theorem of [38]; see Theorem 5.7 below for a more precise statement and a proof.

1.5. Proof of stability

We present here a simple proof of stability in Theorem 1.4, using a few classical facts on weights from [33] and [7]. The case of A_∞ weights in Lemma 1.8 below is folklore from decades ago, but seems to have first been recorded in Hytönen and Lacey [17], where they also prove a sharp dependence on the characteristics using much deeper tools. We begin with the following lemma of Neugebauer.

Lemma 1.7 (Theorem 3 in [33]). *Let (u, v) be a pair of nonnegative functions. Then there exists $W \in A_p$ with $c_1 u \le W \le c_2 v$ if and only if there is $r > 1$ such that*

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q u^r\Big)\Big(\frac{1}{|Q|}\int_Q v^{r(1-p')}\Big)^{p-1} < \infty.$$

Recall that a weight w is a weak A_∞ weight, written $w \in$ weak A_∞, if any of the following equivalent conditions hold for all cubes Q and subsets E (see, e.g., [35]):

- (C1) there exists $R < \infty$ and $\phi(t) \nearrow$ with $\lim_{t \searrow 0} \phi(t) = 0$ such that $\frac{|E|_w}{|RQ|_w} \le \phi(\frac{|E|}{|Q|})$,
- (C2) for all $R > 1$, there exists $C, \varepsilon > 0$ such that $\frac{|E|_w}{|RQ|_w} \le C(\frac{|E|}{|Q|})^\varepsilon$,
- (C3) there exists $r > 1$ such that $(\fint_Q w^r)^{1/r} \le \frac{1}{|2Q|}\int_{2Q} w$.

Lemma 1.8 (Theorem 1.2 in [17]). *Suppose that T is a sufficiently regular*[4] *Calderón–Zygmund operator, and that both ω and σ are* weak A_∞ *weights. Then T satisfies the two-weight norm inequality*

$$\|T_\sigma f\|^2_{L^2(\omega)} \le C\|f\|^2_{L^2(\sigma)}$$

if $A_2(\sigma, \omega) < \infty$.

Proof. Since σ and ω each satisfy the weak reverse Hölder condition (C3) for some $r > 1$, we have

$$\begin{aligned} A_{2,r}(\sigma,\omega) &\equiv \sup_Q \Big(\frac{1}{|Q|}\int_Q \omega^r\Big)^{1/r}\Big(\frac{1}{|Q|}\int_Q \sigma^r\Big)^{1/r} \\ &\lesssim \sup_Q \Big(\frac{1}{|2Q|}\int_{2Q} \omega\Big)\Big(\frac{1}{|2Q|}\int_{2Q} \sigma\Big) = A_2(\sigma,\omega). \end{aligned}$$

[4] See Section 6.13 on p. 221 of [41] for definitions, and for the nature of the ‘sufficiently regular’ assumption.

Now we apply Neugebauer's lemma with $p = 2$ to the weight pair $(u, v) = (\omega, \sigma^{-1})$ to obtain that there exists $W \in A_2$ with $c_1\omega(x) \leq W(x) \leq c_2\sigma(x)^{-1}$. Then the extension of the weighted inequality of Coifman and Fefferman [7] for Calderón–Zygmund operators given in Section 6.13 on p. 221 of [41] shows that

$$\|T_\sigma f\|^2_{L^2(\omega)} \leq c_1^{-1}\|T_\sigma f\|^2_{L^2(w)} \leq Cc_1^{-1}\|f\sigma\|^2_{L^2(w)} \leq Cc_1^{-1}c_2\|f\sigma\|^2_{L^2(\sigma^{-1})},$$

i.e.,

$$\|T_\sigma f\|^2_{L^2(\omega)} \leq Cc_1^{-1}c_2\|f\|^2_{L^2(\sigma)},$$

for all Calderón–Zygmund operators T. ∎

Remark 1.9. We say that a measure pair (σ, ω) is *universal* (for boundedness of smooth Stein-elliptic Calderón–Zygmund operators) if a smooth Stein-elliptic Calderón–Zygmund operator T is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if all such operators are also bounded. Lemma 1.8 above shows that pairs of A_∞ weights are universal, and Theorem 1.4 above shows that not all pairs of doubling measures are universal.

Proof of stability in Theorem 1.4. Let us suppose that the norm inequality $\|T_\sigma f\|^2_{L^2(\omega)} \leq \mathfrak{N}_T(\sigma, \omega)^2\|f\|^2_{L^2(\sigma)}$ holds for a Calderón–Zygmund operator T associated with a kernel K, and a pair of A_∞ weights (σ, ω). Since (1.4) implies the biLipschitz stability of $A_2(\sigma, \omega)$, and since the A_∞-characteristics $[\sigma]_{A_\infty}$ and $[\omega]_{A_\infty}$ are easily seen to be biLipschitz stable as well (in fact, they are stable under the more general class of quasiconformal change of variables, Theorem 2 of [46]), we conclude that the norm inequality also holds for the Calderón–Zygmund operator T_Φ with kernel

$$K_\Phi(x, y) \equiv K(\Phi(x), \Phi(y)).$$

As mentioned at the beginning of Section 1.4, T_Φ is a Calderón–Zygmund operator whenever T is, i.e., when it satisfies (1.5) and is bounded on unweighted $L^2(\mathbb{R}^n)$. Thus, we conclude from Lemma 1.8 that T is bounded on the weight pair $(\Phi_*\sigma, \Phi_*\omega)$.

We can also be more precise in our proof of stability, since Theorem 1.2 of [17] implies that the function

$$\mathcal{G}(w, x, y, z) \equiv Cw^{\alpha_{\mathcal{X}}}x(y^{\beta_{\mathcal{X}}} + z^{\beta_{\mathcal{X}}})$$

satisfies (1.3) for the functional $\mathcal{F} = \mathfrak{N}_T(\sigma, \omega)$, where $\alpha_{\mathcal{X}}$ and $\beta_{\mathcal{X}}$ are appropriately chosen exponents. ∎

Remark 1.10. Let T be a strongly elliptic vector of Calderón–Zygmund operators as in Theorem 2.6 of [37]. Then two-weight boundedness of T implies the two-weight A_2 condition, see Lemma 4.1 in [37]. Thus, if σ and ω are weak A_∞ weights, then Lemma 1.8 shows that the two-weight norm inequality for T holds if and only if the A_2 condition holds. It follows that T is biLipschitz stable on

$$\mathcal{S}_{\text{weak } A_\infty} \equiv \{\mu \in \mathcal{M} : d\mu(x) = u(x)\,dx \text{ with } u \in \text{weak } A_\infty\}.$$

We do not know if all Stein elliptic Calderón–Zygmund operators are biLipschitz stable on $\mathcal{S}_{\text{weak } A_\infty}$.

The proof of instability in Theorem 1.4 is much more complicated.

- In Section 2, we show there exist dyadically doubling weights U, V on $[0, 1]^n$ which fail a square function testing condition.

- In Section 3, we describe Nazarov's "supervisor" disarrangement of the weights U, V into doubling weights u, v on $[0,1]^n$, and we see how the weights u, v are a linear combination of the oscillatory functions $s_k^{\mathrm{hor},P}$.
- In Section 4, we study how the Riesz transforms interact with these oscillatory functions.
- In Section 5, we show that the norm inequality for R_1 fails on the weights (v, u) by showing the testing condition on $[0,1]$ is at least as large as the square function testing condition for (V, U), while the dyadic testing conditions for R_2 holds for the weight pair (u, v). We then extend u, v to all of $\mathbb{R}^n$, and using that u, v are doubling with doubling constant close to that of Lebesgue measure, we get that dyadic testing for R_2 implies the norm inequality for R_2.
- In Section 6, we then extend our results to show that individual iterated Riesz transforms of odd order are rotationally unstable.

1.6. Open problems

The question of stability of operator norms for singular integrals on weighted spaces is in general wide open. Here are two instances that might be more accessible.

(1) Only iterated Riesz transforms of *odd* order are treated in Theorem 1.4. Are Riesz transforms of even order, such as the real and imaginary parts of the Beurling transform, stable under rotations, or more generally biLipschitz pushforwards?

(2) While the individual Riesz transforms R_j are unstable under rotations of $\mathbb{R}^n$, the vector Riesz transform $\mathbf{R} = (R_1, R_2, \dots, R_n)$ is clearly rotationally stable since it is invariant under rotations. In fact, as mentioned at the beginning of the paper, Dąbrowski and Tolsa (see the top of page 6 of [9], and [43]) have demonstrated biLipschitz stability in the Ahlfors–David one-weight setting for the 1-fractional vector Riesz transform $\mathbf{R}^{1,n}$. This motivates the question of whether or not the vector Riesz transform $\mathbf{R}$ of fractional order 0 is biLipschitz stable on $\mathcal{S}_{\mathrm{doub}}$ in the two-weight setting.

2. Preliminaries: Grids, doubling, telescoping identities and dyadic testing

We begin by introducing some notation, Haar bases and the telescoping identity. Then we recall the beautiful Bellman construction used in [32] to obtain the dyadic weights V, U.

2.1. Notation for grids and cubes

Given a cube J, let $\mathcal{D}(J)$ denote the collection of dyadic subcubes of J, and for each $m \geq 0$ let $\mathcal{D}_m(J)$ denote the dyadic subcubes I of J satisfying $\ell(I) = 2^{-m}\ell(J)$. Let $\mathcal{P}(J)$ denote the collection of subcubes of J with sides parallel to the coordinate axes, and $\mathcal{P}^0 \equiv \mathcal{P}([0,1]^n)$. Unless otherwise specified, any cube mentioned in this paper is assumed to be axis-parallel, and we denote the collection of such cubes in $\mathbb{R}^n$ by $\mathcal{P}^n$. We also define $\mathcal{D}^0 \equiv \mathcal{D}([0,1]^n)$.

Given a cube $I \subset \mathbb{R}^n$, we will use the notational convention

$$I = I_1 \times I_2 \times \cdots \times I_n.$$

Given a cube $I \subset \mathbb{R}^n$, we let $\mathfrak{C}^{(k)}(I)$ denote the k-th generation dyadic grandchildren of I, and $\mathfrak{C}(I) \equiv \mathfrak{C}^{(1)}(I)$. And given a dyadic grid $\mathcal{D}$ and a cube I in the grid, we let $\pi_{\mathcal{D}} I$ denote the parent of I in $\mathcal{D}$. The same notation extends to arbitrary grids $\mathcal{K}$, like in Section 3, where $\pi_{\mathcal{K}} I$ denotes the $\mathcal{K}$-parent of I.

In dimension 1, given an interval $I \subset \mathbb{R}$, let I_- denote the left half and I_+ denote the right half; for convenience, given a cube $I \subset \mathbb{R}^n$, we also let $I_\pm \equiv (I_1)_\pm \times I_2 \times \cdots \times I_n$.

It will also be useful to keep track of the location of the children of I in higher dimensions. In $\mathbb{R}^n$, let Θ denote the 2^n locations a dyadic child cube can be in relative to its parent. For instance, when $n = 2$, we can take $\Theta \equiv \{\mathrm{NW}, \mathrm{NE}, \mathrm{SW}, \mathrm{SE}\}$ the set of four locations of a dyadic square Q within its $\mathcal{D}$-parent $\pi_{\mathcal{D}} Q$, where NW stands for Northwest, NE denotes Northeast, etc. Given a cube I and $\theta \in \Theta$, we adopt the notation that I_θ denotes the dyadic child of I at location θ.

As usual we let $|J|_\mu \equiv \int_J d\mu$ for any positive Borel measure μ in $\mathbb{R}^n$. If μ is not specified in the subscript, then $|J|$ denotes the Lebesgue measure of J. Also we define the expectation $E_J \mu \equiv \frac{1}{|J|} \int_J d\mu$. Given a locally integrable function U in $\mathbb{R}^n$, we often abbreviate the absolutely continuous measure $U(x)\, dx$ by U as well. We call U a *weight* if $0 < U(x) < \infty$ for all $x \in \mathbb{R}^n$.

2.2. Doubling

We say that two distinct cubes Q_1 and Q_2 in $\mathbb{R}^n$ are *adjacent* if there exists a cube Q for which Q_1 and Q_2 are dyadic children of Q.

Definition 2.1. Recall a measure μ on $\mathbb{R}^n$ is *doubling* if there exists a constant C such that

$$\mu(2Q) \le C\mu(Q) \quad \text{for all cubes } Q.$$

The smallest such C is called the doubling constant for μ, denoted C_{doub}. Equivalently, if μ is a doubling measure, then there exists $\lambda \ge 1$ such that for any two dyadic children I and J of an arbitrary cube K

$$\frac{E_I \mu}{E_J \mu} \in (\lambda^{-1}, \lambda).$$

The smallest such λ, denoted λ_{adj}, is referred to as the *doubling ratio* or *adjacency constant* of μ.

One may also consider the *dyadic* adjacency constant $\lambda_{\mathrm{adj}}^{\mathrm{dyad}}$ for a measure μ, which is defined as above except that we that we additionally restrict I, J to belong to a fixed dyadic grid $\mathcal{D}$, the last of which will be clear from context.

Given $\tau \in (0, 1)$, we say a doubling measure μ is τ-flat if its adjacency constant λ satisfies $\lambda, \lambda^{-1} \in (1 - \tau, 1 + \tau)$. One can make a similar definition in the dyadic setting.

For a doubling measure μ on $\mathbb{R}^n$, the closer the doubling ratio of μ is to 1, the closer C_{doub} is to 2^n: more precisely, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all doubling measures μ on $\mathbb{R}^n$, if $|\lambda_{\mathrm{adj}}(\mu) - 1| < \delta$, then $|C_{\mathrm{doub}} - 2^n| < \varepsilon$.

One can make similar definitions replacing $\mathbb{R}^n$ by an open subset, and modifying the definitions accordingly.

2.3. Telescoping identity

2.3.1. Working in the plane. We begin by discussing the telescoping identity in the plane where matters can easily be made more explicit. For each square Q in the plane define the 1-dimensional projection $\mathbb{E}_Q$ by

$$\mathbb{E}_Q f \equiv (E_Q f)\,\mathbf{1}_Q,$$

where $E_Q f \equiv \frac{1}{|Q|}\int_Q f$ is the average of f on Q. Denote the four dyadic children of a square Q in the plane by $Q_{\mathrm{NW}}, Q_{\mathrm{NE}}, Q_{\mathrm{SW}}, Q_{\mathrm{SE}}$, where NW stands for the northwest child, etc. Then define an orthonormal Haar basis $\{h_Q^{\mathrm{hor}}, h_Q^{\mathrm{vert}}, h_Q^{\mathrm{check}}\}$ associated with Q by

$$\begin{aligned}
\sqrt{|Q|}\; h_Q^{\mathrm{hor}} &\equiv +\mathbf{1}_{Q_{\mathrm{NW}}} - \mathbf{1}_{Q_{\mathrm{NE}}} + \mathbf{1}_{Q_{\mathrm{SW}}} - \mathbf{1}_{Q_{\mathrm{SE}}},\\
\sqrt{|Q|}\; h_Q^{\mathrm{vert}} &\equiv -\mathbf{1}_{Q_{\mathrm{NW}}} - \mathbf{1}_{Q_{\mathrm{NE}}} + \mathbf{1}_{Q_{\mathrm{SW}}} + \mathbf{1}_{Q_{\mathrm{SE}}},\\
\sqrt{|Q|}\; h_Q^{\mathrm{check}} &\equiv +\mathbf{1}_{Q_{\mathrm{NW}}} - \mathbf{1}_{Q_{\mathrm{NE}}} - \mathbf{1}_{Q_{\mathrm{SW}}} + \mathbf{1}_{Q_{\mathrm{SE}}},
\end{aligned}$$

where we associate the three matrices $\left[\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}\right], \left[\begin{smallmatrix} - & - \\ + & + \end{smallmatrix}\right], \left[\begin{smallmatrix} + & - \\ - & + \end{smallmatrix}\right]$, with $h_Q^{\mathrm{hor}}, h_Q^{\mathrm{vert}}, h_Q^{\mathrm{check}}$, which change sign *horizontally*, *vertically* and in a *checkerboard* pattern, respectively. Thus, we also refer to these three matrices as the horizontal matrix, vertical matrix and checkerboard matrix. Let $\triangle_Q$ denote Haar projection onto the 3-dimensional space of functions that are constant on children of Q, and that also have mean zero. Then we have the linear algebra formula

$$\begin{aligned}
\triangle_Q f &= \langle f, h_Q^{\mathrm{hor}}\rangle\, h_Q^{\mathrm{hor}} + \langle f, h_Q^{\mathrm{vert}}\rangle\, h_Q^{\mathrm{vert}} + \langle f, h_Q^{\mathrm{check}}\rangle\, h_Q^{\mathrm{check}}\\
&= \triangle_Q^{\mathrm{hor}} f + \triangle_Q^{\mathrm{vert}} f + \triangle_Q^{\mathrm{check}} f,
\end{aligned} \tag{2.1}$$

where $\triangle_Q^{\mathrm{hor}} f$ is the rank one projection $\langle f, h_Q^{\mathrm{hor}}\rangle\, h_Q^{\mathrm{hor}}$, etc.

Now given two cubes P and Q in $\mathcal{D}(P)$, with $Q \subsetneqq P$, define

$$(Q, P] \equiv \{I \in \mathcal{D}(P) : Q \subsetneqq I \subset P\}$$

to be the tower of cubes from Q to P that includes P but not Q. Similarly, define the towers (Q, P), $[Q, P]$, $[Q, P)$. Then, for $(Q, P]$, we have the well-known telescoping identity

$$\begin{aligned}
&(\mathbb{E}_Q f - \mathbb{E}_P f)\,\mathbf{1}_Q = \Big(\sum_{I \in (Q,P]} \triangle_I f\Big)\mathbf{1}_Q\\
&= \Big(\sum_{I \in (Q,P]} \langle f, h_I^{\mathrm{hor}}\rangle h_I^{\mathrm{hor}}\Big)\mathbf{1}_Q + \Big(\sum_{I \in (Q,P]} \langle f, h_I^{\mathrm{vert}}\rangle h_I^{\mathrm{vert}}\Big)\mathbf{1}_Q + \Big(\sum_{I \in (Q,P]} \langle f, h_I^{\mathrm{check}}\rangle h_I^{\mathrm{check}}\Big)\mathbf{1}_Q\\
&= \Big(\sum_{I \in (Q,P]} \triangle_I^{\mathrm{hor}} f\Big)\mathbf{1}_Q + \Big(\sum_{I \in (Q,P]} \triangle_I^{\mathrm{vert}} f\Big)\mathbf{1}_Q + \Big(\sum_{I \in (Q,P]} \triangle_I^{\mathrm{check}} f\Big)\mathbf{1}_Q.
\end{aligned}$$

2.3.2. In higher dimension. Turning now to dimension n, we note that a similar telescoping identity holds in $\mathbb{R}^n$. In particular, given a cube $Q \subset \mathbb{R}^n$, if we let $\triangle_Q$ denote the Haar projection onto the space of functions constant on the dyadic children of Q with mean 0, then

$$\triangle_Q f = \sum_{j=1}^{d(n)} \langle f, h_Q^j \rangle h_Q^j \equiv \sum_{j=1}^{d(n)} \triangle_Q^j f,$$

where $\{h_Q^j\}_{j=1}^{d(n)}$ is a choice of $L^2(Q)$ orthonormal basis for the range of $\triangle_Q$, and $d(n) = 2^n - 1$ is the dimension of this space. One of course has an analogue to the telescoping identity above. In our applications for $n \geq 2$, we will be interested in the case that $h_Q^1 = h_Q^{\mathrm{hor}}$, where for $Q = Q_1 \times \cdots \times Q_n$, we define the horizontal Haar wavelet

$$\sqrt{|Q|} h_Q^{\mathrm{hor}}(x) \equiv \begin{cases} 1 & \text{if } x \in Q_-, \\ -1 & \text{if } x \in Q_+, \\ 0 & \text{otherwise.} \end{cases}$$

We will not care about the choice of $h_Q^2, h_Q^3, \ldots, h_Q^{d(n)}$ for each cube Q, although we could simply take the orthogonal Haar basis $\{h_Q^j\}$ to be the 'standard' Haar basis $\{g_1 \otimes \cdots \otimes g_n\}$ consisting of all product functions $g_1(x_1) \times \cdots \times g_n(x_n)$ in which g_j is either the Haar function h_j on Q_j, or the normalized indicator $\frac{1}{\sqrt{|Q_j|}} \mathbf{1}_{Q_j}$, and where the constant function on Q is discarded. Note that

$$\frac{1}{\sqrt{|Q|}} s_1^{Q,\mathrm{hor}} = h_1 \otimes \frac{1}{\sqrt{|Q_2|}} \mathbf{1}_{Q_2} \otimes \cdots \otimes \frac{1}{\sqrt{|Q_n|}} \mathbf{1}_{Q_n}. \tag{2.2}$$

2.4. Horizontal dyadic testing

Given weights V and U on a cube J, define

$$\gamma^{\mathrm{hor}}(V, U; J) \equiv \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \|\triangle_I^{\mathrm{hor}} V\|_{L^2(\mathbb{R}^n)}^2 E_I U = \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} |\langle V, h_I^{\mathrm{hor}} \rangle|^2 E_I U.$$

If $\mathcal{D}$ is the dyadic grid, define the dyadic horizontal testing constant

$$\mathfrak{T}^{\mathrm{hor}}(v, u) \equiv \sup_{J \in \mathcal{D}} \frac{\gamma^{\mathrm{hor}}(V, U; J)}{E_J V}.$$

Remark 2.2. The testing constant $\mathfrak{T}^{\mathrm{hor}}(v, u)$ is the $L^2(V) \to L^2(U)$ testing condition for the 'localized' horizontal dyadic square function

$$S_J^{\mathrm{hor}} f(x) \equiv \sqrt{\sum_{\substack{I \in \mathcal{D}(J): \\ x \in I}} \frac{\|\triangle_I^{\mathrm{hor}} f\|_{L^2(\mathbb{R}^n)}^2}{|I|}} = \sqrt{\sum_{I \in \mathcal{D}(J)} \|\triangle_I^{\mathrm{hor}} f\|_{L^2(\mathbb{R}^n)}^2 \frac{\mathbf{1}_I(x)}{|I|}}.$$

Indeed, we compute

$$\begin{aligned} \int_J |S_J^{\mathrm{hor}}(\mathbf{1}_J V)(x)|^2 U(x)\, dx &= \int_J \sum_{I \in \mathcal{D}(J)} \|\triangle_I^{\mathrm{hor}}(\mathbf{1}_J V)\|_{L^2(\mathbb{R}^n)}^2 U(x) \frac{\mathbf{1}_I(x)}{|I|}\, dx \\ &= \sum_{I \in \mathcal{D}(J)} \|\triangle_I^{\mathrm{hor}}(\mathbf{1}_J V)\|_{L^2(\mathbb{R}^n)}^2 E_I U = |J| \gamma^{\mathrm{hor}}(V, U, J), \end{aligned}$$

and so the square of the dyadic testing condition for the localized horizontal square function is

$$\sup_{J\in\mathcal{D}} \frac{\int_J |S_J^{\rm hor}(\mathbf{1}_J V)(x)|^2\, U(x)\, dx}{\int_J V(x)\, dx} = \sup_{J\in\mathcal{D}} \frac{\gamma^{\rm hor}(V,U,J)}{E_J V}.$$

2.5. The Bellman construction of the dyadic weights

Definition 2.3. Given weights V and U on a cube J in $\mathbb{R}^d$, we define the dyadic A_2 constant relative to J by

$$A_2^{\rm dyadic}(V,U;J) \equiv \sup_{I\in\mathcal{D}(J)} (E_I U)(E_I V).$$

Following the Bellman construction used in [32] gives the following key result.[5]

Theorem 2.4. *Given a cube J in $\mathbb{R}^n$ and arbitrary constants $\Gamma > 0$, $\tau \in (0,1)$, there exist τ-flat weights V and U on J, with V and U constant on all cubes $I \in \mathcal{D}_m(J)$ for some $m > 0$, such that*

$$A_2^{\rm dyadic}(V,U;J) \le 1, \quad \gamma^{\rm hor}(V,U;J) > \Gamma(E_J V).$$

Furthermore, U and V are in the linear span of the finite set

$$\{\mathbf{1}_J\} \cup \{h_I^{\rm hor}\}_{I\in\mathcal{D}(J),\ell(I)\ge 2^{-(m-1)}\ell(J)}.$$

In particular, when $n = 2$, the last conclusion implies

$$\triangle_I^{\rm vert} U = \triangle_I^{\rm check} U = 0, \quad \triangle_I^{\rm vert} V = \triangle_I^{\rm check} V = 0, \quad I \in \mathcal{D}(J). \tag{2.3}$$

Proof. The dimension $n = 1$ case follows from Nazarov's Bellman argument in [30].[6]

For dimension $n \ge 2$, we show matters reduce to the $n = 1$ case. We show this for dimension $n = 2$, and a similar argument shows the same for dimension $n \ge 3$. Let $J = J_1 \times J_2$ be a square. So, given parameters Γ and τ, suppose our 1-dimensional theorem gives us weights (V_0, U_0) defined on J_1. Define U by $U(x_1,x_2) \equiv \mathbf{1}_{J_2}(x_2)U_0(x_1)$, and similarly for V. Then note that

$$E_I U = E_{I_1} U_0, \quad E_I V = E_{I_1} V_0, \quad \text{for } I \in \mathcal{D}(J).$$

Since U_0 and V_0 are τ-flat, and $A_2^{\rm dyadic}(V_0,U_0;J_1) \le 1$, the above equation shows the same must be true for V and U on J.

Then the 2-dimensional testing is given by

$$\begin{aligned}\gamma^{\rm hor}(V,U;J) &\approx \sum_{I\in\mathcal{D}(J)} \frac{|I|}{|J|} (E_{I_{\rm NW}} V + E_{I_{\rm SW}} V - E_{I_{\rm NE}} V - E_{I_{\rm SE}} V)^2 E_I U \\ &= \sum_{k=0}^{\infty} \sum_{I\in\mathcal{D}_k(J)} 2^{-2k} (E_{I_{\rm NW}} V + E_{I_{\rm SW}} V - E_{I_{\rm NE}} V - E_{I_{\rm SE}} V)^2 E_I U\end{aligned}$$

[5] A simpler Bellman proof is provided in [30]; one can also likely obtain the key result by using the disarrangement argument of [19].

[6] See also Section 3 of [32] for a stronger conclusion not used here, but which requires more difficult Hessian computations, and also requires an argument to show that their set of admissible weight pairs $\mathcal{F}_x$ is nonempty, the details of which can be found in, e.g., an earlier preprint of this article, see Lemma 12 in [1].

$$\approx \sum_{k=0}^{\infty} \sum_{K \in \mathcal{D}_k(J_1)} \sum_{\substack{I \in \mathcal{D}_k(J): \\ I_1 = K}} 2^{-2k} (E_{K_-} V_0 - E_{K_+} V_0)^2 E_K U_0$$
$$= \sum_{k=0}^{\infty} \sum_{K \in \mathcal{D}_k(J_1)} 2^{-k} (E_{K_-} V_0 - E_{K_+} V_0)^2 E_K U_0$$
$$= \sum_{k=0}^{\infty} \sum_{K \in \mathcal{D}_k(J_1)} \frac{|K|}{|J|} (E_{K_-} V_0 - E_{K_+} V_0)^2 E_K U_0$$
$$\approx \gamma^{\mathrm{hor}}(V_0, U_0; J_1),$$

which is at least $\Gamma(E_{J_1} V_0) = \Gamma(E_J V)$, which yields the first conclusions after relabeling Γ.

To see the claim about the span, since U, V are constant on squares in $\mathcal{D}_m(J)$, then U and V are bounded, and so are $L^2(J)$ functions. But the space of $L^2(J)$ functions which are constant on elements of $\mathcal{D}_m(J)$ has orthonormal basis

$$\left\{ \frac{1}{\sqrt{|J|}} \mathbf{1}_J \right\} \cup \{h_I^{\mathrm{hor}}, h_I^{\mathrm{vert}}, h_I^{\mathrm{check}}\}_{I \in \mathcal{D}(J) : \ell(I) \geq 2^{-(m-1)} \ell(J)}.$$

Thus, to show the claim about the span, it suffices to show

$$\langle h_I, U \rangle = \langle h_I, V \rangle = 0,$$

for any function h_I that is orthogonal to h_I^{hor}, has mean 0, is supported on I, and is constant on the dyadic children of I. Let h_I be such a function. Since h_I is piecewise constant on the dyadic children of I, we may expand $\langle U, h_I \rangle$ as

$$\int_I U(x) h_I(x)\,dx = E_{I_{\mathrm{NW}}} U \int_{I_{\mathrm{NW}}} h_I(x)\,dx + E_{I_{\mathrm{SW}}} U \int_{I_{\mathrm{SW}}} h_I(x)\,dx + E_{I_{\mathrm{NE}}} U \int_{I_{\mathrm{NE}}} h_I(x)\,dx + E_{I_{\mathrm{SE}}} U \int_{I_{\mathrm{SE}}} h_I(x)\,dx.$$

Substituting averages of U for averages of U_0, taking $a \equiv E_{(I_1)_-} U_0$ and $b \equiv E_{(I_1)_+} U_0$ for convenience, we get that this equals

$$a \int_{I_-} h_I(x)\,dx + b \int_{I_+} h_I(x)\,dx = \frac{a+b}{2} \int_I h_I(x)\,dx + \frac{b-a}{2} \Big(\int_{I_+} h_I(x)\,dx - \int_{I_-} h_I(x)\,dx \Big).$$

Since h_I has mean 0, the first integral on the right vanishes. Since $\langle h_I, h_I^{\mathrm{hor}} \rangle = 0$, then the last term vanishes too, and thus $\langle U, h_I \rangle = 0$. Similarly for V. ■

We will now adapt the supervisor argument of Nazarov to construct a pair of doubling weights (v, u), first on a cube in $\mathbb{R}^n$ and eventually on the whole space $\mathbb{R}^n$, satisfying $A_2(v, u) \leq 1$ and such that the first Riesz transform R_1 has operator norm $\mathfrak{N}_{R_1}(v, u) > \Gamma$, while the other Riesz transforms R_j, $j \geq 2$, have operator norm $\mathfrak{N}_{R_j}(v,u) \leq 1$. Thus, the individual Riesz transform R_1 is not stable under rotations of doubling weights in the plane. We will view the supervisor map more simply as a transplantation map that readily exploits telescoping properties of projections. To make such conclusions about the norm inequalities, we will compute a testing condition, and if V and U are τ-flat for τ sufficiently small, then the classical pivotal condition holds [12], and so we can apply the $T1$ theorem in [38] in order to deduce $\mathfrak{N}_{R_2}(v,u) \leq 1$ from the testing conditions. See Theorem 5.7 below for more details.

3. The supervisor and transplantation map

We again begin our discussion in the plane where matters are more easily pictured. We will construct our weight pair (v,u) on a square $Q^0 \subset \mathbb{R}^2$ from the dyadic weight pair (v,u) by adapting the supervisor argument of Nazarov [32] as follows.[7] Let $\{k_t\}_{t=1}^{\infty}$ be an increasing sequence of positive integers to be fixed later, and let $\mathcal{D}^0$ denote the collection of dyadic subsquares of Q^0. We denote by $\mathcal{K}_t = \mathcal{K}_t(Q^0)$ the collection of dyadic subsquares Q of Q^0 in $\mathcal{D}^0$ whose side lengths satisfy $\ell(Q) = 2^{-k_1-\cdots-k_t}\ell(Q^0)$, and then define

$$\mathcal{K} = \mathcal{K}(Q_0) = \bigcup_{t\in\mathbb{N}} \mathcal{K}_t(Q_0),$$

a subgrid of the dyadic grid $\mathcal{D}^0$. Recall that we have $\Theta \equiv \{\mathrm{NW}, \mathrm{NE}, \mathrm{SW}, \mathrm{SE}\}$, the set of four locations of a dyadic square Q within its $\mathcal{D}$-parent $\pi_{\mathcal{D}} Q$.

3.1. The informal description of the construction

Here is an informal description of the transplantation argument, that we will give precisely later on. Given a nonnegative integrable function $U \in L^1(Q^0)$ and $t \in \mathbb{N}$, we will define $u_t(x)$ to be a step function on Q^0 that is constant on each square in the t-th level $\mathcal{K}_t$ of $\mathcal{K}$, and where the constants are among the expected values of U on the squares in the t-th level $\mathcal{D}_t^0$ of $\mathcal{D}^0$, but 'scattered' according to the following plan.

To each square Q in $\mathcal{K}_t$, there is associated a unique descending '$\mathcal{K}$-tower' $\mathbf{T} = (T_1, \ldots, T_t) \in \mathcal{K}^t = \mathcal{K} \times \cdots \times \mathcal{K}$, with $T_t = Q$, where the square T_ℓ is the unique square in $\mathcal{K}_\ell$ containing Q. To each component square T_ℓ of $\mathbf{T}$, there is associated a unique $\theta_\ell \in \Theta$, which describes the location of T_ℓ within its $\mathcal{D}$-parent $\pi_{\mathcal{D}} T_\ell$. We then define $\mathcal{S}(Q)$ to be the square L in $\mathcal{D}_t^0$ which is obtained from Q^0 via the following algorithm:

(1) Set $L = Q^0$.

(2) For $\ell = 1, \ldots, t$, replace L by its dyadic child with location θ_ℓ within L.

(3) Output L.

In the terminology of Nazarov [32], $\mathcal{S}(Q)$ is the *supervisor* of Q. We then 'transplant' the expected value $E_{\mathcal{S}(Q)}U$ of U on the supervisor to the cube Q in $\mathcal{K}_t$ that is being

[7] A simpler form of 'disarranging' a weight was used in [34] to provide a counterexample to the conjecture of Muckenhoupt and Wheeden, see p. 281 in [29], that a one-tailed A_p condition was sufficient for the norm inequality for M, but the weights were not doubling.

supervised. For example, when $k_\ell = 1$ for all ℓ, this construction yields the identity

$$u_t = \mathbb{E}_t U \equiv \sum_{Q\in\mathcal{D}:\ell(Q)=2^{-t}\ell(Q^0)} (E_Q U)\mathbf{1}_Q,$$

and when the k_ℓ's are bigger than 1, the values $\frac{1}{|Q|}\int_Q U$ are 'scattered' throughout Q^0. Now we give precise details of the 'scattering' construction.

3.2. The supervisor map

We define a map

$$\mathcal{S} : \mathcal{K}_t \to \mathcal{D}_t^0,$$

for every $t \geq 0$. Given a cube $K \in \mathcal{K}_t$, $\mathcal{S}(K)$ is called the *supervisor* of K. We define it as follows. Let $K \in \mathcal{K}_t$. If $t = 0$, then $K = Q^0$ and so we define $\mathcal{S}(K)$ to be Q^0.

If $t \geq 1$, define $\theta_\ell \in \Theta$, $1 \leq \ell \leq t$, to be the unique location for which the $\mathcal{K}$-parent

$$P_\ell \equiv \pi_{\mathcal{K}}^{(t-\ell)} K$$

satisfies

$$(\pi_{\mathcal{D}} P_\ell)_{\theta_\ell} = P_\ell.$$

Then define

$$\mathcal{S}(K) \equiv (\ldots((Q^0)_{\theta_1})_{\theta_2}\ldots)_{\theta_t},$$

using the notation introduced at the beginning of Section 2.

Note that the supervisor map $\mathcal{S}$ is many-to-one, indeed $Q \in \mathcal{D}_t^0$ has $C_{t,k_1,\ldots,k_t}$ preimages under $\mathcal{S}$. Furthermore, we note that $\mathcal{S}(\pi_{\mathcal{K}} Q) = \pi_{\mathcal{D}}\mathcal{S}(Q)$, i.e., π and $\mathcal{S}$ commute.

3.3. The formal construction in the plane

Let $U \in L^1(Q^0)$ be a nonnegative integrable function, and let $t \in \mathbb{N}$. We construct u_t by 'transplanting' the expected value $E_{\mathcal{S}(Q)}U$ of U on the supervisor $\mathcal{S}(Q) \in \mathcal{D}_t^0$ to the cube $Q \in \mathcal{K}_t$ that is being supervised. Here are the precise formulas written out using the parent grid $\mathcal{P}$, where for convenience we will use superscripts to track the level of a square in the grid $\mathcal{D}$:

$$u_0(x) = (E_{Q^0}U)\,\mathbf{1}_{Q^0}(x), \quad \text{and} \quad u_t(x) = \sum_{Q\in\mathcal{K}_t} (E_{\mathcal{S}(Q)}U)\,\mathbf{1}_Q(x) \quad \text{for } t \geq 1.$$

The weights u_t are nonnegative on Q^0, since u_t is constant on each square Q in $\mathcal{K}_t$, and the value of this constant is the expectation $E_{\mathcal{S}(Q)}U$ of U on the supervisor $\mathcal{S}(Q)$, which is of course nonnegative. We also note the following useful fact: $|u_t|$ is bounded by a constant independent of the choice of $\{k_t\}_{t\geq 0}$, namely, $\|u_t\|_{L^\infty} \leq \|U\|_{L^\infty}$, since the only values u_t can take on are precisely the expectations of U over supervising cubes Q.

Recall the Haar projection $\triangle_Q$ associated with Q satisfies

$$\triangle_Q f \equiv \Big(\sum_{Q'\in\mathfrak{C}(Q)} \mathbb{E}_{Q'} f\Big) - \mathbb{E}_Q f = \Big(\sum_{Q'\in\mathfrak{C}(Q)} (E_{Q'} f)\mathbf{1}_{Q'}\Big) - (E_Q f)\mathbf{1}_Q. \tag{3.1}$$

Given cubes Q and P, let $\phi_{P\to Q}$ denote the unique translation and dilation that takes P to Q, and define

$$h_Q^{\mathrm{hor}}[P](x) \equiv h_Q^{\mathrm{hor}}(\phi_{P\to Q}(x)).$$

Note that this function does *not* have $L^2(P)$ norm equal to 1. We can also make the same definition for $h_Q^{\mathrm{vert}}[P], h_Q^{\mathrm{check}}[P]$. Finally, define

$$\begin{aligned}\triangle_Q[P]f(x) &\equiv (\triangle_Q f)(\phi_{P\to Q}(x))\\ &= \langle f, h_Q^{\mathrm{hor}}\rangle\, h_Q^{\mathrm{hor}}[P](x) + \langle f, h_Q^{\mathrm{vert}}\rangle\, h_Q^{\mathrm{vert}}[P](x) + \langle f, h_Q^{\mathrm{check}}\rangle\, h_Q^{\mathrm{check}}[P](x)\\ &\equiv \triangle_Q^{\mathrm{hor}}[P]f(x) + \triangle_Q^{\mathrm{vert}}[P]f(x) + \triangle_Q^{\mathrm{check}}[P]f(x).\end{aligned}$$

Then, using (3.1) for $t \geq 1$, the first order differences of the weights u_t are given by

$$\begin{aligned}u_{t+1}(x) - u_t(x) &= \sum_{Q\in\mathcal{K}_t}\Big\{\Big(\sum_{P\in\mathfrak{C}^{(k_{t+1}-1)}(Q)}\ \sum_{Q'\in\mathfrak{C}(P)} (E_{\mathcal{S}(Q')}U)\,\mathbf{1}_{Q'}(x)\Big)\\ &\qquad - (E_{\mathcal{S}(Q)}U)\,\mathbf{1}_Q(x)\Big\}\\ &= \sum_{Q\in\mathcal{K}_t}\Big\{\sum_{P\in\mathfrak{C}^{(k_{t+1}-1)}(Q)}\ \sum_{Q'\in\mathfrak{C}(P)} (E_{\mathcal{S}(Q')}U - E_{\mathcal{S}(Q)}U)\,\mathbf{1}_{Q'}(x)\Big\}\\ &= \sum_{Q\in\mathcal{K}_t}\Big\{\sum_{P\in\mathfrak{C}^{(k_{t+1}-1)}(Q)} \triangle_{\mathcal{S}(Q)}[P]U(x)\Big\}.\end{aligned}$$

Let $\mathcal{B}$ denote a set indexing our choice of Haar basis. Since we are working in dimension 2, we take

$$\mathcal{B} \equiv \{\mathrm{hor}, \mathrm{vert}, \mathrm{check}\}.$$

For a square Q and an integer $M \in \mathbb{N}$, we define three alternating functions, one for each pattern $\in \mathcal{B}$:

$$s_M^{Q,\mathrm{pattern}}(x) = \sum_{Q'\in\mathfrak{C}^{(M-1)}(Q)} \sqrt{|Q'|}\, h_{Q'}^{\mathrm{pattern}}, \quad \mathrm{pattern} \in \mathcal{B}. \tag{3.2}$$

Note that each of these three alternating functions is a constant ± 1 on grandchildren $P' \in \mathfrak{C}^{(M)}(Q)$ of depth M, and when restricted to a grandchild $Q' \in \mathfrak{C}^{(M-1)}(Q)$, each alternating function $s_k^{Q,\mathrm{hor}}$, $s_k^{Q,\mathrm{vert}}$ and $s_k^{Q,\mathrm{check}}$ has the arrangement of ± 1, given respectively by $\left[\begin{smallmatrix}+ & -\\ + & -\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}- & -\\ + & +\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}+ & -\\ - & +\end{smallmatrix}\right]$. For instance, $s_k^{Q,\mathrm{hor}}$ is the function on Q consisting of ± 1 arranged in the following fashion:

$$s_k^{Q,\mathrm{hor}} \sim \text{the } 2^k \times 2^k \text{ matrix} \begin{bmatrix} + & - & + & - & \cdots & + & -\\ + & - & + & - & \cdots & + & -\\ + & - & + & - & \cdots & + & -\\ + & - & + & - & \cdots & + & -\\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ + & - & + & - & \cdots & + & -\\ + & - & + & - & \cdots & + & - \end{bmatrix},$$

and similarly,

$$s_k^{Q,\text{vert}} \sim \begin{bmatrix} - & - & - & - & \cdots & - & - \\ + & + & + & + & \cdots & + & + \\ - & - & - & - & \cdots & - & - \\ + & + & + & + & \cdots & + & + \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ - & - & - & - & \cdots & - & - \\ + & + & + & + & \cdots & + & + \end{bmatrix}$$

and

$$s_k^{Q,\text{check}} \sim \begin{bmatrix} + & - & + & - & \cdots & + & - \\ - & + & - & + & \cdots & - & + \\ + & - & + & - & \cdots & + & - \\ - & + & - & + & \cdots & - & + \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ + & - & + & - & \cdots & + & - \\ - & + & - & + & \cdots & - & + \end{bmatrix}.$$

Remark 3.1. Notice that the matrix for $s_k^{Q,\text{hor}}$ is given by transplanting 2^{2k-2} copies of the 2×2 matrix $[\substack{+\,-\\+\,-}]$, which corresponds to the tensor product of a 1-dimensional Haar function with matrix $[+\,-]$, and an indicator function with matrix $[\substack{+\\+}]$.

We now write the projections $\triangle_Q U$ as a sum of the horizontal, vertical and checkerboard components as in (2.1) to obtain, for $t \geq 1$,

$$\begin{aligned} u_{t+1}(x) - u_t(x) &= \sum_{\text{pattern} \in \mathcal{B}} \sum_{Q \in \mathcal{K}_t} \Big\{ \sum_{P \in \mathfrak{C}^{(k_{t+1}-1)}(Q)} \triangle_{\mathcal{S}(Q)}^{\text{pattern}} [P] U(x) \Big\} \\ &= \sum_{\text{pattern} \in \mathcal{B}} \sum_{Q \in \mathcal{K}_t} \Big\{ \sum_{P \in \mathfrak{C}^{(k_{t+1}-1)}(Q)} \langle U, h_{\mathcal{S}(Q)}^{\text{pattern}} \rangle \, h_{\mathcal{S}(Q)}^{\text{pattern}} [P](x) \Big\} \\ &= \sum_{\text{pattern} \in \mathcal{B}} \sum_{Q \in \mathcal{K}_t} \langle U, h_{\mathcal{S}(Q)}^{\text{pattern}} \rangle \frac{1}{\sqrt{|\mathcal{S}(Q)|}} s_{k_{t+1}}^{Q,\text{pattern}}. \end{aligned} \tag{3.3}$$

3.4. The construction in dimension $n = 1$

In dimension $n = 1$, we can do the above transplantation construction, with the simple substitution

$$\mathcal{B} \equiv \{\text{hor}\}.$$

Then the transplantation construction reduces to the ‘supervisor and alternating function’ construction by Nazarov and Volberg [32]. Since there is only one choice of pattern $\mathcal{B}$ in one dimension, or alternatively, only one choice of Haar wavelet basis in one dimension, $\{\pm h^{Q,\text{hor}}\}$, we will use the simplified notation

$$s_k^Q \equiv s_k^{Q,\text{hor}} \tag{3.4}$$

in dimension 1, where $s_k^{Q,\text{hor}}$ is defined as in (3.2).

3.5. The construction in higher dimensions

Turning now to general dimension n, we may define

$$s_k^{Q,\mathrm{hor}}(x) = s_k^{Q_1}(x_1)\,\mathbf{1}_{Q_2\times\cdots\times Q_n}(x_2,\dots,x_n),$$

where s_k^Q is the 1-dimensional alternating function as in (3.4). Again, the horizontal direction indicates the direction of sign change. All of the calculations above extend to dimension n using $s_1^{Q,\mathrm{hor}}$ as part of an otherwise arbitrarily chosen basis of Haar functions for the cube $Q = Q_1 \times \cdots \times Q_n$. Again, we could consider the 'standard' Haar basis $\{g_1 \otimes \cdots \otimes g_n\}$ consisting of all product functions $g_1(x_1) \times \cdots \times g_n(x_n)$ in which g_j is either the Haar function h_j on Q_j, or the normalized indicator $\frac{1}{\sqrt{|Q_j|}}\mathbf{1}_{Q_j}$, and where the constant function on Q is discarded; we recall the definition of the horizontal Haar wavelets (2.2).

4. Weak convergence properties of the Riesz transforms

We let H denote the Hilbert transform on $\mathbb{R}$, i.e.,

$$Hf(x) \equiv \frac{1}{\pi}\ \text{p.v.}\int_{\mathbb{R}} \frac{f(x-y)}{y}\,dy,$$

and we let R_j denote the j-th individual Riesz transform on $\mathbb{R}^n$, i.e.,

$$R_j f(x) \equiv c_n\ \text{p.v.}\int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}}\, f(x-y)\,dy, \quad \text{where } c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}. \tag{4.1}$$

Note that with these choices of constants, the symbols of the operators H and R_j are $-i\,\mathrm{sgn}\,\xi$ and $-i\,\xi_j/|\xi|$, respectively. In what follows, all singular integrals are understood to be taken in the sense of principal values, even when we do not explicitly write p.v. in front of the integral. If we apply the Riesz transform R_j in the plane to the difference $u_{t+1} - u_t$ in (3.3), we obtain

$$R_j(u_{t+1}-u_t) = \sum_{\mathrm{pattern}\in\mathcal{B}}\ \sum_{Q\in\mathcal{K}_t} \langle U, h_{S(Q)}^{\mathrm{pattern}}\rangle \frac{1}{\sqrt{|S(Q)|}}\, R_j s_{k_{t+1}}^{Q,\mathrm{pattern}}$$

and, in particular, if $\triangle_P^{\mathrm{vert}}U$, $\triangle_P^{\mathrm{vert}}V$, $\triangle_P^{\mathrm{check}}U$ and $\triangle_P^{\mathrm{check}}V$ vanish for all P, then we have both

$$R_j(u_{t+1}-u_t) = \sum_{Q\in\mathcal{K}_t} \langle U, h_{S(Q)}^{\mathrm{hor}}\rangle \frac{1}{\sqrt{|S(Q)|}}\, R_j s_{k_{t+1}}^{Q,\mathrm{hor}}, \tag{4.2}$$

$$R_j(v_{t+1}-v_t) = \sum_{Q\in\mathcal{K}_t} \langle V, h_{S(Q)}^{\mathrm{hor}}\rangle \frac{1}{\sqrt{|S(Q)|}}\, R_j s_{k_{t+1}}^{Q,\mathrm{hor}}. \tag{4.3}$$

In Section 5, we will wish to establish three key testing estimates, for an arbitrarily large Γ:

(1) $\sup_Q \frac{1}{|Q|_v}\int_Q |R_1 \mathbf{1}_Q v|^2 u \geq \Gamma$,

(2) $\sup_Q \frac{1}{|Q|_v}\int_Q |R_2 \mathbf{1}_Q v|^2 u \leq 1$,

(3) $\sup_Q \frac{1}{|Q|_u}\int_Q |R_2 \mathbf{1}_Q u|^2 v \leq 1$.

As in [32], the first estimate is accomplished by inductively choosing the rapidly increasing sequence $\{k_t\}_{t=1}^m$ of positive integers so that at each stage of the construction labeled by t, the discrepancy

$$\int |R_1(v_{t+1})|^2 u_{t+1} - \int |R_1(v_t)|^2 u_t$$

looks like $\sum_{\ell(I)=2^t} \|\triangle_I^{\mathrm{hor}} V\|^2 E_I U$, whose sum over t exceeds Γ. As suggested by (4.2) and (4.3), it turns out one must then understand the convergence properties of

$$R_j s_{k_{t+1}}^{Q,\mathrm{hor}} \quad \text{for } j = 1, 2,$$

which we do in this section. For $j = 2$, we show this converges to 0 strongly from an application of the alternating series test to exploit the cancellation in $s_k^{Q,\mathrm{hor}}$. But for $j = 1$, the issue is more subtle. In dimension 1, Nazarov proved $H s_k^I \to 0$ weakly, and other subtle weak convergence properties using holomorphic function theory on the unit disc. To extend these considerations to higher dimensions, we have not managed to escape the need for holomorphic function theory, so instead we reduce the study of $R_1 s_{k_{t+1}}^{Q,\mathrm{hor}}$ to $H s_{k_{t+1}}^Q$ using the alternating series test to exploit cancellation in the $s_k^{Q,\mathrm{hor}}$ functions, from which point we can then use Nazarov's techniques. But a considerable amount of preparation is needed to prove these convergence properties. We begin with a discussion of the notion of weak convergence, which we use in connection with the alternating functions introduced in Section 3.

Given $1 < p < \infty$, recall $f_i \to 0$ weakly in $L^p(\mathbb{R}^n)$ if for all functions $b \in L^{p'}(\mathbb{R}^n)$, we have

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} f_i(x) b(x)\, dx = 0. \tag{4.4}$$

Bounded operators on $L^p(\mathbb{R}^n)$ send weakly convergent sequences to weakly convergent sequences. If $\{f_i\}$ is uniformly bounded in $L^p(\mathbb{R}^n)$ and X is a dense subset of $L^{p'}(\mathbb{R}^n)$, then $f_i \to 0$ weakly if and only if (4.4) holds for all $b \in X$. We will apply this last result when X equals $L^\infty(\mathbb{R}^n) \cap L^{p'}(\mathbb{R}^n)$, or when X is the space of compactly supported functions on $\mathbb{R}^n$ which are constant on dyadic cubes of fixed size.

We now turn to some lemmas in dimension $n = 1$ that we will use for establishing the three key testing estimates listed above.

4.1. Weak convergence properties of the Hilbert transform

In Nazarov's supervisor argument in [32], the weak limits appearing in Lemma 4.2 below, for the alternating functions s_k^I, were proved using holomorphic function theory. While the results of this subsection already appear in [30, 32], to keep this paper self-contained, we provide the proofs here along with details omitted in previous articles.

If $f \in L^p(\mathbb{R})$, then for every $z \in \mathbb{R}^2_+$, define the Poisson extension $\mathbb{P} f(z)$ of f by

$$\mathbb{P} f(z) = \mathbb{P} f(x+iy) \equiv \int_{\mathbb{R}} f(t) P_{x+iy}(t)\, dt,$$

where

$$P_{x+iy}(t) \equiv \frac{y}{(x-t)^2+y^2}$$

is the Poisson kernel. A key observation in [30, 32] was the following lemma.

Lemma 4.1. *Given $p \in (1,\infty)$, let $\{f_k\}_k$ in $L^p(\mathbb{R})$ be a bounded sequence. Then $f_k \to 0$ weakly in $L^p(\mathbb{R})$ if and only if $\lim_{k\to\infty} \mathbb{P} f_k(z) = 0$ for all $z \in \mathbb{R}^2_+$.*

Proof. If $f_k \to 0$ weakly in $L^p(\mathbb{R})$, then it is immediate that $\lim_{k\to\infty} \mathbb{P} f_k(z) = 0$ for all $z \in \mathbb{R}^2_+$.

If, on the other hand, $\lim_{k\to\infty} \mathbb{P} f_k(z) = 0$ for all $z \in \mathbb{R}^2_+$, then because finite linear combinations of Poisson kernels are dense[8] in the dual space $L^{p'}(\mathbb{R})$, and the norms $\|f_k\|_{L^p(\mathbb{R})}$ are uniformly bounded in $L^p(\mathbb{R})$, we get $f_k \to 0$ weakly in $L^p(\mathbb{R})$. ∎

In what follows, given $1 \le p < \infty$, let $H^p(\mathbb{C}^+)$ denote the functions f on $\mathbb{R}$ which are the nontangential boundary values of an analytic function on the upper-half plane

$$\mathbb{C}^+ := \{(x,y) \in \mathbb{R}^2 : y > 0\},$$

which we call f, such that

$$\sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \right)^{1/p} < \infty.$$

If $1 < p < \infty$, and if $f \in L^p(\mathbb{R})$ is real-valued, then

$$f + iHf \in H^p(\mathbb{C}^+).$$

Lemma 4.2 (Section 4 of [32]). *Suppose $p \in (1,\infty)$. With s_k^I as above, we have*

$$s_k^I \to 0, \quad Hs_k^I \to 0, \quad s_k^I Hs_k^I \to 0, \quad s_k^I (Hs_k^I)^2 \to 0, \quad (Hs_k^I)^2 \to \mathbf{1}_I,$$

weakly in $L^p(\mathbb{R})$ as $k \to \infty$. More generally, for nonnegative a, b not both zero, there exist positive constants $c_{a,b}$, with $c_{0,2} = 1$, such that

$$(s_k^I)^a (Hs_k^I)^b \to \begin{cases} 0 & \text{if } a \text{ or } b \text{ is odd,} \\ c_{a,b}\, \mathbf{1}_I & \text{if } a \text{ and } b \text{ are even,} \end{cases} \quad \text{weakly in } L^p(\mathbb{R}) \text{ as } k \to \infty.$$

Proof. Since $\lim_{k\to\infty} \int_{\mathbb{R}} s_k^I(t) g(t) dt = 0$ for all dyadic step functions g on $\mathbb{R}$, and since finite linear combinations of dyadic step functions are dense in $L^p(\mathbb{R})$, we conclude that $s_k^I \to 0$ weakly in $L^p(\mathbb{R})$. Since H is bounded on $L^p(\mathbb{R})$, we also have $Hs_k^I \to 0$ weakly in $L^p(\mathbb{R})$. Let $f_k^I \equiv s_k^I + iHs_k^I \in H^p(\mathbb{C}^+)$. By an application of Lemma 4.1 using

[8] *Hint*: Consider the unit circle $\mathbb{T} = [0, 2\pi)$. Let $f \in C(\mathbb{T})$ and $\varepsilon > 0$. For $r < 1$ sufficiently close to 1, and for n sufficiently large depending on r, we have

$$\left| P_r * f(x) - \sum_{k=0}^{n-1} \left(\int_{2\pi k/n}^{2\pi(k+1)/n} f \right) P_r\left(x - \frac{2\pi k}{n}\right) \right| \le \varepsilon.$$

$f_k^I \to 0$ weakly in $L^p(\mathbb{R})$, followed by the fact that $(\mathbb{P} f_k^I)^2$ is holomorphic and must be the Poisson extension of $(f_k^I)^2$, since they share the same boundary values, and then finally writing $(f_k^I)^2$ in terms of its real and imaginary parts, we get

$$0 = \Big[\lim_{k\to\infty} \mathbb{P} f_k^I(z)\Big]^2 = \lim_{k\to\infty} \mathbb{P}[(f_k^I)^2](z) = \lim_{k\to\infty} \mathbb{P}[(s_k^I)^2 - (Hs_k^I)^2 + i2s_k^I Hs_k^I](z)$$

for all $z \in \mathbb{C}^+$. By Lemma 4.1 again,

$$\begin{aligned} s_k^I Hs_k^I &\to 0 \quad \text{weakly in } L^p(\mathbb{R}),\\ \mathbf{1}_I - (Hs_k^I)^2 = (s_k^I)^2 - (Hs_k^I)^2 &\to 0 \quad \text{weakly in } L^p(\mathbb{R}), \end{aligned}$$

since $(s_k^I)^2 = \mathbf{1}_I$. Similarly, we see that the real part of $(f_k^I)^3$ goes to 0 weakly in $L^p(\mathbb{R})$, i.e.,

$$(s_k^I)^3 - 3(s_k^I)(Hs_k^I)^2 \to 0 \quad \text{weakly in } L^p(\mathbb{R}),$$

which gives $s_k^I(Hs_k^I)^2 \to 0$ weakly in $L^p(\mathbb{R})$, since $(s_k^I)^2 = \mathbf{1}_I$ and $s_k^I \to 0$ weakly in $L^p(\mathbb{R})$.

The more general statement involving powers a and b follows similar arguments. ∎

To carry out Nazarov's supervisor argument in [32], one also needs to understand the weak convergence of mixed terms $s_k^I(Hs_k^J)(Hs_k^K)$, where I, J and K are dyadic intervals of same side length. We will often make use of the trivial observation that if $I_1, I_2, \dots, I_N$ are pairwise disjoint sets, and the functions $a_k^{I_j}$ are supported on I_j, then $\sum_{j=1}^N a_k^{I_j} \to 0$ weakly in $L^p(\mathbb{R})$ as $k \to \infty$ if and only if $a_k^{I_j} \to 0$ weakly in $L^p(\mathbb{R})$ for each $j = 1, 2, \dots, N$.

Lemma 4.3. *Suppose $p \in (1,\infty)$. Let I, J and K be dyadic intervals all of equal side-length. Then*

$$s_k^I(Hs_k^J) \to 0 \quad \text{weakly in } L^p(\mathbb{R}) \text{ as } k \to \infty, \tag{4.5}$$

$$(Hs_k^I)(Hs_k^J) \to 0 \quad \text{weakly in } L^p(\mathbb{R}) \text{ as } k \to \infty \text{ if } I \neq J, \tag{4.6}$$

$$s_k^I(Hs_k^J)(Hs_k^K) \to 0 \quad \text{weakly in } L^p(\mathbb{R}) \text{ ask} \to \infty. \tag{4.7}$$

Proof. Let us first show (4.5). If $I = J$, this follows by Lemma 4.2, so assume $I \neq J$. Write $f_k^I \equiv s_k^I + iHs_k^I$, and similarly for J. Since $f_k^I f_k^J \in H^p(\mathbb{C}^+)$ (because H is bounded on, e.g., $L^{2p}(\mathbb{R})$), the method of proof of Lemma 4.2 combined with Lemma 4.1 implies that the real and imaginary parts of $f_k^I f_k^J$ go to 0 weakly in $L^p(\mathbb{R})$. In particular, since $s_k^I s_k^J = 0$ because of their disjoint support, we get

$$\begin{aligned} -(Hs_k^I)(Hs_k^J) &\to 0 \quad \text{weakly in } L^p(\mathbb{R}),\\ s_k^I Hs_k^J + s_k^J Hs_k^I &\to 0 \quad \text{weakly in } L^p(\mathbb{R}). \end{aligned} \tag{4.8}$$

Then (4.5) follows from the second identity in (4.8); since I and J are disjoint, it follows that $s_k^I Hs_k^J \to 0$ weakly in $L^p(\mathbb{R})$ and $s_k^J Hs_k^I \to 0$ weakly in $L^p(\mathbb{R})$. As for (4.6), it follows immediately from the first identity of (4.8).

Now let us show (4.7). Define f_k^I, f_k^J and f_k^K as above. We will expand $f_k^I f_k^J f_k^K$ into its real and imaginary parts, which by Lemma 4.1 go to 0 weakly in $L^p(\mathbb{R})$. We will consider various cases involving the dyadic intervals I, J and K.

Case 1: $I = J = K$. Then $s_k^I(Hs_k^J)(Hs_k^K) = s_k^I(Hs_k^I)^2 \to 0$ weakly in $L^p(\mathbb{R})$ by Lemma 4.2.

Case 2: $I \neq J = K$. Then using that $|s_k^I|^2 = \mathbf{1}_I$, and similarly for J, we compute the real part

$$\begin{aligned}\mathrm{Re}(f_k^I f_k^J f_k^K) &= \mathrm{Re}(f_k^I (f_k^J)^2) = \mathrm{Re}((s_k^I + iHs_k^I)(s_k^J + iHs_k^J)^2) \\ &= -2(Hs_k^I)s_k^J(Hs_k^J) - s_k^I(Hs_k^J)^2.\end{aligned}$$

Since the real part is the sum of two functions with disjoint support, by Lemma 4.1, $s_k^I(Hs_k^J)^2 \to 0$ weakly in $L^p(\mathbb{R})$.

Case 3: $I = J \neq K$ or $I = K \neq J$. Assume without loss of generality that $I = J \neq K$. Using that $s_k^J s_k^K = 0$ because they have disjoint supports, we get $f_k^I f_k^J f_k^K$ has real part

$$-2s_k^J(Hs_k^J)(Hs_k^K) - s_k^K(Hs_k^J)^2 \to 0 \quad \text{weakly in } L^p(\mathbb{R}),$$

by Lemma 4.1. But the two terms have disjoint support J and K, so each goes to 0 weakly in $L^p(\mathbb{R})$.

Case 4: I, J and K are pairwise disjoint. We compute the real part of $f_k^I f_k^J f_k^K$ equals

$$-s_k^I(Hs_k^J)(Hs_k^K) - (Hs_k^I)s_k^J(Hs_k^K) - s_k^K(Hs_k^I)(Hs_k^J) \to 0 \quad \text{weakly in } L^p(\mathbb{R}),$$

by Lemma 4.1. Since the three terms have pairwise disjoint support, then each individual term goes to 0 weakly in $L^p(\mathbb{R})$. ∎

4.2. From Hilbert to Riesz

In analogy with $(Hs_k^I)^2 \to \mathbf{1}_I$ weakly in $L^2(\mathbb{R}^n)$, we want to show that $(R_1 s_k^{P,\mathrm{hor}})^2 \to c\mathbf{1}_P$ weakly in $L^2(\mathbb{R}^n)$ for some positive constant c, and also that $R_2 s_k^{P,\mathrm{hor}} \to 0$ strongly in L^2, even L^p, as $k \to \infty$. Using real variable techniques, we will calculate matters in such a way that our claim for R_1 reduces to that of the Hilbert transform H, where the holomorphic methods used by Nazarov are available, while the claim for R_2 does not need reduction to H.

The following notation will also be useful.

Notation 4.4. Given a sequence $\{f_k\}_{k=1}^\infty$ of functions in $L^2(\mathbb{R}^n)$, we write

$$f_k = o_{k\to\infty}^{\mathrm{weakly}}(1)$$

if

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(t)\, g(t)\, dt = 0 \quad \text{for all } g \in L^2(\mathbb{R}^n),$$

and we write

$$f_k = o_{k\to\infty}^{\mathrm{strongly}}(1)$$

if

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k(t)|^p\, dt = 0 \quad \text{for all } p \in (1, \infty).$$

We first need an elementary consequence of the alternating series test.

Lemma 4.5. *If b is a bounded function on $[0,1]$ and there exists a partition*

$$\{z_0 \equiv 0 < z_1 < \cdots < z_{N-1} < z_N \equiv 1\}$$

such that b is monotone, and of one sign, on each subinterval (z_j, z_{j+1}), then

$$\left| \int b(x) s_k^{[0,1]}(x)\, dx \right| \le C N 2^{-k} \|b\|_\infty .$$

Proof. If b is monotone on $[0,1]$, and if say $b(0) > b(1) \ge 0$, then

$$\left| \int b(x) s_k^{[0,1]}(x)\, dx \right| = \left| \sum_{j=1}^{2^k} (-1)^j \int_{(j-1)/2^k}^{j/2^k} b(x)\, dx \right| \le \int_0^{1/2^k} |b(x)|\, dx \le 2^{-k} \|b\|_\infty , \tag{4.9}$$

by the alternating series test. More generally, we can apply this argument to the subinterval $[z_{m-1}, z_m]$ if the endpoints lie in $\{j2^{-k}\}_{j=0}^{2^k}$, the points of change in sign of $s_k^{[0,1]}$. In the general case, note that if we denote by $j_{m-1}/2^k$ (or $j_m/2^k$) the leftmost (or rightmost) point of the form $j/2^k$ in $[z_{m-1}, z_m]$, then the integrals in each one of the intervals $[z_{m-1}, j_{m-1}/2^k]$, $[j_{m-1}/2^k, j_m/2^k]$, and $[j_m/2^k, z_m]$ all satisfy the same bound as (4.9). ∎

We will use Lemma 4.5 to prove the following results, which encompass the technical details for the estimates in this section. In particular, Lemmas 4.9 and 4.10 below, while technical, will allow for cleaner proofs of the main results Lemma 4.11 and Lemma 4.12 of this section. The reader should keep in mind Lemmas 4.9 and 4.10 essentially follow from an application of the alternating series test Lemma 4.5. We first need to establish some notation.

Definition 4.6. A function g on $[a,b]$ is M*-piecewise monotone* if there is a partition

$$\{a = t_1 < t_2 < \cdots < t_M = b\}$$

such that g is monotone and of one sign on each subinterval (t_k, t_{k+1}), $1 \le k < M$.

Notation 4.7. For $x \in \mathbb{R}^n$ and $P = P_1 \times \cdots \times P_n$ a cube in $\mathbb{R}^n$, we write

$$x = (x_1, \dots, x_n) = (x_1, x') = (x_1, x_2, x'') = (\hat{x}, x_n) = (x_1, \tilde{x}, x_n),$$
$$P = P_1 \times \cdots \times P_n = P_1 \times P' = P_1 \times P_2 \times P'' = \widehat{P} \times P_n = P_1 \times \widetilde{P} \times P_n .$$

Definition 4.8. The common definition of the δ-halo of a cube P is given by

$$H_\delta^P \equiv \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial P) < \delta \ell(P)\}.$$

Given a cube $Q \supset P$, we define the Q-extended halo of P by

$$H_\delta^{P;Q} \equiv \{x \in Q : \operatorname{dist}(x_j, \partial P_j) < \delta \ell(P) \text{ for some } 1 \le j \le n\}.$$

We also write s_k in place of $s_k^{[-1,1]}$.

Lemma 4.9. *Let $p \in (1, \infty)$ and $M \geq 1$. Let $P = P_1 \times P'$ be a subcube of a cube $Q = Q_1 \times Q' \subset \mathbb{R} \times \mathbb{R}^{n-1}$. Furthermore, suppose that*

$$F : Q \times P_1 \times \widetilde{P} \to \mathbb{R}$$

satisfies the following three properties:

(A1) $y_1 \to F(x, y_1, \tilde{y})$ *is M-piecewise monotone for each* $(x, \tilde{y}) \in (Q \setminus H_\delta^{P;Q}) \times \widetilde{P}$ *for all* $0 < \delta < 1/2$,

(A2) $\sup_{(x,y_1,\tilde{y}) \in (Q \setminus H_\delta^{P;Q}) \times P_1 \times \widetilde{P}} |F(x, y_1, \tilde{y})| \leq C_\delta < \infty$ *for all* $0 < \delta < 1/2$,

(A3) $\mathbf{1}_{H_\delta^{P;Q}}(x) \int_{\widetilde{P}} \int_{P_1} |F(x, y_1, \tilde{y})| \, dy_1 \, d\tilde{y} \to 0$ *strongly in $L^p(Q)$ as $\delta \to 0$.*

Then

$$\int_{\widetilde{P}} \int_{P_1} F(x, y_1, \tilde{y}) s_k(y_1) \, dy_1 \, d\tilde{y} \to 0 \quad \textit{strongly in } L^p(Q) \textit{ as } k \to \infty.$$

Proof. Write

$$\begin{aligned} \int_{\widetilde{P}} \int_{P_1} F(x, y_1, \tilde{y}) s_k(y_1) \, dy_1 \, d\tilde{y} &= \left\{ \mathbf{1}_{H_\delta^{P;Q}}(x) + \mathbf{1}_{Q \setminus H_\delta^{P;Q}}(x) \right\} \\ &\quad \times \int_{\widetilde{P}} \int_{P_1} F(x, y_1, \tilde{y}) s_k(y_1) \, dy_1 \, d\tilde{y}. \end{aligned}$$

For the first term, use

$$\mathbf{1}_{H_\delta^{P;Q}}(x) \left| \int_{\widetilde{P}} \int_{P_1} F(x, y_1, \tilde{y}) s_k(y_1) \, dy_1 \, d\tilde{y} \right| \leq \mathbf{1}_{H_\delta^{P;Q}}(x) \int_{\widetilde{P}} \int_{P_1} |F(x, y_1, \tilde{y})| \, dy_1 \, d\tilde{y}$$

and assumption (A3).

For the second term we will use the alternating series test, Lemma 4.5, adapted to the interval P_1 on the integral $\int_{P_1} F(x, y_1, \tilde{y}) s_k(y_1) dy_1$, together with assumptions (A1) and (A2). Indeed, by (A1) and Lemma 4.5, we have that for $(x, \tilde{y}) \in (Q \setminus H_\delta^{P;Q}) \times \widetilde{P}$, there exists a partition $\{t_0, t_1, \dots, t_M\}$ of P_1 *depending on* $(x, \tilde{y})$, but with M *independent* of $(x, \tilde{y})$, such that

$$\left| \int_{P_1} F(x, y_1, \tilde{y}) s_k(y_1) \, dy_1 \right| \leq \sum_{j=0}^{M-1} \left| \int_{t_j}^{t_{j+1}} F(x, y_1, \tilde{y}) s_k(y_1) \, dy_1 \right| \leq C M C_\delta 2^{-k},$$

where the final inequality follows from assumption (A2). Thus, away from the halo we have uniform convergence to zero, and altogether we obtain the desired conclusion. ∎

We will also need a version of the previous lemma in which some of the y variables have been integrated out.

Lemma 4.10. *Let $p \in (1, \infty)$ and $M \geq 1$. Let $P \equiv [-1, 1]^n$, which we will sometimes write as $P_1 \times P'$, and assume P is subcube of a cube $Q = Q_1 \times Q' \subset \mathbb{R} \times \mathbb{R}^{n-1}$. Furthermore, suppose that*

$$F : Q \times P_1 \to \mathbb{R}$$

can be written as

$$F(x, y_1) = \int_{[-1,1]^{n-2}} F_{y''}(x, y_1) \, dy'',$$

where for each fixed x, the function $y_1 \to F_{y''}(x, y_1)$ does not change sign, and where the following three properties hold:

(A1′) $y_1 \to F_{y''}(x, y_1)$ *is M-piecewise monotone for each* $x \in Q$, $y'' \in [-1,1]^{n-2}$,

(A2′) $\sup_{(x,y_1)\in(Q\setminus H_\delta^{P;Q})\times P_1} |F(x,y_1)| \le C_\delta < \infty$ *for all* $0 < \delta < 1/2$,

(A3′) $\mathbf{1}_{H_\delta^{P;Q}}(x) \int_{P_1} |F(x,y_1)|\, dy_1 \to 0$ *strongly in* $L^p(Q)$ *as* $\delta \to 0$.

Then

$$\int_{P_1} F(x,y_1) s_k(y_1)\, dy_1 \to 0 \quad \textit{strongly in } L^p(Q) \textit{ as } k \to \infty.$$

Proof. This short proof is virtually identical to that of the previous lemma, but we include it for convenience. Write

$$\int_{P_1} F(x,y_1) s_k(y_1)\, dy_1 = \{\mathbf{1}_{H_\delta^{P;Q}}(x) + \mathbf{1}_{Q\setminus H_\delta^{P;Q}}(x)\} \int_{P_1} F(x,y_1) s_k(y_1)\, dy_1.$$

For the first term use

$$\mathbf{1}_{H_\delta^{P;Q}}(x) \Big| \int_{P_1} F(x,y_1) s_k(y_1)\, dy_1 \Big| \le \mathbf{1}_{H_\delta^{P;Q}}(x) \int_{P_1} |F(x,y_1)|\, dy_1$$

and assumption (A3′).

Next, the alternating series test on the integral $\int_{P_1} F_{y''}(x,y_1) s_k(y_1) dy_1$ will be used together with (A1′) and (A2′) for the second term. Indeed, by (A1′), there exists a partition $\{t_0, t_1, \dots, t_M\}$ of P_1 *depending on* x and y'', but with M *independent* of x and y'', and then from Lemma 4.5 we have, for $x \in Q \setminus H_\delta^{P;Q}$, that

$$\begin{aligned}
\Big| \int_{P_1} F(x,y_1) s_k(y_1)\, dy_1 \Big| &= \Big| \int_{[-1,1]^{n-2}} \Big\{ \int_{P_1} F_{y''}(x,y_1) s_k(y_1)\, dy_1 \Big\}\, dy'' \Big| \\
&\le \int_{[-1,1]^{n-2}} \Big| \int_{P_1} F_{y''}(x,y_1) s_k(y_1)\, dy_1 \Big| dy'' \\
&\le \int_{[-1,1]^{n-2}} \sum_{j=1}^{M-1} \Big| \int_{t_j}^{t_{j+1}} F_{y''}(x,y_1) s_k(y_1)\, dy_1 \Big|\, dy'' \\
&\le \int_{[-1,1]^{n-2}} \sum_{j=1}^{M-1} \Big\{ \int_{t_j}^{t_j+2^{1-k}} + \int_{t_{j+1}-2^{1-k}}^{t_{j+1}} \Big\} |F_{y''}(x,y_1)|\, dy_1\, dy'' \\
&\le \sum_{j=1}^{M-1} \Big\{ \int_{t_j}^{t_j+2^{1-k}} + \int_{t_{j+1}-2^{1-k}}^{t_{j+1}} \Big\} |F(x,y_1)|\, dy_1 \le CMC_\delta 2^{-k},
\end{aligned}$$

where the penultimate inequality follows since $F_{y''}$ does not change sign, and the final inequality follows from the second assumption. ■

Here is our main reduction of the action of Riesz transforms on $s_k^{P,\mathrm{hor}}(x)$ to that of the Hilbert transform H on $s_k^{P_1}(x_1)$.

Lemma 4.11. *Given $n \geq 1$, a cube $P \subset \mathbb{R}^n$ and $p \in (1, \infty)$, we have, for $x = (x_1, x') \in \mathbb{R}^1 \times \mathbb{R}^{n-1}$,*

$$R_1 s_k^{P,\mathrm{hor}}(x) = \Pi_n H s_k^{P_1}(x_1)\, \mathbf{1}_{P'}(x') + \mathrm{Err}_k^P(x),$$

where

$$\Pi_n = c_n \Lambda_n \Lambda_{n-1} \cdots \Lambda_1, \quad \Lambda_n \equiv \int_{\mathbb{R}} \frac{1}{(1+z^2)^{(n+1)/2}}\, dz > 0,$$

with c_n as in (4.1), and where the error Err_k^P tends to 0 strongly in $L^p(\mathbb{R}^n)$, i.e.,

$$\lim_{k \to \infty} \|\mathrm{Err}_k^P\|_{L^p(\mathbb{R}^n)} = 0.$$

Proof. We prove the lemma by induction on the dimension $n \geq 1$. Since $\Pi_1 = 1$, the case $n = 1$ is a tautology (with the understanding that $R_1 = H$ on $\mathbb{R}$, note that the constants in front of the integrals match) and so we now suppose that $n \geq 2$, and assume the conclusion of the lemma holds with $n-1$ in place of n.

Let $\varepsilon > 0$. For every $M > 1$, we have

$$R_1 s_k^{P,\mathrm{hor}}(x) = \mathbf{1}_{MP}(x) R_1 s_k^{P,\mathrm{hor}}(x) + \mathbf{1}_{\mathbb{R}^n \setminus MP}(x) R_1 s_k^{P,\mathrm{hor}}(x).$$

We note that the second term $\mathbf{1}_{\mathbb{R}^n \setminus MP}(x) R_1 s_k^{P,\mathrm{hor}}(x)$ goes to 0 strongly in $L^p(\mathbb{R}^n)$ as $M \to \infty$, since

$$\begin{aligned}\int_{\mathbb{R}^n \setminus MP} |R_1 s_k^{P,\mathrm{hor}}(x)|^p\, dx &\leq C \int_{\mathbb{R}^n \setminus MP} \Big(\int_P \frac{1}{|x-y|^n}\, dy \Big)^p dx \\ &\leq C \int_{\mathbb{R}^n \setminus MP} \Big(\frac{|P|}{|\operatorname{dist}(x, P)|^n} \Big)^p dx,\end{aligned}$$

which goes to 0 as $M \to \infty$, uniformly in k. In particular, we choose M such that $\int_{\mathbb{R}^n \setminus MP} |R_1 s_k^{P,\mathrm{hor}}(x)|^p dx < \varepsilon/2$ for all $k \geq 0$. With $Q = MP$, it will suffice to show that $\lim_{k\to\infty} \|\mathrm{Err}_k^P\|_{L^p(Q)} < \varepsilon/2$ for k sufficiently large, where Err_k^P is implicitly defined as in the statement of the lemma.

Without loss of generality, we assume $P = [-1,1]^n$. Recalling that $\widehat{x} = (x_1, \dots, x_{n-1})$, $\widehat{y} = (y_1, \dots, y_{n-1})$, we write

$$\begin{aligned}R_1 s_k^{P,\mathrm{hor}}(x) &= c_n \int_{-1}^{1} \int_{[-1,1]^{n-1}} \frac{(x_1 - y_1) s_k^{[-1,1]}(y_1)}{[(x_1-y_1)^2 + |x' - y'|^2]^{(n+1)/2}}\, dy_1 \cdots dy_{n-1}\, dy_n \\ &\equiv \int_{[-1,1]^{n-1}} \Psi(\widehat{x}, x_n, \widehat{y}) s_k^{[-1,1]}(y_1)\, d\widehat{y},\end{aligned}$$

where, by the change of variables $z = (x_n - y_n)/|\widehat{x} - \widehat{y}|$, we have

$$\begin{aligned}\Psi(\widehat{x}, x_n, \widehat{y}) &= c_n \int_{-1}^{1} \frac{x_1 - y_1}{[|\widehat{x} - \widehat{y}|^2 + |x_n - y_n|^2]^{(n+1)/2}}\, dy_n \\ &= \frac{c_n}{c_{n-1}} K_1^{[n-1]}(\widehat{x} - \widehat{y}) \int_{(x_n - 1)/|\widehat{x} - \widehat{y}|}^{(x_n+1)/|\widehat{x} - \widehat{y}|} \frac{1}{(1+z^2)^{(n+1)/2}}\, dz,\end{aligned}$$

and where $K_1^{[m]}$ is the kernel of the first individual Riesz transform $R_1^{[m]}$ in m dimensions. Note that

$$\Phi^{n-1}(\widehat{x}, x_n, \widehat{y}) \equiv \int_{(x_n-1)/|\widehat{x}-\widehat{y}|}^{(x_n+1)/|\widehat{x}-\widehat{y}|} \frac{1}{(1+z^2)^{(n+1)/2}}\, dz$$

is a bounded function of $(\widehat{x}, x_n, \widehat{y})$, with

$$\|\Phi^{n-1}\|_\infty \le \int_{\mathbb{R}} \frac{1}{(1+z^2)^{(n+1)/2}}\, dz = \Lambda_n > 0.$$

With $l_n(x,\widehat{y}) \equiv (x_n-1)/|\widehat{x}-\widehat{y}|$ and $u_n(x,\widehat{y}) \equiv (x_n+1)/|\widehat{x}-\widehat{y}|$, we may further decompose $\Phi^{n-1}(\widehat{x}, x_n, \widehat{y})$ as

$$\begin{aligned}
&\Big\{\int_{l_n}^{0} + \int_0^{u_n}\Big\}\frac{1}{(1+z^2)^{(n+1)/2}}\, dz \\
&\quad= \Big\{-\operatorname{sgn}(x_n-1)\int_0^{|l_n|} + \operatorname{sgn}(x_n+1)\int_0^{|u_n|}\Big\}\frac{1}{(1+z^2)^{(n+1)/2}}\, dz \\
&\quad= \Lambda_n \mathbf{1}_{P_n}(x_n) - \operatorname{sgn}(x_n-1)\Big(\int_0^{|l_n|}\frac{1}{(1+z^2)^{(n+1)/2}} dz - \frac{\Lambda_n}{2}\Big) \\
&\qquad + \operatorname{sgn}(x_n+1)\Big(\int_0^{|u_n|}\frac{1}{(1+z^2)^{(n+1)/2}}\, dz - \frac{\Lambda_n}{2}\Big) \\
&\quad\equiv \Lambda_n \mathbf{1}_{P_n}(x_n) - \operatorname{sgn}(x_n-1) L_n^1(x,\widehat{y}) + \operatorname{sgn}(x_n+1) L_n^2(x,\widehat{y}).
\end{aligned}$$

Relating the above computations to $R_1^{[n-1]}$ and $R_1^{[n]}$, we obtain

$$\begin{aligned}
R_1^{[n]} s_k^{P,\mathrm{hor}}(x) &= \frac{c_n}{c_{n-1}} \Lambda_n R_1^{[n-1]}(s_k^{[-1,1]} \otimes \mathbf{1}_{P_2\times\cdots\times P_{n-1}})(\widehat{x})\, \mathbf{1}_{P_n}(x_n) \\
&\quad - \frac{c_n}{c_{n-1}} \operatorname{sgn}(x_n-1) \int_{[-1,1]^{n-1}} \frac{x_1-y_1}{|\widehat{x}-\widehat{y}|^n} L_n^1(x,\widehat{y}) s_k^{[-1,1]}(y_1)\, d\widehat{y} \\
&\quad + \frac{c_n}{c_{n-1}} \operatorname{sgn}(x_n+1) \int_{[-1,1]^{n-1}} \frac{x_1-y_1}{|\widehat{x}-\widehat{y}|^n} L_n^2(x,\widehat{y}) s_k^{[-1,1]}(y_1)\, d\widehat{y} \\
&\equiv \frac{c_n}{c_{n-1}} \Lambda_n R_1^{[n-1]}(s_k^{[-1,1]} \otimes \mathbf{1}_{\widetilde{P}})(\widehat{x})\, \mathbf{1}_{P_n}(x_n) + \mathrm{Err}_k^1(x) + \mathrm{Err}_k^2(x).
\end{aligned}$$

We now apply our induction hypothesis to the term $R_1^{[n-1]}(s_k^{[-1,1]} \otimes \mathbf{1}_{\widetilde{P}})(\widehat{x})$ to obtain

$$\begin{aligned}
\frac{c_n}{c_{n-1}} \Lambda_n R_1^{[n-1]}(s_k^{[-1,1]} \otimes \mathbf{1}_{\widetilde{P}})(\widehat{x})\, \mathbf{1}_{P_n}(x_n) &= \Pi_n H s_k^{P_1}(x_1)\, \mathbf{1}_{\widetilde{P}}(x_2,\dots,x_n) \\
&\quad + \frac{c_n}{c_{n-1}} \Lambda_n \mathrm{Err}_k^{\widehat{P}}(\widehat{x})\, \mathbf{1}_{P_n}(x_n),
\end{aligned}$$

where $\mathrm{Err}_k^{\widehat{P}}(\widehat{x})\, \mathbf{1}_{P_n}(x_n)$ tends to 0 strongly in $L^p(Q)$ by the induction hypothesis.

So it remains only to show that both $\mathrm{Err}_k^1(x)$ and $\mathrm{Err}_k^2(x)$ go to 0 strongly in $L^p(Q)$, and by symmetry it suffices to consider just $\mathrm{Err}_k^2(x)$. We have

$$L_n^2(x,\widehat{y}) = \int_0^{|u_n|} \frac{1}{(1+z^2)^{(n+1)/2}}\, dz - \frac{\Lambda_n}{2} = -\int_{|u_n|}^{\infty} \frac{1}{(1+z^2)^{(n+1)/2}}\, dz,$$

where we recall that $|u_n| = |x_n+1|/|\widehat{x}-\widehat{y}|$.

We now see that it suffices to verify (A1)–(A3) of Lemma 4.9 for the cube Q and the function

$$F(x, y_1, \tilde{y}) = \frac{x_1 - y_1}{|\hat{x} - \hat{y}|^n} \int_{|x_n+1|/|\hat{x}-\hat{y}|}^{\infty} \frac{1}{(1+z^2)^{(n+1)/2}}\, dz. \tag{4.10}$$

We first turn to verifying property (A1), and since we only require upper bounds at this point, we will not keep track of absolute constants. The case $n = 2$ turns out to be rather special and easily handled so we dispose of that case first. We have when $n = 2$ that

$$F(x, y_1) = \frac{1}{x_1 - y_1} \int_{|u_2(x,y_1)|}^{\infty} \frac{1}{(1+z^2)^{3/2}}\, dz, \quad \text{where } u_2(x, y_1)| = \frac{|x_2 + 1|}{|x_1 - y_1|}.$$

For any fixed x, $|u_2(x, y_1)|$ is monotone as a function of $|x_1 - y_1|$. We now claim that the function $F(x, y_1)$ is M-piecewise monotone for $M = 7$ as a function of y_1. Since $F(x, y_1)$ only changes sign once, to see this it suffices to show that, with $s = |u_2(x, y_1)|$, the function

$$H_\beta(s) \equiv s \int_s^{\infty} (1 + t^2)^{-\beta}\, dt \quad \text{for } s \in (-\infty, \infty), \beta > \frac{1}{2},$$

has three changes in monotonicity on $(-\infty, \infty)$. We compute

$$H''_\beta(s) = 2\{(\beta - 1)s^2 - 1\}(1 + s^2)^{-\beta-1},$$

which has at most 2 zeros in $(-\infty, \infty)$, hence $H'_\beta(s)$ has at most 3 zeros, which proves our claim.

Now we turn to the more complicated case $n \geq 3$. Let $t = x - y$. Then we may write

$$F(x, y_1, \tilde{y}) = \frac{t_1}{(t_1^2 + |\tilde{t}|^2)^{n/2}} V_n\Big(\frac{|x_n + 1|}{(t_1^2 + |\tilde{t}|^2)^{1/2}}\Big),$$

where

$$V_n(w) \equiv \int_w^{\infty} \frac{1}{(1+z^2)^{(n+1)/2}\, dz}.$$

Note that the antiderivative

$$\int \frac{1}{(1+z^2)^{(n+1)/2}}\, dz = \int \frac{1}{(1+\tan^2\theta)^{(n+1)/2}}\, d\tan\theta = \int \frac{\sec^2\theta}{(\sec^2\theta)^{(n+1)/2}}\, d\theta$$

$$= \int \cos^{n-1}\theta\, d\theta = C_n\theta + R_n(z, \sqrt{1+z^2}), \quad z = \tan\theta, \tag{4.11}$$

where R_n is a rational function of $z = \tan\theta$ and $\sqrt{1+z^2} = \sec\theta$, and $C_n = 0$ when n is even. Indeed, one can use the well-known recursion

$$\int \cos^m\theta\, d\theta = \frac{1}{m}\cos^{m-1}\theta\sin\theta + \frac{m-1}{m}\int \cos^{m-2}\theta\, d\theta$$
$$= \frac{1}{m}\frac{1}{\sec^m\theta}\tan\theta + \frac{m-1}{m}\int \cos^{m-2}\theta\, d\theta.$$

Then, setting

$$z = \tan\theta = \frac{|x_n + 1|}{(t_1^2 + |\tilde{t}|^2)^{1/2}}, \quad E_0 \equiv \Big(\frac{x_n + 1}{|\tilde{t}|}\Big)^2, \quad E_1 \equiv \frac{|\tilde{t}|}{|x_n + 1|^n},$$

and using (4.11), we may write (4.10) as

$$\begin{aligned} F(x, y_1, \tilde{y}) &= \frac{t_1}{(t_1^2 + |\tilde{t}|^2)^{n/2}} \big\{R_n(z, \sqrt{1+z^2}) + C_n\theta + C\big\} \\ &= E_1 \tan^{n-1}\theta \sqrt{E_0 - \tan^2\theta}\big\{R_n(z, \sqrt{1+z^2}) + C_n\theta + C\big\} \equiv D_{x,\tilde{t}}(\theta). \end{aligned}$$

At this point, we employ the convention that R_n, T_n, U_n are rational functions which may change line to line, or instance to instance, but their degree will be bounded a constant depending only on the dimension n, where the degree is the sum of degrees of the numerator and denominator. Similarly, we will take M to be an integer which may change line to line or instance to instance, but will only depend on the dimension n. We also recall the fact that the function $R_n(z, \sqrt{1+z^2})$ can equal 0 or ∞ at most M times: indeed, R_n is a rational function of z and $\sqrt{1+z^2}$, which is in turn a nontrivial rational function of $\sin\theta$ and $\cos\theta$, with degree depending only on n. Thus, the number of zeros or poles it possesses is at most a constant depending only the degree, i.e., a constant which only depends n.

Now fix x and $\tilde{y}$, or equivalently x and $\tilde{t}$, and let us only consider the case when $t_1 = x_1 - y_1 > 0$, as the case $t_1 < 0$ will be similar. Then since $t_1 \mapsto \theta(t_1)$ is a decreasing injective map from $\mathbb{R}_+ \to (0, \pi/2)$, then $y_1 \mapsto F(x, y_1, \tilde{y})$ is M-piecewise monotone on $\{y_1 \in \mathbb{R} : y_1 < x_1\}$ if $\theta \mapsto D_{x,\tilde{t}}(\theta)$ is M-piecewise monotone on $(0, \pi/2)$. Since $t_1 > 0$, then F is positive and so is $D_{x,\tilde{t}}$ when $\theta > 0$, since both functions possess the same sign. Since $u \mapsto u^2$ is increasing for $u > 0$, then $D_{x,\tilde{t}}(\theta)$ is M-piecewise monotone if and only if $D_{x,\tilde{t}}(\theta)^2$ is M-piecewise monotone, which we will now show below.

In the reasoning that follows, we assume all rational functions we consider below are non-constant; in the case one of them is constant or even identically 0, the proof of M-piecewise monotonicity is even simpler than the proof below, the details of which we leave to the reader. We have

$$\begin{aligned} D_{x,\tilde{t}}(\theta)^2 &= E_1^2[E_0 - \tan^2\theta][R_n(z, \sqrt{1+z^2})\tan^{n-1}\theta + (C_n\theta + C)\tan^{n-1}\theta]^2 \\ &= R_n(z, \sqrt{1+z^2})\theta^2 + T_n(z, \sqrt{1+z^2})\theta + U_n(z, \sqrt{1+z^2}). \end{aligned}$$

To check $D_{x,\tilde{t}}(\theta)^2$ is M monotone, it suffices to show $D_{x,\tilde{t}}(\theta)^2$ has at most M critical points. For this we compute

$$\begin{aligned} \frac{d}{d\theta} D_{x,\tilde{t}}(\theta)^2 &= R_n(z, \sqrt{1+z^2})\theta^2 + T_n(z, \sqrt{1+z^2})\theta + U_n(z, \sqrt{1+z^2}) \\ &= R_n(z, \sqrt{1+z^2})\big\{\theta^2 + T_n(z, \sqrt{1+z^2})\theta + U_n(z, \sqrt{1+z^2})\big\}, \end{aligned}$$

which equals 0 or ∞ if

$$R_n(z, \sqrt{1+z^2}) = 0 \quad \text{or} \quad \infty,$$

or

$$\theta^2 + \theta R_n(z, \sqrt{1+z^2}) + T_n(z, \sqrt{1+z^2}) = 0 \quad \text{or} \quad \infty.$$

The first equality can clearly only hold for at most M values of θ. To show that the function

$$\theta^2 + \theta R_n(z, \sqrt{1+z^2}) + T_n(z, \sqrt{1+z^2})$$

can equal 0 or ∞ at most M times, it suffices to show that this function also has at most M critical points.

Its derivative is of the form

$$R_n(z, \sqrt{1+z^2})(\theta + T_n(z, \sqrt{1+z^2})),$$

which we claim equals 0 or ∞ at most M times. Indeed, R_n equals 0 or ∞ at most M times, and the function

$$\theta + T_n(z, \sqrt{1+z^2})$$

equals 0 or ∞ at most M times because its derivative is given by

$$1 + T_n(z, \sqrt{1+z^2}),$$

which in turn equals 0 or ∞ at most M times.

Thus, $y_1 \mapsto F(x, y_1, \tilde{y})$ is M-piecewise monotone for some M depending only on n, and not on the additional parameters x and $y_2, \ldots, y_n$. This completes the verification of property (A1) in Lemma 4.9.

We now verify property (A2). For any $x \in Q$, we have from (4.10) and from $|u_n| = |1 + x_n|/|\widehat{x} - \widehat{y}|$ that

$$\begin{aligned} |F(x, y_1, \tilde{y})| &\le \frac{|x_1 - y_1|}{|\widehat{x} - \widehat{y}|^n} \int_{|u_n|}^{\infty} \frac{1}{(1+z^2)^{(n+1)/2}} \, dz \\ &= \frac{|x_1 - y_1|}{|1 + x_n|^n} |u_n|^n \int_{|u_n|}^{\infty} \frac{1}{(1+z^2)^{(n+1)/2}} \, dz. \end{aligned}$$

We claim that

$$|u_n|^n \int_{|u_n|}^{\infty} \frac{1}{(1+z^2)^{(n+1)/2}} \, dz \le C_n.$$

Indeed, when $|u_n| \le 1$, this follows from integrability of the integrand, and when $|u_n| \ge 1$, this follows from a direct computation using the fact that $(1+z^2)^{(n+1)/2} \approx z^{n+1}$. Thus,

$$|F(x, y_1, \tilde{y})| \le C_n \frac{|x_1 - y_1|}{|1 + x_n|^n} \le C_{n,Q,\delta} \quad \text{when } y \in P, x \in Q \setminus H_\delta^{P;Q}.$$

Finally, we verify property (A3). To show that

$$\mathbf{1}_{H_\delta^{P;Q}}(x) \int_{-1}^{1} \int_{[-1,1]^{n-2}} |F(x, y_1, \tilde{y})| \, d\tilde{y} \, dy_1 \to 0 \quad \text{strongly in } L^p(\mathbb{R}^n) \text{ as } \delta \to 0,$$

we split

$$\begin{aligned} &\mathbf{1}_{H_\delta^{P;Q}}(x) \int_{-1}^{1} \int_{[-1,1]^{n-2}} |F(x, y_1, \tilde{y})| \, d\tilde{y} \, dy_1 \\ &\qquad \le \mathbf{1}_{H_\delta^{P;Q}}(x) \int_{\{\widehat{y} \in [-1,1]^{n-1} : |\widehat{x} - \widehat{y}| > |1 + x_n|\}} |F(x, y_1, \tilde{y})| \, d\widehat{y} \\ &\qquad\quad + \mathbf{1}_{H_\delta^{P;Q}}(x) \int_{\{\widehat{y} \in [-1,1]^{n-1} : |\widehat{x} - \widehat{y}| \le |1 + x_n|\}} |F(x, y_1, \tilde{y})| \, d\widehat{y}. \end{aligned}$$

To bound the first term, we use the estimate

$$|F(x,y_1,\tilde{y})| \le \frac{|x_1-y_1|}{|\hat{x}-\hat{y}|^n}\int_{|u_n|}^{\infty}\frac{1}{(1+z^2)^{(n+1)/2}}\,dz \le \frac{1}{|\hat{x}-\hat{y}|^{n-1}}\int_{|u_n|}^{\infty}\frac{1}{(1+z^2)^{(n+1)/2}}\,dz \le C_n\frac{1}{|\hat{x}-\hat{y}|^{n-1}}$$

and polar coordinates to get

$$\int_{\{\hat{y}\in[-1,1]^{n-1}:|\hat{x}-\hat{y}|>|1+x_n|\}}|F(x,y_1,\tilde{y})|\,d\hat{y} \le C_n\int_{\{\hat{y}\in[-1,1]^{n-1}:|\hat{x}-\hat{y}|>|1+x_n|\}}\frac{1}{|\hat{x}-\hat{y}|^{n-1}}\,d\hat{y} \le C_n\int_{\mathbb{S}^{n-2}}\int_{|1+x_n|}^{c_Q}\frac{1}{r}\,dr\,d\theta \le C_n\ln\frac{1}{\operatorname{dist}(x_n,\partial P_n)},$$

where we have used the fact that $|\hat{x}-\hat{y}|\le c_Q$. Thus,

$$\mathbf{1}_{H_\delta^{P;Q}}(x)\int_{\{\hat{y}\in[-1,1]^{n-1}:|\hat{x}-\hat{y}|>|1+x_n|\}}|F(x,y_1,\tilde{y})|\,d\hat{y}\to 0$$

strongly in $L^p(Q)$ as $\delta\to 0$.

As for the second term, for $|u_n|\ge 1$, we estimate

$$|F(x,y_1,\tilde{y})| \le \frac{|x_1-y_1|}{|\hat{x}-\hat{y}|^n}\int_{|u_n|}^{\infty}\frac{1}{(1+z^2)^{(n+1)/2}}\,dz \le \frac{1}{|\hat{x}-\hat{y}|^{n-1}}\int_{|u_n|}^{\infty}\frac{1}{(1+z^2)^{(n+1)/2}}\,dz = \frac{|u_n|^{n-1}}{|1+x_n|^{n-1}}\int_{|u_n|}^{\infty}\frac{1}{(1+z^2)^{(n+1)/2}}\,dz \le \frac{C_n}{|1+x_n|^{n-1}},$$

and so

$$\int_{\{\hat{y}\in[-1,1]^{n-1}:|\hat{x}-\hat{y}|\le|1+x_n|\}}|F(x,y_1,\tilde{y})|\,d\hat{y} \le C_n\int_{\{\hat{y}\in[-1,1]^{n-1}:|\hat{x}-\hat{y}|\le|1+x_n|\}}\frac{1}{|1+x_n|^{n-1}}\,d\hat{y}\le C_n.$$

Thus,

$$\mathbf{1}_{H_\delta^{P;Q}}(x)\int_{\{\hat{y}\in[-1,1]^{n-1}:|\hat{x}-\hat{y}|\le|1+x_n|\}}|F(x,y_1,\tilde{y})|\,d\hat{y}\to 0$$

strongly in $L^p(Q)$ as $\delta\to 0$. ∎

The next lemma is an extension of the one-dimensional lemma of Nazarov in [32].

Lemma 4.12. *Suppose $p \in (1,\infty)$. Let a and b be nonnegative integers, not both zero. Given a cube $P = P_1 \times P_2 \times \cdots \times P_n \subset \mathbb{R}^n$, we have:*

(1) $\lim_{k\to\infty} \int_{\mathbb{R}^n} (s_k^{P,\mathrm{hor}}(x))^a (R_1 s_k^{P,\mathrm{hor}}(x))^b f(x)\,dx = 0$ *for all functions* $f \in L^p(\mathbb{R}^n)$ *when a or b is odd.*

(2) $\lim_{k\to\infty} \int_{\mathbb{R}^n} (s_k^{P,\mathrm{hor}}(x))^a (R_1 s_k^{P,\mathrm{hor}}(x))^b f(x)\,dx = C_{a,b} \int_P f(x)\,dx$ *for all functions* $f \in L^p(\mathbb{R}^n)$*, when both a and b are even, and* $C_{a,b} > 0$ *and* $C_{0,2} = \Pi_n^2$.

(3) $R_j s_k^{P,\mathrm{hor}}(x)$ *tends to* 0 *strongly in* $L^p(\mathbb{R}^n)$ *as* $k \to \infty$ *for all* $p \in (1,\infty)$*, for all* $2 \le j \le n$.

Remark 4.13. A careful reading of the proofs of Lemma 4.10 and part (3) above show that for all $k \ge 1$ and $M > 1$, we have the pointwise inequality

$$|R_2 s_k^{P,\mathrm{hor}}(x)| \le C \ln \frac{1}{\mathrm{dist}(x_2, \partial P_2)} \mathbf{1}_{\{\mathrm{dist}(x_2,\partial P_2)<\delta\}}(x) + C_\delta 2^{-k} \mathbf{1}_{\{\mathrm{dist}(x_2,\partial P_2)\ge\delta\}}(x),$$

for $x \in MP$.

Proof. (1) and (2): By Lemma 4.11 , we may write

$$\begin{aligned}(s_k^{P,\mathrm{hor}}(x))^a &(R_1 s_k^{P,\mathrm{hor}}(x))^b f(x) \\ &= \Pi_n^b (s_k^{P_1}(x_1))^a (H s_k^{P_1}(x_1))^b f(x)\, \mathbf{1}_{P'}(x') + \mathrm{Err}_k^{P,f,a,b}(x),\end{aligned}$$

where $\mathrm{Err}_k^{P,f,a,b(x)} \to 0$ strongly in $L^1(Q)$, and $P' = P_2 \times \cdots \times P_n$ and $x = (x_2, \ldots, x_n)$. Thus, integrating over $\mathbb{R}^n$ and using Lemma 4.2 yields the conclusions sought.

(3) By permuting variables, we can assume without loss of generality that $j = 2$. Let $\varepsilon > 0$. Arguing as in the proof of Lemma 4.11, for every $M > 1$, we have

$$R_2 s_k^{P,\mathrm{hor}}(x) = R_2 s_k^{P,\mathrm{hor}}(x)\, \mathbf{1}_{MP}(x) + R_2 s_k^{P,\mathrm{hor}}(x)\, \mathbf{1}_{\mathbb{R}^n\setminus MP}(x).$$

We note that the second term $R_2 s_k^{P,\mathrm{hor}}(x)\, \mathbf{1}_{\mathbb{R}^n\setminus MP}(x)$ goes to 0 strongly in $L^p(\mathbb{R}^n)$ as $M \to \infty$, since

$$\begin{aligned}\int_{\mathbb{R}^n\setminus MP} |R_2 s_k^{P,\mathrm{hor}}(x)|^p\,dx &\le C \int_{\mathbb{R}^n\setminus MP} \Big(\int_P \frac{1}{|x-y|^n}\,dy \Big)^p dx \\ &\le C \int_{\mathbb{R}^n\setminus MP} \Big(\frac{|P|}{|\,\mathrm{dist}(x,P)|^n} \Big)^p dx,\end{aligned}$$

which goes to 0 as $M \to \infty$. So choose M such that $\int_{\mathbb{R}^n\setminus MP} |R_2 s_k^{P,\mathrm{hor}}(x)|^p\,dx < \varepsilon/2$. Thus, with $Q = MP$, it remains to show that $\|R_2 s_k^{P,\mathrm{hor}}\|_{L^p(Q)} < \varepsilon/2$ for k sufficiently large, which we show below.

Again we may assume that $P = [-1,1]^n$. We have

$$\begin{aligned}R_2 s_k^{P,\mathrm{hor}}(x) &= \int_{-1}^1 \int_{-1}^1 \cdots \int_{-1}^1 c_n \frac{(x_2-y_2) s_k^{[-1,1]}(y_1)}{[(x_1-y_1)^2 + |x'-y'|^2]^{(n+1)/2}}\,dy_1\,dy' \\ &\equiv \int_{-1}^1 F(x,y_1) s_k^{[-1,1]}(y_1)\,dy_1.\end{aligned}$$

For each fixed $y'' \in [-1,1]^{n-2}$, define the function

$$\begin{aligned}(4.12)\qquad F_{y''}(x,y_1) &\equiv -c_n \int_{x_2+1}^{x_2-1} \frac{t}{[(x_1-y_1)^2+t^2+|x''-y''|^2]^{(n+1)/2}}\,dt \\ &= -c_n \int_{|x_2+1|}^{|x_2-1|} \frac{t}{[(x_1-y_1)^2+t^2+|x''-y''|^2]^{(n+1)/2}}\,dt,\end{aligned}$$

where the second line follows from oddness of the kernel, and thus using the substitution $t = x_2 - y_2$, we have

$$F(x,y_1) = \int_{[-1,1]^{n-2}} F_{y''}(x,y_1)\,dy''.$$

Thus, to show $\|R_2 s_k^{P,\mathrm{hor}}\|_{L^p(Q)} \to 0$ as $k \to \infty$, it suffices to show that $F_{y''}(x,y_1)$ satisfies conditions (A1′)–(A3′) of Lemma 4.10, noting that for each fixed x, this function of y_1 does not change sign.

Condition (A1′). Note that

$$F_{y''}(x,y_1) = -c_n \int_{|x_2+1|}^{|x_2-1|} \frac{t}{[(x_1-y_1)^2+t^2+|x''-y''|^2]^{(n+1)/2}}\,dt,$$

and so differentiating in y_1 yields

$$\frac{\partial}{\partial y_1} F_{y''}(x,y_1) = c_n'(x_1-y_1) \int_{|x_2+1|}^{|x_2-1|} \frac{t}{[(x_1-y_1)^2+t^2+|x''-y''|^2]^{(n+1)/2+1}}\,dt.$$

The integral above is of one sign, and so $\frac{\partial}{\partial y_1} F_{y''}(x,y_1)$ only changes sign at $y_1 = x_1$. Thus, $F_{y''}(x,y_1)$ has at most 1 critical point in y_1, and so is 2-monotone.

Condition (A2′). By (4.12), we have

$$\begin{aligned}&|F(x,y_1)| \\ &\quad\le c_n \int_{[-1,1]^{n-2}} \left\{ \int_{\min\{|x_2+1|,|x_2-1|\}}^{\max\{|x_2+1|,|x_2-1|\}} \frac{t}{[(x_1-y_1)^2+t^2+|x''-y''|^2]^{(n+1)/2}}\,dt \right\} dy'' \\ &\quad\le c_n \int_{[-1,1]^{n-2}} \left\{ \int_{\min\{|x_2+1|,|x_2-1|\}}^{\max\{|x_2+1|,|x_2-1|\}} \frac{t}{\delta^{n+1}}\,dt \right\} dy'',\end{aligned}$$

since if $x \in Q \setminus H_\delta^{P;Q}$, then $t > \delta$ by separation. Thus,

$$|\mathbf{1}_{Q\setminus H_\delta^{P;Q}}(x) F(x,y_1)| \le C\,\frac{1}{\delta^{n+1}}.$$

Condition (A3′). Let

$$\begin{aligned}A_x &\equiv \big\{(y_1,y'') \in [-1,1]^{n-1} : |(x_1-y_1, x''-y'')| > |1-x_2|\big\},\\ B_x &\equiv \big\{(y_1,y'') \in [-1,1]^{n-1} : |(x_1-y_1, x''-y'')| < |1-x_2|\big\},\end{aligned}$$

and assume without loss of generality that $|x_2-1| \le |x_2+1|$.

For $x \in H_\delta^{P;Q}$, we have

$$\begin{aligned}
&\int_{-1}^{1} |F(x,y_1)|\,dy_1 \\
&\lesssim \int_{-1}^{1}\int_{[-1,1]^{n-2}}\Big\{\int_{\min\{|x_2+1|,|x_2-1|\}}^{\infty} \frac{t}{[(x_1-y_1)^2+t^2+|x''-y''|^2]^{(n+1)/2}}\,dt\Big\}dy''dy_1 \\
&= \frac{1}{n-1}\int_{-1}^{1}\int_{[-1,1]^{n-2}} \frac{1}{[(x_1-y_1)^2+(1-x_2)^2+|x''-y''|^2]^{(n-1)/2}}\,dy''\,dy_1 \\
&\le \Big\{\int_{A_x}+\int_{B_x}\Big\}\frac{1}{[(x_1-y_1)^2+(1-x_2)^2+|x''-y''|^2]^{(n-1)/2}}\,d(y_1,y'') \\
&\le \int_{A_x}\frac{1}{[(x_1-y_1)^2+|x''-y''|^2]^{(n-1)/2}}\,d(y_1,y'')+\int_{B_x}\frac{1}{|1-x_2|^{n-1}}\,d(y_1,y'').
\end{aligned}$$

By a crude estimate the second integral is bounded by

$$\int_{B_x}\frac{1}{|1-x_2|^{n-1}}\,d(y_1,y'') \le C_n|B_x|\frac{1}{|1-x_2|^{n-1}} \le C_n.$$

As for the first integral, integration using polar coordinates yields the upper bound

$$c\int_{|1-x_2|}^{c_n}\frac{r^{n-2}}{r^{n-1}}\,dr = c\ln\frac{c_n}{|1-x_2|} \in L^p(Q).$$

Similar estimates hold when $|x_2+1|<|x_2-1|$ and $x\in H_\delta^{P;Q}$. Hence, we can conclude that $\mathbf{1}_{H_\delta^{P;Q}}(x)\int_{-1}^{1}|F(x,y_1)|\,dy_1$ goes to 0 strongly in $L^p(Q)$ as $\delta\to 0$. ∎

Theorem 4.14. *The conclusions of Lemma 4.3, namely,* (4.5), (4.6) *and* (4.7)*, hold if one replaces* H *by* R_1 *and* $s_k^{I_1}$ *by* $s_k^{I,\mathrm{hor}}$*, and similarly for* J, K.

Proof. One argues as previously in the proofs of Lemma 4.12 parts (1) and (2), in particular using Lemmas 4.11 and Lemma 4.3. ∎

5. Boundedness properties of the Riesz transforms

We now are equipped with the convergence results we need to complete the proof of the main theorem by following the supervisor argument of Nazarov in [32]. We begin with a short formal argument, then we adapt Nazarov's supervisor argument for the Hilbert transform to the transplantation of Riesz transforms, and then complete the proof by extending our weights to all of $\mathbb{R}^n$ and showing the Riesz transform R_1 has large norm for this weight pair, while R_2 has small norm.

5.1. A brief overview of the argument

We now take $Q^0 \equiv [0,1]^n$ to be the unit cube in $\mathbb{R}^n$, and let V and U be as in Theorem 2.4. We apply the transplantation argument of Section 3 to V and U to obtain weights v_t and u_t

for all $1 \le t \le m$, with $u \equiv u_m$, $v \equiv v_m$. We will compute the R_1-testing conditions for (v, u) by first estimating them for the pair $(v_{t+1} - v_t, u_t)$. Since both V and U have Haar support on finitely many horizontal Haar wavelets in Q^0, by the estimates of Section 4, we obtain that in the limit only the diagonal terms in $[R_1(v_{t+1} - v_t)]^2$ survive the integration with u_t. Indeed, recall that

$$R_1(v_{t+1} - v_t) = \sum_{Q \in \mathcal{K}_t} \langle V, h^{\mathrm{hor}}_{\mathcal{S}(Q)} \rangle \frac{1}{\sqrt{|\mathcal{S}(Q)|}} R_1 s^{Q,\mathrm{hor}}_{k_{t+1}},$$

and the vanishing weak convergence results of Section 4 yield, for $k_{t+1} \ge C(k_1, \dots, k_t)$ and Q, Q' dyadic subcubes of $[0, 1]^n$,

$$\int R_1 s^{Q,\mathrm{hor}}_{k_{t+1}} R_1 s^{Q',\mathrm{hor}}_{k_{t+1}} u_t \to \begin{cases} 0 & \text{if } Q \ne Q' \\ (\Pi_n)^2 \int_Q u_t & \text{if } Q = Q' \end{cases} \quad \text{on } [0, 1]^n,$$

where Π_n is the constant appearing in Lemma 4.11, and so using once again the vanishing weak convergence results of Section 4, for $k_{t+1} \ge C(k_1, \dots, k_t)$, we get

$$\begin{aligned}
\int [R_1(v_{t+1} - v_t)]^2 u_t &= \int \Big[\sum_{Q \in \mathcal{K}_t} \langle V, h^{\mathrm{hor}}_{\mathcal{S}(Q)} \rangle \frac{1}{\sqrt{|\mathcal{S}(Q)|}} R_1 s^{Q,\mathrm{hor}}_{k_{t+1}} \Big]^2 u_t \\
&= \sum_{Q \in \mathcal{K}_t} \int \langle V, h^{\mathrm{hor}}_{\mathcal{S}(Q)} \rangle^2 [R_1 s^{Q,\mathrm{hor}}_{k_{t+1}}]^2 \frac{1}{|\mathcal{S}(Q)|} u_t + \text{offdiagonal} \\
&\to (\Pi_n)^2 \sum_{Q \in \mathcal{K}_t} \langle V, h^{\mathrm{hor}}_{\mathcal{S}(Q)} \rangle^2 \frac{1}{|\mathcal{S}(Q)|} \int_Q u_t,
\end{aligned}$$

and if we now sum in t, pigeonhole cubes Q based on their supervisor S, use the fact that $E_Q u_t = E_S U$, and finally $\sum_{\substack{Q \in \mathcal{K}_t \\ \mathcal{S}(Q) = S}} \frac{|Q|}{|S|} = 1$, we obtain

$$\begin{aligned}
\int \Big[R_1 \sum_{t=1}^{m-1} (v_{t+1} - v_t) \Big]^2 u_t &\approx \sum_{t=1}^{m-1} \int [R_1(v_{t+1} - v_t)]^2 u_t \\
&\approx (\Pi_n)^2 \sum_{t=1}^{m-1} \sum_{Q \in \mathcal{K}_t} \langle V, h^{\mathrm{hor}}_{\mathcal{S}(Q)} \rangle^2 \frac{1}{|\mathcal{S}(Q)|} \int_Q u_t \\
&= (\Pi_n)^2 \sum_{t=1}^{m-1} \sum_{S \in \mathcal{D}_t} \sum_{\substack{Q \in \mathcal{K}_t \\ \mathcal{S}(Q) = S}} \langle V, h^{\mathrm{hor}}_S \rangle^2 E_Q u_t \frac{|Q|}{|S|} \\
&= (\Pi_n)^2 \sum_{t=1}^{m-1} \sum_{S \in \mathcal{D}_t} \sum_{\substack{Q \in \mathcal{K}_t \\ \mathcal{S}(Q) = S}} \langle V, h^{\mathrm{hor}}_S \rangle^2 E_S U \frac{|Q|}{|S|} \\
&= (\Pi_n)^2 \sum_{t=1}^{m-1} \sum_{S \in \mathcal{D}_t} \langle V, h^{\mathrm{hor}}_S \rangle^2 E_S U > (\Pi_n)^2 \, \Gamma(E_{[0,1]^n} V),
\end{aligned}$$

which shows that testing for R_1 blows up, and hence two-weight norm for R_1 blows up as well. On the other hand, we will see that dyadic testing for R_2 is controlled by the dyadic A_2 condition, namely,

$$\sup_{Q\in\mathcal{D}([0,1]^n)} \frac{1}{|Q|_v}\int_Q |R_2\mathbf{1}_Q v|^2 u + \sup_{Q\in\mathcal{D}([0,1]^n)} \frac{1}{|Q|_u}\int_Q |R_2\mathbf{1}_Q u|^2 v \lesssim A_2^{\text{dyadic}}(u,v;[0,1]^n),$$

for $k_1, k_2, \ldots, k_m$ all chosen large enough in an inductive fashion. To make this formal argument precise in the next subsection, we follow the scheme in [32] for R_1, while the scheme for R_2 is our own.

This gives us weights (v,u) in the unit cube $[0,1]^n$. We then extend these weights periodically to the plane (with an additional small decay term), so that they continue to fail the norm inequality for R_1 (since the testing condition is large), while the *dyadic* testing condition for R_2 holds. However, our weights will be doubling with doubling constant close to Lebesgue measure. So we will be able to leverage the $T1$ theorem of [38] and doubling to show that dyadic testing for R_2 implies that the norm inequality holds for R_2. Thus, we will have constructed a weight pair for which R_2 is norm bounded, but R_1 is not, i.e., this weight pair will be rotationally unstable.

5.2. The Nazarov argument for Riesz transforms

We now continue to carry out our adaptation of Nazarov's supervisor argument to the higher-dimensional setting of the supervisor and transplantation map. Equipped with the supervisor and transplantation map, and the weak convergence results above, this remaining argument follows the proof in [32] for R_1, but we include additional details that were omitted in [32] which will clarify the presentation here. The argument for R_2 is new, however.

Recall that $\{k_t\}_{t=0}^{\infty}$ is a strictly increasing sequence of nonnegative integers $k_t \in \mathbb{Z}_+$ with $k_0 = 0$, and whose members will be chosen sufficiently large in the arguments below. We define

$$\mathcal{K} \equiv \dot{\bigcup_{t=0}^{\infty}} \mathcal{K}_t \quad \text{where } \mathcal{K}_0 = \{Q^0\} = \{[0,1]^n\}$$

and

$$\mathcal{K}_t \equiv \{Q \in \mathcal{D}(Q^0) : \ell(Q) = 2^{-k_1-k_2-\cdots-k_t}\}, \quad t \geq 1.$$

Proposition 5.1 (Nazarov [32] in the case of the Hilbert transform). *For every* $\Gamma > 1$ *and* $0 < \tau < 1$, *there exist positive weights* u, v *on the unit cube* $Q^0 \equiv [0,1]^n$ *satisfying*

$$\int_{[0,1]^n} |R_1 v(x)|^2\, u(x)\, dx \geq \Gamma \int_{[0,1]^n} v(x)\, dx,$$

$$\int_I |R_2 \mathbf{1}_I v(x)|^2\, u(x)\, dx \leq \int_I v(x)\, dx \qquad \text{for all dyadic cubes } I \in \mathcal{D}^0,$$

$$\int_I |R_2 \mathbf{1}_I u(x)|^2 v(x)\, dx \leq \int_I u(x)\, dx \qquad \text{for all dyadic cubes } I \in \mathcal{D}^0,$$

$$\Big(\frac{1}{|I|}\int_I u(x)\, dx\Big)\Big(\frac{1}{|I|}\int_I v(x)\, dx\Big) \leq 1 \quad \text{for all dyadic cubes } I \in \mathcal{D}^0,$$

and

$$(5.1)\qquad 1-\tau<\frac{E_J u}{E_K u},\frac{E_J v}{E_K v}<1+\tau \quad \textit{for adjacent dyadic cubes } J,K\in\mathcal{D}^0,$$

where J *and* K *in* (5.1) *need not be dyadic siblings, only adjacent.*

Proof. Let V and U be as arising from Theorem 2.4 with $\gamma(V,U,Q^0)/(E_{Q^0}V)>\Gamma'$ sufficiently large. We apply the transplantation argument of Section 3 to V and U to obtain nonnegative weights v_t and u_t with $1\le t\le m$, and set

$$u\equiv u_m,\quad v\equiv v_m,$$

where m is as in Theorem 2.4. It will be convenient to denote, respectively, the differences

$$\eta_{t+1}\equiv u_{t+1}-u_t,\quad \delta_{t+1}\equiv v_{t+1}-v_t.$$

Note that, by (2.3) and (3.3), η_t and δ_t are of the form

$$\sum_{Q\in\mathcal{K}_t} c_Q\frac{1}{\sqrt{|\mathcal{S}(Q)|}}s_{k_{t+1}}^{Q,\mathrm{hor}}=o^{\mathrm{weakly}}_{k_{t+1}\to\infty}(1),$$

because the constants c_Q depend only on the levels 1 through t of the construction and the number of terms in the sum only depends on $k_1,\dots,k_t$. We may then write

$$u\equiv(E_{Q^0}U)\mathbf{1}_{Q^0}+\sum_{t=0}^{m-1}\sum_{Q\in\mathcal{K}_t}\langle U,h^{\mathrm{hor}}_{\mathcal{S}(Q)}\rangle\frac{1}{\sqrt{|\mathcal{S}(Q)|}}s_{k_{t+1}}^{Q,\mathrm{hor}},$$

$$v\equiv(E_{Q^0}V)\mathbf{1}_{Q^0}+\sum_{t=0}^{m-1}\sum_{Q\in\mathcal{K}_t}\langle V,h^{\mathrm{hor}}_{\mathcal{S}(Q)}\rangle\frac{1}{\sqrt{|\mathcal{S}(Q)|}}s_{k_{t+1}}^{Q,\mathrm{hor}}.$$

We will now focus on the ‘testing’ constants

$$\frac{1}{|[0,1]^n|_v}\int_{[0,1]^n}|R_1v(x)|^2u(x)\,dx$$

and

$$\sup_{Q\in\mathcal{D}(Q^0)}\frac{1}{|Q|_v}\int_Q|R_2\mathbf{1}_Qv|^2u,\qquad \sup_{Q\in\mathcal{D}(Q^0)}\frac{1}{|Q|_u}\int_Q|R_2\mathbf{1}_Qu|^2v,$$

and show that the first is large, and second and third are small, provided we take the integers k_t sufficiently large in an inductive fashion. To tackle the first testing constant, define the discrepancy for R_1 on $Q^0=[0,1]^n$ by

$$\mathrm{Disc}(t)\equiv\int_{Q^0}(R_1\mathbf{1}_Qv_{t+1}(x))^2u_{t+1}(x)\,dx-\int_{Q^0}(R_1\mathbf{1}_Qv_t(x))^2u_t(x)\,dx.$$

We begin with the decomposition

$$\begin{aligned}
\mathrm{Disc}(t) &= \int_{Q^0} (R_1 \mathbf{1}_{Q^0} \delta_{t+1} + R_1 \mathbf{1}_{Q^0} v_t)^2 u_{t+1} - \int_{Q^0} (R_1 \mathbf{1}_{Q^0} v_t)^2 u_t \\
&= \int_{Q^0} (R_1 \mathbf{1}_{Q^0} \delta_{t+1})^2 u_{t+1} + \int_{Q^0} \{2(R_1 \mathbf{1}_{Q^0} \delta_{t+1})(R_1 \mathbf{1}_{Q^0} v_t)\}(u_t + \eta_{t+1}) \\
&\quad + \int_{Q^0} (R_1 \mathbf{1}_{Q^0} v_t)^2 (u_{t+1} - u_t) \\
&= \langle (R_1 \mathbf{1}_{Q^0} \delta_{t+1})^2, u_{t+1} \rangle_{L^2(Q^0)} + 2\langle (R_1 \mathbf{1}_{Q^0} \delta_{t+1})(R_1 \mathbf{1}_{Q^0} v_t), u_t \rangle_{L^2(Q^0)} \\
&\quad + 2\langle (R_1 \mathbf{1}_{Q^0} \delta_{t+1})(R_1 \mathbf{1}_{Q^0} v_t), \eta_{t+1} \rangle_{L^2(Q^0)} + \langle (R_1 \mathbf{1}_{Q^0} v_t)^2, \eta_{t+1} \rangle_{L^2(Q^0)} \\
&\equiv A + B + C + D.
\end{aligned}$$

We first claim that

$$\mathrm{Disc}(t) = (\Pi_n)^2 \sum_{I \in \mathcal{D}:\ \ell(I) = 2^{-t}} (\triangle_I^{\mathrm{hor}} V)^2 (E_I U) + \sum_{r=0}^{t} o_{k_{r+1} \to \infty}(1).$$

We will see in a moment that A is the main term. Using that v_t, u_t and δ_{t+1}, η_{t+1} are supported in $[0,1]^n$,

$$B = 2\langle (R_1 v_t) u_t, R_1 \delta_{t+1} \rangle_{L^2([0,1]^n)} = -2\langle R_1[(R_1 v_t) u_t], \delta_{t+1} \rangle_{L^2([0,1]^n)} = o_{k_{t+1} \to \infty}(1),$$

since the function $R_1[(R_1 v_t) u_t] \in L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$, and in particular belongs to $L^2(\mathbb{R}^n)$, and is independent of k_{t+1}, and finally since $\delta_{t+1} = o^{\mathrm{weakly}}_{k_{t+1} \to \infty}(1)$. Similarly, since $R_1 v_t \in L^4(\mathbb{R}^2)$, we have

$$D = \langle (R_1 v_t)^2, \eta_{t+1} \rangle_{L^2([0,1]^n)} = o_{k_{t+1} \to \infty}(1).$$

For term C, we have

$$\begin{aligned}
C &= 2\langle (R_1 \delta_{t+1})(R_1 v_t), \eta_{t+1} \rangle_{L^2([0,1]^n)} \\
&= 2 \int_{[0,1]^n} \Big(\sum_{Q \in \mathcal{K}_t} \langle V, h^{\mathrm{hor}}_{\mathcal{S}(Q)} \rangle R_1 \frac{1}{\sqrt{|\mathcal{S}(Q)|}} s^{Q,\mathrm{hor}}_{k_{t+1}} \Big) (R_1 v_t) \\
&\quad \times \Big(\sum_{Q' \in \mathcal{K}_t} \langle U, h^{\mathrm{hor}}_{\mathcal{S}(Q')} \rangle \frac{1}{\sqrt{|\mathcal{S}(Q')|}} s^{Q',\mathrm{hor}}_{k_{t+1}} \Big) \\
&= 2 \sum_{Q, Q' \in \mathcal{K}_t} \langle V, h^{\mathrm{hor}}_{\mathcal{S}(Q)} \rangle \langle U, h^{\mathrm{hor}}_{\mathcal{S}(Q')} \rangle \int_{[0,1]^n} R_1 \frac{1}{\sqrt{|\mathcal{S}(Q)|}} s^{Q,\mathrm{hor}}_{k_{t+1}} (R_1 v_t) \frac{1}{\sqrt{|\mathcal{S}(Q')|}} s^{Q',\mathrm{hor}}_{k_{t+1}} \\
&= o_{k_{t+1} \to \infty}(1),
\end{aligned}$$

by Theorem 4.14, since $\mathcal{K}_t$ and $R_1 v_t$ are both independent of k_{t+1}, while

$$(R_1 s^{Q,\mathrm{hor}}_{k_{t+1}}) s^{Q',\mathrm{hor}}_{k_{t+1}} \to 0 \quad \text{weakly in } L^2(\mathbb{R}^n).$$

Finally, for term A, we have

$$A = \langle (R_1 \delta_{t+1})^2, u_{t+1} \rangle_{L^2([0,1]^n)}$$
$$= \Big\langle \Big(\sum_{Q \in \mathcal{K}_t} \langle V, h^{\text{hor}}_{\mathcal{S}(Q)} \rangle R_1 \frac{1}{\sqrt{|\mathcal{S}(Q)|}} s^{Q,\text{hor}}_{k_{t+1}} \Big)^2, u_{t+1} \Big\rangle_{L^2([0,1]^n)}.$$

We first note that if the sum is taken outside the square, so that we consider only the 'diagonal' terms, we have

$$\Big\langle \sum_{Q \in \mathcal{K}_t} \Big(\langle V, h^{\text{hor}}_{\mathcal{S}(Q)} \rangle R_1 \frac{1}{\sqrt{|\mathcal{S}(Q)|}} s^{Q,\text{hor}}_{k_{t+1}} \Big)^2, u_{t+1} \Big\rangle$$
$$= \sum_{Q \in \mathcal{K}_t} \frac{1}{|\mathcal{S}(Q)|} \langle V, h^{\text{hor}}_{\mathcal{S}(Q)} \rangle^2 \Big\{ \langle (R_1 s^{Q,\text{hor}}_{k_{t+1}})^2, u_t \rangle + \langle (R_1 s^{Q,\text{hor}}_{k_{t+1}})^2, \eta_{t+1} \rangle \Big\}$$
$$= (\Pi_n)^2 \Big\{ \sum_{Q \in \mathcal{K}_t} \langle V, h^{\text{hor}}_{\mathcal{S}(Q)} \rangle^2 \frac{|Q|}{|\mathcal{S}(Q)|} E_{\mathcal{S}(Q)} U \Big\}$$
$$+ \Big\{ \sum_{Q \in \mathcal{K}_t} \frac{1}{|\mathcal{S}(Q)|} \langle V, h^{\text{hor}}_{\mathcal{S}(Q)} \rangle^2 \langle (R_1 s^{Q,\text{hor}}_{k_{t+1}})^2, \eta_{t+1} \rangle \Big\} + o_{k_{t+1} \to \infty}(1)$$
$$\equiv F + G + o_{k_{t+1} \to \infty}(1),$$

by Lemma 4.12 (2) for k_{t+1} sufficiently large, and since $\frac{1}{|Q|} \int_Q u_t = E_{\mathcal{S}(Q)} U$. To compute F, we pigeonhole the cubes $Q \in \mathcal{K}_t$ according to their supervisors $S = \mathcal{S}(Q)$,

$$\frac{F}{\Pi_n^2} = \sum_{S \in \mathcal{D}_t} \sum_{\substack{Q \in \mathcal{K}_t \\ \mathcal{S}(Q) = S}} \langle V, h^{\text{hor}}_{\mathcal{S}(Q)} \rangle^2 \frac{|Q|}{|\mathcal{S}(Q)|} E_{\mathcal{S}(Q)} U$$
$$= \sum_{S \in \mathcal{D}_t} \langle V, h^{\text{hor}}_S \rangle^2 E_S U \sum_{\substack{Q \in \mathcal{K}_t \\ \mathcal{S}(Q) = S}} \frac{|Q|}{|\mathcal{S}(Q)|} = \sum_{S \in \mathcal{D}_t} \langle V, h^{\text{hor}}_S \rangle^2 E_S U.$$

However, to compute G, using the definition $\eta_{t+1} = \sum_{Q \in \mathcal{K}_t} \langle U, h^{\text{hor}}_{\mathcal{S}(Q)} \rangle \frac{1}{\sqrt{|\mathcal{S}(Q)|}} s^{Q,\text{hor}}_{k_{t+1}}$, we have

$$G = \sum_{Q, Q' \in \mathcal{K}_t} \frac{1}{|\mathcal{S}(Q)|} \langle V, h^{\text{hor}}_{\mathcal{S}(Q)} \rangle^2 \langle U, h^{\text{hor}}_{\mathcal{S}(Q')} \rangle \langle (R_1 s^{Q,\text{hor}}_{k_{t+1}})^2, s^{Q',\text{hor}}_{k_{t+1}} \rangle = o_{k_{t+1} \to \infty}(1),$$

by Theorem 4.14, and thus we conclude that the sum of the diagonal terms equals

$$\Pi_n^2 \sum_{S \in \mathcal{D}_t} \langle V, h^{\text{hor}}_S \rangle^2 E_S U + \sum_{r=0}^{t} o_{k_{r+1} \to \infty}(1).$$

Turning now to the sum of the off diagonal terms,

$$\sum_{Q \neq Q' \in \mathcal{K}_t} \frac{1}{\sqrt{|\mathcal{S}(Q)|}} \frac{1}{\sqrt{|\mathcal{S}(Q')|}} \big\langle R_1 [\langle V, h^{\text{hor}}_{\mathcal{S}(Q)} \rangle s^{Q,\text{hor}}_{k_{t+1}}] R_1 [\langle V, h^{\text{hor}}_{\mathcal{S}(Q')} \rangle s^{Q',\text{hor}}_{k_{t+1}}], u_{t+1} \big\rangle,$$

we see that they all tend to 0 weakly as $k_{t+1} \to \infty$ by Theorem 4.14. Indeed, we write $u_{t+1} = u_t + \eta_{t+1}$, and split η_{t+1} into a linear combination of functions $s^{L,\text{hor}}_{k_{t+1}}$, noting that the resulting number of terms in the above display is independent of k_{t+1} and that each such term tends to 0 as $k_{t+1} \to \infty$ by Theorem 4.14. Thus, we can choose the components of the sequence $\{k_t\}_{t=1}^m$ sufficiently large that

$$\int_{[0,1]^n} |R_1 v(x)|^2 u(x)\, dx \geq (\Gamma' - C A_2^{\text{dyadic}}(V, U, [0,1]^n)) \int_{[0,1]^n} v(x)\, dx,$$

since we also have

$$\begin{aligned}\int_{[0,1]^n} |R_1 v_0(x)|^2 u_0(x)\, dx &= \int_{[0,1]^n} |R_1 \mathbf{1}_{[0,1]^n} E_{[0,1]^n} V|^2 \mathbf{1}_{[0,1]^n} E_{[0,1]^n} U\, dx \\ &= (E_{[0,1]^n} V)^2 (E_{[0,1]^n} U) \int_{[0,1]^n} |R_1 \mathbf{1}_{[0,1]^n}|^2\, dx \\ &= C (E_{[0,1]^n} V)^2 (E_{[0,1]^n} U) \leq C A_2^{\text{dyadic}}(V, U; [0,1]) E_{[0,1]^n} V.\end{aligned}$$

Our next task is to show that the two testing conditions for R_2 are finite. They are symmetric, so it suffices to show the bound only for the testing condition with u outside the operator. We will argue so using Lemma 4.12 (3) and Theorem 4.14. Let $Q \in \mathcal{D}^0$, and for convenience let $k_0 \equiv 0$. We first consider the case that there exists $t = t(Q)$ such that $2^{-k_0-k_1-k_2-\cdots-k_t} \leq \ell(Q) < 2^{-k_0-k_1-k_2-\cdots-k_{t-1}}$. We will deal later with the remaining cubes Q for which such a t does not exist. Note that at each stage t, there are only finitely many cubes $Q \in \mathcal{D}^0$ such that $\ell(Q) \geq 2^{-k_0-k_1-k_2-\cdots-k_t}$, and hence will only have to consider finitely many error terms which are $o_{k_{t+1}\to\infty}(1)$. Writing $u = u_t + \sum_{s=t+1}^m \eta_s$ and $v = v_t + \sum_{s=t+1}^m \delta_s$, we then compute

$$\begin{aligned}&\int_Q |R_2 \mathbf{1}_Q v(x)|^2 u(x)\, dx \\ &\quad\lesssim \int_Q |R_2 \mathbf{1}_Q (v_t)(x)|^2 u(x)\, dx + \int_Q \Big| R_2 \mathbf{1}_Q \Big(\sum_{s=t+1}^m \delta_s \Big)(x) \Big|^2 u(x)\, dx \\ &\quad= \int_Q |R_2 \mathbf{1}_Q (v_t)(x)|^2 u_t(x)\, dx + \int_Q |R_2 \mathbf{1}_Q (v_t)(x)|^2 \Big(\sum_{s=t+1}^m \eta_s(x) \Big)\, dx \\ &\qquad+ \int_Q \Big| R_2 \mathbf{1}_Q \Big(\sum_{s=t+1}^m \delta_s \Big)(x) \Big|^2 u(x)\, dx \\ &\quad\equiv |Q|_v (\text{main} + \text{Err}_1 + \text{Err}_2).\end{aligned}$$

We first claim Err_2 can be made arbitrarily small, so long as $k_{t+1}, k_{t+2}, \ldots, k_m$ are all chosen sufficiently large. Indeed, we use $u(x) \leq \|U\|_\infty$ independent of the choice of $k_1, \ldots, k_m$, which gives, using Lemma 4.12 (3),

$$\begin{aligned}\text{Err}_2 &= \frac{1}{|Q|_v} \int_Q \Big| R_2 \mathbf{1}_Q \Big(\sum_{s=t+1}^m \delta_s \Big)(x) \Big|^2 u(x)\, dx \\ &\leq \frac{\|U\|_\infty}{|Q|_v} \int_Q \Big| R_2 \mathbf{1}_Q \Big(\sum_{s=t+1}^m \delta_s \Big)(x) \Big|^2 dx \to 0 \quad \text{as } k_{t+j} \to \infty, j = 1, 2, \ldots, m-t,\end{aligned}$$

where we recall that $t = t(Q)$.

As for Err_1, it too can be made arbitrarily small by choosing k_{t+1} sufficiently large, and using the strong convergence of $\eta_{t+j} \to 0$ in $L^p(\mathbb{R}^n)$ for all $j \geq 1$ by Lemma 4.12 (3), as $R_2 \mathbf{1}_Q v_t$ only depends on $k_0, k_1, \ldots, k_t$ and is hence independent of k_{t+j} for $j \geq 1$.

So we are left with estimating the term main. Note now that $E_Q v_t = E_Q v = E_{\mathcal{S}(Q^*)} V$, where Q^* is the unique cube in $\mathcal{K}_t$ containing Q. Note as well that v_t is constant on each $I \in \mathcal{K}_{t+1}$, and satisfies the pointwise estimate

$$\mathbf{1}_Q(x) v_t(x) \leq (E_{\mathcal{S}(Q^*)} V)(1 + \tau),$$

since v_t inherits dyadic τ-flatness from V; similarly for u_t. Then applying the pointwise estimate to u_t, followed by the estimate $\|R_2 \mathbf{1}_Q v_t\|_{L^2(\mathbb{R}^n)} \leq \|\mathbf{1}_Q v_t\|_{L^2(\mathbb{R}^n)}$ by boundedness of R_2, and then the pointwise estimate applied to v_t, we get

$$\begin{aligned}\int_Q (R_2 \mathbf{1}_Q v_t)^2 u_t \, dx &\leq (1 + \tau)(E_{\mathcal{S}(Q^*)} U) \int_Q (R_2 \mathbf{1}_Q v_t)^2 \, dx \\ &\leq (1 + \tau)(E_{\mathcal{S}(Q^*)} U) \int_Q (v_t)^2 \, dx \leq (1 + \tau)^3 (E_{\mathcal{S}(Q^*)} U)(E_{\mathcal{S}(Q^*)} V)^2 |Q|.\end{aligned}$$

Since $A_2^{\text{dyadic}}(V, U; Q^0) \leq 1$, the above is controlled by

$$(1 + \tau)^3 (E_{\mathcal{S}(Q^*)} V) |Q| = (1 + \tau)^3 (E_Q v) |Q| = (1 + \tau)^3 \int_Q v.$$

Finally, we consider cubes Q for which $t(Q)$ does not exist, i.e., cubes Q such that $\ell(Q) < 2^{-k_0 - k_1 - k_2 - \cdots - k_m}$. Then v, u are constant on Q with $E_Q v = E_{\mathcal{S}(Q^*)} V$ and $E_Q u = E_{\mathcal{S}(Q^*)} U$, where Q^* is the unique cube in $\mathcal{K}_m$ which contains Q. Thus,

$$\int_Q (R_2 \mathbf{1}_Q v)^2 u = (E_{\mathcal{S}(Q^*)} V)^2 (E_{\mathcal{S}(Q^*)} U) \int_Q (R_2 \mathbf{1}_Q)^2 \leq (E_{\mathcal{S}(Q^*)} V) |Q| = \int_Q v,$$

where in the inequality we used $(E_{\mathcal{S}(Q^*)} V)(E_{\mathcal{S}(Q^*)} U) \leq 1$ and $\|R_2\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = 1$.

Since $\tau \in (0, 1)$, we obtain that the dual testing constant for R_2 on dyadic cubes is bounded; similarly for the testing constant on dyadic cubes.

Finally, to remove the restriction that J and K must be dyadic siblings from (5.1), one can modify the transplantation argument following [32], as described in Appendix A.4. However, complete proofs were not provided in [32] and we invite the reader to consult Appendix A.4 for missing details, namely, Lemma A.17; see also [30] and [19]. We also explicitly point out that this modification of transplantation will not affect any of the limiting arguments above involving taking k_t sufficiently large for each t, and by Remark A.16, the dyadic A_2 condition will be unaffected.

Finally, by multiplying v, u by an appropriate (small) positive constant, we obtain the statements in the theorem with the required constants. ∎

Remark 5.2. The weights $u(x), v(x)$ in $[0, 1]^n$ constructed in the proof of Proposition 5.1 depend only on the first variable x_1 of x.

Remark 5.3. A careful reading of the proof shows that our weights v, u satisfy the L^p-testing and dual L^p-testing conditions for the operator R_2 when $p \in (1, \infty)$. Thus, if there was a $T1$ theorem for L^p with doubling weights, our results regarding R_2 would extend to L^p. See [40] for a vector-valued $T1$ theorem, where the norm inequality holds if vector-valued analogues of the testing and A_p conditions hold.

In order to complete the proof of Theorem 1.4, we need to extend our doubling conclusions to classical doubling, and remove the restriction to dyadic cubes in our testing conditions for the weight pair (v, u) in Proposition 5.1.

5.3. Classical doubling, A_2 and dyadic testing in $\mathbb{R}^n$

By Proposition 5.1, we have constructed a pair of weights (v, u) on $Q^0 = [0, 1]^n$, which we relabel here as (σ,ω), that satisfy the flatness condition (5.1) on Q^0, the $A_2^{\mathrm{dyadic}}(\sigma, \omega;[0,1]^n)$ condition as well as the *dyadic* testing conditions

$$\int_{Q^0} |R_1(\mathbf{1}_{Q^0}\omega)|^2\, d\sigma > \Gamma |Q^0|_\omega,$$

and for all $Q \in \mathcal{D}^0$,

$$\int_Q |R_2(\mathbf{1}_Q\sigma)|^2\, d\omega \le |Q|_\sigma, \quad \int_Q |R_2(\mathbf{1}_Q\omega)|^2\, d\sigma \le |Q|_\omega.$$

We extend these measures to the entire space by reflecting in each coordinate separately to obtain an extension to $[0, 2]^n$, and then by adding translates $[0, 2]^n + 2(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha \in \mathbb{Z}^n$, so as to be periodic of period two on the entire space $\mathbb{R}^n$. After this reflection process, note that adjacent cubes from neighboring dyadic cubes of side length 1 also satisfy the adjacent doubling condition, and with constant 1 since they have equal measures by the reflection extension process, and so for *any* adjacent dyadic cubes I_1 and I_2, we have $E_{I_1}\sigma/E_{I_2}\sigma \in (1 - \tau, 1 + \tau)$, and similarly for ω. In particular, one can show this implies that σ and ω are both $o_{\tau\to 0}(1)$ flat, and hence doubling, see Lemma 4.2 in [32]. We also note that after this reflection process, the pair (σ, ω) satisfies the dyadic A_2 condition

$$\left(\frac{1}{|Q|}\int_Q \sigma(x)\right)\left(\frac{1}{|Q|}\int_Q \omega(x)\right) \le 1 \quad \text{for all dyadic cubes } Q \in \mathcal{D}. \tag{5.2}$$

Because σ and ω are doubling, from (5.2) we obtain that $A_2(\sigma, \omega) \lesssim 1$. By multiplying σ and ω by an appropriate constant, we may assume without loss of generality that

$$A_2(\sigma, \omega) \le 1.$$

Furthermore, after this reflection process, the pair (σ, ω) also satisfy the *dyadic* testing conditions for all $\mathcal{D}$-dyadic cubes of side length at most 1. We now set

$$Q_\alpha \equiv [0, 1]^n + (\alpha_1, \alpha_2, \ldots, \alpha_n) \quad \text{for all } \alpha \in \mathbb{Z}^n.$$

Let $\tau \in (0, 1)$ be as in Proposition 5.1 and multiply each of these measures by the factor

$$\varphi_\tau(x) \equiv \sum_\alpha a_\alpha \mathbf{1}_{Q_\alpha}(x),$$

where

$$a_\alpha \equiv \frac{1}{|Q_\alpha|}\int_{Q_\alpha} d\mu_\tau \quad \text{and} \quad d\mu_\tau(x) \equiv \frac{dx}{(1+|x|)^\tau},$$

and consider the measure pairs $(\sigma_\tau, \omega_\tau)$ with $\sigma_\tau \equiv \varphi_\tau(x)\, d\sigma(x)$ and $\omega_\tau \equiv \varphi_\tau(x)\, d\omega(x)$. We set $A \equiv |[0,1]^n|_\sigma$ and $B = |[0,1]^n|_\omega$. Note that $A = |Q_\alpha|_\sigma$ and $B = |Q_\alpha|_\omega$ for all $\alpha \in \mathbb{Z}^n$, and $AB \le A_2(\sigma,\omega) \le 1$.

Lemma 5.4. *The measures σ_τ, ω_τ are both $o_{\tau\to 0}(1)$-flat, i.e., the adjacent doubling constant of each measure tends to* 1 *as $\tau \searrow 0$.*

Proof. If Q_α and $Q_{\alpha'}$ are two adjacent cubes of the form $Q_\alpha \equiv [0,1]^n + (\alpha_1, \alpha_2, \dots, \alpha_n)$, then

$$\frac{\int_{Q_\alpha} \sigma_\tau}{\int_{Q_{\alpha'}} \sigma_\tau} = \frac{a_\alpha \int_{Q_\alpha} \sigma}{a_{\alpha'} \int_{Q_{\alpha'}} \sigma} = \frac{a_\alpha A}{a_{\alpha'} A} = \frac{\int_{Q_\alpha} d\mu_\tau}{\int_{Q_{\alpha'}} d\mu_\tau},$$

tends to 1 as $\tau \searrow 0$ independent of the pair $(Q_\alpha, Q_{\alpha'})$, since μ_τ is a doubling weight on $\mathbb{R}^n$ with adjacent doubling constant roughly $1 + O_{\tau\to 0}(\tau)$. If instead we consider adjacent cubes P and P' that are each a union of cubes Q_α, then

$$\frac{\int_P \sigma_\tau}{\int_{P'} \sigma_\tau} = \frac{\sum_{\alpha: Q_\alpha \subset P} a_\alpha |Q_\alpha|_\sigma}{\sum_{\alpha': Q_{\alpha'} \subset P'} a_{\alpha'} |Q_{\alpha'}|_\sigma} = \frac{\sum_{\alpha: Q_\alpha \subset P} \int_{Q_\alpha} d\mu_\tau}{\sum_{\alpha': Q_{\alpha'} \subset P'} \int_{Q_{\alpha'}} d\mu_\tau} = \frac{\int_P d\mu_\tau}{\int_{P'} d\mu_\tau},$$

which again tends to 1 as $\tau \searrow 0$ independent of the pair (P, P'). Therefore, for any adjacent dyadic cubes I_1 and I_2, we have $E_{I_1}\sigma_\tau / (E_{I_2}\sigma_\tau) \in (1-\tau, 1+\tau)$. A standard argument shows that σ_τ has adjacent doubling constant equal to $1 + o(1)$ as $\tau \searrow 0$, and similarly for ω_τ. ∎

Next we turn to the final task of establishing the testing conditions for R_2 on the doubling measure pair $(\sigma_\tau, \omega_\tau)$ uniformly for any $\tau \in (0,1)$, which then leads to boundedness of R_2 via the main result of Theorem 2 in [38] for $\tau > 0$ sufficiently small, since if a pair doubling measures with doubling constant sufficiently close to Lebesgue satisfies the A_2 condition, then they will satisfy the energy condition, Section 1.7 of [38]. Of course, testing fails for R_1. To state this formally, we will need the definition of a weighted norm inequality as used in [37, 38].

We follow the approach in [39], p. 314. So we suppose that K^α is a standard smooth α-fractional Calderón–Zygmund kernel, and σ, ω are locally finite positive Borel measures on $\mathbb{R}^n$, and we introduce a family $\{\eta^\alpha_{\delta,R}\}_{0<\delta<R<\infty}$ of nonnegative functions on $[0,\infty)$ so that the truncated kernels $K^\alpha_{\delta,R}(x,y) = \eta^\alpha_{\delta,R}(|x-y|) K^\alpha(x,y)$ are bounded with compact support for fixed x or y, and uniformly satisfy the smooth Calderón–Zygmund kernel estimates (1.5). Then the truncated operators

$$T^\alpha_{\sigma,\delta,R} f(x) \equiv \int_{\mathbb{R}^n} K^\alpha_{\delta,R}(x,y) f(y)\, d\sigma(y), \quad x \in \mathbb{R}^n,$$

are pointwise well defined when f is bounded with compact support, and we will refer to the pair

$$(K^\alpha, \{\eta^\alpha_{\delta,R}\}_{0<\delta<R<\infty})$$

as an α-fractional singular integral operator, which we typically denote by T^α, suppressing the dependence on the truncations. In the event that $\alpha = 0$ and T^0 is bounded on unweighted $L^2(\mathbb{R}^n)$, we say that $T = T^0$ is a Calderón–Zygmund operator.

Definition 5.5. An α-fractional singular integral operator $T^\alpha = (K^\alpha, \{\eta^\alpha_{\delta,R}\}_{0<\delta<R<\infty})$ is said to satisfy the norm inequality

$$\|T^\alpha_\sigma f\|_{L^2(\omega)} \le \mathfrak{N}_{T^\alpha}(\sigma,\omega)\|f\|_{L^2(\sigma)}, \quad f \in L^2(\sigma), \tag{5.3}$$

if $\mathfrak{N}_{T^\alpha}(\sigma,\omega)$ is the best constant $\mathfrak{N}$ for which

$$\|T^\alpha_{\sigma,\delta,R} f\|_{L^2(\omega)} \le \mathfrak{N}\|f\|_{L^2(\sigma)}, \quad f \in L^2(\sigma),\ 0 < \delta < R < \infty.$$

Independence of truncations. In the presence of the classical Muckenhoupt condition A^α_2, the norm inequality (5.3) is independent of the choice of truncations used, including *nonsmooth* truncations as well, see Section 2.1 of [21].

Now we introduce the testing conditions for Calderón–Zygmund operators.

Definition 5.6. For an α-fractional singular integral operator $T^\alpha = (K^\alpha, \{\eta^\alpha_{\delta,R}\}_{0<\delta<R<\infty})$, define the testing constants

$$\mathfrak{T}_{T^\alpha}(\sigma,\omega)^2 \equiv \sup_{Q\in\mathcal{P}_n} \frac{1}{|Q|_\sigma}\int_Q |T^\alpha(\mathbf{1}_Q\sigma)|^2\, d\omega,$$
$$\mathfrak{T}_{T^{\alpha,*}}(\omega,\sigma)^2 \equiv \sup_{Q\in\mathcal{P}_n} \frac{1}{|Q|_\omega}\int_Q |T^{\alpha,*}(\mathbf{1}_Q\omega)|^2\, d\sigma.$$

We also define the dyadic testing constants by

$$\mathfrak{T}^{\mathcal{D}}_{T^\alpha}(\sigma,\omega)^2 \equiv \sup_{Q\in\mathcal{D}^0} \frac{1}{|Q|_\sigma}\int_Q |T^\alpha(\mathbf{1}_Q\sigma)|^2\, d\omega < \infty,$$
$$\mathfrak{T}^{\mathcal{D}^0}_{T^{\alpha,*}}(\omega,\sigma)^2 \equiv \sup_{Q\in\mathcal{D}} \frac{1}{|Q|_\omega}\int_Q |T^{\alpha,*}(\mathbf{1}_Q\omega)|^2\, d\sigma < \infty.$$

We say T^α satisfies the (dyadic) testing conditions if both (dyadic) testing constants for each admissible truncation are finite, and the constants are bounded uniformly over all admissible truncations.

The following $T1$ theorem, whose proof we include in Appendix A.5, is a corollary of Theorem 2 in [38]. In particular, if τ is sufficiently small, it can be applied to the measure pair $(\sigma_\tau, \omega_\tau)$. Recall that a Calderón–Zygmund operator T^α is $(1+\delta)$-smooth if in addition to having a kernel K^α satisfying (1.5), we also have

$$|\nabla K^\alpha(x,y) - \nabla K^\alpha(x',y)| \le C_{CZ}\left(\frac{|x-x'|}{|x-y|}\right)^\delta |x-y|^{\alpha-(n+1)}$$

whenever

$$\frac{|x-x'|}{|x-y|} \le \frac{1}{2}.$$

Theorem 5.7 ($T1$ theorem for flat doubling measures). *Suppose σ and ω are doubling measures with doubling constant at most $2^{n+\varepsilon}$ for some $\varepsilon \in (0,1)$, and let T be a $(1+\delta)$-smooth Calderón–Zygmund operator of fractional order* 0. *Then*

$$\mathfrak{N}_T(\sigma,\omega) \lesssim \sqrt{A_2(\sigma,\omega)} + \mathfrak{T}_T(\sigma,\omega) + \mathfrak{T}_{T^*}(\omega,\sigma).$$

Finally, we record an estimate from [36] that will be used in proving the next lemma.

Lemma 5.8 (Lemma 23 in [36]). *If μ is a doubling measure and P is a cube, then for every $\delta \in (0,1/2)$, we have*

$$|\{x \in P : \operatorname{dist}(x,\partial P) < \delta \ell(P)\}|_\mu \lesssim \frac{1}{\ln(1/\delta)} |P|_\mu.$$

Lemma 5.9. *For all $\tau > 0$ sufficiently small, the second Riesz transform R_2 satisfies the norm inequality for the measure pair $(\sigma_\tau, \omega_\tau)$, i.e.,*

$$\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau) \lesssim 1.$$

Proof. Let τ be sufficiently small so that the doubling constants for σ and ω are at most $2^{n+1/2}$, and so Theorem 5.7 applies. Fix a *dyadic* cube $Q \in \mathcal{D}$. If Q has side length at most 1, then Q is contained in one of the cubes Q_α, where we have already shown that the testing conditions for (σ,ω) hold in Proposition 5.1. In particular, we have the following inequality that will be used repeatedly below:

$$\begin{aligned}(5.4)\qquad \int_{Q_\alpha} |R_2(\mathbf{1}_{Q_\alpha}\sigma_\tau)|^2\, d\omega_\tau &= a_\alpha^3 \int_{Q_\alpha} |R_2(\mathbf{1}_{Q_\alpha}\sigma)|^2\, d\omega \\ &\le C_* a_\alpha |Q_\alpha|_\sigma = C_* |Q_\alpha|_{\sigma_\tau}, \quad \alpha \in \mathbb{Z}^n.\end{aligned}$$

Suppose Q has side length 2^k with $k \ge 1$ for some $k \in \mathbb{N}$. Then Q is a finite pairwise disjoint union of cubes Q_β, say $Q = \bigcup_{\beta:|\beta|\le 2^k} Q_\beta$, where $|\beta| \equiv \max\{\beta_1,\beta_2,\dots,\beta_n\}$. We will suppose that $Q = [0,2^k]^n$ as the general case follows the same argument. Finally, we note that

$$a_\alpha \approx \frac{1}{(1+|\alpha|)^\tau}.$$

Now we write

$$\begin{aligned}(5.5)\quad \int_Q |R_2(\mathbf{1}_Q\sigma_\tau)|^2\, d\omega_\tau &= \sum_{\substack{\alpha_1,\alpha_2,\alpha_3\in\mathbb{Z}^n\\ 0\le|\alpha_j|\le 2^k}} \int_{Q_{\alpha_1}} R_2(\mathbf{1}_{Q_{\alpha_2}}\sigma_\tau) R_2(\mathbf{1}_{Q_{\alpha_3}}\sigma_\tau)\, d\omega_\tau \\ &\lesssim \sum_{\substack{\alpha_1,\alpha_2,\alpha_3\in\mathbb{Z}^n\\ 0\le|\alpha_j|\le 2^k}} \int_{Q_{\alpha_1}} \frac{|R_2(\mathbf{1}_{Q_{\alpha_2}}\sigma)|\,|R_2(\mathbf{1}_{Q_{\alpha_3}}\sigma)|}{(1+|\alpha_2|)^\tau(1+|\alpha_3|)^\tau} \frac{d\omega}{(1+|\alpha_1|)^\tau}.\end{aligned}$$

We split the sum into several different configurations of $(\alpha_1,\alpha_2,\alpha_3)$, which we consider separately. In what follows, we will *not* specify the configurations considered explicitly within the sum, instead we mention in words which configuration we sum over before estimating the sum.

First, assume that we only sum over the configuration of multi-indices $(\alpha_1, \alpha_2, \alpha_3)$ satisfying $|\alpha_2 - \alpha_1| \geq 2$ and $|\alpha_3 - \alpha_1| \geq 2$, so that what we need to bound is

$$\sum_{\substack{\alpha_1,\alpha_2,\alpha_3 \in \mathbb{Z}^n \\ 0 \leq |\alpha_j| \leq 2^k}} \frac{(1+|\alpha_2-\alpha_1|)^{-n}(1+|\alpha_3-\alpha_1|)^{-n}}{(1+|\alpha_2|)^{\tau}(1+|\alpha_3|)^{\tau}} \frac{|Q_{\alpha_2}|_\sigma |Q_{\alpha_3}|_\sigma |Q_{\alpha_1}|_\omega}{(1+|\alpha_1|)^{\tau}},$$

where we suppress the specified conditions $|\alpha_2 - \alpha_1| \geq 2$ and $|\alpha_3 - \alpha_1| \geq 2$ in the sum. Summing first over α_3 and using $|Q_{\alpha_3}|_\sigma = A$, we deduce that the above term is dominated by

$$\begin{aligned}
&\sum_{\substack{\alpha_1,\alpha_2,\alpha_3 \in \mathbb{Z}^n \\ 0 \leq |\alpha_j| \leq 2^k}} \frac{(1+|\alpha_2-\alpha_1|)^{-n}(1+|\alpha_3-\alpha_1|)^{-n}}{(1+|\alpha_2|)^{\tau}(1+|\alpha_3|)^{\tau}} \frac{A|Q_{\alpha_2}|_\sigma |Q_{\alpha_1}|_\omega}{(1+|\alpha_1|)^{\tau}} \\
&\leq A \sum_{\substack{\alpha_1,\alpha_2 \in \mathbb{Z}^n \\ 0 \leq |\alpha_j| \leq 2^k}} \Bigg[\sum_{\substack{\alpha_3 \in \mathbb{Z}^n \\ 0 \leq |\alpha_3| \leq 2^k}} \frac{(1+|\alpha_3-\alpha_1|)^{-n}}{(1+|\alpha_3|)^{\tau}} \Bigg] \frac{(1+|\alpha_2-\alpha_1|)^{-n}}{(1+|\alpha_2|)^{\tau}} \frac{|Q_{\alpha_2}|_\sigma |Q_{\alpha_1}|_\omega}{(1+|\alpha_1|)^{\tau}} \\
&= A \sum_{\substack{\alpha_1,\alpha_2 \in \mathbb{Z}^n \\ 0 \leq |\alpha_j| \leq 2^k}} \Bigg[\Bigg\{ \sum_{\substack{\alpha_3 \in \mathbb{Z}^n \\ |\alpha_3| < \frac{1}{2}|\alpha_1|}} + \sum_{\substack{\alpha_3 \in \mathbb{Z}^n \\ \frac{1}{2}|\alpha_1| \leq |\alpha_3| \leq 2|\alpha_1|}} + \sum_{\substack{\alpha_3 \in \mathbb{Z}^n \\ 2|\alpha_1| < |\alpha_3|}} \Bigg\} \frac{(1+|\alpha_3-\alpha_1|)^{-n}}{(1+|\alpha_3|)^{\tau}} \Bigg] \\
&\quad \times \frac{(1+|\alpha_2-\alpha_1|)^{-n}}{(1+|\alpha_2|)^{\tau}} \frac{|Q_{\alpha_2}|_\sigma |Q_{\alpha_1}|_\omega}{(1+|\alpha_1|)^{\tau}} \\
&\lesssim A \sum_{\substack{\alpha_1,\alpha_2 \in \mathbb{Z}^n \\ 0 \leq |\alpha_j| \leq 2^k}} \Bigg[\frac{\ln(2+|\alpha_1|)}{(1+|\alpha_1|)^{\tau}} \Bigg] \frac{(1+|\alpha_2-\alpha_1|)^{-n}}{(1+|\alpha_2|)^{\tau}} \frac{|Q_{\alpha_2}|_\sigma |Q_{\alpha_1}|_\omega}{(1+|\alpha_1|)^{\tau}}.
\end{aligned}$$

Now summing over α_1, using that $|Q_{\alpha_1}|_\omega = B$ and that $AB \leq A_2(\sigma, \omega)$, we obtain in a similar way that the final line above is at most a constant times

$$\begin{aligned}
A_2(\sigma,\omega) \sum_{\substack{\alpha_2 \in \mathbb{Z}^n \\ 0 \leq |\alpha_2| \leq 2^k}} \Bigg[\frac{\ln(2+|\alpha_2|)}{(1+|\alpha_2|)^{3\tau}} \Bigg] |Q_{\alpha_2}|_\sigma &= A_2(\sigma,\omega) \sum_{\substack{\alpha_2 \in \mathbb{Z}^n \\ 0 \leq |\alpha_2| \leq 2^k}} \Bigg[\frac{\ln(2+|\alpha_2|)}{(1+|\alpha_2|)^{2\tau}} \Bigg] |Q_{\alpha_2}|_{\sigma_\tau} \\
&\leq C A_2(\sigma,\omega) \sum_{\substack{\alpha_2 \in \mathbb{Z}^n \\ 0 \leq |\alpha_2| \leq 2^k}} |Q_{\alpha_2}|_{\sigma_\tau} = C A_2(\sigma,\omega) |Q|_{\sigma_\tau},
\end{aligned}$$

where we used that $AB \leq A_2(\sigma_\tau, \omega_\tau)$.

The relatively simple case we just proved is case (6) in the following exhaustive list of cases, which we delineate based on the relationship of the indices α_2 and α_3 to the distinguished index α_1:

(1) $\alpha_1 = \alpha_2 = \alpha_3$,

(2) $\alpha_1 = \alpha_2$ and $Q_{\alpha_1}, Q_{\alpha_3}$ are separated,

(3) $\alpha_1 = \alpha_3$ and $Q_{\alpha_1}, Q_{\alpha_2}$ are separated,

(4) $Q_{\alpha_1}, Q_{\alpha_2}$ are adjacent and $Q_{\alpha_1}, Q_{\alpha_3}$ are separated,

(5) $Q_{\alpha_1}, Q_{\alpha_3}$ are adjacent and $Q_{\alpha_1}, Q_{\alpha_2}$ are separated,

(6) $Q_{\alpha_1}, Q_{\alpha_2}$ are separated and $Q_{\alpha_1}, Q_{\alpha_3}$ are separated,

(7) and finally,

$$\begin{cases} \alpha_1 = \alpha_2 \text{ and } Q_{\alpha_1}, Q_{\alpha_3} \text{ are adjacent,} \\ \alpha_1 = \alpha_3 \text{ and } Q_{\alpha_1}, Q_{\alpha_2} \text{ are adjacent,} \\ Q_{\alpha_1}, Q_{\alpha_2} \text{ are adjacent and } Q_{\alpha_1}, Q_{\alpha_3} \text{ are adjacent.} \end{cases}$$

where we say that Q_{α_1} and Q_{α_2} are *separated* if $|\alpha_1 - \alpha_2| \geq 2$, and of course Q_{α_1} and Q_{α_2} are adjacent if and only if $|\alpha_1 - \alpha_2| = 1$.

In the first of these seven cases, the right-hand side of (5.5) equals

$$\sum_{\substack{\alpha \in \mathbb{Z}^n \\ 0 \leq |\alpha| \leq 2^k}} \int_{Q_\alpha} |R_2(\mathbf{1}_{Q_\alpha} \sigma_\tau)|^2 \, d\omega_\tau \leq C_* \sum_{|\alpha|=1}^{2^k} |Q_\alpha|_{\sigma_\tau} = C_* |Q|_{\sigma_\tau},$$

independent of $\tau \in (0,1)$ by (5.4).

In the second of these cases, we will use the separation between Q_{α_1} and Q_{α_3}, as well as the fact that

$$\begin{aligned} (5.6) \qquad \left| \int_{Q_{\alpha_1}} R_2(\mathbf{1}_{Q_{\alpha_1}} \sigma_\tau) \, d\omega_\tau \right| &\leq \left(\int_{Q_{\alpha_1}} |R_2(\mathbf{1}_{Q_{\alpha_1}} \sigma_\tau)|^2 \, d\omega_\tau \right)^{1/2} \sqrt{|Q_{\alpha_1}|_{\omega_\tau}} \\ &\leq \sqrt{C_*} \sqrt{|Q_{\alpha_1}|_{\sigma_\tau}} \sqrt{|Q_{\alpha_1}|_{\omega_\tau}} \lesssim \sqrt{C_*} \frac{AB}{(1+|\alpha_1|)^\tau}, \end{aligned}$$

where the second inequality follows from reasoning using (5.4), similar to the previous display. Thus, recalling that $AB \leq A_2(\sigma, \omega)$, we dominate the right-hand side of (5.5), using (5.6), by

$$\begin{aligned} &\sum_{\substack{\alpha_1, \alpha_3 \in \mathbb{Z}^n \\ 0 \leq |\alpha_j| \leq 2^k}} \int_{Q_{\alpha_1}} \frac{|R_2(\mathbf{1}_{Q_{\alpha_1}} \sigma)| (1 + |\alpha_3 - \alpha_1|)^{-n} |Q_{\alpha_3}|_\sigma}{(1+|\alpha_1|)^\tau (1+|\alpha_3|)^\tau} \frac{d\omega}{(1+|\alpha_1|)^\tau} \\ &\leq A_2(\sigma,\omega) \sqrt{C_*} \sum_{\substack{\alpha_3 \in \mathbb{Z}^n \\ 0 \leq |\alpha_3| \leq 2^k}} \frac{|Q_{\alpha_3}|_\sigma}{(1+|\alpha_3|)^\tau} \sum_{|\alpha_1|=0}^{2^k} \frac{(1+|\alpha_3 - \alpha_1|)^{-n}}{(1+|\alpha_1|)^\tau} \\ &\leq A_2(\sigma,\omega) \sqrt{C_*} \sum_{\substack{\alpha_3 \in \mathbb{Z}^n \\ 0 \leq |\alpha_3| \leq 2^k}} \frac{|Q_{\alpha_3}|_\sigma \ln(2 + |\alpha_3|)}{(1+|\alpha_3|)^{2\tau}} \leq C A_2 \sqrt{C_*} |Q|_{\sigma_\tau}. \end{aligned}$$

To handle the cases where Q_{α_1} is adjacent to one of the cubes Q_{α_2} or Q_{α_3} or both, we use Lemma 5.8, i.e., that doubling measures charge halos with reciprocal log control. Indeed, in the fourth case above, namely, $|\alpha_1 - \alpha_2| = 1$ and $|\alpha_1 - \alpha_3| \geq 2$, we follow the same argument just used except that in place of the testing condition in (5.6), we use

$$\int_{Q_{\alpha_1}} R_2(\mathbf{1}_{Q_{\alpha_2}} \sigma_\tau) \, d\omega_\tau = \left\{ \int_{(1-\delta)Q_{\alpha_1}} + \int_{Q_{\alpha_1} \setminus (1-\delta) Q_{\alpha_1}} \right\} R_2(\mathbf{1}_{Q_{\alpha_2}} \sigma_\tau) \, d\omega_\tau \equiv \mathrm{I} + \mathrm{II}.$$

We control the first term I by δ-separation between $(1-\delta)Q_{\alpha_1}$ and Q_{α_2}:

$$\begin{aligned}|\mathrm{I}| &\leq \int_{(1-\delta)Q_{\alpha_1}} C\,\frac{1}{\delta^n}|Q_{\alpha_2}|_{\sigma_\tau}\,d\omega_\tau \leq C\,\frac{1}{\delta^n}|Q_{\alpha_2}|_{\sigma_\tau}|Q_{\alpha_1}|_{\omega_\tau}\\ &= C\,\frac{1}{\delta^n}\,\frac{AB}{(1+|\alpha_1|)^\tau(1+|\alpha_2|)^\tau} \leq C\,\frac{1}{\delta^n}\,\frac{A_2}{(1+|\alpha_1|)^\tau(1+|\alpha_2|)^\tau}.\end{aligned}$$

We control the second term II by using Lemma 5.8:

$$\begin{aligned}|\mathrm{II}| &\leq \int_{Q_{\alpha_1}\setminus(1-\delta)Q_{\alpha_1}} |R_2(\mathbf{1}_{Q_{\alpha_2}}\sigma_\tau)|\,d\omega_\tau\\ &\leq \mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau)\sqrt{|Q_{\alpha_2}|_{\sigma_\tau}|Q_{\alpha_1}\setminus(1-\delta)Q_{\alpha_1}|_{\omega_\tau}}\\ &\leq \frac{C}{\sqrt{\ln(1/\delta)}}\,\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau)\,\frac{\sqrt{A}\sqrt{B}}{(1+|\alpha_1|)^{\tau/2}(1+|\alpha_2|)^{\tau/2}}\\ &\leq \frac{C}{\sqrt{\ln(1/\delta)}}\,\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau)\,\frac{\sqrt{A_2}}{(1+|\alpha_1|)^{\tau/2}(1+|\alpha_2|)^{\tau/2}}.\end{aligned}$$

Altogether, our replacement for (5.6) is

$$\left|\int_{Q_{\alpha_1}} R_2(\mathbf{1}_{Q_{\alpha_2}}\sigma_\tau)\,d\omega_\tau\right| \leq \left(C_\delta\sqrt{A_2} + \frac{C}{\sqrt{\ln(1/\delta)}}\,\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau)\right)\frac{\sqrt{A_2}}{(1+|\alpha_1|)^\tau}, \tag{5.7}$$

since $|\alpha_1-\alpha_2|=1$. Now the previous argument can continue using (5.7) in place of (5.6), which proves the fourth case since there are just 3^n-1 points α_2 for each fixed point α_1. Indeed, we have

$$\begin{aligned}&\sum_{\substack{\alpha_1,\alpha_3\in\mathbb{Z}^n\\ 0\leq|\alpha_j|\leq 2^k}} \int_{Q_{\alpha_1}} \frac{|R_2(\mathbf{1}_{Q_{\alpha_1}}\sigma)|(1+|\alpha_3-\alpha_1|)^{-n}|Q_{\alpha_3}|_\sigma}{(1+|\alpha_1|)^\tau(1+|\alpha_3|)^\tau}\,\frac{d\omega}{(1+|\alpha_1|)^\tau}\\ &\leq \left(C_\delta\sqrt{A_2} + \frac{C}{\ln(1/\delta)}\,\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau)\right)\sum_{\substack{\alpha_3\in\mathbb{Z}^n\\ 0\leq|\alpha_3|\leq 2^k}}\frac{|Q_{\alpha_3}|_\sigma}{(1+|\alpha_3|)^\tau}\sum_{\substack{\alpha_1\in\mathbb{Z}^n\\ 0\leq|\alpha_1|\leq 2^k}}\frac{(1+|\alpha_3-\alpha_1|)^{-n}}{(1+|\alpha_1|)^\tau}\\ &\leq \left(C_\delta\sqrt{A_2} + \frac{C}{\ln(1/\delta)}\,\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau)\right)|Q|_{\sigma_\tau}.\end{aligned}$$

The third and fifth cases are symmetric to those just handled. So it remains to consider the remaining seventh case, where one of the following three subcases holds:

$$\begin{aligned}\alpha_1 = \alpha_2 \quad &\text{and} \quad |\alpha_1-\alpha_3| = 1,\\ \alpha_1 = \alpha_3 \quad &\text{and} \quad |\alpha_1-\alpha_2| = 1,\\ |\alpha_1-\alpha_2| = 1 \quad &\text{and} \quad |\alpha_1-\alpha_3| = 1.\end{aligned}$$

In all three of these subcases, there is essentially only the sum over α_1, since for each fixed α_1, there are at most 3^{2n} pairs (α_2,α_3) satisfying one of the three subcases. If both Q_{α_2} and Q_{α_3} are adjacent to Q_{α_1}, we write

$$\begin{aligned}\int_{Q_{\alpha_1}} R_2(\mathbf{1}_{Q_{\alpha_2}}\sigma_\tau)R_2(\mathbf{1}_{Q_{\alpha_3}}\sigma_\tau)\,d\omega_\tau &= \int_{Q_{\alpha_1}} R_2(\mathbf{1}_{(1-\delta)Q_{\alpha_2}}\sigma_\tau)R_2(\mathbf{1}_{(1-\delta)Q_{\alpha_3}}\sigma_\tau)\,d\omega_\tau \\ &\quad+\int_{Q_{\alpha_1}} R_2(\mathbf{1}_{(1-\delta)Q_{\alpha_2}}\sigma_\tau)R_2(\mathbf{1}_{Q_{\alpha_3}\setminus(1-\delta)Q_{\alpha 3}}\sigma_\tau)\,d\omega_\tau \\ &\quad+\int_{Q_{\alpha_1}} R_2(\mathbf{1}_{Q_{\alpha_2}\setminus(1-\delta)Q_{\alpha_2}}\sigma_\tau)R_2(\mathbf{1}_{Q_{\alpha_3}}\sigma_\tau)\,d\omega_\tau.\end{aligned}$$

The first term of the right-hand side is handled by the δ-separation between Q_{α_1} and $(1-\delta)Q_{\alpha_2}$, as well as between Q_{α_1} and $(1-\delta)Q_{\alpha_3}$, together with the A_2 condition $AB \le 1$, to obtain

$$\Big|\int_{Q_{\alpha_1}} R_2(\mathbf{1}_{(1-\delta)Q_{\alpha_2}}\sigma_\tau)R_2(\mathbf{1}_{(1-\delta)Q_{\alpha_3}}\sigma_\tau)\,d\omega_\tau\Big| \le C\,\frac{1}{\delta^{2n}}\int_{Q_{\alpha_1}} |Q_{\alpha_2}|_{\sigma_\tau}|Q_{\alpha_3}|_{\sigma_\tau}\,d\omega_\tau$$

and

$$C\,\frac{1}{\delta^{2n}}|Q_{\alpha_3}|_{\sigma_\tau}AB \le C\,\frac{1}{\delta^{2n}}|Q_{\alpha_3}|_\sigma A_2(\sigma,\omega),$$

and since for each fixed α_3, there are at most 3^{2n} pairs (α_1,α_2), we can sum to obtain the bound

$$C\,\frac{1}{\delta^{2n}}|Q|_{\sigma_\tau}.$$

To handle the terms involving a halo $Q_{\alpha_j}\setminus(1-\delta)Q_{\alpha_j}$, we use Lemma 5.8 together with the norm constant $\mathfrak{N}_{R_2}=\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau)$. For example,

$$\begin{aligned}&\Big|\int_{Q_{\alpha_1}} R_2(\mathbf{1}_{Q_{\alpha_2}\setminus(1-\delta)Q_{\alpha_2}}\sigma_\tau)\,R_2(\mathbf{1}_{Q_{\alpha_3}}\sigma_\tau)\,d\omega_\tau\Big| \\ &\qquad\le\Big(\int_{Q_{\alpha_1}}|R_2(\mathbf{1}_{Q_{\alpha_2}\setminus(1-\delta)Q_{\alpha_2}}\sigma_\tau)|^2\,d\omega_\tau\Big)^{1/2}\Big(\int_{Q_{\alpha_1}}|R_2(\mathbf{1}_{Q_{\alpha_3}}\sigma_\tau)|^2\,d\omega_\tau\Big)^{1/2} \\ &\qquad\le \mathfrak{N}_{R_2}\sqrt{|Q_{\alpha_2}\setminus(1-\delta)Q_{\alpha_2}|_{\sigma_\tau}}\,\mathfrak{N}_{R_2}(|Q_{\alpha_3}|_{\sigma_\tau})^{1/2} \\ &\qquad= (\mathfrak{N}_{R_2})^2\frac{C}{\sqrt{\ln(1/\delta)}}\sqrt{|Q_{\alpha_2}|_{\sigma_\tau}|Q_{\alpha_3}|_{\sigma_\tau}},\end{aligned}$$

and again we can sum to obtain the bound

$$(\mathfrak{N}_{R_2})^2\frac{C}{\ln(1/\delta)}|Q|_{\sigma_\tau}$$

because the indices α_j are at distance one from each other. The other terms are handled similarly and we thus obtain in this seventh case that

$$\sum_{|\alpha_1|,|\alpha_2|,|\alpha_3|=0}^{2^k}\Big|\int_{Q_{\alpha_1}} R_2(\mathbf{1}_{Q_{\alpha_2}}\sigma_\tau)R_2(\mathbf{1}_{Q_{\alpha_3}}\sigma_\tau)\,d\omega_\tau\Big| \le C\Big(\frac{1}{\delta^{2n}}A_2+\frac{(\mathfrak{N}_{R_2})^2}{\sqrt{\ln(1/\delta)}}\Big)|Q|_{\sigma_\tau}$$

The cases where just one of the cubes is adjacent to Q_{α_1} are handled similarly. Altogether we now have

$$\mathfrak{T}^{\mathcal{D}}_{R_2}(\sigma_\tau,\omega_\tau)^2 \equiv \sup_{Q\in\mathcal{D}} \frac{1}{|Q|_{\sigma_\tau}} \int_Q |R_2(\mathbf{1}_Q\sigma_\tau)|^2\, d\omega_\tau \le C_\tau + \frac{1}{\delta_1^{2n}} A_2 + \frac{(\mathfrak{N}_{R_2})^2}{\sqrt{\ln(1/\delta_1)}},$$

for any choice of $\delta_1 \in (0,1)$, where the constant C_* arises in (5.4).

Now we turn to the case of a general cube Q. In this case we first fix $M \in \mathbb{N}$ large to be chosen later, and write Q as a union of roughly 2^{Mn} dyadic subcubes $\{Q_\alpha\}_\alpha$ of side length $\delta_2 \equiv \ell(Q)/2^M > 0$, in such a way that the remaining portion of Q is contained in the $5\delta_2$-halo of Q. Then the above argument shows that the testing condition holds except for the terms that arise from the halo. But by Lemma 5.8 these leftover terms in $(\int_Q |R_2(\mathbf{1}_Q\sigma_\tau)|^2\, d\omega_\tau)^{1/2}$ are dominated by

$$\begin{aligned}
(5.8)\quad & \mathfrak{T}_{R_2}(\sigma_\tau,\omega_\tau) \\
&\le C_{\delta_2}\mathfrak{T}^{\mathcal{D}}_{R_2}(\sigma_\tau,\omega_\tau) + C\frac{1}{\sqrt[4]{\ln(1/\delta_2)}}\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau) \\
&\quad + C_{\delta_2}\Big(C_* + C_*C_\tau + \frac{1}{\delta_1^{2n}}A_2 + \frac{(\mathfrak{N}_{R_2})^2}{\sqrt{\ln(1/\delta_1)}}\Big)^{1/2} + C\frac{\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau)}{\sqrt[4]{\ln(1/\delta_2)}} \\
&\le C_{\delta_2,\tau}\sqrt{C_*} + C_{\delta_2}\frac{1}{\delta_1^n}\sqrt{A_2} + \Big(\frac{C_{\delta_2}}{\sqrt[4]{\ln(1/\delta_1)}} + \frac{C}{\sqrt[4]{\ln(1/\delta_2)}}\Big)\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau).
\end{aligned}$$

Note that the two-weight norm $\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau)$ is finite, as both weights σ_τ,ω_τ are bounded step functions, and so by the boundedness of the principal value interpretation of R_2 on Lebesgue spaces, we have

$$\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau) \le \|\sigma_\tau\|_\infty\|\omega_\tau\|_\infty < \infty.$$

Thus, by boundedness of maximal truncations (see e.g., Proposition 1 on p. 31 of [41]) together with the independence of truncations mentioned above, the above arguments actually prove that (5.8) holds *uniformly* over all admissible truncations of R_2, which is the hypothesis used in [4, 37, 38]. Thus, noting Definition 5.5, we can apply Theorem 5.7 to obtain

$$\begin{aligned}
\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau) &\le C\sqrt{A_2(\sigma_\tau,\omega_\tau)} + C\mathfrak{T}_{R_2}(\sigma_\tau,\omega_\tau) + C\mathfrak{T}_{R_2}(\omega_\tau,\sigma_\tau) \\
&\le C\sqrt{A_2(\sigma_\tau,\omega_\tau)} + 2\Big\{C_{\delta_2,\tau}\sqrt{C_*} + C_{\delta_2}\frac{1}{\delta_1^2}\sqrt{A_2(\sigma_\tau,\omega_\tau)} \\
&\quad + \Big(\frac{C_{\delta_2}}{\sqrt{\ln(1/\delta_1)}} + \frac{C}{\sqrt{\ln(1/\delta_2)}}\Big)\mathfrak{N}_{R_2}(\sigma_\tau,\omega_\tau)\Big\},
\end{aligned}$$

for any admissible truncation of R_2. Thus, with $\delta_2 > 0$ chosen sufficiently small that $C/\sqrt{\ln(1/\delta_2)} < 1/4$, and then $\delta_1 > 0$ chosen sufficiently small that $C_{\delta_2}/\sqrt{\ln(1/\delta_1)} < 1/4$, an absorption completes the proof that the norm inequality for R_2 holds (recall that truncations of R_2 are *a priori* bounded). ∎

We have thus proved the following special case of Theorem 1.4 for the individual Riesz transforms R_1 and R_2.

Proposition 5.10. *For every $\Gamma > 1$ and $0 < \tau < 1$, there is a pair of positive weights (σ, ω) in $\mathbb{R}^n$ satisfying*

$$\begin{aligned}
&\int_{\mathbb{R}^n} |R_1(\mathbf{1}_{[0,1]^n}\sigma)(x)|^2(x)\, d\omega(x) \geq \Gamma \int_{[0,1]^n} d\sigma(x),\\
&\int_I |R_2 \mathbf{1}_I \sigma(x)|^2\, d\omega(x) \leq \int_I d\sigma(x) \quad \textit{for all cubes } I \in \mathcal{P}^n,\\
&\int_I |R_2 \mathbf{1}_I \omega(x)|^2\, d\sigma(x) \leq \int_I d\omega(x) \quad \textit{for all cubes } I \in \mathcal{P}^n,\\
&\Big(\frac{1}{|I|}\int_I d\sigma\Big)\Big(\frac{1}{|I|}\int_I d\omega\Big) \leq 1 \qquad \textit{for all cubes } I \in \mathcal{P}^n,\\
&1-\tau < \frac{E_J\sigma}{E_K\sigma}, \frac{E_J\omega}{E_K\omega} < 1+\tau \quad \textit{for arbitrary adjacent cubes } J, K \in \mathcal{P}^n.
\end{aligned}$$

The argument used in proving this proposition also shows that in any two-weight $T1$ theorem for doubling pairs (σ, ω), the testing may be carried out over only cubes in any fixed dyadic grid $\mathcal{D}$, and here is one possible formulation of this improvement.

Theorem 5.11. *Suppose $0 \leq \alpha < n$, and let T^α be an α-fractional Calderón–Zygmund singular integral operator on $\mathbb{R}^n$ with a smooth α-fractional kernel K^α. Assume that σ and ω are doubling measures on $\mathbb{R}^n$. Finally, fix a dyadic grid $\mathcal{D}$ on $\mathbb{R}^n$.*

If the two-weight norm $\mathfrak{N}_{T^\alpha}(\sigma, \omega)$ satisfies

$$\mathfrak{N}_{T^\alpha}(\sigma, \omega) \leq C_{\alpha,n}\big(\sqrt{A_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{(T^\alpha)^*}\big),$$

where A_2^α is the classical Muckenhoupt constant and the constant $C_{\alpha,n}$ depends on the Calderón–Zygmund kernel and the doubling constants of the measures σ and ω, then

$$\mathfrak{N}_{T^\alpha} \leq C'_{\alpha,n}\big(\sqrt{A_2^\alpha} + \mathfrak{T}^{\mathcal{D}}_{T^\alpha} + \mathfrak{T}^{\mathcal{D}}_{(T^\alpha)^*}\big),$$

where the constant $C'_{\alpha,n}$ also depends on the Calderón–Zygmund kernel and the doubling constants of σ and ω, and $\mathfrak{T}^{\mathcal{D}}_{T^\alpha}$, $\mathfrak{T}^{\mathcal{D}}_{(T^\alpha)^*}$ are the $\mathcal{D}$-dyadic testing constants.

In order to complete the proof of Theorem 1.4, we need to consider iterated Riesz transforms.

6. Iterated Riesz transforms

Throughout Section 4 and 5, we considered Riesz transforms of order 1. However, our results extend to arbitrary iterated Riesz transforms of odd order in $\mathbb{R}^n$. We will extend the results of Section 4 to their appropriate analogues to make the reasoning of Section 5 hold for the appropriate iterated Riesz transforms, and we begin by establishing the following theorem.

Theorem 6.1. *The odd order pure iterated Riesz transforms R_1^{2m+1} are unstable on $\mathbb{R}^n$ for pairs of doubling measures under* 90° *rotations in any coordinate plane. In fact, there exists a measure pair of doubling measures on which R_1^{2m+1} is unbounded, and all iterated Riesz transforms of order $2m+1$ that are* not *a pure power of R_1, are bounded.*

Proof. Recall the notation $T_\sigma f = T(f\sigma)$. We begin first by considering Riesz transforms of arbitrary order, even or odd. Using the identity

$$R_1^2 + \cdots + R_n^2 = -I,$$

and for $N \geq 2$, we have for an arbitrary positive measure σ that

$$(R_1^N)_\sigma = (R_1^{N-2} R_1^2)_\sigma = -(R_1^{N-2})_\sigma - \sum_{j=2}^{n} (R_1^{N-2} R_j^2)_\sigma.$$

Iteration then yields, for $N \geq 1$,

$$(6.1) \qquad (R_1^N)_\sigma = \begin{cases} \pm I_\sigma + \sum_{k=0}^{m} \Big[\pm \sum_{j=2}^{n} (R_1^{N-2k} R_j^2)_\sigma \Big] & \text{if } N = 2m \text{ is even,} \\ \pm (R_1)_\sigma + \sum_{k=0}^{m} \Big[\pm \sum_{j=2}^{n} (R_1^{N-2k} R_j^2)_\sigma \Big] & \text{if } N = 2m+1 \text{ is odd.} \end{cases}$$

For the weight pairs $(\sigma_\tau, \omega_\tau)$ constructed in Section 5, and with $N = 2m+1$ odd, the second line in (6.1) yields

$$\begin{aligned} &\|(R_1^N)_{\sigma_\tau}\|_{L^2(\sigma_\tau)\to L^2(\omega_\tau)} \\ &\qquad \geq \|(R_1)_{\sigma_\tau}\|_{L^2(\sigma_\tau)\to L^2(\omega_\tau)} - \sum_{k=0}^{m}\sum_{j=2}^{n} \|(R_1^{N-2k} R_j^2)_{\sigma_\tau}\|_{L^2(\sigma_\tau)\to L^2(\omega_\tau)} \\ &\qquad \geq \Gamma - \sum_{k=0}^{m}\sum_{j=2}^{n} \|(R_1^{N-2k} R_j^2)_{\sigma_\tau}\|_{L^2(\sigma_\tau)\to L^2(\omega_\tau)}, \end{aligned}$$

where Γ is the constant in the construction of the weight pair $(\sigma_\tau, \omega_\tau)$. Note that the operator norm dominates the testing constant, which was shown to exceed Γ.

We now claim that the double sum of the operator norms on the right-hand side is bounded independently of Γ, i.e.,

$$\sum_{k=0}^{m}\sum_{j=2}^{n} \|(R_1^{N-2k} R_j^2)_{\sigma_\tau}\|_{L^2(\sigma_\tau)\to L^2(\omega_\tau)} = O(1).$$

In fact, if $j \geq 2$ and $R^\alpha = R_1^{\alpha_1} R_2^{\alpha_2} \cdots R_n^{\alpha_n}$ with $\alpha_j > 0$, then, by Lemma 4.12 (3),

$$\begin{aligned} &\limsup_{k\to\infty} \int |R_j R^\alpha s_k^{P,\mathrm{hor}}(x)|^2\, dx \\ &= \limsup_{k\to\infty} \Big| \int (R_j s_k^{P,\mathrm{hor}})(x)(R_j R^{2\alpha} s_k^{P,\mathrm{hor}})(x)\, dx \Big| \\ &\leq \sqrt{\limsup_{k\to\infty} \int |R_j s_k^{P,\mathrm{hor}}(x)|^2\, dx}\ \sqrt{\limsup_{k\to\infty} \int |R_j R^{2\alpha} s_k^{P,\mathrm{hor}}(x)|^2\, dx} \\ &\leq \sqrt{\limsup_{k\to\infty} \int |R_j s_k^{P,\mathrm{hor}}(x)|^2\, dx} \cdot \|R_j R^{2\alpha}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \sqrt{|P|} = 0 \quad \text{for all } N \in \mathbb{N}. \end{aligned}$$

Therefore, the reasoning in Proposition 5.1 and Lemma 5.9 shows that iterated Riesz transforms of order N, which are *not* pure powers of R_1, have dyadic testing constants on the weight pairs $(\sigma_\tau, \omega_\tau)$ that are $O(1)$. Then Theorem 5.11 shows that the operator norms of such operators, including $R_1^{N-2k} R_j^2$, are $O(1)$, which proves our claim, and completes the proof of the second assertion of the theorem. The first assertion regarding R_1^{2m+1} now follows from the fact that a rotation in the (x_1, x_j)-plane interchanges R_1^{2m+1} and R_j^{2m+1}. ∎

The key to our proof of Theorem 6.1 is the construction of weight pairs $(\sigma_\tau, \omega_\tau)$ satisfying the inequality

$$\|(R_1^N)_{\sigma_\tau}\|_{L^2(\sigma_\tau)\to L^2(\omega_\tau)} \geq \Gamma \quad \text{for } \Gamma \text{ arbitrarily large,} \tag{6.2}$$

when N is odd. In fact, the inequality (6.2) actually *fails* for the weight pairs we construct when N is even. Indeed, from the first line in (6.1), and the fact that the proof of Theorem 6.1 shows that

$$\sum_{k=0}^{m} \sum_{j=2}^{n} \|(R_1^{N-2k} R_j^2)_{\sigma_\tau}\|_{L^2(\sigma_\tau)\to L^2(\omega_\tau)} = O(1),$$

we get

$$\|(R_1^N)_{\sigma_\tau}\|_{L^2(\sigma_\tau)\to L^2(\omega_\tau)} \leq \|I_{\sigma_\tau}\|_{L^2(\sigma_\tau)\to L^2(\omega_\tau)} + O(1).$$

The right-hand side of the above display is bounded since the operator norm of I_{σ_τ} is bounded by $A_2(\sigma_\tau, \omega_\tau)$. Indeed, when σ and ω are weights, we have $\|\sigma\omega\|_\infty \leq A_2(\sigma, \omega)$ by the Lebesgue differentiation theorem, and so

$$\|I_\sigma f\|_{L^2(\omega)}^2 = \int_{\mathbb{R}^n} f^2 \sigma^2 \omega \leq A_2(\sigma, \omega) \int_{\mathbb{R}^n} f^2 \sigma = \|f\|_{L^2(\sigma)}^2 .$$

Moreover, it is easily shown that $\|I_\sigma\|_{L^2(\sigma)\to L^2(\omega)} = A_2(\sigma, \omega)$ for arbitrary weights σ and ω. Thus, R_1^N must then satisfy the testing conditions for the measure pair (σ, ω).

In the next subsection we show that every odd order iterated Riesz transform $R^\beta = R_1^{\beta_1} R_2^{\beta_2} \cdots R_n^{\beta_n}$ is unstable under rotations, by showing that $R_1^{\beta_1} R_2^{\beta_2} \cdots R_n^{\beta_n}$ is some rotation of $R^{(|\beta|,0,\ldots,0)}$ whenever $\beta \neq |\beta| e_k$ for some k. When $\beta = |\beta| e_k$ some k, then we may assume without loss of generality that $k = 2$.

6.1. Rotations

Let β be a multi-index of length $|\beta| = N$. The symbol of the iterated Riesz transform $R^\beta = R_1^{\beta_1} R_2^{\beta_2} \cdots R_n^{\beta_n}$ is

$$i^N \frac{\xi_1^{\beta_1} \xi_2^{\beta_2} \cdots \xi_n^{\beta_n}}{|\xi|^N}.$$

We already know that $R^{(N,0,\ldots,0)}$ is unstable, and the following lemma will be used to show all R^β are unstable.

Lemma 6.2. *If $P(\xi)$ is a nontrivial homogeneous polynomial of degree N that does not contain the monomial ξ_1^N, then there is a set of rotations of full-measure Λ, and for any rotation $\Theta \in \Lambda$, we have $\xi = \Theta\eta$ such that $P(\Theta\eta)$ contains the monomial η_1^N.*

Proof. In dimension $n = 2$, we have

$$P(\xi_1, \xi_2) = \sum_{m=1}^{N} c_m \, \xi_1^m \, \xi_2^{N-m}, \quad \text{where not all } c_m = 0,$$

and the restriction of this polynomial to the unit circle cannot vanish identically (otherwise P itself would vanish identically by homogeneity, a contradiction). Thus, there is $\theta \in [0, 2\pi)$ such that

$$0 \neq P(\cos\theta, \sin\theta) = \sum_{m=1}^{N} c_m \cos^m\theta \sin^{N-m}\theta.$$

However, if we make the rotational change of variable, i.e.,

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \cos\theta - \eta_2 \sin\theta \\ \eta_1 \sin\theta + \eta_2 \cos\theta \end{pmatrix},$$

then

$$\begin{aligned} P(\xi_1, \xi_2) = \sum_{m=1}^{N} c_m \xi_1^m \xi_2^{N-m} &= \sum_{m=1}^{N} c_m (\eta_1 \cos\theta - \eta_2 \sin\theta)^m (\eta_1 \sin\theta + \eta_2 \cos\theta)^{N-m} \\ &= \eta_1^N \sum_{m=1}^{N} c_m \cos^m\theta \sin^{N-m}\theta + \sum_{\beta \neq \mathbf{e}_1 : |\beta| = N} \eta^\beta f_\beta(\theta) \end{aligned}$$

where

$$\sum_{m=1}^{N} c_m \cos^m\theta \sin^{N-m}\theta \neq 0.$$

The case $n \geq 3$ is similar. ■

6.2. Completion of proofs of main Theorems 1.4 and 1.5

To complete the proof of Theorem 1.4 we use the above Lemma, together with Proposition 5.10, and we see that any iterated Riesz transform R^β of odd order $N = |\beta|$ with $\beta \neq (N, 0, \ldots, 0)$, is bounded on the higher-dimensional analogue of the weight pair (σ, ω) constructed in Proposition 5.10, and can be rotated into a sum S of iterated Riesz transforms that includes $R^{(N,0,\ldots,0)}$, and hence S is unbounded on the weight pair (σ, ω). Since stability under rotational change of variables is unaffected by rotation of the operator, this completes our proof that all iterated Riesz transforms R^β of odd order are unstable under rotational changes of variable, even when the measures are doubling with adjacency constant λ_{adj} arbitrarily close to 1. This completes the proof of the main Theorem 1.4.

To prove Theorem 1.5, suppose R^β is an odd order iterated Riesz transform; without loss of generality, assume that $R^\beta \neq R_1^{|\beta|}$. Then, by Lemma 6.2, there is a set Λ of rotations of full measure such that for each $\Theta \in \Lambda$, Θ rotates R^β to $c(\Theta) R_1^{|\beta|}$ plus mixed iterated Riesz transforms, where $c(\Theta) \neq 0$. Then our construction yields a weight pair (σ, ω) for which the norm inequality for R^β is bounded, but the norm inequality for the rotated operator can be made arbitrarily large.

A. Appendix

We begin by using the counterexamples in [22] to show that the Hilbert transform is two-weight norm biLipschitz unstable on $\mathcal{S}_{\mathrm{lfpB}}$. Then we demonstrate that the notion of stability that is maximal for preserving the classical A_2 condition, is that of *biLipschitz* stability. Next, we show that all sparse bump functionals are biLipschitz stable on the pairs of doubling measures. After that, we give the details for arguments surrounding classical doubling which were omitted from [32]. And finally, we give the proof of the $T1$ Theorem 5.7.

A.1. BiLipschitz instability of the Hilbert transform for arbitrary weight pairs

Here we show that the Hilbert transform H is two-weight norm unstable under biLipschitz transformations. We consider the measure pairs (σ, ω) and $(\ddot{\sigma}, \omega)$ constructed in [22], where (σ, ω) satisfies the two-weight norm inequality for H, while $(\ddot{\sigma}, \omega)$ does not, although it continues to satisfy the two-tailed Muckenhoupt $\mathcal{A}_2$ condition. The measure ω is the standard Cantor measure on $[0,1]$ supported in the middle-third Cantor set E. The measures $\sigma = \sum_{k,j} s_j^k \delta_{z_j^k}$ and $\ddot{\sigma} = \sum_{k,j} s_j^k \delta_{\ddot{z}_j^k}$ are sums of weighted point masses located at positions z_j^k and $\ddot{z}_j^k$ within the component G_j^k removed at the k-th stage of the construction of E, and satisfy

$$0 < c_1 < \frac{\operatorname{dist}(z_j^k, \partial G_j^k)}{|G_j^k|}, \frac{\operatorname{dist}(\ddot{z}_j^k, \partial G_j^k)}{|G_j^k|} < c_2 < 1, \tag{A.1}$$

independent of k, j. See [22] for notation and proofs.

It remains to construct a biLipschitz map $\Phi\colon \mathbb{R} \to \mathbb{R}$ such that $(\ddot{\sigma}, \omega) = (\Phi_* \sigma, \Phi_* \omega)$. For this, we first define biLipschitz maps $\Phi\colon \overline{G}_j^k \to \overline{G}_j^k$ so that Φ fixes the endpoints of $\overline{G}_j^k$ and $\ddot{z}_j^k = \Phi(z_j^k)$, and note that this can be done with bounds independent of k, j by (A.1). Now we extend the definition of Φ to all of $\mathbb{R}$ by the identity map, and it is evident that Φ is biLipschitz and pushes (σ, ω) forward to $(\ddot{\sigma}, \omega)$.

A.2. Beyond biLipschitz maps for A_2 stability

Here we initiate an investigation of how general a map can be, and still preserve the two-weight A_2 condition for all pairs of measures (σ, ω). We begin by defining some of the terminology we will use in this subsection.

Definition A.1. Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$. Let $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be a Borel measurable function. We define the pushforward of the measure μ by the map Φ as the unique measure $\Phi_* \mu$ such that

$$\int_E \Phi_* \mu = \int_{\Phi^{-1}(E)} \mu \quad \text{for all Borel sets } E \subset \mathbb{R}^n.$$

In the case $d\mu(x) = w(x)\,dx$ is absolutely continuous, its pushforward for Φ sufficiently smooth is given by

$$(\Phi_* \mu)(y) \equiv w(\Phi(y)) \left| \det \frac{\partial \Phi}{\partial x}(y) \right|.$$

Definition A.2. A map $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is A_2*-stable* if there exists a constant $C > 0$ such that for every pair of locally finite positive Borel measures σ, ω, we have

$$A_2(\Phi_*\sigma, \Phi_*\omega) \leq C A_2(\sigma, \omega).$$

Definition A.3. A map $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ (not necessarily invertible) is *shape-preserving* if there exists $K \geq 1$ such that for every cube $Q \subset \mathbb{R}^n$, we can find cubes Q_{small} and Q_{big} with the following properties:

$$Q_{\text{small}} \subset \Phi^{-1}(Q) \subset Q_{\text{big}} \quad \text{and} \quad \frac{\ell(Q_{\text{big}})}{\ell(Q_{\text{small}})} \leq K.$$

We call such a set $\Phi^{-1}(Q)$ an *almost cube*.

Note that homeomorphisms on the real line are automatically shape-preserving, as are quasiconformal maps in $\mathbb{R}^n$, see Lemma 3.4.5 in [5].

Theorem A.4. *Let $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be shape-preserving and Borel-measurable. Then the following two conditions are equivalent*:

(1) *There exists a constant $C_1 > 0$ such that $|\Phi^{-1}(Q)| \leq C_1|Q|$ for every cube Q.*

(2) *Φ is A_2-stable.*

Remark A.5. If Φ is sufficiently regular that the usual change of variables formula holds, e.g., Φ^{-1} is locally Lipschitz, then condition (1) becomes $|\det D\Phi^{-1}| \lesssim 1$.

Proof. Assume condition (1) holds, where Φ is shape-preserving with constant K, and let Q be an arbitrary cube in $\mathbb{R}^n$. Then

$$\begin{aligned} A_2(\Phi_*\sigma, \Phi_*\omega) &= \sup_Q \Big(\frac{\int_Q d\Phi_*\sigma}{|Q|}\Big)\Big(\frac{\int_Q d\Phi_*\omega}{|Q|}\Big) = \sup_Q \Big(\frac{\int_{\Phi^{-1}(Q)} d\sigma}{|Q|}\Big)\Big(\frac{\int_{\Phi^{-1}(Q)} d\omega}{|Q|}\Big) \\ &\leq C_1^2 \sup_Q \Big(\frac{\int_{\Phi^{-1}(Q)} d\sigma}{|\Phi^{-1}Q|}\Big)\Big(\frac{\int_{\Phi^{-1}(Q)} d\omega}{|\Phi^{-1}Q|}\Big) \\ &\leq C_1^2 K^{2n} \sup_Q \Big(\frac{\int_{Q_{\text{big}}} d\sigma}{|Q_{\text{big}}|}\Big)\Big(\frac{\int_{Q_{\text{big}}} d\omega}{|Q_{\text{big}}|}\Big) \leq C_1^2 K^{2n} A_2(\sigma, \omega). \end{aligned}$$

Conversely, if condition (2) holds, then with both measures σ and ω equal to Lebesgue measure, and for any cube Q, we have

$$\Big(\frac{|\Phi^{-1}(Q)|}{|Q|}\Big)^2 = \Big(\frac{\int_{\Phi^{-1}(Q)} dx}{|Q|}\Big)\Big(\frac{\int_{\Phi^{-1}(Q)} dx}{|Q|}\Big) = \Big(\frac{\int_Q d\Phi_*\sigma}{|Q|}\Big)\Big(\frac{\int_Q d\Phi_*\omega}{|Q|}\Big) \leq C. \quad \blacksquare$$

Remark A.6. If the pair $(\Phi_*\sigma, \Phi_*\omega)$ is in A_2 for the *single* choice of weights $d\sigma(x) = d\omega(x) = dx$, then the above proof shows that Φ preserves all A_2 pairs under the side assumption of shape-preservation.

Corollary A.7. *Assume $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a shape-preserving invertible Lipschitz map with $\|D\Phi\|_\infty \leq 1$. Then Φ is A_2-stable if and only if Φ is biLipschitz.*

Proof. Using Theorem A.4 and Remark A.5, we see that Φ is A_2-stable if and only if $|\det D\Phi| \gtrsim 1$. But then $1 \le C|\det D\Phi| \le C'|D\Phi|^n$, together with $\|D\Phi\|_\infty \le 1$, shows that Φ is A_2-stable if and only if Φ is biLipschitz. ∎

Corollary A.8. *Assume $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is Borel-measurable and invertible, and that both Φ and Φ^{-1} are shape-preserving. Then both Φ and Φ^{-1} are A_2-stable* if and only if *Φ is biLipschitz.*

Proof. If both Φ and Φ^{-1} are A_2-stable, then from Theorem A.4, we obtain that

$$\begin{aligned} |\Phi^{-1}(Q)| &\le C_1|Q| \quad \text{for every cube } Q, \\ |\Phi(Q)| &\le C_1|Q| \quad \text{for every cube } Q. \end{aligned}$$

Thus, if Q is a minimal cube containing both x and y, then the almost cube $\Phi^{-1}(Q)$ contains both $\Phi^{-1}(x)$ and $\Phi^{-1}(y)$, and so

$$\frac{|\Phi^{-1}(x) - \Phi^{-1}(y)|}{|x-y|} \lesssim \frac{\operatorname{diam} \Phi^{-1}(Q)}{\operatorname{diam} Q} \lesssim \frac{|\Phi^{-1}(Q)|}{|Q|} \le C_1,$$

and since the almost cube $\Phi(Q)$ contains both $\Phi(x)$ and $\Phi(y)$,

$$\frac{|\Phi(x) - \Phi(y)|}{|x-y|} \lesssim \frac{\operatorname{diam} \Phi(Q)}{\operatorname{diam} Q} \lesssim \frac{|\Phi(Q)|}{|Q|} \le C_1. \qquad ∎$$

A.3. Stability and sparse operators

Recall that a grid of dyadic cubes $\mathcal{S}$ is called η-sparse, $0 < \eta < 1$, if for every $Q \in \mathcal{S}$, there are subsets $E_Q \subset Q$ such that $|E_Q| \ge \eta|Q|$ and the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint. Note that such an $\mathcal{S}$ satisfies the following $\frac{1}{\eta}$-Carleson condition:

$$\begin{aligned} \sum_{Q'\in\mathcal{S}:Q'\subset Q} |Q'| &\le \frac{1}{\eta} \sum_{Q'\in\mathcal{S}:Q'\subset Q} |E_{Q'}| \le \frac{1}{\eta}|Q| \quad \text{for all } Q \in \mathcal{S}, \\ \sum_{Q'\in\mathcal{S}:Q'\subset \Omega} |Q'| &\le \frac{1}{\eta}|\Omega| \quad \text{for all open sets } \Omega. \end{aligned}$$

Conversely, if $\mathcal{S}$ satisfies the Λ-Carleson condition

$$\sum_{Q'\in\mathcal{S}:Q'\subset Q} |Q'| \le \Lambda|Q| \quad \text{for all } Q \in \mathcal{S},$$

then $\mathcal{S}$ is $\frac{1}{\Lambda}$-sparse, see e.g., [27].

Definition A.9. Given a sparse grid of cubes $\mathcal{S}$, we define the associated sublinear *sparse operator* S by

$$Sf(x) \equiv \sum_{Q\in\mathcal{S}} \left(\frac{1}{|Q|}\int_Q |f|\right) \mathbf{1}_Q(x), \quad x \in \mathbb{R}^n, \tag{A.2}$$

and we say that S is η-sparse if $\mathcal{S}$ is η-sparse.

Definition A.10. Let $\mathcal{U}$ be a biLipschitz invariant set of locally finite positive Borel measures on $\mathbb{R}^n$. A functional $\mathcal{B}(\sigma, \omega)$ on pairs of measures (σ, ω) is called a *sparse bump functional* on $\mathcal{U}$ if for every $\eta \in (0, 1)$, there exists a continuous increasing function $\Gamma_\eta \colon (0, \infty) \to (0, \infty)$ such that for all η-sparse operators S,

$$\mathfrak{N}_S(\sigma, \omega) \leq \Gamma_\eta(\mathcal{B}(\sigma, \omega)) \quad \text{for all} (\sigma, \omega) \in \mathcal{U} \times \mathcal{U}.$$

Clearly, no biLipschitz stable (bump) condition can characterize a biLipschitz unstable weighted norm inequality. Here we will show that no sparse bump functional can either. Note that it is shown in [26] that all (separated) Orlicz or entropy bump conditions, that are currently known to imply boundedness of singular integrals, are sparse bump functionals on any such $\mathcal{U}$. Here is the main result of this section.

Theorem A.11. *Let $\mathcal{U}_{\text{doub}}$ be the biLipschitz invariant set of doubling measures on $\mathbb{R}^n$ (called $\mathcal{S}_{\text{doub}}$ in the introduction), and let $\mathcal{B}(\sigma, \omega)$ be a sparse bumpfunctional on $\mathcal{U}_{\text{doub}}$. Then for any smooth Calderón–Zygmund operator T that is biLipschitz unstable on pairs of doubling weights, there is no continuous increasing function $\Gamma \colon (0, \infty) \to (0, \infty)$, such that*

$$\mathcal{B}(\sigma, \omega) \leq \Gamma(\mathfrak{N}_T(\sigma, \omega)) \quad \textit{for all } (\sigma, \omega) \in \mathcal{U}_{\text{doub}}. \tag{A.3}$$

In particular, by Theorem 1.4, we can take T to be an iterated Riesz transform of odd order.

Remark A.12. This theorem, together with Theorem A.13 below, shows that no sparse bump functional $\mathcal{B}(\sigma, \omega)$ can characterize the two-weight norm inequality for an iterated Riesz transform of odd order on doubling measures.

To prove Theorem A.11, we will use a special case of the groundbreaking sparse domination principle of Lerner. Recall that a Dini-regular Calderón–Zygmund operator T with kernel K is an operator where the kernel, rather than satisfying the size and smoothness estimates (1.5), instead satisfies

$$\begin{aligned} &|K(x, y)| \leq C_{CZ} |x - y|^{-n}, \\ &|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq f_{CZ}\Big(\frac{|x - x'|}{|x - y|}\Big) |x - y|^{-n}, \end{aligned}$$

where the nonnegative function f_{CZ} satisfies the Dini condition

$$\int_0^1 f_{CZ}(t) \, \frac{dt}{t} < \infty.$$

Theorem A.13 (Lerner [25]). *Let T be a Dini-regular Calderón–Zygmund operator, and let $f \in L^1(\mathbb{R}^n)$ be compactly supported. Then with $\eta_n = \frac{1}{2(5\sqrt{n})^n}$ there is an η_n-sparse grid $\mathcal{S}$ depending on f, such that*

$$|Tf(x)| \leq C_{n,T} \, Sf(x) \quad \textit{for a.e. } x \in \mathbb{R}^n,$$

where Sf is as in (A.2).

Now we can give the proof of Theorem A.11.

Proof of Theorem A.11. Suppose in order to derive a contradiction that (A.3) holds for some sparse bump functional $\mathcal{B}(\sigma,\omega)$ on $\mathcal{U}$ in $\mathbb{R}^n$. Then, for any BiLipschitz map Φ, we have $\mathfrak{N}_T(\Phi_*\sigma,\Phi_*\omega)=\mathfrak{N}_{\Phi_*T}(\sigma,\omega)$, and so if a compactly supported function $f\in L^2(\sigma)$ is chosen to be a near extremizer for the norm $\mathfrak{N}_{\Phi_*T}(\sigma,\omega)$, we have from Lerner's theorem, applied to the Dini-regular Calderón–Zygmund operator Φ_*T, that there is an η_n-sparse operator S, such that

$$\begin{aligned}\mathfrak{N}_{\Phi_*T}(\sigma,\omega) &\le 2\,\frac{\|\Phi_*T(f\sigma)\|_{L^2(\omega)}}{\|f\|_{L^2(\sigma)}} \le 2C_{n,T,\|\Phi\|}\,\frac{\|S(f\sigma)\|_{L^2(\omega)}}{\|f\|_{L^2(\sigma)}}\\ &\le 2C_{n,T,\|\Phi\|}\,\mathfrak{N}_S(\sigma,\omega) \le C_{n,T,\|\Phi\|}\,\Gamma_{\eta_n}(\mathcal{B}(\sigma,\omega))\\ &\le C_{n,T,\|\Phi\|}\,\Gamma_{\eta_n}(\Gamma(\mathfrak{N}_T(\sigma,\omega))),\end{aligned}$$

where the first line uses Theorem A.13, the second line uses the definition of sparse bump functional and the assumed inequality (A.3). Thus, two-weight norm inequalities for Dini-regular operators are biLipschitz stable, as defined in Definition 1.3. But by Theorem 1.4, the inequality for T equal an individual Riesz transform cannot be biLipschitz stable. This contradiction proves the theorem. ∎

A.4. Modification of transplantation to achieve classical doubling

In Section 3, we constructed functions v,u on a cube Q^0 such that both v,u are dyadically τ-flat on Q_0. However, dyadic doubling does not imply continuous doubling on Q^0. As such, we will need to modify the transplantation argument to smooth out v,u into weights v',u' which are classically doubling, as done in [32]. See also [19, 30].

We will describe how to attain u' from u, as the process for v' and v is identical. Recall, in Proposition 5.1, we define u by

$$u=(E_{Q^0}U)\mathbf{1}_{Q^0}+\sum_{t=0}^{m-1}\sum_{Q\in\mathcal{K}_t}\langle U,h^{\text{hor}}_{\mathcal{S}(Q)}\rangle\frac{1}{\sqrt{|\mathcal{S}(Q)|}}s^{Q,\text{hor}}_{k_{t+1}},$$

where $s_{k_{t+1}}$ is constant on cubes in $\mathcal{K}_{t+1}$.

Define the grid $\widehat{\mathcal{K}}$ from $\mathcal{K}$ inductively as follows. First set $\widehat{\mathcal{K}}_0\equiv\mathcal{K}_0$. Now given $Q\in\widehat{\mathcal{K}}_t$, a cube $R\in\mathcal{K}_{t+1}$ is called a *transition cube* for Q if $Q=\pi_{\mathcal{K}}R$ and $(\partial\pi_{\mathcal{D}}R)\cap\partial Q$ is non-empty; as such, define $\widehat{\mathcal{K}}_{t+1}$ to consist of the cubes $P\in\mathcal{K}_{t+1}$ such that $\pi_{\mathcal{K}}P\in\widehat{\mathcal{K}}_t$ and P is *not* a transition cube. Finally, set $\widehat{K}\equiv\bigcup_t\widehat{\mathcal{K}}_t$.

One can see that $\widehat{\mathcal{K}}$ consists of the cubes in $\mathcal{K}$ not contained in a transition cube. This implies that if R is a transition cube, then $\pi_{\mathcal{K}}R\in\mathcal{K}$. It also implies that no two transition cubes have overlapping interiors. Visually, the union of the transition cubes for a cube Q forms a "halo" for Q. Recalling that two distinct dyadic cubes in $\mathcal{D}$ of the same size are *adjacent* if their boundaries intersect, we then note that two adjacent cubes in $\widehat{\mathcal{K}}$ must then have the same $\mathcal{K}$-parent, and so are close to each other in the tree distance of $\mathcal{K}$. The proof of the following lemma is left to the reader, who is encouraged to draw a picture. It helps to note that in $\mathbb{R}$, if two transition intervals R_1 and R_2 are at levels s and $s+2$, then there must be a transition interval R at level $s+1$ such that R lies between R_1 and R_2.

Lemma A.14. *Let $R_1 \in \mathcal{K}_s$ be a transition cube.*

(1) *If $R_2 \in \mathcal{K}_t$ is a transition cube such that the interiors of R_2 and R_1 are disjoint, but not their closures, then $t \in \{s-1, s, s+1\}$.*

(2) *If $K \in \widehat{\mathcal{K}}_t$ is such that the interiors of K and R_1 are disjoint, but not their closures, then $t \in \{s-1, s\}$. And if $t = s$, then $\pi_{\mathcal{K}} K = \pi_{\mathcal{K}} R_1$.*

With this in mind, given $Q \in \widehat{\mathcal{K}}_t$, define

$$r_{k_{t+1}}^{Q,\mathrm{hor}}(x) \equiv \begin{cases} s_{k_{t+1}}^{Q,\mathrm{hor}}(x) & \text{if } x \text{ is not contained in a transition cube for } Q, \\ 0 & \text{otherwise.} \end{cases}$$

Now we may define

$$u'_\ell \equiv (E_{Q^0} U)\,\mathbf{1}_{Q^0} + \sum_{t=0}^{\ell-1} \sum_{Q \in \widehat{\mathcal{K}}_t} \langle U, h_{\mathcal{S}(Q)}^{\mathrm{hor}} \rangle \frac{1}{\sqrt{|\mathcal{S}(Q)|}}\, r_{k_{t+1}}^{Q,\mathrm{hor}}, \quad 0 \le \ell \le m,$$

$$u' \equiv u'_m \quad \text{and} \quad v' \equiv v'_m.$$

Given $x \in Q^0$ and $\ell \le m$, if we define

$$t(x) \equiv \begin{cases} t & \text{if } x \text{ is contained in a transition cube belonging to} \mathcal{K}_t \text{ for some } t < \ell, \\ \ell & \text{otherwise,} \end{cases}$$

then pointwise we have

$$u'_\ell(x) = (E_{Q^0} U)\,\mathbf{1}_{Q^0}(x) + \sum_{t=0}^{t(x)-1} \sum_{Q \in \widehat{\mathcal{K}}_t} \langle U, h_{\mathcal{S}(Q)}^{\mathrm{hor}} \rangle \frac{1}{\sqrt{|\mathcal{S}(Q)|}}\, s_{k_{t+1}}^{Q,\mathrm{hor}}(x), \quad 0 \le \ell \le m.$$

The function u' is nearly a transplantation of U, as exhibited by the following lemma, whose proof we leave to the reader. The reader should note that for each cube contained in a transition cube, the value of u'_ℓ is equal to its average on the transition cube containing it.

Lemma A.15. *Let $\mathcal{K}$ be as above.*

(1) *If $P \in \mathcal{K}$ is not contained in a transition cube, then $E_P u'_\ell = E_{\mathcal{S}(P)} U$.*

(2) *If $P \in \mathcal{K}$ is contained in a transition cube R, then $E_P u'_\ell = E_{\mathcal{S}(\pi_{\mathcal{K}} R)} U$.*

(3) *If $P \in \mathcal{D}$ is a cube for which $K_{t+1} \subsetneq P \subset K_t$, where $K_{t+1} \in \mathcal{K}_{t+1}$ and $K_t \in \mathcal{K}_t$, then $E_P u'_\ell = E_{K_t} u'_\ell$.*

Remark A.16. From the above lemma, it follows that

$$A_2^{\mathrm{dyadic}}(u'_\ell, v'_\ell) \le A_2^{\mathrm{dyadic}}(U, V) \le 1.$$

Lemma A.17. *If P_1, P_2 are adjacent dyadic subcubes of Q^0, then $\frac{E_{P_1} u'}{E_{P_2} u'} \in (1-\tau, 1+\tau)$. Similarly, for v'.*

Proof of Lemma A.17. Let P_1, P_2 be adjacent dyadic subcubes of Q^0. By Lemma A.15 part (3), it suffices to check the case when $P_1, P_2 \in \mathcal{K}$. We consider various cases.

Case 1. Neither P_1 nor P_2 is contained in a transition cube, i.e., both belong to $\widehat{\mathcal{K}}$. Then P_1 and P_2 must have a common $\mathcal{K}$- parent, meaning

$$\pi_{\mathcal{D}} \mathcal{S}(P_1) = \mathcal{S}(\pi_{\mathcal{K}} P_1) = \mathcal{S}(\pi_{\mathcal{K}} P_2) = \pi_{\mathcal{D}} \mathcal{S}(P_2)$$

and so $\mathcal{S}(P_1)$ and $\mathcal{S}(P_2)$ must be equal or dyadic siblings. From the first formula of Lemma A.15, we get $E_{P_1} u'/E_{P_2} u' \in (1-\tau, 1+\tau)$.

Case 2. Exactly one of the cubes, say P_1, is contained in a transition cube R_1. Since P_2 is not in a transition cube, then the only way for P_1, P_2 to be adjacent is for both to have the same $\mathcal{K}$-parent. And since P_2 is not contained in a transition cube, then R_1 must in fact equal P_1, i.e., P_1 is a transition cube. Indeed, if P_1 were a level below R_1 in the grid $\mathcal{K}$, then the only way P_2 can be adjacent to P_1 is by being in a transition cube adjacent to R_1 or in R_1 itself, but the latter cannot happen by assumption on P_2.

Altogether, the above yields that $\mathcal{S}(\pi_{\mathcal{K}} P_1) = \mathcal{S}(\pi_{\mathcal{K}} P_2) = \pi_{\mathcal{D}} \mathcal{S}(P_2)$. Thus, by Lemma A.15 parts (1) and (2), dyadic τ-flatness of U, and the fact that P_1 is a transition cube, we have

$$\frac{E_{P_1} u'}{E_{P_2} u'} = \frac{E_{\mathcal{S}(\pi_{\mathcal{K}} P_1)} U}{E_{\mathcal{S}(P_2)} U} = \frac{E_{\pi_{\mathcal{D}} \mathcal{S}(P_2)} U}{E_{\mathcal{S}(P_2)} U} \in (1-\tau, 1+\tau).$$

Case 3. Both P_1 and P_2 are contained within transition cubes, say R_1 and R_2, respectively. Using Lemma A.15, it suffices to show the ratio

$$\frac{E_{P_1} u'}{E_{P_2} u'} = \frac{E_{\mathcal{S}(\pi_{\mathcal{K}} R_1)} U}{E_{\mathcal{S}(\pi_{\mathcal{K}} R_2)} U}$$

lies between $1-\tau$ and $1+\tau$. Note that adjacency of P_1, P_2 implies R_1 and R_2 have disjoint interiors, but not closures, or they are equal.

Case 3 (a). $R_1 = R_2$. Then we get $E_{P_1} u'/E_{P_2} u' = 1$.

Case 3 (b). R_1 and R_2 are of the same sidelength, but $R_1 \neq R_2$. Then both R_1 and R_2 are adjacent, and so $\mathcal{S}(\pi_{\mathcal{K}} R_1)$ and $\mathcal{S}(\pi_{\mathcal{K}} R_2)$ must be equal or dyadic siblings. In either case, by the formula above, $E_{P_1} u'/E_{P_2} u' \in (1-\tau, 1+\tau)$.

Case 3 (c). R_1 and R_2 are of different sidelengths, say $\ell(R_1) > \ell(R_2)$. Since P_1 and P_2 are adjacent, then R_1 and R_2 have disjoint interiors, but not closures. It follows that if $R_1 \in \mathcal{K}_t$, then $R_2 \in \mathcal{K}_{t+1}$, by Lemma A.14. Thus, R_1 is adjacent to $\pi_{\mathcal{K}} R_2$. In fact, since R_1 is a transition cube but $\pi_{\mathcal{K}} R_2$ is not, then by Lemma A.14 (2), we have that $\pi_{\mathcal{K}} R_1 = \pi_{\mathcal{K}}^{(2)} R_2$, and so

$$\mathcal{S}(\pi_{\mathcal{K}} R_1) = \mathcal{S}(\pi_{\mathcal{K}}^{(2)} R_2) = \pi_{\mathcal{D}} \mathcal{S}(\pi_{\mathcal{K}} R_2).$$

Thus,

$$\frac{E_{P_1} u'}{E_{P_2} u'} = \frac{E_{\mathcal{S}(\pi_{\mathcal{K}} R_1)} U}{E_{\mathcal{S}(\pi_{\mathcal{K}} R_2)} U} = \frac{E_{\pi_{\mathcal{D}} \mathcal{S}(\pi_{\mathcal{K}} R_2)} U}{E_{\mathcal{S}(\pi_{\mathcal{K}} R_2)} U} \in (1-\tau, 1+\tau).$$

This completes the proof. ∎

Showing u' has relative adjacency constant $1 + o_{\tau \to 1}(1)$ as $\tau \to 0$ on Q^0 follows from Lemma A.17 and a standard argument, and similarly for v'.

A.5. Proof of the T1 Theorem 5.7

By Theorem 2 in [38], we have

$$\mathfrak{N}_T(\sigma,\omega) \lesssim \sqrt{\mathcal{A}_2(\sigma,\omega)} + \mathfrak{T}_T(\sigma,\omega) + \mathfrak{T}_{T^*}(\omega,\sigma) + \mathcal{E}(\sigma,\omega) + \mathcal{E}(\omega,\sigma),$$

where the two-tailed $\mathcal{A}_2$ condition is given by

$$\mathcal{A}_2(\sigma,\omega) \equiv \sup_{Q\in\mathcal{P}^n} \Big(\int_{\mathbb{R}^n} \Big(\frac{\ell(Q)}{(\ell(Q)+|x-c_Q|)^2}\Big)^n d\sigma\Big)\Big(\int_{\mathbb{R}^n} \Big(\frac{\ell(Q)}{(\ell(Q)+|x-c_Q|)^2}\Big)^n d\omega\Big),$$

and the energy condition is defined by

$$\mathcal{E}(\sigma,\omega)^2 \equiv \sup_{I=\dot\bigcup_r J_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \mathrm{P}(J_r,\sigma)^2 |J_r|_\omega \,\mathrm{E}(J_r,\omega)^2,$$

where the supremum is taken over all cubes $I \in \mathcal{P}^n$ and all disjoint decompositions of $I \in \mathcal{P}^n$ into disjoint cubes $\dot\bigcup_r J_r$. Within the energy condition we also have the Poisson average $\mathrm{P}(J,\sigma)$, which is defined by

$$\mathrm{P}(J,\sigma) \equiv \int_{\mathbb{R}^n} \frac{\ell(J)}{(\ell(J)+|x-c_J|)^{n+1}}\, d\sigma,$$

and we also define

$$\mathrm{E}(J_r,\omega)^2 \equiv \frac{1}{|J_r|_\omega}\int_{J_r} \left|\frac{x-A}{\ell(J_r)}\right|^2 d\omega(x) \quad \text{and} \quad A \equiv \frac{1}{|J_r|_\omega}\int_{J_r} z\, d\omega(z).$$

Since $\mathrm{E}(J_r,\omega)^2 \le 1$, the energy condition is bounded by the pivotal condition

$$\mathcal{V}(\sigma,\omega)^2 \equiv \sup_{I=\dot\bigcup_r J_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \mathrm{P}(J_r,\sigma)^2 |J_r|_\omega,$$

By Theorem 4 of [12], if σ and ω are doubling, then the tailed $\mathcal{A}_2$ condition is equivalent to the classical A_2 condition, i.e.,

$$\mathcal{A}_2(\sigma,\omega) \lesssim A_2(\sigma,\omega);$$

see also Proposition 39 of [2] for further details.

As for the pivotal condition, a dyadic decomposition yields that the Poisson average of σ on Q is controlled by the expectation of σ on Q, i.e.,

$$\begin{aligned}
\mathrm{P}(Q,\sigma) &\lesssim \frac{|Q|_\sigma}{|Q|} + \sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^k Q} \frac{\ell(Q)}{(\ell(Q)+|x-c_Q|)^{n+1}}\, d\sigma \\
&\lesssim \sum_{k=0}^{\infty} |2^k Q|_\sigma \frac{\ell(Q)}{(2^k\ell(Q))^{n+1}} \lesssim |Q|_\sigma \sum_{k=0}^{\infty} 2^{(n+\varepsilon)k} \frac{\ell(Q)}{(2^k\ell(Q))^{n+1}} \\
&= \frac{|Q|_\sigma}{|Q|} \sum_{k=0}^{\infty} 2^{-(1-\varepsilon)k} \lesssim \frac{|Q|_\sigma}{|Q|},
\end{aligned}$$

where the third inequality follows by the hypothesis on the doubling constants, and the last inequality follows because $\varepsilon < 1$ implies the geometric series converges. Thus, we can estimate

$$\mathcal{V}(\sigma,\omega)^2 \lesssim \sup_{I=\dot\bigcup_r J_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \mathrm{P}(J_r,\sigma)^2 |J_r|_\omega \lesssim \sup_{I=\dot\bigcup_r J_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \frac{|J_r|_\sigma^2}{|J_r|^2} |J_r|_\omega$$
$$\lesssim A_2(\sigma,\omega) \sup_{I=\dot\bigcup_r J_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} |J_r|_\sigma \lesssim A_2(\sigma,\omega).$$

Combining all the above estimates with the corresponding dual estimates yields the theorem.

Alternatively, rather than applying Theorem 2 in [38], one can apply Theorem 2.6 (1) in [37], and then in our particular situation where both measures σ and ω are doubling, one can dispose of the weak-boundedness property using an argument similar to that of Lemma 2.4 in [16], or to that in the proof of Lemma 14 in [2].

Acknowledgments. We thank D. Cruz-Uribe, K. Moen and X. Tolsa for valuable comments. We also thank two referees for thoughtful feedback, which helped make the final version of this paper much more accessible.

Funding. E. T. Sawyer is partially supported by a grant from the National Research Council of Canada. I. Uriarte-Tuero has been partially supported by grant PID2019-106870GB-I00 (MICINN, Spain), and is partially supported by a grant from the National Research Council of Canada.

References

[1] Alexis, M., Luna-Garcia, J. L., Sawyer, E. T., and Uriarte-Tuero, I.: Stability of weighted norm inequalities. Preprint 2022, arXiv: 2208.08400v2.

[2] Alexis, M., Luna-Garcia, J. L., Sawyer, E. T., and Uriarte-Tuero, I.: The scalar $T1$ theorem for pairs of doubling measures fails for Riesz transforms when p not 2. Preprint 2024, arXiv: 2308.15739v3.

[3] Alexis, M., Sawyer, E. T., and Uriarte-Tuero, I.: Tops of dyadic grids and $T1$ theorems. Preprint 2022, arXiv: 2201.02897v1.

[4] Alexis, M., Sawyer, E. T. and Uriarte-Tuero, I.: A $T1$ theorem for general smooth Calderón–Zygmund operators with doubling weights, and optimal cancellation conditions, II. *J. Funct. Anal.* **285** (2023), no. 11, article no. 110139, 52 pp. Zbl 1523.42018 MR 4642561

[5] Astala, K., Iwaniec, T. and Martin, G.: *Elliptic partial differential equations and quasiconformal mappings in the plane*. Princeton Mathematical Series 48, Princeton University Press, Princeton, NJ, 2009. Zbl 1182.30001 MR 2472875

[6] Bourgain, J.: Some remarks on Banach spaces in which martingale difference sequences are unconditional. *Ark. Mat.* **21** (1983), no. 2, 163–168. Zbl 0533.46008 MR 727340

[7] Coifman, R. R. and Fefferman, C.: Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.* **51** (1974), 241–250. Zbl 0291.44007 MR 358205

[8] Cruz-Uribe, D. and Pérez, C.: On the two-weight problem for singular integral operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **1** (2002), no. 4, 821–849. Zbl 1072.42010 MR 1991004

[9] Dąbrowski, D., and Tolsa, X.: The measures with L^2 bounded Riesz transform satisfying a subcritical Wolff-type energy condition. Preprint 2021, arXiv: 2106.00303v1.

[10] Fefferman, C. and Muckenhoupt, B.: Two nonequivalent conditions for weight functions. *Proc. Amer. Math. Soc.* **45** (1974), 99–104. Zbl 0318.26010 MR 360952

[11] Grafakos, L.: *Classical Fourier analysis*. Second edition. Graduate Texts in Mathematics 249, Springer, New York, 2008. Zbl 1220.42001 MR 2445437

[12] Grigoriadis, C.: Necessary/sufficient conditions in weighted theory. Preprint 2020, arXiv: 2009.12091v1.

[13] Grigoriadis, C. and Paparizos, M.: Counterexample to the off-testing condition in two dimensions. *Colloq. Math.* **167** (2022), no. 2, 171–185. Zbl 1486.42018 MR 4358444

[14] Havin, V. P. and Nikolski, N. K. (eds.): *Linear and complex analysis. Problem book 3. Part I*. Lecture Notes in Mathematics 1573, Springer, Berlin, 1994. Zbl 0893.30036 MR 1334345

[15] Hytönen, T. P.: The sharp weighted bound for general Calderón–Zygmund operators. *Ann. of Math. (2)* **175** (2012), no. 3, 1473–1506. Zbl 1250.42036 MR 2912709

[16] Hytönen, T. P.: The two-weight inequality for the Hilbert transform with general measures. *Proc. Lond. Math. Soc. (3)* **117** (2018), no. 3, 483–526. Zbl 1420.42010 MR 3857692

[17] Hytönen, T. P. and Lacey, M. T.: The A_p-A_∞ inequality for general Calderón–Zygmund operators. *Indiana Univ. Math. J.* **61** (2012), no. 6, 2041–2052. Zbl 1290.42037 MR 3129101

[18] Johnson, R. and Neugebauer, C. J.: Homeomorphisms preserving A_p. *Rev. Mat. Iberoam.* **3** (1987), no. 2, 249–273. Zbl 0677.42019 MR 990859

[19] Kakaroumpas, S. and Treil, S.: "Small step" remodeling and counterexamples for weighted estimates with arbitrarily "smooth" weights. *Adv. Math.* **376** (2021), article no. 107450, 52 pp. Zbl 1453.42011 MR 4178922

[20] Lacey, M., Sawyer, E. T. and Uriarte-Tuero, I.: A characterization of two weight norm inequalities for maximal singular integrals with one doubling measure. *Anal. PDE* **5** (2012), no. 1, 1–60. Zbl 1279.42016 MR 2957550

[21] Lacey, M. T., Sawyer, E. T., Shen, C.-Y. and Uriarte-Tuero, I.: Two-weight inequality for the Hilbert transform: a real variable characterization, I. *Duke Math. J.* **163** (2014), no. 15, 2795–2820. Zbl 1312.42011 MR 3285857

[22] Lacey, M. T., Sawyer, E. T. and Uriarte-Tuero, I.: A two weight inequality for the Hilbert transform assuming an energy hypothesis. *J. Funct. Anal.* **263** (2012), no. 2, 305–363. Zbl 1252.42018 MR 2923415

[23] Lacey, M. T. and Spencer, S.: On entropy bumps for Calderón–Zygmund operators. *Concr. Oper.* **2** (2015), no. 1, 47–52. Zbl 1333.42021 MR 3357767

[24] Lacey, M. T. and Wick, B. D.: Two weight inequalities for Riesz transforms: uniformly full dimension weights. Preprint 2016, arXiv:1312.6163v4.

[25] Lerner, A. K.: On pointwise estimates involving sparse operators. *New York J. Math.* **22** (2016), 341–349. Zbl 1347.42030 MR 3484688

[26] Lerner, A. K.: On separated bump conditions for Calderón–Zygmund operators. *Proc. Amer. Math. Soc.* **150** (2022), no. 3, 1197– 1208. Zbl 1486.42023 MR 4375714

[27] Lerner, A. K. and Nazarov, F.: Intuitive dyadic calculus: the basics. *Expo. Math.* **37** (2019), no. 3, 225–265. Zbl 1440.42062 MR 4007575

[28] Muckenhoupt, B.: Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* **165** (1972), 207–226. Zbl 0236.26016 MR 293384

[29] Muckenhoupt, B. and Wheeden, R. L.: Two weight function norm inequalities for the Hardy–Littlewood maximal function and the Hilbert transform. *Studia Math.* **55** (1976), no. 3, 279–294. Zbl 0336.44006 MR 417671

[30] Nazarov, F.: A counterexample to Sarason's conjecture. Unpublished manuscript 1997, available at http://users.math.msu.edu/users/fedja/prepr.html, visited on 27 January 2025.

[31] Nazarov, F., Treil, S., and Volberg, A.: Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures. Preprint 2010, arXiv:1003.1596v1.

[32] Nazarov, F. and Volberg, A.: The Bellman function, the two-weight Hilbert transform, and embeddings of the model spaces K_θ. *J. Anal. Math.* **87** (2002), 385–414. MR 1945290 Zbl 1035.42010

[33] Neugebauer, C. J.: Inserting A_p-weights. *Proc. Amer. Math. Soc.* **87** (1983), no. 4, 644–648. Zbl 0521.42019 MR 687633

[34] Sawyer, E. T.: A characterization of a two-weight norm inequality for maximal operators. *Studia Math.* **75** (1982), no. 1, 1–11. Zbl 0508.42023 MR 676801

[35] Sawyer, E. T.: Two weight norm inequalities for certain maximal and integral operators. In *Harmonic analysis (Minneapolis, Minn., 1981)*, pp. 102–127. Lecture Notes in Math. 908, Springer, Berlin-New York, 1982. Zbl 0508.42024 MR 654182

[36] Sawyer, E. T.: A $T1$ theorem for general Calderón–Zygmund operators with comparable doubling weights, and optimal cancellation conditions. *J. Anal. Math.* **146** (2022), no. 1, 205–297. Zbl 07572388 MR 4467998

[37] Sawyer, E. T., Shen, C.-Y. and Uriarte-Tuero, I.: A two weight theorem for α-fractional singular integrals with an energy side condition. *Rev. Mat. Iberoam.* **32** (2016), no. 1, 79–174. Zbl 1344.42014 MR 3470665

[38] Sawyer, E. T., Shen, C.-Y. and Uriarte-Tuero, I.: A good-λ lemma, two weight $T1$ theorems without weak boundedness, and a two weight accretive global Tb theorem. In *Harmonic analysis, partial differential equations and applications*, pp. 125–164. Applied and Numerical Harmonic Analysis, Birkhäuser, Cham, 2017. Zbl 1380.42015 MR 3642742

[39] Sawyer, E. T., Shen, C.-Y. and Uriarte-Tuero, I.: A two weight fractional singular integral theorem with side conditions, energy and k-energy dispersed. In *Harmonic analysis, partial differential equations Banach spaces, and operator theory. Vol. 2*, pp. 305–372. Assoc. Women Math. Ser. 5, Springer, Cham, 2017. (See also arXiv:1603.04332v2.) Zbl 1387.42013 MR 3688149

[40] Sawyer, E. T., and Wick, B.: Two weight L^p inequalities for fractional vector Riesz transforms and doubling measures. *J. Geom. Anal.* **35** (2025), article no. 44, 77 pp. Zbl 07965351 MR 4843036

[41] Stein, E. M.: *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Princeton Mathematical Series 43, Princeton University Press, Princeton, NJ, 1993. Zbl 0821.42001 MR 1232192

[42] Tolsa, X.: Bilipschitz maps, analytic capacity, and the Cauchy integral. *Ann. of Math. (2)* **162** (2005), no. 3, 1243–1304. Zbl 1097.30020 MR 2179730

[43] Tolsa, X.: The measures with L^2-bounded Riesz transform and the Painlevé problem for Lipschitz harmonic functions. Preprint 2021, arXiv:2106.00680v2.

[44] Treil, S. and Volberg, A.: Completely regular multivariate stationary processes and the Muckenhoupt condition. *Pacific J. Math.* **190** (1999), no. 2, 361–382. Zbl 1006.60027 MR 1722900

[45] Treil, S. and Volberg, A.: Entropy conditions in two weight inequalities for singular integral operators. *Adv. Math.* **301** (2016), 499–548. Zbl 1377.42024 MR 3539383

[46] Uchiyama, A.: Weight functions of the class (A_∞) and quasi-conformal mappings. *Proc. Japan Acad.* **51** (1975), 811–814. Zbl 0362.30023 MR 442226

Received May 1, 2023; revised April 10, 2024.

Michel Alexis
Department of Mathematics & Statistics, McMaster University
1280 Main Street West, Hamilton, L8S 4K1, Canada;
alexism@mcmaster.ca, micalexis.math@gmail.com

José Luis Luna-Garcia
Department of Mathematics & Statistics, McMaster University
1280 Main Street West, Hamilton, L8S 4K1, Canada;
lunagaj@mcmaster.ca

Eric Sawyer
Department of Mathematics & Statistics, McMaster University
1280 Main Street West, Hamilton, L8S 4K1, Canada;
sawyer@mcmaster.ca

Ignacio Uriarte-Tuero
Department of Mathematics, University of Toronto
40 St. George Street, Toronto, L81 4K1, Canada;
Department of Mathematics, Michigan State University
619 Red Cedar Road, East Lansing, MI 48824, USA;
ignacio.uriartetuero@utoronto.ca