
Short note On the equation $x/y + y/z + z/x = 4^m$

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Abstract. We provide an elementary proof that the equation in the title has no solutions in the positive integers. This extends recent work of the author [Elem. Math. 78 (2023), 168–170] from $m = 1$ to arbitrary positive integers m .

1 Introduction

Spierpiński [2, page 80] remarked that it was not known if the equation

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 4 \tag{1}$$

has solutions in the positive integers. Erik [1] showed that the equation

$$a^3 + b^3 + c^3 = nabc \tag{2}$$

has no solutions in the *integers* except $a = b = c = 0$ for several values of n , including 4. By the substitution $x = a^2b$, $y = b^2c$, $z = c^2a$, equation (1) transforms into equation (2). Then it follows from Erik's work that equation (1) has no solutions in the *integers*. However, Erik used a fair amount of algebraic number theory, and an elementary argument to Spierpiński's remark is desirable. This was done by the author [3]. The modern approach to the equation

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = n \tag{3}$$

is to transform it into an elliptic curve. By cyclic permutations of (x, y, z) , we can assume that $x \geq z$. From (3), we have

$$\left(\frac{x}{y} - \frac{y}{z}\right)^2 = \left(\frac{x}{y} + \frac{y}{z}\right)^2 - \frac{4x}{z} = \left(n - \frac{z}{x}\right)^2 - \frac{4x}{z}. \tag{4}$$

Multiplying both sides of (4) with $16x^2/z^2$ gives

$$\left(\frac{4x(xz - y^2)}{yz^2}\right)^2 = \left(\frac{-4x}{z}\right)^3 + \left(\frac{-4nx}{z} + 4\right)^2.$$

Note that $x \geq z$, so $-4x/z \leq -4$. Therefore, a solution (x, y, z) of (3) in the positive integers with $x \geq z$ gives a rational point

$$(X, Y) = \left(\frac{-4x}{z}, \frac{4x(xz - y^2)}{yz^2}\right)$$

on the elliptic curve

$$Y^2 = X^3 + (nX + 4)^2 \quad (5)$$

with $X \leq -4$.

Conversely, assume that (X, Y) is a rational point on (5) with $X \leq -4$. Let

$$\begin{cases} A = -\frac{4}{X}, \\ B = \frac{1}{2}\left(n + \frac{4}{X} - \frac{Y}{X}\right), \\ C = \frac{1}{2}\left(n + \frac{4}{X} + \frac{Y}{X}\right). \end{cases} \quad (6)$$

Then

$$\begin{aligned} ABC &= -\frac{1}{X} \left(\left(n + \frac{4}{X} \right)^2 - \frac{Y^2}{X^2} \right) \\ &= -\frac{(nX + 4)^2 - Y^2}{X^3} \\ &= 1 \quad (\text{since } Y^2 = X^3 + (nX + 4)^2). \end{aligned}$$

Since $X \leq -4$, it is clear from (6) that $0 < A \leq 1$. Since $ABC = 1$, it follows that $BC > 0$. In addition, since $X \leq -4$ and $n \geq 1$, we have

$$B + C = n + \frac{4}{X} \geq 0.$$

So B, C are positive. Thus, A, B, C are positive rational numbers satisfying

$$ABC = 1. \quad (7)$$

From (6), we have

$$A + B + C = n. \quad (8)$$

By writing $A = z/x$, $B = x/y$, where x, y, z are positive integers, then (7) forces $C = y/z$. By (8), (x, y, z) is a solution of (3) in the positive integers. Since $A \leq 1$, we have $x \geq z$. So a rational point (X, Y) on (5) with $X \leq -4$ gives a solution (x, y, z) of (3) in the positive integers with $x \geq z$.

The goal of this note is to extend the argument in [3] for the equation in the title from $m = 1$ to arbitrary positive integers $m \geq 1$, thus providing an elementary argument that the family of elliptic curves

$$Y^2 = X^3 + (4^m X + 4)^2, \quad m = 1, 2, 3, \dots,$$

has no rational points with $X \leq -4$. The main result is the following.

Theorem 1. *Let m be a positive integer. Then the equation*

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 4^m \quad (9)$$

has no solutions in the positive integers.

2 A proof of Theorem 1

The case $m = 1$ was done in [3], so it suffices to consider $m > 1$. Assume that there exist positive integers x, y, z satisfying (9). If $xy - z^2 = 0$, $yz - x^2 = 0$, and $zx - y^2 = 0$, then $xyz = x^3 = y^3 = z^3$ so that $x = y = z$, which does not satisfy (9). So at least one of $xy - z^2$, $yz - x^2$, and $zx - y^2$ is non-zero. Assume that $xz - y^2 \neq 0$. From (9), we have $x/y + y/z = 4^m - z/x$. Hence, $4^m x - z > 0$. We also have

$$\left(\frac{x}{y} - \frac{y}{z}\right)^2 = \left(\frac{x}{y} + \frac{y}{z}\right)^2 - 4\frac{x}{z} = \left(4^m - \frac{z}{x}\right)^2 - 4\frac{x}{z} = \frac{z(4^m x - z)^2 - 4x^3}{x^2 z}.$$

Thus, $(z(4^m x - z)^2 - 4x^3)/z$ is a non-zero rational square. Therefore, it follows that $z(z(4^m x - z)^2 - 4x^3)$ is a non-zero perfect square. Let $d = \gcd(x, z)$. Let $x = da$ and $z = db$, where $a, b \in \mathbb{Z}^+$ with $\gcd(a, b) = 1$. Then $d^4 b(b(4^m a - b)^2 - 4a^3)$ is a non-zero perfect square. Hence, $b(b(4^m a - b)^2 - 4a^3)$ is a non-zero perfect square. Note that $4^m a - b = (4^m x - z)/d > 0$. Let

$$b(b(4^m a - b)^2 - 4a^3) = f^2, \quad f \in \mathbb{Z}^+.$$

Let $k = \gcd(b, b(4^m a - b)^2 - 4a^3)$. As $\gcd(a, b) = 1$, we have $k \mid 4$. Hence, $k \in \{1, 2, 4\}$.

Case 1: $k = 1$. Then

$$b = r^2, \quad b(4^m a - b)^2 - 4a^3 = s^2, \quad f = rs, \quad r, s \in \mathbb{Z}^+, \quad \gcd(r, s) = 1, 2 \nmid s. \quad (10)$$

Since $4^m a - b = 4^m a - r^2 \equiv 3 \pmod{4}$ and $4^m a - b > 0$, there exists a prime divisor p of $4^m a - b$ such that $p \equiv 3 \pmod{4}$. From (10), we have $-4a^3 \equiv s^2 \pmod{p}$. Therefore,

$$-4r^6 \equiv -4(4^m a)^3 \equiv -4^{3m+1}a^3 \equiv 4^{3m}s^2 \pmod{p}. \quad (11)$$

Since $p \mid 4^m a - b$ and $\gcd(a, b) = 1$, we have $p \nmid a$, $p \nmid r$, and $p \nmid s$. So (11) implies that -1 is a square mod p , which is impossible since $p \equiv 3 \pmod{4}$.

Case 2: $k = 2$. Then

$$b = 2r^2, \quad b(4^m a - b)^2 - 4a^3 = 2s^2, \quad f = 2rs, \quad r, s \in \mathbb{Z}^+, \quad \gcd(r, s) = 1.$$

Since $2 \mid b$, we have $2 \nmid a$. Now

$$s^2 = 4r^2(2^{2m-1}a - r^2)^2 - 2a^3.$$

Therefore, $2 \mid s$. So $4 \mid s^2$. Thus, $4 \mid 2a^3$, which is impossible since $2 \nmid a$.

Case 3: $k = 4$. Then

$$b = 4r^2, \quad b(4^m a - b)^2 - 4a^4 = 4s^2, \quad f = 4rs, \quad r, s \in \mathbb{Z}^+, \quad \gcd(r, s) = 1.$$

Then

$$16r^2(4^{m-1}a - r^2)^2 - a^3 = s^2. \quad (12)$$

Since $2 \mid b$, we have $2 \nmid a$. From (12), we have $2 \nmid s$. Taking (12) mod 8 gives $a \equiv -1 \pmod{8}$.

Assume the following claim is correct:

there exists a prime divisor p of $4^{m-1}a - r^2$ such that $p \equiv 3 \pmod{4}$. (*)

Let p be a prime in (*). Since $\gcd(a, b) = 1$, we have $p \nmid a$, $p \nmid r$, and $p \nmid s$. Taking (12) mod p gives $-a^3 \equiv s^2 \pmod{p}$. Therefore,

$$-r^6 \equiv (-4^{m-1}a)^3 \equiv 4^{3(m-1)}s^2 \pmod{p}.$$

Since $p \nmid r, s$, we have that -1 is a square mod p , which is impossible since $p \equiv 3 \pmod{4}$.

Now we will prove (*).

Case 3.1: $2 \nmid r$. Then $4^{m-1}a - r^2 \equiv 3 \pmod{4}$. Since $4^{m-1}a - r^2 = (4^m a - b)/4 > 0$, there exists a prime divisor p of $4^{m-1}a - r^2$ such that $p \equiv 3 \pmod{4}$.

Case 3.2: $2 \mid r$. Let $r = 2^h r_1$, where $h, r_1 \in \mathbb{Z}^+$ and $2 \nmid r_1$. Then

$$4^{m-1}a - r^2 = 2^{2(m-1)}a - 2^{2h}r_1^2. \quad (13)$$

- $h = m - 1$. From (13), we have

$$4^{m-1}a - r^2 = 2^{2h}(a - r_1^2).$$

Since $a \equiv -r_1^2 \equiv -1 \pmod{8}$, we have $a - r_1^2 \equiv -2 \pmod{8}$. Thus, $a - r_1^2 = 2t$, with $t \equiv 3 \pmod{4}$. Since $4^{m-1}a - r^2 > 0$, we have $a - r_1^2 > 0$. So $t > 0$. Therefore, there exists a prime divisor p of t such that $p \equiv 3 \pmod{4}$. Of course, $p \mid 4^{m-1}a - r^2$.

- $h < m - 1$. From (13), we have

$$4^{m-1}a - r^2 = 2^{2h}(2^{2m-2h-2}a - r_1^2).$$

Then $2^{2m-2h-2}a - r_1^2 > 0$ and $2^{2m-2h-2}a - r_1^2 \equiv -1 \pmod{4}$. Therefore, there exists a prime divisor p of $2^{2m-2h-2}a - r_1^2$ such that $p \equiv 3 \pmod{4}$. Of course, $p \mid 4^{m-1}a - r^2$.

- $h > m - 1$. From (13), we have

$$4^{m-1}a - r^2 = 2^{2m-2}(a - 2^{2h-2m+2}r_1^2).$$

Then $a - 2^{2h-2m+2}r_1^2 > 0$ and $a - 2^{2h-2m+2}r_1^2 \equiv -1 \pmod{4}$. Therefore, there exists a prime divisor p of $a - 2^{2h-2m+2}r_1^2$ such that $p \equiv 3 \pmod{4}$. Of course, $p \mid 4^{m-1}a - r^2$.

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References

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