EMS Surv. Math. Sci. (Online first)
DOI 10.4171/EMSS/94
© 2025 European Mathematical Society
Published by EMS Press

Transformation of the Gibbs measure of the cubic NLS and fractional NLS under an approximated Birkhoff map

Giuseppe Genovese, Renato Lucà, and Riccardo Montalto

Abstract. We study the Gibbs measure associated to the periodic cubic nonlinear Schrödinger equation. We establish a change of variable formula for this measure under the first step of the Birkhoff normal form reduction. We also consider the case of fractional dispersion.

Dedicated to the memory of Thomas Kappeler

1. Introduction

We study the cubic non-linear Schrödinger equation with fractional dispersion $\alpha > 0$

$$\partial_t u = i(|D_x|^{2\alpha} u + \sigma |u|^2 u), \quad x \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}), \tag{1.1}$$

where $\sigma = \pm 1$, depending on the defocusing or focusing character of the equation: for $\sigma = 1$, equation (1.1) is defocusing, whereas for $\sigma = -1$, it is focusing. The operator $|D_x|^\alpha$ is the Fourier multiplier defined by $|D_x|^\alpha (e^{\mathrm{inx}}) = |n|^\alpha e^{\mathrm{inx}}$, $n \in \mathbb{Z}$. One is typically interested in the regime $\frac{1}{2} < \alpha \le 1$, with $\alpha = 1$ being the usual cubic NLS equation and $\alpha = \frac{1}{2}$ the cubic half wave equation. The main results of this paper are for $\alpha \in [\overline{\alpha}, 1]$, where $\overline{\alpha} := \frac{1+\sqrt{97}}{12} \sim 0.9$.

As α varies, equation (1.1) describes a Hamiltonian (for $\alpha = 1$ in fact integrable) PDE with energy

$$H^{(\alpha)}(u) := \int_{\mathbb{T}} ||D_x|^\alpha u|^2 + \frac{\sigma}{2} \int_{\mathbb{T}} |u|^4 dx. \tag{1.2}$$

To these energies one associates infinite dimensional Gibbs measures ρ_α absolutely continuous with respect to the Gaussian measures $\widetilde{\gamma}_\alpha(du)$ restricted to some L^2-ball:

$$\rho_\alpha(du) := e^{-\frac{\sigma}{2}\|u\|_{L^4}^4} \widetilde{\gamma}_\alpha(du); \tag{1.3}$$

see (1.6) below and the surrounding discussion for the definition of $\widetilde{\gamma}_\alpha(du)$. These are central objects in this note. The construction of ρ_1 was achieved in the seminal papers of

Mathematics Subject Classification 2020: 35Q55.
Keywords: cubic Schrödinger equations, Gibbs measure, quasi-invariance, Birkhoff normal form.

Lebowitz–Rose–Speer [26] and Bourgain [8]. For fractional α, one can follow the same procedure, the only delicate point being the integrability of the term $e^{-\frac{\sigma}{2}\|u\|_{L^4}^4}$ with respect to $\widetilde{\gamma}_\alpha$ in the focusing case. This is achieved in the subsequent Proposition 5.8.

The aim of this paper is to study how the measures ρ_α transform under the action of a given canonical transformation which removes the non-resonant part of the Hamiltonian (1.2) up to terms of order $|u|^6$. We call the reduced Hamiltonian *Birkhoff normal form* and the reducing transformation *approximate Birkhoff map*. For general Hamiltonian systems a classical theorem of Birkhoff establishes the existence of a canonical transformations putting the Hamiltonian in normal form up to a remainder of a given arbitrary order (see, for instance, [25, Theorem G.1]). The construction of such maps has been exploited in infinite dimension for many Hamiltonian PDEs in different contexts, starting from the pioneering papers [1,4,9]. Without trying to be exhaustive, we also mention several more recent extensions to PDEs in higher space dimension and to quasi-linear PDEs, see [2,3, 5–7, 11–13, 15, 23, 28].

One nice feature of the approximate Birkhoff map is that it can be expressed as a Hamiltonian flow. Let us introduce it. Consider

$$\Phi_t^N : E_N \to E_N, \quad E_N := \operatorname{span}\{e^{ijx} : |j| \le N\}$$

defined by the system of ODEs

$$\frac{d}{dt}(\Phi_t^N(u))(n) = \sum_{\substack{|j_1|,|j_2|,|j_3|\le N \\ j_1+j_2-j_3=n \\ |j_1|^{2\alpha}+|j_2|^{2\alpha}-|j_3|^{2\alpha}-|n|^{2\alpha}\neq 0}} \frac{-\sigma}{(|j_1|^{2\alpha}+|j_2|^{2\alpha}-|j_3|^{2\alpha}-|n|^{2\alpha})} \Phi_t^N(u)(j_1)\Phi_t^N(u)(j_2)\overline{\Phi_t^N(u)(j_3)}.$$

We are interested in the Birkhoff map/flow $\Phi_t := \Phi_t^\infty$ and, more specifically, in the 1-time shift Φ_1, that we abbreviate to Φ in order to simplify the notations. Clearly, the Birkhoff map also depends on α, but for notational simplicity, we will not keep track of that in the manuscript. This transformation acts on the Hamiltonian as follows:

$$H^{(\alpha)}[\Phi u] = \||D_x|^\alpha u\|_{L^2}^2 + \frac{\sigma}{2}\|u\|_{L^2}^4 + R^{(\alpha)}[u],$$

with $R^{(\alpha)}[u] = O(|u|^6)$ being a remainder which has a zero of order six at $u = 0$. This identity can be easily justified for sufficiently regular functions, for instance, for $u \in H^s$, $s > 1/2$, see, e.g., [4].

Next, we shortly introduce $\widetilde{\gamma}_\alpha$. Let $\{g_n\}_{n\in\mathbb{Z}}$ be a sequence of independent, identically distributed complex centred Gaussian random variables with unitary variance. We consider the random Fourier series

$$\sum_{n\in\mathbb{Z}} \frac{g_n}{(1+|n|^{2\alpha})^{\frac{1}{2}}} e^{\mathrm{inx}}. \tag{1.4}$$

If $\alpha > 1/2$, this defines a function on $L^2(\mathbb{T})$ for almost all realisations of the sequence $\{g_n\}_{n\in\mathbb{Z}}$. Thanks to separability and the isomorphism between $\mathbb{C}^{2N+1}$ and

$$E_N := \mathrm{span}_{\mathbb{C}}\{e^{\mathrm{inx}} : |n| \leq N\}, \tag{1.5}$$

the space $L^2(\mathbb{T})$ inherits the measurable-space structure by a standard limit procedure and we will denote by $\mathcal{B}(L^2(\mathbb{T}))$ the Borel σ-algebra on $L^2(\mathbb{T})$. The Gaussian measure on $\mathcal{B}(L^2(\mathbb{T}))$ induced by (1.4) is denoted by γ_α. The triple $(L^2(\mathbb{T}), \mathcal{B}(L^2(\mathbb{T})), \gamma_\alpha)$ is a Gaussian probability space satisfying the concentration properties:

$$\widetilde{\gamma}_\alpha\Big(\bigcap_{s<\alpha-\frac{1}{2}} H^s(\mathbb{T})\Big) = 1, \quad \widetilde{\gamma}_\alpha\big(H^{\alpha-\frac{1}{2}}(\mathbb{T})\big) = 0.$$

The expectation value with respect to γ_α is always indicated by E_α.

Finally, we introduce the restriction of γ_α to a ball of $L^2(\mathbb{T})$ as

$$\widetilde{\gamma}_\alpha(A) := \gamma_\alpha(A \cap \{\|u\|_{L^2} \leq R\}), \quad A \in \mathcal{B}(L^2(\mathbb{T})) \tag{1.6}$$

for some $R > 0$. The measure $\widetilde{\gamma}_\alpha$ proves useful since the L^2 norm is preserved by equation (1.1).

Now, we are ready to give our main theorem (recall (1.3)).

Theorem 1.1. *Let $\overline{\alpha} := \frac{1+\sqrt{97}}{12} \sim 0.9$, $\alpha \in (\overline{\alpha}, 1]$. The approximated Birkhoff map Φ_t is well-posed for all $t \in [0,1]$ for $\widetilde{\gamma}_\alpha$-almost all initial data. Moreover, $\Phi = \Phi_1$ leaves quasi-invariant the Gibbs measure ρ_α. More precisely, given $s \in (2-2\alpha, \alpha - \frac{1}{2})$ and $A \subseteq H^s(\mathbb{T})$ such that $A \in \mathcal{B}(L^2(\mathbb{T}))$, the quantity*

$$\int_A \exp\Big(-\int_0^1 \frac{d}{d\tau}(H^{(\alpha)}[\Phi_\tau(u)])d\tau\Big)\rho_\alpha(du) \tag{1.7}$$

is well defined and equals $\rho_\alpha(\Phi(A))$.

The proof is based on the Tzvetkov argument for quasi-invariance [29] and the method of [19] for the transported density (for other ways of determining the density, see [10, 17]). In particular, we exploit crucially that the approximated Birkhoff map is given by a Hamiltonian flow, a property which is not enjoyed by the global Birkhoff map.

The restriction on α in Theorem 1.1 arises from the probabilistic analysis (see Proposition 5.1), but it is necessary in order to justify the dynamics as well (see Proposition 6.3). We do not assign any special meaning to this range $(\overline{\alpha}, 1]$, $\overline{\alpha} := \frac{1+\sqrt{97}}{12} \sim 0.9$.

The restriction $s > 2 - 2\alpha$ is used in order to prove local well-posedness in H^s of the truncated (over the Fourier modes) Birkhoff map with estimates that are uniform with respect to the truncation (see Lemma 4.1). It is not unusual when dealing with fractional dispersion: similar conditions appear in the case of the flow of the fractional Schrödinger equation also in [14, 16–18]. This assumption plays no role in the first part of our Theorem 1.1, where the flow is proved to be almost surely well-posed, as H^s for

$2-2\alpha < s < \alpha - \frac{1}{2}$ has full $\widetilde{\gamma}_\alpha$ measure. However, it is relevant to determine the right σ-field we are allowed to use for quasi-invariance in the second part of Theorem 1.1. Our result can be probably extended to include the case $s = 2-2\alpha$. For simplicity, we do not attempt that here, but remark that from our proof it follows that $s \geq 0$ if $\alpha = 1$.

We stress that if $\alpha < 1$, showing that the flow of the approximate Birkhoff map is globally well defined is quite a non-trivial task. By a direct application of the Bourgain probabilistic globalisation method [8], we prove here global well-posedness of the flow-map for almost all data in $H^{(\alpha-1/2)^-}$ with respect to the Gibbs measure; however, it is not clear to us how to do that deterministically, since we cannot prove local well-posedness for data in L^2 and the L^2 norm is the sole conserved quantity at our disposal when $\alpha < 1$.

Equation (1.1) is completely integrable for $\alpha = 1$. In the defocusing case, exploiting the complete integrability it is possible to construct a map, so-called *global Birkhoff map*, which trivialises equation (1.1) and transforms the energies into non-linear Sobolev norms [22] (alternatively the method of the paper [27] can also be applied to this context). Our approximate Birkhoff map can be though of as a local approximation of the global Birkhoff map, which is accurate about the origin. Constructing the global Birkhoff map for a Hamiltonian PDE is a very challenging task, which has been achieved only in few cases, namely, by Kappeler and Pöschel for the KdV equation [25], by Grebért and Kappeler for the defocussing NLS equation [22], by Gérard, Kappeler, and Topalov for the Benjamin–Ono equation [20, 21]. In the recent preprint [30], Tzvetkov exhibits a large class of invariant measures with respect to the Benjamin–Ono flow (including Gaussian-based measures), using the global Birkhoff map of Gérard, Kappeler and Topalov. However, these measures are constructed on the image space of the map (that is, they are given in terms of Birkhoff coordinates) and the author conjectures that the Birkhoff map acts on Gibbs measure in a quasi-invariant way (see [30, Conjecture 3.1]). In the same spirit, we conjecture that a change of variable formula similar to the one of Theorem 1.1 can be proved for map reducing the Hamiltonian in normal forms of any order. We plan to further investigate this topic in future works.

The paper is organised as follows. The Birkhoff normal form reduction is given in Section 2 along with a crucial estimate on the derivative along the Birkhoff flow of the Hamiltonian at time zero. In Section 3, we give the necessary deterministic estimates on the Birkhoff flow for $\alpha = 1$, while the ones for fractional α are given in Section 4. In Section 5, we give the necessary probabilistic estimates. Section 6 is devoted to the proof of the quasi-invariance and here we also prove the probabilistic global well-posedness of the Birkhoff flow for $\alpha > \bar{\alpha}$, $\bar{\alpha} := \frac{1+\sqrt{97}}{12} \sim 0.9$. We establish the formula for the transported density in Section 7.

Notations

Given a function $u : \mathbb{T} \to \mathbb{R}$, we denote by $f(n)$ its Fourier coefficient

$$u(n) := \frac{1}{2\pi} \int_{\mathbb{T}} u(x) e^{-\mathrm{inx}} dx, \quad n \in \mathbb{Z}.$$

We define the Sobolev norms H^s of f as

$$\|u\|_{H^s}^2 := \sum_{n\in\mathbb{Z}}(1+|n|^{2s})|u(n)|^2.$$

We also define the Fourier–Lebesgue norm for any $p \geq 1$

$$\|u\|_{FL^{0,p}} = \Big(\sum_{n\in\mathbb{Z}}|u(n)|^p\Big)^{\frac{1}{p}}.$$

A ball of radius R and centred in zero in the H^s topology is denoted by $B_s(R)$. We drop the subscript for $s=0$ (ball of L^2). We write $\langle\cdot\rangle := (1+|\cdot|^2)^{1/2}$. We write $X \lesssim Y$ if there is a constant $c>0$ such that $X \leq cY$ and $X \simeq Y$ if $Y \lesssim X \lesssim Y$. We underscore the dependency of c on the additional parameter a writing $X \lesssim_a Y$. C, c always denote constants that often vary from line to line within a calculation. $A+B$ is the Minkowski sum of the sets A, B, namely, $A+B := \{x+y : x\in A, y\in B\}$. We denote by Π_N the standard orthogonal projection

$$\Pi_N(u) := \sum_{|n|\leq N} u(n)e^{\mathrm{inx}}, \quad \Pi_N^{\perp} := \mathrm{Id} - \Pi_N,$$

where $u(n)$ is the nth Fourier coefficient of $u \in L^2$. Also, we denote the Littlewood–Paley projector by $\Delta_0 := \Pi_1$, $\Delta_j := \Pi_{2^j} - \Pi_{2^{j-1}}$, $j \in \mathbb{N}$.

We will use the well-known tail bounds for sequences of independent centred Gaussian random variables $X_1, \dots, X_d$ (see, for instance, [31]): the Hoeffding inequality

$$P\Big(\sum_{i=1}^{d}|X_i| \geq \lambda\Big) \leq C\exp\Big(-c\frac{\lambda^2}{d}\Big) \tag{1.8}$$

and the Bernstein inequality

$$P\bigg(\bigg|\sum_{i=1}^{d}|X_i|^2 - E\bigg[\sum_{i=1}^{d}|X_i|^2\bigg]\bigg| \geq \lambda\bigg) \leq C\exp\Big(-c\min\Big(\lambda, \frac{\lambda^2}{d}\Big)\Big). \tag{1.9}$$

2. The Birkhoff Normal form transformation

In this section, we introduce the finite dimensional approximated Birkhoff transformation studied in this paper. To do so, we present some more notations.

Given the Hamiltonian function $\mathcal{F}$ the associated Hamilton equations are written as

$$\partial_t U = X_{\mathcal{F}}(U),$$

where $U := (u, \bar{u}) \in \mathcal{L}^2(\mathbb{T}) := L^2(\mathbb{T}) \times L^2(\mathbb{T})$ and the Hamiltonian vector field $X_{\mathcal{F}}$ is given by

$$\begin{aligned} X_{\mathcal{F}}(U) &:= \begin{pmatrix} i\nabla_{\bar{u}}\mathcal{F}(U) \\ -i\nabla_u \mathcal{F}(U) \end{pmatrix} = J\nabla_U \mathcal{F}(U), \\ \nabla_U \mathcal{F}(U) &= (\nabla_u \mathcal{F}(U), \nabla_{\bar{u}}\mathcal{F}(U)), \quad J := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \end{aligned} \tag{2.1}$$

We also define, for any $s \geq 0$, $\mathcal{H}^s(\mathbb{T}) := H^s(\mathbb{T}) \times H^s(\mathbb{T})$. Given two Hamiltonian functions $\mathcal{F}, \mathcal{G} : \mathcal{L}^2(\mathbb{T}) \to \mathbb{R}$, we define the Poisson bracket

$$\{\mathcal{F}, \mathcal{G}\}(U) := D_U \mathcal{F}(U)[J\nabla \mathcal{G}(U)]. \tag{2.2}$$

Given $\mathcal{G} : \mathcal{L}^2(\mathbb{T}) \to \mathbb{R}$ such that its Hamiltonian vector field $X_{\mathcal{F}} : \mathcal{H}^s(\mathbb{T}) \to \mathcal{H}^s(\mathbb{T})$, $s \geq 0$, we denote by $\Phi_t^{\mathcal{F}}$ its Hamiltonian flow, namely,

$$\begin{cases} \partial_t \Phi_t^{\mathcal{F}}(U) = X_{\mathcal{F}}(\Phi_t^{\mathcal{F}}(U)), \\ \Phi_0^{\mathcal{F}}(U) = U. \end{cases}$$

Given a function $\mathcal{G}$, one has that

$$\mathcal{G} \circ \Phi_t^{\mathcal{F}} = \mathcal{G} + t\{\mathcal{G}, \mathcal{F}\} + R_t, \quad R_t := \int_0^t (t-\tau)\{\{\mathcal{G}, \mathcal{F}\}, \mathcal{F}\} \circ \Phi_\tau^{\mathcal{F}} \, d\tau. \tag{2.3}$$

Given a Hamiltonian $\mathcal{F} : \mathcal{H}^s(\mathbb{T}) \to \mathbb{R}$, we define the truncated Hamiltonian $\mathcal{F}_N := \mathcal{F}_{|\mathcal{E}_N}$, $\mathcal{E}_N := E_N \times E_N$ (recall (1.5)), and the truncated Hamiltonian vector field is given by

$$X_{\mathcal{F}_N}(U) := \Pi_N X_{\mathcal{F}}(\Pi_N U). \tag{2.4}$$

The truncated Hamiltonian flow $\Phi_t^{\mathcal{F}_N} : \mathcal{E}_N \to \mathcal{E}_N$ then solves

$$\begin{cases} \partial_t \Phi_t^{\mathcal{F}_N}(U) = X_{\mathcal{F}_N}(\Phi_t^{\mathcal{F}_N}(U)), \\ \Phi_0^{\mathcal{F}_N}(U) = U, \end{cases} \qquad U \in \mathcal{E}_N.$$

Let us now define the Birkhoff map. We set

$$\begin{aligned} &\mathcal{F}_N \equiv \mathcal{F}_N^{(\alpha)}, \\ &\mathcal{F}_N^{(\alpha)} := \\ &\sum_{\substack{|n_1|,|n_2|,|m_1|,|m_2| \leq N \\ n_1+n_2=m_1+m_2 \\ |n_1|^{2\alpha}+|n_2|^{2\alpha} \neq |m_1|^{2\alpha}+|m_2|^{2\alpha}}} \frac{-\sigma}{2i(|n_1|^{2\alpha}+|n_2|^{2\alpha}-|m_1|^{2\alpha}-|m_2|^{2\alpha})} u(n_1)u(n_2)\bar{u}(m_1)\bar{u}(m_2). \end{aligned} \tag{2.5}$$

The flow-map $\Phi_t^N := \Phi_t^{\mathcal{F}_N} : \mathcal{E}_N \to \mathcal{E}_N$ defined by the system of ODEs

$$\begin{aligned} &\frac{d}{dt}(\Phi_t^N(u))(n) \\ &= \sum_{\substack{|j_1|,|j_2|,|j_3| \leq N \\ j_1+j_2-j_3=n \\ |j_1|^{2\alpha}+|j_2|^{2\alpha} \neq |j_3|^{2\alpha}+|n|^{2\alpha}}} \frac{-\sigma}{(|j_1|^{2\alpha}+|j_2|^{2\alpha}-|j_3|^{2\alpha}-|n|^{2\alpha})} u(j_1)u(j_2)\bar{u}(j_3) \end{aligned} \tag{2.6}$$

is Hamiltonian with respect to $\mathcal{F}_N$ (with canonical Poisson brackets). We call this finite dimensional Birkhoff flow and we set

$$\Phi^N = (\Phi_t^N)_{t=1}.$$

Recall that we omit the dependence on α in the energy and in the Birkhoff flow map.

The following lemma is a classical fact.

Lemma 2.1. *Let* $\alpha \in (1/2, 1]$. *The Hamiltonian* $\mathcal{F}_N$ *satisfies*

$$\{\||D_x|^\alpha \Pi_N u\|_{L^2}, \mathcal{F}_N(u)\} + \frac{\sigma}{2}\|\Pi_N u\|_{L^4}^4 = \sigma\|\Pi_N u\|_{L^2}^4. \tag{2.7}$$

Proof. A direct computation gives

$$\begin{aligned}\{\||D_x|^\alpha \Pi_N u\|_{L^2}^2, \mathcal{F}_N\} &+ \frac{\sigma}{2}\|\Pi_N u\|_{L^4}^4 \\ &= \frac{\sigma}{2} \sum_{\substack{|j_1|,\dots,|j_4|\le N \\ j_1+j_2=j_3+j_4 \\ |j_1|^{2\alpha}+|j_2|^{2\alpha}=|j_3|^{2\alpha}+|j_4|^{2\alpha}}} u(j_1)u(j_2)\bar{u}(j_3)\bar{u}(j_4).\end{aligned} \tag{2.8}$$

For $\alpha = 1$, we note that under the constraint $j_1 + j_2 = j_3 + j_4$, one obtains that

$$j_1^2 + j_2^2 - j_3^2 - j_4^2 = 2(j_3 - j_2)(j_1 - j_3).$$

Therefore, if $j_1^2 + j_2^2 - j_3^2 - j_4^2 = 0$, then either $j_2 = j_3$ or $j_1 = j_3$. If $j_2 = j_3$, then $j_1 + j_2 - j_3 - j_4 = 0$ gives $j_1 = j_4$. Similarly, if $j_1 = j_3$, then $j_2 = j_4$. The same extends to fractional $\alpha \in (1/2, 1]$ thanks to [16, Lemma 2.4]. Indeed, this lemma implies that if $j_1 + j_2 - j_3 - j_4 = 0$, one has the lower bound

$$\begin{aligned}&\big||j_1|^{2\alpha} + |j_2|^{2\alpha} - |j_3|^{2\alpha} - |j_4|^{2\alpha}\big| \\ &\qquad \ge C|j_1 - j_4||j_2 - j_4| j_{\max}^{2\alpha-2}, \quad j_{\max} := \max\{|j_1|, |j_2|, |j_3|, |j_4|\}\end{aligned}$$

for some constant $C > 0$. This latter bound implies that if $|j_1|^{2\alpha} + |j_2|^{2\alpha} - |j_3|^{2\alpha} - |j_4|^{2\alpha} = 0$, then $\{j_1, j_2\} = \{j_3, j_4\}$. Therefore, the right-hand side of (2.8) takes the form

$$\sigma \sum_{|j|,|j'|\le N} |u(j)|^2|u(j')|^2 = \sigma\|\Pi_N u\|_{L^2}^4. \qquad \blacksquare$$

3. The flow of the Birkhoff map for the cubic NLS

Here, we analyse the well-posedness of the Birkhoff flow. The case $\alpha = 1$ is easier to deal with and will be presented first. We first provide some estimates on the Hamiltonian vector field $X_{\mathcal{F}_N}$.

Lemma 3.1. *Let $\alpha = 1$ and $N \in \mathbb{N} \cup \{\infty\}$.*

(i) *For any $\sigma \geq 0$ and for any $u \in H^\sigma(\mathbb{T})$, it is*

$$\|X_{\mathcal{F}_N}(u)\|_{H^\sigma} \lesssim \|u\|_{L^2}^2 \|u\|_{H^\sigma}. \tag{3.1}$$

(ii) *For any $\sigma \geq 0$ and for any $u, v \in H^\sigma(\mathbb{T})$, it is*

$$\|X_{\mathcal{F}_N}(u) - X_{\mathcal{F}_N}(v)\|_{H^\sigma} \lesssim \|u - v\|_{H^\sigma}(\|u\|_{H^\sigma}^2 + \|v\|_{H^\sigma}^2).$$

Proof. The Hamiltonian vector field is

$$X_{\mathcal{F}_N} = (i\nabla_{\bar{u}}\mathcal{F}_N, -i\nabla_u \mathcal{F}_N)$$

and one computes, by recalling formula (2.5), that

$$\begin{aligned}
\partial_{\bar{u}(n)}\mathcal{F}_N(u) &= \sum_{\substack{|j_1|,|j_2|,|j_3| \leq N \\ j_1+j_2-j_3=n \\ j_1^2+j_2^2 \neq j_3^2+n^2}} \frac{\sigma i}{2(j_1^2 + j_2^2 - j_3^2 - n^2)} u(j_1)u(j_2)\bar{u}(j_3) \\
&= \sum_{\substack{|j_1|,|j_2|,|j_3| \leq N \\ j_1+j_2-j_3=n}} \frac{\sigma i}{2(j_3 - j_2)(j_1 - j_3)} u(j_1)u(j_2)\bar{u}(j_3).
\end{aligned}$$

Then, in order to deduce the desired estimates in (i)–(ii), it is enough to prove that the trilinear form $\mathcal{T}$ defined by

$$\begin{aligned}
\mathcal{T}[u, v, \varphi] &:= \sum_{n \in \mathbb{Z}} \mathcal{T}_n[u, v, \varphi] e^{\mathrm{inx}}, \\
\mathcal{T}_n[u, v, \varphi] &:= \sum_{\substack{|j_1|,|j_2|,|j_3| \leq N \\ j_1+j_2-j_3=n}} \frac{\sigma i}{2(j_3 - j_2)(j_1 - j_3)} u(j_1)v(j_2)\varphi(j_3)
\end{aligned} \tag{3.2}$$

is continuous on $H^\sigma(\mathbb{T})$ and satisfies

$$\begin{aligned}
\|\mathcal{T}[u, v, \varphi]\|_{H^\sigma} \lesssim_\sigma\ & \|u\|_{H^\sigma}\|v\|_{L^2}\|\varphi\|_{L^2} + \|u\|_{L^2}\|v\|_{H^\sigma}\|\varphi\|_{L^2} \\
& + \|u\|_{L^2}\|v\|_{L^2}\|\varphi\|_{H^\sigma} \quad \forall u, v, \varphi \in H^\sigma(\mathbb{T}).
\end{aligned} \tag{3.3}$$

One has

$$\begin{aligned}
\|\mathcal{T}[u, v, \varphi]\|_{H^\sigma}^2 &= \sum_{n \in \mathbb{Z}} \langle n \rangle^{2\sigma} |\mathcal{T}_n[u, v, \varphi]|^2 \\
&\lesssim \sum_{n \in \mathbb{Z}} \Bigg(\sum_{\substack{|j_1|,|j_2|,|j_3| \leq N \\ j_1+j_2-j_3=n}} \frac{\langle n \rangle^\sigma}{|j_3 - j_2||j_1 - j_3|} |u(j_1)||v(j_2)||\varphi(j_3)| \Bigg)^2.
\end{aligned}$$

By using that $n = j_1 + j_2 - j_3$, one gets that $\langle n\rangle^\sigma \lesssim_\sigma \langle j_1\rangle^\sigma + \langle j_2\rangle^\sigma + \langle j_3\rangle^\sigma$, implying that

$$
\begin{aligned}
&\|\mathcal{T}[u,v,\varphi]\|_{H^\sigma}^2 \lesssim_\sigma T_1 + T_2 + T_3,\\
&\qquad T_1 := \sum_{n\in\mathbb{Z}} \Bigg(\sum_{\substack{|j_1|,|j_2|,|j_3|\le N\\ j_1+j_2-j_3=n}} \frac{1}{|j_3-j_2||j_1-j_3|} \langle j_1\rangle^\sigma |u(j_1)||v(j_2)||\varphi(j_3)| \Bigg)^2,\\
&\qquad T_2 := \sum_{n\in\mathbb{Z}} \Bigg(\sum_{\substack{|j_1|,|j_2|,|j_3|\le N\\ j_1+j_2-j_3=n}} \frac{1}{|j_3-j_2||j_1-j_3|} |u(j_1)\langle j_2\rangle^\sigma ||v(j_2)||\varphi(j_3)| \Bigg)^2,\\
&\qquad T_3 := \sum_{n\in\mathbb{Z}} \Bigg(\sum_{\substack{|j_1|,|j_2|,|j_3|\le N\\ j_1+j_2-j_3=n}} \frac{1}{|j_3-j_2||j_1-j_3|} |u(j_1)||v(j_2)|\langle j_3\rangle^\sigma |\varphi(j_3)| \Bigg)^2.
\end{aligned}
$$

The estimate of T_1, T_2, T_3 is similar, hence, we concentrate on the estimate for T_1. By using the Cauchy–Schwarz inequality, one obtains that, for any $n \in \mathbb{Z}$,

$$
\begin{aligned}
&\sum_{j_1,j_2\in\mathbb{Z}} \frac{1}{\langle j_1-n\rangle\langle j_2-n\rangle} \langle j_1\rangle^\sigma |u(j_1)||v(j_2)||\varphi(j_1+j_2-n)|\\
&\lesssim \Bigg(\sum_{j_1,j_2\in\mathbb{Z}} \langle j_1\rangle^{2\sigma} |u(j_1)|^2 |v(j_2)|^2 |\varphi(j_1+j_2-n)|^2 \Bigg)^{\frac12} \Bigg(\sum_{j_1,j_2\in\mathbb{Z}} \frac{1}{\langle j_1-n\rangle^2\langle j_2-n\rangle^2} \Bigg)^{\frac12}\\
&\lesssim \Bigg(\sum_{j_1,j_2\in\mathbb{Z}} \langle j_1\rangle^{2\sigma} |u(j_1)|^2 |v(j_2)|^2 |\varphi(j_1+j_2-n)|^2 \Bigg)^{\frac12} \Bigg(\sum_{k_1,k_2\in\mathbb{Z}} \frac{1}{\langle k_1\rangle^2\langle k_2\rangle^2} \Bigg)^{\frac12}\\
&\lesssim \Bigg(\sum_{j_1,j_2\in\mathbb{Z}} \langle j_1\rangle^{2\sigma} |u(j_1)|^2 |v(j_2)|^2 |\varphi(j_1+j_2-n)|^2 \Bigg)^{\frac12}.
\end{aligned}
$$

Thus, T_1 can be estimated as

$$
\begin{aligned}
T_1 &\lesssim \sum_{n\in\mathbb{Z}} \sum_{j_1,j_2\in\mathbb{Z}} \langle j_1\rangle^{2\sigma} |u(j_1)|^2 |v(j_2)|^2 |\varphi(j_1+j_2-n)|^2\\
&\lesssim \sum_{j_1\in\mathbb{Z}} \langle j_1\rangle^{2\sigma} |u(j_1)|^2 \sum_{j_2\in\mathbb{Z}} |v(j_2)|^2 \sum_{n\in\mathbb{Z}} |\varphi(j_1+j_2-n)|^2\\
&\overset{j_1+j_2-n=k}{\lesssim} \sum_{j_1\in\mathbb{Z}} \langle j_1\rangle^{2\sigma} |u(j_1)|^2 \sum_{j_2\in\mathbb{Z}} |v(j_2)|^2 \sum_{k\in\mathbb{Z}} |\varphi(k)|^2\\
&\lesssim \|u\|_{H^\sigma}^2 \|v\|_{L^2}^2 \|\varphi\|_{L^2}^2.
\end{aligned}
$$

By similar arguments, one can show that

$$
T_2 \lesssim \|u\|_{L^2}^2 \|v\|_{H^\sigma}^2 \|\varphi\|_{L^2}^2, \quad T_3 \lesssim \|u\|_{L^2}^2 \|v\|_{L^2}^2 \|\varphi\|_{H^\sigma}^2,
$$

implying the estimate (3.3). Hence, the items (i) and (ii) follow since $\nabla_{\bar{u}}\mathcal{F}_N = \mathcal{T}[u,u,\bar{u}]$, and thus,

$$\begin{aligned}&\nabla_{\bar{u}}\mathcal{F}_N(u) - \nabla_{\bar{v}}\mathcal{F}_N(v)\\ &\quad = \mathcal{T}[u,u,\bar{u}] - \mathcal{T}[v,v,\bar{v}] = \mathcal{T}[u-v,u,\bar{u}] + \mathcal{T}[v,u-v,\bar{u}] + \mathcal{T}[v,v,\overline{u-v}]. \quad \blacksquare\end{aligned}$$

The foregoing lemma implies that the flow $\Phi_t^N = \Phi_t^{\mathcal{F}_N}$ is well defined in $H^\sigma(\mathbb{T})$ for any $\sigma \geq 0$. This follows by a standard Picard iteration. In particular, the following lemma holds.

Lemma 3.2. *Let $N \in \mathbb{N} \cup \{\infty\}$. For any $u_0 \in L^2(\mathbb{T})$, there exists a unique local solution $u \in C^1(\mathbb{R}, L^2(\mathbb{T}))$ which solves the Cauchy problem*

$$\begin{cases} \partial_t u(t) = X_{\mathcal{F}_N}(u(t)), \\ u(0) = u_0. \end{cases}$$

Thus, the flow $\Phi_t^N : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a well defined C^1 map. Moreover, for all $\sigma \geq 0$, $u_0 \in H^\sigma(\mathbb{T})$ and all $T > 0$ the H^σ norm is controlled as $\|u(t)\|_{H^\sigma} \leq Ce^{CT}\|u_0\|_{H^\sigma}$, we have $u \in C^1([-T,T], H^\sigma(\mathbb{T}))$ and the flow $\Phi^t_{\mathcal{F}_N} : H^\sigma(\mathbb{T}) \to H^\sigma(\mathbb{T})$ is a well defined C^1 map for any $\tau \in [-T,T]$.

Proof. The local existence follows by a standard fixed point argument on the Volterra integral operator

$$u(t) \mapsto \mathcal{S}(u)(t) := u_0 + \int_0^t X_{\mathcal{F}_N}(u(\tau))d\tau$$

in the closed ball

$$\Big\{u \in C^0([-\delta,\delta], H^\sigma(\mathbb{T})) : \|u\|_{L^\infty_T H^\sigma_x} := \sup_{t\in[-\delta,\delta]} \|u(t)\|_{H^\sigma} \leq R\Big\}.$$

This fixed point argument requires that R is larger than $\|u_0\|_{H^\sigma}$ and $\delta R^2 \ll 1$. If u is a fixed point for $\mathcal{S}$, one also immediately gets that $u \in C^1([-\delta,\delta], H^\sigma(\mathbb{T}))$. For L^2 initial data, the argument to globalize the solution is standard. The exponential control of the H^σ norm up to arbitrary time $T > 0$ follows again looking at the Volterra integral operator and using (3.1). Once we know that the H^σ norm does not blow-up in finite time we can also extend the local H^σ flow to arbitrary times $T > 0$. ■

Lemma 3.3. *Let $s \geq 0$, $T > 0$, $R > 0$, $N \in \mathbb{N} \cup \{\infty\}$. Assume that $\|w^N\|_{L^2}, \|v^N\|_{L^2} \leq R$. Then, for $|t| \lesssim T$ it holds*

$$\|\Phi_t^N(v) - \Phi_t^N(w)\|_{H^s} \lesssim_{R,T} \|v-w\|_{H^s}. \tag{3.4}$$

Proof. The argument easily follows using Lemma 3.2, the inequalities of Lemma 3.1 and the Duhamel representation of truncated flows. By time reversibility, the same bound holds for the inverse flow. This ends the proof. ■

We now prove the convergence of the flow of the truncated vector field to the one of the non-truncated vector field. The flow $\Phi_t = \Phi_t^{\mathcal{F}}$ is the flow of the Hamiltonian vector field $X_{\mathcal{F}}$, whereas the flow of $\Phi_t^N = \Phi_t^{\mathcal{F}_N}$ is the flow associated to the Hamiltonian vector field $X_{\mathcal{F}_N}$ (recall (2.1)) where $\mathcal{F}_N(u) = \mathcal{F}(\Pi_N u)$.

Lemma 3.4. *Let $u_0 \in L^2(\mathbb{T})$. It holds*

$$\sup_{\tau\in[-1,1]} \|\Phi_\tau(u_0) - \Phi_\tau^N(\Pi_N u_0)\|_{L^2} \to 0 \quad \text{as } N \to +\infty. \tag{3.5}$$

Moreover, if $K \subset L^2(\mathbb{T})$ is compact, then

$$\sup_{u_0\in K}\sup_{\tau\in[-1,1]} \|\Phi_\tau(u_0) - \Phi_\tau^N(\Pi_N u_0)\|_{L^2} \to 0 \quad \text{as } N \to +\infty. \tag{3.6}$$

Proof. Recalling (2.4) and by setting $u_N := \Pi_N u$, one has

$$\begin{aligned} X_{\mathcal{F}}(u) - X_{\mathcal{F}_N}(u_N) &= X_{\mathcal{F}}(u) - \Pi_N X_{\mathcal{F}}(u_N) \\ &\overset{\mathrm{Id}=\Pi_N+\Pi_N^\perp}{=} \Pi_N^\perp X_{\mathcal{F}}(u) + \Pi_N X_{\mathcal{F}}(u) - \Pi_N X_{\mathcal{F}}(u_N) \\ &= \mathcal{R}'_N(u) + \mathcal{R}_N(u), \quad \text{where} \\ \mathcal{R}'_N(u) := \Pi_N X_{\mathcal{F}}(u) - \Pi_N X_{\mathcal{F}}(u_N), &\quad \mathcal{R}_N(u) := \Pi_N^\perp X_{\mathcal{F}}(u). \end{aligned} \tag{3.7}$$

The latter computation is needed in order to estimate the difference of the solutions of the following Cauchy problem. Given $U_0 \in L^2(\mathbb{T})$, we consider

$$\begin{cases} \partial_t u = X_{\mathcal{F}}(u), \\ u(0) = u_0, \end{cases} \qquad \begin{cases} \partial_t u_N = X_{\mathcal{F}_N}(u_N), \\ u_N(0) = \Pi_N u_0, \end{cases} \tag{3.8}$$

with $\sup_{\tau\in[-1,1]} \|u_N\|_{L^2} \le R$ and $\sup_{\tau\in[-1,1]} \|u\|_{L^2} \le R$ for some $R \ge 0$. By (3.7), one obtains that $\delta_N := u - u_N$ satisfies the following problem:

$$\begin{cases} \partial_\tau \delta_N = \mathcal{R}'_N(u) + \mathcal{R}_N(u), \\ \delta_N(0) = \Pi_N^\perp u_0. \end{cases} \tag{3.9}$$

This implies that

$$\delta_N(\tau) = \Pi_N^\perp u_0 + \int_0^\tau \mathcal{R}'_N(u(z,\cdot))dz + \int_0^\tau \mathcal{R}_N(u(z,\cdot))dz,$$

since $\sup_{\tau\in[-1,1]} \|u_N\|_{L^2} \le R$ and $\sup_{\tau\in[-1,1]} \|u\|_{L^2} \le R$, by applying Lemma 3.1 item (ii), one obtains that for any $z \in [-1, 1]$

$$\|\mathcal{R}'_N(u(z,\cdot))\|_{L^2} \lesssim \Big(\sup_{\tau\in[-1,1]} \|u_N\|_{L^2} + \sup_{\tau\in[-1,1]} \|u\|_{L^2}\Big)^2 \|\delta_N\|_{L^2} \lesssim R^2 \|\delta_N\|_{L^2},$$

implying that $\delta_N(\tau)$ satisfies the integral inequality

$$\|\delta_N(\tau)\|_{L^2} \le \|\Pi_N^\perp u_0\|_{L^2} + \int_{-1}^1 \|\mathcal{R}_N(u(z,\cdot))\|_{L^2} dz + CR^2 \Big| \int_0^\tau \|\delta_N(z)\|_{L^2} dz \Big|.$$

By the Grönwall inequality, one then gets that

$$\sup_{\tau\in[-1,1]} \|\delta_N\|_{L^2} = \sup_{\tau\in[-1,1]} \|\delta_N(\tau)\|_{L^2} \lesssim_R \|\Pi_N^\perp u_0\|_{L^2} + \int_{-1}^{1} \|\mathcal{R}_N(u(z,\cdot))\|_{L^2} dz.$$

Clearly,

$$\Pi_N^\perp u_0 \to 0 \quad \text{as } N \to \infty \text{ in } L^2(\mathbb{T}) \tag{3.10}$$

in $L^2(\mathbb{T})$. We will show that

$$\int_{-1}^{1} \|\mathcal{R}_N(u(z,\cdot))\|_{L^2} dz \to 0 \tag{3.11}$$

as $N \to +\infty$. This can be proved by the Lebesgue dominate convergence. Indeed, again by applying Lemma 3.1 (i), one gets that for any $\tau \in [-1,1]$

$$\|X_{\mathcal{F}}(u(\tau))\|_{L^2} \lesssim \|u(\tau)\|_{L^2}^3 \lesssim R^3.$$

This implies that, for any $\tau \in [-1,1]$,

$$\Pi_N^\perp X_{\mathcal{F}}(u(\tau)) \to 0 \quad \text{as } N \to \infty \text{ in } L^2(\mathbb{T}) \tag{3.12}$$

and

$$\sup_{N\in\mathbb{N}} \sup_{\tau\in[-1,1]} \|\mathcal{R}_N(u(\tau))\|_{L^2} \lesssim R^3.$$

Therefore, equation (3.11) follows by dominated convergence. This completes the proof of (3.5). To show (3.6), we note that the sequences (3.10)–(3.12) are monotone and that the functions

$$(U_0, \tau) \in (K, [-1,1]) \to \Phi_t(u_0) - \Phi_t^N(u_0)$$

are continuous (by Lemma 4.3), thus, the point-wise convergence to zero is promoted to uniform convergence by using the Dini criterion. ∎

Recall that we are abbreviating $\Phi = (\Phi_t)|_{t=1}$ and $\Phi^N = (\Phi_t^N)|_{t=1}$.

Lemma 3.5. *Let A be a compact set in the $L^2(\mathbb{T})$ topology. For all $\varepsilon > 0$, we can find N_ε sufficiently large so that*

$$\Phi(A) \subset \Phi^N(A + B(\varepsilon)) \quad \forall N > N_\varepsilon. \tag{3.13}$$

Proof. Set

$$\Psi_N(v) := (\Phi^N)^{-1}\Phi(v).$$

Then, equation (4.15) follows once we prove that for all $v \in A$ and all $\varepsilon > 0$ there is N_ε large enough such that

$$\|v - \Psi_N(v)\|_{L^2(\mathbb{T})} < \varepsilon \quad \text{for all } N > N_\varepsilon. \tag{3.14}$$

Indeed, assuming (3.14), we have for any $v \in A$

$$\Psi_N(v) \in v + B(\varepsilon) \subset A + B(\varepsilon);$$

thus,

$$\Phi(v) = \Phi^N \Psi_N(v) \in \Phi^N(v + B(\varepsilon)),$$

which readily implies (3.13). The proof of (3.14):

$$\begin{aligned}\|v - \Psi_N(v)\|_{L^2(\mathbb{T})} &= \|(\Phi^N)^{-1}\Phi^N(v) - (\Phi^N)^{-1}\Phi(v)\|_{L^2(\mathbb{T})} \\ &\lesssim_R \|\Phi^N(v) - \Phi(v)\|_{L^2(\mathbb{T})},\end{aligned}$$

where we used Lemma 3.3 and the fact that the L^2 norms of $\Phi^N(v)$ and of $\Phi(v)$ are bounded by $\|v\|_{L^2} = R$. By Lemma 3.4 for all $\varepsilon > 0$, there is a N_ε such that

$$\|\Phi^N(v) - \Phi(v)\|_{L^2(\mathbb{T})} \le \varepsilon,$$

and (4.16) is proved. ■

4. The flow of the Birkhoff map for the fractional NLS

In this section, we prove local well-posedness for the flow of the Birkhoff map associated to the fractional NLS, i.e., $\alpha \neq 1$. We will do it for $\alpha > \frac{3}{4}$, that is sufficient for our purposes.

Let us shorten

$$\Phi(j_1, j_2, j_3) := j_1^{2\alpha} + j_2^{2\alpha} - j_3^{2\alpha} - n^{2\alpha} \quad \text{with } n = j_1 + j_2 - j_3.$$

We will crucially use the following lower bound (see, for instance, [16, Lemma 2.4]):

$$|\Phi(j_1, j_2, j_3)| \gtrsim |j_1 - j_3||j_2 - j_3|(\langle j_1\rangle + \langle j_2\rangle + \langle j_3\rangle)^{-(2-2\alpha)}. \tag{4.1}$$

Lemma 4.1. *Let $\alpha > \frac{3}{4}$ and $N \in \mathbb{N} \cup \{\infty\}$. Then, the following holds.*

(i) *For any $s \ge 2 - 2\alpha$, for any $u \in H^s(\mathbb{T})$,*

$$\|X_{\mathcal{F}_N}(u)\|_{H^s} \lesssim_s \|u\|_{H^s}^2 \|u\|_{L^2}. \tag{4.2}$$

(ii) *For any $s \ge 2 - 2\alpha$, for any $u, v \in H^s(\mathbb{T})$, one has*

$$\|X_{\mathcal{F}_N}(u) - X_{\mathcal{F}_N}(v)\|_{H^s} \lesssim \|u - v\|_{H^s}(\|u\|_{H^s}^2 + \|v\|_{H^s}^2).$$

Proof. The Hamiltonian vector field is $X_{\mathcal{F}_N} = (i\nabla_{\bar{u}}\mathcal{F}_N, -i\nabla_u\mathcal{F}_N)$, and one computes that

$$\partial_{\bar{u}_n}\mathcal{F}_N = \sum_{\substack{|j_1|,|j_2|,|j_3|\le N \\ j_1+j_2-j_3=n \\ j_1^{2\alpha}+j_2^{2\alpha}\neq j_3^{2\alpha}+n^{2\alpha}}} \frac{\sigma i}{2\Phi(j_1, j_2, j_3)} u(j_1)u(j_2)\bar{u}(j_3).$$

Then, in order to deduce the desired estimates in (i)-(ii), it is enough to prove that the trilinear form $\mathcal{T}$ defined by

$$\mathcal{T}[u,v,\varphi] := \sum_{n\in\mathbb{Z}} \mathcal{T}_n[u,v,\varphi]e^{\mathrm{inx}},$$
$$\mathcal{T}_n[u,v,\varphi] := \sum_{\substack{|j_1|,|j_2|,|j_3|\le N\\ j_1+j_2-j_3=n}} \frac{\sigma i}{2\Phi(j_1,j_2,j_3)} u(j_1)v(j_2)\varphi(j_3)$$

is continuous on H^s for $s \ge a = 2-2\alpha$, namely,

$$\|\mathcal{T}[u,v,\varphi]\|_{H^s} \lesssim_s \|u\|_{H^s}\|v\|_{H^s}\|\varphi\|_{L^2} + \|u\|_{H^s}\|v\|_{L^2}\|\varphi\|_{H^s} + \|u\|_{L^2}\|v\|_{H^s}\|\varphi\|_{H^s}. \tag{4.3}$$

We have

$$\begin{aligned}
&\|\mathcal{T}[u,v,\varphi]\|_{H^s}^2\\
&\quad = \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |\mathcal{T}_n[u,v,\varphi]|^2\\
&\quad \overset{(4.1)}{\lesssim} \sum_{n\in\mathbb{Z}} \Bigg(\sum_{\substack{|j_1|,|j_2|,|j_3|\le N\\ j_1+j_2-j_3=n}} \frac{\langle n\rangle^{\sigma}(\langle j_1\rangle + \langle j_2\rangle + \langle j_3\rangle)^a}{|j_3-j_2||j_1-j_3|} |u(j_1)||v(j_2)||\varphi(j_3)| \Bigg)^2 .
\end{aligned}$$

By using that $n = j_1 + j_2 - j_3$, one gets that $\langle n\rangle^s \lesssim_\sigma \langle j_1\rangle^s + \langle j_2\rangle^s + \langle j_3\rangle^s$, implying that

$$\begin{aligned}
&\|\mathcal{T}[u,v,\varphi]\|_{H^\sigma}^2 \lesssim_s T_1 + T_2 + T_3,\\
&T_1 := \sum_{n\in\mathbb{Z}} \Bigg(\sum_{\substack{|j_1|,|j_2|,|j_3|\le N\\ j_1+j_2-j_3=n}} \frac{(\langle j_1\rangle + \langle j_2\rangle + \langle j_3\rangle)^a \langle j_1\rangle^s}{|j_3-j_2||j_1-j_3|} |u(j_1)||v(j_2)||\varphi(j_3)| \Bigg)^2,\\
&T_2 := \sum_{n\in\mathbb{Z}} \Bigg(\sum_{\substack{|j_1|,|j_2|,|j_3|\le N\\ j_1+j_2-j_3=n}} \frac{(\langle j_1\rangle + \langle j_2\rangle + \langle j_3\rangle)^a \langle j_2\rangle^s}{|j_3-j_2||j_1-j_3|} |u(j_1)||v(j_2)||\varphi(j_3)| \Bigg)^2,\\
&T_3 := \sum_{n\in\mathbb{Z}} \Bigg(\sum_{\substack{|j_1|,|j_2|,|j_3|\le N\\ j_1+j_2-j_3=n}} \frac{(\langle j_1\rangle + \langle j_2\rangle + \langle j_3\rangle)^a \langle j_3\rangle^s}{|j_3-j_2||j_1-j_3|} |u(j_1)||v(j_2)||\varphi(j_3)| \Bigg)^2.
\end{aligned}$$

The estimates of T_1, T_2, T_3 can be done similarly, hence, we estimate only the term T_1. The term T_1 can be split into three terms, namely,

$$\begin{aligned}
&T_1 \lesssim A_1 + A_2 + A_3,\\
&A_1 := \sum_{n\in\mathbb{Z}} \Bigg(\sum_{\substack{|j_1|,|j_2|,|j_3|\le N\\ j_1+j_2-j_3=n}} \frac{\langle j_1\rangle^{s+a}}{|j_3-j_2||j_1-j_3|} |u(j_1)||v(j_2)||\varphi(j_3)| \Bigg)^2,
\end{aligned}$$

$$A_2 := \sum_{n\in\mathbb{Z}} \Bigg(\sum_{\substack{|j_1|,|j_2|,|j_3|\le N\\ j_1+j_2-j_3=n}} \frac{\langle j_1\rangle^s \langle j_2\rangle^a}{|j_3-j_2||j_1-j_3|} |u(j_1)||v(j_2)||\varphi(j_3)| \Bigg)^2,$$

$$A_3 := \sum_{n\in\mathbb{Z}} \Bigg(\sum_{\substack{|j_1|,|j_2|,|j_3|\le N\\ j_1+j_2-j_3=n}} \frac{\langle j_1\rangle^s \langle j_3\rangle^a}{|j_3-j_2||j_1-j_3|} |u(j_1)||v(j_2)||\varphi(j_3)| \Bigg)^2.$$

The terms A_1, A_2, A_3 are estimated by similar techniques, hence, we only provide an estimate of A_1. By the triangular inequality, one has

$$\begin{aligned}\langle j_1\rangle^{s+a} &= \langle j_1\rangle^s \langle j_1\rangle^a \lesssim \langle j_1\rangle^s (\langle j_3\rangle^a + \langle j_1-j_3\rangle^a)\\ &\lesssim \langle j_1\rangle^s \langle j_3\rangle^a \langle j_1-j_3\rangle^a.\end{aligned}$$

The latter inequality, together with the Cauchy–Schwarz inequality implies that for any $n\in\mathbb{Z}$

$$\begin{aligned}
&\sum_{j_1,j_2\in\mathbb{Z}} \frac{1}{\langle j_1-n\rangle\langle j_2-n\rangle^{1-a}} \langle j_1\rangle^s |u(j_1)||v(j_2)|\langle j_1+j_2-n\rangle^a |\varphi(j_1+j_2-n)|\\
&\lesssim \Bigg(\sum_{j_1,j_2\in\mathbb{Z}} \langle j_1\rangle^{2s}|u(j_1)|^2|v(j_2)|^2\langle j_1+j_2-n\rangle^{2a}|\varphi(j_1+j_2-n)|^2 \Bigg)^{\frac12}\\
&\quad\times \Bigg(\sum_{j_1,j_2\in\mathbb{Z}} \frac{1}{\langle j_1-n\rangle^2\langle j_2-n\rangle^{2(1-a)}} \Bigg)^{\frac12}\\
&\lesssim \Bigg(\sum_{j_1,j_2\in\mathbb{Z}} \langle j_1\rangle^{2s}|u(j_1)|^2|v(j_2)|^2\langle j_1+j_2-n\rangle^{2a}|\varphi(j_1+j_2-n)|^2 \Bigg)^{\frac12}\\
&\quad\times \Bigg(\sum_{k_1,k_2\in\mathbb{Z}} \frac{1}{\langle k_1\rangle^2\langle k_2\rangle^{2(1-a)}} \Bigg)^{\frac12}\\
&\lesssim \Bigg(\sum_{j_1,j_2\in\mathbb{Z}} \langle j_1\rangle^{2s}|u(j_1)|^2|v(j_2)|^2\langle j_1+j_2-n\rangle^{2a}|\varphi(j_1+j_2-n)|^2 \Bigg)^{\frac12}
\end{aligned}$$

by using that $2(1-a) > 1$. Thus, A_1 can be estimated as

$$\begin{aligned}
A_1 &\lesssim \sum_{n\in\mathbb{Z}} \sum_{j_1,j_2\in\mathbb{Z}} \langle j_1\rangle^{2s}|u(j_1)|^2|v(j_2)|^2\langle j_1+j_2-n\rangle^{2a}|\varphi(j_1+j_2-n)|^2\\
&\lesssim \sum_{j_1\in\mathbb{Z}} \langle j_1\rangle^{2s}|u(j_1)|^2 \sum_{j_2\in\mathbb{Z}} |v(j_2)|^2 \sum_{n\in\mathbb{Z}} \langle j_1+j_2-n\rangle^{2a}|\varphi(j_1+j_2-n)|^2\\
&\overset{j_1+j_2-n=k}{\lesssim} \sum_{j_1\in\mathbb{Z}} \langle j_1\rangle^{2s}|u(j_1)|^2 \sum_{j_2\in\mathbb{Z}} |v(j_2)|^2 \sum_{k\in\mathbb{Z}} \langle k\rangle^{2a}|\varphi(k)|^2\\
&\lesssim \|u\|_{H^s}^2 \|v\|_{L^2}^2 \|\varphi\|_a^2 \overset{s\ge a}{\lesssim} \|u\|_{H^s}\|v\|_{L^2}\|\varphi\|_{H^s}.
\end{aligned}$$

By similar arguments, one can estimate A_2, A_3, T_2, T_3, and hence, one obtains the bound (4.3). Hence, the items (i) and (ii) follow since $\nabla_{\bar{u}}\mathcal{F}_N = \mathcal{T}[u,u,\bar{u}]$, and thus,

$$\begin{aligned}\nabla_{\bar{u}}\mathcal{F}_N(u) - \nabla_{\bar{u}}\mathcal{F}_N(v) &= \mathcal{T}[u,u,\bar{u}] - \mathcal{T}[v,v,\bar{v}]\\ &= \mathcal{T}[u-v,u,\bar{u}] + \mathcal{T}[v,u-v,\bar{u}] + \mathcal{T}[v,v,\overline{u-v}]. \qquad \blacksquare\end{aligned}$$

The latter lemma implies that the flow $\Phi_t^N = \Phi_t^{\mathcal{F}_N}$ is well defined in $H^s(\mathbb{T})$ for any $s \geq 2-2\alpha$ when $\alpha > 3/4$. This follows by a standard Picard iteration. More precisely, the following lemma holds.

Lemma 4.2. *Let $\alpha > 3/4$ and $N \in \mathbb{N}\cup\{\infty\}$. For any $u_0 \in H^s(\mathbb{T})$, $s \geq 2-2\alpha$, $\|u_0\|_{H^s} \leq R$, there exist a time $T_\alpha = T_\alpha(R) = \frac{\mathsf{c}_\alpha}{R^2} > 0$, $\mathsf{c}_\alpha \ll 1$, and a unique local solution $u \in C^1([-T_\alpha, T_\alpha], H^s(\mathbb{T}))$ which solves the Cauchy problem*

$$\begin{cases}\partial_t u(t) = X_{\mathcal{F}_N}(u(t)),\\ u(0) = u_0.\end{cases}$$

Thus, the flow $\Phi_t^N : H^s(\mathbb{T}) \to H^s(\mathbb{T})$, $t \in [-T_\alpha, T_\alpha]$ is a well defined C^1 map and $\|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s}$. In the case where $\alpha = 1$, one has that $T_\alpha = 1$.

Proof. The local existence follows by a standard fixed point argument on the Volterra integral operator

$$u(t) \mapsto \mathcal{S}(u)(t) := u_0 + \int_0^t X_{\mathcal{F}_N}(u(\tau))d\tau$$

in the closed ball

$$\Big\{u \in C^0([-T_\alpha, T_\alpha], H^s(\mathbb{T})) : \sup_{|t|<T_\alpha} \|u\|_{H^s} := \sup_{t\in[-T_\alpha,T_\alpha]} \|u(t)\|_{H^s} \leq R\Big\}.$$

This fixed point argument requires that R is larger than $\|u_0\|_{H^s}$ and $TR^2 = \mathsf{c}_\alpha$ with $\mathsf{c}_\alpha \ll 1$ small enough. If u is a fixed point for $\mathcal{S}$, then it must be $u \in C^1([-T_\alpha, T_\alpha], H^s(\mathbb{T}))$. ■

Lemma 4.3. *Let $\alpha > 3/4$, $s \geq 2-2\alpha$, $N \in \mathbb{N}\cup\{\infty\}$. Assume that $\|w\|_{H^s}, \|v\|_{H^s} \leq R$. It is*

$$\sup_{|t|<T_\alpha} \|\Phi_t^N(v) - \Phi_t^N(w)\|_{H^s} \lesssim_{R,T_\alpha,s} \|v-w\|_{H^s}. \tag{4.4}$$

Proof. The argument easily follows using Lemma 4.2, the inequalities of Lemma (4.1) and the Duhamel representation of truncated flows. ■

We now prove the convergence of the flow of the truncated vector field, to the one of the non-truncated vector field. The flow $\Phi_t = \Phi_t^{\mathcal{F}}$ is the flow of the Hamiltonian vector field $X_{\mathcal{F}}$ whereas the flow of $\Phi_N^t = \Phi_{\mathcal{F}_N}^t$ is the flow associated to the Hamiltonian vector field $X_{\mathcal{F}_N}$ where $\mathcal{F}_N(u) = \mathcal{F}(\Pi_N u)$.

Lemma 4.4. *Let $\alpha > 3/4$, $2 - 2\alpha \le s' < s$, $R > 1$ and $N \in \mathbb{N} \cup \{\infty\}$. Then,*

$$\sup_{\|u_0\|_{H^s} \le R} \sup_{\tau \in [-T_\alpha, T_\alpha]} \|\Phi_\tau(u_0) - \Phi_\tau^N(u_0)\|_{H^{s'}} \lesssim R N^{-(s-s')}. \tag{4.5}$$

Proof. Given $u_0 \in H^s(\mathbb{T})$, we consider

$$\begin{cases} \partial_t u = X_{\mathcal{F}}(u), \\ u(0) = u_0, \end{cases} \qquad \begin{cases} \partial_t u_N = X_{\mathcal{F}_N}(u_N), \\ u_N(0) = \Pi_N u_0. \end{cases}$$

If $\|u_0\|_{H^s} \le R$, then the corresponding solutions satisfy

$$\sup_{\tau \in [-T_\alpha, T_\alpha]} \|u_N(\tau)\|_{H^s}, \quad \sup_{\tau \in [-T_\alpha, T_\alpha]} \|u(\tau)\|_{H^s} \lesssim R.$$

By (3.7), one obtains that $v_N := u - u_N$ satisfies the following problem:

$$\begin{cases} \partial_\tau v_N = \mathcal{A}(u, u_N) v_N + \mathcal{R}_N(u), \\ v_N(0) = \Pi_N^\perp u_0. \end{cases}$$

This implies that

$$v_N(\tau) = \Pi_N^\perp u_0 + \int_0^\tau \mathcal{R}_N(u)(z) dz + \int_0^\tau \mathcal{A}(u, u_N)(z) v_N(z) dz.$$

We have that $\|u\|_{L_\tau^\infty H_x^s}, \|u_N\|_{L_\tau^\infty H_x^s} \le R$ and we need to estimate $\|v_N(t)\|_{H^{s'}}$ with $s' < s$. By applying Lemma 4.1 item (ii), one obtains that for any $z \in [-T_\alpha, T_\alpha]$

$$\begin{aligned} \|\mathcal{A}(u, u_N)(z)[\varphi]\|_{H^{s'}} &\lesssim \Big(\sup_{\tau \in [-T_\alpha, T_\alpha]} \|u(\tau)\|_{H^{s'}} + \sup_{\tau \in [-T_\alpha, T_\alpha]} \|u_N(\tau)\|_{H^{s'}} \Big)^2 \|\varphi\|_{H^{s'}} \\ &\lesssim R^2 \|\varphi\|_{H^{s'}}, \end{aligned}$$

implying that $V_N(\tau)$ satisfies the integral inequality

$$\|v_N(\tau)\|_{H^{s'}} \le \|\Pi_N^\perp u_0\|_{H^{s'}} + \int_{-T_\alpha}^{T_\alpha} \|\mathcal{R}_N(u)(z)\|_{H^{s'}} dz + C R^2 \left| \int_0^\tau \|v_N(z)\|_{H^{s'}} dz \right| \tag{4.6}$$

for all $\tau \in [-T_\alpha, T_\alpha]$. We have (recall $T_\alpha = \frac{\mathtt{c}_\alpha}{R^2}$)

$$\left| \int_0^\tau \|v_N(z)\|_{H^{s'}} dz \right| \le \frac{\mathtt{c}_\alpha}{R^2} \sup_{|\tau| \le T_\alpha} \|v_N(z)\|_{H^{s'}}.$$

Plugging this into inequality (4.6) and then taking the sup over $\tau \in [-T_\alpha, T_\alpha]$ of the new inequality, we arrive at

$$\sup_{|\tau| \le T_\alpha} \|v_N(\tau)\|_{H^{s'}} \le \|\Pi_N^\perp u_0\|_{H^{s'}} + \int_{-T_\alpha}^{T_\alpha} \|\mathcal{R}_N(u)(z)\|_{H^{s'}} dz + C \mathtt{c}_\alpha \sup_{|\tau| \le T_\alpha} \|v_N(z)\|_{H^{s'}}.$$

Since $c_\alpha \ll 1$, we can reabsorb the last term on the right-hand side into the left-hand side and we arrive at

$$\sup_{|\tau|\le T_\alpha} \|v_N(\tau)\|_{H^{s'}} \lesssim \|\Pi_N^{\perp} u_0\|_{H^{s'}} + \int_{-T_\alpha}^{T_\alpha} \|\mathcal{R}_N(u)(z)\|_{H^{s'}} dz.$$

By standard smoothing properties, one has that

$$\|\Pi_N^{\perp} u_0\|_{H^{s'}} \lesssim N^{-(s-s')} \|u_0\|_{H^s} \lesssim R N^{-(s-s')},$$

and by using also Lemma 4.1 (i), one gets that

$$\begin{aligned}
&\int_{-T_\alpha}^{T_\alpha} \|\mathcal{R}_N(u)(z)\|_{H^{s'}} dz \\
&\quad = \int_{-T_\alpha}^{T_\alpha} \|\Pi_N^{\perp} X_{\mathcal{F}}(u(z))\|_{H^{s'}} dz \lesssim N^{-(s-s')} \int_{-T_\alpha}^{T_\alpha} \|X_{\mathcal{F}}(u(z))\|_{H^s} dz \\
&\quad \lesssim N^{-(s-s')} \int_{-T_\alpha}^{T_\alpha} \|u(z)\|_{H^s}^3 dz \lesssim N^{-(s-s')} \frac{\mathsf{c}_\alpha}{R^2} R^3 \lesssim c_\alpha R N^{-(s-s')}.
\end{aligned}$$

This implies

$$\sup_{|\tau|\le T_\alpha} \|v_N(\tau)\|_{H^{s'}} \lesssim R N^{-(s-s')}$$

from which we deduce the statement. ■

We need an approximation result that allows to construct a flow Φ_t on $t \in [-1,1]$ once we have suitable estimates on the approximated flow Φ_t^N that are uniform over $N \in \mathbb{N}$.

Proposition 4.5. *Let $\alpha > 3/4$ and $2 - 2\alpha \le s' < s$, $R > 0$ and $\varepsilon > 0$. Let $A \subset B_s(R)$. There exists N sufficiently large (depending on $\alpha, s, s', \varepsilon, R$) such that the following holds. If*

$$\sup_{t\in[-1,1]} \sup_{u_0\in A} \|\Phi_t^N(u_0)\|_{H^s} \le R, \tag{4.7}$$

then the flow $\Phi_t(u_0)$ is well defined on $t \in [-1,1]$ for all $u_0 \in A$. Moreover,

$$\sup_{t\in[-1,1]} \|\Phi_t(u_0) - \Phi_t^N(u_0)\|_{H^{s'}} \le \varepsilon \quad \forall u_0 \in A. \tag{4.8}$$

Remark 4.6. The assumption (4.7) on the truncated flow Φ_t^N will be verified in Proposition 6.3.

Proof. Recall that $\mathsf{c}_\alpha = T_\alpha R \ll 1$, where T_α is the local existence time of Φ_t (see Lemma 4.2). Let J be the smallest integer such that $J \frac{\mathsf{c}_\alpha}{2(R^2+1)} \ge 1$. We have

$$\frac{2(R^2+1)}{\mathsf{c}_\alpha} \le J < \frac{2(R^2+1)}{\mathsf{c}_\alpha} + 1.$$

We partition the interval $[0,1]$ into $J-1$ intervals of length $\frac{c\alpha}{2(R^2+1)}$ and a last, possibly smaller interval. We will compare the approximated flow Φ_t^N and Φ_t (which exists only for small times) on these small intervals and we will glue the local solutions. Let

$$s' < s_J < \cdots < s_2 < s_1 < s_0 = s.$$

We proceed by induction over $j = 0, \dots, J$. Assuming that Φ_t is well defined on $[0, (j+1)\frac{c\alpha}{2(R^2+1)}]$ and that

$$\sup_{t\in[0, j\frac{c\alpha}{2(R^2+1)}]} \|\Phi_t(u_0) - \Phi_t^N(u_0)\|_{H^{s_j}} \le N^{-\kappa_j} \tag{4.9}$$

for some $\kappa_j > 0$, we will show that $\Phi_t(u_0)$ is well defined on $[0, (j+2)\frac{c\alpha}{2(R^2+1)}]$ and that

$$\sup_{t\in[0,(j+1)\frac{c\alpha}{2(R^2+1)}]} \|\Phi_t(u_0) - \Phi_t^N(u_0)\|_{H^{s_{j+1}}} \le N^{-\kappa_{j+1}}, \tag{4.10}$$

for a suitable $\kappa_{j+1} > 0$, provided N is sufficiently large. In particular, we take N so large in such a way that we also have $N^{-\kappa_{j+1}} < \varepsilon$. Using the induction procedure up to $j = J$, the statement would then follow.

The induction base $j = 0$ is covered by Lemmas 4.2 and 4.4 and by the fact that $A \subset B_s(R)$.

Regarding the induction step, we first prove (4.10). Then, using the assumption (4.7) and the triangle inequality, we get

$$\sup_{t\in[0,(j+1)\frac{c\alpha}{2(R^2+1)}]} \|\Phi_t(u_0)\|_{H^{s_{j+1}}} \le R + N^{-\kappa_{j+1}} < 2R. \tag{4.11}$$

By (4.11), we then use Lemma 4.2 (with $2R$ in place of R) to show that $\Phi_t(u_0)$ is well defined on $[0, (j+2)\frac{c\alpha}{2(R^2+1)}]$.

Now, we show (4.10). If the sup in (4.10) is attained for $t \in [0, j\frac{c\alpha}{2(R^2+1)}]$, then (4.10) follows by (4.9) simply taking $\kappa_{j+1} = \kappa_j$. On the other hand, if the sup is attained for $t \in [j\frac{c\alpha}{2(R^2+1)}, (j+1)\frac{c\alpha}{2(R^2+1)}]$, using the group property of the flow, we need to prove

$$\sup_{t\in[j\frac{c\alpha}{2(R^2+1)},(j+1)\frac{c\alpha}{2(R^2+1)}]} \|\Phi_t\Phi_{j\frac{c\alpha}{2(R^2+1)}}(u_0) - \Phi_t^N\Phi^N_{j\frac{c\alpha}{2(R^2+1)}}(u_0)\|_{H^{s_{j+1}}} \le N^{-\kappa_{j+1}}.$$

To do so, we decompose

$$\begin{aligned}
&\|\Phi_t\Phi_{j\frac{c\alpha}{2(R^2+1)}}(u_0) - \Phi_t^N\Phi^N_{j\frac{c\alpha}{2(R^2+1)}}(u_0)\|_{H^{s_{j+1}}} \\
&\le \|\Phi_t\Phi_{j\frac{c\alpha}{2(R^2+1)}}(u_0) - \Phi_t\Phi^N_{j\frac{c\alpha}{2(R^2+1)}}(u_0)\|_{H^{s_{j+1}}} && (4.12)\\
&\quad + \|\Phi_t\Phi^N_{j\frac{c\alpha}{2(R^2+1)}}(u_0) - \Phi_t^N\Phi^N_{j\frac{c\alpha}{2(R^2+1)}}(u_0)\|_{H^{s_{j+1}}}, && (4.13)
\end{aligned}$$

and we will handle these two terms separately.

To bound (4.12), we first note that by the induction assumption (4.9) and by the assumption (4.7), we have for N large enough

$$\|\Phi_{j\frac{c\alpha}{2(R^2+1)}}(u_0)\|_{H^{s_j}} \leq R + N^{-\kappa_j} < R + \varepsilon.$$

Using this fact, the assumption (4.7) and the fact that the stability estimate (4.4) is time-translation invariant, we apply (4.4) to get

$$\begin{aligned}
&\sup_{t\in[j\frac{c\alpha}{2(R^2+1)},(j+1)\frac{c\alpha}{2(R^2+1)}]} \|\Phi_t\Phi_{j\frac{c\alpha}{2(R^2+1)}}(u_0) - \Phi_t\Phi^N_{j\frac{c\alpha}{2(R^2+1)}}(u_0)\|_{H^{s_{j+1}}} \\
&\quad\lesssim_{R+\varepsilon,s} \|\Phi_{j\frac{c\alpha}{2(R^2+1)}}(u_0) - \Phi^N_{j\frac{c\alpha}{2(R^2+1)}}(u_0)\|_{H^{s_{j+1}}} \leq N^{-\kappa_{j+1}},
\end{aligned}$$

where in the last step we used the induction assumption (4.9), where $0 > \kappa_{j+1} > s_{j+1} - s_j$ and N sufficiently large.

In order to bound the term (4.13), we use the stability estimate (3.4) and initial datum $\Phi^N_{j\frac{c\alpha}{2(R^2+1)}}(u_0)$, that is allowed recalling the assumption (4.7). Thus, for κ_{j+1} as above we arrive to

$$\begin{aligned}
&\sup_{t\in[j\frac{c\alpha}{2(R^2+1)},(j+1)\frac{c\alpha}{2(R^2+1)}]} \|\Phi_t\Phi^N_{j\frac{c\alpha}{2(R^2+1)}}(u_0) - \Phi^N_t\Phi^N_{j\frac{c\alpha}{2(R^2+1)}}(u_0)\|_{H^{s_{j+1}}} \\
&\quad\lesssim_R N^{s_{j+1}-s_j} < \frac{1}{2}N^{-\kappa_{j+1}},
\end{aligned}$$

provided that N is sufficiently large. This concludes the proof. ∎

Recall that we are abbreviating $\Phi = (\Phi_t)\big|_{t=1}$ and $\Phi^N = (\Phi^N_t)\big|_{t=1}$.

Lemma 4.7. *Let $\alpha > 3/4$ and $s \geq 2 - 2\alpha$, $R > 0$ and $\varepsilon > 0$. Let $A \subset B_s(R)$. Assume that*

$$\sup_{N\in\mathbb{N}}\ \sup_{t\in[-1,1]}\ \sup_{u_0\in A} \|\Phi^N_t(u_0)\|_{H^s} \leq R. \tag{4.14}$$

Then, if $u_0, v_0 \in A$, then

$$\|\Phi^N_t(u_0) - \Phi^N_t(v_0)\|_{H^s} \lesssim_R \|u_0 - v_0\|_{H^s}, \quad t \in [-1,1].$$

Proof. By the Duhamel representation of the solution, one has

$$\begin{aligned}
&\Phi^N_t(u_0) - \Phi^N_t(v_0) = u_0 - v_0 + \int_0^t \delta_N(\tau)d\tau, \\
&\qquad\qquad \delta_N(\tau) := X_{\mathcal{F}_N}(\Phi^N_\tau(u_0)) - X_{\mathcal{F}_N}(\Phi^N_\tau(v_0)), \quad \tau \in [-1,1].
\end{aligned}$$

By the estimates of Lemma 4.1 and by using the assumption (4.14), we get

$$\|\delta_N(\tau)\|_{H^s} \lesssim_s R^2\|\Phi^N_\tau(u_0) - \Phi^N_\tau(v_0)\|_{H^s} \quad \forall\tau \in [-1,1];$$

and hence,

$$\|\Phi_t^N(u_0) - \Phi_t^N(v_0)\|_{H^s} \leq \|u_0 - v_0\|_{H^s} + CR^2 \left| \int_0^t \|\Phi_\tau^N(u_0) - \Phi_\tau^N(v_0)\|_{H^s} d\tau \right|.$$

This implies the claimed bound by using the Grönwall inequality. ∎

Lemma 4.8. *Let $\alpha > 3/4$, $2 - 2\alpha \leq s' < s$, $R > 0$ and $\varepsilon > 0$. Let $A \subset B_s(R)$. Assume that*

$$\sup_{N \in N} \sup_{t \in [-1,1]} \sup_{u_0 \in A} \|\Phi_t^N(u_0)\|_{H^s} \leq R.$$

Let $E \subset A$ be a compact set in the $H^s(\mathbb{T})$ topology. Then, for all $\varepsilon > 0$, $2 - 2\alpha \leq s' < s$ there exists N_ε sufficiently large so that

$$\Phi(E) \subset \Phi^N(E + B_{s'}(\varepsilon)) \quad \forall N > N_{\varepsilon'}. \tag{4.15}$$

Proof. Set

$$\Psi_N v := (\Phi^N)^{-1} \Phi(v).$$

Then, equation (4.15) follows once we prove for all $v \in E$ and all $\varepsilon > 0$ there is N_ε large enough such that

$$\|v - \Psi_N(v)\|_{H^{s'}} < \varepsilon \quad \text{for all } N > N_\varepsilon. \tag{4.16}$$

Indeed, assuming (4.16), we have for any $v \in E$

$$\Psi_N(v) \in v + B_{s'}(\varepsilon) \subset E + B_{s'}(\varepsilon),$$

thus,

$$\Phi(v) = \Phi^N \Psi_N(v) \in \Phi^N(v + B_{s'}(\varepsilon)),$$

which readily implies (4.15). The proof of (4.16):

$$\begin{aligned} \|v - \Psi_N(v)\|_{H^{s'}} &= \|(\Phi^N)^{-1}\Phi^N(v) - (\Phi^N)^{-1}\Phi(v)\|_{H^{s'}} \\ &\overset{\text{by Lemma 4.7}}{\lesssim_R} \|\Phi^N(v) - \Phi(v)\|_{H^{s'}}. \end{aligned}$$

Finally, by Proposition 4.5, for all $\varepsilon > 0$, there is an N_ε such that

$$\|\Phi^N(v) - \Phi(v)\|_{H^{s'}} \leq \varepsilon,$$

(4.16) is proved, and the proof is concluded. ∎

5. Main probabilistic estimates

The main result of this section is the following.

Proposition 5.1. *Let $1 \geq \alpha > \frac{1+\sqrt{97}}{12} \sim 0.9$ and set*

$$\zeta(\alpha) := \min\left(\frac{1}{3} + \frac{2\alpha - 1}{3(1-\alpha)}, \frac{2\alpha(\alpha+1)}{4 - 4\alpha^2 + 3\alpha}\right) > 1. \tag{5.1}$$

For all $p \geq 1$, it holds

$$\left\| \frac{d}{dt} H[\Phi_t^N(u)]\big|_{t=0} \right\|_{L^p(\tilde{\gamma}_\alpha)} \lesssim_R p^{\frac{1}{\zeta(\alpha)}}. \tag{5.2}$$

Remark 5.2. Note that $\zeta(1) = 4/3$.

We start with the following deterministic estimate.

Lemma 5.3. *For all $\alpha \in (3/4, 1]$, it holds*

$$\begin{aligned}\left| \frac{d}{dt} H^{(\alpha)}[\Phi_t^N(u)]\Big|_{t=0} \right| &\lesssim \|\Pi_N u\|_{L^2}^4 + \|\Pi_N u\|_{L^2}^2 \|\Pi_N u\|_{FL^{0,1}}^2 \\ &\quad + \|\Pi_N u\|_{L^2} \|\Pi_N u\|_{H^{2-2\alpha}}^2 \|\Pi_N u\|_{FL^{0,1}}^3.\end{aligned}$$

Proof. We have

$$\begin{aligned}\frac{d}{dt} H^{(\alpha)}[\Phi_t^N(u)]\big|_{t=0} &= \left\{H^{(\alpha)}[\Pi_N u], \mathcal{F}_N\right\} \\ &= \left\{\| |D_x|^\alpha \Pi_N u\|_{L^2}^2, \mathcal{F}_N\right\} + \frac{\sigma}{2}\left\{\|\Pi_N u\|_{L^4}^4, \mathcal{F}_N\right\} \\ &\overset{(2.7)}{=} -\frac{\sigma}{2}\|\Pi_N u\|_{L^4}^4 + \sigma \|\Pi_N u\|_{L^2}^4 + \frac{\sigma}{2}\left\{\|\Pi_N u\|_{L^4}^4, \mathcal{F}_N\right\}.\end{aligned} \tag{5.3}$$

The first summand in (5.3) is bounded by

$$\|\Pi_N u\|_{L^4}^4 \leq \|\Pi_N u\|_{L^2}^2 \left(\sum_{|n| \leq N} |u(n)|\right)^2 \leq \|\Pi_N u\|_{L^2}^2 \|\Pi_N u\|_{FL^{0,1}}^2. \tag{5.4}$$

Hence, we need to estimate only the term $\{\|\Pi_N u\|_{L^4}^4, \mathcal{F}_N\}$. By the definition of the Poisson brackets given in (2.2), one has that

$$\left\{\|\Pi_N u\|_{L^4}^4, \mathcal{F}_N(u)\right\} = D_U \mathcal{G}(U)[X_{\mathcal{F}_N}(U)],$$

where

$$\mathcal{G}(U) := \int_{\mathbb{T}} u^2 \bar{u}^2 dx.$$

We write

$$\begin{aligned}&D_U \mathcal{G}(U)[h] = \mathcal{T}_1 h + \mathcal{T}_2 \bar{h}, \\ &\qquad \mathcal{T}_1 h := 2\int_{\mathbb{T}} u \bar{u}^2 h dx, \quad \mathcal{T}_2 \bar{h} := 2\int_{\mathbb{T}} u^2 \bar{u} \bar{h} dx.\end{aligned}$$

We have

$$\begin{aligned} |\mathcal{T}_1 h| &\lesssim \sum_{j_1+j_2-j_3-j_4=0} |h(j_1)||u(j_2)||u(j_3)||u(j_4)| \\ &\lesssim \sum_{j_2,j_3,j_4} |h(j_3+j_4-j_2)||u(j_2)||u(j_3)||u(j_4)| \\ &\lesssim \|h\|_{L^2} \sum_{j_2,j_3,j_4} |u(j_2)||u(j_3)||u(j_4)| \lesssim \|h\|_{L^2}\|u\|^3_{FL^{0,1}} \end{aligned}$$

and similarly $|\mathcal{T}_2 \bar{h}| \lesssim \|h\|_{L^2}\|u\|^3_{FL^{0,1}}$.

Therefore,

$$|D_U \mathcal{G}(U)[X_{\mathcal{F}_N}(U)]| \lesssim \|u\|^3_{FL^{0,1}} \|X_{\mathcal{F}_N}(U)\|_{L^2}. \tag{5.5}$$

Combining (5.5) and (4.2) (recall that $2-2\alpha \geq 0$) gives

$$\begin{aligned} |\{\|\Pi_N u\|^4_{L^4}, \mathcal{F}_N(u)\}| &= |D_U \mathcal{G}(U)[X_{\mathcal{F}_N}(U)]| \lesssim \|u\|^3_{FL^{0,1}}\|X_{\mathcal{F}_N}(U)\|_{L^2} \\ &\lesssim \|u\|^3_{FL^{0,1}}\|X_{\mathcal{F}_N}(U)\|_{2-2\alpha} \lesssim \|u\|^3_{FL^{0,1}}\|u\|^2_{2-2\alpha}\|u\|_{L^2}. \end{aligned} \tag{5.6}$$

Using (5.4) and (5.6) in (5.3), we finish the proof. ■

We now establish the following tail estimates.

Lemma 5.4. *Let $s < \alpha - \frac{1}{2}$. There is $c(R) > 0$ such that for all $t > 0$*

$$\widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|_{H^s} \geq t) \leq \exp(-c(R)t^{\frac{2\alpha}{s}}).$$

Proof. Since

$$\|\Pi_N u\|_{H^s} \lesssim \sum_{j\in\mathbb{N}} 2^{js}\|\Delta_j \Pi_N u\|_{L^2},$$

we have

$$\widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|_{H^s} \geq t) \lesssim \widetilde{\gamma}_{\alpha,N}\bigg(\sum_{j\in\mathbb{N}} 2^{js}\|\Delta_j \Pi_N u\|_{H^s} \geq t\bigg). \tag{5.7}$$

Let now

$$j_t := \min\left\{j \in \mathbb{N} : 2^j \geq (t/R)^{\frac{1}{s}}\right\} \tag{5.8}$$

so that we have

$$2^{js} < t/R \quad \text{for } j < j_t.$$

Hence,

$$\sum_{0\leq j<j_t} 2^{js}\|\Delta_j \Pi_N u\|_{L^2} \lesssim 2^{j_t s}R \lesssim \frac{t}{R}R = t,$$

$\widetilde{\gamma}_{\alpha,N}$-a.s., therefore,

$$\widetilde{\gamma}_{\alpha,N}\bigg(\sum_{0\leq j<j_t} 2^{js}\|\Delta_j P_N u\|_{L^2} \geq t\bigg) = 0. \tag{5.9}$$

Thus, (5.7) and (5.9) give

$$\widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|_{H^s} \geq t) \lesssim \widetilde{\gamma}_{\alpha,N}\bigg(\sum_{j\geq j_t} 2^{js}\|\Delta_j \Pi_N u\|_{L^2} \geq t\bigg).$$

Let $c_0 > 0$ small enough in such a way that

$$\sigma_j := c_0(j - j_t + 1)^{-2}, \quad \sum_{j\in\mathbb{N}} \sigma_j \leq 1.$$

Therefore, we can bound

$$\widetilde{\gamma}_{\alpha,N}\bigg(\sum_{j\geq j_t} 2^{js}\|\Delta_j \Pi_N u\|_{L^2} \geq t\bigg) \leq \sum_{j\geq j_t} \widetilde{\gamma}_{\alpha,N}(\|\Delta_j \Pi_N u\|_{L^2} \geq \sigma_j 2^{-js} t). \tag{5.10}$$

For each term of this sum, we have ($\{g_j\}_{j\in\mathbb{Z}}$ indicates a sequence of $\mathcal{N}(0,1)$ random variables)

$$\begin{aligned}
\widetilde{\gamma}_{\alpha,N}\big(\|\Delta_j \Pi_N u\|_{L^2} \geq \sigma_j 2^{-js} t\big) &= \widetilde{\gamma}_{\alpha,N}\big(\|\Delta_j \Pi_N u\|_{L^2}^2 \geq 2^{-2js}\sigma_j^2 t^2\big) \\
&\leq P\bigg(\sum_{j\sim 2^j} |g_j|^2 \geq 2^{2j(\alpha-s)}\sigma_j^2 t^2\bigg) \\
&= P\bigg(\sum_{j\sim 2^j} (|g_j|^2 - 1) \geq 2^{2j(\alpha-s)}\sigma_j^2 t^2 - 2^j\bigg) \\
&\leq P\bigg(\sum_{j\sim 2^j} (|g_j|^2 - 1) \geq c 2^{2j(\alpha-s)}\sigma_j^2 t^2\bigg),
\end{aligned}$$

where $c > 0$ is a suitable small constant and we used the fact that $2(\alpha - s) > 1$ in the last inequality. Then, from the Bernstein inequality, we get

$$P\bigg(\sum_{j\sim 2^j} (|g_j|^2 - 1) \geq c 2^{2j(\alpha-s)}\sigma_j^2 t^2\bigg) \leq e^{-c 2^{2j(\alpha-s)}\sigma_j^2 t^2}.$$

Thus, recalling (5.8), we have arrive at the desired estimate:

$$\begin{aligned}
\text{right-hand side of (5.10)} &\leq \sum_{j>j_t} e^{-c 2^{2j(\alpha-s)}\sigma_j^2 t^2} \\
&\leq e^{-c 2^{j_t(\alpha-s)}\sigma_{j_t}^2 t^2} \\
&\leq \exp(-c(R) t^{\frac{2\alpha}{s}}).
\end{aligned}$$

∎

Lemma 5.5. *There is $c > 0$ such that*

$$\widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|_{FL^{0,1}} \geq t) \leq 2\exp\bigg(-c\frac{t^{2+2\alpha}}{R^{2\alpha}}\bigg)$$

for all $t \gtrsim (\log N)^2$ if $\alpha = 1$ or $t \gtrsim N^{1-\alpha}$ for $\alpha < 1$.

Proof. We have

$$\|u\|_{FL^{0,1}} = \sum_{j\in\mathbb{N}}\sum_{n\sim 2^j}|u(n)| \le \sum_{j\in\mathbb{N}}\|\Delta_j u\|_{FL^{0,1}}.$$

Then,

$$\widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|_{FL^{0,1}} \ge t) \le \widetilde{\gamma}_{\alpha,N}\Big(\sum_{j\in\mathbb{N}}\|\Delta_j\Pi_N u\|_{FL^{0,1}} \ge t\Big). \tag{5.11}$$

We note that

$$\|\Delta_j u\|_{FL^{0,1}} \le 2^{\frac{j}{2}}\|\Delta_j u\|_{L^2}. \tag{5.12}$$

Let now

$$j_t := \min\left\{j \in \mathbb{N} : R2^{\frac{j}{2}} \ge t\right\}$$

so that we have

$$R2^{\frac{j}{2}} < t \quad \text{for } j < j_t. \tag{5.13}$$

Therefore, using (5.12) and (5.13), we get

$$\sum_{0\le j<j_t}\|\Delta_j\Pi_N u\|_{FL^{0,1}} \le 2^{\frac{j_t}{2}}R < t$$

for any element of $B(R)$, therefore,

$$\widetilde{\gamma}_{\alpha,N}\Big(\sum_{0\le j<j_t}\|\Delta_j\Pi_N u\|_{FL^{0,1}} \ge t\Big) = 0. \tag{5.14}$$

Thus, (5.11) and (5.14) give

$$\widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|_{FL^{0,1}} \ge t) \le \widetilde{\gamma}_{\alpha,N}\Big(\sum_{j\ge j_t}\|\Delta_j\Pi_N u\|_{FL^{0,p}} \ge t\Big).$$

Let $c_0 > 0$ small enough in such a way that

$$\sigma_j := c_0(j - j_t + 1)^{-2}, \quad \sum_{j\in\mathbb{N}}\sigma_j \le 1.$$

We bound

$$\widetilde{\gamma}_{\alpha,N}\Big(\sum_{j\ge j_t}\|\Delta_j\Pi_N u\|_{FL^{0,1}} \ge t\Big) \le \sum_{j\ge j_t}\widetilde{\gamma}_{\alpha,N}(\|\Delta_j\Pi_N u\|_{FL^{0,1}} \ge \sigma_j t). \tag{5.15}$$

We introduce the centred sub-Gaussian random variables

$$Y_j := \|\Delta_j\Pi_N u\|_{FL^{0,1}} - E_\alpha[\|\Delta_j\Pi_N u\|_{FL^{0,1}}] = \sum_{n\sim 2^j}\frac{|g_n| - E[|g_n|]}{\langle n\rangle^\alpha}$$

and we have

$$\widetilde{\gamma}_{\alpha,N}(\|\Delta_j \Pi_N u\|_{FL^{0,1}} \geq \sigma_j t) = \widetilde{\gamma}_{\alpha,N}(Y_j \geq \sigma_j t - E[\|\Delta_j \Pi_N u\|_{FL^{0,1}}]). \tag{5.16}$$

Note that

$$E_\alpha[\|\Delta_j \Pi_N u\|_{FL^{0,1}}] = \sum_{n\sim 2^j} \frac{E[|g_n|]}{\langle n\rangle^\alpha} \simeq \sum_{n\sim 2^j} \frac{1}{\langle n\rangle^\alpha}.$$

We set

$$s_j := \sigma_j t - E_\alpha[\|\Delta_j \Pi_N u\|_{FL^{0,1}}].$$

It follows that there is $C > 0$ large enough such that for all $t > C(\log_2 N)^2$ for $\alpha = 1$ or for $t > CN^{1-\alpha}$ for $\alpha \neq 1$

$$\inf_{j\in\{j_t,\dots,\log N\}} s_j > 0. \tag{5.17}$$

We have by the Hoeffding inequality

$$\gamma_{\alpha,N}(Y_j \geq s_j) \leq 2\exp(-c2^{\alpha j} s_j^2). \tag{5.18}$$

Thus,

$$\begin{aligned}\text{right-hand side of (5.16)} &\leq \sum_{j\geq j_t} 2\exp(-c2^{\alpha j} s_j^2) \leq 2\exp(-c2^{\alpha j_t} s_{j_t}^2)\\ &\leq 2\exp\Big(-c\frac{t^4}{R^2}\Big).\end{aligned}$$

■

In the remaining part of this section, it is convenient to shorten

$$G_N := \frac{d}{dt} H[\Pi_N \Phi_t^N(u)]\big|_{t=0}. \tag{5.19}$$

We have the following lemma.

Lemma 5.6. *It holds for all $M < N \in \mathbb{N}$*

$$\|G_N - G_M\|_{L^p(\widetilde{\gamma}_\alpha)} \lesssim \frac{p^3}{M^{2\alpha-1}}. \tag{5.20}$$

Proof. We will prove

$$\|G_N - G_M\|_{L^2(\widetilde{\gamma}_\alpha)} \lesssim \frac{1}{M^{2\alpha-1}}. \tag{5.21}$$

The assertion will follow by the standard hyper-contractivity estimate [24, Theorem 5.10, Remark 5.11], noting that G_N is a multilinear form of at most 6 factors.

We use formula (5.3) in the proof of Lemma 5.3. The first summand is bounded in the support of $\widetilde{\gamma}_\alpha$ by R^4. For the second and the third summand, we estimate the $L^2(\gamma_\alpha)$ norm and use that it controls the $L^2(\widetilde{\gamma}_\alpha)$ norm.

We have

$$\|G_N - G_M\|_{L^2(\tilde{\gamma}_\alpha)} \le \big\|\|\Pi_N u\|_{L^4}^4 - \|\Pi_M u\|_{L^4}^4\big\|_{L^2(\gamma_\alpha)} + \big\|\{\|\Pi_N u\|_{L^4}^4, \mathcal{F}_N\} - \{\|\Pi_M u\|_{L^4}^4, \mathcal{F}_M\}\big\|_{L^2(\gamma_\alpha)}.$$

By a direct calculation, one has that, for any $N \in \mathbb{N}$,

$$\{\|\Pi_N u\|_{L^4}^4, \mathcal{F}_N\} = \sum_{\substack{|n_i|,|m_i|\le N,\\ \sum_{i=1}^3 n_i = \sum_{i=1}^3 m_i}} \mathtt{c}(n_1,n_2,n_3,m_1,m_2,m_3) u(n_1)u(n_2)u(n_3)\bar{u}(m_1)\bar{u}(m_2)\bar{u}(m_3), \tag{5.22}$$

where the coefficients $\mathtt{c}(n_1,n_2,n_3,m_1,m_2,m_3)$ are such that

$$|\mathtt{c}(n_1,n_2,n_3,m_1,m_2,m_3)| \lesssim 1 \quad \forall\, n_1,n_2,n_3,m_1,m_2,m_3 \in \mathbb{Z}. \tag{5.23}$$

In what follows, we will use the Wick formula for expectation values of multilinear forms of Gaussian random variables [24, Theorem 1.28] in the following form. Let $\ell \in \mathbb{N}$ and S_ℓ be the symmetric group on $\{1,\dots,\ell\}$, whose elements are denoted by σ. We have

$$E_\alpha\Big[\prod_{j=1}^{\ell} u(n_j)\bar{u}(m_j)\Big] = \sum_{\sigma\in S_\ell}\prod_{j=1}^{\ell}\frac{\delta_{m_j,n_{\sigma(j)}}}{1+|n_j|^{2\alpha}} \simeq \sum_{\sigma\in S_\ell}\prod_{j=1}^{\ell}\frac{\delta_{m_j,n_{\sigma(j)}}}{\langle n_j\rangle^{2\alpha}}, \tag{5.24}$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$. We convey that the labels m_i (respectively, n_i) are associated to the Fourier coefficients of $\bar{u}$ (respectively, u). We say that σ contracts the pairs of indexes $(m_j, n_{\sigma(j)})$, and we shorten for any $\Omega \subset \mathbb{Z}^\ell \times \mathbb{Z}^\ell$

$$\sigma(\Omega) := \Omega \cap \big\{m_i = n_{\sigma(i)}, i = 1,\dots,\ell\big\}, \quad \sigma \in S_\ell.$$

We also define the set $\overline{\Omega}$ to be obtained by Ω swapping the role of n_i and m_i $i = 1,\dots,\ell$.

Let $N > M$ and define for $a, b \in \mathbb{N}$

$$A_{N,M}^{a,b} := \big\{|n_{a,b}|, |m_{a,b}| \le N, n_a + n_b = m_a + m_b, \max(|m_{a,b}|, |n_{a,b}|) > M\big\}.$$

Squaring and using (5.24) with $\ell = 4$, we have

$$\begin{aligned}\big\|\|\Pi_N u\|_{L^4}^4 - \|\Pi_M u\|_{L^4}^4\big\|_{L^2(\gamma_\alpha)} &= \sum_{A_{N,M}^{1,2}\times A_{N,M}^{3,4}} E_\alpha\Big[\prod_{i=1}^4 u(n_i)\bar{u}(m_i)\Big] \\ &= \sum_{\sigma\in S_4}\sum_{\sigma(A_{N,M}^{1,2})\times\sigma(A_{N,M}^{3,4})}\frac{1}{\langle n_1\rangle^{2\alpha}\langle n_2\rangle^{2\alpha}\langle n_3\rangle^{2\alpha}\langle n_4\rangle^{2\alpha}} \lesssim \frac{1}{M^{4\alpha-2}}.\end{aligned}$$

Similarly,

$$B^{a,b,c}_{N,M} := \big\{|n_{a,b,c}|, |m_{a,b,c}| \le N, n_a + n_b + n_c = m_a + m_b + m_c,$$
$$\max(|m_{a,b,c}|, |n_{a,b,c}|) > M m_c \neq n_b, n_c\big\}$$

and

$$\begin{aligned}
&\|\{\|\Pi_N u\|^4_{L^4}, \mathcal{F}_N\} - \{\|\Pi_N u\|^4_{L^4}, \mathcal{F}_N\}\|_{L^2(\gamma_1)} \\
&\lesssim \sum_{B^{1,2,3}_{N,M} \times \bar{B}^{4,5,6}_{N,M}} E_\alpha\Big[\prod_{i=1}^{6} u(n_i)\bar{u}(m_i)\Big] \\
&\lesssim \sum_{\sigma \in S_6} \sum_{\sigma(B^{1,2,3}_{N,M}) \times \sigma(\bar{B}^{4,5,6}_{N,M})} \prod_{i=1}^{6} \frac{1}{\langle n_i\rangle^{2\alpha}} \lesssim \frac{1}{M^{4\alpha-2}}.
\end{aligned}$$

■

Then, we can immediately establish the following result.

Proposition 5.7. *There are $C, c > 0$ such that*

$$\gamma_\alpha(|G_N - G| \ge t) \le C e^{-c t^{\frac{1}{3}} N^{\frac{2\alpha-1}{3}}}. \tag{5.25}$$

Proof. Having bounded all the moments as in Lemma 5.6, we can also bound the fractional exponential moment

$$E_\alpha[\exp(c|G_N - G_M|^{\frac{1}{3}} N^{\frac{2\alpha}{3}-\frac{1}{3}})] < \infty, \tag{5.26}$$

for a suitable constant $c > 0$. From (5.26), we obtain (5.25) in the standard way using Markov inequality. ■

Proof of Proposition 5.1. We will prove that there is $c(R) > 0$ such that for all $N \in \mathbb{N} \cup \{\infty\}$,

$$\widetilde{\gamma}_{\alpha,N}(G_N \ge t) \lesssim e^{-c(R) t^{\xi(\alpha)}}. \tag{5.27}$$

Then, the fact that (5.2) follows from (5.27) is standard.

By Lemma 5.3, we see that for $u \in B(R)$,

$$\begin{aligned}
|G_N| &\le R^4 + R^2 \|\Pi_N u\|^2_{FL^{0,1}} + R \|\Pi_N u\|^2_{H^{2-2\alpha}} \|\Pi_N u\|^3_{FL^{0,1}} \\
&\lesssim C(R) \|\Pi_N u\|^2_{H^{2-2\alpha}} \|\Pi_N u\|^3_{FL^{0,1}}.
\end{aligned}$$

Therefore,

$$\widetilde{\gamma}_{\alpha,N}(G_N \ge t) \le \widetilde{\gamma}_{\alpha,N}\big(\|\Pi_N u\|^2_{H^{2-2\alpha}} \|\Pi_N u\|^3_{FL^{0,1}} \ge C(R) t\big).$$

Lemma 5.5 yields for $\alpha = 1$

$$\widetilde{\gamma}_{\alpha,N}(G_N \ge t) \lesssim e^{-c(R) t^{\frac{4}{3}}} \quad \text{for } t \gtrsim (\log N)^2. \tag{5.28}$$

For $\alpha < 1$, we estimate

$$\begin{aligned}&\widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|^2_{H^{2-2\alpha}}\|\Pi_N u\|^3_{FL^{0,1}} \geq C(R)t)\\&\quad\leq \widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|^3_{FL^{0,1}} \geq c(R)t^{k_1}) + \widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|^2_{H^{2-2\alpha}} \geq c(R)t^{k_2})\end{aligned}$$

with

$$k_1 + k_2 = 1. \tag{5.29}$$

Then, using Lemma 5.5, we bound

$$\widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|^3_{FL^{0,1}} \geq t^{k_1}) \lesssim \exp\big(-c(R)t^{\frac{(2+2\alpha)k_1}{3}}\big) \quad \text{for } t \gtrsim N^{1-\alpha},$$

and using Lemma 5.4, we have

$$\widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|^2_{H^{2-2\alpha}} \geq t^{k_2}) \lesssim \exp\big(-c(R)t^{\frac{\alpha k_2}{2-2\alpha}}\big).$$

We optimise choosing k_1, k_2 such that

$$\frac{\alpha k_2}{2-2\alpha} = \frac{(2+2\alpha)k_1}{3}.$$

Together with (5.29), this leads to the choice

$$k_2 = \frac{4-4\alpha^2}{4-4\alpha^2+3\alpha}.$$

It is clear that $k_2 \in (0,1)$ when $\alpha \in (0,1)$, thus, this choice is admissible. We finally arrive at

$$\widetilde{\gamma}_{\alpha,N}(G_N \geq t) \leq \exp\big(-c(R)t^{\frac{2\alpha(\alpha+1)}{4-4\alpha^2+3\alpha}}\big). \tag{5.30}$$

Note that

$$\frac{2\alpha(\alpha+1)}{4-4\alpha^2+3\alpha} > 1 \quad \text{for } \alpha > \frac{1+\sqrt{97}}{12}. \tag{5.31}$$

Note also that, in order to use Lemma 5.4, we must have $2-2\alpha < \alpha - 1/2$, namely, $\alpha > 5/6$. This is compatible with (5.31).

When $\alpha \neq 1$, $t \leq N^{1-\alpha}$. We set $T := \lfloor t^{\frac{1}{1-\alpha}} \rfloor$. We bound

$$\widetilde{\gamma}_{\alpha,N}(|G_N| \geq t) \leq \widetilde{\gamma}_{\alpha,N}(|G_T| \geq t/2) + \widetilde{\gamma}_{\alpha,N}(|G_N - G_T| \geq t/2). \tag{5.32}$$

Since $t \geq T^{1-\alpha}$, the first term can be estimated again by Lemma 5.5

$$\widetilde{\gamma}_{\alpha,N}(|G_T| \geq t/2) \lesssim \exp\big(-c(R)t^{\frac{2\alpha(\alpha+1)}{4-4\alpha^2+3\alpha}}\big). \tag{5.33}$$

For the second summand of (5.32), we observe that $N > T$, so Proposition 5.7 gives

$$\widetilde{\gamma}_{\alpha,N}(|G_N - G_T| \geq t/2) \leq \gamma_{\alpha,N}(|G_N - G_T| \geq t/2) \leq Ce^{-ct^{\frac{1}{3}}T^{\frac{2\alpha-1}{3}}} \leq Ce^{-ct^{\frac{1}{3}+\frac{2\alpha-1}{3(1-\alpha)}}}.$$

Combining that with (5.30), (5.32), and (5.33) gives (5.27) for $\alpha \neq 1$.

Finally, we take $\alpha = 1$. Consider $t \leq N^{\varepsilon}$ for some $\varepsilon > 0$, set $T := \lfloor t^{\frac{1}{\varepsilon}} \rfloor$. We bound again as in (5.32). Since $t \geq T^{\varepsilon}$, we have

$$\widetilde{\gamma}_{\alpha,N}(|G_T| \geq t/2) \lesssim 2\exp(-c(R)t^{\frac{4}{3}}). \tag{5.34}$$

Since $N > T$, by Proposition 5.7, we get

$$\widetilde{\gamma}_{\alpha,N}(|G_N - G_T| \geq t/2) \leq \gamma_{\alpha,N}(|G_N - G_T| \geq t/2) \leq Ce^{-ct^{\frac{1}{3}}T^{\frac{2\alpha-1}{3}}} \leq Ce^{-c(R)t^{\frac{1}{3}+\frac{1}{3\varepsilon}}}.$$

Combining that with equations (5.28), (5.34), and taking ε small enough gives (5.27) for $\alpha = 1$. ∎

The next result also follows by the considerations in this section.

Proposition 5.8. *Let $\alpha > 3/4$ and $\lambda \in \mathbb{R}$. The quantity $E_\alpha[1_{\{\|\Pi_N u\|_{L^2} \leq R\}} e^{\lambda \|\Pi_N u\|_{L^4}^4}]$ is bounded uniformly in N.*

Proof. Set

$$L_N := \begin{cases} N^{1-\alpha} & \alpha \neq 1, \\ (\log N)^2 & \alpha = 1. \end{cases}$$

We write

$$\begin{aligned} E_\alpha[1_{\{\|\Pi_N u\|_{L^2} \leq R\}} e^{\lambda \|\Pi_N u\|_{L^4}^4}] &= \int_0^\infty dt e^{t\lambda} \widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|_{L^4}^4 \geq t) \\ &= 1 + \int_1^{L_{\alpha,N}} dt e^{t\lambda} \widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|_{L^4}^4 \geq t) \qquad (5.35) \\ &\quad + \int_{L_{\alpha,N}}^\infty dt e^{t\lambda} \widetilde{\gamma}_{\alpha,N}(\|\Pi_N u\|_{L^4}^4 \geq t). \qquad (5.36) \end{aligned}$$

By Lemma 5.5, we have

$$\begin{aligned} (5.36) &\leq \int_{L_{\alpha,N}}^\infty dt e^{t\lambda} \widetilde{\gamma}_{\alpha,N}(R^2 \|\Pi_N u\|_{FL^{0,1}}^2 \geq t) \\ &\leq 2\int_{L_{\alpha,N}}^\infty dt \exp(\lambda t - c(R)t^{1+\alpha}) < \infty. \end{aligned}$$

Set $T := \lfloor t^{\frac{1}{1-\alpha}} \rfloor$ for $\alpha \neq 1$ or $T := \lfloor t^{\frac{1}{\varepsilon}} \rfloor$ for some $\varepsilon > 0$ if $\alpha = 1$. We bound

$$\begin{aligned} (5.35) \leq 1 &+ \int_1^{L_{\alpha,N}} dt e^{\lambda t} \widetilde{\gamma}_{\alpha,N}(\|\Pi_T u\|_{L^4}^4 \geq t/2) \qquad (5.37) \\ &+ \int_1^{L_{\alpha,N}} dt e^{\lambda t} \widetilde{\gamma}_{\alpha,N}(|\|\Pi_N u\|_{L^4}^4 - \|\Pi_T u\|_{L^4}^4| \geq t/2). \qquad (5.38) \end{aligned}$$

Since $t \geq L_{\alpha,T}$ again by Lemma 5.5, we have

$$(5.37) \leq 1 + \int_1^{L_{\alpha,N}} dt \widetilde{\gamma}_{\alpha,N}(R^2 \|\Pi_N u\|^2_{FL^{0,1}} \geq t) \leq 1 + 2\int_1^{\infty} dt \exp(\lambda t - c(R)t^{1+\alpha}) < \infty.$$

It remains to bound (5.38). The same proof of Proposition 5.7 gives for $\alpha \neq 1$

$$(5.38) \leq C \int_1^{\infty} dt e^{\lambda t - t^{\frac{1}{3}} T^{\frac{2\alpha-1}{3}}} \leq C \int_1^{\infty} dt e^{\lambda t - t^{\frac{1}{3} + \frac{2\alpha-1}{3(1-\alpha)}}} < \infty$$

(since $\alpha > 3/4$) and for $\alpha = 1$

$$(5.38) \leq C \int_1^{\infty} dt e^{\lambda t - t^{\frac{1}{3}} T^{\frac{2\alpha-1}{3\varepsilon}}} \leq C \int_1^{\infty} dt e^{\lambda t - t^{\frac{1}{3} + \frac{2\alpha-1}{3\varepsilon}}} < \infty$$

for ε sufficiently small (these estimates are loose but sufficient to our end). This ends the proof. ∎

6. Quasi-invariance

The main result of this section is the following.

Proposition 6.1. *Let $\alpha \in (\overline{\alpha}, 1]$, $\overline{\alpha} := \frac{1+\sqrt{97}}{12} \sim 0.9$ and $t \in [-1, 1]$. There exists $\bar{f}(t,\cdot) \in L^p(\rho_\alpha)$ for all $p \geq 1$, such that for any measurable set A*

$$\rho_\alpha(\Phi_t(A)) = \int_A \bar{f}(t,u)\rho_\alpha(du).$$

In other words, we have

$$\bar{f}(t,\cdot) := \frac{d(\rho_\alpha \circ \Phi_t)}{d\rho_\alpha} \in L^p(\rho_\alpha)$$

for the Radon–Nikodim derivative $\frac{d(\rho_\alpha \circ \Phi_t)}{d\rho_\alpha}$ of $\rho_\alpha \circ \Phi_t$ with respect to ρ_α.

Let

$$\rho_{\alpha,N}(du) := e^{-\frac{\sigma}{2}\|\Pi_N u\|^4_{L^4}} \widetilde{\gamma}_{\alpha,N}(du).$$

The convergence $\rho_{\alpha,N} \to \rho_\alpha$ was proven in [8, 26]. In particular, we have that for any measurable set A for every $\varepsilon > 0$ there is $\overline{N} \in \mathbb{N}$ such that for all $N > \overline{N}$

$$|\rho_\alpha(A) - \rho_{\alpha,N}(A)| < \varepsilon. \tag{6.1}$$

We also set $\gamma_N^{\perp}$ the Gaussian measure induced by the random Fourier series

$$\sum_{|n|>N} \frac{g_n}{(1+|n|^2)^{\frac{1}{2}}} e^{\mathrm{inx}}.$$

We define the Lebesgue measure on $\mathbb{C}^N \simeq \mathbb{R}^{2N}$ as

$$L_N(d\Pi_N u) = \prod_{|n|\le N} d(\operatorname{Re} u(n))d(\operatorname{Im} u(n)),$$

using the standard isomorphism between u and its Fourier coefficients. Since the flow is Hamiltonian, we have $L_N(\Pi_N \Phi_1^N(E)) = L_N(E)$.

We start by proving the quasi-invariance of ρ_α with respect to the truncated flow, which is defined for all t.

Proposition 6.2. *Let $\alpha \in (\overline{\alpha}, 1]$, $\overline{\alpha} := \frac{1+\sqrt{97}}{12} \sim 0.9$ and recall $\zeta(\alpha) > 1$ defined in (5.1). We have*

$$\rho_\alpha(\Phi_t^N(A)) \lesssim (\rho_\alpha(A))^{1-\varepsilon} \exp(C(R,\varepsilon)(1+|t|^{\frac{\zeta(\alpha)}{\zeta(\alpha)-1}})) \tag{6.2}$$

for all $N \in \mathbb{N}$ and $\varepsilon > 0$.

Proof of Proposition 6.2. We compute

$$\begin{aligned}
\rho_{\alpha,N}(\Phi_t^N(A)) &= \int_{\Phi_t^N(A)} e^{-\frac{\sigma}{2}\|\Pi_N u\|_{L^4}^4} \widetilde{\gamma}_\alpha(du) \\
&= \int_{\Phi_t^N(A)\cap B(R)} L_N(d\Pi_N u)\gamma_{\alpha,N}^{\perp}(dP_{>N}u) \exp(-H^{(\alpha)}[\Pi_N u]) \\
&= \int_{A\cap B(R)} L_N(d\Pi_N u)\gamma_{\alpha,N}^{\perp}(dP_{>N}u) \exp(-H^{(\alpha)}[\Pi_N \Phi_t^N(u)]).
\end{aligned}$$

Taking the derivative in time and evaluating it at $t = 0$, we get

$$\begin{aligned}
&\frac{d}{dt}\rho_{\alpha,N}(\Phi_t^N(A))\Big|_{t=0} \\
&= -\int_{A\cap B(R)} L_N(d\Pi_N u)\gamma_{\alpha,N}^{\perp}(dP_{>N}u) \exp(-H^{(\alpha)}[\Pi_N u])\Big(\frac{d}{dt}H^{(\alpha)}[\Pi_N \Phi_t^N(u)]_{t=0}\Big) \\
&= -\int_A \rho_{\alpha,N}(du)G_N(u),
\end{aligned} \tag{6.3}$$

where (recall (5.3) and (5.19))

$$G_N(u) = -\frac{\sigma}{2}\|\Pi_N u\|_{L^4}^4 + \sigma\|\Pi_N u\|_{L^2}^4 + \frac{\sigma}{2}\{\|\Pi_N u\|_{L^4}^4, \mathcal{F}_N\}.$$

Since $t \in (\mathbb{R},+) \to \Phi_t^N$ is a one parameter group of transformations, we can easily check that

$$\frac{d}{dt}(\rho_{\alpha,N} \circ \Phi_t^N(A))\Big|_{t=t^*} = \frac{d}{dt}(\rho_{\alpha,N} \circ \Phi_t^N(\Phi_{t^*}^N(A)))\Big|_{t=0}. \tag{6.4}$$

Using (6.3) and (6.4) under the choice $E = \Phi_t^N A$, we arrive at

$$\frac{d}{dt}(\rho_{\alpha,N} \circ \Phi_t^N(A))\Big|_{t=t^*} = -\int_{\Phi_{t^*}^N A} \rho_{\alpha,N}(du)G_N(u).$$

Thus, combining the Hölder inequality and (5.2), we get

$$\left|\frac{d}{dt}(\rho_{\alpha,N}\circ\Phi_t^N(A))\Big|_{t=t^*}\right| \leq \|G_N\|_{L^p}\rho_{\alpha,N}(\Phi_{t^*}^N(A))^{1-\frac{1}{p}} \lesssim_R \rho_{\alpha,N}(\Phi_{t^*}^N(A))^{1-\frac{1}{p}}\, p^{\frac{1}{\zeta(\alpha)}}, \tag{6.5}$$

whence

$$\frac{d}{dt}((\rho_{\alpha,N}\circ\Phi_t^N(A)))^{1/p} \lesssim_R p^{\frac{1}{\zeta(\alpha)}-1},$$

whence

$$\rho_{\alpha,N}(\Phi_t^N(A)) \leq \rho_{\alpha,N}(A)\exp(p\log(1+C(R)|t|p^{\frac{1}{\zeta(\alpha)}-1}(\rho_{\alpha,N}(A))^{-\frac{1}{p}})). \tag{6.6}$$

We can now show (6.2). We may and will assume $\rho(A)>0$. Consider

$$p=\log\Big(\frac{1}{2\rho(A)}\Big), \tag{6.7}$$

and note that

$$(2\rho(A))^{-\frac{1}{p}}=e. \tag{6.8}$$

By (6.1), we have that there is $\bar{N}=\bar{N}(A)$ such that $\rho_{\alpha,N}(A)\leq 2\rho(A)$ for all $N>\bar{N}$. Thus, for sufficiently large N (6.6) reads

$$\begin{aligned}
\rho_{\alpha,N}(\Phi_t^N(A)) &\leq 2\rho_\alpha(A)\exp\big(p\log\big(1+C(R)|t|p^{\frac{1}{\zeta(\alpha)}-1}(2\rho_\alpha(A))^{-\frac{1}{p}}\big)\big)\\
&= 2\rho_\alpha(A)\exp\big(p\log\big(1+C(R)|t|p^{\frac{1}{\zeta(\alpha)}-1}\big)\big)\\
&\leq 2\rho_\alpha(A)\exp\big(C(R)|t|p^{\frac{1}{\zeta(\alpha)}}\big)\\
&\leq 2\rho_\alpha(A)\exp\bigg(C(R)|t|\bigg(\ln\Big(\frac{1}{2\rho(A)}\Big)\bigg)^{\frac{1}{\zeta(\alpha)}}\bigg)\\
&\leq 2\rho_\alpha(A)\exp\bigg(\varepsilon\bigg(\ln\Big(\frac{1}{2\rho(A)}\Big)\bigg)\bigg)\exp\big(C(R,\varepsilon)|t|^{\frac{\zeta(\alpha)}{\zeta(\alpha)-1}}\big)\\
&\lesssim \rho_\alpha(A)^{1-\varepsilon}\exp\big(C(R,\varepsilon)\big(1+|t|^{\frac{\zeta(\alpha)}{\zeta(\alpha)-1}}\big)\big)
\end{aligned}$$

for all $\varepsilon>0$, where we used (6.8) in the second line and the Young inequality in the penultimate line. Then, equation (6.2) follows by (6.1). ∎

For $\alpha\neq 1$, we extend the flow of the Birkhoff map globally in time using the Bourgain probabilistic argument [8].

Proposition 6.3. *Let $\alpha\in(\bar{\alpha},1)$, $\bar{\alpha}:=\frac{1+\sqrt{97}}{12}\sim 0.9$, $s<\alpha-\frac{1}{2}$ and $t\in[-1,1]$. Then, the Birkhoff flow map (2.6) is globally well defined for ρ_α-almost all initial data. Moreover, there exists $\gamma,c,C>0$ such that*

$$\sup_{t\in[-1,1]}\|\Phi_t(u)\|_{H^s}\leq K \tag{6.9}$$

and

$$\sup_{t\in[-1,1]} \|\Phi_t(u) - \Phi_t^N(u)\|_{H^{s'}} \lesssim KN^{-(s-s')}, \quad 0 \le s' < s \tag{6.10}$$

hold for all u *outside an exceptional set of* ρ_α*-measure* $\le CK^2 e^{-cK^\gamma}$.

Proof. Let $K > 0$. We partition $[0, 1]$ into J intervals of size at most

$$\tau_K := \frac{\mathsf{c}_\alpha}{K^2+1},$$

where c_α is given by Lemma 4.2. Clearly,

$$J \le \mathsf{c}_\alpha^{-1}(K^2+1)+1. \tag{6.11}$$

We take any $s > 2-2\alpha$ (note that $s < \alpha - 1/2$ for our choice of α) and set

$$\begin{aligned} E_{K,N} := &\left\{u \notin B_s(K/2)\right\} \cup \left\{u \notin \Phi^N_{-\tau_K}(B_s(K/2))\right\} \cup \left\{u \notin \Phi^N_{-2\tau_K}(B_s(K/2))\right\} \\ &\cdots \cup \left\{u \notin \Phi^N_{-(J-1)\tau_K}(B_s(K/2))\right\} \cup \left\{u \notin \Phi^N_{-J\tau_K}(B_s(K/2))\right\}. \end{aligned}$$

We will show that the ρ_α measure of these sets vanishes in the limit $K \to \infty$ (and $\tau_K \to 0$).

First, we record for later use that since the density of ρ_α is in $L^2(\widetilde{\gamma}_\alpha)$ (the proof is done by an elementary adaption of [8, Lemma 3.1]) we can use the Cauchy–Schwarz inequality to obtain

$$\rho_\alpha(B_s(K/2)^C) \lesssim \sqrt{\widetilde{\gamma}_\alpha(\|u\|_{H^s} \ge K/2)} \lesssim e^{-cK^\gamma}, \quad \gamma > 2,$$

where we used Lemma 5.4 for the last bound.

Using (6.2), we have for some $\varepsilon > 0$

$$\begin{aligned} \rho_\alpha(E_{K,N}) &\le \sum_{j=0}^{J} \rho_\alpha(\Phi^N_{-\tau_K}(B_s(K/2))^C) \\ &\le \sum_{j=0}^{J} (\rho_\alpha(B_s(K/2))^C)^{1-\varepsilon} \lesssim_{R,\varepsilon} Je^{-cK^\gamma} \lesssim_{R,\varepsilon} K^2 e^{-cK^\gamma}, \end{aligned} \tag{6.12}$$

where we used (6.11) in the last inequality.

Let now $\{N_K\}_{K\in\mathbb{N}}$ be a diverging sequence and

$$E := \bigcap_{K\in\mathbb{N}} E_{K,N_K}.$$

Using (6.12) and Proposition 4.5, we will first show that the flow Φ_t is well defined for all $t \in [0, 1]$ and for all initial data in E_T^C. Then, we will show that $\rho(E_T) = 0$, concluding the proof of the statement (negative times are covered just by time reversibility).

Let us consider

$$\begin{aligned} E^C_{K,N} := &\left\{u \in B_s(K/2)\right\} \cap \left\{u \in \Phi^N_{-\tau_K}(B_s(K/2))\right\} \cap \left\{u \in \Phi^N_{-2\tau_K}(B_s(K/2))\right\} \\ &\cdots \cap \left\{u \in \Phi^N_{-(J-1)\tau_K}(B_s(K/2))\right\} \cap \left\{u \in \Phi^N_{-J\tau_K}(B_s(K/2))\right\}. \end{aligned}$$

Since

$$\Phi^N_{j\tau_K}(E^C_{K,N}) \subset B_s(K/2), \quad j = 0, \dots, J+1,$$

by the group property of the flow, we can apply Lemma 4.2 on each time interval $[j\tau_K, (j+1)\tau_K]$ so that we have

$$\sup_{t\in[0,1]} \sup_{N\in\mathbb{N}} \sup_{u\in E^C_{K,N}} \|\Phi^N_t(u)\|_{H^s} \le K. \tag{6.13}$$

The bound (6.13) can be extended to times $t \in [-1, 1]$ by the reversibility of the flow. Thus, for all $K > 1$, we can pick N_K sufficiently large and invoke Proposition 4.5, to show that Φ_t is well defined for times $t \in [-1, 1]$ and data in

$$E^C = \bigcup_{K\in\mathbb{N}} E_{K,N_K}.$$

On the other hand, by (6.12), we have

$$\rho_\alpha(E) = \rho_\alpha\Big(\bigcap_{K\in\mathbb{N}} E_{K,N_K,T}\Big) \le \lim_{K\to\infty} \rho_\alpha(E_{K,N_K,T}) = 0,$$

as claimed.

So, the flow Φ_t is defined for $t \in [-1, 1]$ for all data outside an exceptional sets $E_{K,N}$ of measure smaller than $CK^2e^{-cK^\gamma}$ (recall (6.12)). Therefore, (6.13) implies (4.8) by Proposition 4.5 and then (6.9) easily follows. Moreover, (6.9) implies the global approximation bound (6.10). Indeed thanks to (6.9) the local approximation bound (4.5) applies for all data outside the exceptional set starting by any initial time $t \in [-1, 1]$. This proves the (6.10) and concludes the proof. ■

Proof of Proposition 6.1. Proceeding exactly as in the proof of [19, Lemma 6.2], we obtain that, given any $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon, R) > 0$ such that for all A

$$\rho_{\alpha,N}(A) \le \delta \Longrightarrow \rho_{\alpha,N}(\Phi^N_t(A)) \le \varepsilon. \tag{6.14}$$

Next, we pass to the limit $N \to \infty$ in this inequality, showing the quasi invariance of the measure $\rho_{\alpha,N}$. It suffices to consider only compact sets $A \subset B_s(R)$. The argument then extends to Borel sets using the inner regularity of the Gaussian measure $\widetilde{\gamma}$ and the continuity of the flow map, that is, Lemma (4.3) (see [29, Lemma 8.1]).

Assume $\rho_\alpha(A) \le \delta/2$ with A compact. We have then for all sufficiently small δ' (that will depend on A and δ)

$$\rho_\alpha(A + B(\delta')) \le \delta.$$

Thus, by (6.14), we have

$$\rho_{\alpha,N}(\Phi^N_t(A + B(\delta'))) \le \varepsilon. \tag{6.15}$$

For N large enough, we have

$$\rho_\alpha(\Phi_t(A)) \le \rho_\alpha(\Phi^N_t(A + B(\delta'))) \le \rho_{\alpha,N}(\Phi^N_t(A + B(\delta'))) + \varepsilon \le 2\varepsilon,$$

where we used Lemma 4.8 in the first inequality, (6.1) in the second one and (6.15) in the last step. Thus, absolutely, continuity of $\rho \circ \Phi_t$ with respect to ρ is proved. ■

7. Density of the transported measure

In this section, we show that the density $\bar{f}$ of Proposition 6.1 can be obtained as a limit of finite dimensional approximations. We define the finite dimensional approximated densities as

$$f_N(s,u) := \exp\Big(- \int_0^s \frac{d}{d\tau}(H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)])d\tau \Big). \tag{7.1}$$

Following the notation $\Phi_\tau(u) := \Pi_\infty \Phi_\tau^\infty(u)$ used in the rest of the paper, we can write the limit density as

$$f_\infty(s,u) := \exp\Big(- \int_0^s \frac{d}{d\tau}(H^{(\alpha)}[\Phi_\tau(u)])d\tau \Big). \tag{7.2}$$

The main result of this section is the following.

Proposition 7.1. *Let $\alpha \in (\overline{\alpha}, 1]$, $\overline{\alpha} := \frac{1+\sqrt{97}}{12} \sim 0.9$. The sequence $\{f_N\}_{N\in\mathbb{N}}$ defined by (7.1) converges in $L^p(\rho_\alpha)$ to $f_\infty(s,u)$ and it holds $f_\infty(s,u) := \bar{f}(s,u)$ for ρ_α-almost all u, where $\bar{f}(s,\cdot)$ is the transported density from Proposition 6.1.*

Combining Propositions 6.1 and 7.1, we complete the proof of Theorem 1.1.

To prove Proposition 7.1, first we show that this sequence has a limit in $L^p(\rho_\alpha)$. This is a consequence of the following lemmas.

Lemma 7.2. *Let $\alpha \in (\overline{\alpha}, 1]$, $\overline{\alpha} := \frac{1+\sqrt{97}}{12} \sim 0.9$. We have for all $p \in [1,\infty)$*

$$\sup_{N\in\mathbb{N}} \|f_N\|_{L^p(\rho_\alpha)} < \infty. \tag{7.3}$$

Proof. Let $p \geq 1$. We write

$$\|f_N\|^p_{L^p(\rho_\alpha)} = \int_0^\infty dt e^\eta \rho_\alpha(f_N(s,u) \geq t^{1/p}),$$

and changing variables $t = e^\eta$, we can bound

$$\begin{aligned}
\|f_N\|^p_{L^p(\rho_\alpha)} &\leq e + \int_1^\infty d\eta e^\eta \rho_\alpha(f_N(s,u) \geq e^{\frac{\eta}{p}}) \\
&= e + \int_1^\infty d\eta e^\eta \rho_\alpha\Big(\int_0^s \frac{d}{d\tau}(H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)])d\tau \geq \frac{\eta}{p} \Big).
\end{aligned} \tag{7.4}$$

Now, we note

$$\begin{aligned}
\Big| \int_0^s \frac{d}{d\tau} H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)]d\tau \Big| &\leq s \max_{\tau\in[0,s]} \Big| \frac{d}{d\tau} H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)] \Big| \\
&=: s \Big| \frac{d}{d\tau} H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)] \Big|_{\tau=\tau^*} \Big|
\end{aligned}$$

for some $\tau^* \in [0, s]$. Therefore,

$$\rho_\alpha\left(\int_0^s \frac{d}{d\tau}(H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)])d\tau \geq \frac{\eta}{p}\right) \\ \leq \rho_\alpha\left(\left|\frac{d}{d\tau}H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)]\Big|_{\tau=\tau^*}\right| \geq \frac{\eta}{ps}\right). \tag{7.5}$$

Let

$$A = \left\{u : \left|\frac{d}{d\varepsilon}H^{(\alpha)}[\Pi_N \Phi_\varepsilon^N(u)]\Big|_{\varepsilon=0}\right| > \frac{\eta}{ps}\right\}.$$

Note that if $u \in A$ then $v = \Phi_{-\tau*}^N(u)$ satisfies

$$\frac{d}{d\tau}H^{(\alpha)}[\Pi_N \Phi_\tau^N(v)]\Big|_{\tau=\tau^*} = \lim_{\varepsilon\to 0} \varepsilon^{-1}\big(H^{(\alpha)}[\Pi_N \Phi_\varepsilon^N \Phi_{\tau^*}^N(v)] - H^{(\alpha)}[\Pi_N \Phi_{\tau^*}^N(v)]\big) \\ = \lim_{\varepsilon\to 0} \varepsilon^{-1}\big(H^{(\alpha)}[\Pi_N \Phi_\varepsilon^N(u)] - H^{(\alpha)}[\Pi_N u]\big) = \frac{d}{d\varepsilon}H^{(\alpha)}[\Pi_N \Phi_\varepsilon^N(u)]\Big|_{\varepsilon=0};$$

hence,

$$\Phi_{-\tau*}(A) = \left\{v : \left|\frac{d}{d\tau}H^{(\alpha)}[\Pi_N \Phi_\tau^N(v)]\Big|_{\tau=\tau^*}\right| \geq \frac{\eta}{ps}\right\}.$$

Thus, using Proposition 6.2, we can continue the estimate (7.5) as follows:

$$\rho_\alpha\left(\left|\frac{d}{d\tau}H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)]\Big|_{\tau=\tau^*}\right| \geq \frac{\eta}{ps}\right) \\ \lesssim_{\alpha,\varepsilon,R} \left(\rho_\alpha\left(\left|\frac{d}{d\varepsilon}H^{(\alpha)}[\Pi_N \Phi_\varepsilon^N(u)]\Big|_{\varepsilon=0}\right| \geq \frac{\eta}{ps}\right)\right)^{1-\varepsilon}$$

for all $\varepsilon > 0$. Using (5.27), we have

$$\left(\rho_\alpha\left(\left|\frac{d}{d\varepsilon}H^{(\alpha)}[\Pi_N \Phi_\varepsilon^N(u)]\Big|_{\varepsilon=0}\right| \geq \frac{\eta}{ps}\right)\right)^{1-\varepsilon} \lesssim_{s,\varepsilon} e^{-C(R)(\frac{\eta}{ps})^{\zeta(\alpha)}}.$$

Plugging this into (7.4), we have

$$\|f_N\|_{L^p(\rho_\alpha)}^p \leq e + \int_1^\infty d\eta e^{\eta - C(R)(\frac{\eta}{ps})^{\zeta(\alpha)}} \lesssim C(R, p, \alpha, s),$$

since we have $\zeta(\alpha) > 1$ (recall (5.1)). So, equation (7.3) follows. ∎

We will also need to show that the sequence $f_N(t,\cdot)$ converges in measure, for all $t \in [-1, 1]$ and, in particular, the limit is $f_\infty(t,\cdot)$.

Lemma 7.3. *Let $\alpha \in (\overline{\alpha}, 1)$, $\overline{\alpha} := \frac{1+\sqrt{97}}{12} \sim 0.9$. For all $t \in [-1, 1]$, we have that $f_N(t,\cdot) \to f_\infty(t,\cdot)$ in ρ_α-measure as $N \to \infty$.*

Proof. By the continuity of the exponential function, it is sufficient to show

$$\int_0^s \frac{d}{d\tau}(H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)])d\tau \to \int_0^s \frac{d}{d\tau}(H^{(\alpha)}[\Phi_\tau(u)])d\tau \tag{7.6}$$

in ρ_α-measure as $N \to \infty$.

Since

$$\begin{aligned}\left|\int_0^s \frac{d}{d\tau}(H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)])d\tau - \int_0^s \frac{d}{d\tau}(H^{(\alpha)}[\Phi_\tau(u)])d\tau\right| \\ \leq |s| \sup_{\tau\in[0,s]}\left|\frac{d}{d\tau}(H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)]) - \frac{d}{d\tau}(H^{(\alpha)}[\Phi_\tau(u)])\right|,\end{aligned}$$

we can deduce (7.6) from

$$\sup_{\tau\in[-1,1]}\left|\frac{d}{d\tau}(H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)]) - \frac{d}{d\tau}(H^{(\alpha)}[\Phi_\tau(u)])\right| \to 0$$

in ρ_α-measure as $N \to \infty$.

We now compute $\frac{d}{d\tau}H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)]$. One has

$$\begin{aligned}\frac{d}{d\tau}H^{(\alpha)}[\Pi_N \Phi_\tau^N(u)] &= \{H^{(\alpha)}, \mathcal{F}_N\}(\Phi_\tau^N(u)) \\ &\overset{(5.3)}{=} -\frac{\sigma}{2}\|\Pi_N \Phi_\tau^N(u)\|_{L^4}^4 + \sigma\|\Pi_N \Phi_\tau^N(u)\|_{L^2}^4 \\ &\quad + \frac{\sigma}{2}\{\|\Pi_N \Phi_\tau^N(u)\|_{L^4}^4, \mathcal{F}_N \circ \Phi_\tau^N\}.\end{aligned} \tag{7.7}$$

We will consider the three contributions to (7.7) separately. For the first one, we must estimate

$$|\|\Pi_N \Phi_\tau^N(u)\|_{L^4}^4 - \|\Pi_N \Phi_\tau(u)\|_{L^4}^4| = |L(g_N, \dots, g_N) - L(g, \dots, g)|, \tag{7.8}$$

where we have defined

$$L(h_1, h_2, h_3, h_4,) = \int h_1 h_2 \bar{h}_3 \bar{h}_4$$

and

$$g_N(\tau, u) = \Pi_N \Phi_\tau^N(u), \quad g = \Phi_\tau(u).$$

When it does not create confusion, we will abbreviate $g_N(\tau, u)$ to g_N in order to simplify the notations. We decompose telescopically

$$\begin{aligned}L(g_N, \dots, g_N) - L(g, \dots, g) = L(g_N, \dots, g_N) - L(g, g_N, \dots, g_N) \\ + L(g, g_N, \dots, g_N) - L(g, g, g_N, \dots, g) \\ + \cdots \\ + L(g, g, g, g_N) - L(g, \dots, g).\end{aligned}$$

We only show how to handle the first one, as the other ones require a similar procedure. We have

$$
\begin{aligned}
|L(g_N,\dots,g_N) - L(g,g_N,\dots,g_N)| &= \left| \int (g_N - g) g_N \bar{g}_N \bar{g}_N \right| \\
&\le \|g_N - g\|_{L^4} \|g_N\|_{L^4}^3 \\
&\lesssim \|g_N - g\|_{H^{\frac14}} \|g_N\|_{H^{\frac14}}^3 \\
&\le \|g_N - g\|_{H^{\frac14}} \|g_N\|_{H^s}^3,
\end{aligned}
\tag{7.9}
$$

where we used the Sobolev embedding. Here, we restrict to $\frac14 \le s < \alpha - \frac12$, coherently with Proposition 6.3. Note that for all $\alpha \in [\overline{\alpha}, 1]$ we have $\frac14 < \alpha - \frac12$, so the set of possible s is non empty. Taking $K = N^{\frac{4s-1}{32}}$ in (6.9) and (6.10) gives

$$
\sup_{t\in[-1,1]} (\|g(t,u)\|_{H^s} + \|g(t,u)\|_{H^s}) \le 2N^{\frac{4s-1}{32}}
$$

and

$$
\sup_{t\in[-1,1]} \|g(t,u) - g_N(t,u)\|_{H^{\frac14}} \lesssim N^{-\frac{7}{32}(4s-1)}, \quad s > \frac14
$$

for u outside an exceptional set of ρ_α-measure smaller than $CN^{\frac{4s-1}{16}} e^{-cN^{\gamma \frac{4s-1}{16}}}$. The two displays above combined with the (7.9) imply that

$$
(7.8) \le \frac{1}{N^{\frac{2s-1}{8}}}
$$

with probability at least $1 - CN^{\frac{4s-1}{16}} e^{-cN^{\gamma \frac{4s-1}{16}}}$, that is, it converges in measure to zero as $N \to \infty$.

The analysis of the second contribution is similar (actually easier since we simply need to control the L^2 norm of the evolution rather than the L^4).

For the last contribution, we must control $L(g_N,\dots,g_N) - L(g,\dots,g)$, where now we redefine

$$
\begin{aligned}
&L(h_1,h_2,h_3,h_4,h_5,h_6) \\
&\quad := \left\{ \|\Pi_N u\|_{L^4}^4, \mathcal{F}_N \right\} \\
&\quad \overset{(5.22)}{=} \sum_{\substack{|n_i|,|m_i|\le N,\\ \sum_{i=1}^3 n_i = \sum_{i=4}^6 n_i}} \mathsf{c}(n_1,n_2,n_3,n_4,n_5,n_6) h(n_1)h(n_2)h(n_3)\bar{h}(n_4)\bar{h}(n_5)\bar{h}(n_6)
\end{aligned}
$$

with $|\mathsf{c}(n_1,n_2,n_3,n_4,n_5,n_6)| \lesssim 1$ (recall (5.23)). We do the same decomposition as before (but of course in this case we have six differences to handle rather than four) and we explain how to estimate the first contribution:

$$
\begin{aligned}
&|L(g_N,\dots,g_N)-L(g,g_N,\dots,g_N)|\\
&\lesssim \sum_{\substack{|n_i|\le N,\\ \sum_{i=1}^3 n_i=\sum_{i=4}^6 n_i}} |g_N(n_1)-g(n_1)||g_N(n_2)||g_N(n_3)||g_N(n_4)||g_N(n_5)||g_N(n_6)|\\
&\lesssim \sum_{\substack{|n_i|\le N,\\ \sum_{i=1}^6 n_i=0}} |g_N(n_1)-g(n_1)||g_N(n_2)||g_N(n_3)||g_N(-n_4)||g_N(-n_5)||g_N(-n_6)|.
\end{aligned}
\tag{7.10}
$$

After spotting the convolution structure of (7.10) and recalling the inequality

$$
\|(a_1*a_2*a_3*a_4*a_5*a_6)_n\|_{\ell^\infty_n}\le\prod_{j^1}^{6}\|(a_j)_n\|_{\ell_n^{\frac{6}{5}}},
$$

we can further estimate

$$
(7.10)\le\|g_N-g\|_{FL^{0,\frac{6}{5}}}\|g\|^5_{FL^{0,\frac{6}{5}}}.
$$

Then, using the inequality (for sequencies) $\|a_n\|_{\ell_n^{\frac{6}{5}}}\le(\sum_{n\in\mathbb{Z}}\langle n\rangle^{2s}|a_n|^2)^{\frac{1}{2}}$ valid for $s>1/3$, we further estimate

$$
(7.10)\le\|g_N(\tau,u)-g(\tau,u)\|_{H^{1/3+}}\|g\|^5_{H^{1/3+}},
$$

where here we restrict to $\frac{1}{3}<s<\alpha-\frac{1}{2}$, coherently with Proposition 6.3. Note that for all $\alpha\in[\bar{\alpha},1]$ we have $\frac{1}{3}<\alpha-\frac{1}{2}$, so the set of possible s is non empty. From here, we can proceed exactly as before (from (7.9) onward) to show that also this last summand of (7.7) converges in measure to zero as $N\to\infty$. This implies the convergence in measure of $f_N(t,\cdot)$ to $f_\infty(t,\cdot)$ for all $t\in[-1,1]$, so the proof is concluded. ■

Proof of Proposition 7.1. By Lemmas 7.2 and 7.3, we obtain that, for all $p\ge1$, the sequence f_N converges in $L^p(\rho_\alpha)$. More precisely, the uniform $L^p(\rho_\alpha)$ bounds at a fixed p and the convergence in measure of the sequence guarantee the convergence in $L^{p'}(\rho_\alpha)$ for all $p'<p$ (see, for instance, [19, Lemma 3.7]) to a certain $L^{p'}(\rho_\alpha)$ function. Moreover, this limit must coincide ρ_α-a.s. with f_∞ by Lemma by 7.3. ■

Proof of Theorem 1.1. Once we have identified the $L^p(\rho_\alpha)$ limit f_∞, in order to complete the proof of Theorem 1.1, we need to show that $f_\infty=\bar{f}$ ρ_α-a.s., where we recall that $\bar{f}$ is the density of the transport of the measure ρ_α under the flow. The almost sure identity $f=\bar{f}$ follows by an abstract argument which can be adapted line by line from [19, Proposition 7.2]. ■

Acknowledgements. This work is dedicated to the memory of our colleague and dearest friend Thomas Kappeler. Discussing with him through the past years greatly inspired us to start the study of the invariance and quasi-invariance properties of Birkhoff maps in the context of dispersive PDEs, of which this paper is the first step. We thank Nikolay Tzvetkov for a stimulating discussion.

Funding. Renato Lucà was supported by the Basque Government under the (BCAM) BERC Programme 2022–2025 and by the Spanish Ministry of Science, Innovation and Universities through the (BCAM) Severo Ochoa accreditation CEX2021-001142-S and the project PID2021-123034NB-I00. Renato Lucà was also supported by the Ramon y Cajal fellowship RYC2021-031981-I. Riccardo Montalto was supported by the ERC Starting Grant "Hamiltonian Dynamics, Normal Forms and Water Waves" (HamDyWWa), project number: 101039762. Views and opinions expressed are however, those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them. Riccardo Montalto was also supported by INDAM-GNFM.

References

[1] D. Bambusi, Birkhoff normal form for some nonlinear PDEs. *Comm. Math. Phys.* **234** (2003), no. 2, 253–285 Zbl 1032.37051 MR 1962462

[2] D. Bambusi, J.-M. Delort, B. Grébert, and J. Szeftel, Almost global existence for Hamiltonian semilinear Klein–Gordon equations with small Cauchy data on Zoll manifolds. *Comm. Pure Appl. Math.* **60** (2007), no. 11, 1665–1690 Zbl 1170.35481 MR 2349351

[3] D. Bambusi, R. Feola, and R. Montalto, Almost global existence for some Hamiltonian PDEs with small Cauchy data on general tori. *Comm. Math. Phys.* **405** (2024), no. 1, article no. 15 Zbl 1537.35319 MR 4694422

[4] D. Bambusi and B. Grébert, Birkhoff normal form for partial differential equations with tame modulus. *Duke Math. J.* **135** (2006), no. 3, 507–567 Zbl 1110.37057 MR 2272975

[5] M. Berti and J. M. Delort, *Almost global existence of solutions for capillarity-gravity water waves equations with periodic spatial boundary conditions*. Springer, Cham, 2018

[6] M. Berti, R. Feola, and F. Pusateri, Birkhoff normal form and long time existence for periodic gravity water waves. *Comm. Pure Appl. Math.* **76** (2023), no. 7, 1416–1494 Zbl 1531.76016 MR 4588319

[7] M. Berti, A. Maspero, and F. Murgante, Hamiltonian Birkhoff normal form for gravity-capillary water waves with constant vorticity: almost global existence. *Ann. PDE* **10** (2024), no. 2, article no. 22 Zbl 07949347 MR 4803410

[8] J. Bourgain, Periodic nonlinear Schrödinger equation and invariant measures. *Comm. Math. Phys.* **166** (1994), no. 1, 1–26 Zbl 0822.35126 MR 1309539

[9] J. Bourgain, Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations. *Geom. Funct. Anal.* **6** (1996), no. 2, 201–230 Zbl 0872.35007 MR 1384610

[10] A. Debussche and Y. Tsutsumi, Quasi-invariance of Gaussian measures transported by the cubic NLS with third-order dispersion on **T**. *J. Funct. Anal.* **281** (2021), no. 3, article no. 109032 MR 4247030

[11] J.-M. Delort, A quasi-linear Birkhoff normal forms method. Application to the quasi-linear Klein–Gordon equation on $\mathbb{S}^1$. *Astérisque* (2012), no. 341, vi+113 Zbl 1243.35123 MR 2952065

[12] J.-M. Delort, Quasi-linear perturbations of Hamiltonian Klein–Gordon equations on spheres. *Mem. Amer. Math. Soc.* **234** (2015), no. 1103, vi+80 MR 3288855

[13] J.-M. Delort and R. Imekraz, Long-time existence for the semilinear Klein–Gordon equation on a compact boundary-less Riemannian manifold. *Comm. Partial Differential Equations* **42** (2017), no. 3, 388–416 Zbl 1365.37056 MR 3620892

[14] S. Demirbas, M. B. Erdoğan, and N. Tzirakis, Existence and uniqueness theory for the fractional Schrödinger equation on the torus. In *Some topics in harmonic analysis and applications*, pp. 145–162, Adv. Lect. Math. (ALM) 34, International Press, Somerville, MA, 2016 MR 3525558

[15] R. Feola and R. Montalto, Quadratic lifespan and growth of Sobolev norms for derivative Schrödinger equations on generic tori. *J. Differential Equations* **312** (2022), 276–316 Zbl 1483.35209 MR 4358597

[16] J. Forlano and K. Seong, Transport of Gaussian measures under the flow of one-dimensional fractional nonlinear Schrödinger equations. *Comm. Partial Differential Equations* **47** (2022), no. 6, 1296–1337 Zbl 1490.35414 MR 4432957

[17] J. Forlano and L. Tolomeo, Quasi-invariance of Gaussian measures of negative regularity for fractional nonlinear Schrödinger equations. 2022, arXiv:2205.11453v1

[18] J. Forlano and W. J. Trenberth, On the transport of Gaussian measures under the one-dimensional fractional nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **36** (2019), no. 7, 1987–2025 Zbl 1427.35253 MR 4020531

[19] G. Genovese, R. Lucà, and N. Tzvetkov, Transport of Gaussian measures with exponential cut-off for Hamiltonian PDEs. *J. Anal. Math.* **150** (2023), no. 2, 737–787 Zbl 1529.35438 MR 4645055

[20] P. Gérard and T. Kappeler, On the integrability of the Benjamin–Ono equation on the torus. *Comm. Pure Appl. Math.* **74** (2021), no. 8, 1685–1747 Zbl 1452.37070 MR 4275336

[21] P. Gérard, T. Kappeler, and P. Topalov, Sharp well-posedness results of the Benjamin–Ono equation in $H^s(\mathbb{T}, \mathbb{R})$ and qualitative properties of its solutions. *Acta Math.* **231** (2023), no. 1, 31–88 Zbl 1533.35061 MR 4652410

[22] B. Grébert and T. Kappeler, *The defocusing NLS equation and its normal form*. EMS Ser. Lect. Math., European Mathematical Society (EMS), Zürich, 2014 Zbl 1298.35002 MR 3203027

[23] A. D. Ionescu and F. Pusateri, Long-time existence for multi-dimensional periodic water waves. *Geom. Funct. Anal.* **29** (2019), no. 3, 811–870 Zbl 1420.35243 MR 3962880

[24] S. Janson, *Gaussian Hilbert spaces*. Cambridge Tracts in Math. 129, Cambridge University Press, Cambridge, 1997 Zbl 0887.60009 MR 1474726

[25] T. Kappeler and J. Pöschel, *KdV & KAM*. Ergeb. Math. Grenzgeb. (3) 45, Springer, Berlin, 2003 MR 1997070

[26] J. L. Lebowitz, H. A. Rose, and E. R. Speer, Statistical mechanics of the nonlinear Schrödinger equation. *J. Statist. Phys.* **50** (1988), no. 3-4, 657–687 Zbl 0925.35142 MR 0939505

[27] F. Planchon, N. Tzvetkov, and N. Visciglia, Modified energies for the periodic generalized KdV equation and applications. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **40** (2023), no. 4, 863–917 Zbl 1527.35360 MR 4620335

[28] G. Staffilani and B. Wilson, Stability of the cubic nonlinear Schrödinger equation on an irrational torus. *SIAM J. Math. Anal.* **52** (2020), no. 2, 1318–1342 Zbl 1434.37043 MR 4078804

[29] N. Tzvetkov, Quasiinvariant Gaussian measures for one-dimensional Hamiltonian partial differential equations. *Forum Math. Sigma* **3** (2015), article no. e28 MR 3482274

[30] N. Tzvetkov, New non degenerate invariant measures for the Benjamin–Ono equation. *C. R. Math. Acad. Sci. Paris* **362** (2024), 77–86 Zbl 1535.35165 MR 4703677

[31] R. Vershynin, *High-dimensional probability. An introduction with applications in data science, with a foreword by Sara van de Geer*. Camb. Ser. Stat. Probab. Math. 47, Cambridge University Press, Cambridge, 2018 MR 3837109

Received 20 December 2023; revised 1 October 2024.

Giuseppe Genovese
Department of Mathematics, University of British Columbia, 228-1984 Mathematics Road, Vancouver, BC, V6T 1Z2, Canada; giuseppe.genovese@math.ubc.ca

Renato Lucà
Institut Denis Poisson CNRS UMR 7013, Université d'Orléans, Rue de Chartres, 45100 Orléans, France; renato.luca@univ-orleans.fr

Riccardo Montalto
Dipartimento di Matematica Federigo Enriques, Universitá Statale di Milano, Via Saldini 50, 20133 Milan, Italy; riccardo.montalto@unimi.it