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Yi Xie · Boyu Zhang

Classification of links with Khovanov homology of minimal rank

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Abstract. If L is an oriented link with n components, then the rank of its Khovanov homology is at least 2^n. We classify all links that achieve this lower bound and show that such links can be obtained by iterated connected sums and disjoint unions of Hopf links and unknots. This gives a positive answer to a question asked by Batson and Seed (2015).

Keywords. Khovanov homology, instanton Floer homology, knot theory, low-dimensional topology, gauge theory

Contents

1. Introduction

Let L be an oriented link in S^3 and R be a ring. Khovanov homology [14] assigns a bi-graded R-module $\mathrm{Kh}(L;R)$ to the link L. When R is an integral domain, the Euler

Yi Xie: Beijing International Center for Mathematical Research, Peking University, Beijing 100871, P.R. China; yixie@pku.edu.cn

Boyu Zhang: Department of Mathematics, University of Maryland at College Park, College Park, MD 20742, USA; bzh@umd.edu

Mathematics Subject Classification (2020): 57K18 characteristics of $\mathrm{Kh}(L; R)$ are given by the coefficients of the unreduced Jones polynomial of L. If L has n components, then the value of the unreduced Jones polynomial at $t = 1$ equals $(-2)^n$. Therefore,

$$\mathrm{rank}_R \mathrm{Kh}(L; R) \geq 2^n. \tag{1.1}$$

The rank of $\mathrm{Kh}(L; R)$ is independent of the orientation of L. This paper classifies all links L such that equality is achieved in (1.1).

If L is the unlink, then $\mathrm{rank}_R \mathrm{Kh}(L; R) = 2^n$ and (1.1) attains equality. However, there are other examples where equality holds in (1.1). In graph theory, a finite simple graph is called a *forest* if it contains no cycles. Given a forest G, we define the link L_G by placing an unknot at each vertex of G and linking two unknots as a Hopf link whenever there is an edge connecting the corresponding vertices (see Figures 1–3 for examples). The link L_G is called the *forest of unknots* defined by G. By definition, every forest of unknots can be obtained by iterated connected sums and disjoint unions of Hopf links and unknots. By [1, Corollary 6.6], if L_G is a forest of unknots with n components, then $\mathrm{rank}_R \mathrm{Kh}(L_G; R) = 2^n$.

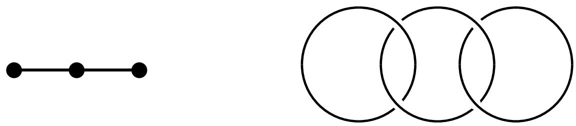

Fig. 1. Example of a forest of unknots.

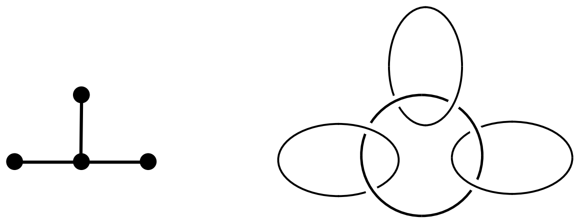

Fig. 2. Example of a forest of unknots.

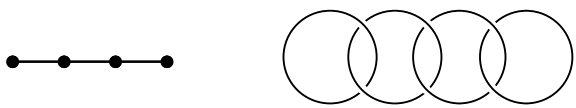

Fig. 3. Example of a forest of unknots.

The following question was asked by Batson and Seed:

Question 1.1 ([5, Question 7.2]). *Are forests of unknots the only n-component links with Khovanov homology of rank 2^n in $\mathbb{Z}/2$-coefficients?*

The main result of the paper gives an affirmative answer to the above question.

Theorem 1.2. *If L is an n-component link such that* $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L;\mathbb{Z}/2) = 2^n$, *then L is a forest of unknots.*

Remark 1.3. The special case of Theorem 1.2 when L is alternating was proved by Shumakovitch [30, Lemma 3.3.C].

The detection property of Khovanov homology has been a central question since the introduction of the theory. The first breakthrough in this field was the landmark paper by Kronheimer and Mrowka [20], which proved that Khovanov homology detects the unknot (see also [11,12]). Since then, several other detection results have been proved. It is known that Khovanov homology detects the unlink [5, 13], the trefoil [3] and the Hopf link [4].

Theorem 1.2 recovers all the previous detection results on links [4, 5, 13]. In fact, we have the following two corollaries of Theorem 1.2. They will be proved in Section 12.

Corollary 1.4. *Suppose L_1 and L_2 are two oriented links with n components and L_1 is a forest of unknots. If there exist $e, f \in \mathbb{Z}$ such that*

$$\operatorname{Kh}^{i,j}(L_1;\mathbb{Z}/2) \cong \operatorname{Kh}^{i+e,j+f}(L_2;\mathbb{Z}/2)$$

for all $i, j \in \mathbb{Z}$, then L_2 is isotopic to a forest of unknots whose graph has the same number of edges as the graph of L_1. In particular, if

$$\operatorname{Kh}^{i,j}(L_1;\mathbb{Z}/2) \cong \operatorname{Kh}^{i,j}(L_2;\mathbb{Z}/2)$$

for all $i, j \in \mathbb{Z}$, and if L_1 is the unlink, or a Hopf link, or the disjoint union of a Hopf link and the unlink, or a connected sum of two Hopf links, then L_2 is isotopic to L_1.

Given a link L with n components, one can equip $\operatorname{Kh}(L;\mathbb{Z}/2)$ with a module structure over the ring

$$R_n := (\mathbb{Z}/2)[X_1,\ldots,X_n]/(X_1^2,\ldots,X_n^2).$$

For the definition of the module structure, the reader may refer to [13, Section 2] and [15, Section 3].

Corollary 1.5. *Suppose L_1 and L_2 are two links with n components, and suppose L_1 is a forest of unknots with graph G_1. If* $\operatorname{Kh}(L_1;\mathbb{Z}/2)$ *is isomorphic to* $\operatorname{Kh}(L_2;\mathbb{Z}/2)$ *as R_n-modules, then L_2 is isotopic to a forest of unknots with graph G_2 such that*

(1) *there is a one-to-one correspondence between the connected components of G_1 and the connected components of G_2;*

(2) *the corresponding components of G_1 and G_2 have the same number of vertices.*

If we further assume the number of vertices of every connected component of G_1 is less than or equal to 3, *then L_2 is isotopic to L_1 as unoriented links.*

Remark 1.6. Suppose L_1 and L_2 are two forests of unknots which are associated to trees with the same number of vertices (see Figures 2 and 3 for examples). We may also orient L_1, L_2 so that the linking number of any two components is non-negative. Then $\operatorname{Kh}(L_1;\mathbb{Z}/2)$ and $\operatorname{Kh}(L_2;\mathbb{Z}/2)$ are isomorphic as bi-graded vector spaces according to

[1, Theorem 6.2]. Moreover, the module structures of $\mathrm{Kh}(L_1;\mathbb{Z}/2)$ and $\mathrm{Kh}(L_2;\mathbb{Z}/2)$ are also isomorphic according to Lemma 12.1.

The proof of Theorem 1.2 consists of five main steps. By Kronheimer–Mrowka's spectral sequence [20] and Batson–Seed's inequality, if L satisfies the condition of Theorem 1.2, then all the components of L are unknots. Using instanton Floer homology and various topological arguments, we establish the following results:

(1) The linking number of any two components of L is 0 or 1. This is proved in Section 6 using a braid-detection result for annular instanton homology by the authors [34, Corollary 8.4] together with an Alexander polynomial argument for exchangeably braided links.

(2) Let G be the graph such that each vertex of G corresponds to a component of L and there is an edge between two vertices of G if and only if the linking number of the corresponding components of L is non-zero. In Section 5, we show that if G is a forest, then L must be a forest of unknots. This uses the authors' previous result on the relationship between the generalized Thurston norm and singular instanton Floer homology [34, Theorem 8.2].

(3) If G is not a forest, we may assume without loss of generality that G is a cycle by passing to a sublink. By step (2), deleting a component of L gives a connected sum of Hopf links which is fibered. In Section 7.2, we show that the deleted component can be made disjoint from a fiber. This step uses the properties of instanton homology with local coefficients developed in Section 3, and it is the most difficult gauge-theoretic step in the proof.

(4) It can be shown that the removed component in step (3) is the boundary of a disk that intersects the fiber in a single arc. We explicitly determine the arc using results in Sections 8 and 9. This step does not use gauge theory but depends on an in-depth analysis of the fundamental group of the complement of the connected sum of Hopf links.

(5) The link is now known explicitly enough to be ruled out by hand using a computation in Section 10.

The method in step (1) above can also be used to prove other new detection results for links with small Khovanov homology. The proof of Theorem 1.7 below is given in Section 6.

Theorem 1.7. *Let L_1 be the oriented link given by Figure 4, and let L_2 be the disjoint union of a trefoil and an unknot. Let $L = K_1 \cup K_2$ be a 2-component oriented link.*

(1) *If*
$$\mathrm{Kh}(L;\mathbb{Z}/2) \cong \mathrm{Kh}(L_1;\mathbb{Z}/2)$$
as bi-graded vector spaces, then L is isotopic to L_1.

(2) *Let $q \in K_2$ and p be a basepoint on the unknotted component of L_2. If*
$$\mathrm{Khr}(L,q;\mathbb{Z}) \cong \mathrm{Khr}(L_2,p;\mathbb{Z})$$

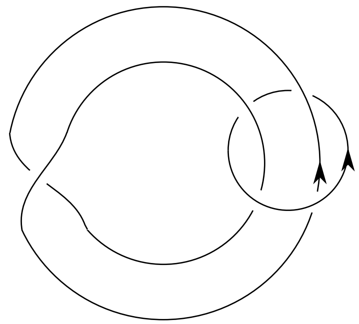

Fig. 4. The link L4a1 in the Thistlethwaite table with an orientation.

as bi-graded abelian groups, then L splits into the disjoint union of a trefoil K_1 and an unknot K_2.

Remark 1.8. By the trefoil detection result of Baldwin and Sivek [3], the essential content of part (2) of Theorem 1.7 is the splitness of L. Shortly after the first version of this paper was posted to arXiv, Lipshitz and Sarkar [24] proved that the splitness of a general link can be detected by the module structure of Khovanov homology.

The proof of Theorem 1.2 does not immediately apply to other coefficient rings because it relies on [30, Corollary 3.2.C], which only holds for $\mathbb{Z}/2$-coefficients. In Section 12, we will use a purely algebraic argument to extend Theorem 1.2 to arbitrary coefficient rings and prove the following theorem.

Theorem 1.9. *Suppose R is an integral domain. If L is an n-component link such that* $\operatorname{rank}_R \operatorname{Kh}(L; R) = 2^n$, *then L is a forest of unknots.*

2. Annular instanton Floer homology

The singular instanton Floer homology theory was introduced by Kronheimer and Mrowka [20,21]. Let (Y, L, ω) be a triple where Y is a closed oriented 3-manifold, $L \subset Y$ is a link and $\omega \subset Y$ is an embedded 1-manifold such that $\partial\omega = \omega \cap L$. The triple (Y, L, ω) is called *admissible* if there is an embedded closed surface $\Sigma \subset Y$ satisfying either one of the following conditions:

- Σ is disjoint from L and the intersection number of ω and Σ is odd,
- the intersection number of L and Σ is odd.

If (Y, L, ω) is admissible, the instanton Floer homology $\mathrm{I}(Y, L, \omega)$ is defined to be a Morse homology of the Chern–Simons functional on a certain space of orbifold SO(3)-connections over Y, where Y is equipped with an orbifold structure with cone angle π along L, and ω represents the second Stiefel–Whitney class of the SO(3)-bundle. In this article, we will always take $\mathbb{C}$-coefficients for instanton Floer homology.

The homology group $\mathrm{I}(Y, L, \omega)$ carries a relative $\mathbb{Z}/4$-homological grading. Given an embedded closed surface $F \subset Y$, there is an operator $\mu^{\mathrm{orb}}(F)$ defined on $\mathrm{I}(Y, L, \omega)$ with degree 2. For more details the reader may refer to, for example, [34, Section 2] or [31, Section 2.3.2].

The rest of this section gives a brief review of the annular instanton Floer homology introduced in [33]. Let L be a link in the solid torus $S^1 \times D^2$. The annular instanton Floer homology $\mathrm{AHI}(L)$ is defined by the following procedure:

(1) Let $\mathcal{K}_2$ be the product link $S^1 \times \{p_1, p_2\}$ in $S^1 \times D^2$, and let u be an arc in $S^1 \times D^2$ connecting $S^1 \times \{p_1\}$ and $S^1 \times \{p_2\}$.

(2) Form the new link $L \cup \mathcal{K}_2$ in

$$S^1 \times S^2 = S^1 \times D^2 \cup_{S^1 \times S^1} S^1 \times D^2,$$

where L lies in the first copy of $S^1 \times D^2$, and $\mathcal{K}_2$ lies in the second copy.

(3) Define

$$\mathrm{AHI}(L) := \mathrm{I}(S^1 \times S^2, L \cup \mathcal{K}_2, u).$$

The vector space $\mathrm{AHI}(L)$ is equipped with an absolute $\mathbb{Z}$-grading (called the *f-grading*). By definition, the component of $\mathrm{AHI}(L)$ with f-degree i is given by the generalized eigenspace of $\mu^{\mathrm{orb}}(S^2)$ for the eigenvalue i, and is denoted by $\mathrm{AHI}(L, i)$. Since $\mu^{\mathrm{orb}}(S^2)$ has degree 2 with respect to the $\mathbb{Z}/4$-homological grading of $\mathrm{AHI}(L)$, the subspace $\mathrm{AHI}(L, i)$ carries a $\mathbb{Z}/2$-homological grading, and we have

$$\mathrm{AHI}(L, i) \cong \mathrm{AHI}(L, -i). \tag{2.1}$$

There is a product formula for split links in $S^1 \times D^2$.

Proposition 2.1 ([33, Proposition 4.3]). *Suppose L_1 and L_2 are two links in $S^1 \times D_1$ and $S^1 \times D_2$ respectively, where D_1 and D_2 are disjoint subdisks of D^2. Then*

$$\mathrm{AHI}(L_1 \cup L_2) \cong \mathrm{AHI}(L_1) \otimes \mathrm{AHI}(L_2).$$

Moreover, the isomorphism above preserves the f-gradings.

In the following, we will use $\mathcal{U}_n$ to denote the unlink with n components in $S^1 \times D^2$, and use $\mathcal{K}_n$ to denote the closure of the trivial braid with n strands in $S^1 \times D^2$. We will use $\mathcal{U}_k \cup \mathcal{K}_l$ to denote the union of $\mathcal{U}_k$ and $\mathcal{K}_l$ such that $\mathcal{U}_k$ is included in a solid 3-ball disjoint from $\mathcal{K}_l$.

Example 2.2 ([33, Example 4.2]). The critical set of the unperturbed Chern–Simons functional for $\mathrm{AHI}(\mathcal{U}_1)$ is diffeomorphic to S^2, and after perturbation the critical set consists of two points whose homological degrees differ by 2. Therefore there is no differential, and we have

$$\mathrm{AHI}(\mathcal{U}_1) \cong \mathbb{C} \oplus \mathbb{C}.$$

The vector space $\mathrm{AHI}(\mathcal{U}_1)$ is supported in f-grading 0.

The critical set for $\mathrm{AHI}(\mathcal{K}_1)$ consists of two points whose homological degrees differ by 2, so there is no differential and

$$\mathrm{AHI}(\mathcal{K}_1) \cong \mathbb{C} \oplus \mathbb{C}.$$

The vector space $\mathrm{AHI}(\mathcal{K}_1)$ is supported in f-gradings ± 1.

By Proposition 2.1, we have

$$\mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l) \cong \mathrm{AHI}(\mathcal{U}_1)^{\otimes k} \otimes \mathrm{AHI}(\mathcal{K}_1)^{\otimes l},$$

and the above isomorphism preserves the f-gradings.

We also have $\mathrm{AHI}(\emptyset) \cong \mathbb{C}$, because the critical set consists of a single point.

Definition 2.3. A properly embedded, connected, oriented surface $S \subset S^1 \times D^2$ is called a *meridional surface* if ∂S is a meridian of $S^1 \times D^2$.

The annular instanton Floer homology detects the generalized Thurston norm of meridional surfaces.

Theorem 2.4 ([34, Theorem 8.2]). *Fix a link L in $S^1 \times D^2$ and suppose S is a meridional surface that intersects L transversely. Let g be the genus of S and let $n := |S \cap L|$. If S minimizes the value of $2g + n$ among meridional surfaces, then*

$$\begin{aligned} &\mathrm{AHI}(L, i) = 0 \quad \textit{for all } |i| > 2g + n, \\ &\mathrm{AHI}(L, \pm(2g + n)) \neq 0. \end{aligned}$$

We also need the following result.

Proposition 2.5 ([34, Corollary 8.4]). *Let L be a link in $S^1 \times D^2$. Then L is isotopic to the closure of a braid with n strands if and only if the top f-grading of $\mathrm{AHI}(L)$ is n and $\mathrm{AHI}(L, n) \cong \mathbb{C}$.*

Although annular instanton Floer homology is defined for links in the solid torus, it can be used to study links in S^3. Let L be a link in S^3 and let p be a basepoint on L. In [20], Kronheimer and Mrowka defined the link invariant

$$\mathrm{I}^\natural(L, p) := \mathrm{I}(S^3, L \cup m, u),$$

where m is a small meridian of L around p and u is an arc joining m and p. The following result is a consequence of the excision property of instanton Floer homology.

Proposition 2.6 ([33, Section 4.3]). *Suppose L has an unknotted component U and let $p \in U$. Let $N(U)$ be a tubular neighborhood of U. Then $L_0 := L - U$ is a link in the solid torus $S^3 - N(U)$, and*

$$\mathrm{AHI}(L_0) \cong \mathrm{I}^\natural(L, p). \tag{2.2}$$

The above isomorphism does not preserve the f-grading of $\mathrm{AHI}(L_0)$ since there is no such grading on $\mathrm{I}^\natural(L, p)$. Notice that a meridional surface in the solid torus $S^3 - N(U)$ is a Seifert surface of U.

3. Local coefficients

This section reviews the singular instanton Floer homology theory with local coefficients, which was introduced in [21, Section 3.9] (see also [22, Section 3]). Let $\mathcal{B}(Y, L, \omega)$ be the space of gauge-equivalence classes of orbifold connections over (Y, L, ω). Let $\mathcal{R}$ be the ring

$$\mathcal{R} := \mathbb{C}[t, t^{-1}],$$

and suppose

$$\mu : \mathcal{B}(Y, L, \omega) \to \mathbb{R}/\mathbb{Z}$$

is a continuous function. For each $a \in \mathcal{B}(Y, L, \omega)$, define a rank-1 free $\mathcal{R}$-module by the formal multiplication

$$\Gamma_a^\mu := t^{\tilde{\mu}(a)} \cdot \mathcal{R},$$

where $\tilde{\mu}(a)$ is a lift of $\mu(a)$ in $\mathbb{R}$. Let $\mathrm{Crit}(CS)$ be the set of critical points of the (perturbed) Chern–Simons functional CS, and define a free $\mathcal{R}$-module $\mathbf{C}^\mu$ by

$$\mathbf{C}^\mu := \bigoplus_{\alpha \in \mathrm{Crit}(CS)} \Gamma_\alpha^\mu.$$

To make $\mathbf{C}^\mu$ a chain complex, we need to define a differential on it. For $\alpha, \beta \in \mathrm{Crit}(CS)$, let $M_d(\alpha, \beta)$ be the d-dimensional moduli space of trajectories of CS from α to β. This space carries an $\mathbb{R}$-action and we denote the quotient space by

$$\breve{M}_d(\alpha, \beta) := M_d(\alpha, \beta)/\mathbb{R}.$$

A trajectory $z \in M_d(\alpha, \beta)$ determines a path p_z in $\mathcal{B}(Y, L, \omega)$ from α to β. The map

$$\mu \circ p_z : [0, 1] \to \mathbb{R}/\mathbb{Z}$$

can be lifted to a map

$$\widetilde{\mu \circ p_z} : [0, 1] \to \mathbb{R}.$$

Although $\widetilde{\mu \circ p_z}$ is not unique, the difference

$$\nu(z) := \widetilde{\mu \circ p_z}(1) - \widetilde{\mu \circ p_z}(0)$$

is well-defined. We define an $\mathcal{R}$-module homomorphism by

$$d_\beta^\alpha : \Gamma_\alpha^\mu \to \Gamma_\beta^\mu, \quad t^s \mapsto \sum_{[z] \in \breve{M}_1(\alpha,\beta)} \mathrm{sign(z)} \cdot t^{s+\nu(z)}.$$

The differential D on $\mathbf{C}^\mu$ is then given by

$$D := \bigoplus_{\alpha,\beta \in \mathrm{Crit}(CS)} d_\beta^\alpha,$$

and the instanton Floer homology with local coefficients is defined by

$$\mathrm{I}(Y, L, \omega; \Gamma^\mu) := H^*(\mathbf{C}^\mu, D). \tag{3.1}$$

If $F \subset Y$ is an embedded closed surface, the operator $\mu^{\mathrm{orb}}(F)$ can be defined in the setting with local coefficients. Roughly speaking, the surface F defines a two-dimensional cohomology class on $\mathcal{B}(Y, L, \omega)$ (see [34, formula (10)]), and its Poincaré dual is given by a linear combination of divisors on $\mathcal{B}(Y, L, \omega)$ as

$$\sum a_i V_i, \quad a_i \in \mathbb{Q}.$$

There is a map from $M_d(\alpha, \beta)$ to $\mathcal{B}(Y, L, \omega)$ by restricting the trajectories at time 0. The divisors V_i can be chosen to be generic in the sense that they are transverse to the restriction map $M_d(\alpha, \beta) \to \mathcal{B}(Y, L, \omega)$ for all $\alpha, \beta \in \mathrm{Crit}(CS)$ and $d \in \mathbb{N}$. We define an $\mathcal{R}$-module homomorphism by

$$f^{\alpha}_{\beta} : \Gamma^{\mu}_{\alpha} \to \Gamma^{\mu}_{\beta}, \quad t^s \mapsto \sum_i a_i \sum_{z \in M_2(\alpha,\beta) \cap V_i} \mathrm{sign}(z) \cdot t^{s+\nu(z)}.$$

A standard argument shows that the map

$$\bigoplus_{\alpha,\beta \in \mathrm{Crit}(CS)} f^{\alpha}_{\beta} : \mathbf{C}^{\mu} \to \mathbf{C}^{\mu} \tag{3.2}$$

is a chain map. The map (3.2) induces the operator $\mu^{\mathrm{orb}}(F)$ on $\mathrm{I}(Y, L, \omega; \Gamma^{\mu})$. When it does not cause confusion, we will also use $\mu^{\mathrm{orb}}(F)$ to denote the chain map (3.2) by abuse of notation.

The tensor products

$$\big(\mathbf{C}^{\mu} \otimes_{\mathcal{R}} \mathcal{R}/(t-1), D \otimes_{\mathcal{R}} \mathcal{R}/(t-1)\big), \quad \mu^{\mathrm{orb}}(F) \otimes_{\mathcal{R}} \mathcal{R}/(t-1)$$

recover the ordinary Floer chain complex (C, d) and the chain map defining the ordinary operator $\mu^{\mathrm{orb}}(F)$ on $\mathrm{I}(Y, L, \omega)$ with $\mathbb{C}$-coefficients.

Suppose there is a component $K \subset L$ such that $K \cap \omega = \emptyset$. Fix an orientation and a framing of K. We can define a continuous map

$$\mu_K : \mathcal{B}(Y, L, \omega) \to U(1) = \mathbb{R}/\mathbb{Z}$$

by taking the limit holonomy of the orbifold connections along the longitude of K. The map μ_K then gives a local system. The local systems defined by different framings of K are isomorphic via multiplications by powers of t, therefore the choice of the framing is not important. More generally, suppose there is a sublink $L' = K_1 \cup \cdots \cup K_l$ of L such that $\omega \cap L' = \emptyset$. We can choose a framing for each K_j and define the map μ_{K_j} as above, and hence we obtain a local system Γ associated with L' defined by

$$\mu_{L'} := \mu_{K_1} \cdots \mu_{K_l}.$$

If L' is the empty link, then $\mu_{L'} = 1$, thus the local system Γ is the trivial system with $\mathcal{R}$-coefficients. In this case, we have

$$\mathrm{I}(Y, L, \omega; \Gamma) = \mathrm{I}(Y, L, \omega) \otimes_{\mathbb{C}} \mathcal{R}. \tag{3.3}$$

Suppose (Y_0, L_0, ω_0) and (Y_1, L_1, ω_1) are two admissible triples with local systems Γ_0 and Γ_1 associated with oriented sublinks $L_0' \subset L_0$ and $L_1' \subset L_1$ respectively. Let

$$(W, S, \eta) = (W, S' \sqcup S'', \eta) : (Y_0, L_0, \omega_0) \to (Y_1, L_1, \omega_1)$$

be a cobordism such that $\partial S' = L_0' \cup L_1'$ and $\eta \cap S' = \emptyset$. Then (W, S, η) induces a map

$$\mathrm{I}(W, S, \eta) : \mathrm{I}(Y_0, L_0, \omega_0; \Gamma_0) \to \mathrm{I}(Y_1, L_1, \omega_1; \Gamma_1).$$

This makes the instanton Floer homology with local coefficients a functor. By the definition of cobordism of triples, S and η are required to be embedded surfaces in W. We can also consider the situation where S is an immersed surface with transverse double points, as discussed in [21, Section 5] and [18]. In this situation, one can blow up W at the self-intersection points of S to resolve the double points and obtain an ordinary cobordism $(\widetilde{W}, \widetilde{S}, \eta)$, and then define

$$\mathrm{I}(W, S, \eta) := \mathrm{I}(\widetilde{W}, \widetilde{S}, \eta).$$

Now suppose $S = S' \sqcup S''$ and $\hat{S} = \hat{S}' \sqcup \hat{S}''$ are two immersed surfaces with transverse double points in W such that $\eta \cap S' = \hat{\eta} \cap \hat{S}' = \emptyset$, $\eta = \hat{\eta}$, $\partial S = \partial \hat{S} = L_0 \cup L_1$, and $\partial S' = \partial \hat{S}' = L_0' \cup L_1'$. We consider the following five situations:

(i) $\hat{S}$ is obtained from S by an ambient isotopy;

(ii) $\hat{S}'' = S''$, and $\hat{S}'$ is obtained from S' by a twist move introducing a positive double point (see [10, Section 1.3] for the definition of twist move);

(iii) $\hat{S}'' = S''$, and $\hat{S}'$ is obtained from S' by a twist move introducing a negative double point;

(iv) $\hat{S}'' = S''$, and $\hat{S}'$ is obtained from S' by a finger move introducing two double points of opposite signs (see Figure 5 for a schematic picture and see [10, Section 1.5] for the precise definition of finger move);

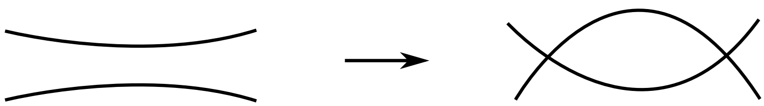

Fig. 5. A schematic picture of the finger move.

(v) $\hat{S}$ is obtained from S by a finger move introducing two double points of opposite signs in $\hat{S}' \cap \hat{S}''$.

All the moves and isotopies above are assumed to be supported outside a neighborhood of $\eta = \hat{\eta}$.

Proposition 3.1. *Let $(Y_0, L_0, \omega_0), (Y_1, L_1, \omega_1), W, S, \hat{S}, \eta, \Gamma_0, \Gamma_1$ be as above, and let*

$$\mathrm{I}(W, S, \eta) : \mathrm{I}(Y_0, L_0, \omega_0; \Gamma_0) \to \mathrm{I}(Y_1, L_1, \omega_1; \Gamma_1),$$
$$\mathrm{I}(W, S', \eta) : \mathrm{I}(Y_0, L_0, \omega_0; \Gamma_0) \to \mathrm{I}(Y_1, L_1, \omega_1; \Gamma_1)$$

be the induced cobordism maps. For the five cases listed above, the following equations hold respectively:

(i) $\mathrm{I}(W, \hat{S}, \eta) = \mathrm{I}(W, S, \eta)$;

(ii) $\mathrm{I}(W, \hat{S}, \eta) = (1 - t^2)\,\mathrm{I}(W, S, \eta)$;

(iii) $\mathrm{I}(W, \hat{S}, \eta) = \mathrm{I}(W, S, \eta)$;

(iv) $\mathrm{I}(W, \hat{S}, \eta) = (1 - t^2)\,\mathrm{I}(W, S, \eta)$;

(v) $\mathrm{I}(W, \hat{S}, \eta) = \theta(t)\,\mathrm{I}(W, S, \eta)$ *for a universal non-zero polynomial* $\theta(t) \in \mathcal{R}$.

Proof. Part (i) is trivial. (ii)–(iv) are from [22, Proposition 3.1]. The proof of (v) is similar to that of (iv). We first review Kronheimer and Mrowka's proof of (iv) briefly. For simplicity, consider a special case that (W, S, η) is closed, thus it can be viewed as a cobordism from the empty set to the empty set. We also assume $S'' = \hat{S}'' = \emptyset$. In this case,

$$\mathrm{I}(W, S, \eta) \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}, \mathcal{R}) = \mathcal{R}$$

is the singular Donaldson invariant (from 0-dimensional moduli spaces) introduced by Kronheimer and Mrowka [19]. More precisely, we have

$$\mathrm{I}(W, S, \eta) = \sum_{k,l} q_{k,l}(W, S, \eta) t^{-l}, \tag{3.4}$$

where $q_{k,l}$ denotes the singular Donaldson invariant defined by counting the number of points in the 0-dimensional moduli spaces over all orbifold bundles with instanton number k and monopole number l. The restriction of an orbifold SO(3)-bundle to S has a reduction to $K \oplus \underline{\mathbb{R}}$ where K is an SO(2)-bundle. By definition, the monopole number l is given by

$$l = -\tfrac{1}{2} e(K)[S].$$

If $(W, \hat{S})$ is obtained from (W, S) by a finger move, [18, Proposition 3.1] uses a gluing argument to prove that

$$q_{k,l}(W, \hat{S}, \eta) = q_{k,l}(W, S, \eta) - q_{k-1,l+2}(W, S, \eta). \tag{3.5}$$

When (W, S, η) is closed and $S'' = \hat{S}'' = \emptyset$, part (iv) follows immediately from (3.4) and (3.5). Since the gluing argument only depends on the local structure of the finger move, it is straightforward to extend the argument to the general (relative) case.

To prove (v), we first assume (W, S, η) is closed and $S'' \neq \emptyset$. We give a refined definition of the monopole number l by taking

$$l_0 := -\tfrac{1}{2} e(K)[S'], \quad l_1 := -\tfrac{1}{2} e(K)[S''].$$

It is clear from the definition that $l = l_0 + l_1$. With this definition, we have a refined singular Donaldson invariant q_{k,l_0,l_1}. Similar to (3.4), we define the polynomial

$$Q(W, S, \eta)(t_0, t_1) := \sum_{k,l_0,l_1} q_{k,l_0,l_1}(W, S, \eta) t_0^{-l_0} t_1^{-l_1}.$$

If $\hat{S}$ is obtained by a finger move that introduces intersection points between S' and S'', then the proof of [18, equation (23)] shows that there exist universal constants $a_{i,j} \in \mathbb{Z}$ such that

$$q_{k,l_0,l_1}(W, \hat{S}, \eta) = \sum_{2|i+j} a_{i,j} q_{k-\frac{i+j}{2},l_0+i,l_1+j}(W, S, \eta).$$

By the Uhlenbeck compactness theorem, only finitely many $a_{i,j}$'s are non-zero. This implies that

$$Q(W, \hat{S}, \eta)(t_0, t_1) = P(t_0, t_1) Q(W, S, \eta)(t_0, t_1) \tag{3.6}$$

for a universal polynomial $P(t_0, t_1) \in \mathbb{C}[t_0, t_0^{-1}, t_1, t_1^{-1}]$. Notice that

$$q_{k,l} = \sum_{l_0+l_1=l} q_{k,l_0,l_1},$$

therefore (3.5) implies

$$P(t, t) = 1 - t^2. \tag{3.7}$$

We also have $P(t_0, t_1) = P(t_1, t_0)$ because there is no difference between the roles of S' and S'' in the finger move.

We claim that

$$P(t, 1) = P(1, t) \neq 0. \tag{3.8}$$

In fact, suppose the contrary; then

$$(t_0 - 1) \mid P(t_0, t_1), \quad (t_1 - 1) \mid P(t_0, t_1),$$

thus we have

$$(t_0 - 1)(t_1 - 1) \mid P(t_0, t_1), \quad \text{therefore} \quad (t - 1)(t - 1) \mid P(t, t),$$

which contradicts (3.7), hence the claim is proved.

Now, let $\theta(t) := P(t, 1)$. We have

$$\mathrm{I}(W, S, \eta)(t) = Q(W, S, \eta)(t, 1),$$

therefore in the closed case, part (v) of the proposition follows from (3.6) and (3.8). By the gluing argument, the same result holds for the non-closed case. ■

Suppose (Y, L_0, ω) is an admissible triple, and let L_0' be a sublink of L_0 such that $L_0' \cap \omega = \emptyset$. Fix an orientation of L_0'. By the previous discussion, L_0' defines a local system Γ_0 with $\mathcal{R}$-coefficients. Suppose L_1 is obtained from L_0 by a local crossing change in $Y - \omega$, where the crossing is either within L_0' or between L_0' and $L_0 - L_0'$, and let Γ_1 be the local system of (Y, L_1, ω) associated with the image of L_0' after the crossing change.

The crossing change induces an immersed cobordism $S : L_0 \to L_1$, where S is an immersed surface in $[0, 1] \times Y$ with one double point. Reversing S, we obtain an immersed cobordism $\overline{S} : L_1 \to L_0$ with one double point. The composition $S \cup \overline{S} \subset$

$[0,2] \times Y$ can be obtained from the product cobordism $[0,2] \times L_0$ by a finger move decribed by case (iv) or case (v) of Proposition 3.1. Therefore by Proposition 3.1, the map

$$\mathrm{I}([0,2] \times Y, S \cup \overline{S}, [0,2] \times \omega) : \mathrm{I}(Y, L_0, \omega_0; \Gamma_0) \to \mathrm{I}(Y, L_0, \omega_0; \Gamma_0) \tag{3.9}$$

is equal to $(1-t^2)\,\mathrm{id}$ or $\theta(t)\,\mathrm{id}$. Similarly, the map

$$\mathrm{I}([0,2] \times Y, \overline{S} \cup S, [0,2] \times \omega) : \mathrm{I}(Y, L_1, \omega; \Gamma_1) \to \mathrm{I}(Y, L_1, \omega; \Gamma_1) \tag{3.10}$$

is equal to $(1-t^2)\,\mathrm{id}$ or $\theta(t)\,\mathrm{id}$. As a consequence, we have the following result.

Proposition 3.2. *Suppose (Y, L_0, ω) is an admissible triple and $L_0' \subset L_0$ is a sublink with $L_0 \cap \omega = \emptyset$. Fix an orientation on L_0' and let Γ_0 be the local system of (Y, L_0, ω) defined by L_0'. Suppose L_1' is an oriented link that is homotopic to L_0' in $Y - \omega$ and is disjoint from $L_0 - L_0'$. Let $L_1 := L_1' \cup (L_0 - L_0')$. Let Γ_1 be the local system of (Y, L_1, ω) defined by L_1'. Then*

$$T^{-1}\,\mathrm{I}(Y, L_0, \omega; \Gamma_0) \cong T^{-1}\,\mathrm{I}(Y, L_1, \omega; \Gamma_1),$$

where T is the multiplicative system generated by $(1-t^2)\theta(t)$.

Proof. Since L_0' is homotopic to L_1' in $Y - \omega$, the link L_1 can be obtained from L_0 by a finite sequence of crossing changes in $Y - \omega$ such that the sublink $L_0 - L_0'$ remains fixed. Without loss of generality, we may assume that L_1 is obtained from L_0 by one such crossing change. Let $S \subset Y \times [0,1]$ be the immersed cobordism from L_0 to L_1 given by the crossing change, and let $\bar{S}$ be the reverse of S. By (3.9), (3.10), and the functoriality of the instanton Floer homology with local coefficients, we have

$$\mathrm{I}(Y \times [0,1], \overline{S}, \omega \times [0,1]) \circ \mathrm{I}(Y \times [0,1], S, \omega \times [0,1]) = (1-t^2)\,\mathrm{id} \ \text{ or } \theta(t)\,\mathrm{id}$$

on $\mathrm{I}(Y, L_0, \omega; \Gamma_0)$, and

$$\mathrm{I}(Y \times [0,1], S, \omega \times [0,1]) \circ \mathrm{I}(Y \times [0,1], \overline{S}, \omega \times [0,1]) = (1-t^2)\,\mathrm{id} \ \text{ or } \theta(t)\,\mathrm{id}$$

on $\mathrm{I}(Y, L_1, \omega; \Gamma_1)$. Hence $T^{-1}\,\mathrm{I}(Y, L_0, \omega; \Gamma_0)$ and $T^{-1}\,\mathrm{I}(Y, L_1, \omega; \Gamma_1)$ are isomorphic. ■

Corollary 3.3. *Let $Y, \omega, L_0, L_1, \Gamma_0, \Gamma_1$ be as in Proposition 3.2. Then*

$$\operatorname{rank}_{\mathcal{R}} \mathrm{I}(Y, L_0, \omega; \Gamma_0) = \operatorname{rank}_{\mathcal{R}} \mathrm{I}(Y, L_1, \omega; \Gamma_1). \qquad ■$$

Given an oriented link L in $S^1 \times D^2$, we define the annular instanton Floer homology with local coefficients by

$$\mathrm{AHI}(L; \Gamma) := \mathrm{I}(S^1 \times S^2, L \cup \mathcal{K}_2, u; \Gamma),$$

where Γ is the local system associated with L. The operator $\mu^{\mathrm{orb}}(S^2)$ on $\mathrm{AHI}(L; \Gamma)$ is now an $\mathcal{R}$-module homomorphism instead of a $\mathbb{C}$-linear map, therefore $\mathrm{AHI}(L; \Gamma)$ no longer carries the f-grading. The torus excision theorem [20, Theorem 5.6] still holds for instanton Floer homology with local coefficients, as long as the excision surface is disjoint from the sublink defining the local system. Therefore, Proposition 2.1 still holds for the annular instanton Floer homology with local coefficients, except that there is no f-grading anymore.

Example 3.4. By Example 2.2, the critical points of the (perturbed) Chern–Simons functional for $\mathrm{AHI}(\mathcal{U}_1)$ (or $\mathrm{AHI}(\mathcal{K}_1)$) consist of two points whose homological degrees differ by 2. Therefore there are no differentials in the Floer chain complex, and we have

$$\mathrm{AHI}(\mathcal{U}_1;\Gamma) \cong \mathcal{R} \oplus \mathcal{R}, \quad \mathrm{AHI}(\mathcal{K}_1;\Gamma) \cong \mathcal{R} \oplus \mathcal{R}.$$

By Proposition 2.1,

$$\mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l;\Gamma) \cong \mathcal{R}^{2^{k+l}}.$$

Proposition 2.6 follows from the torus excision theorem, therefore it also works in the case with local coefficients. Let $L \subset S^1 \times D^2$ be a link with n components and view $S^1 \times D^2$ as the complement of a neighborhood of the unknot U in S^3, let $p \in U$ and let Γ_L be the local system associated with L. Then

$$\mathrm{AHI}(L;\Gamma) \cong \mathrm{I}^\natural(L \cup U, p; \Gamma_L).$$

Suppose the annular link L has n components. Then the embedded image of L in S^3 is homotopic to the embedded image of $\mathcal{U}_n$ in S^3. By Corollary 3.3, we have

$$\mathrm{rank}_{\mathcal{R}}\, \mathrm{I}^\natural(L \cup U, p; \Gamma_L) = \mathrm{rank}_{\mathcal{R}}\, \mathrm{I}^\natural(\mathcal{U}_n \cup U, p; \Gamma_{\mathcal{U}_n}).$$

By Proposition 2.6 again,

$$\mathrm{I}^\natural(\mathcal{U}_n \cup U, p; \Gamma) \cong \mathrm{AHI}(\mathcal{U}_n;\Gamma) \cong \mathcal{R}^{2^n}.$$

In conclusion, we obtain

$$\mathrm{rank}_{\mathcal{R}}\, \mathrm{AHI}(L;\Gamma) = 2^n. \tag{3.11}$$

By the universal coefficient theorem,

$$\mathrm{rank}_{\mathbb{C}}\, \mathrm{I}^\natural(L \cup U, p) = \mathrm{rank}_{\mathbb{C}}\, \mathrm{AHI}(L) \ge \mathrm{rank}_{\mathcal{R}}\, \mathrm{AHI}(L;\Gamma) = 2^n. \tag{3.12}$$

4. Limits of chain complexes

This section discusses a simple observation from linear algebra and its consequences in instanton Floer homology.

Suppose $\{C_n\}_{n\in\mathbb{Z}}$ is a sequence of finite-dimensional complex vector spaces. For each $n \in \mathbb{Z}$ and $k \ge 0$, and let $\partial_n^{(k)}$ be a $\mathbb{C}$-linear map from C_n to C_{n-1}, let $f_n^{(k)}$ be an endomorphism of C_n. Suppose that for each pair (n,k) we have $\partial_n^{(k)} \circ \partial_{n+1}^{(k)} = 0$ and $\partial_n^{(k)} \circ f_n^{(k)} = f_{n-1}^{(k)} \circ \partial_n^{(k)}$. Moreover, suppose that for each n the limits

$$\lim_{k\to\infty} f_n^{(k)} \quad \text{and} \quad \lim_{k\to\infty} \partial_n^{(k)}$$

exist. Let

$$\partial_n := \lim_{k\to\infty} \partial_n^{(k)}, \quad f_n := \lim_{k\to\infty} f_n^{(k)},$$
$$H_n^{(k)} := \ker \partial_n^{(k)} / \operatorname{Im} \partial_{n+1}^{(k)}, \quad H_n := \ker \partial_n / \operatorname{Im} \partial_{n+1}.$$

The maps $f_n^{(k)}$ and f_n induce maps on $H_n^{(k)}$ and H_n respectively. For $\Lambda \subset \mathbb{C}$, define $E_{n,\Lambda}^{(k)} \subset H_n^{(k)}$ to be the direct sum of the generalized eigenspaces of $f_n^{(k)}$ with eigenvalues in Λ. Similarly, define $E_{n,\Lambda} \subset H_n$ to be the direct sum of the generalized eigenspaces of f_n with eigenvalues in Λ.

Lemma 4.1. *Let C_n, $\partial_n^{(k)}$, ∂_n, $f_n^{(k)}$, f_n, $H_n^{(k)}$, H_n, $E_{n,\Lambda}^{(k)}$, and $E_{n,\Lambda}$ be as above.*

(1) *If $\Lambda \subset \mathbb{C}$ is a closed subset, then*

$$\dim E_{n,\Lambda} \geq \limsup_{k\to\infty} \dim E_{n,\Lambda}^{(k)}.$$

(2) *For $\varepsilon > 0$, let $N(\Lambda, \varepsilon)$ be the closed ε-neighborhood of Λ. If Λ is closed, then*

$$\dim E_{n,\Lambda} \geq \lim_{\varepsilon\to 0} \limsup_{k\to\infty} \dim E_{n,N(\Lambda,\varepsilon)}^{(k)}.$$

(3) *If we further assume that* $\dim H_n^{(k)} = \dim H_n$ *for all k, then*

$$\dim E_{n,\Lambda} = \lim_{\varepsilon\to 0} \limsup_{k\to\infty} \dim E_{n,N(\Lambda,\varepsilon)}^{(k)}.$$

Proof. (1) Let $Z_n := \ker \partial_n^{(k)}$, $B_n^{(k)} := \operatorname{Im} \partial_{n+1}^{(k)}$, $Z_n := \ker \partial_n$, $B_n := \operatorname{Im} \partial_{n+1}$. After taking a subsequence, we may assume that the dimensions of $Z_n^{(k)}$ and $B_n^{(k)}$ are independent of k, and that they are convergent in the corresponding Grassmannians as $k \to \infty$. The spectrum of f_n (with multiplicities) on

$$\Big(\lim_{k\to\infty} Z_n^{(k)}\Big) / \Big(\lim_{k\to\infty} B_n^{(k)}\Big) \tag{4.1}$$

is the limit of the spectra of $f_n^{(k)}$ on $Z_n^{(k)}/B_n^{(k)}$ as $k \to \infty$. Let

$$E'_{n,\Lambda} \subset \Big(\lim_{k\to\infty} Z_n^{(k)}\Big) / \Big(\lim_{k\to\infty} B_n^{(k)}\Big)$$

be the direct sum of the generalized eigenspaces of f_n for the eigenvalues in Λ. Since Λ is closed, the previous argument implies

$$\dim E'_{n,\Lambda} \geq \limsup_{k\to\infty} \dim E_{n,\Lambda}^{(k)}.$$

On the other hand, we have

$$Z_n \supset \lim_{k\to\infty} Z_n^{(k)}, \quad B_n \subset \lim_{k\to\infty} B_n^{(k)},$$

therefore (4.1) is a subquotient of $H_n = Z_n/B_n$, and hence

$$\dim E_{n,\Lambda} \geq \dim E'_{n,\Lambda} \geq \limsup_{k\to\infty} \dim E^{(k)}_{n,\Lambda},$$

which completes the proof.

(2) Let ε_0 be sufficiently small such that $E_{n,N(\Lambda,\varepsilon_0)} = E_{n,\Lambda}$. By (1), we have

$$\begin{aligned}\dim E_{n,\Lambda} &= \dim E_{n,N(\Lambda,\varepsilon_0)} \\ &\geq \limsup_{k\to\infty} \dim E^{(k)}_{n,N(\Lambda,\varepsilon_0)} \geq \lim_{\varepsilon\to 0} \limsup_{k\to\infty} \dim E^{(k)}_{n,N(\Lambda,\varepsilon)}.\end{aligned}$$

(3) Let ε_0 be sufficiently small such that $E_{n,N(\Lambda,\varepsilon_0)} = E_{n,\Lambda}$. Suppose $\varepsilon < \varepsilon_0$. Then $E_{n,N(\Lambda,\varepsilon)} = E_{n,\Lambda}$. Let $\Lambda_1 := \overline{\mathbb{C} - N(\Lambda,\varepsilon)}$. By the condition on ε,

$$E_{n,\partial N(\Lambda,\varepsilon)} = E_{n,\partial\Lambda_1} = \{0\}.$$

Hence by (1), for k sufficiently large we have

$$\dim E^{(k)}_{n,\partial N(\Lambda,\varepsilon)} = \dim E^{(k)}_{n,\partial\Lambda_1} = 0.$$

Applying (1) again on $N(\Lambda,\varepsilon)$ and Λ_1, we deduce that if k is sufficiently large, then $\dim E_{n,N(\Lambda,\varepsilon)} \geq \dim E^{(k)}_{n,N(\Lambda,\varepsilon)}$ and $\dim E_{n,\Lambda_1} \geq \dim E^{(k)}_{n,\Lambda_1}$. Therefore

$$\begin{aligned}\dim H_n &= \dim E_{n,N(\Lambda,\varepsilon)} + \dim E_{n,\Lambda_1} \\ &\geq \dim E^{(k)}_{n,N(\Lambda,\varepsilon)} + \dim E^{(k)}_{n,\Lambda_1} \\ &= \dim H^{(k)}_n = \dim H_n.\end{aligned}$$

As a consequence, for k sufficiently large, $\dim E_{n,N(\Lambda,\varepsilon)} = \dim E^{(k)}_{n,N(\Lambda,\varepsilon)}$, and hence

$$\dim E_{n,\Lambda} = \dim E_{n,N(\Lambda,\varepsilon)} = \limsup_{k\to\infty} \dim E^{(k)}_{n,N(\Lambda,\varepsilon)}.$$

Since the above equation holds for all $\varepsilon < \varepsilon_0$, part (3) of the lemma is proved. ∎

Recall that given an admissible triple (Y, L, ω) and a continuous function $\mu : \mathcal{B}(Y, L, \omega) \to \mathbb{R}/\mathbb{Z}$, there is a local system Γ^μ on $\mathcal{B}(Y, L, \omega)$ defined by μ. The Floer chain complex $\mathbf{C}^\mu$ is a finitely generated free $\mathcal{R}$-module, where $\mathcal{R} = \mathbb{C}[t, t^{-1}]$. The differential D is an $\mathcal{R}$-endomorphism of $\mathbf{C}^\mu$. For $h \in \mathbb{C} - \{0\}$, define

$$(C_h, d_h) := (\mathbf{C}^\mu \otimes_{\mathcal{R}} \mathcal{R}/(t-h), D \otimes_{\mathcal{R}} \mathrm{id}_{\mathcal{R}/(t-h)}).$$

Notice that $\mathcal{R}/(t-h) \cong \mathbb{C}$ via the map $t \mapsto h$, so (C_h, d_h) is a finite-dimensional chain complex over $\mathbb{C}$. Let $C := \mathbb{C}^{\mathrm{rank}_{\mathcal{R}} \mathbf{C}^\mu}$. We identify C_h with C using the isomorphism $\mathcal{R}/(t-h) \cong \mathbb{C}$. The differentials d_h become a continuous family of linear maps on C. Given an embedded surface $F \subset Y$, define

$$\mu^{\mathrm{orb}}(F)_h := \mu^{\mathrm{orb}}(F) \otimes_{\mathcal{R}} \mathcal{R}/(t-h).$$

Then $\mu^{\text{orb}}(F)_h$ is continuous with respect to h and is a chain map on (C, d_h). Therefore, for each $h \in \mathbb{C} - \{0\}$, the map $\mu^{\text{orb}}(F)_h$ induces a map on the Floer homology

$$\mathrm{I}(Y, L, \omega; \Gamma^{\mu} \otimes_{\mathcal{R}} \mathcal{R}/(t-h)) = H^*(C, d_h).$$

To simplify notations, we will use $\Gamma^{\mu}(h)$ to denote $\Gamma^{\mu} \otimes_{\mathcal{R}} \mathcal{R}/(t-h)$ for the rest of this article. If $h = 1$, then $\mathrm{I}(Y, L, \omega; \Gamma^{\mu}(1))$ is the ordinary instanton Floer homology without local coefficients, and $\mu^{\text{orb}}(F)_1$ coincides with the ordinary μ map.

Proposition 4.2. *Let $Y, \omega, L_0, L_1, L_0', L_1', \Gamma_0, \Gamma_1$ be as in Proposition 3.2. Let $\theta(t)$ be the polynomial given by Proposition 3.1* (v). *Suppose $h \in \mathbb{C} - \{0\}$ satisfies*

$$(1 - h^2)\theta(h) \neq 0. \tag{4.2}$$

Then

$$\mathrm{I}(Y, L_0, \omega; \Gamma_0(h)) \cong \mathrm{I}(Y, L_1, \omega; \Gamma_1(h)). \tag{4.3}$$

Moreover, if $F \subset Y$ is a closed embedded surface in Y, then the isomorphism (4.3) *intertwines with $\mu^{\text{orb}}(F)_h$.*

Proof. Let $T \subset \mathcal{R}$ be the multiplicative system generated by $(1-t^2)\theta(t)$ as in Proposition 3.2. By (4.2), the elements of T have non-zero images in $\mathcal{R}/(t-h) \cong \mathbb{C}$, hence $\mathcal{R}/(t-h)$ is isomorphic to $(T^{-1}\mathcal{R})/(t-h)$. Therefore, for $i = 0, 1$, we have

$$\mathrm{I}(Y, L_i, \omega; \Gamma_i(h)) \cong \mathrm{I}\big(Y, L_i, \omega; \Gamma_i \otimes_{\mathcal{R}} T^{-1}\mathcal{R} \otimes_{T^{-1}\mathcal{R}} T^{-1}\mathcal{R}/(t-h)\big). \tag{4.4}$$

On the other hand, since localization is an exact functor, we have

$$\mathrm{I}(Y, L_i, \omega; \Gamma_i \otimes_{\mathcal{R}} T^{-1}\mathcal{R}) \cong T^{-1}\,\mathrm{I}(Y, L_i, \omega; \Gamma_i).$$

Therefore by Proposition 3.2,

$$\mathrm{I}(Y, L_0, \omega; \Gamma_0 \otimes_{\mathcal{R}} T^{-1}\mathcal{R}) \cong \mathrm{I}(Y, L_1, \omega; \Gamma_1 \otimes_{\mathcal{R}} T^{-1}\mathcal{R}). \tag{4.5}$$

Since $\mathcal{R}$ is a principal ideal domain, the localization $T^{-1}\mathcal{R}$ is also a principal ideal domain, hence (4.3) follows from the universal coefficient theorem and the isomorphisms (4.4) and (4.5).

It remains to prove that (4.3) intertwines with $\mu^{\text{orb}}(F)_h$. Since the isomorphism (4.5) is induced by a cobordism in which the two copies of the surface F on the two ends are homologous, it intertwines the $\mu^{\text{orb}}(F)_h$ on the in-coming end with the $\mu^{\text{orb}}(F)_h$ on the out-going end, hence the statement is proved. ∎

Lemma 4.1 and Proposition 4.2 have the following application.

Proposition 4.3. *Suppose $L \subset S^1 \times D^2$ is an oriented link such that every component of L has winding number* 0 *or* ± 1. *Assume there are k components with winding number* 0 *and l components with winding number* ± 1. *Then*

$$\dim_{\mathbb{C}} \mathrm{AHI}(L, i) \geq \dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l, i) \quad \textit{for all } i \in \mathbb{Z}.$$

Proof. For $\lambda \in \mathbb{C}$, let $N(\lambda, \varepsilon)$ be the closed ε-neighborhood of λ in $\mathbb{C}$. Given a vector space V over $\mathbb{C}$, a linear map $f : V \to V$, and a subset $\Lambda \subset \mathbb{C}$, we use $E(V, f, \Lambda)$ to denote the direct sum of the generalized eigenspaces of f with eigenvalues in Λ.

Recall that $\mathrm{AHI}(L; \Gamma)$ is defined to be the instanton Floer homology

$$\mathrm{I}(S^1 \times S^2, L \cup \mathcal{K}_2, u; \Gamma),$$

where Γ is the local coefficient system associated with L. For $h \in \mathbb{C} - \{0\}$, recall that $\Gamma(h)$ is the local system over $\mathbb{C}$ given by $\Gamma \otimes_{\mathcal{R}} \mathcal{R}/(t-h)$. For every $i \in \mathbb{Z}$, Lemma 4.1 (2) and Proposition 4.2 give

$$\begin{aligned}\dim \mathrm{AHI}(L, i) &\geq \lim_{\varepsilon \to 0} \limsup_{h \to 1} E\big(\mathrm{AHI}(L; \Gamma(h)), \mu^{\mathrm{orb}}(S^2)_h, N(i, \varepsilon)\big)\\ &= \lim_{\varepsilon \to 0} \limsup_{h \to 1} E\big(\mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l; \Gamma(h)), \mu^{\mathrm{orb}}(S^2)_h, N(i, \varepsilon)\big).\end{aligned}$$

According to Example 3.4, $\mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l; \Gamma)$ is a free $\mathcal{R}$-module of rank 2^{k+l}. By the universal coefficient theorem, $\dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l; \Gamma(h)) = 2^{k+l}$ for all $h \in \mathbb{C} - \{0\}$. Therefore Lemma 4.1 (3) gives

$$\lim_{\varepsilon \to 0} \limsup_{h \to 1} E\big(\mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l; \Gamma(h)), \mu^{\mathrm{orb}}(F)_h, N(i, \varepsilon)\big) = \dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l, i),$$

and the proposition is proved. ∎

Corollary 4.4. *Suppose $L \subset S^1 \times D^2$ is an oriented link such that every component of L has winding number 0 or ± 1. Assume there are k components with winding number 0 and l components with winding number ± 1. Moreover, assume*

$$\dim_{\mathbb{C}} \mathrm{AHI}(L) = 2^{k+l}. \tag{4.6}$$

Then there exists a meridional disk S in $S^1 \times D^2$ such that S intersects every component of L with winding number ± 1 transversely in one point, and S is disjoint from every component of L with winding number 0.

Proof. By Example 3.4, $\dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l) = 2^{k+l}$, therefore (4.6) and Proposition 4.3 imply

$$\dim_{\mathbb{C}} \mathrm{AHI}(L, i) = \dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l, i)$$

for all $i \in \mathbb{Z}$. The top f-grading of $\mathrm{AHI}(\mathcal{U}_k \cup \mathcal{K}_l, i)$ is l. By Theorem 2.4, there exists a meridional surface S with genus g such that S intersects L transversely in n points, and $2g + n = l$. On the other hand, every component with a non-zero winding number must intersect S, therefore we have $g = 0$ and $n = l$, and the surface S is the desired meridional disk. ∎

5. Linking numbers and forests of unknots

This section proves a weaker version of Theorem 1.2:

Theorem 5.1. *Suppose $L = K_1 \cup \cdots \cup K_n$ is an oriented link with n components in S^3 such that*

(1) $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 2^n$*;*

(2) *there exists a forest of unknots $L_G = K_1' \cup \cdots \cup K_n'$ such that*

$$\operatorname{lk}(K_i, K_j) = \operatorname{lk}(K_i', K_j') \quad \text{for all } i \neq j.$$

Then L is isotopic to L_G.

Before starting the proof, we need to make some preparations. Notice that Batson and Seed's work [5] implies the following useful result.

Proposition 5.2 ([5]). *If L is a link in S^3 with n components and*

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 2^n,$$

then $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L_0; \mathbb{Z}/2) = 2^{|L_0|}$ *for every sublink L_0 of L, where $|L_0|$ is the number of components of L_0.*

Proof. Suppose $L = K_1 \cup \cdots \cup K_n$. Let I be a subset of $\{1, \ldots, n\}$ with $|I|$ components. By [5, Theorem 1.1] (cf. the proof of [5, Proposition 7.1]), we have

$$\begin{aligned} 2^n &= \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(K; \mathbb{Z}/2) \\ &\geq \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}\Big(\bigcup_{i \notin I} K_i; \mathbb{Z}/2\Big) \cdot \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}\Big(\bigcup_{i \in I} K_i; \mathbb{Z}/2\Big) \\ &\geq 2^{n-|I|} \cdot \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}\Big(\bigcup_{i \in I} K_i; \mathbb{Z}/2\Big) \geq 2^n. \end{aligned}$$

Hence the inequalities above are equalities, and we have

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}\Big(\bigcup_{i \in I} K_i; \mathbb{Z}/2\Big) = 2^{|I|}. \qquad \blacksquare$$

The above result together with Kronheimer–Mrowka's unknot detection theorem in [20] implies the following proposition.

Proposition 5.3 ([5, Proposition 7.1]). *If L is a link in S^3 with n components and*

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 2^n,$$

then each component of L is an unknot. ■

Proposition 5.4. *Suppose L is a link in S^3 with n components and*

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L; \mathbb{Z}/2) = 2^n.$$

Then for every point $p \in L$, we have $\dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, p) = 2^{n-1}$.

Proof. Given a point $p \in L$, we use $\mathrm{Khr}(L, p)$ to denote the reduced Khovanov homology with basepoint p. By [30, Corollary 3.2.C],

$$\mathrm{rank}_{\mathbb{Z}/2}\, \mathrm{Khr}(L, p; \mathbb{Z}/2) = \tfrac{1}{2}\, \mathrm{rank}_{\mathbb{Z}/2}\, \mathrm{Kh}(L; \mathbb{Z}/2) = 2^{n-1}.$$

By the universal coefficient theorem,

$$\mathrm{rank}_{\mathbb{Q}}\, \mathrm{Khr}(L, p; \mathbb{Q}) \leq \mathrm{rank}_{\mathbb{Z}/2}\, \mathrm{Khr}(L, p; \mathbb{Z}/2) = 2^{n-1}.$$

Let $\overline{L}$ be the mirror image of L. By [14, Corollary 11],

$$\mathrm{rank}_{\mathbb{Q}}\, \mathrm{Khr}(\overline{L}, p; \mathbb{Q}) = \mathrm{rank}_{\mathbb{Q}}\, \mathrm{Khr}(L, p; \mathbb{Q}) \leq 2^{n-1}.$$

Using Kronheimer–Mrowka's spectral sequence [20, Theorem 8.2] whose E_2-page is $\mathrm{Khr}(\overline{L}, p; \mathbb{Z})$ and which converges to $\mathrm{I}^\natural(L, p; \mathbb{Z})$, we obtain

$$\dim_{\mathbb{C}} \mathrm{I}^\natural(L, p) = \mathrm{rank}_{\mathbb{Z}}\, \mathrm{I}^\natural(L, p; \mathbb{Z}) \leq \mathrm{rank}_{\mathbb{Z}}\, \mathrm{Khr}(\overline{L}, p; \mathbb{Z}) \leq 2^{n-1}.$$

On the other hand, Proposition 5.3 and (3.12) imply that $\dim_{\mathbb{C}} \mathrm{I}^\natural(L, p) \geq 2^{n-1}$. Therefore we obtain $\dim_{\mathbb{C}} \mathrm{I}^\natural(L, p) = 2^{n-1}$. ∎

Proof of Theorem 5.1. We prove the theorem by induction on n. When $n = 1$, it is the unknot detection theorem of Kronheimer and Mrowka [20].

Assume the theorem holds when the number of components is smaller than n. Since G is a forest, we can find a vertex of G with degree less than or equal to 1. We discuss two cases.

Case 1: There is a vertex of G with degree 1. Without loss of generality, assume this vertex corresponds to the component K_n' of L_G. By the assumption of Theorem 5.1, there exists $i \in \{1, \dots, n-1\}$ such that $\mathrm{lk}(K_i, K_n) = \pm 1$ and $\mathrm{lk}(K_j, K_n) = 0$ when $1 \leq j \leq n-1, j \neq i$.

Pick a basepoint $p \in K_n$ and use L' to denote $K_1 \cup \dots \cup K_{n-1}$. According to Proposition 5.3, K_n is an unknot. Let $N(K_n)$ be a tubular neighborhood of K_n. Then L' can be viewed as a link in the solid torus $S^3 - N(K_n)$. By Propositions 2.6 and 5.4 we have

$$\mathrm{AHI}(L') \cong \mathrm{I}^\natural(L, p) \cong \mathbb{C}^{2^{n-1}}$$

According to Corollary 4.4, we can find a meridional disk S in the solid torus $S^3 - N(K_n)$ which intersects K_i in a single point and is disjoint from the other components. The meridional disk S is a Seifert disk of K_n. By the induction hypothesis, L' is a forest of unknots. We can shrink K_n via S to a small meridian of K_i. Therefore L is also a forest of unknots. Since the linking numbers uniquely determine a forest of unknots, we conclude that L is isotopic to L_G.

Case 2: There is a vertex of G with degree 0. Without loss of generality, assume this vertex corresponds to the component K_n' of L_G. By the assumption of Theorem 5.1, we have $\mathrm{lk}(K_j, K_n) = 0$ for all $1 \leq j \leq n-1$. Let $L' := K_1 \cup \dots \cup K_{n-1}$ and let $N(K_n)$ be a tubular neighborhood of K_n. We can view L' as a link in the solid torus $S^3 - N(K_n)$, and the same argument as above gives $\mathrm{AHI}(L') \cong \mathbb{C}^{2^{n-1}}$. By Proposition 4.4, we can find a meridional disk S in the solid torus $S^3 - N(K_n)$ which is disjoint from L'. Therefore L is the disjoint union of L' and the unknot, and the result follows from the induction hypothesis on L'. ■

6. The case of 2-component links

6.1. The linking numbers of 2-component links

This subsection proves that condition (2) of Theorem 5.1 is implied by condition (1) when $n = 2$. The main result of this subsection is the following lemma.

Lemma 6.1. *Suppose $L = K_1 \cup K_2$ is a link with two components such that*

$$\operatorname{rank}_{\mathbb{Z}/2} \mathrm{Kh}(L; \mathbb{Z}/2) = 4.$$

Then $|\mathrm{lk}(K_1, K_2)| \leq 1$.

Combining this lemma with Theorem 5.1, we have the following corollary.

Corollary 6.2. *Suppose L is a link with two components and* $\operatorname{rank}_{\mathbb{Z}/2} \mathrm{Kh}(L; \mathbb{Z}/2) = 4$. *Then L is either the 2-component unlink or the Hopf link.*

We start the proof of Lemma 6.1 with the following lemma.

Lemma 6.3. *Suppose $L = K_1 \cup K_2$ satisfies the assumption of Lemma 6.1, and suppose L is not the unlink. Then K_1 is an unknot, and K_2 is a braid closure with axis K_1. Similarly, K_2 is an unknot, and K_1 is a braid closure with axis K_2.*

Proof. Proposition 5.3 implies that K_1 and K_2 are both unknots. Let $N(K_1)$ be a tubular neighborhood of K_1. Then K_2 is a knot in the solid torus $S^3 - N(K_1)$. Proposition 5.4 yields

$$\dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, p) = 2$$

for every $p \in L$. By Proposition 2.6,

$$\dim_{\mathbb{C}} \mathrm{AHI}(K_2) = \dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, p) = 2. \tag{6.1}$$

If $\mathrm{AHI}(K_2)$ is supported in f-degree 0, then by Theorem 2.4, there exists a meridional disk which is disjoint from K_2. This means K_2 is included in a 3-ball in the solid torus $S^3 - N(K_1)$, hence K_1 and K_2 are split, and therefore the link L is the unlink, contradicting the assumption. Therefore $\mathrm{AHI}(K_2)$ is supported in f-degrees $\pm l$ for $l > 0$. By (6.1), we have $\mathrm{AHI}(K_2, \pm l) \cong \mathbb{C}$ and $\mathrm{AHI}(K_2)$ vanishes in all the other f-degrees. According to Proposition 2.5, K_2 is the closure of an l-braid in $S^3 - N(K_1)$.

The same argument for $S^3 - N(K_2)$ proves the second half of the lemma. ■

Remark 6.4. A link described by the conclusion of Lemma 6.3 is called an *exchangeably braided link*. This concept was first introduced and studied by Morton [25].

Let $l > 1$ be an integer. Recall that the braid group B_l is given by

$$B_l = \langle \sigma_1, \dots, \sigma_{l-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\, \sigma_i\sigma_j = \sigma_j\sigma_i \ (j - i \geq 2)\rangle.$$

The reduced Burau representation (see [6]) is a group homomorphism

$$\rho : B_l \to \mathrm{GL}(l-1, \mathbb{Z}[t, t^{-1}])$$

defined by

$$\rho(\sigma_i) := \begin{pmatrix} I_{i-2} & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{l-i-2} \end{pmatrix}, \quad 2 \leq i \leq l-2,$$

$$\rho(\sigma_1) := \begin{pmatrix} -t & 1 & \\ 0 & 1 & \\ & & I_{l-3} \end{pmatrix}, \quad \rho(\sigma_{n-1}) := \begin{pmatrix} I_{l-3} & & \\ & 1 & 0 \\ & t & -t \end{pmatrix}$$

for $l > 2$, while for $l = 2$ it is defined by $\rho(\sigma_1) := (-t)$. Notice that for every $\beta \in B_l$, there exists an integer a such that

$$\det(\rho(\beta)) = \pm t^a. \tag{6.2}$$

We also need the follow result by Morton.

Theorem 6.5 ([25, Theorem 3]). *Suppose $L = U \cup \widehat{\beta}$ is a 2-component link where U is the unknot and $\widehat{\beta}$ is the closure of a braid $\beta \in B_l$ with axis U. Then the multi-variable Alexander polynomial $\Delta_L(x, t)$ of L is given by*

$$\Delta_L(x, t) \doteq \det(xI - \rho(\beta)(t)),$$

where x and t are variables corresponding to U and $\hat{\beta}$ respectively.

Remark 6.6. The sign "$\doteq$" in Theorem 6.5 means the two sides are equal up to multiplication by $\pm x^a t^b$. This is necessary because the multi-variable Alexander polynomial is only defined up to multiplication by $\pm x^a t^b$.

Lemma 6.7. *Suppose $L = K_1 \cup K_2$ is an exchangeably braided link with linking number $l \geq 2$. Let $\Delta_L(x, y)$ be the multi-variable Alexander polynomial of L. Then the expansion of the Laurent polynomial $(x-1)(y-1)\Delta_L(x, y)$ has (strictly) more than four terms.*

Proof. Without loss of generality, assume x and y are the variables corresponding to K_1 and K_2 respectively. Let $\beta \in B_l$ be the braid whose closure is isotopic to K_2 as a link in the solid torus $S^3 - K_1$. By (6.2) and Theorem 6.5, we have

$$\begin{aligned}\Delta_L(x,y) &\doteq (-1)^{l-1}\det(\rho(\beta)(y)) + f_1(y)x + \cdots + f_{l-2}(y)x^{l-2} + x^{l-1} \\ &= \pm y^a + f_1(y)x + \cdots + f_{l-2}(y)x^{l-2} + x^{l-1} \end{aligned} \tag{6.3}$$

for some $a \in \mathbb{Z}$ and $f_i \in \mathbb{Z}[y, y^{-1}]$.

Switching the roles of K_1 and K_2, we have

$$\Delta_L(x,y) \doteq \pm x^b + g_1(x)y + \cdots + g_{l-2}(x)y^{l-2} + y^{l-1} \tag{6.4}$$

for some $b \in \mathbb{Z}$ and $g_i(x) \in \mathbb{Z}[x, x^{-1}]$.

By (6.3), we have

$$\begin{aligned}&(y-1)\Delta_L(x,y) \\ &\quad\doteq \pm(y-1)y^a + (y-1)f_1(y)x + \cdots + (y-1)f_{l-2}(y)x^{l-2} + (y-1)x^{l-1},\end{aligned}$$

hence we have the following expansion in increasing powers of x:

$$(x-1)(y-1)\Delta_L(x,y) \doteq \pm(y-1)y^a + h_1(y)x + \cdots + h_{l-1}(y)x^{l-1} + (y-1)x^l.$$

The right-hand side has at least four terms after expansion, which come from the lowest and highest powers of x. Suppose it has only four terms in total; then all the terms in between must vanish, thus we have

$$(x-1)(y-1)\Delta_L(x,y) \doteq \pm(y-1)y^a + (y-1)x^l. \tag{6.5}$$

Plugging in $x = 1$, we have

$$0 = \pm(y-1)y^a + (y-1),$$

therefore $a = 0$, and (6.5) gives

$$\Delta_L(x,y) \doteq \frac{-(y-1) + (y-1)x^l}{(x-1)(y-1)} = 1 + x + \cdots + x^{l-1},$$

which contradicts (6.4) when $l \geq 2$. ∎

Proof of Lemma 6.1. Suppose $l \geq 2$. We use $\widehat{HFK}$ and $\widehat{HFL}$ to denote the Heegaard knot Floer homology [26, 28] and link Floer homology [27] respectively. The link Floer homology was originally defined only for $\mathbb{Z}/2$-coefficients, and was generalized to $\mathbb{Z}$-coefficients in [29]. It is known that

$$\operatorname{rank}_{\mathbb{Q}} \widehat{HFK}(L;\mathbb{Q}) = \operatorname{rank}_{\mathbb{Q}} \widehat{HFL}(L;\mathbb{Q}),$$

but $\widehat{HFL}$ carries more refined gradings.

By [7, Corollary 1.7], we have

$$\operatorname{rank}_{\mathbb{Q}} \widehat{HFK}(L;\mathbb{Q}) \leq 2\operatorname{rank}\operatorname{Khr}(L;\mathbb{Q}) \leq 2\operatorname{rank}\operatorname{Khr}(L;\mathbb{Z}/2) = 4. \tag{6.6}$$

On the other hand, let $\Delta_L(x, y)$ be the multi-variable Alexander polynomial of L. It was proved in [27] that the graded Euler characteristic of $\widehat{HFL}(L;\mathbb{Q})$ satisfies

$$\chi(\widehat{HFL}(L;\mathbb{Q})) \doteq (x-1)(y-1)\Delta_L(x, y).$$

By Lemma 6.7, we have

$$\operatorname{rank}_{\mathbb{Q}} \widehat{HFK}(L;\mathbb{Q}) = \operatorname{rank}_{\mathbb{Q}} \widehat{HFL}(L;\mathbb{Q}) > 4,$$

which contradicts (6.6). ■

Remark 6.8. The proof of Lemma 6.1 relies on Dowlin's inequality [7, Corollary 1.7] and the fact that the graded Euler characteristic of link Floer homology recovers the multi-variable Alexander polynomial [27]. The instanton analogue of Dowlin's inequality is proved in [32]. When the first version of this paper was written, it was not clear how to recover the multi-variable Alexander polynomial from instanton knot homology. Recently, such a result has been established by Zhenkun Li and Fan Ye [23, Theorem 1.4]. Therefore, Lemma 6.1 can also be proved using instanton Floer homology by a similar argument using results from [23, 32] instead.

We introduce the following condition on a link $L \subset S^3$:

Condition 6.9. (1) *L has $n \geq 3$ connected components.*

(2) *The rank of* $\mathrm{Kh}(L;\mathbb{Z}/2)$ *is* 2^n.

(3) *The components of L can be arranged as a sequence $K_1, \ldots, K_n$ such that the linking number of K_i and K_j $(i \neq j)$ is ± 1 when $|i-j| = 1$ or $n-1$, and is zero otherwise.*

Theorem 5.1 and Lemma 6.1 have the following consequence.

Lemma 6.10. *If L_0 is an m-component link with* $\operatorname{rank}_{\mathbb{Z}/2} \mathrm{Kh}(L_0;\mathbb{Z}/2) = 2^m$, *then either L_0 is a forest of unknots, or L_0 contains a sublink L satisfying Condition 6.9.*

Proof. Let $K_1, \ldots, K_m$ be the components of L_0. By Proposition 5.2 and Lemma 6.1, for each pair $i \neq j$, the linking number of K_i and K_j is equal to 0 or ± 1. Let G be a simple graph with m vertices $p_1, \ldots, p_m$ such that p_i and p_j are connected by an edge if and only if $|\mathrm{lk}(K_i, K_j)| = \pm 1$. If G is a forest, then Theorem 5.1 implies that L_0 is a forest of unknots. If G contains a cycle, then the vertices of the shortest cycle of G correspond to a sublink of L_0 satisfying Condition 6.9. ■

The next subsection gives a proof of Theorem 1.7. The rest of this article is devoted to proving that there is no link satisfying Condition 6.9, therefore Theorem 1.2 will follow from Lemma 6.10.

6.2. Some 2-component links with small Khovanov homology

This subsection gives a proof of Theorem 1.7, and shows that the bi-graded Khovanov homology detects some simple 2-component links other than the unlink and the Hopf link. The results of this subsection will not be used in the proof of Theorem 1.2.

Recall that the internal grading of the Khovanov homology of a link L is introduced in [5, Section 2] as $h - q$, where h is the homological grading and q is the quantum grading. The following is a special case of a more general result due to Batson and Seed.

Theorem 6.11 ([5, Corollary 4.4]). *Suppose $L = K_1 \cup K_2$ is a 2-component oriented link. Then*

$$\operatorname{rank}^l_{\mathbb{F}} \operatorname{Kh}(L;\mathbb{F}) \geq \operatorname{rank}^{l+2\operatorname{lk}(K_1,K_2)}_{\mathbb{F}}(\operatorname{Kh}(K_1;\mathbb{F}) \otimes \operatorname{Kh}(K_2;\mathbb{F})),$$

where $\mathbb{F}$ is an arbitrary field and rank^k denotes the rank of the summand with internal grading k.

Let L_1 be the oriented link given in Figure 4. Then its Khovanov homology is given by

$$\operatorname{Kh}(L_1;\mathbb{Z}) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(2)} \oplus (\mathbb{Z}/2)_{(3)} \oplus \mathbb{Z}^2_{(4)} \oplus \mathbb{Z}_{(6)}, \tag{6.7}$$

where the subscripts represent the internal gradings.

Theorem 6.12. *Let $L = K_1 \cup K_2$ be a 2-component oriented link. Suppose*

$$\operatorname{Kh}(L;\mathbb{Z}/2) \cong \operatorname{Kh}(L_1;\mathbb{Z}/2)$$

as bi-graded vector spaces. Then L is isotopic to L_1.

Proof. To simplify notation, we use $\mathbb{F}$ to denote the field $\mathbb{Z}/2$. By (6.8) and the universal coefficient theorem, we have

$$\operatorname{Kh}(L;\mathbb{F}) = \mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)} \oplus \mathbb{F}^2_{(2)} \oplus \mathbb{F}_{(3)} \oplus \mathbb{F}^2_{(4)} \oplus \mathbb{F}_{(6)}, \tag{6.8}$$

where the subscripts represent the internal grading. By Theorem 6.11, we have

$$8 = \operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(L;\mathbb{F}) \geq \operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_1;\mathbb{F}) \cdot \operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_2;\mathbb{F}),$$

hence $\operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_i;\mathbb{F}) \leq 4$. On the other hand, $\operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_i;\mathbb{F}) = 2\operatorname{rank}_{\mathbb{F}} \operatorname{Khr}(K_i;\mathbb{F})$, and $\operatorname{rank}_{\mathbb{F}} \operatorname{Khr}(K_i;\mathbb{F})$ is always odd for knots. Therefore

$$\begin{aligned}\operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_1;\mathbb{F}) &= \operatorname{rank}_{\mathbb{F}} \operatorname{Kh}(K_2;\mathbb{F}) = 2,\\ \operatorname{rank}_{\mathbb{F}} \operatorname{Khr}(K_1;\mathbb{F}) &= \operatorname{rank}_{\mathbb{F}} \operatorname{Khr}(K_2;\mathbb{F}) = 1.\end{aligned}$$

By Kronheimer–Mrowka's unknot detection theorem, both K_1 and K_2 are unknots. Hence

$$\operatorname{Kh}(K_1;\mathbb{F}) \otimes \operatorname{Kh}(K_2;\mathbb{F}) = \mathbb{F}_{(-2)} \oplus \mathbb{F}^2_{(0)} \oplus \mathbb{F}_{(2)}.$$

By Theorem 6.11 and (6.8), we have $\operatorname{lk}(K_1, K_2) = 1$ or 2, hence K_1 is homotopic to the closure of a 1-braid or a 2-braid in the solid torus $S^3 - N(K_2)$. Let $l := \operatorname{lk}(K_1, K_2)$, and let $\hat{\beta}_l$ be the closure of an arbitrary l-braid subject to the condition that $\hat{\beta}_l$ is connected. By [33, Section 4.4],

$$\dim_{\mathbb{C}} \operatorname{AHI}(K_1, l) \equiv \dim_{\mathbb{C}} \operatorname{AHI}(\hat{\beta}_l, l) = 1 \bmod 2, \tag{6.9}$$

$$\dim_{\mathbb{C}} \operatorname{AHI}(K_1, j) \equiv \dim_{\mathbb{C}} \operatorname{AHI}(\hat{\beta}_l, j) = 0 \bmod 2 \quad \text{if } j > l. \tag{6.10}$$

By Proposition 2.6 and Kronheimer–Mrowka's spectral sequence [20, Theorem 8.2], we have

$$4 = \mathrm{rank}_{\mathbb{F}}\, \mathrm{Khr}(L_1, p; \mathbb{F}) \geq \dim_{\mathbb{C}} \mathrm{AHI}(K_1), \tag{6.11}$$

where $p \in K_2$.

If $l = 1$, we have $\mathrm{AHI}(K_1, \pm 1) = \mathbb{C}$ by (6.9) and (6.11). By (6.10) and (6.11), we have $\mathrm{AHI}(K_1, j) = 0$ for all $j > l$. Hence the top f-grading of $\mathrm{AHI}(K_1)$ is 1, and K_1 is the closure of a 1-braid by Theorem 2.5. This implies that L is a Hopf link, whose Khovanov homology is different from $\mathrm{Kh}(L_1)$, which yields a contradiction. Hence we must have $l = 2$. By similar arguments, we obtain $\mathrm{AHI}(K_1, \pm 2) = \mathbb{C}$ and the top f-grading of $\mathrm{AHI}(K_1)$ is 2. Therefore K_1 is the closure of a 2-braid. Since K_1 is an unknot, this 2-braid must be given by a generator of the 2-braid group. The proof is then completed by directly checking all the possible choices of the generator and orientations. ∎

Now we prove the second part of Theorem 1.7. Let T be the left-handed trefoil. We have

$$\mathrm{Kh}(T; \mathbb{Z}) \cong \mathbb{Z}_{(-3,-9)} \oplus \mathbb{Z}_{(-2,-5)} \oplus \mathbb{Z}_{(0,-3)} \oplus \mathbb{Z}_{(0,-1)} \oplus (\mathbb{Z}/2)_{(-2,-7)},$$

where the subscripts represent the (h, q)-bigrading. Let L_2 be the disjoint union of T and an unknot U. Then

$$\mathrm{Khr}(L_2, p; \mathbb{Z}) \cong \mathrm{Kh}(T; \mathbb{Z}),$$

where the basepoint p is in U.

Theorem 6.13. *Let $L = K_1 \cup K_2$ be a 2-component link with a basepoint $q \in K_2$. Suppose*

$$\mathrm{Khr}(L, q; \mathbb{Z}) \cong \mathrm{Khr}(L_2, p; \mathbb{Z})$$

as bi-graded abelian groups. Then the link L splits into a left-handed trefoil K_1 and an unknot K_2.

Proof. By Kronheimer–Mrowka's spectral sequence [20, Theorem 8.2], we have

$$4 = \mathrm{rank}_{\mathbb{Z}}\, \mathrm{Khr}(L, q; \mathbb{Z}) \geq \dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, q). \tag{6.12}$$

Let Γ be the local system associated with K_1. We have

$$\dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, q) \geq \mathrm{rank}_{\mathcal{R}}\, \mathrm{I}^{\natural}(L, q; \Gamma).$$

By Corollary 3.3,

$$\mathrm{rank}_{\mathcal{R}}\, \mathrm{I}^{\natural}(L, q; \Gamma) = \mathrm{rank}_{\mathcal{R}}\, \mathrm{I}^{\natural}(K_2 \cup U, q; \Gamma).$$

By excision,

$$\mathrm{rank}_{\mathcal{R}}\, \mathrm{I}^{\natural}(K_2 \cup U, q; \Gamma) = 2 \dim_{\mathbb{C}} \mathrm{I}^{\natural}(K_2, q).$$

Hence $\dim_{\mathbb{C}} \mathrm{I}^{\natural}(K_2, q) \leq 2$. We know that $\dim_{\mathbb{C}} \mathrm{I}^{\natural}(K_2, q)$ is always odd since crossing changes do not change the parity of $\mathrm{I}^{\natural}$ and $\mathrm{I}^{\natural}(\text{unknot}) \cong \mathbb{C}$. Hence K_2 is the unknot by

[20, Proposition 1.4]. Let $\mathbb{F} = \mathbb{Z}/2$. The universal coefficient theorem implies

$$\begin{aligned}\mathrm{Khr}(L,q;\mathbb{F}) &\cong \mathrm{Khr}(L_2,p;\mathbb{F})\\ &\cong \mathbb{F}_{(-3,-9)} \oplus \mathbb{F}_{(-2,-5)} \oplus \mathbb{F}_{(0,-3)} \oplus \mathbb{F}_{(0,-1)} \oplus \mathbb{F}_{(-2,-7)} \oplus \mathbb{F}_{(-3,-7)}\\ &\cong \mathbb{F}_{(6)} \oplus \mathbb{F}_{(5)} \oplus \mathbb{F}_{(4)} \oplus \mathbb{F}^2_{(3)} \oplus \mathbb{F}_{(1)},\end{aligned}$$

where the subscripts in the second to last row represent the (h,q)-bigrading, and the subscripts in the last row represent the internal grading. According to [30, Corollary 3.2.C],

$$\mathrm{Kh}^{i,j}(L;\mathbb{F}) \cong \mathrm{Khr}^{i,j-1}(L,q;\mathbb{F}) \oplus \mathrm{Khr}^{i,j+1}(L,q;\mathbb{F}).$$

Hence

$$\mathrm{Kh}(L;\mathbb{F}) \cong \mathbb{F}_{(0)} \oplus \mathbb{F}^2_{(2)} \oplus \mathbb{F}_{(3)} \oplus \mathbb{F}^3_{(4)} \oplus \mathbb{F}^2_{(5)} \oplus \mathbb{F}_{(6)} \oplus \mathbb{F}_{(7)}. \tag{6.13}$$

By Theorem 6.11,

$$12 = \mathrm{rank}_{\mathbb{F}}\,\mathrm{Kh}(L;\mathbb{F}) \ge \mathrm{rank}_{\mathbb{F}}\,\mathrm{Kh}(K_1;\mathbb{F})\,\mathrm{rank}_{\mathbb{F}}\,\mathrm{Kh}(K_2;\mathbb{F}) = 2\,\mathrm{rank}_{\mathbb{F}}\,\mathrm{Kh}(K_1;\mathbb{F}).$$

Hence $\mathrm{rank}_{\mathbb{F}}\,\mathrm{Kh}(K_1;\mathbb{F}) \le 6$ and $\mathrm{rank}_{\mathbb{F}}\,\mathrm{Khr}(K_1;\mathbb{F}) \le 3$. Kronheimer–Mrowka's unknot detection theorem [20] and Baldwin–Sivek's trefoil detection theorem [3] imply K_1 is either an unknot or a trefoil.

Suppose K_1 is an unknot. Then $\mathrm{lk}(K_1, K_2) = 1$ or 2 by Theorem 6.11 and (6.13). Now K_1 is a knot in the solid torus $S^3 - N(K_2)$ with winding number 1 or 2, and by Proposition 2.6,

$$4 \ge \dim_{\mathbb{C}} \mathrm{AHI}(K_1).$$

The argument in the proof of Theorem 6.12 shows that K_1 is either the closure of the 1-braid or the closure of a generator of the 2-braid group. In either case, $\mathrm{Khr}(L,q;\mathbb{Z})$ is not isomorphic to $\mathrm{Khr}(L_2,p;\mathbb{Z})$, which is a contradiction.

By the discussion above, K_1 must be a trefoil, hence Theorem 6.11 and (6.13) imply $\mathrm{lk}(K_1, K_2) = 0$. Hence K_1 is homotopic to the unknot in $S^3 - N(K_2)$. By [33, Section 4.4], $\dim_{\mathbb{C}} \mathrm{AHI}(K_1, j)$ is even for all $|j| > 0$. We also have $\dim_{\mathbb{C}} \mathrm{AHI}(K_1, 0) \ge \dim_{\mathbb{C}} \mathrm{AHI}(\mathcal{U}_1, 0) = 2$ by Proposition 4.3. By (6.12), $\dim_{\mathbb{C}} \mathrm{AHI}(K_1) = \dim_{\mathbb{C}} \mathrm{I}^{\natural}(L,q) \le 4$, hence the argument above implies that the top f-grading of $\mathrm{AHI}(K_1)$ is 0. By Theorem 2.4, K_2 has a Seifert disk which is disjoint from K_1, hence L is a split link. ■

7. Topological properties from instanton Floer homology

From now on, let $L = K_1 \cup \cdots \cup K_n$ be a hypothetical link that satisfies Condition 6.9. The goal is to deduce a contradiction from Condition 6.9. This section derives several topological properties of L using instanton Floer homology.

In Section 7.1, we show that every K_i bounds a disk D_i such that when $i \neq j$, the disk D_i intersects K_j transversely in $|\mathrm{lk}(K_i, K_j)|$ points. If $n \ge 4$, we show that this property implies that (after isotopy) the link L has the form given by Figure 8. In Section 7.2, we show that K_n can be isotoped in the complement of $K_1 \cup \cdots \cup K_{n-1}$ in such a way that it becomes disjoint from the surface shown in Figure 11 or the surface shown in Figure 12.

Let $L' = K_1 \cup \cdots \cup K_{n-1}$. In Sections 8 and 9, we will use the two topological properties summarized above to study the possible isotopy classes of K_n in $S^3 - L'$. The results proved in this section will imply that the *homotopy* class of K_n in $S^3 - L'$ has a specific form and that K_n bounds a disk that intersects a fibered Seifert surface of $K_1 \cup \cdots \cup K_{n-1}$ in an arc. Sections 8 and 9 will find all the possible isotopy classes of the intersection arc by solving the word problem in $\pi_1(S^3 - L')$. This will imply that L must be isotopic to the link $L_{n,1-n}$ or $L_{n,2-n}$ defined in Section 10. We will then show that the ranks of the Khovanov homology of $L_{n,1-n}$ and $L_{n,2-n}$ are both greater than 2^n, which yields a contradiction.

7.1. Seifert surfaces of K_i

Proposition 7.1. *For each K_i, there exists an embedded disk D_i such that*

(1) *$\partial D_i = K_i$;*

(2) *for each $j \neq i$, if $|i-j| = 1$ or $n-1$, then D_i intersects K_j transversely in one point; otherwise, the disk D_i is disjoint from K_j.*

Proof. Pick a basepoint $p \in K_i$. By Proposition 5.4, we have $\dim \mathrm{I}^\natural(L,p) = 2^{n-1}$. By Proposition 5.3, K_i is an unknot. Let $N(K_i)$ be a tubular neighborhood of K_i, and view $L - K_i$ as a link in the solid torus $S^3 - N(K_i)$. By Proposition 2.6,

$$\dim \mathrm{AHI}(L - K_i) = \mathrm{I}^\natural(L,p) = 2^{n-1}.$$

By Corollary 4.4, there exists a meridional disk $\hat{D}_i$ which is disjoint from K_j if $|i-j| \neq 1$ or $n-1$ and intersects K_j transversely at one point if $|i-j| = 1$ or $n-1$. The meridional disk $\hat{D}_i$ extends to the desired Seifert disk of K_i. ■

Definition 7.2. Let $D_1, \ldots, D_n$ be a sequence of immersed disks in $\mathbb{R}^3$ such that $\partial D_i = K_i$ for all i. We say that the sequence $D_1, \ldots, D_n$ is *generic* if every self-intersection point of $\bigsqcup D_i$ is locally diffeomorphic to one of the following models in $\mathbb{R}^3$ at $(0,0,0)$:

(1) the intersection of $\{(x,y,z) \mid z = 0,\ y \geq 0\}$ and the yz-plane;

(2) the intersection of the xy-plane and the yz-plane;

(3) the intersection of the xy-, yz-, and xz-planes.

If $D = (D_1, \ldots, D_n)$ is generic, let $\Sigma_1(D)$, $\Sigma_2(D)$, $\Sigma_3(D)$ be the sets of self-intersection points described by (1), (2), (3) above respectively.

Definition 7.3. If $D = (D_1, \ldots, D_n)$ is generic, define the *complexity* of D to be the number of components of $\Sigma_2(D)$.

Notice that if D is generic, then the complexity of D is greater than or equal to $\frac{1}{2}\,\#\,\Sigma_1(D)$, which is at least n.

Definition 7.4. We say that the sequence $D = (D_1, \dots, D_n)$ is *admissible* if

(1) D is generic;

(2) $\#\Sigma_1(D) = 2n$;

(3) every point in $\Sigma_3(D)$ is contained in at least two different disks in D.

Remark 7.5. In the definitions above, the disks $\{D_i\}$ are only required to be immersed. Condition (2) in the definition above is equivalent to the following statement: for each $j \neq i$, if $|i - j| = 1$ or $n - 1$, then D_i intersects K_j transversely at one point; otherwise, the immersed disk D_i is disjoint from K_j. Moreover, the interior of D_i is disjoint from K_i.

Proposition 7.6. *If $n \geq 4$, then there exists a sequence $D = (D_1, \dots, D_n)$ of disks such that $\partial D_i = K_i$ for all i, and D is admissible with complexity n.*

Proof. By Proposition 7.1, there exists a sequence of disks $\hat{D} = (\hat{D}_1, \dots, \hat{D}_n)$ such that for all i, $\hat{D}_i$ is embedded, $\partial \hat{D}_i = K_i$, and $\#\Sigma_1(\hat{D}) = 2n$. Perturb $\hat{D}_1, \dots, \hat{D}_n$ in such a way that they are generic. Since all the disks are embedded, every point in $\Sigma_3(\hat{D})$ is contained in three different disks. Therefore $\hat{D}$ is admissible, so admissible configurations exist. Let $D = (D_1, \dots, D_n)$ be an admissible configuration with minimal complexity.

We first show that all the D_i's are embedded. Suppose there exists i such that D_i is not embedded. Then by admissibility, D_i does not have triple self-intersections, and the self-intersection locus of D_i is a disjoint union of circles. Let $\gamma \subset D_i$ be a circle in the self-intersection of D_i.

Let B^2 be the unit disk in $\mathbb{R}^2$, and let $f_i : B^2 \to \mathbb{R}^3$ be an immersion that parametrizes D_i. Then $f_i^{-1}(\gamma)$ is a double cover of γ. There are three possibilities:

Case 1: $f_i^{-1}(\gamma)$ is a disjoint union of two circles, and they bound disjoint disks B_1 and B_2. In this case, take a diffeomorphism ι from B_1 to B_2 such that

$$(f_i \circ \iota)|_{\partial B_1} = f_i|_{\partial B_1}.$$

Define

$$f_i'(p) := \begin{cases} f_i(p) & \text{if } p \notin B_1 \cup B_2, \\ f_i(\iota(p)) & \text{if } p \in B_1, \\ f_i(\iota^{-1}(p)) & \text{if } p \in B_2. \end{cases}$$

By smoothing f_i', we obtain an immersed disk with the same boundary as D_i but has fewer self-intersection components. Figure 6 shows a local picture of f_i' after the

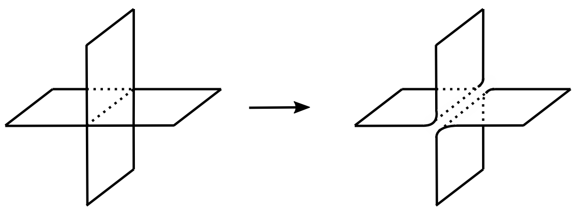

Fig. 6. The local construction of f_i' after smoothing.

smoothing. Replacing D_i by the image of the smoothed f_i' decreases the complexity of D and preserves the admissibility condition.

Case 2: $f_i^{-1}(\gamma)$ is a disjoint union of two circles, and they bound disks B_1 and B_2 with $B_1 \supset B_2$. In this case, take a diffeomorphism ι from B_1 to B_2 such that

$$(f_i \circ \iota)|_{\partial B_1} = f_i|_{\partial B_1}.$$

Define

$$f_i'(p) := \begin{cases} f_i(p) & \text{if } p \notin B_1, \\ f_i(\iota(p)) & \text{if } p \in B_1. \end{cases}$$

Replacing f_i by the smoothed version of f_i' gives an admissible configuration with smaller complexity.

Case 3: $f_i^{-1}(\gamma)$ is one circle, and it bounds a disk B_1. We show that this case is impossible. Let $\tau : f_i^{-1}(\gamma) \to f_i^{-1}(\gamma)$ be the deck transformation of the double cover. Fix an orientation of $f_i^{-1}(\gamma)$. For each $p \in f_i^{-1}(\gamma)$, let $v(p)$ be the unit tangent vector of $f_i^{-1}(\gamma)$ with positive orientation, and let $w(p)$ be the unit normal vector of $f_i^{-1}(\gamma)$ pointing outward of B_1. Then the images of $v(p), w(p), w(\tau(p))$ under the tangent map of f_i form a basis of $\mathbb{R}^3$. However, isotoping the point p to $\tau(p)$ on $f_i^{-1}(\gamma)$ reverses the orientation of the basis, which yields a contradiction.

Since D is assumed to have minimal complexity among admissible configurations, we conclude that D_i has to be embedded.

The intersection of D_i and D_j $(i \neq j)$ is a disjoint union of compact 1-manifolds, possibly with boundary. Assume there exist $i \neq j$ such that the intersection of D_i and D_j contains a circle γ. The circle γ bounds a disk B_1 in D_i, and bounds a disk B_2 in D_j. Let

$$D_i' := (D_i - B_i) \cup B_j, \quad D_j' := (D_j - B_j) \cup B_i.$$

Replacing D_i and D_j by D_i' and D_j' and smoothing the corners, we obtain a generic configuration with smaller complexity. Since neither D_i' nor D_j' has triple self-intersection points, the new configuration is still admissible, contradicting the definition of D. We conclude that the intersection of D_i and D_j $(i \neq j)$ does not contain any circle.

By the admissibility assumption, the intersection $D_i \cap D_j$ $(i \neq j)$ consists of circles and at most one arc, and there is an arc if and only if $|i - j| = 1$ or $n - 1$. As a consequence, D_i and D_j are disjoint if $|i - j| \neq 1, n - 1$, and $D_i \cap D_j$ is an arc if $|i - j| = 1$ or $n - 1$. Since $n \geq 4$, this implies that $D_i \cap D_{i+1}$ is disjoint from $D_j \cap D_{j+1}$ whenever $i \neq j$ (where the subscripts are taken modulo n), so the complexity of D is n. ■

Let $L' := K_1 \cup \cdots \cup K_{n-1}$. Proposition 7.6 has the following corollary.

Corollary 7.7. *The link L' is a connected sum of $n - 2$ Hopf links as given by Figure 7. If $n \geq 4$, then the link L has a diagram described by Figure 8.*

Proof. The first part of the statement follows from Theorem 5.1 and Proposition 5.2. For the second part, by Proposition 7.6, there exists a sequence of disks $D_1, \ldots, D_{n-1}$ such that (1) D_i is embedded and $\partial D_i = K_i$ for $i = 1, \ldots, n-1$, (2) if $|i - j| = 1$, the disks D_i

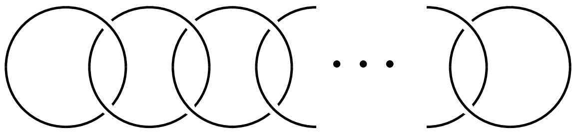

Fig. 7. The link L'.

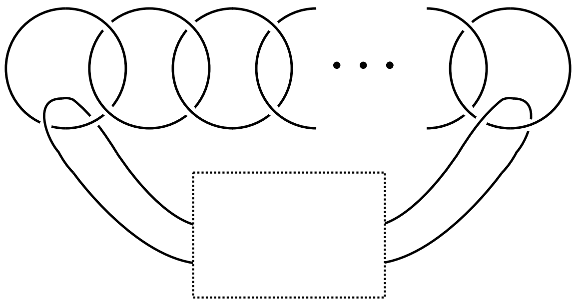

Fig. 8. The link L.

and D_j intersect in an arc, (3) if $i \neq j$ and $|i - j| \neq 1$, the disks D_i and D_j are disjoint. It follows that if we arrange Figure 7 in such a way that each component is contained in a (flat) plane, then after an isotopy, the link L' is given by Figure 7, and D_i is the disk bounded by K_i on the corresponding plane. Moreover, by Proposition 7.6 again, we may assume that K_n bounds an embedded disk D_n which intersects D_1 and D_{n-1} respectively in an arc and is disjoint from $D_2 \cup \cdots \cup D_{n-2}$, so L is isotopic to a link described by Figure 8. ∎

7.2. Seifert surfaces of L'

We recall the following property of fibered links.

Lemma 7.8. *Suppose L_1 and L_2 are two oriented fibered links with oriented Seifert surfaces S_1 and S_2 respectively. Let $f_1 : S_1 \to S_1$ and $f_2 : S_2 \to S_2$ be the monodromies. Take $p_1 \in L_1$, $p_2 \in L_2$, and form the connected sum $L_1 \# L_2$ and the boundary connected sum $S_1 \#_b S_2$ with respect to (p_1, p_2). Then $L_1 \# L_2$ is a fibered link with Seifert surface $S_1 \#_b S_2$ and monodromy $f_1 \#_b f_2$.*

Proof. Given a compact surface S with boundary, and given a diffeomorphism $f : S \to S$ that restricts to the identity on a neighborhood of ∂S, define

$$\mathcal{M}_f := S \times [0, 1]/\sim,$$

where $\sim$ is defined by $(x, 0) \sim (f(x), 1)$ for $x \in S$, and $(x, t_1) \sim (x, t_2)$ for $x \in \partial S$, $t_1, t_2 \in [0, 1]$. By the assumptions of the lemma,

$$\mathcal{M}_{f_1} \cong S^3, \quad \mathcal{M}_{f_2} \cong S^3,$$

and the images of ∂S_1 and ∂S_2 are isotopic to L_1 and L_2 respectively. Therefore,

$$\mathcal{M}_{f_1 \#_b f_2} \cong \mathcal{M}_{f_1} \# \mathcal{M}_{f_2} \cong S^3,$$

and the image of $\partial(S_1 \#_b S_2)$ is isotopic to $L_1 \# L_2$. ∎

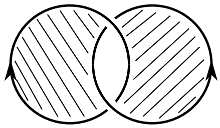

Fig. 9. Seifert surface of the Hopf link with linking number -1.

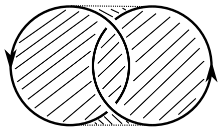

Fig. 10. Seifert surface of the Hopf link with linking number 1.

Notice that the Hopf link is fibered. Depending on the orientations of the components, the corresponding Seifert surface is given by Figure 9 or Figure 10. Both Seifert surfaces are diffeomorphic to the annulus, and the monodromies are Dehn twists along the core circles.

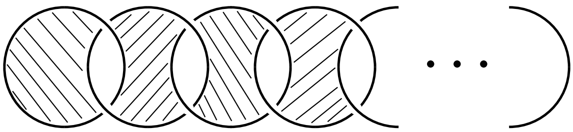

Fig. 11. The Seifert surface S_1.

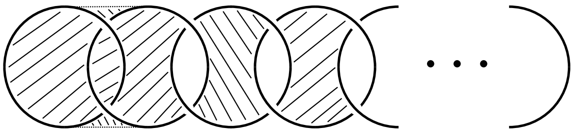

Fig. 12. The Seifert surface S_2.

Let S_1 and S_2 be the Seifert surfaces of L' given by Figure 11 and Figure 12 respectively. By Lemma 7.8, the link L' is fibered with respect to both S_1 and S_2. For each $j = 1, 2$, endow the components $K_1, \ldots, K_{n-1}$ with the boundary orientation of S_j and choose an arbitrary orientation for K_n. Then the algebraic intersection number of K_n and S_j is equal to the sum of the linking numbers $\sum_{i=1}^{n-1} \operatorname{lk}(K_n, K_i)$. Therefore, Condition 6.9 (3) implies that there exists exactly one $j \in \{1, 2\}$ such that the algebraic intersection number of S_j and K_n is zero. The main result of this subsection is the following proposition.

Proposition 7.9. *Suppose $j \in \{1, 2\}$ and the algebraic intersection number of K_n with S_j is zero. Then there exists a knot K'_n such that K'_n is disjoint from S_j, and K_n is isotopic to K'_n in $\mathbb{R}^3 - L'$.*

Before proving Proposition 7.9, we need to prove some results on instanton Floer homology. Let U be an unknot included in a 3-ball which is disjoint from L', let m_i be a small meridian circle around K_i $(1 \le i \le n-1)$ and u_i be a small arc joining K_i and m_i.

Lemma 7.10. *We have*

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) = 2^{2n-4}, \tag{7.1}$$

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) = 2^{2n-3}, \tag{7.2}$$

and

$$\mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i \cup U, \sum_{i=1}^{n-1} u_i; \Gamma_U\Big) \cong \mathcal{R}^{2^{2n-3}}, \tag{7.3}$$

where Γ_U is the local system associated with U.

Proof. Picking a crossing between m_1 and K_1 and applying Kronheimer–Mrowka's unoriented skein exact triangle [20, Section 6], we obtain a 3-periodic exact sequence

$$\cdots \to \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) \to \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=2}^{n-1} m_i, \sum_{i=2}^{n-1} u_i\Big)$$
$$\to \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=2}^{n-1} m_i, \sum_{i=2}^{n-1} u_i\Big) \to \cdots .$$

See [32, Section 3] for more details. The above exact triangle implies

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) \le 2 \dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=2}^{n-1} m_i, \sum_{i=2}^{n-1} u_i\Big).$$

Repeating this argument for the other meridians, we obtain

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big)$$
$$\le 2^{n-2} \dim_{\mathbb{C}} \mathrm{I}(S^3, L' \cup m_{n-1}, u_{n-1}) = 2^{n-2} \dim_{\mathbb{C}} \mathrm{I}^{\natural}(L', p).$$

By Propositions 5.2 and 5.4, $\dim_{\mathbb{C}} \mathrm{I}^{\natural}(L', p) = 2^{n-2}$, therefore

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) \le 2^{2(n-2)}. \tag{7.4}$$

A similar earrings-removal argument yields

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) \le 2^{n-2} \dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, p) = 2^{2n-3}. \tag{7.5}$$

We recall some properties of the instanton knot Floer homology KHI for oriented links, which was introduced in [17, Definition 2.4]. Given an oriented link $M \subset S^3$, the homology group $\mathrm{KHI}(M)$ carries an Alexander $\mathbb{Z}$-grading and a homological $\mathbb{Z}/2$-grading. The rank of $\mathrm{KHI}(M)$ does not depend on the orientation of M. We use $\mathrm{KHI}(M, i)$ to denote the summand of $\mathrm{KHI}(M)$ with Alexander degree i, and use $\chi(\mathrm{KHI}(M, i))$ to denote its Euler characteristic with respect to the homological grading. Recall that we always take coefficients in $\mathbb{C}$ for instanton Floer homology in this article. According to [17, Theorem 3.6 and (14)], we have

$$\sum_i \chi(\mathrm{KHI}(M, i)) t^i = \pm (t^{1/2} - t^{-1/2})^{|M|-1} \Delta_M(t),$$

where $\Delta_M(t)$ denotes the single-variable Alexander polynomial of M. Notice that the Alexander polynomial for L' satisfies $|\Delta_{L'}(-1)| = 2^{n-2}$ for every orientation of L'. Therefore, taking $M = L'$, we have

$$\dim_{\mathbb{C}} \mathrm{KHI}(L') \ge 2^{n-2} |\Delta_{L'}(-1)| = 2^{2n-4}.$$

By [32, Proposition 5.1],

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) = \dim_{\mathbb{C}} \mathrm{KHI}(L') \ge 2^{2n-4}. \tag{7.6}$$

Inequalities (7.4) and (7.6) imply (7.1).

Consider the two admissible triples

$$\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i \cup U, \sum_{i=1}^{n-1} u_i\Big), \quad (S^1 \times S^2, S^1 \times \{p_1, p_2\}, v),$$

where v is an arc joining the two components of $S^1 \times \{p_1, p_2\}$. Let $N(K_1)$ be a small tubular neighborhood of K_1 in the first triple, and deform U into $N(K_1)$ by an isotopy. Let $N(S^1 \times \{p_1\})$ be a small tubular neighborhood of $S^1 \times \{p_1\}$ in the second triple. Cutting out $N(K_1)$ and $N(S^1 \times \{p_1\})$, exchanging them, and gluing back, we obtain two new triples

$$\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big), \quad (S^1 \times S^2, S^1 \times \{p_1, p_2\} \cup U', v),$$

where U' is an unknot included in a 3-ball disjoint from $S^1 \times \{p_1, p_2\}$. By the torus excision theorem and the definition of AHI, we have

$$\mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i \cup U, \sum_{i=1}^{n-1} u_i; \Gamma_U\Big) \otimes_{\mathcal{R}} \mathrm{AHI}(\emptyset; \Gamma) \\ \cong \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i; \Gamma_\emptyset\Big) \otimes_{\mathcal{R}} \mathrm{AHI}(U'; \Gamma), \quad (7.7)$$

where $\Gamma_\emptyset$ is the trivial local system with coefficients $\mathcal{R}$. By (3.3) and Example 3.4, $\mathrm{AHI}(\emptyset; \Gamma) \cong \mathcal{R}$ and $\mathrm{AHI}(U'; \Gamma) \cong \mathcal{R}^2$. By (3.3) and (7.1),

$$\mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i; \Gamma_\emptyset\Big) \cong \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) \otimes_{\mathbb{C}} \mathcal{R} \cong \mathcal{R}^{2^{2n-4}}.$$

Therefore by (7.7),

$$\mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i \cup U, \sum_{i=1}^{n-1} u_i; \Gamma_U\Big) \cong \mathcal{R}^{2^{2n-4}} \otimes_{\mathcal{R}} \mathcal{R}^2 \cong \mathcal{R}^{2^{2n-3}}.$$

This completes the proof of (7.3).

Let Γ_{K_n} be the local system on $\mathcal{R}(S^3, L \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i)$ associated with K_n. By Corollary 3.3 and the universal coefficient theorem, we have

$$\dim_{\mathbb{C}} \mathrm{I}\Big(S^3, L \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i\Big) \geq \mathrm{rank}_{\mathcal{R}}\, \mathrm{I}\Big(S^3, L \cup \bigcup_{i=1}^{n-1} m_i, \sum_{i=1}^{n-1} u_i; \Gamma_{K_n}\Big) \\ = \mathrm{rank}_{\mathcal{R}}\, \mathrm{I}\Big(S^3, L' \cup \bigcup_{i=1}^{n-1} m_i \cup U, \sum_{i=1}^{n-1} u_i; \Gamma_U\Big) = 2^{2n-3}.$$

The above inequality together with (7.5) implies (7.2). ■

Choose $j \in \{1, 2\}$ such that the algebraic intersection number of S_j and K_n is zero. Choose an orientation of S_j, and endow L' with the boundary orientation. For each $i = 1, \dots, n-1$, let $N(K_i)$ be a sufficiently small tubular neighborhood of K_i that is disjoint from m_i. Cut $N(K_i)$ from S^3 and glue it back using a diffeomorphism that identifies the meridian of $N(K_i)$ to $S_j \cap \partial N(K_i)$. Since $S^3 - L'$ is fibered over S^1 with fiber S_j, the manifold obtained from the cutting-and-pasting (which are, of course, Dehn surgeries) is fibered over S^1 with fiber S^2. Since the orientation-preserving mapping class group of S^2 is trivial, the resulting manifold is diffeomorphic to $S^1 \times S^2$ with the product fibration. Let

$$\hat{K}_1, \dots, \hat{K}_n, \hat{m}_1, \dots, \hat{m}_{n-1}, \hat{U} \subset S^1 \times S^2$$

be the images of $K_1, \dots, K_n, m_1, \dots, m_{n-1}, U$ respectively by the cutting-and-pasting. Let $\hat{L}' := \hat{K}_1 \cup \dots \cup \hat{K}_{n-1}$ be the image of L', let $m := m_1 \cup \dots \cup m_{n-1}$ be the union of the earrings, and let $\hat{m} := \hat{m}_1 \cup \cdots \cup \hat{m}_{n-1}$ be the image of m. We further require that the diffeomorphism used to glue back $N(K_i)$ fixes $u_i \cap \partial N(K_i)$ for all $i = 1, \ldots, n-1$, so the image of u_i is an arc connecting $\hat{K}_i$ and $\hat{m}_i$, and we denote the image of u_i by $\hat{u}_i$.

Given a $\mathbb{C}$-vector space V, a linear map $f : V \to V$, and $\lambda \in \mathbb{C}$, we will use $E(V, f, \lambda)$ to denote the generalized eigenspace of f with eigenvalue λ.

Lemma 7.11. *For all $\lambda \in \mathbb{C}$, we have*

$$\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), \lambda\Big) = \dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), \lambda\Big),$$

where the operator $\mu^{\mathrm{orb}}(S^2)$ is the μ-map defined by $\{p\} \times S^2 \subset S^1 \times S^2$ for an arbitrary $p \in S^1$.

Proof. By the torus excision theorem and Lemma 7.10, we have

$$\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big) \cong \mathrm{I}\Big(S^3, L \cup m, \sum_{i=1}^{n-1} u_i\Big) \cong \mathbb{C}^{2^{2n-3}}, \tag{7.8}$$

and

$$\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{U}}\Big) \cong \mathrm{I}\Big(S^3, L' \cup m \cup U, \sum_{i=1}^{n-1} u_i; \Gamma_U\Big) \cong \mathcal{R}^{2^{2n-3}}, \tag{7.9}$$

where $\Gamma_{\hat{U}}$ is the local system associated with $\hat{U}$, and Γ_U is the local system associated with U. Since the algebraic intersection number of K_n and S_j is zero, we conclude that $\hat{K}_n$ is homotopic to $\hat{U}$ in $S^1 \times S^2 - \bigcup_{i=1}^{n-1} \hat{u}_i$. Let $\Gamma_{\hat{K}_n}$ be the local system associated with $\hat{K}_n$. By Proposition 4.2, we have

$$\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{K}_n}(h)\Big) \cong \mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{U}}(h)\Big) \tag{7.10}$$

for every $h \in \mathbb{C} - \{0\}$ satisfying $(1-h^2)\theta(h) \neq 0$, and this isomorphism commutes with $\mu^{\mathrm{orb}}(S^2)$. As a consequence, for every $\lambda \in \mathbb{C}$ and $h \in \mathbb{C} - \{0\}$ satisfying $(1-h^2)\theta(h) \neq 0$, we have

$$\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{K}_n}(h)\Big), \mu^{\mathrm{orb}}(S^2), \lambda\Big) = \dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i; \Gamma_{\hat{U}}(h)\Big), \mu^{\mathrm{orb}}(S^2), \lambda\Big). \tag{7.11}$$

When $h(1-h^2)\theta(h) \neq 0$, the universal coefficient theorem and (7.9), (7.10) imply

$$\mathrm{I}\Big(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n, \sum_{i=1}^{n-1}\hat{u}_i;\Gamma_{\hat{K}_n}(h)\Big) \cong \mathrm{I}\Big(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{U}, \sum_{i=1}^{n-1}\hat{u}_i;\Gamma_{\hat{U}}(h)\Big) \cong \mathbb{C}^{2^{2n-3}}.$$

On the other hand, notice that when $h=1$, the local systems $\Gamma_{\hat{K}_n}(h)$ and $\Gamma_{\hat{U}}(h)$ become the trivial system with coefficients $\mathbb{C}$, hence by (7.8) and (7.9),

$$\mathrm{I}\Big(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n, \sum_{i=1}^{n-1}\hat{u}_i;\Gamma_{\hat{K}_n}(1)\Big) \cong \mathbb{C}^{2^{2n-3}} \cong \mathrm{I}\Big(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{U}, \sum_{i=1}^{n-1}\hat{u}_i;\Gamma_{\hat{U}}(1)\Big).$$

Therefore the desired result follows from (7.11) by taking the limit $h\to 1$ and invoking Proposition 4.1 (3). ∎

Notice that $\hat{L}'\cup\hat{m}$ is a braid in $S^1\times S^2$. In fact, the projection of $\hat{L}'\cup\hat{m}$ to S^1 is a diffeomorphism on each component. Therefore, after an isotopy, we may write $S^1\times S^2$ as $A_0\cup_{S^1\times S^1} A_1$, where A_0, A_1 are diffeomorphic to $S^1\times D^2$, such that

(1) $\hat{K}_1$, $\hat{m}_1$ are included in A_0 and are given by $S^1\times\{p_1\}$ and $S^2\times\{p_2\}$ with p_1, p_2 in D^2,

(2) $\hat{u}_1$ is an arc connecting $\hat{K}_1$ and $\hat{m}_1$, and $\hat{u}_1$ is included in A_0;

(3) $\hat{K}_2,\dots,\hat{K}_n,\hat{m}_2,\dots,\hat{m}_{n-1},\hat{U}$ are included in A_1.

Let

$$\mathcal{L}_0 := \bigcup_{i=2}^{n-1}\hat{K}_i \cup \bigcup_{i=2}^{n-1}\hat{m}_i\cup\hat{K}_n, \tag{7.12}$$

$$\mathcal{L}_1 := \bigcup_{i=2}^{n-1}\hat{K}_i \cup \bigcup_{i=2}^{n-1}\hat{m}_i\cup\hat{U}. \tag{7.13}$$

Then $\mathcal{L}_0$ and $\mathcal{L}_1$ are two annular links in A_1. By the definition of annular instanton Floer homology, we have

$$\mathrm{AHI}(\mathcal{L}_0) \cong \mathrm{I}(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n, \hat{u}_1), \tag{7.14}$$

$$\mathrm{AHI}(\mathcal{L}_1) \cong \mathrm{I}(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{U}, \hat{u}_1). \tag{7.15}$$

Lemma 7.12. *Assume there exists a connected oriented Seifert surface $S\subset S^3$ of L' such that S is compatible with the orientation of L', and S has genus g and is disjoint from K_n. Then*

$$\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n, \sum_{i=1}^{n-1}\hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), 2g+2n-4\Big) = \dim_{\mathbb{C}} E\big(\mathrm{I}(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n, \hat{u}_1), \mu^{\mathrm{orb}}(S^2), 2g+2n-4\big),$$

and

$$E\Big(\mathrm{I}\Big(S^1\times S^2,\hat{L}'\cup\hat{m}\cup\hat{K}_n,\sum_{i=1}^{n-1}\hat{u}_i\Big),\mu^{\mathrm{orb}}(S^2),j\Big)=0$$

for all integers $j>2g+2n-4$.

Proof. After an isotopy, we may assume that S intersects each m_i transversely in one point. The image of $S-\bigcup_{i=1}^{n-1}N(K_i)$ in $S^1\times S^2$ is a connected surface with $n-1$ boundary components, where the boundary components are given by the meridians of $\hat{K}_1,\dots,\hat{K}_{n-1}$. Therefore we can glue disks to the boundary of the image of $S-\bigcup_{i=1}^{n-1}N(K_i)$ and obtain a connected closed surface in $S^1\times S^2$ with genus g that is disjoint from $\hat{K}_n$ and intersects each of $\hat{K}_1,\dots,\hat{K}_{n-1},\hat{m}_1,\dots,\hat{m}_{n-1}$ transversely in one point. Denote this surface by $\hat{S}$. After a further isotopy, we may assume that the arcs $\hat{m}_1,\dots,\hat{m}_{n-1}$ lie on $\hat{S}$.

Recall that $\hat{K}_1$ and $\hat{m}_1$ are contained in $A_0\cong S^1\times D^2$ and are given by $S^1\times\{p_1\}$ and $S^1\times\{p_2\}$ for $p_1,p_2\in D^2$. Take a point $p_0\in D^2-\{p_1,p_2\}$, and let $\hat{K}_0\subset A_0$ be the knot $S^1\times\{p_0\}$. After a further isotopy, we may assume that $\hat{S}$ intersects $\hat{K}_0$ transversely in one point. Let c be a simple closed curve on D^2 such that p_0,p_1 are inside c and p_2 is outside. Let $T_1\subset A_0$ be the torus given by $T_1:=S^1\times c$.

Notice that $\hat{S}$ is homologous to the slice of S^2 in $S^1\times S^2$, therefore $\mu^{\mathrm{orb}}(\hat{S})=\mu^{\mathrm{orb}}(S^2)$. The surface $\hat{S}$ intersects $\hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0$ transversely in $2n-1$ points. Applying [34, Theorem 6.1] to the surface $\hat{S}$, we deduce that the set of eigenvalues of $\mu^{\mathrm{orb}}(S^2)$ on

$$\mathrm{I}\Big(S^1\times S^2,\hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0,\sum_{i=1}^{n-1}\hat{u}_i\Big)$$

is included in

$$\{-(2g+2n-3),-(2g+2n-5),\dots,2g+2n-5,2g+2n-3\}.$$

Consider the triple $(S^1\times S^2,S^1\times\{q_1,q_2\},v)$, where $q_1,q_2\in S^2$ and v is an arc connecting $S^1\times\{q_1\}$ and $S^1\times\{q_2\}$. Let T_2 be a torus given by the boundary of a tubular neighborhood of $S^1\times\{q_1\}$. Recall that $T_1\subset A_0$ is the torus $S^1\times c$ as defined above. Applying the torus excision on the triple

$$\Big(S^1\times S^2,\hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0,\sum_{i=1}^{n-1}\hat{u}_i\Big)\sqcup(S^1\times S^2,S^1\times\{q_1,q_2\},v)$$

along $T_1\cup T_2$ yields

$$\begin{aligned}\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1\times S^2,\hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0,\sum_{i=1}^{n-1}\hat{u}_i\Big),\mu^{\mathrm{orb}}(S^2),\lambda\Big)\\ =\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1\times S^2,\hat{L}'\cup\hat{m}\cup\hat{K}_n,\sum_{i=1}^{n-1}\hat{u}_i\Big),\mu^{\mathrm{orb}}(S^2),\lambda-1\Big)\\ +\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1\times S^2,\hat{L}'\cup\hat{m}\cup\hat{K}_n,\sum_{i=1}^{n-1}\hat{u}_i\Big),\mu^{\mathrm{orb}}(S^2),\lambda+1\Big)\end{aligned}$$

for all $\lambda \in \mathbb{C}$. Therefore

$$\begin{aligned}&\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n, \sum_{i=1}^{n-1}\hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), 2g+2n-4\Big)\\ &= \dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0, \sum_{i=1}^{n-1}\hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), 2g+2n-3\Big) \qquad (7.16)\end{aligned}$$

and

$$\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n, \sum_{i=1}^{n-1}\hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), j\Big) = 0 \qquad (7.17)$$

for all integers $j > 2g+2n-4$.

Similarly, applying torus excision on the triple

$$(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0, \hat{u}_1) \sqcup (S^1\times S^2, S^1\times\{q_1,q_2\}, v)$$

along $T_1\cup T_2$ yields

$$\begin{aligned}&\dim_{\mathbb{C}} E\big(\mathrm{I}(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0, \hat{u}_1), \mu^{\mathrm{orb}}(S^2), 2g+2n-3\big)\\ &\qquad = \dim_{\mathbb{C}} E\big(\mathrm{I}(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n, \hat{u}_1), \mu^{\mathrm{orb}}(S^2), 2g+2n-4\big). \qquad (7.18)\end{aligned}$$

Let $\{z_1,\dots,z_{2n-1}\}\subset\hat{S}$ be the intersection of $\hat{S}$ with $\hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0$. Apply the singular excision theorem [34, Theorem 6.4] on the triple

$$(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0, \hat{u}_1)\sqcup\Big(S^1\times\hat{S}, S^1\times\{z_1,\dots,z_{2n-1}\}, \sum_{i=2}^{n-1}\hat{u}_i\Big)$$

along $\hat{S}$ in the first component, and a slice of $\hat{S}$ in the second component, and invoke [34, Proposition 6.7], to obtain

$$\begin{aligned}&\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0, \sum_{i=1}^{n-1}\hat{u}_i\Big), \mu^{\mathrm{orb}}(\hat{S}_0), 2g+2n-3\Big)\\ &= \dim_{\mathbb{C}} E\big(\mathrm{I}(S^1\times S^2, \hat{L}'\cup\hat{m}\cup\hat{K}_n\cup\hat{K}_0, \hat{u}_1), \mu^{\mathrm{orb}}(\hat{S}_0), 2g+2n-3\big). \qquad (7.19)\end{aligned}$$

Since $\mu^{\mathrm{orb}}(\hat{S}_0) = \mu^{\mathrm{orb}}(S^2)$, the first part of the lemma is proved by (7.16), (7.18), and (7.19). The second part of the lemma is proved by (7.17). ∎

Lemma 7.13. *Let $\mathcal{L}_0\subset A_1$ be the annular link defined by (7.12). Suppose there exists a meridional surface S (cf. Definition 2.3) with genus g such that S intersects $\mathcal{L}_0$ transversely at m points. Then there exists a connected Seifert surface $\hat{S}$ of L' in S^3 such that $\hat{S}$ is compatible with the orientation of L' and is disjoint from K_n, and the genus of $\hat{S}$ is equal to*

$$g + m/2 - n + 2.$$

Proof. Suppose there is a component K of $\mathcal{L}_0$ whose intersection with S has different signs. Then we can attach a tube to S along a segment of K to decrease the value of m by 2 and increase the value of g by 1. Repeating this process until the number of intersection points of S with each component of $\mathcal{L}_1$ equals the absolute value of their algebraic intersection number, we obtain a new meridional surface $S' \subset A_1$ such that

(1) the genus of S' equals $g + (m - 2n + 4)/2$;

(2) S' intersects each of $\hat{K}_2, \dots, \hat{K}_{n-1}, \hat{m}_2, \dots, \hat{m}_{n-1}$ transversely in one point;

(3) S' is disjoint from $\hat{K}_n$.

Since S' is a meridional surface, by attaching a disk in A_0, we can complete the surface S' to a closed surface with the same genus that intersects each of $\hat{K}_1, \dots, \hat{K}_{n-1}$ transversely in one point and is disjoint from $\hat{K}_n$, therefore it gives rise to a Seifert surface of L' in S^3 with the same genus that is disjoint from K_n, hence the lemma is proved. ∎

Corollary 7.14. *Let g_0 be the smallest integer with the property that there exists a connected oriented Seifert surface $S \subset S^3$ of L' that is compatible with the orientation of L', has genus g_0 and is disjoint from K_n. Then*

$$\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), 2g_0 + 2n - 4\Big) \neq 0,$$

and

$$E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), j\Big) = 0$$

for all integers $j > 2g_0 + 2n - 4$.

Proof. By Theorem 2.4 and (7.14), there are integers g, m such that there exists a meridional surface in A_1 with genus g and intersecting $\mathcal{L}_1$ transversely in m points such that

$$\dim_{\mathbb{C}} E\big(\mathrm{I}(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \hat{u}_1), \mu^{\mathrm{orb}}(S^2), 2g + m\big) > 0. \tag{7.20}$$

Let $g' := g + m/2 - n + 2$. By Lemma 7.13, we may choose g, m such that there exists a connected oriented Seifert surface of L' in S^3 that is compatible with the orientation of L', has genus g', and is disjoint from K_n. Since $2g + m = 2g' + 2n - 4$, by Lemma 7.12 and (7.20) we have

$$\begin{aligned} &\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), 2g' + 2n - 4\Big) \\ &\qquad = \dim_{\mathbb{C}} E\big(\mathrm{I}(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \hat{u}_1), \mu^{\mathrm{orb}}(S^2), 2g + m\big) > 0. \end{aligned} \tag{7.21}$$

By the definition of g_0, we have $g_0 \leq g'$. On the other hand, the second part of Lemma 7.12 implies that

$$\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{K}_n, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), j\Big) = 0 \tag{7.22}$$

for all integers $j > 2g_0 + 2n - 4$. Therefore by (7.21) and (7.22), we have $g_0 \geq g'$. In conclusion, we must have $g = g_0$, and the lemma follows from (7.21) and (7.22). ∎

Replacing $\hat{K}_n$ with $\hat{U}$ in the previous arguments, we also have the following lemma.

Lemma 7.15. *Let g_1 be the smallest integer with the property that there exists a connected oriented Seifert surface $S \subset S^3$ of L' that is compatible with the orientation of L', has genus g_1 and is disjoint from U. Then*

$$\dim_{\mathbb{C}} E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), 2g_1 + 2n - 4\Big) \neq 0,$$

and

$$E\Big(\mathrm{I}\Big(S^1 \times S^2, \hat{L}' \cup \hat{m} \cup \hat{U}, \sum_{i=1}^{n-1} \hat{u}_i\Big), \mu^{\mathrm{orb}}(S^2), j\Big) = 0$$

for all integers $j > 2g_1 + 2n - 4$. ■

Proof of Proposition 7.9. It is obvious that the minimal genus g_1 in Lemma 7.15 is zero, therefore by Lemma 7.11, Corollary 7.14, and Lemma 7.15, the genus g_0 in Corollary 7.14 is also zero. As a result, there exists a connected oriented Seifert surface $S \subset S^3$ for L' with genus zero that is disjoint from K_n and is compatible with the orientation of L'. Since the minimal-genus Seifert surface for an oriented fibered link is unique up to isotopy, we conclude that there exists an ambient isotopy of S^3 that fixes L' and takes S to S_j. This ambient isotopy gives the desired isotopy from K_n to K_n'. ■

8. The fundamental group of $\mathbb{R}^3 - L'$

This section takes a detour to study the properties of $\pi_1(\mathbb{R}^3 - L')$. The results in this section (or more precisely, Lemma 8.12 and Corollary 8.14) will be used in the proof of the non-existence of L.

By the Wirtinger presentation, $\pi_1(\mathbb{R}^3 - L')$ is generated by

$$g_1, \dots, g_{n-1}, g_2', \dots, g_{n-2}'$$

as shown in Figure 13, where the basepoint is taken to be a point above and far away from the diagram. Notice that g_i' and g_i are homotopic relative to the basepoint because one

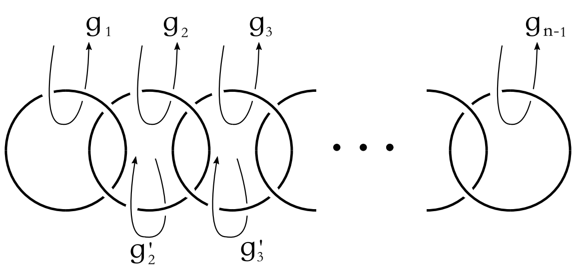

Fig. 13. Generators of $\pi_1(\mathbb{R}^3 - L')$.

can shrink $K_1 \cup \cdots \cup K_{i-1}$ into a small neighborhood of K_i. Therefore $\pi_1(\mathbb{R}^3 - L')$ is generated by $g_1, \ldots, g_{n-1}$, and the Wirtinger presentation gives

$$\pi_1(\mathbb{R}^3 - L') = \langle g_1, \ldots, g_{n-1} \mid [g_i, g_{i+1}] = 1 \text{ for } i = 1, \ldots, n-2 \rangle.$$

To simplify notation, for the rest of this section we will use m to denote $n-1$, and use G to denote the group $\pi_1(\mathbb{R}^3 - L')$. For $i = 1, \ldots, m$, define the set

$$C_i := \begin{cases} \{g_1, g_2, g_1^{-1}, g_2^{-1}\} & \text{if } i = 1, \\ \{g_{i-1}, g_i, g_{i+1}, g_{i-1}^{-1}, g_i^{-1}, g_{i+1}^{-1}\} & \text{if } i = 2, \ldots, m-1, \\ \{g_{m-1}, g_m, g_{m-1}^{-1}, g_m^{-1}\} & \text{if } i = m. \end{cases}$$

The first part of this section solves the word problem for G.

Definition 8.1. A *word* is a sequence $(x_1, \ldots, x_N)$ such that

$$x_i \in \{g_1, \ldots, g_m, g_1^{-1}, \ldots, g_m^{-1}\} \quad \text{for all } i.$$

We call $x_1, \ldots, x_N$ the *letters* of the word $(x_1, x_2, \ldots, x_N)$.

Definition 8.2. The word $(x_1, x_2, \ldots, x_N)$ is called *reduced* if for every pair $u < v$ with $(x_u, x_v) = (g_i, g_i^{-1})$ or (g_i^{-1}, g_i), there exists w such that $u < w < v$ and $x_w \notin C_i$.

Definition 8.3. Define an equivalence relation $\sim$ on the set of words using the following relations as generators:

$$(x_1, \ldots, x_k, g_i, g_{i+1}, x_{k+3}, \ldots, x_N) \sim (x_1, \ldots, x_k, g_{i+1}, g_i, x_{k+3}, \ldots, x_N),$$
$$(x_1, \ldots, x_k, g_i^{-1}, g_{i+1}, x_{k+3}, \ldots, x_N) \sim (x_1, \ldots, x_k, g_{i+1}, g_i^{-1}, x_{k+3}, \ldots, x_N),$$
$$(x_1, \ldots, x_k, g_i, g_{i+1}^{-1}, x_{k+3}, \ldots, x_N) \sim (x_1, \ldots, x_k, g_{i+1}^{-1}, g_i, x_{k+3}, \ldots, x_N),$$
$$(x_1, \ldots, x_k, g_i^{-1}, g_{i+1}^{-1}, x_{k+3}, \ldots, x_N) \sim (x_1, \ldots, x_k, g_{i+1}^{-1}, g_i^{-1}, x_{k+3}, \ldots, x_N).$$

It is straightforward to verify that if two words are equivalent and one of them is reduced, then the other is also reduced. Therefore $\sim$ defines an equivalence relation on the set of reduced words.

Every word $(x_1, \ldots, x_N)$ represents an element of G by taking the product $x_1 \cdots x_N$. By the definition of G, equivalent words represent the same element.

Proposition 8.4. *Every element of G is represented by a reduced word. Two reduced words represent the same element if and only if they are equivalent.*

Proof. Define another group $\widetilde{G}$ as follows. The elements of $\widetilde{G}$ are the equivalence classes of reduced words. If $(x_1, \ldots, x_N)$ is a reduced word, we use $[x_1, \ldots, x_N] \in \widetilde{G}$ to denote the equivalence class of $(x_1, \ldots, x_N)$. Let $(x_1, \ldots, x_N)$ be a reduced word, and let $y \in \{g_1, \ldots, g_m, g_1^{-1}, \ldots, g_m^{-1}\}$. If the word $(x_1, \ldots, x_N, y)$ is reduced, define

$$[x_1, \ldots, x_N] \cdot [y] := [x_1, \ldots, x_N, y]. \tag{8.1}$$

If the word $[x_1,\dots,x_N,y]$ is not reduced, then there exists u such that $x_u y = 1$ and every letter in $(x_{u+1},\dots,x_N)$ commutes with both x_u and y. In this case, define

$$[x_1,\dots,x_N]\cdot[y] := [x_1,\dots,x_{u-1},x_{u+1},\dots,x_N]. \tag{8.2}$$

For different choices of x_u, the right-hand side of (8.2) gives the same equivalence class. Moreover, if we take a different representative of $[x_1,\dots,x_N]$, the right-hand sides of (8.1) and (8.2) remain the same. It is also straightforward to verify that if y_1 and y_2 are commutative generators of G, then

$$[[x_1,\dots,x_N]\cdot y_1]\cdot y_2 = [[x_1,\dots,x_N]\cdot y_2]\cdot y_1. \tag{8.3}$$

Therefore, we obtain a well-defined product operator on $\widetilde{G}$ defined inductively by

$$[x_1,\dots,x_N]\cdot[y_1,\dots,y_M] := [[x_1,\dots,x_N]\cdot[y_1,\dots,y_{M-1}]]\cdot y_M.$$

We show that

$$(a\cdot b)\cdot c = a\cdot(b\cdot c) \tag{8.4}$$

for all $a,b,c\in\widetilde{G}$ by induction on the length of the reduced words representing c.

If c is given by a word with length 1, write $b=[x_1,\dots,x_N]$ and $c=[y_1]$. We discuss two cases. If $(x_1,\dots,x_N,y_1)$ is a reduced word, then (8.4) follows from the definition. If $(x_1,\dots,x_N,y_1)$ is not reduced, let y_1^{-1} be the reciprocal of y_1; then there exists $(x_1',\dots,x_{N-1}')$ such that $b=[x_1',\dots,x_{N-1}',y_1^{-1}]$. Let $b' := [x_1',\dots,x_{N-1}']$. By definition and (8.3),

$$(a\cdot b)\cdot c = ((a\cdot b')\cdot y_1^{-1})\cdot y_1 = a\cdot b' = a\cdot(b\cdot c).$$

Hence (8.4) is proved if c is given by a word with one letter.

In general, if $c=[x_1,\dots,x_N]$ with $N\ge 2$, let $c' := [x_1,\dots,x_{N-1}]$. By definition and the induction hypothesis,

$$a\cdot(b\cdot c) = a\cdot((b\cdot c')\cdot x_N) = (a\cdot(b\cdot c'))\cdot x_N = ((a\cdot b)\cdot c')\cdot x_N = (a\cdot b)\cdot c.$$

In conclusion, we have proved that $\widetilde{G}$ is associative.

For an element $[x_1,\dots,x_N]\in\widetilde{G}$, we have $[x_1,\dots,x_N]\cdot[x_N^{-1},\dots,x_1^{-1}]=1$, so every element in $\widetilde{G}$ has an inverse, therefore $\widetilde{G}$ is a group.

By the universal property, there is a unique homomorphism φ from G to $\widetilde{G}$ defined by $\varphi(g_i) := [g_i]$. We also have a map ψ from $\widetilde{G}$ to G defined by

$$\psi([x_1,\dots,x_N]) := x_1\cdots x_N.$$

Since

$$\psi([x_1,\dots,x_N]\cdot[y_1,\dots,y_M]) = \psi([x_1,\dots,x_N])\cdot\psi([y_1,\dots,y_M]),$$

the map ψ is a group homomorphism. It is obvious from the definitions that φ and ψ are inverse to each other, therefore φ and ψ are isomorphisms, and the proposition is proved. ∎

Definition 8.5. If $(x_1, \dots, x_N)$ is a reduced word and $w = x_1 \cdots x_N \in G$, we call $x_1 \cdots x_N$ a *reduced presentation* of w.

Definition 8.6. For $w \in G$, define length(w) to be the length of a reduced presentation of w.

By Proposition 8.4, length$(\cdot)$ does not depend on the choice of the reduced presentation, so it is well-defined.

Lemma 8.7. *For $C \subset \{g_1, \dots, g_m, g_1^{-1}, \dots, g_m^{-1}\}$, let G_C be the subgroup of G generated by C. Suppose $(x_1, \dots, x_N)$ is a reduced word. Then $x_1 \cdots x_N \in G_C$ if and only if $x_i \in C \cup C^{-1}$ for all i.*

Proof. Let $w = x_1 \cdots x_N$. Suppose $w \in G_C$. Then $w = y_1 \cdots y_M$ with $y_i \in C \cup C^{-1}$ for all i. If $(y_1, \dots, y_M)$ is not reduced, there exist letters y_u and y_v such that $y_u y_v = 1$ and every letter between y_u and y_v commutes with both y_u and y_v. Removing y_u and y_v from the word yields a shorter word representing the same element of G. Repeating this process, we obtain a reduced word representing w which is given by a subsequence of $y_1, \dots, y_M$. By Proposition 8.4, this word is equivalent to $(x_1, \dots, x_N)$, and hence $x_i \in C \cup C^{-1}$ for all i.

The other direction of the lemma is obvious. ∎

Lemma 8.8. *For each i, the centralizer of g_i in G is generated by C_i.*

Proof. Suppose there exists an element w in the centralizer of g_i that is not generated by C_i; choose such a w with $N :=$ length(w) smallest possible. Let $w = x_1 \cdots x_N$ be a reduced presentation of w. Then there exists u such that $x_u \notin C_i$. If $x_1 \in C_i$, then $x_2 \cdots x_N$ is an element in the centralizer of g_i, and by Lemma 8.7, the element $x_2 \cdots x_N$ is not generated by C_i, which contradicts the minimality of N. Therefore $x_1 \notin C_i$. Similarly, $x_N \notin C_i$. Moreover, the same property holds for every reduced word that is equivalent to $(x_1, \dots, x_N)$. Therefore $(x_1, \dots, x_N, g_i, x_N^{-1}, \dots, x_1^{-1})$ is a reduced word. By Proposition 8.4, $x_1 \cdots x_N g_i x_N^{-1} \cdots x_1^{-1} \neq g_i$, therefore w is not in the centralizer of g_i, contradicting the assumption. ∎

Lemma 8.9. *If $m \geq 4$, then the only element that commutes with both g_1 and g_m is 1.*

Proof. The lemma is an immediate consequence of Proposition 8.4, Lemma 8.7 and Lemma 8.8. ∎

Lemma 8.10. *Suppose $(x_1, \dots, x_N)$ and $(y_1, \dots, y_N)$ are reduced words such that x_1, y_1 do not commute. Then $x_1 \cdots x_N \neq y_1 \cdots y_N$.*

Proof. By Proposition 8.4, we only need to show that the two words $(x_1, \dots, x_N)$ and $(y_1, \dots, y_N)$ are not equivalent. Assume the contrary, and let w_x be the word obtained by removing all the letters that are not equal to x_1 or y_1 from $(x_1, \dots, x_N)$. Similarly, let w_y be the word obtained by removing all the letters that are not equal to x_1 or y_1 from $(y_1, \dots, y_N)$. Since x_1 and y_1 do not commute with each other, w_x must be equal to w_y

if $(x_1, \dots, x_N)$ and $(y_1, \dots, y_N)$ are equivalent. On the other hand, w_x starts with x_1, and w_y starts with y_1, so $w_x \neq w_y$, a contradiction. ■

Lemma 8.11. *Suppose $m \geq 4$. Then the centralizer of $g_1 g_m$ is generated by $g_1 g_m$.*

Proof. Let $w = x_1 \cdots x_N$ be an element in the centralizer of $g_1 g_m$, and assume that $(x_1, \dots, x_N)$ is a reduced word. We use induction on N to show that w is a power of $g_1 g_m$. If $N = 0$, then $w = 1$ and the property is trivial. From now, assume $N > 0$, and assume that the claim is proved when $\text{length}(w) < N$.

By the assumptions on w, we have $g_1 g_m w g_m^{-1} g_1^{-1} = w$, so the word

$$(g_1, g_m, x_1, \dots, x_N, g_m^{-1}, g_1^{-1})$$

is not reduced. Hence there are three possibilities:

(a1) g_m^{-1} is a letter in $(x_1, \dots, x_N)$, and every letter before the first appearance of g_m^{-1} in $(x_1, \dots, x_N)$ is in C_m;

(a2) g_m is a letter in $(x_1, \dots, x_N)$, and every letter after the last appearance of g_m in $(x_1, \dots, x_N)$ is in C_m;

(a3) every letter in $(x_1, \dots, x_N)$ is contained in C_m.

Case (a3) implies $[w, g_m] = 1$, therefore $[w, g_1] = 1$, and by Lemma 8.9, $w = 1$. Since we are assuming $N > 0$, case (a3) is impossible.

Similarly, since $g_m^{-1} g_1^{-1} w g_1 g_m = w$, the word

$$(g_m^{-1}, g_1^{-1}, x_1, \dots, x_N, g_1, g_m)$$

is not reduced. Applying the same argument as before, we conclude that there are two possibilities:

(b1) g_1 is a letter in $(x_1, \dots, x_N)$, and every letter before the first appearance of g_1 in $(x_1, \dots, x_N)$ is in C_1;

(b2) g_1^{-1} is a letter in $(x_1, \dots, x_N)$, and every letter after the last appearance of g_1^{-1} in $(x_1, \dots, x_N)$ is in C_1;

Since $m \geq 4$, we have $C_1 \cap C_m = \emptyset$, therefore (a1) and (b1) are mutually exclusive, and (a2) and (b2) are mutually exclusive. Hence either (a2) and (b1) hold, or (a1) and (b2) hold.

If (a2) and (b1) hold, then $(x_1, \dots, x_N)$ is equivalent to a reduced word of the form $(g_1, x'_2, \dots, x'_{N-1}, g_m)$. Let $w' := x'_2 \cdots x'_{N-1}$. Then $[g_1 w' g_m, g_1 g_m] = [w, g_1 g_m] = 1$, and hence $[w', g_m g_1] = 1$. Let $\sigma : G \to G$ be the isomorphism of G defined by $\sigma(g_k) := g_{m+1-k}$. Then $[\sigma(w'), g_1 g_m] = [\sigma(w'), \sigma(g_m g_1)] = 1$. By the induction hypothesis, $\sigma(w')$ is a power of $g_1 g_m$, therefore w' is a power of $g_m g_1$, so $w = g_1 w' g_m$ is a power of $g_1 g_m$.

If (a1) and (b2) hold, then $(x_1, \dots, x_N)$ is equivalent to a reduced word of the form $(g_m^{-1}, x'_2, \dots, x'_{N-1}, g_1^{-1})$, and the result follows from a similar argument. ■

Lemma 8.12. *Suppose $m \geq 4$. The solutions $u, v \in G$ to the equation*

$$ug_1u^{-1} \cdot vg_mv^{-1} = g_1g_m \tag{8.5}$$

are given by

$$u = (g_1g_m)^k u', \tag{8.6}$$

$$v = (g_1g_m)^k v', \tag{8.7}$$

where $k \in \mathbb{Z}$, u' is in the centralizer of g_1, and v' is in the centralizer of g_m.

Remark 8.13. The expressions on the right-hand side of (8.6) and (8.7) are not required to be reduced. For example, we may have $k = 1$, $u' = 1$, $v' = g_m^{-1}$.

Proof. It is clear that every pair (u, v) given by (8.6) and (8.7) is a solution to (8.5). To prove the reverse, we use induction on $\text{length}(u) + \text{length}(v)$. If $\text{length}(u) + \text{length}(v) = 0$, then $u = v = 1$, and the result is obvious.

Suppose $\text{length}(u) + \text{length}(v) = N > 0$, and assume the result is proved when $\text{length}(u) + \text{length}(v) < N$. We can write u as $u = u_1u_2$ with the following properties:

- $\text{length}(u) = \text{length}(u_1) + \text{length}(u_2)$;
- u_2 is in the centralizer of g_1;
- u_1 does not have a reduced presentation that ends with a letter in C_1.

Notice that (u_1, v) is also a solution to (8.5), so the result is proved by the induction hypothesis if $u_2 \neq 1$.

Similarly, we can write v as $v = v_1v_2$ with the following properties:

- $\text{length}(v) = \text{length}(v_1) + \text{length}(v_2)$;
- v_2 is in the centralizer of g_m,
- v_1 does not have a reduced presentation that ends with a letter in C_m.

Since (u, v_2) is also a solution to (8.5), the result is proved by the induction hypothesis if $v_2 \neq 1$.

From now on, we assume $u_2 = v_2 = 1$. This implies $u = u_1$, $v = v_1$, so both ug_1u^{-1} and $vg_m^{-1}v^{-1}$ are reduced presentations, and we have

$$\text{length}(ug_1u^{-1}) = 2\,\text{length}(u) + 1,$$

$$\text{length}(vg_m^{-1}v^{-1}) = 2\,\text{length}(v) + 1.$$

As a result,

$$\text{length}(g_1^{-1}ug_1u^{-1}) = 2\,\text{length}(u) + 2 \text{ or } 2\,\text{length}(u),$$

$$\text{length}(g_mvg_m^{-1}v^{-1}) = 2\,\text{length}(v) + 2 \text{ or } 2\,\text{length}(v).$$

By (8.5),

$$g_1^{-1}ug_1u^{-1} = g_mvg_m^{-1}v^{-1},$$

so there are four possibilities:

Case 1: length(u) = length(v) + 1, the expression $g_m v g_m^{-1} v^{-1}$ is reduced, and $g_1^{-1} u g_1 u^{-1}$ is not reduced. By the previous assumption on u, the element u cannot be represented by a reduced word that ends with a letter in C_1, so for $g_1^{-1} u g_1 u^{-1}$ not to be reduced, u must have a presentation of the form $u = g_1 \hat{u}$, where length$(\hat{u})$ = length$(u) - 1$. Thus

$$g_m v g_m^{-1} v^{-1} = \hat{u} g_1 \hat{u}^{-1} g_1^{-1}. \tag{8.8}$$

Since the left-hand side of (8.8) is reduced, and the right-hand side of (8.8) is given by a word with the same length, the right-hand side of (8.8) is also reduced. Therefore, by Proposition 8.4, the corresponding words given by the two sides of (8.8) are equivalent. By the assumption that $v = v_1$, every reduced presentation of $g_m v g_m^{-1} v^{-1}$ has the property that the product of the first length$(v) + 1$ terms is $g_m v$. Similarly, since $u = u_1$, every reduced presentation of $\hat{u} g_1 \hat{u}^{-1} g_1^{-1}$ has the property that the product of the first length$(\hat{u}) + 1$ terms is $\hat{u} g_1$. Therefore

$$g_m v = \hat{u} g_1, \quad g_m^{-1} v^{-1} = \hat{u}^{-1} g_1^{-1}.$$

Hence $g_1 \hat{u} = v g_m$, and

$$g_1 g_m v = g_1 \hat{u} g_1 = v g_m g_1,$$

and so $[g_1 g_m, v g_m] = 1$. By Lemma 8.11, we have $v = (g_1 g_m)^k g_m^{-1}$ for some integer k. By the previous equations, $\hat{u} = g_1^{-1} (g_m g_1)^k$, and $u = g_1 \hat{u} = (g_1 g_m)^k$, so the desired result is proved.

Case 2: length(v) = length(u) + 1, and $g_1^{-1} u g_1 u^{-1}$ is reduced, while $g_m v g_m^{-1} v^{-1}$ is not. This case follows from the same argument as Case 1.

Case 3: length(u) = length(v), and both $g_1^{-1} u g_1 u^{-1}$ and $g_m v g_m^{-1} v^{-1}$ are reduced. This is impossible by Lemma 8.10.

Case 4: length(u) = length(v), neither $g_1^{-1} u g_1 u^{-1}$ nor $g_m v g_m^{-1} v^{-1}$ is reduced. By the previous assumption that $u = u_1$, for $g_1^{-1} u g_1 u^{-1}$ not to be reduced, u must have a presentation of the form $u = g_1 \hat{u}$, where length$(\hat{u})$ = length$(u) - 1$. Similarly by the assumption that $v = v_1$, there is a presentation of v given by $v = g_m^{-1} \hat{v}$, where length$(\hat{v})$ = length$(v) - 1$. Equation (8.5) gives

$$\hat{u} g_1 \hat{u}^{-1} g_1^{-1} g_m^{-1} \hat{v} g_m \hat{v}^{-1} = 1,$$

therefore

$$\hat{v} g_m \hat{v}^{-1} \hat{u} g_1 \hat{u} = g_m g_1.$$

Let $\sigma : G \to G$ be the isomorphism of G defined by $\sigma(g_k) := g_{m+1-k}$. Then

$$\sigma(\hat{v}) g_1 \sigma(\hat{v})^{-1} \sigma(\hat{u}) g_m \sigma(\hat{u})^{-1} = g_1 g_m.$$

By the induction hypothesis, $\sigma(\hat{v}) = (g_1 g_m)^k \hat{v}'$, $\sigma(\hat{u}) = (g_1 g_m)^k \hat{u}'$, where $k \in \mathbb{Z}$, $\hat{v}'$ is in the centralizer of g_1, and $\hat{u}'$ is in the centralizer of g_m. Therefore

$$u = g_1 \hat{u} = g_1 (g_m g_1)^k \sigma(\hat{u}') = (g_1 g_m)^k g_1 \sigma(\hat{u}'),$$
$$v = g_m^{-1} \hat{v} = g_m^{-1} (g_m g_1)^k \sigma(\hat{v}') = (g_1 g_m)^k g_m^{-1} \sigma(\hat{v}').$$

Since $g_1\sigma(\hat{u}')$ is in the centralizer of g_1, and $g_m^{-1}\sigma(\hat{v}')$ is in the centralizer of g_m, the desired result is proved.

In conclusion, every solution of (8.5) can be written as (8.6) and (8.7). ∎

Corollary 8.14. *Suppose $m \geq 4$. The solutions $u, v \in G$ to the equation*

$$ug_1u^{-1} \cdot vg_m^{-1}v^{-1} = g_1g_m^{-1} \tag{8.9}$$

are given by

$$u = (g_1g_m^{-1})^k u', \tag{8.10}$$

$$v = (g_1g_m^{-1})^k v', \tag{8.11}$$

where $k \in \mathbb{Z}$, u' is in the centralizer of g_1, and v' is in the centralizer of g_m.

Proof. Notice that there is an isomorphism $\sigma : G \to G$ defined by $\sigma(g_i) := g_i$ for $i < m$, and $\sigma(g_m) := g_m^{-1}$. Applying σ to the formulas of Lemma 8.12 yields the result. ∎

9. Arcs on compact surfaces

This section collects several results about arcs on surfaces that will be used later. We first recall the following result of Feustel.

Proposition 9.1 ([9]). *Let S be a smooth compact surface with boundary, and let γ_1, γ_2 be two smoothly embedded arcs in S such that $\gamma_i \cap \partial S = \partial\gamma_i$ and γ_i is transverse to ∂S for $i = 1, 2$. Suppose γ_1 and γ_2 are homotopic to each other in S relative to $\partial\gamma_1 = \partial\gamma_2$. Then γ_1 and γ_2 are isotopic to each other in S relative to $\partial\gamma_1 = \partial\gamma_2$.*

We also need the following result in Section 11.

Lemma 9.2. *Let $p \in S^1$, and let D be a closed disk in $(S^1 - \{p\}) \times (0, 1)$. Let $S := S^1 \times [0, 1] - D$ and $\gamma_0 := \{p\} \times [0, 1]$. Let $f_1 : S \to S$ be the Dehn twist along a curve parallel to $S^1 \times \{0\}$, and let $f_2 : S \to S$ be the Dehn twist along a curve parallel to $S^1 \times \{1\}$. Suppose γ is an arc on S from $(p, 0)$ to $(p, 1)$. Then there exist integers u and v such that γ is isotopic to $f_1^u f_2^v(\gamma_0)$ in S relative to ∂S.*

Proof. Notice that S can be embedded in $\mathbb{R}^2$. By the Jordan curve theorem, cutting S open along γ yields a closed annulus with corners, which is diffeomorphic to the manifold obtained by cutting S open along γ_0. Hence there exists an orientation-preserving diffeomorphism $\varphi : S \to S$ such that $\varphi(\gamma_0) = \gamma$ and $\varphi|_{\partial S} = \mathrm{id}$. Since

- the mapping class group of S is generated by Dehn twists (see, for example, [2] or [8, Corollary 4.16]);
- every simple closed curve on S is parallel to the boundary;
- Dehn twists along curves parallel to ∂D preserve the isotopy class of γ_0,

the desired result is proved. ∎

The rest of this section studies arcs on Seifert surfaces.

Definition 9.3. Let L_0 be a link in $\mathbb{R}^3$, let S be a Seifert surface of L_0, and let γ be an arc on S such that γ intersects ∂S transversely in S and $\gamma \cap \partial S = \partial\gamma$. Define $K(S,\gamma)$ to be the knot in $\mathbb{R} - L_0$ which bounds an embedded disk D in $\mathbb{R}^3$ that intersects S transversely in γ and intersects ∂S transversely in $\partial\gamma$.

Remark 9.4. Since $K(S,\gamma)$ can be isotoped to a neighborhood of γ in $D - \gamma$, the knot $K(S,\gamma)$ satisfying Definition 9.3 is unique up to isotopy in $\mathbb{R}^3 - L_0$. An example of $K(S,\gamma)$ can be constructed as follows. Let S' be an extension of S to a slightly bigger embedded surface such that S is in the interior of S'. Let $N(S') \subset \mathbb{R}^3$ be a small neighborhood of the zero section of the normal bundle of S'. Then $N(S')$ is a neighborhood of S. Let $\pi : N(S') \to S'$ be the bundle projection. Let γ' be an extension of γ in S'. Then $K(S,\gamma)$ can be taken to be the boundary of a neighborhood of γ in $\pi^{-1}(\gamma')$.

By definition, $K(S,\gamma)$ is always an unknot in $\mathbb{R}^3$.

Lemma 9.5. *Let S, γ be as in Definition 9.3. Suppose K is a knot in $\mathbb{R}^3 - L_0$ such that K bounds an embedded disk D in $\mathbb{R}^3$. Moreover, assume D intersects both ∂S and S transversely, and that $D \cap S$ is the disjoint union of γ and a family of circles. Then K is isotopic to $K(S,\gamma)$ in $\mathbb{R}^3 - L_0$.*

Proof. By the assumptions, γ is an arc in the interior of D, and $D \cap L_0 = \partial(D \cap S) = \partial\gamma$. Let K' be the boundary of a small neighborhood of γ in D. Then K is isotopic to K' in $D - \gamma$. Since $(D - \gamma) \cap L_0 = \emptyset$, the isotopy remains in $\mathbb{R}^3 - L_0$. By the definition of $K(S,\gamma)$, the knot K' is isotopic to $K(S,\gamma)$ in $\mathbb{R}^3 - L_0$, hence the lemma is proved. ∎

Lemma 9.6. *Let S, γ be as in Definition 9.3. Suppose L_0 is a fibered link with respect to the Seifert surface S and with monodromy $f : S \to S$. Then $K(S,\gamma)$ is isotopic to $K(S, f(\gamma))$ in $\mathbb{R}^3 - L_0$.*

Proof. By the definition of monodromy, there exists an isotopy $\tau : S \times [0,1] \to \mathbb{R}^3$ such that

- $\tau(x,t)$ is independent of t for $x \in \partial S$;
- $\tau(x,0) = x$ for all $x \in S$;
- $\tau(x,1) = f(x)$ for all $x \in S$.

The map τ induces an isotopy from $K(S,\gamma)$ to $K(S, f(\gamma))$ in $\mathbb{R}^3 - L_0$ by the family of knots $K(\tau(S,t), \tau(\gamma,t))$. ∎

10. The link $L_{u,v}$

This section defines a family of links $L_{u,v}$ and computes their Jones polynomials at $t = -1$. The computation will be used in the proof of the non-existence of the hypothetical link L satisfying Condition 6.9.

Definition 10.1. For a pair of integers (u, v) with $u \geq 3$, we define a link $L_{u,v}$ as follows. If $v \geq 0$, define $L_{u,v}$ to be the link given by Figure 14 with u components such that there are v crossings in the dotted rectangle. If $v < 0$, define $L_{u,v}$ to be the link given by Figure 15 with u components such that there are $|v|$ crossings in the dotted rectangle.

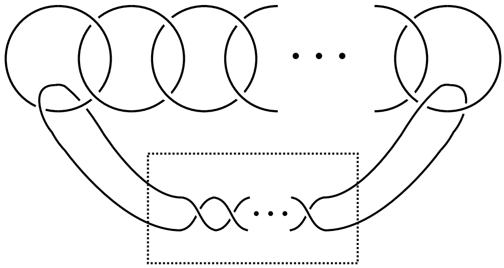

Fig. 14. $L_{u,v}$ when $v \geq 0$.

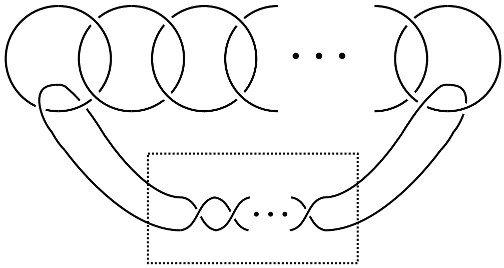

Fig. 15. $L_{u,v}$ when $v < 0$.

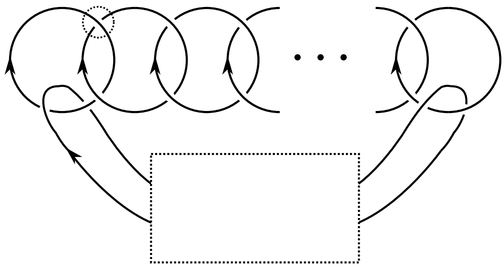

Fig. 16. An orientation of $L_{u,v}$.

The only difference between Figures 14 and 15 is that the crossings in the dotted rectangles are reversed. Notice that $L_{u,v}$ is alternating if $v \geq 0$.

Let $V(L_{u,v})$ be the reduced Jones polynomial of $L_{u,v}$ with the orientation given by Figure 16. The Jones polynomial is normalized so that if U is the unknot then $V(U) = 1$. Let $V_{u,v}$ be the value of $V(L_{u,v})$ when plugging in $t^{1/2} = -i$. Although this will not be used in the proofs, we remark that $|V_{u,v}|$ is equal to the determinant of $L_{u,v}$.

Notice that the Hopf link with linking number $+1$ has Jones polynomial $-t^{1/2} - t^{5/2}$, and the Hopf link with linking number -1 has Jones polynomial $-t^{-1/2} - t^{-5/2}$. Moreover, the Jones polynomial of the connected sum of links is the product of the Jones polynomials of the summands. Therefore, if v is even, by the skein relation at the dotted circle in Figure 16, we have

$$(t^{1/2} - t^{-1/2})V(L_{u-1,v}) = t^{-1}V(L_{u,v}) - t(-t^{1/2} - t^{5/2})^{u-1}.$$

If v is odd, then the skein relation gives

$$(t^{1/2} - t^{-1/2})V(L_{u-1,v}) = t^{-1}V(L_{u,v}) - t(-t^{1/2} - t^{5/2})^{u-2}(-t^{-1/2} - t^{-5/2}).$$

Hence

$$V_{u,v} = \begin{cases} (2i)V_{u-1,v} + (2i)^{u-1} & \text{if } v \text{ is even,} \\ (2i)V_{u-1,v} - (2i)^{u-1} & \text{if } v \text{ is odd.} \end{cases}$$

On the other hand, if v is even, the skein relation at a crossing in the dotted box in Figure 16 yields

$$(t^{1/2} - t^{-1/2})(-t^{1/2} - t^{5/2})^{u} = t^{-1}V(L_{u,v-2}) - tV(L_{u,v}).$$

If v is odd, then the skein relation gives

$$(t^{1/2} - t^{-1/2})(-t^{1/2} - t^{5/2})^{u-1}(-t^{-1/2} - t^{-5/2}) = t^{-1}V(L_{u,v-2}) - tV(L_{u,v}).$$

Therefore

$$V_{u,v} = \begin{cases} V_{u,v-2} - (2i)^{u+1} & \text{if } v \text{ is even,} \\ V_{u,v-2} + (2i)^{u+1} & \text{if } v \text{ is odd.} \end{cases}$$

It can be directly computed that

$$\begin{aligned} V(L_{3,-1}) &= 2 + t^2 + t^4, \\ V(L_{3,0}) &= t^7 - t^6 + 3t^5 - t^4 + 3t^3 - 2t^2 + t, \end{aligned}$$

therefore

$$V_{3,-1} = 4, \quad V_{3,0} = -12.$$

Combining the computations above, we have

$$V_{u,v} = (-1)^v(2i)^{u-1}(u + 2v). \tag{10.1}$$

As a consequence, we have the following result.

Corollary 10.2. *If* $|u + 2v| > 1$, *then* $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L_{u,v}; \mathbb{Z}/2) > 2^u$.

Proof. Since the coefficients of the Jones polynomial $V(L_{u,v})$ are the Euler characteristics of $\operatorname{Khr}(L_{u,v})$ at different q-gradings, we have

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(L_{u,v}; \mathbb{Z}/2) \geq |V_{u,v}| = |2^{u-1}(u + 2v)|.$$

If $|u + 2v| > 1$, then

$$\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L_{u,v}; \mathbb{Z}/2) = 2 \cdot \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Khr}(L_{u,v}; \mathbb{Z}/2) > 2^u. \qquad \blacksquare$$

11. The non-existence of L

This section combines the results from Sections 7–10 to prove that the hypothetical link L satisfying Condition 6.9 does not exist. We will proceed by showing more properties of L and eventually deduce a contradiction. By Lemma 6.10, this will finish the proof of Theorem 1.2.

Recall that the components of L are $K_1, \dots, K_n$, and $L' = K_1 \cup \cdots \cup K_{n-1}$. We have defined S_1 and S_2 to be the Seifert surfaces of L' given by Figures 11 and 12 respectively. By the conditions on the linking numbers of L, there are two possibilities:

Case 1: The algebraic intersection number of S_1 and K_n is zero.

Case 2: The algebraic intersection number of S_2 and K_n is zero.

By Proposition 7.9, for $j \in \{1, 2\}$, if the algebraic intersection number of S_j and K_n is zero, then K_n can be isotopically deformed in $\mathbb{R}^3 - L'$ into $\mathbb{R}^3 - S_j$. The first half of this section will focus on Case 1. The argument for Case 2 is similar and will be sketched afterwards.

Let γ_0 be the arc on S_1 as shown in Figure 17, where γ_0 starts from a point $p_1 \in K_1$ and travels from left to right, goes through the crossings of L' in an alternating way, and ends at a point $p_2 \in K_{n-1}$.

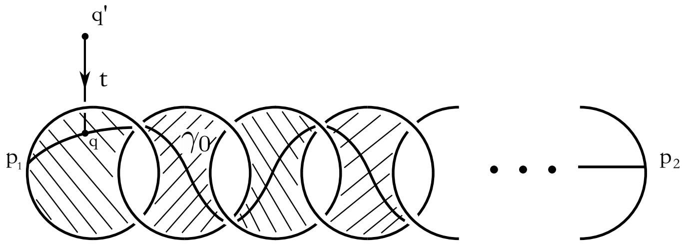

Fig. 17. The arc γ_0 on S_1.

Lemma 11.1. *Suppose Case* 1 *holds. Then there exists an arc $\gamma \subset S_1$ from p_1 to p_2 such that K_n is isotopic to $K(S_1, \gamma)$ in $\mathbb{R}^3 - L'$.*

Proof. By Proposition 7.9, there exists a knot $K_n' \subset \mathbb{R}^3 - S_1$ such that K_n is isotopic to K_n' in $\mathbb{R}^3 - L'$. By Proposition 7.1, K_n' bounds a disk D_n such that D_n intersects K_1 and K_{n-1} in one point each, and is disjoint from $K_2 \cup \cdots \cup K_{n-2}$. After a further isotopy, we may assume that $D_n \cap L' = \{p_1, p_2\}$, and that D_n intersects S_1 transversely. Therefore $D_n \cap S_1$ consists of an arc $\gamma \subset S_1$ from p_1 to p_2 and a union of circles. By Lemma 9.5, K_n' is isotopic to $K(S_1, \gamma)$ in $\mathbb{R}^3 - L'$. ■

Lemma 11.2. *Suppose Case* 1 *holds. Fix an orientation on S_1, let f_1, f_2 be the Dehn twists on S_1 along an oriented curve parallel to K_1 and an oriented curve parallel to K_{n-1} respectively, and let $f_3 : S_1 \to S_1$ be the monodromy of the fibered structure of L'. Let γ be the arc given by Lemma 11.1. Then there exist integers a, b, c such that γ is isotopic to $f_1^a f_2^b f_3^c(\gamma_0)$ relative to $\{p_1, p_2\}$ on S_1.*

Proof. If $n = 3$, then S_1 is an annulus, and every arc from p_1 to p_2 is isotopic to $f_1^a \gamma_0$ for some integer a. If $n = 4$, then S_1 is an annulus with a disk removed, and the result follows from Lemma 9.2 with $c = 0$. From now, we assume $n \geq 5$.

Fix a point q in the interior of γ_0 as shown in Figure 17. Let γ_1 be the subarc of γ_0 from p_1 to q, and let γ_2 be the subarc of γ_0 from q to p_2. Then there exists a closed curve w in the interior of S_1, based at q, such that γ is homotopic to $\gamma_1 \cdot w \cdot \gamma_2$ relative to $\{p_1, p_2\}$ on S_1. The loop w is not necessarily simple.

Let $g_1, \ldots, g_{n-1}$ be the generators of $\pi_1(\mathbb{R}^3 - L', q')$ defined in Section 8, where q' is their basepoint. Fix an arc t from q' to q as given by Figure 17, let t^{-1} be the same arc with the reversed orientation, and let $[w] \in \pi_1(\mathbb{R}^3 - L', q')$ be the homotopy class of $t \cdot w \cdot t^{-1}$.

Every oriented knot in $\mathbb{R}^3 - L'$ defines a conjugacy class in $\pi_1(\mathbb{R}^3 - L', q')$. By Corollary 7.7, the conjugacy class defined by K_n has the form $g_1^a g_{n-1}^b$, where $a, b \in \{-1, 1\}$ depend on the signs of the linking numbers and the orientation of K_n. On the other hand, under a suitable orientation, the conjugacy class defined by $K(S_1, \gamma)$ is given by $g_1[w]g_{n-1}^{b'}[w]^{-1}$, where $b' = (-1)^{n+1}$. Therefore, there exists $r \in \pi_1(\mathbb{R}^3 - L', q')$ such that

$$r g_1 [w] g_{n-1}^{b'} [w]^{-1} r^{-1} = g_1^a g_{n-1}^b.$$

Comparing the images of both sides in $H_1(\mathbb{R}^3 - L'; \mathbb{Z})$ yields $a = 1$, $b = b'$, thus the equation can be rewritten as

$$r g_1 r^{-1} \cdot (r[w]) g_{n-1}^{b'} (r[w])^{-1} = g_1 g_{n-1}^{b'}.$$

Applying Lemma 8.12 and Corollary 8.14 for $u = r$, $v = r[w]$, and invoking Lemma 8.8, we have

$$[w] = g_1^{\alpha} g_2^{\beta} g_{n-2}^{\delta} g_{n-1}^{\eta}$$

for some $\alpha, \beta, \delta, \eta \in \mathbb{Z}$. Notice that the image of $H_1(\text{interior}(S_1); \mathbb{Z})$ in $H_1(\mathbb{R}^3 - L'; \mathbb{Z})$ is generated by $[g_1] + [g_2], [g_2] + [g_3], \ldots, [g_{n-2}] + [g_{n-1}]$, therefore we have

$$\alpha - \beta + (-1)^{n-1}\delta + (-1)^n \eta = 0,$$

and hence

$$[w] = (g_1 g_2)^{\beta} g_1^{\theta} (g_{n-1}^{(-1)^{n-1}})^{\theta} (g_{n-2} g_{n-1})^{\delta}, \tag{11.1}$$

where $\theta := \alpha - \beta = (-1)^{n-1}(\eta - \delta)$.

We construct a set of generators of $\pi_1(\text{interior}(S_1), q)$ as follows. Let $u_1, \ldots, u_{n-2}$ be the oriented simple closed curves on S_1 as given by Figure 18. Each u_i intersects γ_0 at

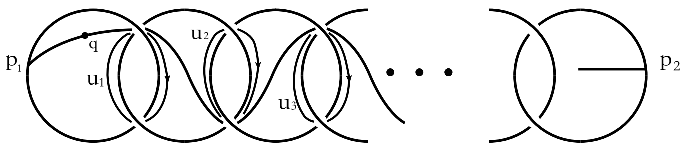

Fig. 18. The generators of $\pi_1(\text{interior}(S_1), q)$.

one point near one of the crossings of L'. Let q_i be the intersection point of u_i and γ_0, let v_i be the subarc of γ_0 from q to q_i, and let v_i^{-1} be the same arc with reverse orientation. Let u_i' be the loop based at q defined by $v_i \cdot u_i \cdot v_i^{-1}$. Then $\pi_1(\text{interior}(S_1), q)$ is a free group generated by $[u_1'], \dots, [u_{n-2}']$. Equation (11.1) implies that w is based homotopic to

$$u_1'^{\beta} \cdot (u_1'^{\theta} u_2'^{-\theta} u_3'^{\theta} \cdots u_{n-2}'^{(-1)^{n-1}\theta}) \cdot u_{n-2}'^{\delta} \tag{11.2}$$

in $\pi_1(\mathbb{R}^3 - L', q)$.

Since $\mathbb{R}^3 - L'$ is a fiber bundle over S^1 with a fiber being the interior of S_1, the map from $\pi_1(\text{interior}(S_1), q)$ to $\pi_1(\mathbb{R}^3 - L', q)$ is injective, hence w is based homotopic to (11.2) in S_1. By Lemma 7.8, the monodromy f_3 is given by the composition of the Dehn twists along $u_1, \dots, u_{n-2}$. Therefore, under a suitable choice of the orientation for the monodromy f_3, the image of γ_0 under f_3^c is homotopic to $\gamma_1 \cdot (u_1'^c u_2'^{-c} u_3'^c \cdots u_{n-2}'^{(-1)^{n-1}c}) \cdot \gamma_2$ relative to $\{p_1, p_2\}$, where the alternating signs in front of c come from the fact that the normal vector field of S_1 switches directions at each crossing of the diagram. As a consequence, γ is homotopic to $f_1^a f_2^b f_3^c(\gamma_0)$ relative to $\{p_1, p_2\}$ on S_1 with $a = \pm\beta$, $b = \pm\delta$, $c = \theta$, where the signs depend on the orientations of the Dehn twists in the definitions of f_1, f_2. By Proposition 9.1, γ_0 is isotopic to $f_1^a f_2^b f_3^c(\gamma_0)$ relative to $\{p_1, p_2\}$ on S_1. ■

Corollary 11.3. *Under the condition of Case* 1*, the knot* K_n *is isotopic to* $K(S_1, \gamma_0)$ *in* $\mathbb{R}^3 - L'$.

Proof. Let f_1, f_2, f_3 be as in Lemma 11.2. By Lemmas 11.1 and 11.2, there exist integers a, b, c such that K_n is isotopic to $K(S_1, \gamma)$ in $\mathbb{R}^3 - L'$, where γ is an arc on S_1 that is isotopic to $f_1^a f_2^b f_3^c(\gamma_0)$ relative to $\{p_1, p_2\}$. Therefore γ is isotopic to $f_3^c(\gamma_0)$ on S_1 if we allow its boundary points to move on ∂S_1. Hence $K(S_1, \gamma)$ is isotopic to $K(S_1, f_3^c(\gamma_0))$ in $\mathbb{R}^3 - L'$. By Lemma 9.6, $K(S_1, f_3^c(\gamma_0))$ is isotopic to $K(S_1, \gamma_0)$ in $\mathbb{R}^3 - L'$, hence the result is proved. ■

Recall that for a pair of integers u, v with $u \geq 3$, the link $L_{u,v}$ is defined by Definition 10.1.

Lemma 11.4. *The link* $L' \cup K(S_1, \gamma_0)$ *is isotopic to* $L_{n,1-n}$.

Proof. Notice that (S_1, γ_0) is isotopic to Figure 19. Removing the bands in the dotted boxes in Figure 19 from S_1 yields a disk, so the surface S_1 is given by a disk with $n - 1$

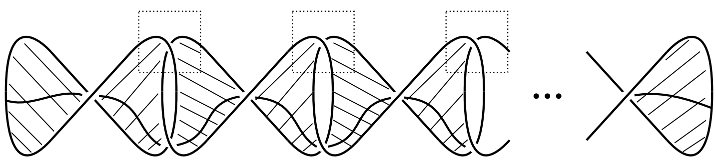

Fig. 19. Another diagram for S_1 and γ_0.

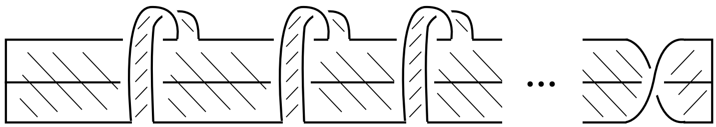

Fig. 20

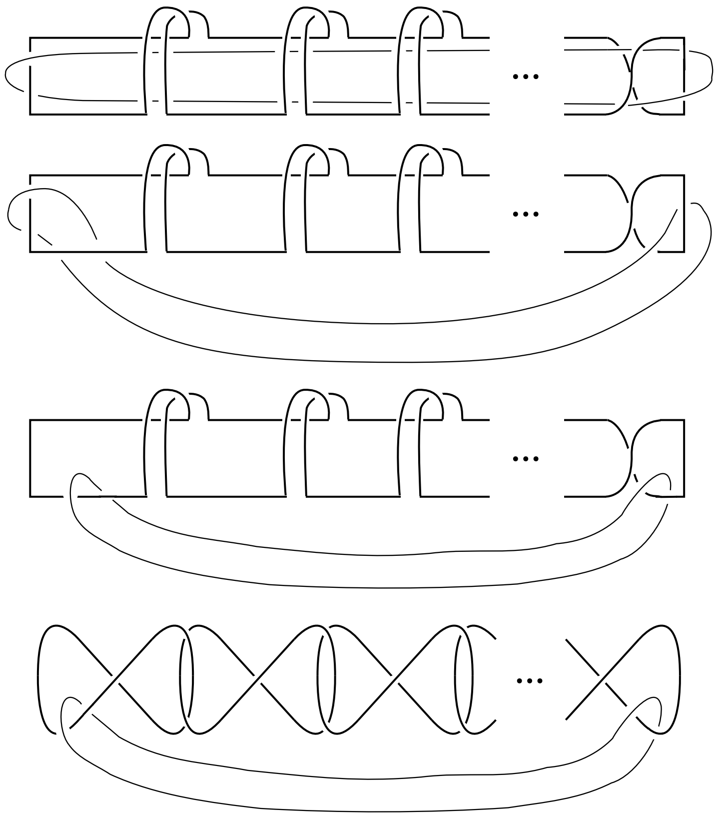

Fig. 21. Isotopy from $L' \cup K(S_1, \gamma_0)$ to $L_{n,1-n}$.

bands attached, and one can isotope Figure 19 to Figure 20. Figure 21 then shows an isotopy from $L' \cup K(S_1, \gamma_0)$ to $L_{n,1-n}$. ■

By Corollary 10.2, if $n \geq 4$, then $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L_{n,1-n}; \mathbb{Z}/2) > 2^n$. It can be directly verified that $L_{3,-2}$ is isotopic (up to mirror image) to the link L6n1 in the Thistlethwaite link table, and the rank of $\operatorname{Kh}(L_{3,-2}; \mathbb{Z}/2)$ equals 12. Therefore the links $L_{n,1-n}$ all fail to satisfy Condition 6.9 (2). This proves the non-existence of L for Case 1.

To prove the statement for Case 2, let γ_0 be the arc on S_2 given by Figure 22. Then the same argument as for Lemma 11.2 and Corollary 11.3 shows that K_n is iso-

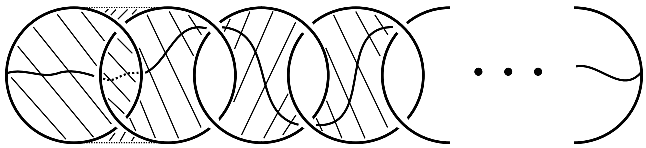

Fig. 22. The arc γ_0 on S_2.

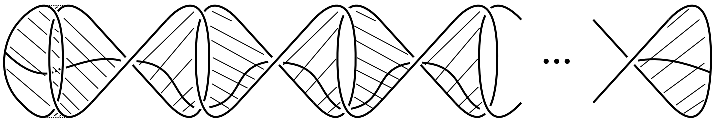

Fig. 23. Another diagram for S_2 and γ_0.

topic to $K(S_2, \gamma_0)$ in $\mathbb{R}^3 - L'$. A similar argument to the one for Lemma 11.4 shows that $L' \cup K(S_2, \gamma_0)$ is isotopic to $L_{n,2-n}$. By Corollary 10.2, when $n \geq 6$, we have $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L_{n,2-n}; \mathbb{Z}/2) > 2^n$. The link $L_{3,-1}$ is isotopic up to mirror image to the link L6n1 in the Thistlethwaite link table, and the rank of $\operatorname{Kh}(L_{3,-1}; \mathbb{Z}/2)$ equals 12. The link $L_{4,-2}$ is isotopic up to mirror image to L8n8, and the rank of $\operatorname{Kh}(L_{4,-2}; \mathbb{Z}/2)$ equals 24. The link $L_{5,-3}$ is isotopic up to mirror image to L10n113, and the rank of $\operatorname{Kh}(L_{5,-3}; \mathbb{Z}/2)$ equals 60. Therefore, $\operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}(L_{n,2-n}; \mathbb{Z}/2) > 2^n$ for all n, and this proves the desired result for Case 2.

In conclusion, we have proved that the link L satisfying Condition 6.9 does not exist, therefore Theorem 1.2 follows from Lemma 6.10.

12. Algebraic results

In this section, we use algebraic arguments to prove Corollary 1.4, Corollary 1.5 and Theorem 1.9.

Proof of Corollary 1.4. Suppose T is a tree with k vertices and let L_T be the forest of unknots given by T. If L_T is oriented such that the linking numbers between the components of L_T are all non-negative, then by [1, Corollary 6.6] we have

$$\begin{aligned} P(L_T) &:= \sum_{i,j} t^i q^j \operatorname{rank}_{\mathbb{Z}/2} \operatorname{Kh}^{i,j}(L_T; \mathbb{Z}/2) \\ &= t^{k-1} q^{3(k-1)} (q + q^{-1})(tq^2 + t^{-1} q^{-2})^{k-1}. \end{aligned}$$

Changing the orientation of L_T will change $P(L_T)$ by multiplication by $\pm t^r q^s$, which is a unit in the ring $\mathbb{Z}[t, t^{-1}, q, q^{-1}]$. Theorem 1.2 implies L_2 is a forest of unknots. Let $G = T_1 \sqcup \cdots \sqcup T_l$ be the graph of L_2, where T_i is a tree with k_i vertices and $n = \sum k_i$. Then the Künneth formula shows that

$$P(L_G) = t^a q^b (q + q^{-1})^l (tq^2 + t^{-1} q^{-2})^{n-l},$$

where a and b are integers depending on the orientaion of L_2. Since $q + q^{-1}$ and $tq^2 + t^{-1}q^{-2}$ are irreducible polynomials in the unique factorization domain $\mathbb{Z}[t, t^{-1}, q, q^{-1}]$, the value of $n - l$ is determined by $P(L_G)$. Therefore the first part of the corollary is proved. For the second part, notice that in these four cases the graph for L_1 is uniquely determined by the number of edges. ■

Lemma 12.1. *Suppose L is an oriented link and $L' = L \cup m$ where m is a meridian near a point $p \in L$. Then*

$$\mathrm{Kh}(L'; \mathbb{Z}/2) \cong \mathrm{Kh}(L; \mathbb{Z}/2) \oplus \mathrm{Kh}(L; \mathbb{Z}/2) \tag{12.1}$$

as un-graded $\mathbb{Z}/2$-vector spaces. Given any point $r \in L$, this isomorphism intertwines the basepoint operators X_r on the two sides. If $q \in m$, then the isomorphism intertwines X_q on the left-hand side with X_p on the right-hand side.

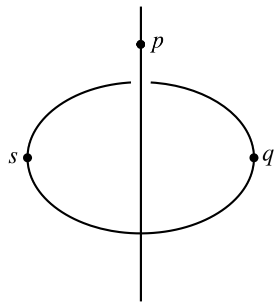

Fig. 24. The diagram $D' = D \cup m$

Proof. In the proof we set $R = \mathbb{Z}/2$. Let D be a diagram of L, and C be the associated Khovanov chain complex. The meridian m is added to D with two new crossings introduced as in Figure 24. We may also assume the point r is away from the region drawn in Figure 24. According to Figure 25, the Khovanov chain complex C' for $D \cup m$ is

$$\begin{array}{ccc} C \otimes_R R[X_s]/(X_s^2) & \xrightarrow{\ \mu\ } & C \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle -\Delta} \\ C & \xrightarrow{\ \Delta\ } & C \otimes_R R[X_q]/(X_q^2) \end{array}$$

where $\mu(\alpha \otimes 1) = \alpha,\ \mu(\alpha \otimes X_s) = X_p\alpha$, and $\Delta(\alpha) = \alpha \otimes X_q + X_p\alpha \otimes 1$. It is clear that the subcomplex

$$\begin{array}{ccc} C \otimes_R R\{1\} & \xrightarrow{\ \mu\ } & C \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle -\Delta} \\ C & \xrightarrow{\ \Delta\ } & \mathrm{Im}\,\Delta \end{array}$$

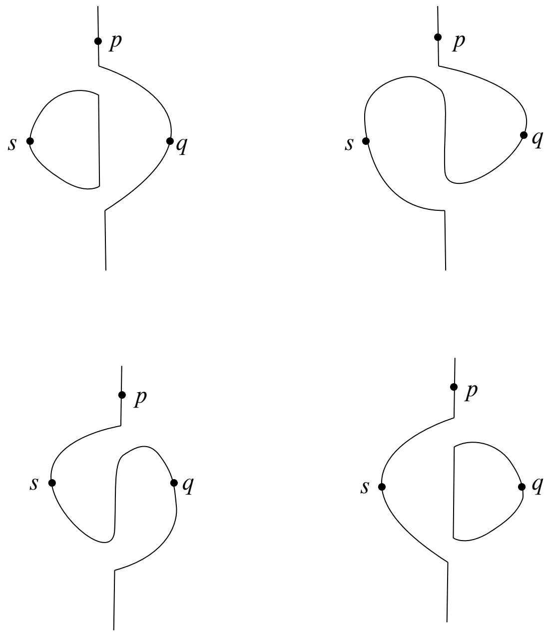

Fig. 25. Four resolutions of $D' = D \cup m$.

is acyclic. Therefore the quotient complex C'' given by

$$\begin{array}{ccc} C \otimes_R R\{X_s\} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \frac{C \otimes_R R[X_q]/(X_q^2)}{\operatorname{Im} \Delta} \end{array}$$

is quasi-isomorphic to C'. This quasi-isomorphism respects the actions of X_p, X_q, X_r since the subcomplex is indeed an $R[X]/(X^2)$-submodule for $X = X_p, X_q, X_r$. Now

$$H(C') \cong H(C'') \cong H(C \otimes_R R\{X_s\}) \oplus H\left(\frac{C \otimes_R R[X_q]/(X_q^2)}{\operatorname{Im} \Delta}\right) \cong H(C) \oplus H(C).$$

The actions of X_p and X_q on $C \otimes_R R\{X_s\}$ are the same since p and q lie on the same component of the resolved diagram in Figure 25. On the quotient

$$\frac{C \otimes_R R[X_q]/(X_q^2)}{\operatorname{Im} \Delta}$$

the actions of X_p and X_q are also the same. Therefore the actions of X_p, X_q on $H(C'')$ also coincide. This completes the proof. ∎

Remark 12.2. The change of grading in (12.1) is computed in [1, Theorem 6.2].

Proof of Corollary 1.5. Let U_n be the n-component unlink and H be the Hopf link. It is straightforward to calculate that

$$\mathrm{Kh}(U_n;\mathbb{Z}/2) \cong R_n \quad \text{and} \quad \mathrm{Kh}(H;\mathbb{Z}/2) \cong R_2/(X_1 - X_2) \oplus R_2/(X_1 - X_2).$$

For $k \in \mathbb{Z}^+$, let H_{k-1} be a forest of unknots with k components whose graph is a tree. Then for $k \geq 2$, the link H_{k-1} is given by a connected sum of $k-1$ Hopf links. Lemma 12.1 implies that

$$\mathrm{Kh}(H_{k-1};\mathbb{Z}/2) \cong [R_k/(X_1 = \cdots = X_k)]^{\oplus 2^{k-1}}. \tag{12.2}$$

The Khovanov module of the disjoint union of two links is the tensor product of the Khovanov modules of the two links over $\mathbb{Z}/2$. Theorem 1.2 implies L_2 is a forest of unknots with a graph G_2. It is clear from the above discussion that the module structure of $\mathrm{Kh}(L_2;\mathbb{Z}/2)$ determines the number of vertices in each component of G_2, and hence the corollary is proved. ■

Now we prove Theorem 1.9. The proof of Theorem 1.2 does not immediately apply to the case of arbitrary coefficient rings because we have used the equation

$$\mathrm{rank}_{\mathbb{Z}/2} \mathrm{Khr}(L, p;\mathbb{Z}/2) = \tfrac{1}{2} \mathrm{rank}_{\mathbb{Z}/2} \mathrm{Kh}(L;\mathbb{Z}/2)$$

in the proof of Proposition 5.4, and the above equation only holds for $\mathbb{Z}/2$-coefficients. For $\mathbb{Q}$-coefficients, the same proof would only give the following result.

Theorem 12.3. *If L is an n-component link such that*

$$\mathrm{rank}_{\mathbb{Q}} \mathrm{Kh}(L;\mathbb{Q}) = 2^n,$$
$$\mathrm{rank}_{\mathbb{Q}} \mathrm{Khr}(L, p;\mathbb{Q}) = 2^{n-1} \quad \textit{for all basepoints } p,$$

then L is a forest of unknots.

Sketch of proof. By Batson–Seed's inequality, we have $\mathrm{rank}_{\mathbb{Q}} \mathrm{Kh}(K;\mathbb{Q}) = 2$ for every component K of L. Therefore, Kronheimer–Mrowka's unknot detection theorem implies that every component of L is an unknot. The assumption $\mathrm{rank}_{\mathbb{Q}} \mathrm{Khr}(L, p;\mathbb{Q}) = 2^{n-1}$ implies that $\dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, p) \leq 2^{n-1}$ by Kronheimer–Mrowka's spectral sequence. By (3.12), we also have $\dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, p) \geq 2^{n-1}$, therefore

$$\dim_{\mathbb{C}} \mathrm{I}^{\natural}(L, p) = 2^{n-1} \quad \text{for every basepoint } p \in L.$$

The proof then proceeds as for Theorem 1.2 to reduce to the three links $L_{3,-2}$, $L_{4,-2}$ and $L_{5,-3}$. Since

$$\mathrm{rank}_{\mathbb{Q}}(L_{3,-2};\mathbb{Q}) = 10 > 2^3,$$
$$\mathrm{rank}_{\mathbb{Q}}(L_{4,-2};\mathbb{Q}) = 20 > 2^4,$$
$$\mathrm{rank}_{\mathbb{Q}}(L_{5,-3};\mathbb{Q}) = 46 > 2^5,$$

all the three links are eliminated. ■

Using the equivariant Khovanov homology introduced in [16], we have the following lemma.

Lemma 12.4. *If L is an n-component link such that* $\operatorname{rank}_{\mathbb{Q}} \mathrm{Kh}(L;\mathbb{Q}) = 2^n$, *then*

$$\operatorname{rank}_{\mathbb{Q}} \mathrm{Khr}(L,p;\mathbb{Q}) = 2^{n-1} \quad \textit{for all basepoints } p.$$

Proof. Given a diagram D for L, a chain complex $\mathcal{F}_3(D)$ of free $\mathbb{Z}[t]$-modules is introduced in [16]. The tensor product $C_t(D) := \mathcal{F}_3(D) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a chain complex of free $\mathbb{Q}[t]$-modules. Its homology $\mathcal{H}(L)$ is a $\mathbb{Q}[t]$-module called the *equivariant Khovanov homology*. If a basepoint $p \in L$ is chosen, then $C_t(D)$ (hence $\mathcal{H}(L)$) becomes a $\mathbb{Q}[X]$-module, where the action of X depends on p and satisfies $X^2 = t$. The tensor product $C_t(D) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t]/(t)$ is the chain complex defining $\mathrm{Kh}(L;\mathbb{Q})$. The tensor product $C_t(D) \otimes_{\mathbb{Q}[X]} \mathbb{Q}[X]/(X)$ is the chain complex defining $\mathrm{Khr}(L,p;\mathbb{Q})$. By [16, Proposition 7] (and the discussion before it), we have

$$\mathcal{H}(L) \cong \mathbb{Q}[t]^{\oplus 2^n} \oplus T,$$

where T is a direct sum of torsion modules of the form $\mathbb{Q}[t]/(t^l)$. Since $\mathrm{Kh}(L;\mathbb{Q}) \cong \mathbb{Q}^{2^n}$, the universal coefficient theorem implies that $T = 0$. Therefore

$$\mathcal{H}(L) \cong \mathbb{Q}[X]^{\oplus 2^{n-1}}$$

as a $\mathbb{Q}[X]$-module since $X^2 = t$. Now applying the universal coefficient theorem again we obtain

$$\mathrm{Khr}(L,p;\mathbb{Q}) \cong \mathbb{Q}^{\oplus 2^{n-1}} \quad \text{for all basepoints } p. \qquad \blacksquare$$

Theorem 12.3 and Lemma 12.4 imply Theorem 1.9.

Theorem 1.9. *Suppose R is an integral domain. If L is an n-component link such that* $\operatorname{rank}_R \mathrm{Kh}(L;R) = 2^n$, *then L is a forest of unknots.*

Proof. By the universal coefficient theorem,

$$2^n = \operatorname{rank}_R \mathrm{Kh}(L;R) \geq \operatorname{rank}_{\mathbb{Z}} \mathrm{Kh}(L;\mathbb{Z}) = \operatorname{rank}_{\mathbb{Q}} \mathrm{Kh}(L;\mathbb{Q}).$$

Therefore by (1.1), we have $\operatorname{rank}_{\mathbb{Q}} \mathrm{Kh}(L;\mathbb{Q}) = 2^n$. Lemma 12.4 and Theorem 12.3 then imply that L is a forest of unknots. ■

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