J. Eur. Math. Soc. 27, 269–278 (2025) DOI 10.4171/JEMS/1378

Wentao Zhang · Hehui Wu

The total Betti number of the independence complex of ternary graphs

Received May 17, 2021; revised October 14, 2021

Abstract. Given a graph G, the *independence complex* $I(G)$ is the simplicial complex whose faces are the independent sets of $V(G)$. Let $\tilde{b}_i$ denote the i-th reduced Betti number of $I(G)$, and let $b(G)$ denote the sum of the $\tilde{b}_i(G)$'s. A graph is ternary if it does not contain induced cycles with length divisible by 3. Kalai and Meshulam conjectured that $b(G) \leq 1$ whenever G is ternary. We prove this conjecture. This extends a recent result proved by Chudnovsky, Scott, Seymour and Spirkl that for any ternary graph G, the number of independent sets with even cardinality and the number of independent sets with odd cardinality differ by at most 1.

Keywords. Total Betti number, independence complex, graph

1. Introduction

A graph is *ternary* if it has no induced cycle of length divisible by 3. Ternary graphs are also called trinity graphs [2, 11]. Given a graph G, let f_G be the sum of $(-1)^{|A|}$ over all independent sets A of vertices. Recently, Chudnovsky, Scott, Seymour and Spirkl [5] proved an intriguing conjecture on the independent sets (or stable sets) of ternary graphs proposed by Kalai and Meshulam (see [11]) in the late 1990s.

Theorem 1.1. *If G is a graph with no induced cycle of length divisible by* 3*, then* $|f_G| \leq 1$.

A stronger version of the conjecture of Kalai and Meshulam concerns the total Betti number of the independence complex of a ternary graph, which builds a connection between algebraic topology and graph theory.

The *independence complex* $I(G)$ of a graph G is the simplicial complex whose faces are the independent sets of $V(G)$. $\tilde{H}_i(I(G))$ is the i-th reduced homology group of $I(G)$, and $\tilde{b}_i(I(G)) = \dim \tilde{H}_i(I(G))$ is the i-th reduced Betti number of $I(G)$. Note that the

Wentao Zhang: Shanghai Center for Mathematical Sciences, Fudan University, Shanghai, China; wtzhang20@fudan.edu.cn

Hehui Wu: Shanghai Center for Mathematical Sciences, Fudan University, Shanghai, China; hhwu@fudan.edu.cn

Mathematics Subject Classification (2020): Primary 05Cxx; Secondary 05E45

Betti number b_i of a simplicial complex equals the reduced Betti number, except for the 0-th Betti number, which is one more than $\tilde{b}_0$. By convention, when G is a null graph (with no vertex), we let $b_0(I(G)) = 0$ and $\tilde{b}_0(I(G)) = -1$.

Let $b(G)$ denote the sum of the $\tilde{b}_i(I(G))$'s, called the *total Betti number* of $I(G)$. For a simplicial complex, the Euler characteristic can be defined as $\sum_i (-1)^i b_i$, which is $1 + \sum_i (-1)^i \tilde{b}_i$. From a basic theorem in homology theory, we know that the Euler characteristic of $I(G)$ also equals $\sum (-1)^{|A|-1}$, the sum being over all the non-empty independent sets A in G (see [10]). It immediately follows that $f_G = \sum_{i=0}^{\infty} (-1)^{i+1} \tilde{b}_i(G)$, and so $|f_G| \leq b(G)$.

Note that $b(G) \geq |f_G| = 2$ when G is a cycle of length divisible by 3. A question was asked by Kalai and Meshulam on the total Betti number of graphs without induced cycle of length divisible by 3. The purpose of the paper is to prove the conjecture of Kalai–Meshulam (see [11]), which is a stronger version of Theorem 1.1.

Theorem 1.2. *If G is a graph with no induced cycle of length divisible by 3, then $b(G) \leq 1$.*

Analogously, a *clique complex* of a graph G is the simplicial complex whose faces are the cliques of G. In an abstract simplicial complex, a set S of vertices that is not itself a face of the complex, but such that each pair of vertices in S belongs to some face in the complex, is called an *empty simplex*. A *flag complex* is an abstract simplicial complex that has no empty simplex. As any flag complex is the clique complex of its 1-skeleton, and the clique complex of a graph G is the independence complex of the complement of G, the above theorem gives a full characterization of minimal flag complexes with total Betti number 2.

If we further forbid any C_{3k} as a subgraph instead of an induced subgraph, the following result was proved by Engström [7], which extends a result of Gauthier [9] on f_G of such graphs:

Theorem 1.3. *If G is a graph without cycles of length divisible by 3, then $I(G)$ is contractible or homotopy equivalent to a sphere.*

In the same paper, Engström also asked whether for any ternary graph, $I(G)$ is contractible or homotopy equivalent to a sphere. Very recently, based on our proof, Kim [12] confirmed that this is indeed the case.

There are some other conjectures asked simultaneously by Kalai and Meshulam (see [11]), relating chromatic numbers, the Euler characteristc or the total Betti number of the independence complex, and ternary graphs. Some of them have been answered recently. See the papers of Bonamy, Charbit and Thomassé [2], of Scott and Seymour [14], and some others [4, 8].

Our proof is inspired by the proof of Theorem 1.1 by Chudnovsky, Scott, Seymour and Spirkl [5]. The proof could be shorter if we used their results directly. But we prefer to give a full and independent proof, as the total Betti number gives us more details on the induction process than the Euler characteristic, and the proofs are smoother and shorter than in the original paper after the system is set up.

Given a graph G and vertex sets X, Y let $f_G(X, Y)$ be the sum of $(-1)^{|A|}$ where A runs over all independent sets that include X and are disjoint from Y. The proof in [5] is based on the recursive formula $f_G(X, Y) = f_G(X \cup \{v\}, Y) + f_G(X, Y \cup \{v\})$ for every $v \in V(G)$. To recursively calculate $\tilde{b}_i(I(G))$, we will instead use a formula coming from the Mayer–Vietoris sequence, which is a powerful tool in calculation of homology groups.

2. Mayer–Vietoris sequence

As some graph theorists may not be very familiar with homology theory, we first introduce some prerequisites from homology theory. In this part we follow a paper of Delfinado and Edelsbrunner [6].

An *abstract simplicial complex* K is a family of sets that is closed under taking subsets. Each element in such a set is called a *vertex* and each finite set in K is called a *face*. An n*-face* is a face of size $n + 1$. Each n-face can be oriented with a linear order of its vertices, denoted by $[v_0, \dots, v_n]$. The chain group $C_n(K)$ is the free abelian group generated by the oriented n-faces of K, and the boundary map $\partial_n : C_n(K) \to C_{n-1}(K)$ is defined by

$$\partial_n[v_0, \dots, v_n] = \sum_{j=0}^{n} (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_n]$$

where $\hat{v}_j$ means v_j is omitted.

The reduced homology groups $\tilde{H}_i(K)$ are the homology groups of the augmented chain complex

$$\cdots \to C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\varepsilon : C_0(K) \to \mathbb{Z}$ is the augmentation map defined by $\varepsilon(v) = 1$ for each vertex v of K. We have $\partial_i \circ \partial_{i+1} = 0$ and $\varepsilon \circ \partial_1 = 0$. The i-th *reduced homology group* $\tilde{H}_i(K)$ of K is the quotient group $\ker \partial_i / \operatorname{im} \partial_{i+1}$ for positive i, and $\tilde{H}_0(K) = \ker \varepsilon / \operatorname{im} \partial_1$. The i-th *reduced Betti number* $\tilde{\beta}_i(K)$ of K is the rank of $\tilde{H}_i(K)$.

Let K' and K'' be subcomplexes such that $K = K' \cup K''$ and let $L = K' \cap K''$. A chain complex is *exact* if $\operatorname{im} \partial_{i+1} = \ker \partial_i$ for all i. There is an exact sequence of reduced homology groups called the *Mayer–Vietoris sequence* [13]:

$$\begin{aligned} \cdots \to \tilde{H}_i(L) \xrightarrow{\lambda_i} \tilde{H}_i(K') \oplus \tilde{H}_i(K'') \to \tilde{H}_i(K) \\ \to \tilde{H}_{i-1}(L) \xrightarrow{\lambda_{i-1}} \tilde{H}_{i-1}(K') \oplus \tilde{H}_{i-1}(K'') \to \cdots \to \tilde{H}_0(K) \to 0. \end{aligned}$$

It follows that

$$0 \to \operatorname{cok} \lambda_i \to \tilde{H}_i(K) \to \ker \lambda_{i-1} \to 0$$

is a short exact sequence of abelian groups. Let $N_i = \ker \lambda_i$ and let $\beta(N_i)$ be its dimension. We have (see [1])

$$\begin{aligned}\beta_i(K) &= \beta(\operatorname{cok}\lambda_i) + \beta(\ker\lambda_{i-1}) \\ &= \beta(\tilde{H}_i(K') \oplus \tilde{H}_i(K'')/\operatorname{im}\lambda_i) + \beta(N_{i-1}) \\ &= \beta_i(K') + \beta_i(K'') - \beta_i(L) + \beta(N_i) + \beta(N_{i-1}) \end{aligned} \tag{2.1}$$

Note that $\beta(N_i) \le \beta_i(L)$ as $N_i \subseteq \tilde{H}_i(L)$ for each i.

Given a graph G, let $N(v)$ denote the set of neighbors of a vertex v in G, let $N(X) = \bigcup_{v\in X} N(v)$ denote the open neighborhood of a vertex set X, and let $N[X] = N(X) \cup X$ denote the closed neighborhood of X. Suppose X is an independent set of G and Y is a vertex set disjoint from X, and let $G(X \mid Y)$ be the subgraph induced by $V(G) - N[X] - Y$. If the elements of X or Y are listed we omit the braces for simplicity. Also for simplicity, when $G(X \mid Y)$ is not a null graph, we write $I(X \mid Y)$ and $b(X \mid Y)$ for $I(G(X \mid Y))$ and $b(G(X \mid Y))$ when G is known. Similarly, we define $\tilde{b}_i(X, Y)$. For the intuition of the construction, note that faces of $I(X \mid Y)$ are order isomorphic with independent sets of $V(G)$ containing X and disjoint from Y.

If v is a vertex of G, take $K = I(G)$, $K' = I(G - v)$ and $K'' = I(G - N(v))$ in (2.1). Then $K = K' \cup K''$ and $L = I(v \mid \emptyset)$, so we have

$$\tilde{b}_i(G) = \tilde{b}_i(G(\emptyset \mid v)) + \tilde{b}_i(G - N(v)) - \tilde{b}_i(G(v \mid \emptyset)) + \beta(N_i) + \beta(N_{i-1}).$$

Note that $I(G - N(v))$ is the collection of all simplices of the form $va_0 \cdots a_p$ where $a_0 \cdots a_p$ is a simplex of $I(v \mid \emptyset)$, along with all faces of such simplices. That is, $I(G - N(v))$ is the cone on $I(v \mid \emptyset)$ with vertex v, denoted by $v * I(v \mid \emptyset)$. It is an elementary fact in topology that a cone has zero reduced homology groups [13]:

$$\tilde{H}_i(v * I(v \mid \emptyset)) = 0 \quad \text{for all } i.$$

That is, we have the following proposition:

Proposition 2.1. *If H has an isolated vertex, then $b(H) = 0$.*

So the above equation is reduced to

$$\tilde{b}_i(G) = \tilde{b}_i(\emptyset \mid v) - \tilde{b}_i(v \mid \emptyset) + \beta(N_i) + \beta(N_{i-1}), \quad \forall i. \tag{2.2}$$

Similarly, if we replace G by $G - N[X] - Y$ for any vertex sets X and Y, we have

$$\tilde{b}_i(X, Y) = \tilde{b}_i(X \mid Y \cup \{v\}) - \tilde{b}_i(X \cup \{v\} \mid Y) + \beta(N_i') + \beta(N_{i-1}'), \quad \forall i. \tag{2.3}$$

Here N_i' is a subgroup of $\tilde{H}_i(I_G(X \cup \{v\} \mid Y))$. We have

$$\beta(N_i) \le \tilde{b}_i(v \mid \emptyset) \quad \text{and} \quad \beta(N_i') \le \tilde{b}_i(X \cup \{v\} \mid Y). \tag{2.4}$$

Our proof of the main result is based on the above recursion formulas.

3. Proof of Main Theorem

We are going to prove Theorem 1.2: If $b(G) \ge 2$ and $b(H) \le 1$ for every induced subgraph H of G, then $G = C_{3k}$ for some integer k.

In the remaining part of the paper, we fix G to be the one in the main result, unless it is specified otherwise.

Suppose $G(X \mid Y)$ is well defined and not a null graph. By assumption, $b(X \mid Y)$ is 0 or 1 if $X \cup Y \neq \emptyset$. If $b(H) = 1$ for some graph H, we denote by $d(H)$ the dimension where the reduced Betti number takes value 1. That is, $d(G) = i$ if $\tilde{b}_i(G) = 1$ and $\tilde{b}_j(G) = 0$ for $j \neq i$. If $b(G) = 0$ then we define $d(G)$ to be '$*$'. Note that for $b_G \geq 2$, $d(G)$ is not defined. For simplicity, we write $d(X \mid Y)$ for $d(G(X \mid Y))$. Additionally, when X is not independent, we let $d(X \mid Y) = *$, and if $G(X|Y)$ is a null graph, then $d(X \mid Y) = -1$. That is, we consider the empty simplex as the (-1)-dimensional sphere.

Lemma 3.1. *For any disjoint vertex sets X and Y in G with $X \cup Y \neq \emptyset$ and a vertex v not in X or Y, the triple $(d(X \mid Y), d(X \cup \{v\} \mid Y), d(X \mid Y \cup \{v\}))$ fits into one of the following four patterns: $(k, *, k)$, $(*, *, *)$, $(*, k, k)$ and $(k+1, k, *)$ for some integer k.*

This is illustrated in the triangle diagram below:

(X, Y)		k		$*$		$*$		$k+1$
$(X \cup \{v\} \mid Y)$	$(X \mid Y \cup \{v\})$	$*$	k	$*$	$*$	k	k	k	$*$

We say such triples (triangles) are *legal* and others are *illegal*. Note that if we know two corners of a legal triangle then we can determine the third.

Proof of Lemma 3.1. If $X \cup \{v\}$ is not independent, then $d(X \cup \{v\} \mid Y) = *$, and $G - N[X] - \{v\} - Y = G - N[X] - Y$, hence $d(X \mid Y \cup \{v\}) = d(X \mid Y)$. The triple must be $(k, *, k)$ or $(*, *, *)$. So we may assume $X \cup \{v\}$ is an independent set, and v is a vertex in $G(X \mid Y)$.

If $V(G(X \mid Y)) = \{v\}$, then the triple is $(*, -1, -1)$. If $V(G(X \mid Y)) \neq \{v\}$, and $G(X \cup \{v\} \mid Y)$ is null, then v adjacent to all vertices in $G(X \mid Y)$. By the minimality of G, we may assume G does not contain triangles, hence $G(X \mid Y)$ is a star centered at v. As $b_0(X \mid Y)$ is the number of components of $I(X \mid Y)$ and v is isolated in $I(X \mid Y)$, we have $b(X \mid Y) = 0$. Furthermore, $G(X \mid Y \cup \{v\})$ is a graph with no edge, and so $b(X \mid Y \cup \{v\}) = *$ by Proposition 2.1. Therefore the triple will be $(0, -1, *)$. So we may assume $G(X \cup \{v\} \mid Y)$ and $G(X \mid Y \cup \{v\})$ are not null.

By (2.4), if $d(X \cup \{v\} \mid Y) = *$, then $\beta(N'_i) = 0$ for all i, and by (2.3), $\tilde{b}_i(X \mid Y) = \tilde{b}_i(G(X \mid Y \cup \{v\}))$. Thus $(d(X \mid Y), d(X \cup \{v\} \mid Y), d(X \mid Y \cup \{v\}))$ is $(k, k, *)$ for some k, or $(*, *, *)$.

If $d(X \cup \{v\} \mid Y) = k$ then $\beta(N'_k) = 0$ or 1 and $\beta(N'_i) = 0$ for $i \neq k$. In the case $\beta(N'_k) = 0$, by (2.3), $\tilde{b}_k(X \mid Y) = \tilde{b}_k(X \mid Y \cup \{v\})) - 1$, which should be non-negative. So $\tilde{b}_k(G(X \mid Y \cup \{v\})) = 1$ and $\tilde{b}_k(X \mid Y) = 0$, and also $\tilde{b}_i(X \mid Y) = \tilde{b}_i(X \mid Y \cup \{v\})) - \tilde{b}_i(X \cup \{v\} \mid Y) = 0$ for $i \neq k$, hence $(d(X, Y), d(X \cup \{v\} \mid Y), d(X \mid Y \cup \{v\})) = (*, k, k)$.

If $\beta(N'_k) = 1$, by (2.3) we have

$$\begin{aligned}\tilde{b}_{k+1}(X \mid Y) &= \tilde{b}_{k+1}(X \mid Y \cup \{v\}) - \tilde{b}_{k+1}(G(X \cup \{v\} \mid Y)) + \beta(N'_{k+1}) + \beta(N'_k) \\ &= \tilde{b}_{k+1}(X \mid Y \cup \{v\}) + 1.\end{aligned}$$

So $\tilde{b}_{k+1}(X \mid Y) = 1$, and $\tilde{b}_i(X \mid Y) = 0$ for $i \neq k+1$. Now that we know $d(X, Y)$, $d(X \cup \{v\} \mid Y)$ and all $\beta(N_i')$'s, from (2.3) we have $\tilde{b}_i(G(X \mid Y \cup \{v\})) = 0$ for all i. So $(d(X \mid Y), d(X \cup \{v\} \mid Y), d(X \mid Y \cup \{v\})) = (k+1, k, *)$. ■

Lemma 3.2. *Suppose X, Y are vertex sets of G with $d(X \mid Y) = k$ for some integer k. If v_1, v_2 are two vertices not in $X \cup Y$ with $d(X \cup \{v_1\} \mid Y\} = k-1$ and $d(X \cup \{v_2\} \mid Y) = *$, then $d(X \cup \{v_1, v_2\} \mid Y) = *$.*

Proof. By Lemma 3.1, we should have

$$\big(d(X \mid Y), d(X \cup \{v_1\} \mid Y), d(X \mid Y \cup \{v_1\})\big) = (k, k-1, *),$$
$$\big(d(X \mid Y), d(X \cup \{v_2\} \mid Y), d(X \mid Y \cup \{v_2\})\big) = (k, *, k).$$

Suppose $d(X \cup \{v_1, v_2\} \mid Y) \neq *$. Then

$$\big(d(X \cup \{v_1\} \mid Y), d(X \cup \{v_1, v_2\} \mid Y), d(X \cup \{v_1\} \mid Y \cup \{v_2\})\big) = (k-1, k-2, *),$$
$$\big(d(X \cup \{v_2\} \mid Y), d(X \cup \{v_1, v_2\} \mid Y), d(X \cup \{v_2\} \mid Y \cup \{v_1\})\big) = (*, k-2, k-2).$$

Now to calculate $d(X \mid Y \cup \{v_1, v_2\})$, we should have

$$\big(d(X \mid Y \cup \{v_1\}), d(X \cup \{v_2\} \mid Y \cup \{v_1\}, d(X \mid Y \cup \{v_1, v_2\})\big) = (*, k-2, k-2),$$
$$\big(d(X \mid Y \cup \{v_2\}), d(X \cup \{v_1\} \mid Y \cup \{v_2\}), d(X \mid Y \cup \{v_1, v_2\})\big) = (k, *, k),$$

which conflict at the value of $d(X \mid Y \cup \{v_1, v_2\})$. ■

Claim 3.3. *There is some $k \geq 0$ such that $\tilde{b}_k(G) = 2$ and $\tilde{b}_i(G) = 0$ for all $i \neq k$. Furthermore, for every vertex v, $d(v \mid \emptyset) = k-1$ and $d(\emptyset \mid v) = k$.*

Proof. Note that $N_{-1} = 0$ in (2.2), so for any vertex v,

$$\begin{aligned} b(G) &= b(\emptyset \mid v) - b(v \mid \emptyset) + 2\sum_{i \geq 0} \beta(N_i) \\ &\leq b(\emptyset \mid v) - b(v \mid \emptyset) + 2b(v \mid \emptyset) \\ &= b(\emptyset \mid v) + b(v \mid \emptyset). \end{aligned}$$

By the assumption on G, we have $b(\emptyset \mid v) = b(v \mid \emptyset) = 1$ and $b(G) = 2$. Also, we must have $\beta(N_i) = \tilde{b}_i(G(v \mid \emptyset))$ for all i and (2.2) is reduced to

$$\begin{aligned} \tilde{b}_i(G) &= \tilde{b}_i(G(\emptyset \mid v)) - \tilde{b}_i(G(v \mid \emptyset)) + \beta(N_i) + \beta(N_{i-1}) \\ &= \tilde{b}_i(G(\emptyset \mid v)) + \tilde{b}_{i-1}(G(v \mid \emptyset)). \end{aligned} \tag{3.1}$$

Suppose $\tilde{b}_k(G) = \tilde{b}_l(G) = 1$ for some integers k, l with $k < l$. By (3.1), for each vertex v, either

$$v \in V_1 = \{u : d(u \mid \emptyset) = k-1,\ d(\emptyset \mid u) = l\},$$

or

$$v \in V_2 = \{u : d(u \mid \emptyset) = l-1,\ d(\emptyset \mid u) = k\}.$$

We claim that for disjoint subsets X, Y of V_1 such that $X \cup Y \neq \emptyset$, we have

$$d(X \mid Y) = \begin{cases} k - |X|, & Y = \emptyset, \\ *, & X, Y \neq \emptyset, \\ l, & Y = \emptyset, \end{cases}$$

We illustrate this in the triangle diagram below:

$(|X|, |Y|)$:

$$\begin{array}{ccccccc} & & (1,0) & & (0,1) & & \\ & (2,0) & & (1,1) & & (0,2) & \\ (3,0) & & (2,1) & & (1,2) & & (0,3) \\ & & & \vdots & & & \\ (t,0) & (t-1,1) & \cdots & & \cdots & (1,t-1) & (0,t) \end{array}$$

$d(X \mid Y)$:

$$\begin{array}{ccccccc} & & k-1 & & l & & \\ & k-2 & & * & & l & \\ k-3 & & * & & * & & l \\ & & & \vdots & & & \\ k-t & * & \cdots & & \cdots & * & l \end{array}$$

We argue by induction on $t = |X \cup Y|$. The case $t = 1$ holds by definition of V_1.

Suppose we have proved this for $|X \cup Y| \leq t - 1$ for $t \geq 2$. Note that for the first $t - 1$ rows, $d(X \mid Y)$ is determined by $(|X|, |Y|)$, so we may use $d(|X|, |Y|)$ to denote $d(X \mid Y)$. For $W = X \cup Y$ given, by repeatedly using Lemma 3.1, the t-th row is determined by $d(W \mid \emptyset)$, and $d(X \mid Y)$ is also determined by $(|X|, |Y|)$. Furthermore, the triple $(d(X \mid Y), d(X \cup \{v\} \mid Y), d(X \mid Y \cup \{v\}))$ in Lemma 3.1 can also be replaced by $(d(|X|, |Y|), d(|X| + 1, |Y|), d(|X|, |Y| + 1))$, which forms a small triangle in the triangle diagram above.

By Lemma 3.1, there are at most two possible lists of values on the t-th row, depending on $d(t, 0)$:

$$(k - t, *, \ldots, *, l) \quad \text{or} \quad (*, k - t + 1, k - t + 1, \ldots, k - t + 1, *).$$

The latter is legal only when $k = l + t - 2$, which conflicts with the assumption that $k < l$. Therefore the t-th row must be $(k - t, *, \cdots, *, l)$ for any $W = X \cup Y$ with size t. By induction, the claim is true for all rows.

Moreover, we have $d(\emptyset \mid V_1) = l$. Using the same argument as above with $t = 2$, as $l \neq k + t - 2 = k$, we can find that for all $u, v \in V_2$ we have $d(u, v \mid \emptyset) = l - 2$, which implies that no two vertices u, v in V_2 are adjacent, hence V_2 is an independent set. However, as $G(\emptyset \mid V_1) = G[V_2]$, by Proposition 2.1, $b(\emptyset \mid V_1) = 0$, contradicting $d(\emptyset \mid V_1) = l$.

So we have $\tilde{b}_k(G) = 2$ for some k. By (3.1), we have $d(v \mid \emptyset) = k - 1$ and $d(\emptyset \mid v) = k$ for every vertex v. ■

Throughout the rest of this article, we use k to denote the integer we obtained in the above theorem.

Let u, v be two vertices of G. Since $d(u \mid \emptyset) = k-1$, by Lemma 3.1 the triple $(d(u \mid \emptyset), d(u, v \mid \emptyset), d(u \mid v))$ is either $(k-1, k-2, *)$ or $(k-1, *, k-1)$. We construct a new graph H on $V(G)$ such that u, v are adjacent if and only if $d(u, v \mid \emptyset) = k-2$. The following propositions on H immediately follow:

Proposition 3.4. *In H, any two vertices u, v satisfy:*

(1) *If $u \sim v$ in H, then $u \nsim v$ in G. That is, $E(G) \cap E(H) = \emptyset$.*

(2) *If $u \sim v$ in H, then $d(u, v \mid \emptyset) = k-2$, $d(u \mid v) = d(v \mid u) = *$, and $d(\emptyset \mid u, v) = k$,*

(3) *If $u \nsim v$ in H, then $d(u, v \mid \emptyset) = d(\emptyset \mid u, v) = *$ and $d(u \mid v) = d(v \mid u) = k-1$.*

The following proposition is a key feature of H.

Lemma 3.5. *Every component C of H is a complete graph. Furthermore, for any disjoint subsets X and Y of $V(C)$ with $X \cup Y \neq \emptyset$, we have*

$$d(X \mid Y) = \begin{cases} k - |X|, & Y = \emptyset, \\ *, & X, Y \neq \emptyset, \\ k, & X = \emptyset. \end{cases}$$

We can illustrate this in the triangle diagram below:

$(|X|, |Y|)$:

$$\begin{array}{ccccccc} & & & (1,0) \quad (0,1) & & & \\ & & (2,0) & (1,1) & (0,2) & & \\ & (3,0) & (2,1) & & (1,2) & (0,3) & \\ & & & \vdots & & & \\ (t,0) & (t-1,1) & \cdots & & \cdots & (1,t-1) & (0,t) \end{array}$$

$d(X, Y)$:

$$\begin{array}{ccccccc} & & & k-1 \quad k & & & \\ & & k-2 & * & k & & \\ & k-3 & * & & * & k & \\ & & & \vdots & & & \\ k-t & * & \cdots & & \cdots & * & k \end{array}$$

Proof. Suppose C is not complete. Then there must exist three distinct vertices u, v and w in C such that $u \sim v$, $v \sim w$ but $u \nsim w$ in H. Since $d(u \mid \emptyset) = k-1$, $d(u, v \mid \emptyset) = k-2$, $d(u, w \mid \emptyset) = *$, by Lemma 3.2 we have $d(u, v, w \mid \emptyset) = *$. So

$$(d(u, v \mid \emptyset), d(u, v, w \mid \emptyset), d(u, v \mid w)) = (k-2, *, k-2).$$

But as $d(\emptyset \mid w) = k$, $d(u \mid w) = k-1$, $d(v \mid w) = *$, by Lemma 3.2 we should have $d(u, v \mid w) = *$, a contradiction.

So C must be complete, which implies the first two rows of the triangle diagram. The remaining level can be proved inductively just as in Theorem 3.3, with $k = l$ and $k \neq l + t - 2$ when $t \geq 3$. ■

The following result follows immediately.

Claim 3.6. *There does not exist a vertex v with all neighbors in G located in one component of H.*

Proof. Suppose there is a component C of H such that $N_G(v) \subseteq C$. By Lemma 3.5, $d(\emptyset \,|\, C) = d(G - C) = k$; but v is an isolated vertex in $G - C$, so we have $b(G - C) = 0$ by Proposition 2.1, a contradiction. ∎

Lemma 3.7. *There do not exist two edges $v_1 v_2, v_3 v_4$ in G with v_1, v_2, v_3, v_4 located in four distinct components of H.*

Proof. Suppose the vertices are located in distinct components of H. We consider $d(X \mid Y)$ for disjoint subsets X, Y in $\{v_1, v_2, v_3, v_4\}$ with $X \cup Y \neq \emptyset$. We claim to have the following triangle diagram:

$$
\begin{array}{c|c}
(|X|, |Y|): & d(X \mid Y): \\
(1,0) \quad (0,1) & k-1 \quad k \\
(2,0) \quad (1,1) \quad (0,2) & * \quad k-1 \quad * \\
(3,0) \quad (2,1) \quad (1,2) \quad (0,3) & * \quad * \quad k-1 \quad k-1 \\
(4,0) \quad (3,1) \quad (2,2) \quad (1,3) \quad (0,4) & * \quad * \quad * \quad ?
\end{array}
$$

The first row is implied by Claim 3.3. The second row follows from the assumption that each pair of the v_i's belongs to a different component of H. Note that when $|X| \geq 3$, X is not an independent set, hence $d(X \mid Y) = *$. Therefore the first terms of the third row and the fourth row are $*$. And we can get the rest of the third row using Theorem 3.1. Similarly the first three terms of the fourth row are '$*$'. But there is no value for $d(X \mid Y)$ with $(|X|, |Y|) = (1, 3)$ that fits Lemma 3.1. ∎

Now we are ready to complete the proof of the main theorem.

Proof of Theorem 1.2. As $b(C_{3k}) \geq |f_{C_{3k}}| = 2$ for any k, we just need to show that G contains an induced C_{3k} for some k.

First each component of H is an independent set in G. By Lemma 3.6, the neighbors of any vertex in G are located in at least two components of H; and by Lemma 3.7 no two edges have all ends in four distinct components. Altogether it is easy to deduce that H has exactly three components C_0, C_1, C_2, and every vertex in C_i has neighbours in C_{i-1} and C_{i+1} for each i (indices modulo 3). We orient all the edges of G from C_i to C_{i+1} for $i = 0, 1, 2$. Then every vertex has positive out-degree, so there is an induced directed cycle $[v_1, \ldots, v_r]$, which must have length divisible by 3. ∎

Acknowledgments. The first author would like to thank his colleagues Guozhen Wang and Xiping Zhang in Shanghai Center for Mathematical Sciences for sharing their knowledge in algebraic topology, and to thank Qiqin Xie for the early discussion on this problem.

Funding. H. W. was supported in part by National Natural Science Foundation of China grant 11931006, National Key Research and Development Program of China (Grant No. 2020YFA0713200), and the Shanghai Dawn Scholar Program grant 19SG01.

References

[1] Alexandroff, P., Hopf, H.: Topologie. I. Die Grundlehren der mathematischen Wissenschaften 44, Springer, Berlin (1935) JFM 61.0602.07 MR 0345087

[2] Bonamy, M., Charbit, P., Thomassé, S.: Graphs with large chromatic number induce $3k$-cycles. arXiv:1408.2172 (2014)

[3] Bousquet-Mélou, M., Linusson, S., Nevo, E.: On the independence complex of square grids. J. Algebraic Combin. **27**, 423–450 (2008) Zbl 1202.05095 MR 2393250

[4] Chen, G., Saito, A.: Graphs with a cycle of length divisible by three. J. Combin. Theory Ser. B **60**, 277–292 (1994) Zbl 0799.05036 MR 1271275

[5] Chudnovsky, M., Scott, A., Seymour, P., Spirkl, S.: Proof of the Kalai–Meshulam conjecture. Israel J. Math. **238**, 639–661 (2020) Zbl 1450.05024 MR 4145813

[6] Delfinado, C. J. A., Edelsbrunner, H.: An incremental algorithm for Betti numbers of simplicial complexes on the 3-sphere. Computer Aided Geom. Design **12**, 771–784 (1995) Zbl 0873.55007 MR 1365107

[7] Engström, A.: Graph colouring and the total Betti number. arXiv:1412.8460 (2014)

[8] Engström, A.: On the topological Kalai–Meshulam conjecture. arXiv:2009.11077 (2020)

[9] Gauthier, G.: Graphs with no cycle length divisible by three. Ph.D. thesis, Princeton University (2017)

[10] Hatcher, A.: Algebraic topology. Cambridge University Press, Cambridge (2002) Zbl 1044.55001 MR 1867354

[11] Kalai, G.: When do a few colors suffice? https://gilkalai.wordpress.com/2014/12/19/when-a-few-colors-suffice

[12] Kim, J.: The homotopy type of the independence complex of graphs with no induced cycles of length divisible by 3. Eur. J. Combin. **104**, art. 103534, 9 pp. (2022) Zbl 1492.05169 MR 4403044

[13] Munkres, J. R.: Elements of algebraic topology. Addison-Wesley, Menlo Park, CA (1984) Zbl 0673.55001 MR 755006

[14] Scott, A., Seymour, P.: Induced subgraphs of graphs with large chromatic number. X. Holes of specific residue. Combinatorica **39**, 1105–1132 (2019) Zbl 1449.05110 MR 4039603