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Subconvexity bounds for GL(3) × GL(2) L-functions in GL(2) spectral aspect

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Abstract. Let π be a Hecke–Maass cusp form for $\mathrm{SL}(3,\mathbb{Z})$ and f be a holomorphic cusp form for $\mathrm{SL}(2,\mathbb{Z})$ of weight k or a Hecke–Maass cusp form corresponding to the Laplacian eigenvalue $1/4+k^2$, $k \geq 1$, for $\mathrm{SL}(2,\mathbb{Z})$. In this paper, we prove the following subconvexity bound:

$$L(1/2,\pi \times f) \ll_{\pi,\varepsilon} k^{3/2-1/51+\varepsilon}.$$

Keywords. Subconvexity, Rankin–Selberg L-functions, Hecke–Maass forms

1. Introduction

A degree d automorphic L-function $L(s,F)$ associated to an automorphic form F is a Dirichlet series with an Euler product of degree d and satisfying some 'nice' analytic properties. In fact, it has a meromorphic continuation to the whole complex plane $\mathbb{C}$ and its completed L-function satisfies a functional equation relating its value at s to the value of the corresponding dual L-function at $1-s$. One may apply the Phragmén–Lindelöf principle together with the functional equation to get an upper bound $L(1/2+it,F) \ll_{d,\varepsilon} (C(F,t))^{1/4+\varepsilon}$, for any $\varepsilon > 0$, on the critical line $\Re(s) = 1/2$. Here $C(F,t)$ is a quantity, called the analytic conductor, which measures the complexity of the L-function and encapsulates the main parameters (level, spectral parameters, etc.) attached to the form F. The resulting bound is usually referred to as the *convexity bound* (or the trivial bound). The famous generalised Lindelöf Hypothesis (GLH) predicts that the exponent $1/4$ should be 0. While the GLH seems very far from reach with the current methods and technology, breaking the convexity bound, i.e, reducing the exponent $1/4$ by any small quantity, known as the *subconvexity problem* (ScP), is still a challenging problem. One is often interested in resolving the ScP with respect to a single 'family' associated to the form F. For instance, if the GL(1) family $|\cdot|^{it}$ (or χ, a Dirichlet character) associated to the form F vary, we call it the t (or twist) aspect ScP, or if spectral

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parameters (or level) of F vary, we call it the spectral (or level) aspect ScP. We refer to [24] for a vast introduction to the subconvexity problem.

For degree one L-functions ($\zeta(s)$ and $L(s,\chi)$), such estimates are due to Weyl ([35]) and Hardy–Littlewood in the t-aspect and due to Burgess [4] in the level aspect. For degree two L-functions, the first subconvexity bound was achieved by Good [10] in the t-aspect, by Duke–Friedlander–Iwaniec [6–8] in the level aspect and by Iwaniec [14] in the spectral aspect. For degree three L-functions attached to self-dual forms, such estimates were first obtained by Li [21] in the t-aspect in a groundbreaking work. Li's work was generalised to all GL(3) forms by Munshi [27], by introducing a novel delta method which he also applied in resolving the subconvexity problem for GL(3) L-functions in the twist aspect [28]. In GL(3) spectral aspect, when spectral parameters of a GL(3) form, π, say, are in 'generic' position, subconvexity estimates for $L(1/2,\pi)$ were obtained by Blomer–Buttcane [2].

For higher degree L-functions, the subconvexity problem becomes more challenging, and hence it is mostly open except for a few particular cases of Rankin–Selberg convolution L-functions. For the Rankin–Selberg L-functions on $\mathrm{GL}(2)\times\mathrm{GL}(2)$, subconvexity bounds are due to Michel–Venkatesh [25] in the t-aspect, Sarnak [32], and Lau–Liu–Ye [19] in the spectral aspect, and Kowalski–Michel–Vanderkam [16], Michel [23] and Harcos–Michel [11] in the level aspect. Some impressive subconvexity estimates were obtained by Bernstein–Reznikov [1] and Venkatesh [34] for the Rankin–Selberg triple L-functions on GL(2). In a recent breakthrough, Nelson [30] resolved the spectral aspect ScP (away from conductor dropping scenario) for the L-function on $\mathrm{U}(n+1)\times\mathrm{U}(n)$, for any n, where $\mathrm{U}(n)$ is the unitary group.

We will now discuss a few known results for degree six Rankin–Selberg L-functions on $\mathrm{GL}(3)\times\mathrm{GL}(2)$. To start with, let π be a normalized Hecke–Maass cusp form of type (ν_1,ν_2) for $\mathrm{SL}(3,\mathbb{Z})$. Let f be a holomorphic cusp form of weight k or a Hecke–Maass cusp form corresponding to the Laplace eigenvalue $1/4+k^2$ for $\mathrm{SL}(2,\mathbb{Z})$. The associated Rankin–Selberg L-series is given by

$$L(s,\pi\times f)=\sum\sum_{n,r\geq 1}\frac{\lambda_\pi(n,r)\lambda_f(n)}{(nr^2)^s},\quad \Re(s)>1. \tag{1.1}$$

In a pioneering work, Li [21] studied the above series and obtained subconvexity for $L(1/2,\pi\times f)$ in the GL(2) spectral aspect as well as subconvexity for $L(1/2+it,\pi)$ for a self-dual form π in the t-aspect (also mentioned above). Her main theorem was the following:

Theorem (X. Li). *Let π be a fixed self-dual Hecke–Maass cusp form for $\mathrm{SL}(3,\mathbb{Z})$ and u_j be an orthonormal basis of even Hecke–Maass cusp form for $\mathrm{SL}(3,\mathbb{Z})$ corresponding to the Laplacian eigenvalue $1/4+t_j^2$ with $t_j\geq 0$. Then for $\varepsilon>0$, T large and $T^{3/8+\varepsilon}\leq M\leq T^{1/2}$, we have*

$$\sum_j{}' e^{-\frac{(t_j-T)^2}{M^2}}L(1/2,\pi\times u_j)+\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{-\frac{(t_j-T)^2}{M^2}}|L(1/2-it,\pi)|^2\,\mathrm{d}t\ll_{\varepsilon,\pi}T^{1+\varepsilon}M,$$

where the prime means summing over the orthonormal basis of even Hecke–Maass cusp forms.

As a corollary, she obtained

$$L(1/2, \pi \times u_j) \ll_{\varepsilon,\pi} (1 + |t_j|)^{3/2-1/8+\varepsilon}. \tag{1.2}$$

She adapted Conrey–Iwaniec's moment method approach (see [5]) to prove the above theorem. The fact $L(1/2, \pi \times u_j) \geq 0$ plays a crucial role in her approach, that is why she could deal with self-dual forms only. Recently, Munshi [29], using the delta method, obtained subconvexity for $L(s, \pi \times f)$ in the t-aspect proving the following result:

$$L(1/2 + it, \pi \times f) \ll_{\varepsilon,f,\pi} (1 + |t|)^{3/2-1/51+\varepsilon}.$$

His method is insensitive to the self-duality of GL(3) forms. Thus he obtained the above result for any GL(3) form. Using a similar approach, Sharma [33] and the author along with Mallesham and Singh [17] proved subconvexity in the twist and the GL(3) spectral aspect (in some non-generic cases) respectively.

In this article, we vary the GL(2) family and establish subconvexity for $L(1/2, \pi \times f)$ in the GL(2) spectral aspect. Our main theorem is the following:

Theorem 1. *Let π be a fixed Hecke–Maass cusp form for* $\mathrm{SL}(3,\mathbb{Z})$ *and f be a holomorphic cusp form of weight k or a Hecke–Maass cusp form corresponding to the Laplacian eigenvalue* $1/4 + k^2$, $k \geq 1$, *for* $\mathrm{SL}(2,\mathbb{Z})$. *Then for any* $\varepsilon > 0$, *we have*

$$L(1/2, \pi \times f) \ll_{\pi,\varepsilon} k^{3/2-1/51+\varepsilon}.$$

Remark 1. We generalise the bound (1.2) of Li [21] to any GL(3) form. Although our bound is weaker than hers, it yields subconvexity.

The arguments in the proof work for both Maass and holomorphic forms. For the exposition of the method, we will give details for holomorphic forms only. Our method also works for any fixed central value $1/2 + it$. In this case, the implied constant will depend polynomially on t. For simplicity, we take $t = 0$ in the proof. If we take π to be the minimal Eisenstein series with Langlands parameters $(\alpha_1, \alpha_2, \alpha_3)$ for $\mathrm{SL}(3,\mathbb{Z})$ in (1.1), we observe that (see [9, p. 314])

$$L(s, \pi) = \prod_p \prod_{i=1}^{3} (1 - p^{\alpha_i - s})^{-1} = \zeta(s - \alpha_1)\zeta(s - \alpha_2)\zeta(s - \alpha_3).$$

It is also well-known that

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \prod_{j=1}^{2} (1 - \beta_{p,j} p^{-s})^{-1},$$

where $\beta_{p,1}\beta_{p,2} = 1$, $\beta_{p,1} + \beta_{p,2} = \lambda_f(p)$, and $\lambda_f(n)$ denote the normalised Fourier coefficients of f. Using Rankin–Selberg theory (see [9, p. 379]), we get

$$\begin{aligned} L(s, \pi \times f) &= \prod_p \prod_{i=1}^{3} (1 - \beta_{p,1} p^{\alpha_i - s})^{-1} (1 - \beta_{p,2} p^{\alpha_i - s})^{-1} \\ &= L(s - \alpha_1, f) L(s - \alpha_2, f) L(s - \alpha_3, f). \end{aligned}$$

Our method also applies to the above L-function. Hence we also obtain the following result:

Theorem 2. *Let f be a holomorphic cusp form of weight k or a Hecke–Maass cusp form corresponding to the Laplacian eigenvalue $1/4 + k^2$, $k \geq 1$, for* $\mathrm{SL}(2, \mathbb{Z})$. *Then for $\varepsilon > 0$, we have*

$$L(1/2, f) \ll_\varepsilon k^{1/2 - 1/153 + \varepsilon}.$$

Discussion of the method

We follow Munshi's delta symbol approach (see [29]) to prove our theorem. So far this approach has been successful to resolve the subconvexity problems where either a GL(1) form varies or the higher degree automorphic form admits a varying GL(1) factor (see [27–29]). In this article we take a step further by implementing the delta method to tackle subconvexity when the higher degree form (GL(2) form in our case) is allowed to vary.

After an application of the approximate functional equation, our problem boils down to getting non-trivial cancellations in the smooth sum

$$\sum_{n \sim k^3} \lambda_\pi(n, 1) \lambda_f(n),$$

where $\lambda_f(n)$'s are oscillatory. Munshi's separation of oscillation method (along with the 'conductor lowering trick') works here mainly due to the GL(3) × GL(2) structure. As such, any GL(1) analytic twist of $\lambda_f(n)$ by n^{it} with $|t| < k^{1-\varepsilon}$ does not alter the 'conductor' of the associated L-function. We benefit from this fact while applying the GL(2) Voronoi formula. A crucial observation, which was also present in [17, 29, 33], that the GL(2) and GL(3) Voronoi formulae together transform the Ramanujan sum

$$\sum_{a \bmod q}^{\star} e\left(\frac{(n-m)a}{q}\right),$$

arising from the DFI delta method, into

$$\sum_{a \bmod q}^{\star} S(\overline{a}, n; q) e\left(\frac{\overline{a}m}{q}\right),$$

which boils down to an additive character $qe(\overline{m}n/q)$ with respect to n, also plays a vital role in proving our main theorem.

One of the main hurdles in spectral aspect subconvexity problems (in comparison with the t-aspect) is the analysis of complicated integral transforms involving various Bessel functions in the 'transition' range. With no surprise, we also encounter such transforms, which we analyse (see Section 6) by applying stationary phase analysis.

Remark 2 (Notation). Throughout the paper $e(x)$ means $e^{2\pi i x}$. By negligibly small we mean $O(k^{-A})$ for any large positive constant $A > 0$. In particular, we take $A = 2020$. The letter ε denotes an arbitrarily small constant, not necessarily the same at different occurrences. The notation $\alpha \ll A$ will mean that for any $\varepsilon > 0$, there is a constant c such that $|\alpha| \leq cAk^{\varepsilon}$. We also ignore the dependence of the constant on π and ε, whenever it occurs. By $\alpha \asymp A$ we mean that $k^{-\varepsilon}A \leq \alpha \leq k^{\varepsilon}A$; also $\alpha \sim A$ means $A \leq \alpha < 2A$.

2. Preliminaries

In this section we recall some well-known results which we need in the proof.

2.1. *Holomorphic cusp forms on* GL(2)

Let f be a holomorphic Hecke eigenform of weight k for the full modular group $\mathrm{SL}(2,\mathbb{Z})$. The Fourier expansion of f at the cusp ∞ is given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz), \quad z \in \mathbb{H}.$$

We assume that f is normalised so that $\lambda_f(1) = 1$. We have the well-known Deligne bound $|\lambda_f(n)| \leq d(n)$, $n \geq 1$, where $d(n)$ is the divisor function. However, in our proof, we only need the Ramanujan bound on average:

$$\sum_{n \leq X} |\lambda_f(n)|^2 \ll_{\varepsilon} X^{1+\varepsilon}, \tag{2.1}$$

for any $\varepsilon > 0$. We now recall the Voronoi summation formula for the form f, which will be crucially used in our proof.

Lemma 1 (see [16, Theorem A.4]). *Let $\lambda_f(n)$ be as above and g be a smooth, compactly supported function on $(0,\infty)$. Let $a, q \in \mathbb{Z}$ with $(a,q) = 1$. Then*

$$\sum_{n=1}^{\infty} \lambda_f(n) e\left(\frac{an}{q}\right) g(n) = \frac{1}{q} \sum_{n=1}^{\infty} \lambda_f(n) e\left(-\frac{dn}{q}\right) h\left(\frac{n}{q^2}\right),$$

where $ad \equiv 1 \bmod q$ and

$$h(y) = 2\pi i^k \int_0^{\infty} g(x) J_{k-1}(4\pi\sqrt{xy})\, \mathrm{d}x,$$

where J_{k-1} is the usual J-Bessel function of order $k-1$.

2.2. *Maass cusp forms for* GL(2)

Let f be a Hecke–Maass eigenform for $SL(2,\mathbb{Z})$ with Laplace eigenvalue $1/4+\nu^2$, $\nu > 0$. The Fourier series expansion of f at the cusp ∞ is given by

$$f(z) = \sqrt{y}\sum_{n\neq 0}\lambda_f(n)K_{i\nu}(2\pi|n|y)e(nx),$$

where $K_{i\nu}(y)$ is the Bessel function of the third kind and f is normalised so that $\lambda_f(1) = 1$. The Ramanujan–Petersson conjecture, which asserts that $|\lambda_f(n)| \leq d(n)$, has not been confirmed yet. However, we do not need such an individual bound for our proof. Rather, the following Ramanujan bound on average (see [14, Lemma 1])

$$\sum_{1\leq n\leq X}|\lambda_f(n)|^2 \ll_\varepsilon \nu^\varepsilon X^{1+\varepsilon}, \tag{2.2}$$

for any $\varepsilon > 0$, is sufficient for our purpose. We also have the following Voronoi summation formula for the Maass cusp forms, which is similar to the case of holomorphic cusp forms.

Lemma 2 (see [16, Theorem A.4]). *Let $\lambda_f(n)$ be as above and g be a smooth, compactly supported function on $(0,\infty)$. Let a, $q \in \mathbb{Z}$ with $(a,q) = 1$. Then*

$$\sum_{n=1}^{\infty}\lambda_f(n)e\left(\frac{an}{q}\right)g(n) = \frac{1}{q}\sum_{\pm}\sum_{n=1}^{\infty}\lambda_f(n)e\left(\mp\frac{dn}{q}\right)H^{\pm}\left(\frac{n}{q^2}\right),$$

where $ad \equiv 1 \bmod q$ and

$$H^-(y) = \frac{-\pi}{\sin(\pi i\nu)}\int_0^\infty g(x)\{J_{2i\nu} - J_{-2i\nu}\}(4\pi\sqrt{xy})\,dx,$$
$$H^+(y) = 4\varepsilon_f\cosh(\pi\nu)\int_0^\infty g(x)K_{2i\nu}(4\pi\sqrt{xy})\,dx.$$

Here ε_f is the eigenvalue of f under the reflection operator.

2.3. *Automorphic forms on* GL(3)

This section, except for the notations, is taken from [21]. Let π be a Hecke–Maass cusp form of type (ν_1,ν_2) for $SL(3,\mathbb{Z})$. Let $\lambda_\pi(n,r)$ denote the normalised Fourier coefficients of π. Let

$$\alpha_1 = -\nu_1 - 2\nu_2 + 1, \quad \alpha_2 = -\nu_1 + \nu_2 \quad \text{and} \quad \alpha_3 = 2\nu_1 + \nu_2 - 1$$

be the spectral parameters for π (see [9]). Let g be a compactly supported smooth function on $(0,\infty)$ and

$$\tilde{g}(s) = \int_0^\infty g(x)x^{s-1}\,dx$$

be its Mellin transform. For $\ell = 0, 1$, we define

$$\gamma_\ell(s) := \frac{\pi^{-3s-3/2}}{2}\prod_{i=1}^{3}\frac{\Gamma\left(\frac{1+s+\alpha_i+\ell}{2}\right)}{\Gamma\left(\frac{-s-\alpha_i+\ell}{2}\right)}. \tag{2.3}$$

Set $\gamma_{\pm}(s) = \gamma_0(s) \mp \gamma_1(s)$ and let

$$G_{\pm}(y) = \frac{1}{2\pi i} \int_{(\sigma)} y^{-s} \gamma_{\pm}(s) \tilde{g}(-s)\, \mathrm{d}s, \tag{2.4}$$

where $\sigma > -1 + \max\{-\Re(\alpha_1), -\Re(\alpha_2), -\Re(\alpha_3)\}$.

Lemma 3 (see [26]). *Let $g(x)$ and $\lambda_\pi(n,r)$ be as above. Let $a, q \in \mathbb{Z}$ with $q \geq 1, (a,q) = 1$, and $a\bar{a} \equiv 1 \bmod q$. Then*

$$\sum_{n=1}^{\infty} \lambda_\pi(n,r) e\left(\frac{an}{q}\right) g(n) = q \sum_{\pm} \sum_{n_1 | qr} \sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1,n_2)}{n_1 n_2} S(r\bar{a}, \pm n_2; qr/n_1) G_{\pm}\left(\frac{n_1^2 n_2}{q^3 r}\right),$$

where $S(a,b;q)$ is the Kloosterman sum which is defined as follows:

$$S(a,b;q) = \sideset{}{^\star}\sum_{x \bmod q} e\left(\frac{ax + b\bar{x}}{q}\right).$$

The following lemma extracts the oscillations of $G_{\pm}$.

Lemma 4 (see [20, Lemma 6.1]). *Let $G_{\pm}(x)$ be as above, and $g(x) \in C_c^\infty(X, 2X)$. Then for any fixed integer $K \geq 1$ and $xX \gg 1$, we have*

$$G_{\pm}(x) = x \int_0^\infty g(y) \sum_{j=1}^{K} \frac{c_j(\pm) e(3(xy)^{1/3}) + d_j(\pm) e(-3(xy)^{1/3})}{(xy)^{j/3}}\, \mathrm{d}y$$
$$+ O((xX)^{(-K+5)/3}),$$

where $c_j(\pm)$ and $d_j(\pm)$ are some absolute constants depending on α_i, $i = 1, 2, 3$.

The following lemma is the well-known Ramanujan bound on average.

Lemma 5. *We have*

$$\sum\sum_{n_1^2 n_2 \leq x} |\lambda_\pi(n_1,n_2)|^2 \ll_\pi x, \tag{2.5}$$

where the implied constant depends on the form π.

Proof. For the proof, we refer to Goldfeld's book [9]. ∎

2.4. *The delta method*

Let $\delta : \mathbb{Z} \to \{0,1\}$ be defined by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The above function can be used to separate the oscillations involved in a sum

$$\sum_{n \sim X} a(n) b(n),$$

say, where $\{a(n)\}$ and $\{b(n)\}$ are two sequences of arithmetic interest. Furthermore, we seek a 'nice' Fourier expansion of $\delta(n)$. We mention here an expansion due to Duke, Friedlander and Iwaniec (see [15, Chapter 20]). Let $L \geq 1$ be a large real number. For $n \in [-2L, 2L]$, we have

$$\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sideset{}{^\star}\sum_{a \bmod q} e\left(\frac{na}{q}\right) \int_{\mathbb{R}} g(q,x) e\left(\frac{nx}{qQ}\right) \mathrm{d}x, \tag{2.6}$$

where $Q = 2L^{1/2}$. The $\star$ on the sum indicates that the sum over a is restricted by the condition $(a,q) = 1$. The function g is the only part in the above formula which is not explicitly given. Nevertheless, we only need the following properties of g in our analysis. For any $B > 1$, we have (see [29, pp. 5–6])

$$\begin{aligned}
&(1)\ g(q,x) = 1 + h(q,x) \quad \text{with} \quad h(q,x) = O\left(\frac{Q}{q}\left(\frac{q}{Q} + |x|\right)^B\right),\\
&(2)\ x^j \frac{\partial^j}{\partial x^j} g(q,x) \ll \log Q \min\left\{\frac{Q}{q}, \frac{1}{|x|}\right\}, \quad j \geq 1,\\
&(3)\ g(q,x) \ll |x|^{-B},\\
&(4)\ \int_{\mathbb{R}} (|g(q,x)| + |g(q,x)|^2)\,\mathrm{d}x \ll Q^{\varepsilon}.
\end{aligned} \tag{2.7}$$

Using the third property we observe that the effective range of the x-integral in (2.6) is $[-Q^{\varepsilon},\ Q^{\varepsilon}]$. We record the above observations in the following lemma.

Lemma 6 (see [15, Chapter 20] and [12, Lemma 15]). *Let δ be as above. Let $L \geq 1$ be a large parameter. Then, for $n \in [-2L, 2L]$, we have*

$$\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sideset{}{^\star}\sum_{a \bmod q} e\left(\frac{na}{q}\right) \int_{\mathbb{R}} W(x/Q^{\varepsilon}) g(q,x) e\left(\frac{nx}{qQ}\right) \mathrm{d}x + O(L^{-2020}),$$

where $Q = 2L^{1/2}$, g is a function satisfying (2.7) and $W(x)$ is a non-negative smooth bump function supported in $[-2,2]$, with $W(x) = 1$ for $x \in [-1,1]$ and $W^{(j)}(x) \ll_j 1$ for $j \geq 0$.

2.5. *Bessel functions*

In this subsection, we will recall some well-known expansions of Bessel functions of first kind. For $k \geq 2$ an integer, let $J_{k-1}(x)$ be the Bessel function of the first kind and of order $k-1$, which is defined as

$$J_{k-1}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e\left(\frac{(k-1)\tau - x \sin \tau}{2\pi}\right) \mathrm{d}\tau \tag{2.8}$$

for any $x \in \mathbb{R}$. In the analysis of integral transforms, we require a uniform asymptotic expansion of $J_{k-1}(x)$ for large values of k and x. The following lemma provides one such asymptotic expansion.

Lemma 7. *Let $x \geq (k-1)^{1+\varepsilon/2}$ be a positive real number. Then, as $k \to \infty$, we have*

$$J_{k-1}(x) = g(k,w)\Big[\cos\Big((k-1)(w-\tan^{-1} w) - \frac{\pi}{4}\Big)\sum_{j=0}^{\infty} \frac{P_j\big(\frac{1}{w-\tan^{-1} w}\big)}{(k-1)^j}\Big] + g(k,w)\Big[\sin\Big((k-1)(w-\tan^{-1} w) - \frac{\pi}{4}\Big)\sum_{j=1}^{\infty} \frac{P_j\big(\frac{1}{w-\tan^{-1} w}\big)}{(k-1)^j}\Big], \quad (2.9)$$

where

$$g(k,w) = \left(\frac{2}{\pi(k-1)w}\right)^{1/2}, \quad w = \left(\frac{x^2}{(k-1)^2} - 1\right)^{1/2},$$

and P_j is a polynomial of degree j with coefficients which are bounded functions of $k-1$ and $\log(x/(k-1))$ with $P_0 \equiv 1$.

Proof. Let $x = (k-1)\sec\beta$ with $0 < \beta < \pi/2$. Thus, as $x \geq (k-1)^{1+\varepsilon/2}$, we have $\sec\beta \geq (k-1)^{\varepsilon/2}$ and

$$\xi := (k-1)(\tan\beta - \beta) \geq (k-1)(\sqrt{(k-1)^{\varepsilon} - 1} - \pi/2).$$

Thus, on using formula (63) of [18, p. 58], we get

$$J_{k-1}((k-1)\sec\beta) = \left(\frac{2}{\pi(k-1)\tan\beta}\right)^{1/2}\Big[\cos f_1(\beta)\sum_{j=0}^{\infty}\frac{P_j\big(\frac{1}{\tan\beta-\beta}\big)}{(k-1)^j}\Big] + \left(\frac{2}{\pi(k-1)\tan\beta}\right)^{1/2}\Big[\sin f_1(\beta)\sum_{j=1}^{\infty}\frac{P_j\big(\frac{1}{\tan\beta-\beta}\big)}{(k-1)^j}\Big],$$

where $f_1(\beta) = (k-1)(\tan\beta - \beta) - \pi/4$, and P_j represents a polynomial of degree j with coefficients which are bounded functions of $k-1$ and $\log\sec\beta$ with $P_0 \equiv 1$. Now substituting $(k-1)\sec\beta = x$ and $\tan\beta = w$, we get the lemma. ∎

The expansion (2.9) can be truncated at any stage to get

Corollary 1. *Under the assumptions of Lemma 7, we have*

$$J_{k-1}(x) = \sum_{\pm}\sum_{j=0}^{2019} \frac{e\big(\pm\frac{(k-1)(w-\tan^{-1} w)}{2\pi}\big)P_j\big(\frac{1}{w-\tan^{-1} w}\big)}{\sqrt{\pi}\, w^{1/2}(k-1)^{j+1/2}} + O\Big(\frac{1}{k^{2020}}\Big).$$

Proof. The statement follows directly from Lemma 7. ∎

For $0 < x \leq (k-1)^{1-\varepsilon/2}$, we have the following lemma.

Lemma 8. *Let $x = (k-1)z$ with $0 < z \ll (k-1)^{-\varepsilon/2}$. Then, as $k \to \infty$, we have*

$$J_{k-1}(x) \ll \exp\{-(k-1)/6\}.$$

Proof. By [31, Lemma 4.2], we have

$$|J_{k-1}((k-1)z)| \leq A_1(k-1)^{-1/2}(1-z^2)^{-1/4} \exp\bigl\{-\tfrac{1}{3}(k-1)(1-z^2)^{3/2}\bigr\}$$

for $0 < z \leq \sqrt{1-(k-1)^{-2/3}}$, $k \geq 16$ and some absolute constant A_1. Note that, by assumption, $z \leq (k-1)^{-\varepsilon}$. Thus, $1 - z^2 \geq 1/2^{2/3}$ as $k \to \infty$, and we get

$$|J_{k-1}((k-1)z)| \leq A_1 2^{1/6} \exp\bigl\{-\tfrac{1}{6}(k-1)\bigr\}.$$

Hence the lemma follows. ∎

2.6. *Stationary phase analysis*

In this subsection we will recall some facts about the exponential integrals of the form

$$I = \int_a^b g(x) e(f(x))\, \mathrm{d}x,$$

where f and g are smooth real valued functions on $[a, b]$.

Lemma 9 (see [27, Section 2.2] and [13, Lemma 5.1.4]). *Let I, f and g be as above. Let $V(g)$ denote the total variation of $g(x)$ on $[a, b]$ plus the maximum modulus of $g(x)$ on $[a, b]$. Then, if f' is monotone and $|f'(x)| \geq \mu_1 > 0$ for $x \in [a, b]$, we have $I \ll V(g)/\mu_1$. For $r > 1$, let $|f^{(r)}(x)| \geq \mu_r > 0$. Then we have $I \ll_r V(g)/\mu_r^{1/r}$. Moreover, let $f'(x) \geq B$ and $f^{(j)}(x) \ll B^{1+\varepsilon}$ for $j \geq 2$ together with* $\operatorname{supp}(g) \subset (a, b)$ *and $g^{(j)}(x) \ll_{a,b,j} 1$. Then*

$$I \ll_{a,b,j,\varepsilon} B^{-j+\varepsilon}.$$

We apply the above lemma for $r = 1$ whenever the phase function f does not have any stationary point. We will also apply it for $r = 2, 3$. In case there is a unique stationary point, we use the following stationary phase expansion.

Lemma 10 (see [3, Lemma 8.1]). *Let I, f and g be as above. Let $0 < \delta < 1/10$, $X, Y, U, Q > 0$, $Z := Q + X + Y + b - a + 1$, and assume that*

$$Y \geq Z^{3\delta}, \quad b - a \geq U \geq \frac{QZ^{\delta/2}}{\sqrt{Y}}.$$

Further, assume that g satisfies

$$g^{(j)}(x) \ll_j X/U^j \quad \text{for } j = 0, 1, \ldots.$$

Suppose that there exists a unique $x_0 \in [a, b]$ such that $f'(x_0) = 0$, and the function f satisfies

$$f''(x) \gg \frac{Y}{Q^2}, \quad f^{(j)}(x) \ll_j \frac{Y}{Q^j} \quad \text{for } j = 1, 2, \ldots.$$

Then

$$I = \frac{e(f(x_0))}{\sqrt{f''(x_0)}} \sum_{n=0}^{3\delta^{-1}A} p_n(x_0) + O_{A,\delta}(Z^{-A}),$$
$$p_n(x_0) = \frac{e^{\pi i/4}}{n!}\left(\frac{i}{2f''(x_0)}\right)^n G^{(2n)}(x_0),$$

where $A > 0$ *is arbitrary, and*

$$G(x) = g(x)e(F(x)), \quad F(x) = f(x) - f(x_0) - \tfrac{1}{2} f''(x_0)(x - x_0)^2.$$

Furthermore, each p_n *is a rational function in* $f', f'', \ldots,$ *satisfying*

$$\frac{d^j}{dx_0^j} p_n(x_0) \ll_{j,n} X\left(\frac{1}{U^j} + \frac{1}{Q^j}\right)\left(\left(\frac{U^2Y}{Q^2}\right)^{-n} + Y^{-n/3}\right).$$

3. The set-up and outline of proof

Let π and f be defined as in Theorem 1. Let $\lambda_\pi(n,r)$ denote the normalised Fourier coefficients of the form π (see [9, Chapter 6]) and let $\lambda_f(n)$ denote the normalised Fourier coefficients of the form f (see [15, Chapter 14]). We are interested in analysing the Rankin–Selberg L-series $L(s, \pi \times f)$ (defined in (1.1)) attached to π and f at the central point $1/2$. To study $L(1/2, \pi \times f)$, we first express it as a weighted Dirichlet series.

Lemma 11. *Let* $0 < \theta < 3/2$. *Then, as* $k \to \infty$,

$$L(1/2, \pi \times f) \ll k^\varepsilon \sup_{r \le k^\theta} \sup_{k^{3-\theta}/r^2 \le N \le k^{3+\varepsilon}/r^2} \frac{|S_r(N)|}{N^{1/2}} + k^{(3-\theta)/2+\varepsilon}, \tag{3.1}$$

where

$$S_r(N) := \sum_{n=1}^{\infty} \lambda_\pi(n,r)\lambda_f(n)V\left(\frac{n}{N}\right) \tag{3.2}$$

for some smooth function V *supported in* $[1,2]$, *satisfying* $V^{(j)}(x) \ll_j 1$ *for* $j \ge 0$ *and normalised so that* $\int V(y)\,\mathrm{d}y = 1$.

Proof. Use the template of [15, Theorem 5.3]; see also [29, pp. 1546–1547]. ■

Remark 3. Upon estimating $S_r(N)$ using Cauchy's inequality and the Ramanujan bound on average (see (2.1), (2.2), (2.5)), we see that $L(1/2, \pi \times f) \ll_{\pi,\varepsilon} k^{3/2+\varepsilon}$. Hence, to establish subconvexity, we need to get some cancellations in the sum $S_r(N)$ for N, roughly, of size k^3. To this end we will analyse $S_r(N)$ in the rest of the paper.

3.1. An application of the delta method

As a first step, following Munshi [29], we separate the oscillatory terms $\lambda_\pi(n,r)$ and $\lambda_f(n)$ involved in the sum $S_r(N)$. We use the delta method of Duke, Friedlander and Iwaniec as a device to separate these terms. We also apply the conductor lowering trick introduced by Munshi [27]. For this purpose we introduce an extra t-integral. In fact, we express $S_r(N)$ as

$$\frac{1}{T}\int_{\mathbb{R}} V\left(\frac{t}{T}\right) \sum\sum_{\substack{n,m=1\\ n=m}}^{\infty} \lambda_\pi(n,r)\lambda_f(m)\left(\frac{n}{m}\right)^{it} V\left(\frac{n}{N}\right)U\left(\frac{m}{N}\right) \mathrm{d}t$$
$$= \frac{1}{T}\int_{\mathbb{R}} V\left(\frac{t}{T}\right) \sum\sum_{n,m=1}^{\infty} \delta(n-m)\lambda_\pi(n,r)\lambda_f(m)\left(\frac{n}{m}\right)^{it} V\left(\frac{n}{N}\right)U\left(\frac{m}{N}\right) \mathrm{d}t, \tag{3.3}$$

where $k^\varepsilon < T < k^{1-\varepsilon}$ is a parameter of the form $k^{1-\eta}$, for $\eta > 0$, which will be chosen later optimally, and U is a smooth function supported in $[1/2, 5/2]$ with $U(x) = 1$ for $x \in [1,2]$, and $U^{(j)}(x) \ll_j 1$ for any integer $j \geq 0$. Consider the t-integral

$$\int_{\mathbb{R}} V\left(\frac{t}{T}\right)\left(\frac{m}{n}\right)^{it} \mathrm{d}t.$$

On applying integration by parts repeatedly, we observe that the above integral is negligibly small unless $|n-m| \ll k^\varepsilon N/T$. Thus the t-integral reduces the size of the equation $n = m$. Thus, on applying Lemma 6 to (3.3) with $L = k^\varepsilon N/T$, and $Q = k^\varepsilon \sqrt{N/T}$, we see that $S_r(N)$ is transformed into

$$S_r(N) = \frac{1}{QT}\int_{\mathbb{R}} W(x/Q^\varepsilon)\int_{\mathbb{R}} V\left(\frac{t}{T}\right)$$
$$\times \sum_{1\leq q\leq Q} \frac{g(q,x)}{q} \sideset{}{^\star}\sum_{a \bmod q} \sum_{n=1}^{\infty} \lambda_\pi(n,r) e\left(\frac{an}{q}\right) e\left(\frac{nx}{qQ}\right) n^{it} V\left(\frac{n}{N}\right)$$
$$\times \sum_{m=1}^{\infty} \lambda_f(m) m^{-it} e\left(\frac{-am}{q}\right) e\left(\frac{-mx}{qQ}\right) U\left(\frac{m}{N}\right) \mathrm{d}t\,\mathrm{d}x + O(k^{-2020}). \tag{3.4}$$

3.2. Sketch of proof

In this subsection, we will discuss rough ideas to get non-trivial cancellations in $S_r(N)$ given in (3.4). For simplicity, we consider the generic case, i.e., $N = k^3$, $r = 1$ and $q \sim Q = \sqrt{N/T} = k^{3/2}/T^{1/2}$. Thus $S_r(N)$ is roughly given by

$$\frac{1}{QT}\int_T^{2T} \sum_{q\sim Q} \frac{1}{q} \sideset{}{^\star}\sum_{a \bmod q} \sum_{n\sim N} \lambda_\pi(n,1) n^{it} e\left(\frac{an}{q}\right) \sum_{m\sim N} \lambda_f(m) m^{-it} e\left(\frac{-am}{q}\right) \mathrm{d}t.$$

Note that we have ignored the x-integral, as it does not contribute in the generic case, and we have also suppressed all the weight functions. On estimating the above sum using

Cauchy's inequality and the Rankin–Selberg bound, we get $S_r(N) \ll N^{2+\varepsilon}$. Our goal is to save N plus a little more, say, k^{δ}. In other words, we need to show $S_r(N) \ll N^2/(Nk^{\delta})$ for some $\delta > 0$.

In the next step we dualize the sum over n and m (see Section 4 for full details). Consider the sum over n

$$\mathrm{S}_3 = \sum_{n \sim N} \lambda_\pi(n,1) n^{it} e\left(\frac{an}{q}\right).$$

On applying the GL(3) Voronoi summation formula to the above sum we arrive at (see Lemma 12)

$$\mathrm{S}_3 \approx \frac{N^{2/3}}{q} \sum_{n_2 \sim Q^3T^3/N} \frac{\lambda_\pi(1,n_2)}{n_2^{1/3}} S(\bar{a}, \pm n_2; q)\, \mathrm{I}_3(\dots),$$

where $\mathrm{I}_3(\dots)$ is an integral transform in which we need to get square root cancellations, i.e., we need to show $\mathrm{I}_3(\dots) \ll 1/\sqrt{T}$. Next we apply the GL(2) Voronoi formula to the sum over m and we get (see Lemma 13 for details)

$$\sum_{m \sim N} \lambda_f(m) m^{-it} e\left(\frac{-am}{q}\right) \approx \frac{N}{q} \sum_{m \sim Q^2k^2/N} \lambda_f(m) e\left(\frac{\bar{a}m}{q}\right) \mathrm{I}_2(\dots),$$

where $\mathrm{I}_2(\dots)$ is an integral transform in which we need to get full cancellations, i.e., we need to show $\mathrm{I}_2(\dots) \ll 1/k$. Next we analyse the sum over a which is given by

$$\mathfrak{C} = \sideset{}{^*}\sum_{a \bmod q} S(\bar{a}, n_2; q) e\left(\frac{\bar{a}m}{q}\right) \approx q e\left(-\frac{\bar{m}n_2}{q}\right).$$

We observe that the above sum becomes an additive character with respect to n_2 (which saves us extra q when we apply the Poisson summation formula after Cauchy's inequality). Thus we arrive at the following expression:

$$\frac{1}{QT} \frac{N}{Q^2T} \frac{N}{Q} \sum_{q \sim Q} \sum_{n_2 \sim T^{3/2}N^{1/2}} \lambda_\pi(1,n_2) \sum_{m \sim k^2/T} \lambda_f(m) e\left(-\frac{\bar{m}n_2}{q}\right) \mathfrak{J},$$

where $\mathfrak{J}$ is an integral transform involving the t-integral, $\mathrm{I}_2(\dots)$ and $\mathrm{I}_3(\dots)$. We analyse it in Section 6. We observe that

$$\mathfrak{J} \ll T \frac{1}{\sqrt{T}} \frac{1}{\sqrt{T}} \frac{1}{k}.$$

Note that a saving of $\sqrt{T}$ comes from the t-integral, another saving of $\sqrt{T}$ comes from the GL(3)-integral and the saving of k comes from the GL(2)-integral. The factor T reflects the length of the t-integral. On plugging it in place of $\mathfrak{J}$ we see that

$$\begin{aligned} S_r(N) &\ll \sum_{q \sim Q} \sum_{n_2 \sim T^{3/2}N^{1/2}} |\lambda_\pi(1,n_2)| \left| \sum_{m \sim k^2/T} \lambda_f(m) e\left(-\frac{\bar{m}n_2}{q}\right) \mathfrak{J} \right| \\ &\ll QT^{3/2}N^{1/2} \frac{k^2}{T} \frac{1}{k} \ll Nk. \end{aligned}$$

Thus we now need to save $k^{1+\delta}$. We apply Cauchy's inequality to the sum over n_2 to get rid of the GL(3) coefficients. Thus we arrive at (see Section 5.1)

$$(T^{3/2}N^{1/2})^{1/2}\Bigg(\sum_{n_2\sim T^{3/2}N^{1/2}}\Bigg|\sum_{q\sim Q}\sum_{m\sim k^2/T}\lambda_f(m)e\left(-\frac{\bar{m}n}{q}\right)\mathfrak{I}\Bigg|^2\Bigg)^{1/2}.$$

The end game strategy is to apply the Poisson formula to the sum over n_2 (see Section 5.2). Opening the absolute value square followed by the Poisson formula, we observe that we save the whole length, i.e., k^2Q/T in the zero frequency ($n_2 = 0$ case) which suffices if $k^2Q/T > k^2$ which implies that $T < k$. On the other hand, in the non-zero frequencies ($n_2 \neq 0$), we save

$$\frac{T^{3/2}N^{1/2}}{(Q^2T)^{1/2}}.$$

Here the factor Q^2T in the denominator reflects the size of the conductor, which is given by

$$\text{arithmetic conductor} \times \text{analytic conductor}.$$

Note that the arithmetic conductor is of size Q^2 and the analytic conductor is of size T (because $\mathfrak{I}$ oscillates like n_2^{iT} with respect to n_2). We also save Q due to the presence of the additive character $e(-\bar{m}n/q)$. Thus the total saving in the non-zero frequencies turns out to be

$$\frac{T^{3/2}N^{1/2}}{(Q^2T)^{1/2}} \times Q = TN^{1/2},$$

which suffices if $TN^{1/2} > k^2$ which boils down to $T > k^{1/2}$. Hence we get the restriction $k^{1/2} < T < k$. In fact, the optimal choice for T turns out to be $k^{41/51}$, and Theorem 1 follows.

4. Applications of Voronoi formulae

In this section we will analyse the sum over n and m in (3.4) using Voronoi summation formulae.

4.1. GL(3) *Voronoi*

Let us consider the sum over n

$$\mathrm{S}_3 := \sum_{n=1}^{\infty}\lambda_\pi(n,r)e\left(\frac{an}{q}\right)e\left(\frac{nx}{qQ}\right)n^{it}V\left(\frac{n}{N}\right). \tag{4.1}$$

Recall that $N = 2^\alpha$, $\alpha \in [-1,\infty) \cap \mathbb{Z}$, is such that $N \leq k^{3+\varepsilon}/r^2$. We analyse S_3 using the GL(3) Voronoi summation formula (see Lemma 3). In the present set-up, we have

$g(n) = e(nx/(qQ))n^{it}V(n/N)$ and $X = N$. Thus, on applying Lemma 3 to the above sum, we get

$$\mathrm{S}_3 = q \sum_{\pm} \sum_{n_1|qr} \sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1,n_2)}{n_1 n_2} S(r\bar{a}, \pm n_2; qr/n_1) G_\pm(n_2^\star), \tag{4.2}$$

where $n_2^\star := n_1^2 n_2/(q^3 r)$ and $G_\pm(n_2^\star)$ is the integral transform defined in (2.4). Next we extract the oscillations of the integral transform $G_\pm(n_2^\star)$ using Lemma 4, which gives us the following expression for $G_\pm(n_2^\star)$ in the range $n_2^\star N \gg k^\varepsilon$:

$$G_\pm(n_2^\star) = n_2^\star \int_0^\infty g(z) \sum_{j=1}^{K_0} \frac{c_j(\pm)e(3(n_2^\star z)^{1/3}) + d_j(\pm)e(-3(n_2^\star z)^{1/3})}{(n_2^\star z)^{j/3}} \, \mathrm{d}z + O(k^{-2020}),$$

where $K_0 = [\frac{6060}{\varepsilon} + 5] + 1$ with $[\cdot]$ denoting the greatest integer function. From now on, we will continue our analysis with the terms corresponding to $j = 1$, as the other terms can be treated in a similar way and in fact, give us better estimates. Thus, on plugging the contribution corresponding to $j = 1$ into (4.2), we arrive at

$$\frac{N^{2/3+it}}{qr^{2/3}} \sum_{\pm} \sum_{n_1|qr} n_1^{1/3} \sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1,n_2)}{n_2^{1/3}} S(r\bar{a}, \pm n_2; qr/n_1) \, \mathrm{I}_3(n_1^2 n_2, q, x),$$

where

$$\mathrm{I}_3(n_1^2 n_2, q, x) := \int_0^\infty V(z) z^{it} e\left(\frac{Nxz}{qQ} \pm \frac{3(Nn_1^2 n_2 z)^{1/3}}{qr^{1/3}}\right) \mathrm{d}z. \tag{4.3}$$

On applying the change of variable $z \mapsto z^3$ followed by integration by parts (differentiating $3z^2 V(z^3) z^{i3t} e(Nxz^3/(qQ))$ and integrating $e(\pm 3(Nn_1^2 n_2)^{1/3} z/(qr^{1/3})))$ j-times to the above integral, we observe that

$$|\mathrm{I}_3(n_1^2 n_2, q, x)| \ll_j \left(1 + T + \frac{N|x|}{qQ}\right)^j \left(\frac{qr^{1/3}}{(Nn_1^2 n_2)^{1/3}}\right)^j$$

for any integer $j \geq 0$, and it is negligibly small if

$$n_1^2 n_2 \gg k^\varepsilon \max\{q^3 T^3 r/N, T^{3/2} N^{1/2} r\} =: N_0. \tag{4.4}$$

Now it remains to analyse $G_\pm(n_2^\star)$ for $n_2^\star N \ll k^\varepsilon$, which is given as

$$\begin{aligned} G_\pm(n_2^\star) &= \frac{1}{2\pi i} \int_{(\sigma)} (n_2^\star)^{-s} \gamma_\pm(s) \tilde{g}(-s) \, \mathrm{d}s \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (n_2^\star)^{-\sigma - i\tau} \gamma_\pm(\sigma + i\tau) \tilde{g}(-\sigma - i\tau) \, \mathrm{d}\tau. \end{aligned} \tag{4.5}$$

We will analyse this case in Section 8.3. We conclude this subsection by summarising the above discussion in the following lemma.

Lemma 12. *Let* S_3 *be as in* (4.1)*. Then, for* $n_2^\star N = n_1^2 n_2 N/(q^3 r) \gg k^\varepsilon$*, we have*

$$\mathrm{S}_3 = \frac{N^{2/3+it}}{qr^{2/3}} \sum_{\pm} \sum_{n_1|qr} n_1^{1/3} \sum_{n_2 \ll N_0/n_1^2} \frac{\lambda_\pi(n_1,n_2)}{n_2^{1/3}} S(r\bar{a}, \pm n_2; qr/n_1)\, \mathrm{I}_3(n_1^2 n_2, q, x)$$
$$+ \text{ other lower order terms} + O(k^{-2020}), \tag{4.6}$$

where $\mathrm{I}_3(n_1^2 n_2, q, x)$ *is the integral transform defined in* (4.3) *and* N_0 *is as defined in* (4.4)*. For the non-generic case* $n_2^\star N \ll k^\varepsilon$*, we have*

$$\mathrm{S}_3 = q \sum_{\pm} \sum_{n_1|qr} \sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1,n_2)}{n_1 n_2} S(r\bar{a}, \pm n_2; qr/n_1) G_\pm(n_2^\star), \tag{4.7}$$

where $G_\pm(n_2^\star)$ *is as defined in* (4.5)*.*

From now on we will proceed with the main term of (4.6).

4.2. GL(2) *Voronoi*

We now consider the sum over m in (3.4), which is given as

$$\mathrm{S}_2 := \sum_{m=1}^{\infty} \lambda_f(m) m^{-it} e\left(\frac{-am}{q}\right) e\left(\frac{-mx}{qQ}\right) U\left(\frac{m}{N}\right). \tag{4.8}$$

On applying the GL(2) Voronoi summation formula (see Lemma 1) to the above sum with $g(m) = m^{-it} e(-mx/(qQ)) U(m/N)$, we get

$$\mathrm{S}_2 = \frac{2\pi i^k N^{1-it}}{q} \sum_{m=1}^{\infty} \lambda_f(m) e\left(\frac{\bar{a}m}{q}\right) \mathrm{I}_2(m, q, x),$$

where

$$\mathrm{I}_2(m, q, x) := \int_0^\infty U(y) y^{-it} e\left(\frac{-Nxy}{qQ}\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \mathrm{d}y. \tag{4.9}$$

We now analyse the above integral to determine the range of m. We claim that $\mathrm{I}_2(m, q, x)$ is negligibly small unless

$$M := \frac{q^2 (k-1)^2 k^{-\varepsilon}}{N} \le m \le k^\varepsilon \max\left\{\frac{(k-1)^2 q^2}{N}, T\right\} =: M_0. \tag{4.10}$$

In fact, in the range $m < M$, we have

$$4\pi\sqrt{mNy}/q < 4\pi\sqrt{5/2}\,(k-1)^{1-\varepsilon/2} \ll (k-1)^{1-\varepsilon/2}.$$

Thus, by Lemma 8, $\mathrm{I}_2(m, q, x)$ is negligibly small.

Next we consider the range $m > M_0$ and we claim that $\mathrm{I}_2(m, q, x)$ is also negligibly small. We note that $4\pi\sqrt{mNy}/q > (k-1)^{1+\varepsilon/2}$. Thus we apply Langer's expansion (see Lemma 7) for J_{k-1}. On applying Corollary 1 with $x = 4\pi\sqrt{mNy}/q$, we see that $\mathrm{I}_2(m,q,x)$, up to a negligible error term, is given by

$$\sum_{j=0}^{2019} \frac{1}{(k-1)^{j+1/2}} \int_0^\infty U_j(y) y^{-it} e\left(\frac{-Nxy}{qQ}\right) e\left(\pm\frac{(k-1)(w-\tan^{-1} w)}{2\pi}\right) \mathrm{d}y,$$

where $U_j(y) = U(y)P_j((w-\tan^{-1} w)^{-1})w^{-1/2}$ with

$$w = \left(\frac{x^2}{(k-1)^2} - 1\right)^{1/2} = \left(\frac{16\pi^2 mNy}{q^2(k-1)^2} - 1\right)^{1/2},$$

and P_j is a polynomial of degree j with coefficients which are bounded functions of k. Note that $w > ((k-1)^\varepsilon - 1)^{1/2}$. Thus

$$w - \tan^{-1} w = w - \frac{\pi}{2} + \tan^{-1}\frac{1}{w} \asymp w,$$

and $U_j^{(i)}(y) \ll_i k^{\varepsilon i}$ for any integer $i \geq 0$. Next we apply integration by parts i-times to the y-integral and we get

$$\begin{aligned}|\mathrm{I}_2(m,q,x)| &\ll_i \left(k^\varepsilon + T + \frac{N|x|}{qQ}\right)^i \left(\frac{1}{(k-1)\sqrt{mN}/(q(k-1))}\right)^i \\ &\ll \left(\frac{Tq}{\sqrt{M_0 N}} + \frac{N}{Q\sqrt{M_0 N}}\right)^i \ll \left(\frac{k^\varepsilon T}{k} + \frac{1}{k^\varepsilon}\right)^i \\ &\ll \frac{1}{k^{\varepsilon i}}.\end{aligned}$$

Upon taking i sufficiently large, we get the claim. We end this subsection by summarizing the above arguments in the following lemma.

Lemma 13. *Let* S_2 *be the sum over m as given in* (4.8). *Then*

$$\mathrm{S}_2 = \frac{2\pi i^k N^{1-it}}{q} \sum_{M \leq m \leq M_0} \lambda_f(m) e\left(\frac{\bar{a}m}{q}\right) \mathrm{I}_2(m,q,x) + O(k^{-2020}), \tag{4.11}$$

where

$$\mathrm{I}_2(m,q,x) = \int_0^\infty U(y) y^{-it} e\left(\frac{-Nxy}{qQ}\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \mathrm{d}y,$$

and M *and* M_0 *are the ranges of m defined in* (4.10).

5. Cauchy and Poisson

After the applications of the Voronoi formulae and applying Lemmas 12 and 13 to (3.4), we find that the expression in (3.4), up to an error term to be treated in Section 8.3, has been essentially reduced to

$$\frac{N^{5/3}}{QTr^{2/3}} \sum_{1\le q\le Q} \frac{1}{q^3} \sum_{a \bmod q}^{\star} \sum_{\pm} \sum_{n_1|qr} n_1^{1/3} \sum_{n_2 \ll N_0/n_1^2} \frac{\lambda_\pi(n_1,n_2)}{n_2^{1/3}} S(r\bar{a}, \pm n_2; qr/n_1) \times \sum_{M\ll m\ll M_0} \lambda_f(m) e\left(\frac{\bar{a}m}{q}\right) \mathrm{J}_\pm(m, n_1^2 n_2, q), \tag{5.1}$$

where

$$\mathrm{J}_\pm(m,n_1^2n_2,q) = \int_{\mathbb{R}}\int_{\mathbb{R}} W(x/Q^\varepsilon) g(q,x) \mathrm{I}_2(m,q,x) \mathrm{I}_3(n_1^2n_2,q,x)\, V\left(\frac{t}{T}\right) \mathrm{d}t\, \mathrm{d}x. \tag{5.2}$$

In this section we will analyse (5.1) using Cauchy's inequality and the Poisson summation formula.

5.1. *Cauchy's inequality*

Splitting the sum over q into dyadic blocks $q \sim C$, i.e., $C \le q < 2C$, $C \ll Q$ and writing $q = q_1q_2$ with $q_1 \mid (n_1r)^\infty$, $(n_1r, q_2) = 1$, we see that the expression in (5.1) is dominated by

$$\sup_{C\ll Q} \frac{N^{5/3}\log Q}{QTr^{2/3}C^3} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty} \sum_{n_2\ll N_0/n_1^2} \frac{|\lambda_\pi(n_1,n_2)|}{n_2^{1/3}} \times \left| \sum_{q_2\sim C/q_1} \sum_{M\ll m\ll M_0} \lambda_f(m) \mathcal{C}_\pm(q,n_2,m)\, \mathrm{J}_\pm(m,n_1^2n_2,q)\right|, \tag{5.3}$$

where the character sum $\mathcal{C}_\pm(q,n_2,m) = \mathcal{C}_\pm(\dots)$ is defined as

$$\mathcal{C}_\pm(\dots) := \sum_{a \bmod q}^{\star} S(r\bar{a}, \pm n_2; qr/n_1) e\left(\frac{\bar{a}m}{q}\right) = \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sum_{\substack{\alpha \bmod qr/n_1 \\ n_1\alpha\equiv -m \bmod d}}^{\star} e\left(\pm\frac{\bar{\alpha}n_2}{qr/n_1}\right).$$

Next we analyse the expression inside $|\ |$. We first split the sum over m into dyadic blocks $m \sim M_1$, $M \ll M_1 \ll M_0$ and then apply Cauchy's inequality to the sum over n_2 in (5.3) to arrive at

$$S_r(N) \ll \sup_{\substack{M\ll M_1\ll M_0 \\ C\ll Q}} \frac{N^{5/3}(QM_0)^\varepsilon}{QTr^{2/3}C^3} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty} \sqrt{\Omega_\pm}, \tag{5.4}$$

where

$$\Theta = \sum_{n_2\ll N_0/n_1^2} \frac{|\lambda_\pi(n_1,n_2)|^2}{n_2^{2/3}}, \tag{5.5}$$

$$\Omega_\pm = \sum_{n_2\ll N_0/n_1^2} \left| \sum_{q_2\sim C/q_1} \sum_{m\sim M_1} \lambda_f(m)\mathcal{C}_\pm(q,n_2,m)\, \mathrm{J}_\pm(m,n_1^2n_2,q)\right|^2, \tag{5.6}$$

with

$$\frac{(k-1)^2C^2}{N}k^{-\varepsilon} = M \ll M_1 \ll M_0 = k^{\varepsilon}\max\left\{\frac{(k-1)^2C^2}{N}, T\right\},$$
$$N_0 = k^{\varepsilon}\max\left\{\frac{(CT)^3 r}{N}, T^{3/2}N^{1/2}r\right\}. \tag{5.7}$$

5.2. *Poisson summation*

Next we apply the Poisson summation formula to the sum over n_2 with the modulus $\mathfrak{q} := q_1 q_2 q_2' r/n_1$ in (5.6). To this end we first split the sum over n_2 into dyadic blocks $n_2 \sim \tilde{N}/n_1^2$, $\tilde{N} \ll N_0$. Then opening the absolute value square in (5.6), we arrive at

$$\Omega_{\pm} = \sum\sum_{q_2, q_2' \sim C/q_1} \sum\sum_{m, m' \sim M_1} \lambda_f(m)\,\overline{\lambda_f(m')}\,\Delta_{\pm},$$

where

$$\Delta_{\pm} = \sum_{\tilde{N}} \sum_{n_2 \in \mathbb{Z}} \phi\left(\frac{n_1^2 n_2}{\tilde{N}}\right) \mathcal{C}_{\pm}(q, n_2, m)\,\overline{\mathcal{C}_{\pm}(q', n_2, m')}\,\mathrm{J}_{\pm}(m, n_1^2 n_2, q)\,\overline{\mathrm{J}_{\pm}(m', n_1^2 n_2, q')},$$

$q' = q_1 q_2'$ and $\phi(w)$ is a non-negative smooth function supported on $[2/3, 3]$ with $\phi(w) = 1$ for $w \in [1, 2]$ and $\phi^{(j)}(w) \ll_j 1$. Now applying the change of variable

$$n_2 \to n_2\mathfrak{q} + \beta, \quad 0 \le \beta < \mathfrak{q},$$

we get the following expression for $\Delta_{\pm}$:

$$\Delta_{\pm} = \sum_{\tilde{N}} \sum_{\beta \bmod \mathfrak{q}} \mathcal{C}_{\pm}(q, \beta, m)\,\overline{\mathcal{C}_{\pm}(q', \beta, m')}$$
$$\times \sum_{n_2 \in \mathbb{Z}} \phi\left(\frac{n_2\mathfrak{q} + \beta}{\tilde{N}/n_1^2}\right) \mathrm{J}_{\pm}(m, n_1^2(n_2\mathfrak{q} + \beta), q)\,\overline{\mathrm{J}_{\pm}(m', n_1^2(n_2\mathfrak{q} + \beta), q')}.$$

On applying the Poisson summation formula to the sum over n_2, we see that

$$\Omega_{\pm} = \sum_{\tilde{N}} \frac{\tilde{N}}{n_1^2} \sum\sum_{q_2, q_2' \sim C/q_1} \sum\sum_{m, m' \sim M_1} \lambda_f(m)\,\overline{\lambda_f(m')} \sum_{n_2 \in \mathbb{Z}} \mathfrak{C}_{\pm}\mathcal{J}_{\pm}, \tag{5.8}$$

where

$$\mathfrak{C}_{\pm} = \frac{1}{\mathfrak{q}} \sum_{\beta \bmod \mathfrak{q}} \mathcal{C}_{\pm}(q, \beta, m)\,\overline{\mathcal{C}_{\pm}(q', \beta, m')}\,e\left(\frac{n_2\beta}{\mathfrak{q}}\right)$$
$$= \sum\sum_{\substack{d|q \\ d'|q'}} d d' \mu\left(\frac{q}{d}\right)\mu\left(\frac{q'}{d'}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1 \\ n_1\alpha \equiv -m \bmod d}} \sideset{}{^\star}\sum_{\substack{\alpha' \bmod q'r/n_1 \\ n_1\alpha' \equiv -m' \bmod d' \\ \pm\bar{\alpha}q_2' \mp \bar{\alpha}' q_2 \equiv -n_2 \bmod \mathfrak{q}}} 1 \tag{5.9}$$

and

$$\mathcal{J}_{\pm} = \int_{\mathbb{R}} \phi(w)\ \mathrm{J}_{\pm}(m, \tilde{N}w, q)\ \overline{\mathrm{J}_{\pm}(m', \tilde{N}w, q')}\, e\left(-\frac{n_2 \tilde{N} w}{q_1 q_2 q_2' r n_1}\right) \mathrm{d}w. \tag{5.10}$$

On estimating the sum over $\tilde{N}$, we get

$$\Omega_{\pm} \ll k^{\varepsilon} \sup_{\tilde{N} \ll N_0} \frac{\tilde{N}}{n_1^2} \sum\sum_{q_2, q_2' \sim C/q_1} \sum\sum_{m, m' \sim M_1} |\lambda_f(m)|\,|\lambda_f(m')| \sum_{n_2 \in \mathbb{Z}} |\mathfrak{C}_{\pm}|\,|\mathcal{J}_{\pm}|. \tag{5.11}$$

6. Estimates for the integral transform

In this section we will analyse the integral transform

$$\mathcal{J}_{\pm} = \int_{\mathbb{R}} \phi(w)\ \mathrm{J}_{\pm}(m, \tilde{N}w, q)\, \overline{\mathrm{J}_{\pm}(m', \tilde{N}w, q')}\, e\left(-\frac{n_2 \tilde{N} w}{q_1 q_2 q_2' r n_1}\right) \mathrm{d}w, \tag{6.1}$$

where (see (5.2))

$$\begin{aligned} \mathrm{J}_{\pm}(m, \tilde{N}w, q) &= \int_{\mathbb{R}} \int_{\mathbb{R}} W(x/Q^{\varepsilon}) g(q,x)\, \mathrm{I}_2(m,q,x)\, \mathrm{I}_3(\tilde{N}w, q, x)\, V\left(\frac{t}{T}\right) \mathrm{d}t\, \mathrm{d}x \\ &= \int_{\mathbb{R}} W(x/Q^{\varepsilon}) g(q,x) \int_{\mathbb{R}} V\left(\frac{t}{T}\right) \int_0^{\infty} U(y) y^{-it} \int_0^{\infty} V(z) z^{it} \\ &\quad \times e\left(\frac{Nx(z-y)}{qQ} \pm \frac{3(N\tilde{N}wz)^{1/3}}{qr^{1/3}}\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}t\, \mathrm{d}x. \end{aligned} \tag{6.2}$$

and $\mathrm{J}_{\pm}(m', \tilde{N}w, q')$ is similarly defined. We first analyse $\mathrm{J}_{\pm}(m, \tilde{N}w, q)$.

Lemma 14. *Let $\mathrm{J}_{\pm}(m, \tilde{N}w, q)$ be as above. Then*

$$\mathrm{J}_{\pm}(m, \tilde{N}w, q) = \int_{\mathbb{R}} V\left(\frac{t}{T}\right) \int_{u \ll \frac{k^{\varepsilon} C}{QT}} \mathrm{I}_u\, \mathrm{I}_{\pm}(m, \tilde{N}w, q)\, \mathrm{d}u\, \mathrm{d}t + O(k^{-2020}), \tag{6.3}$$

where I_u and $\mathrm{I}_{\pm}(m, \tilde{N}w, q)$ are the integrals defined in (6.6) and (6.7) respectively, with the weight function $U_{u,t}$ satisfying $U_{u,t}^{(j)}(y) \ll_j k^{\varepsilon j}$ for $j \geq 0$.

Proof. We consider two cases.

Case 1: $q \sim C \ll Q^{1-\varepsilon}$. Consider the integral over x in (6.2) which is given by

$$\begin{aligned} \mathrm{I}_{z-y} &:= \int_{\mathbb{R}} W(x/Q^{\varepsilon}) g(q,x) e\left(\frac{Nx(z-y)}{qQ}\right) \mathrm{d}x \\ &= Q^{\varepsilon} \int_{\mathbb{R}} W(x) g(q, xQ^{\varepsilon}) e\left(\frac{NxQ^{\varepsilon}(z-y)}{qQ}\right) \mathrm{d}x. \end{aligned}$$

We split the above integral as follows:

$$\int_{\mathbb{R}} \dots \,\mathrm{d}x = \int_{-Q^{-2\varepsilon}}^{Q^{-2\varepsilon}} \dots \,\mathrm{d}x + \int_{D} \dots \,\mathrm{d}x,$$

where $D = [-2,2] \setminus [-Q^{-2\varepsilon}, Q^{-2\varepsilon}]$. Note that, for $x \in [-Q^{-2\varepsilon}, Q^{-2\varepsilon}]$, we have

$$g(q, xQ^{\varepsilon}) = 1 + h(q, xQ^{\varepsilon}) = 1 + O\left(\frac{Q}{q}\left(\frac{q}{Q} + |x|Q^{\varepsilon}\right)^{B}\right) = 1 + O(Q^{-2020}).$$

Thus, in this range, we can replace $g(q, xQ^{\varepsilon})$ by 1 at the cost of a negligible error term. Then by repeated integration by parts we see that the integral is negligibly small unless

$$|z - y| \ll k^{\varepsilon} C/(QT). \tag{6.4}$$

Now we consider the complementary range, i.e., $x \in D$. Note that, using the second property (see (2.7)) of $g(q, x)$, we have

$$x^{j} \frac{\partial^{j}}{\partial x^{j}} g(q, x) \ll \log Q \min\left\{\frac{Q}{q}, \frac{1}{|x|}\right\} \ll Q^{2\varepsilon}.$$

Thus, on using integration by parts repeatedly, we see that the integral is negligibly small unless (6.4) holds true.

Case 2: $q \sim C \gg Q^{1-\varepsilon}$. In this case we consider the t-integral in (6.2), which is given by

$$\int_{\mathbb{R}} V\left(\frac{t}{T}\right)\left(\frac{z}{y}\right)^{it} \mathrm{d}t.$$

On applying the change of variable $t \to tT$ followed by integration by parts repeatedly, we conclude that the t-integral is negligibly small unless

$$|z - y| \ll k^{\varepsilon}/T \ll k^{\varepsilon} C/(QT).$$

Next writing $z - y = u$ with $u \ll k^{\varepsilon} C/(QT)$ in (6.2), we see that

$$\mathrm{J}_{\pm}(m, \tilde{N} w, q) = \int_{\mathbb{R}} V\left(\frac{t}{T}\right) \int_{u \ll \frac{k^{\varepsilon} C}{QT}} \mathrm{I}_u \, \mathrm{I}_{\pm}(m, \tilde{N} w, q) \,\mathrm{d}u \,\mathrm{d}t + O(k^{-2020}), \tag{6.5}$$

where

$$\mathrm{I}_u = \int_{\mathbb{R}} W(x/Q^{\varepsilon}) g(q, x) e\left(\frac{Nxu}{qQ}\right) \mathrm{d}x, \tag{6.6}$$

and

$$\mathrm{I}_{\pm}(m, \tilde{N} w, q) = \int_0^{\infty} U_{u,t}(y) e\left(\pm \frac{3(N\tilde{N} w(y+u))^{1/3}}{q r^{1/3}}\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \mathrm{d}y \tag{6.7}$$

with $U_{u,t}(y) = U(y)V(y+u)(1+u/y)^{it}$. Note that

$$\frac{\partial^j}{\partial y^j}\left(1+\frac{u}{y}\right)^{it} = \frac{\partial^j}{\partial y^j}\exp\left(it\log\left(1+\frac{u}{y}\right)\right) \ll_j k^{\varepsilon j}, \quad j \geq 0.$$

Thus $U_{u,t}^{(j)}(y) \ll_j k^{\varepsilon j}$ for $j \geq 0$. Hence the lemma follows. ∎

The analysis for $\mathrm{J}_{\pm}(m', \tilde{N}w, q')$ is exactly the same. Thus on plugging the expression of $\mathrm{J}_{\pm}(m, \tilde{N}w, q)$ from (6.3) and a corresponding expression of $\mathrm{J}_{\pm}(m', \tilde{N}w, q')$ into (6.1), we see that

$$\mathcal{J}_{\pm} = \int_{\mathbb{R}}\int_{\mathbb{R}} V\left(\frac{t}{T}\right)V\left(\frac{t'}{T}\right)\int_{u \ll \frac{k^{\varepsilon}C}{QT}}\int_{u' \ll \frac{k^{\varepsilon}C}{QT}} \mathrm{I}_u\,\bar{\mathrm{I}}_{u'}\,\mathfrak{J}_{\pm}\,\mathrm{d}u'\,\mathrm{d}u\,\mathrm{d}t'\,\mathrm{d}t + O(k^{-2020}), \tag{6.8}$$

where

$$\mathfrak{J}_{\pm} := \int_{\mathbb{R}} \phi(w)\,\mathrm{I}_{\pm}(m, \tilde{N}w, q)\,\overline{\mathrm{I}_{\pm}(m', \tilde{N}w, q')}\,e\left(-\frac{n_2\tilde{N}w}{q_2q_2'q_1rn_1}\right)\mathrm{d}w, \tag{6.9}$$

which we will analyse now. We have the following proposition.

Proposition 1. *Let $\mathfrak{J}_{\pm}$ be as above. Then $\mathfrak{J}_{\pm}$ is negligibly small unless*

$$n_2 \ll k^{\varepsilon}\frac{CN^{1/3}r^{2/3}n_1}{q_1\tilde{N}^{2/3}} =: N_2, \tag{6.10}$$

in which case

$$\mathfrak{J}_{\pm} \ll \frac{k^{\varepsilon}C^2}{M_1N}. \tag{6.11}$$

Furthermore, if $q \sim C \gg k^{1+\varepsilon}$ and $n_2 \neq 0$, then

$$\mathfrak{J}_{\pm} \ll \frac{Cr^{1/3}k^{2/3}}{k^2(N\tilde{N})^{1/3}}. \tag{6.12}$$

Before proving the proposition, we will analyse $\mathrm{I}_{\pm}(m, \tilde{N}w, q)$ and $\mathrm{I}_{\pm}(m', \tilde{N}w, q')$. We have the following lemma.

Lemma 15. *Let $\mathrm{I}_{\pm}(m, \tilde{N}w, q)$ be as in (6.7). Let $\mathfrak{b} = 4\pi\sqrt{mN}/q$ and $\mathfrak{a} = \mathfrak{a}(q, r) := 3(N\tilde{N})^{1/3}/(qr^{1/3}) \gg k^{\varepsilon}$. Then $\mathrm{I}_{\pm}(m, \tilde{N}w, q)$ is negligibly small unless $\mathfrak{a} \leq k^{\varepsilon}\mathfrak{b}$. In the case when $\mathfrak{a} \leq k^{-\varepsilon}\mathfrak{b}$, we have*

$$\mathrm{I}_{\pm}(m, \tilde{N}w, q) \ll k^{\varepsilon}/\mathfrak{b}.$$

Furthermore, if $q \sim C \gg k^{1+\varepsilon}$, then $\mathfrak{b} \asymp k$ and

$$\mathrm{I}_{\pm}(m, \tilde{N}w, q) = \frac{e(f(\tau_0))}{\sqrt{f''(\tau_0)}}\,\frac{c_3\mathfrak{a}^{9/2}w^{3/2}}{\mathfrak{b}^5\tau_0^5\sqrt{1-\tau_0^2}}U_{u,t}\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\tau_0}\right)^6\right)$$
$$+ \text{lower order terms} + O(k^{-2020}), \tag{6.13}$$

where τ_0 *is the stationary point of the phase function*

$$f(\tau) = \frac{(k-1)\sin^{-1}\tau}{2\pi} + \frac{16\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2\tau^2},$$

which is given by (6.24) *and* $c_3 = c_2 e(1/8) = 3\sqrt{2}\,(4\pi/3)^5 e(1/4)$. *In the remaining case, i.e.,* $k^{-\varepsilon}\mathfrak{b} \le \mathfrak{a} \le k^{\varepsilon}\mathfrak{b}$, $\mathrm{I}_{\pm}(m, \tilde{N}w, q)$ *essentially looks like*

$$\frac{c_2\mathfrak{a}^{9/2}w^{3/2}}{\mathfrak{b}^5}\int_{b_1/2}^{1}\frac{1}{\tau^5\sqrt{1-\tau^2}}U_{u,t}\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\tau}\right)^6\right)e(f(\tau))\,\mathrm{d}\tau,$$

where $b_1 := 4\pi(2/3)^{1/3}\mathfrak{a}/(3(2.5)^{1/6}\mathfrak{b})$.

Proof. Recall from (6.7) that

$$\mathrm{I}_{\pm}(m, \tilde{N}w, q) = \int_{1/2}^{5/2} U_{u,t}(y)\, e(\pm\mathfrak{a}w^{1/3}(y+u)^{1/3}) J_{k-1}(\mathfrak{b}\sqrt{y})\,\mathrm{d}y. \tag{6.14}$$

Consider the term $e(\pm\mathfrak{a}w^{1/3}(y+u)^{1/3})$. It can be written as

$$e(\pm\mathfrak{a}w^{1/3}(y+u)^{1/3}) = e(\pm\mathfrak{a}w^{1/3}y^{1/3})e\big(\pm\mathfrak{a}w^{1/3}y^{1/3}((1+u/y)^{1/3}-1)\big).$$

Note that

$$\frac{\partial^j}{\partial y^j}e\big(\pm\mathfrak{a}w^{1/3}y^{1/3}((1+u/y)^{1/3}-1)\big) \ll_j k^{\varepsilon j}, \quad j \ge 0.$$

This is obvious for $j = 0$. We will verify it for $j = 1$ (for other j, a similar calculation will follow). Let $h(y, w) := \pm\mathfrak{a}w^{1/3}y^{1/3}((1+u/y)^{1/3}-1)$. Thus for $j = 1$ we have

$$\frac{\partial}{\partial y}e(h(y,w)) = e(h(y,w))(\pm\mathfrak{a})w^{1/3}\left(\frac{(1+u/y)^{1/3}-1}{3y^{2/3}} - \frac{u}{3y^{5/3}(1+u/y)^{2/3}}\right).$$

Thus, using $y, w \asymp 1$ and $(1+u/y)^{1/3} - 1 \ll |u|$, we see that

$$\frac{\partial}{\partial y}e(h(y,w)) \ll \mathfrak{a}|u| \ll \frac{(N\tilde{N})^{1/3}}{Cr^{1/3}}\frac{Ck^{\varepsilon}}{QT} \ll \frac{(NN_0)^{1/3}}{Qr^{1/3}}\frac{Qk^{\varepsilon}}{QT} \ll k^{\varepsilon},$$

where we have used (5.7) to estimate N_0. Hence we can insert $e(h(y, w))$ into the weight function $U_{u,t}(y)$. Thus we arrive at the following expression:

$$\mathrm{I}_{\pm} := \mathrm{I}_{\pm}(m, \tilde{N}w, q) = \int_{1/2}^{5/2} U_{u,t}(y)\, e(\pm\mathfrak{a}w^{1/3}y^{1/3}) J_{k-1}(\mathfrak{b}\sqrt{y})\,\mathrm{d}y. \tag{6.15}$$

Notice the slight abuse of notation: the weight function $U_{u,t}$ in the above expression is different from the one in (6.14). To analyse (6.15) further, we use an integral representation of the Bessel function J_{k-1}. On applying (2.8) to the Bessel function J_{k-1} we see that

$$\mathrm{I}_{\pm} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-1)\tau}\int_{1/2}^{5/2} U_{u,t}(y)e\big(\pm\mathfrak{a}w^{1/3}y^{1/3} - \mathfrak{b}\sqrt{y}(\sin\tau)/(2\pi)\big)\,\mathrm{d}y\,\mathrm{d}\tau.$$

We now split the τ-integral as follows:

$$\int_{-\pi}^{\pi} \ldots \, \mathrm{d}\tau = \int_{0}^{\pi/2} \ldots \, \mathrm{d}\tau + \int_{\pi/2}^{\pi} \ldots \, \mathrm{d}\tau + \int_{-\pi/2}^{0} \ldots \, \mathrm{d}\tau + \int_{-\pi}^{-\pi/2} \ldots \, \mathrm{d}\tau.$$

Let $\mathrm{I}_{\pm}^{(i)}$ denote the i-th integral on the right hand side of the above expression for $i = 1, 2, 3$ and 4. Let us first consider $\mathrm{I}_{\pm}^{(1)}$ which is defined as follows:

$$\mathrm{I}_{\pm}^{(1)} = \frac{1}{2\pi} \int_{0}^{\pi/2} e^{i(k-1)\tau} \int_{1/2}^{5/2} U_{u,t}(y) e\big(\pm \mathfrak{a} w^{1/3} y^{1/3} - \mathfrak{b}\sqrt{y}(\sin\tau)/(2\pi)\big) \, \mathrm{d}y \, \mathrm{d}\tau. \tag{6.16}$$

Next we apply stationary phase analysis to the y-integral. By the change of variable $y \to y^3$, we arrive at the following expression of the y-integral:

$$\int_{\sqrt[3]{1/2}}^{\sqrt[3]{5/2}} 3y^2 U_{u,t}(y^3) e\big(\pm \mathfrak{a} w^{1/3} y - \mathfrak{b} y^{3/2}(\sin\tau)/(2\pi)\big) \, \mathrm{d}y.$$

Note that if we have $\mathfrak{a}$ with the minus sign, then the above integral is negligibly small by Lemma 9. Thus we proceed with the y-integral of $\mathrm{I}_{+}^{(1)}$, which is given by

$$\int_{\sqrt[3]{1/2}}^{\sqrt[3]{5/2}} 3y^2 U_{u,t}(y^3) e(\mathfrak{a} w^{1/3} y - \mathfrak{b} y^{3/2}(\sin\tau)/(2\pi)) \, \mathrm{d}y.$$

Here the phase function is given by $f_1(y) = \mathfrak{a} w^{1/3} y - \mathfrak{b} y^{3/2}(\sin\tau)/(2\pi)$. On computing the first order derivative, we see that the stationary point occurs at $y_0 = \left(\frac{4\pi\mathfrak{a} w^{1/3}}{3\mathfrak{b}\sin\tau}\right)^2$. Note that

$$\sqrt[3]{1/2} \leq y_0 \leq \sqrt[3]{5/2}, \quad \text{i.e.,} \quad \frac{4\pi}{3} \frac{\mathfrak{a} w^{1/3}}{\mathfrak{b}(2.5)^{1/6}} \leq \sin\tau \leq \frac{4\pi}{3} \frac{\mathfrak{a} w^{1/3}}{\mathfrak{b}(0.5)^{1/6}}.$$

Let $b_1 := \frac{4\pi}{3} \frac{\mathfrak{a}(2/3)^{1/3}}{\mathfrak{b}(2.5)^{1/6}}$ and $b_2 := \frac{4\pi}{3} \frac{3^{1/3}\mathfrak{a}}{\mathfrak{b}(0.5)^{1/6}}$. We consider three cases.

Case 1: $\mathfrak{a} \geq k^{\varepsilon}\mathfrak{b}$. In this case $b_1 \geq 2$. Thus there is no stationary point in the range $[(1/2)^{1/3}, (5/2)^{1/3}]$. Moreover,

$$f_1'(y) = \mathfrak{a} w^{1/3} - 3\mathfrak{b}\sqrt{y}(\sin\tau)/(4\pi) \gg \mathfrak{b}, \quad f_1^{(j)}(y) \ll \mathfrak{b}, \ j \geq 2.$$

Hence, by Lemma 9, the integral is negligibly small. This proves the first part of the lemma.

Case 2: $\mathfrak{a} \leq k^{-\varepsilon}\mathfrak{b}$. In this case $0 < b_1/2 < 2b_2 \ll k^{-\varepsilon} < 1$. We now split the τ-integral in (6.16) as follows:

$$\int_{0}^{\pi/2} \ldots \, \mathrm{d}\tau = \int_{0}^{\sin^{-1}(b_1/2)} \ldots \, \mathrm{d}\tau + \int_{\sin^{-1}(b_1/2)}^{\sin^{-1}(2b_2)} \ldots \, \mathrm{d}\tau + \int_{\sin^{-1}(2b_2)}^{\pi/2} \ldots \, \mathrm{d}\tau.$$

Note that the first and the third integrals of the right side of the above expression are negligibly small due to absence of the stationary point. Hence it boils down to analyse the second integral which is given by

$$\int_{\sin^{-1}(b_1/2)}^{\sin^{-1} 2b_2} e^{i(k-1)\tau} \int_{\sqrt[3]{1/2}}^{\sqrt[3]{5/2}} 3y^2 U_{u,t}(y^3) e\big(\mathfrak{a} w^{1/3} y - \mathfrak{b} y^{3/2} (\sin\tau)/(2\pi)\big)\, \mathrm{d}y\, \mathrm{d}\tau. \quad (6.17)$$

On applying the stationary phase analysis (see Lemma 10) to the y-integral, we see that it is given by

$$\frac{c_1 y_0^2 U_{u,t}(y_0^3) e(f_1(y_0))}{\sqrt{|f_1''(y_0)|}} + \text{lower order terms} + O(k^{-2020}),$$

where $c_1 = 3e(1/8)$, $y_0 = (\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\sin\tau})^2$ and $f_1(y) = \mathfrak{a}w^{1/3}y - \mathfrak{b}y^{3/2}(\sin\tau)/(2\pi)$. We will proceed with the main term, as the other terms can be analysed similarly, and in fact give better bounds. Hence, on plugging in the values of y_0, $f_1(y_0)$ and $f_1''(y_0)$, we essentially get the following expression for the y-integral:

$$\frac{c_2 \mathfrak{a}^{9/2} w^{3/2}}{\mathfrak{b}^5 \sin^5\tau} U_{u,t}\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\sin\tau}\right)^6\right) e\left(\frac{16\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2\sin^2\tau}\right), \quad (6.18)$$

where $c_2 = c_1\sqrt{2}\,(4\pi/3)^5$. On plugging the above expression in place of the y-integral into (6.17), we arrive at

$$\frac{c_2 \mathfrak{a}^{9/2} w^{3/2}}{\mathfrak{b}^5} \int_{\sin^{-1}(b_1/2)}^{\sin^{-1} 2b_2} \frac{1}{\sin^5\tau} U_{u,t}\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\sin\tau}\right)^6\right) e\left(\frac{(k-1)\tau}{2\pi} + \frac{16\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2\sin^2\tau}\right) \mathrm{d}\tau.$$

On applying the change of variable $\sin\tau \to \tau$, we arrive at

$$\frac{c_2 \mathfrak{a}^{9/2} w^{3/2}}{\mathfrak{b}^5} \int_{b_1/2}^{2b_2} \frac{1}{\tau^5\sqrt{1-\tau^2}} U_{u,t}\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\tau}\right)^6\right) e\left(\frac{(k-1)\sin^{-1}\tau}{2\pi} + \frac{16\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2\tau^2}\right) \mathrm{d}\tau. \quad (6.19)$$

Next we apply the second derivative bound to the above integral. Here the phase function is given by

$$f(\tau) = \frac{(k-1)\sin^{-1}\tau}{2\pi} + \frac{16\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2\tau^2}.$$

Computing the first and the second order derivatives, we see that

$$f'(\tau) = \frac{k-1}{2\pi\sqrt{1-\tau^2}} - \frac{32\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2\tau^3},$$

$$f''(\tau) = \frac{(k-1)\tau}{2\pi(1-\tau^2)^{3/2}} + \frac{32\pi^2\mathfrak{a}^3 w}{9\mathfrak{b}^2\tau^4} \gg \frac{\mathfrak{a}^3}{\mathfrak{b}^2\tau^4} \gg \frac{\mathfrak{b}^2}{\mathfrak{a}}. \quad (6.20)$$

Thus on applying Lemma 9 to (6.19), we see that it is bounded above by

$$\frac{\operatorname{Var} g + \max|g|}{\min\sqrt{f''(\tau)}} \ll \frac{k^\varepsilon \mathfrak{a}^{9/2}}{\mathfrak{b}^5(\mathfrak{a}/\mathfrak{b})^5\sqrt{\mathfrak{b}^2/\mathfrak{a}}} = \frac{k^\varepsilon}{\mathfrak{b}},$$

where Var g denotes the total variation of the weight function

$$g(\tau) = \frac{c_2 \mathfrak{a}^{9/2} w^{3/2} U_{u,t}\left((4\pi \mathfrak{a} w^{1/3}/(3\mathfrak{b}\tau))^6\right)}{\mathfrak{b}^5 \tau^5 \sqrt{1-\tau^2}}.$$

Hence, $\mathrm{I}_{\pm}^{(1)} \ll k^{\varepsilon}/\mathfrak{b}$. Analysing other $\mathrm{I}_{\pm}^{(i)}$'s in a similar fashion, we get

$$\mathrm{I}_{\pm} = \mathrm{I}_{\pm}(m, N_0 w, q) \ll k^{\varepsilon}/\mathfrak{b}.$$

Now we proceed to prove (6.13). We will give the details for $\mathrm{I}_{\pm}^{(1)}$ only, as the analysis for other $\mathrm{I}_{\pm}^{(i)}$ is similar. Let $q \sim C \gg k^{1+\varepsilon}$. Note that this condition ensures that $\mathfrak{b} \asymp k$, because, by (5.7), we have

$$k^{-\varepsilon}(k-1)^2 C^2/N \ll M_1 \ll k^{\varepsilon} \max\{(k-1)^2 C^2/N, T\} \ll k^{\varepsilon}(k-1)^2 C^2/N, \tag{6.21}$$

and hence

$$\mathfrak{a} = \frac{3(N\tilde{N})^{1/3}}{q r^{1/3}} \ll \frac{(N N_0)^{1/3}}{q r^{1/3}} \ll (kT)^{1/2} = k^{1-\eta/2} < k \asymp \mathfrak{b}, \tag{6.22}$$

since $T = k^{1-\eta} < k$. We now apply the stationary phase analysis to (6.19). The stationary point of the phase function $f(\tau)$ occurs at τ_0 where τ_0 satisfies

$$\frac{k-1}{2\pi\sqrt{1-\tau_0^2}} = \frac{32\pi^2 \mathfrak{a}^3 w}{27\mathfrak{b}^2 \tau_0^3}, \quad \text{i.e.,} \quad \frac{\tau_0^3}{\sqrt{1-\tau_0^2}} = \left(\frac{4\pi}{3}\right)^3 \frac{\mathfrak{a}^3 w}{\mathfrak{b}^2(k-1)}.$$

Simplifying it further, we see that τ_0 satisfies

$$\tau^6 - \mathfrak{c}^2(1-\tau^2) = 0,$$

where $\mathfrak{c} = \mathfrak{c}(w) := (\frac{4\pi}{3})^3 \frac{\mathfrak{a}^3 w}{\mathfrak{b}^2(k-1)}$. Upon letting $\tau^2 = \tau_1$, the above equation reduces to the cubic polynomial equation $\tau_1^3 - \mathfrak{c}^2(1-\tau_1) = 0$, which can be solved using Cardano's method. In fact, as the discriminant of the cubic is negative, it has only one real root which can be found as follows: Let $\theta_1 + \theta_2$ be the real root. Upon substituting it into the cubic, we get

$$\theta_1^3 + \theta_2^3 + (3\theta_1\theta_2 + \mathfrak{c}^2)(\theta_1 + \theta_2) - \mathfrak{c}^2 = 0,$$

which leads to the following system of equations:

$$3\theta_1\theta_2 + \mathfrak{c}^2 = 0, \quad \theta_1^3 + \theta_2^3 - \mathfrak{c}^2 = 0.$$

Now using the formula

$$(\theta_1^3 - \theta_2^3)^2 = (\theta_1^3 + \theta_2^3)^2 - 4\theta_1^3\theta_2^3,$$

we see that the real root $\theta_1 + \theta_2$ is given by

$$\sqrt[3]{\frac{\mathfrak{c}^2}{2} + \sqrt{\frac{\mathfrak{c}^4}{4} + \frac{\mathfrak{c}^6}{27}}} + \sqrt[3]{\frac{\mathfrak{c}^2}{2} - \sqrt{\frac{\mathfrak{c}^4}{4} + \frac{\mathfrak{c}^6}{27}}}.$$

Hence we get

$$\tau_0 = \tau_0(w) = \left(\sqrt[3]{\frac{\mathfrak{c}^2}{2} + \sqrt{\frac{\mathfrak{c}^4}{4} + \frac{\mathfrak{c}^6}{27}}} + \sqrt[3]{\frac{\mathfrak{c}^2}{2} - \sqrt{\frac{\mathfrak{c}^4}{4} + \frac{\mathfrak{c}^6}{27}}}\right)^{1/2}$$
$$= \sqrt[6]{\frac{\mathfrak{c}^2}{2} + \sqrt{\frac{\mathfrak{c}^4}{4} + \frac{\mathfrak{c}^6}{27}}}\left(1 - \frac{3}{\mathfrak{c}^2}\left(\sqrt{\frac{\mathfrak{c}^4}{4} + \frac{\mathfrak{c}^6}{27}} - \frac{\mathfrak{c}^2}{2}\right)^{2/3}\right)^{1/2}. \tag{6.23}$$

Now expanding the above expression using the binomial theorem, we see that

$$\tau_0 = \tau_0(w) = c_1\mathfrak{h}(w) + c_3(\mathfrak{h}(w))^3 + c_3(\mathfrak{h}(w))^5 + \cdots + c_{2n-1}(\mathfrak{h}(w))^{2n-1} + \cdots, \tag{6.24}$$

where c_i's, $i = 1, 3, 5, \ldots$, are some non-zero explicit absolute constants and

$$\mathfrak{h}(w) = \frac{\mathfrak{a}w^{1/3}}{\mathfrak{b}^{2/3}(k-1)^{1/3}}.$$

Note that the series in (6.24) is convergent and each binomial expansion in (6.23) is justified as $\mathfrak{c} \ll \mathfrak{a}^3/(\mathfrak{b}^2(k-1)) \ll k^{-3\eta/2}$. Next we analyse the higher order derivatives of the phase function $f(\tau)$. Using (6.20) and computing other higher order derivatives of $f(\tau)$, we get

$$f''(\tau) \asymp \mathfrak{b}^2/\mathfrak{a} = \mathfrak{a}(\mathfrak{a}/\mathfrak{b})^{-2}, \quad f'(\tau) \ll \mathfrak{a}(\mathfrak{a}/\mathfrak{b})^{-1},$$
$$f^{(j)}(\tau) = \frac{k-1}{2\pi}\frac{\mathrm{d}^{j-2}}{\mathrm{d}\tau^{j-2}}\frac{\tau}{(1-\tau^2)^{3/2}} + \frac{32\pi^2\mathfrak{a}^3 w}{9\mathfrak{b}^2}\frac{\mathrm{d}^{j-2}(\tau^{-4})}{\mathrm{d}\tau^{j-2}}$$
$$\ll \mathfrak{a}(\mathfrak{a}/\mathfrak{b})^{-j}, \quad j = 3, 4, \ldots,$$

where we have used the fact that $\mathfrak{a} \ll \mathfrak{b} \asymp k$ and

$$\frac{\mathrm{d}^{j-2}}{\mathrm{d}\tau^{j-2}}\frac{\tau}{(1-\tau^2)^{3/2}} \ll_j 1.$$

On computing derivatives of the weight function

$$g(\tau) = \frac{c_2\mathfrak{a}^{9/2}w^{3/2}U_{u,t}\big((4\pi\mathfrak{a}w^{1/3}/(3\mathfrak{b}\tau))^6\big)}{\mathfrak{b}^5\tau^5\sqrt{1-\tau^2}},$$

since $\tau \asymp \mathfrak{a}/\mathfrak{b}$, we see that

$$g^{(i)}(\tau) \ll \mathfrak{a}^{-1/2}(\mathfrak{a}/\mathfrak{b})^{-i}, \quad i = 0, 1, 2, \ldots.$$

Thus, on applying Lemma 10 with $X = \mathfrak{a}^{-1/2}$, $Q = U = \mathfrak{a}/\mathfrak{b}$ and $Y = \mathfrak{a}$ to the τ-integral in (6.19), we get (6.13).

Case 3: $k^{-\varepsilon}\mathfrak{b} \le \mathfrak{a} \le k^{\varepsilon}\mathfrak{b}$. In this case we can assume that $b_1/2 < 1$, otherwise, we get back to the starting point of the discussion in Case 1. Consider

$$\mathrm{I}_{\pm}^{(1)} = \frac{1}{2\pi}\int_0^{\pi/2} e^{i(k-1)\tau}\int_{1/2}^{5/2} U_{u,t}(y)e\big(\pm\mathfrak{a}w^{1/3}y^{1/3} - \mathfrak{b}\sqrt{y}(\sin\tau)/(2\pi)\big)\,\mathrm{d}y\,\mathrm{d}\tau. \tag{6.25}$$

We split the τ-integral as follows:

$$\int_0^{\pi/2} \ldots \, \mathrm{d}\tau = \int_0^{\sin^{-1}(b_1/2)} \ldots \, \mathrm{d}\tau + \int_{\sin^{-1}(b_1/2)}^{\pi/2} \ldots \, \mathrm{d}\tau.$$

The first integral on the right side is negligibly small due to absence of the stationary point. Consider the second integral, which is given by

$$\int_{\sin^{-1}(b_1/2)}^{\pi/2} e^{i(k-1)\tau} \int_{\sqrt[3]{1/2}}^{\sqrt[3]{5/2}} 3y^2 U_{u,t}(y^3) e\big(\mathfrak{a} w^{1/3} y - \mathfrak{b} y^{3/2} (\sin\tau)/(2\pi)\big) \, \mathrm{d}y \, \mathrm{d}\tau. \tag{6.26}$$

On analysing the y-integral as in Case 2, we get the lemma. ■

Proof of Proposition 1. Recall from (6.15) that

$$\mathrm{I}_{\pm}(m, \tilde{N} w, q) = \int_{1/2}^{5/2} U_{u,t}(y) \, e(\pm \mathfrak{a} w^{1/3} y^{1/3}) J_{k-1}(\mathfrak{b}\sqrt{y}) \, \mathrm{d}y.$$

Note that

$$\frac{\partial^j}{\partial w^j} \mathrm{I}_{\pm}(m, \tilde{N} w, q) \ll \mathfrak{a}^j, \quad j \geq 0.$$

Similarly it follows that

$$\frac{\partial^j}{\partial w^j} \mathrm{I}_{\pm}(m', \tilde{N} w, q') \ll \mathfrak{a}'^j, \quad j \geq 0.$$

Hence, on applying integration by parts j-times to the w-integral in (6.9), we see that

$$\begin{aligned}
\mathfrak{J}_{\pm} &\ll (k^{\varepsilon} + \mathfrak{a} + \mathfrak{a}')^j \left(\frac{q_2 q_2' q_1 r n_1}{n_2 \tilde{N}}\right)^j \\
&\ll \left(\frac{(N\tilde{N})^{1/3}}{C r^{1/3}}\right)^j \left(\frac{C^2 r n_1}{q_1 n_2 \tilde{N}}\right)^j = \left(\frac{N^{1/3} C r^{2/3} n_1}{q_1 n_2 \tilde{N}^{2/3}}\right)^j.
\end{aligned}$$

Thus $\mathfrak{J}_{\pm}$ is negligibly small if

$$\frac{N^{1/3} C r^{2/3} n_1}{q_1 n_2 \tilde{N}^{2/3}} \ll \frac{1}{k^{\varepsilon}}, \quad \text{i.e.,} \quad n_2 \gg k^{\varepsilon} \frac{C N^{1/3} r^{2/3} n_1}{q_1 \tilde{N}^{2/3}}.$$

Next we prove

$$\mathfrak{J}_{\pm} \ll k^{\varepsilon} C^2/(M_1 N).$$

Case 1: $\mathfrak{a} \not\asymp \mathfrak{b}$, i.e., $\mathfrak{a}' \asymp \mathfrak{a} \ll k^{-\varepsilon}\mathfrak{b} \asymp k^{-\varepsilon}\mathfrak{b}'$ or $\mathfrak{a}' \asymp \mathfrak{a} \gg k^{\varepsilon}\mathfrak{b} \asymp k^{\varepsilon}\mathfrak{b}'$. When $\mathfrak{a} \gg k^{\varepsilon}\mathfrak{b}$, on applying Lemma 15 to $\mathrm{I}_{\pm}(m, \tilde{N} w, q)$, we see that $\mathfrak{J}_{\pm}$ is negligibly small. In the other case, i.e., $\mathfrak{a}' \asymp \mathfrak{a} \ll k^{-\varepsilon}\mathfrak{b} \asymp k^{-\varepsilon}\mathfrak{b}'$, on applying Lemma 15 to (6.9), we get

$$\mathfrak{J}_{\pm} \ll \int_{\mathbb{R}} \phi(w) |\mathrm{I}_{\pm}(m, \tilde{N} w, q)| \, |\overline{\mathrm{I}_{\pm}(m', \tilde{N} w, q')}| \, \mathrm{d}w \ll \frac{k^{\varepsilon}}{\mathfrak{b}\mathfrak{b}'} \ll \frac{k^{\varepsilon} C^2}{M_1 N}. \tag{6.27}$$

Case 2: $\mathfrak{a} \asymp \mathfrak{b}$, i.e., $k^{-\varepsilon}\mathfrak{b} \le \mathfrak{a} \le k^{\varepsilon}\mathfrak{b}$. On applying the last part of Lemma 15 to (6.9), we see that

$$\mathfrak{J}_{\pm} \ll \frac{(\mathfrak{a}\mathfrak{a}')^{9/2}}{(\mathfrak{b}\mathfrak{b}')^5} \int_{b_1/2}^{1}\int_{b_1'/2}^{1} \frac{1}{\tau^5\sqrt{1-\tau^2}}\,\frac{1}{\tau'^5\sqrt{1-\tau'^2}} \times \left|\int_{2/3}^{3} g_3(\tau,\tau',w)e(wf_3(\tau,\tau'))\,\mathrm{d}w\right| \mathrm{d}\tau\,\mathrm{d}\tau',$$

where

$$f_3(\tau,\tau') = \frac{16\pi^2\mathfrak{a}^3}{27\mathfrak{b}^2\tau^2} - \frac{16\pi^2\mathfrak{a}'^3}{27\mathfrak{b}'^2\tau'^2} - \frac{n_2\tilde{N}}{q_2q_2'q_1rn_1},$$
$$g_3(\tau,\tau',w) = \phi(w)w^3U_{u,t}\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\tau}\right)^6\right)\bar{U}_{u',t'}\left(\left(\frac{4\pi\mathfrak{a}'w^{1/3}}{3\mathfrak{b}'\tau'}\right)^6\right).$$

On applying the change of variable $\tau \to 1/\sqrt{\tau}$, $\tau' \to 1/\sqrt{\tau'}$, we arrive at

$$\mathfrak{J}_{\pm} \ll \frac{(\mathfrak{a}\mathfrak{a}')^{9/2}}{(\mathfrak{b}\mathfrak{b}')^5} \int_{1}^{4/b_1^2}\int_{1}^{4/b_1'^2} \frac{\tau^{3/2}}{2\sqrt{\tau-1}}\,\frac{\tau'^{3/2}}{2\sqrt{\tau'-1}} \times \left|\int_{2/3}^{3} g_3(1/\sqrt{\tau},1/\sqrt{\tau'},w)e\left(\frac{16\pi^2\mathfrak{a}^3w}{27\mathfrak{b}^2}f_4(\tau,\tau')\right)\mathrm{d}w\right| \mathrm{d}\tau\,\mathrm{d}\tau', \tag{6.28}$$

where

$$f_4(\tau,\tau') = \tau - \frac{\mathfrak{a}'^3\mathfrak{b}^2}{\mathfrak{a}^3\mathfrak{b}'^2}\tau' - \frac{27n_2\tilde{N}\mathfrak{b}^2}{16\pi^2q_2q_2'q_1rn_1\mathfrak{a}^3}.$$

Now using the change of variable

$$\frac{\mathfrak{a}'^3\mathfrak{b}^2}{\mathfrak{a}^3\mathfrak{b}'^2}\tau' + \frac{27n_2\tilde{N}\mathfrak{b}^2}{16\pi^2q_2q_2'q_1rn_1\mathfrak{a}^3} \to \tau',$$

we arrive at the w-integral

$$\int_{2/3}^{3} g_3(\ldots,w)e\left(w\frac{16\pi^2\mathfrak{a}^3}{27\mathfrak{b}^2}(\tau-\tau')\right)\mathrm{d}w,$$

where $g_3(\ldots,w)$ is given by

$$\phi(w)w^3U_{u,t}\left(\frac{(4\pi\mathfrak{a})^6\tau^3w^2}{(3\mathfrak{b})^6}\right)\bar{U}_{u',t'}\left(\frac{(4\pi\mathfrak{a}')^6w^2}{(3\mathfrak{b}')^6}\left(\frac{\mathfrak{a}^3\mathfrak{b}'^2}{\mathfrak{a}'^3\mathfrak{b}^2}\tau' - \frac{27n_2\tilde{N}\mathfrak{b}'^2}{16\pi^2q_2q_2'q_1rn_1\mathfrak{a}'^3}\right)^3\right).$$

Note that

$$\frac{\partial^j}{\partial w^j}g_3(\ldots,w) \ll_j k^{\varepsilon j}, \quad j \ge 0, \tag{6.29}$$

as $\mathfrak{a} \asymp \mathfrak{b}$ and

$$\frac{\mathfrak{a}^3\mathfrak{b}'^2}{\mathfrak{a}'^3\mathfrak{b}^2}\tau' - \frac{27n_2\tilde{N}\mathfrak{b}'^2}{16\pi^2q_2q_2'q_1rn_1\mathfrak{a}'^3} \ll k^{\varepsilon} + \frac{(\mathfrak{a}+\mathfrak{a}')\mathfrak{b}'^2}{\mathfrak{a}'^3} \ll k^{\varepsilon},$$

where, in the first inequality, we have used $\frac{n_2\tilde{N}}{q_2q_2'q_1rn_1} \ll \mathfrak{a}+\mathfrak{a}'$, which follows by applying integration by parts to the w-integral in (6.9). On applying integration by parts repeatedly, we see that the above integral is negligibly small unless

$$|\tau-\tau'| \ll k^{\varepsilon}\mathfrak{b}^2/\mathfrak{a}^3.$$

Now writing $\tau-\tau'=\tau_2$ with $\tau_2 \ll k^{\varepsilon}\mathfrak{b}^2/\mathfrak{a}^3$, and estimating all the integrals in (6.28) trivially, we get

$$\mathfrak{J}_{\pm} \ll \frac{(\mathfrak{a}\mathfrak{a}')^{9/2}}{(\mathfrak{b}\mathfrak{b}')^5}\frac{k^{\varepsilon}\mathfrak{b}^2}{\mathfrak{a}^3} \ll \frac{1}{(\mathfrak{b}\mathfrak{b}')^{1/2}}\frac{k^{\varepsilon}}{\mathfrak{b}} \ll \frac{k^{\varepsilon}C^2}{M_1N},$$

where we have used the fact $\mathfrak{a}' \asymp \mathfrak{a} \asymp \mathfrak{b} \asymp \mathfrak{b}'$. Hence we get (6.11).

Now we proceed to prove the last part. Let $q \sim C \gg k^{1+\varepsilon}$. We also have $q' \sim C \gg k^{1+\varepsilon}$. Note that in this situation we have $\mathfrak{a} \ll k^{-\varepsilon}\mathfrak{b}$, $\mathfrak{a}' \ll k^{-\varepsilon}\mathfrak{b}'$ and $\mathfrak{b} \asymp \mathfrak{b}' \asymp k$ (see (6.21) and (6.22)). On substituting the main term of $\mathrm{I}_{\pm}(m,\tilde{N}w,q)$ from (6.13) and a similar expression for $\mathrm{I}_{\pm}(m',\tilde{N}w,q')$ into (6.9), we arrive at the following expression:

$$\frac{c_3^2(\mathfrak{a}\mathfrak{a}')^{9/2}}{(\mathfrak{b}\mathfrak{b}')^5}\int_{\mathbb{R}} \phi_1(w)e(f_5(w))\,\mathrm{d}w, \tag{6.30}$$

where

$$\phi_1(w) = \frac{1}{\sqrt{f''(\tau_0)}}\frac{1}{\tau_0^5\sqrt{1-\tau_0^2}}\frac{1}{\sqrt{f_2''(\tau_0')}}\frac{1}{\tau_0'^5\sqrt{1-\tau_0'^2}} \times U_{u,t}\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\tau_0}\right)^6\right)\bar{U}_{u',t'}\left(\left(\frac{4\pi\mathfrak{a}'w^{1/3}}{3\mathfrak{b}'\tau_0'}\right)^6\right) \tag{6.31}$$

and

$$f_5(w) = \frac{(k-1)(\sin^{-1}\tau_0-\sin^{-1}\tau_0')}{2\pi}+\frac{16\pi^2}{27}\left(\frac{\mathfrak{a}^3w}{\mathfrak{b}^2\tau_0^2}-\frac{\mathfrak{a}'^3w}{\mathfrak{b}'^2\tau_0'^2}\right)-\frac{\tilde{N}n_2w}{q_2q_2'q_1rn_1},$$

to which we apply the third derivative bound. Recall from (6.24) that

$$\tau_0=\tau_0(w)=c_1\mathfrak{h}(w)+c_3(\mathfrak{h}(w))^3+c_3(\mathfrak{h}(w))^5+\cdots+c_{2n-1}(\mathfrak{h}(w))^{2n-1}+\cdots, \tag{6.32}$$

with

$$\mathfrak{h}(w)=\frac{\mathfrak{a}w^{1/3}}{\mathfrak{b}^{2/3}(k-1)^{1/3}},\quad \mathfrak{b}=\frac{4\pi\sqrt{mN}}{q},\quad \mathfrak{a}=\frac{3(N\tilde{N})^{1/3}}{qr^{1/3}},$$

and τ_0' is similarly defined. On applying the change of variable $w\to w^3$ in (6.30), we see that the phase function is given by

$$\frac{(k-1)(\sin^{-1}\tau_0(w^3)-\sin^{-1}\tau_0'(w^3))}{2\pi}+\frac{16\pi^2}{27}\left(\frac{\mathfrak{a}^3w^3}{\mathfrak{b}^2\tau_0^2(w^3)}-\frac{\mathfrak{a}'^3w^3}{\mathfrak{b}'^2\tau_0'^2(w^3)}\right)-\frac{\tilde{N}n_2w^3}{q_2q_2'q_1rn_1}.$$

On applying the Taylor series expansion of $\sin^{-1}\tau_0(w^3)$, we see that

$$\begin{aligned}\sin^{-1}\tau_0(w^3) &= \tau_0(w^3) + (\tau_0(w^3))^3/6 + \cdots \\ &= d_1\mathfrak{h}(w^3) + d_3(\mathfrak{h}(w^3))^3 + \cdots \\ &= d_1\frac{\mathfrak{a}w}{\mathfrak{b}^{2/3}(k-1)^{1/3}} + d_3\frac{\mathfrak{a}^3w^3}{\mathfrak{b}^2(k-1)} + \cdots,\end{aligned}$$

where $d_1, d_3, \ldots$ are some absolute constants. Thus

$$\frac{\partial^3}{\partial w^3}\sin^{-1}\tau_0(w^3) \ll \frac{\mathfrak{a}^3}{\mathfrak{b}^2(k-1)}.$$

Similarly

$$\frac{\partial^3}{\partial w^3}\sin^{-1}\tau_0'(w^3) \ll \frac{\mathfrak{a}'^3}{\mathfrak{b}'^2(k-1)}.$$

Next we consider $\mathfrak{a}^3w^3/(\mathfrak{b}^2\tau_0^2(w^3))$. On applying the Taylor series expansion, we get

$$\begin{aligned}\frac{\mathfrak{a}^3w^3}{\mathfrak{b}^2\tau_0^2(w^3)} = \frac{(k-1)(\mathfrak{h}(w^3))^3}{\tau_0^2(w^3)} &= \frac{(k-1)\mathfrak{h}(w^3)}{c_1^2}\left(1 + \frac{c_3(\mathfrak{h}(w^3))^3}{c_1\mathfrak{h}(w^3)} + \cdots\right)^{-2} \\ &= \frac{(k-1)\mathfrak{h}(w^3)}{c_1^2}\left(1 - \frac{2c_3(\mathfrak{h}(w^3))^3}{c_1\mathfrak{h}(w^3)} - \cdots\right) \\ &= \frac{k-1}{c_1^2}\left(\mathfrak{h}(w^3) - \frac{2c_3(\mathfrak{h}(w^3))^3}{c_1} - \cdots\right).\end{aligned}$$

Thus

$$\frac{\partial^3}{\partial w^3}\frac{\mathfrak{a}^3w^3}{\mathfrak{b}^2\tau_0^2(w^3)} \ll \frac{\mathfrak{a}^3}{\mathfrak{b}^2}.$$

A similar analysis also gives us

$$\frac{\partial^3}{\partial w^3}\frac{\mathfrak{a}'^3w^3}{\mathfrak{b}'^2\tau_0'^2(w^3)} \ll \frac{\mathfrak{a}'^3}{\mathfrak{b}'^2}. \tag{6.33}$$

Hence, upon combining the above estimates, we conclude that

$$\frac{\partial^3 f_5(w^3)}{\partial w^3} = O\left(\frac{\mathfrak{a}^3}{\mathfrak{b}^2} + \frac{\mathfrak{a}'^3}{\mathfrak{b}'^2}\right) - \frac{6\tilde{N}n_2}{q_2q_2'q_1rn_1}.$$

Since $n_2 \neq 0$, we note that

$$\frac{\mathfrak{a}^3}{\mathfrak{b}^2} + \frac{\mathfrak{a}'^3}{\mathfrak{b}'^2} \ll \frac{N\tilde{N}}{C^3rk^2} \ll \frac{(k^3/r^2)\tilde{N}}{C^2rk^{3+\varepsilon}} \ll \frac{\tilde{N}}{k^\varepsilon C^2r(n_1,r)} \ll \frac{\tilde{N}}{k^\varepsilon(C^2/q_1)rn_1} \ll \frac{k^{-\varepsilon}6\tilde{N}|n_2|}{q_2q_2'q_1rn_1}.$$

In the first inequality, we have used the fact that $\mathfrak{a} \asymp \mathfrak{a}'$, $\mathfrak{b} \asymp \mathfrak{b}' \asymp k$. For the second inequality, we have used $Nr^2 \ll k^{3+\varepsilon}$ and $C \gg k^{1+\varepsilon}$, while for the second last inequality, $(n_1, r) \geq n_1/q_1$ has been used. Hence we see that

$$\left|\frac{\partial^3 f_5(w^3)}{\partial w^3}\right| = \left|O\left(\frac{\mathfrak{a}^3}{\mathfrak{b}^2} + \frac{\mathfrak{a}'^3}{\mathfrak{b}'^2}\right) - \frac{6\tilde{N}n_2}{q_2q_2'q_1rn_1}\right| \gg \frac{\mathfrak{a}^3}{\mathfrak{b}^2} + \frac{\mathfrak{a}'^3}{\mathfrak{b}'^2} \asymp \frac{N\tilde{N}}{C^3rk^2}.$$

On computing the variation of $\phi_1(w)$ (see (6.31)), we note that

$$\operatorname{Var}\phi_1 \ll \frac{1}{\sqrt{\mathfrak{b}^2/\mathfrak{a}}}\frac{1}{(\mathfrak{a}/k)^5}\frac{1}{\sqrt{\mathfrak{b}'^2/\mathfrak{a}'}}\frac{1}{(\mathfrak{a}'/k)^5} \ll \frac{1}{\mathfrak{b}^2/\mathfrak{a}}\frac{1}{(\mathfrak{a}/k)^{10}}, \tag{6.34}$$

where we have used $f''(\tau_0) \asymp \mathfrak{b}^2/\mathfrak{a}$, $f_2''(\tau_0') \asymp \mathfrak{b}'^2/\mathfrak{a}'$, $\tau_0 \asymp \mathfrak{a}/(\mathfrak{b}^{2/3}(k-1)^{1/3}) \asymp \mathfrak{a}/k$ and $\tau_0' \asymp \mathfrak{a}'/k$. Hence, on applying the third derivative bound (see Lemma 9) to (6.30), we see that (6.30) is bounded by

$$\frac{c_3^2(\mathfrak{a}\mathfrak{a}')^{9/2}}{(\mathfrak{b}\mathfrak{b}')^5}\frac{\operatorname{Var}\phi_1 + \max|\phi_1|}{\min|f_5(w^3)|^{1/3}} \ll \frac{\mathfrak{a}^9}{\mathfrak{b}^{10}}\frac{1}{\mathfrak{b}^2/\mathfrak{a}}\frac{1}{(\mathfrak{a}/k)^{10}}\frac{(C^3rk^2)^{1/3}}{(N\tilde{N})^{1/3}} \asymp \frac{Cr^{1/3}k^{2/3}}{k^2(N\tilde{N})^{1/3}}.$$

Hence we get Proposition 1. ∎

We conclude this section by giving the final estimation of the main integral $\mathcal{J}_\pm$ defined in (6.1) in the following corollary:

Corollary 2. *Let $\mathcal{J}_\pm$ be the integral transform as defined in* (6.1)*. Then*

$$\mathcal{J}_\pm \ll \frac{k^\varepsilon C^4}{Q^2 M_1 N}. \tag{6.35}$$

Furthermore, if $C \gg k^{1+\varepsilon}$ and $n_2 \neq 0$,

$$\mathcal{J}_\pm \ll \frac{k^\varepsilon C^2}{Q^2}\frac{Cr^{1/3}k^{2/3}}{k^2(N\tilde{N})^{1/3}}. \tag{6.36}$$

Proof. Recall from (6.8) that

$$\mathcal{J}_\pm = \int_{\mathbb{R}}\int_{\mathbb{R}} V\left(\frac{t}{T}\right)V\left(\frac{t'}{T}\right)\int_{u\ll\frac{k^\varepsilon C}{QT}}\int_{u'\ll\frac{k^\varepsilon C}{QT}} \mathrm{I}_u\bar{\mathrm{I}}_{u'}\,\mathfrak{J}_\pm\,\mathrm{d}u'\,\mathrm{d}u\,\mathrm{d}t'\,\mathrm{d}t + O(k^{-2020}),$$

where

$$\mathrm{I}_u = \int_{\mathbb{R}} W(x/Q^\varepsilon)g(q,x)e\left(\frac{Nxu}{qQ}\right)\mathrm{d}x,$$

and $\mathrm{I}_{u'}$ is similarly defined. On applying the bound $\mathfrak{J}_\pm \ll k^\varepsilon C^2/(M_1N)$ from Proposition 1, we see that

$$|\mathcal{J}_\pm| \ll \frac{k^\varepsilon C^2}{M_1N}\int_{\mathbb{R}}\int_{\mathbb{R}} V\left(\frac{t}{T}\right)V\left(\frac{t'}{T}\right)\int_{u\ll\frac{k^\varepsilon C}{QT}}\int_{u'\ll\frac{k^\varepsilon C}{QT}} |\mathrm{I}_u||\bar{\mathrm{I}}_{u'}|\,\mathrm{d}u'\,\mathrm{d}u\,\mathrm{d}t'\,\mathrm{d}t. \tag{6.37}$$

Note that

$$\int_{u\ll\frac{k^\varepsilon C}{QT}}|\mathrm{I}_u|\,\mathrm{d}u \ll \int_{u\ll\frac{k^\varepsilon C}{QT}}\int_{\mathbb{R}} W(x/Q^\varepsilon)|g(q,x)|\,\mathrm{d}x\,\mathrm{d}u \ll \frac{k^\varepsilon C}{QT}Q^\varepsilon,$$

where we have used the property (2.7)(4) of $g(q,x)$. The same bound holds for the u'-integral as well. Thus, on plugging these bounds into (6.37) and estimating the t- and t'-integral trivially, we get (6.35). On analysing the u-, u'-, t- and t'-integrals as above and applying the bound (6.12) from Proposition 1, we get the second part of the corollary. ∎

7. Analysis of the zero frequency: $n_2 = 0$

With all ingredients in hand we now give final estimates for $S_r(N)$, given in (5.4), in the present and coming sections. The zero frequency case, i.e., $n_2 = 0$, has to be analysed differently. Let $\Omega_\pm^0$ denote the contribution of the zero frequency to $\Omega_\pm$, given in (5.8), and let $S_r^0(N)$ denote the contribution of $\Omega_\pm^0$ to $S_r(N)$. We have the following lemma:

Lemma 16. *Let $\Omega_\pm^0$ and $S_r^0(N)$ be defined as above. Then*

$$\Omega_\pm^0 \ll \frac{k^\varepsilon N_0 C^6 r}{q_1 n_1^2 Q^2 N}(C + M_1),$$
$$S_r^0(N) \ll k^\varepsilon r^{1/2} N^{1/2} k^{3/2-\eta/2}.$$

Recall that $T = k^{1-\eta}$.

Proof. Recall from (5.11) that

$$\Omega_\pm^0 \ll k^\varepsilon \sup_{\tilde{N} \ll N_0} \frac{\tilde{N}}{n_1^2} \sum\sum_{q_2, q_2' \sim C/q_1} \sum\sum_{m,m' \sim M_1} |\lambda_f(m)|\,|\lambda_f(m')|\,|\mathfrak{C}_\pm|\,|\mathcal{J}_\pm|. \tag{7.1}$$

Consider the congruence condition

$$\pm\bar{\alpha}q_2' \mp \bar{\alpha}' q_2 \equiv n_2 \bmod q_1 q_2 q_2' r/n_1$$

appearing in the expression (5.9) of $\mathfrak{C}_\pm$. For $n_2 = 0$, it follows that $q_2 = q_2'$ and $\alpha = \alpha'$. Hence

$$\begin{aligned}\mathfrak{C}_\pm &= \sum\sum_{d,d' \mid q} d d' \mu\left(\frac{q}{d}\right)\mu\left(\frac{q}{d'}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1 \\ n_1\alpha \equiv -m \bmod d \\ n_1\alpha \equiv -m' \bmod d'}} 1 \\ &\ll \sum\sum_{\substack{d,d' \mid q \\ (d,d') \mid (m-m')}} d d' \frac{qr}{n_1[d/(n_1,d), d'/(n_1,d')]} \\ &\ll \sum\sum_{\substack{d,d' \mid q \\ (d,d') \mid (m-m')}} d d' \frac{qr}{[d,d']}.\end{aligned}$$

On plugging the above expression and the bound $\mathcal{J}_\pm \ll k^\varepsilon C^4/(Q^2 M_1 N)$ from Corollary 2 into (7.1), we get

$$\Omega_\pm^0 \ll \frac{k^\varepsilon C^4}{Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \frac{\tilde{N}}{n_1^2} \sum_{q_2 \sim C/q_1} qr \sum\sum_{d,d' \mid q} (d,d') \sum\sum_{\substack{m,m' \sim M_1 \\ (d,d') \mid (m-m')}} |\lambda_f(m)|\,|\lambda_f(m')|.$$

We use the inequality

$$|\lambda_f(m)|\,|\lambda_f(m')| \le \tfrac{1}{2}(|\lambda_f(m)|^2 + |\lambda_f(m')|^2) \tag{7.2}$$

to count the number of m and m' as follows:

$$\begin{aligned}\mathop{\sum\sum}_{\substack{m,m'\sim M_1\\(d,d')|(m-m')}} &|\lambda_f(m)\lambda_f(m')|\\ &\ll \sum_{m\sim M_1}|\lambda_f(m)|^2+\mathop{\sum\sum}_{\substack{m,m'\sim M_1\\(d,d')|(m-m'),m\neq m'}}(|\lambda_f(m)|^2+|\lambda_f(m')|^2)\\ &\ll k^{\varepsilon}M_1+\sum_{m\sim M_1}|\lambda_f(m)|^2\sum_{\substack{m'\sim M_1\\(d,d')|(m-m'),m'\neq m}}1\\ &\ll k^{\varepsilon}M_1(1+M_1/(d,d')),\end{aligned}$$

where we have used the Ramanujan bound on average (see (2.1) and (2.2)). Thus

$$\begin{aligned}\Omega_{\pm}^{0} &\ll \frac{k^{\varepsilon}N_0C^4}{n_1^2Q^2M_1N}\sum_{q_2\sim C/q_1}qr\mathop{\sum\sum}_{d,d'|q}(M_1(d,d')+M_1^2)\\ &\ll \frac{k^{\varepsilon}N_0C^4}{n_1^2Q^2M_1N}\sum_{q_2\sim C/q_1}qr(M_1q+M_1^2)\ll\frac{k^{\varepsilon}N_0C^6r}{q_1n_1^2Q^2N}(C+M_1).\end{aligned}$$

Hence we have the first part of the lemma. On substituting the above bound in place of $\Omega_{\pm}$ in (5.4), we get

$$\begin{aligned}S_r^0(N)\ll \sup_{\substack{M_1\ll M_0\\C\ll Q}}\frac{N^{5/3+\varepsilon}}{QTr^{2/3}C^3}&\sum_{\frac{n_1}{(n_1,r)}\ll C}n_1^{1/3}\Theta^{1/2}\\ &\times\sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^{\infty}}\frac{C^3(N_0r)^{1/2}}{n_1q_1^{1/2}Q\sqrt{N}}\big(\sqrt{M_1}+\sqrt{C}\big).\end{aligned}$$

Estimating the q_1-sum trivially and replacing the range for n_1 by the longer range $n_1\ll Cr$, we get

$$S_r^0(N)\ll k^{\varepsilon}\sup_{\substack{M_1\ll M_0\\C\ll Q}}\frac{N^{2/3}(N_0r)^{1/2}}{r^{2/3}\sqrt{N}}\sum_{n_1\ll Cr}\frac{(n_1,r)^{1/2}}{n_1^{7/6}}\Theta^{1/2}\big(\sqrt{M_1}+\sqrt{C}\big).$$

Next we evaluate the n_1-sum, using Cauchy's inequality and the Ramanujan bound on average (see Lemma 5), as follows:

$$\begin{aligned}\sum_{n_1\ll Cr}\frac{(n_1,r)^{1/2}}{n_1^{7/6}}\Theta^{1/2} &\ll\left[\sum_{n_1\ll Cr}\frac{(n_1,r)}{n_1}\right]^{1/2}\left[\mathop{\sum\sum}_{n_1^2n_2\le N_0}\frac{|\lambda_\pi(n_1,n_2)|^2}{(n_1^2n_2)^{2/3}}\right]^{1/2}\\ &\ll_{\pi,\varepsilon} N_0^{1/6+\varepsilon}.\end{aligned}\tag{7.3}$$

Thus we arrive at

$$S_r^0(N) \ll k^\varepsilon \frac{N^{2/3} N_0^{2/3}}{r^{1/6}\sqrt{N}} \big(\sqrt{M_0} + \sqrt{Q}\big). \tag{7.4}$$

Note that

$$Q = k^\varepsilon \sqrt{N/T} \ll k^{3/2+\varepsilon}/\sqrt{T} \ll k^{2+\varepsilon}/T \ll k^{2+\varepsilon} Q^2/N.$$

We also have $M_0 = k^\varepsilon \max\{(k-1)^2 C^2/N, T\} \ll k^{2+\varepsilon} Q^2/N$ and

$$N_0 = k^\varepsilon \max\{(CT)^3 r/N, T^{3/2} N^{1/2} r\} \ll k^\varepsilon (QT)^3 r/N \ll k^\varepsilon T^{3/2} \sqrt{N}\, r.$$

Finally, upon using the above bounds in (7.4), we get

$$S_r^0(N) \ll \frac{k^\varepsilon r^{2/3} T N}{r^{1/6}\sqrt{N}} \frac{kQ}{\sqrt{N}} \ll k^\varepsilon r^{1/2} N^{1/2} k^{3/2-\eta/2}.$$

Hence the lemma follows. ∎

8. Analysis of the non-zero frequencies: $n_2 \neq 0$

It now remains to estimate $S_r(N)$ corresponding to the non-zero frequencies, i.e., $n_2 \neq 0$. We will consider two cases, small q and large q. To begin, we analyse the character sum $\mathfrak{C}_\pm$ given in (5.9). We have the following lemma which is taken from [29].

Lemma 17. *Let $\mathfrak{C}_\pm$ be as in (5.9). Then, for $n_2 \neq 0$, we have*

$$\mathfrak{C}_\pm \ll \frac{q_1^2\, r(m, n_1)}{n_1} \sum\sum_{\substack{d_2 | (q_2, n_1 q_2' \mp m n_2) \\ d_2' | (q_2', n_1 q_2 \pm m' n_2)}} d_2 d_2'.$$

Proof. Recall from (5.9) that

$$\mathfrak{C}_\pm = \sum\sum_{\substack{d|q \\ d'|q'}} d d' \mu\left(\frac{q}{d}\right) \mu\left(\frac{q'}{d'}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1 \\ n_1\alpha \equiv -m \bmod d}} \sideset{}{^\star}\sum_{\substack{\alpha' \bmod q'r/n_1 \\ n_1\alpha' \equiv -m' \bmod d' \\ \pm\bar{\alpha} q_2' \mp \bar{\alpha}' q_2 \equiv -n_2 \bmod q_1 q_2 q_2' r/n_1}} 1.$$

Using the Chinese remainder theorem, we observe that $\mathfrak{C}_\pm$ can be dominated by a product of two sums, $\mathfrak{C}_\pm \ll \mathfrak{C}_\pm^{(1)} \mathfrak{C}_\pm^{(2)}$, where

$$\mathfrak{C}_\pm^{(1)} = \sum\sum_{d_1, d_1' | q_1} d_1 d_1' \sideset{}{^\star}\sum_{\substack{\beta \bmod q_1 r/n_1 \\ n_1\beta \equiv -m \bmod d_1}} \sideset{}{^\star}\sum_{\substack{\beta' \bmod q_1 r/n_1 \\ n_1\beta' \equiv -m' \bmod d_1' \\ \pm\overline{\beta} q_2' \mp \overline{\beta'} q_2 + n_2 \equiv 0 \bmod q_1 r/n_1}} 1$$

and

$$\mathfrak{C}_{\pm}^{(2)} = \sum_{\substack{d_2|q_2 \\ d_2'|q_2'}}\sum d_2 d_2' \mathop{\sum\nolimits^{\star}}_{\substack{\beta \bmod q_2 \\ n_1\beta\equiv -m \bmod d_2}} \mathop{\sum\nolimits^{\star}}_{\substack{\beta' \bmod q_2' \\ n_1\beta'\equiv -m' \bmod d_2' \\ \pm\overline{\beta}q_2' \mp \overline{\beta'}q_2 + n_2 \equiv 0 \bmod q_2 q_2'}} 1.$$

Analysing the second sum $\mathfrak{C}_{\pm}^{(2)}$, we get $\beta \equiv -m\overline{n}_1 \bmod d_2$ and $\beta' \equiv -m'\overline{n}_1 \bmod d_2'$, as $(n_1, q_2q_2') = 1$. Then using the congruence modulo q_2q_2', we conclude that

$$\mathfrak{C}_{\pm}^{(2)} \ll \sum_{\substack{d_2|(q_2, n_1q_2' \mp mn_2) \\ d_2'|(q_2', n_1q_2 \pm m'n_2)}}\sum d_2 d_2'.$$

In the first sum $\mathfrak{C}_{\pm}^{(1)}$, the congruence condition determines β uniquely in terms of β', and hence

$$\mathfrak{C}_{\pm}^{(1)} \ll \sum_{d_1, d_1'|q_1}\sum d_1 d_1' \mathop{\sum\nolimits^{\star}}_{\substack{\beta \bmod q_1 r/n_1 \\ n_1\beta \equiv -m \bmod d_1}} 1 \ll \frac{r\, q_1^2\, (m, n_1)}{n_1}.$$

Hence we have the lemma. ∎

8.1. $S_r(N)$ for small q

In this subsection we will estimate $S_r(N)$ for small values of q. Let $\Omega_{\pm}^{\neq 0}$ denote the part of $\Omega_{\pm}$ (defined in (5.8)) which is complementary to $\Omega_{\pm}^{0}$ (contribution of $n_2 \neq 0$) and let $S_r^{\neq 0}(N)$ denote the part of $S_r(N)$ corresponding to $\Omega_{\pm}^{\neq 0}$. We have the following lemma.

Lemma 18. *Let $\Omega_{\pm}^{\neq 0}$ and $S_r^{\neq 0}(N)$ be as above. Then, for $C \ll k^{1+\varepsilon}$, we have*

$$\Omega_{\pm}^{\neq 0} \ll \frac{k^{\varepsilon} r^2 C^7 (TN)^{1/2}}{n_1^2 q_1 Q^2 M_1 N}\left(\frac{CM_1 n_1}{q_1} + M_1^2\right). \tag{8.1}$$

Furthermore, let $S_{r,\text{small}}^{\neq 0}(N)$ denote the contribution of $C \ll k^{1+\varepsilon}$ to $S_r^{\neq 0}(N)$. Then

$$S_{r,\text{small}}^{\neq 0}(N) \ll r^{1/2} k^{3-\eta/2}. \tag{8.2}$$

Recall that $T = k^{1-\eta}$.

Proof. On applying (7.2) to (5.11), we see that $\Omega_{\pm}^{\neq 0}$ is dominated by

$$k^{\varepsilon} \sup_{\tilde{N} \ll N_0} \frac{\tilde{N}}{n_1^2} \sum_{q_2, q_2' \sim C/q_1}\sum \sum_{m, m' \sim M_1}\sum \left(|\lambda_f(m)|^2 + |\lambda_f(m')|^2\right) \sum_{n_2 \in \mathbb{Z}-\{0\}} |\mathfrak{C}_{\pm}|\, |\mathcal{J}_{\pm}|.$$

We analyse the expression corresponding to $|\lambda_f(m')|^2$ only, since the calculation for the other expression is very much similar. Thus, on applying Lemma 17 and Corollary 2, we arrive at

$$\frac{k^{\varepsilon}q_1^2rC^4}{n_1^3Q^2M_1N}\sup_{\tilde{N}\ll N_0}\tilde{N}\sum\sum_{q_2,q_2'\sim\frac{C}{q_1}}\sum\sum_{\substack{d_2|q_2\\ d_2'|q_2'}}d_2d_2'\sum\sum_{\substack{m,m'\sim M_1\\ n_1q_2'\mp mn_2\equiv 0 \bmod d_2\\ n_1q_2\pm m'n_2\equiv 0 \bmod d_2'}}\sum_{n_2\in\mathbb{Z}-\{0\}}|\lambda_f(m')|^2(m,n_1).$$

Writing q_2d_2 and $q_2'd_2'$ in place of q_2' and q_2' respectively, we arrive at

$$\frac{k^{\varepsilon}q_1^2rC^4}{n_1^3Q^2M_1N}\sup_{\tilde{N}\ll N_0}\tilde{N}\sum\sum_{d_2,d_2'\ll C/q_1}d_2d_2'\sum\sum_{\substack{q_2\sim\frac{C}{d_2q_1}\\ q_2'\sim\frac{C}{d_2'q_1}}}\sum\sum_{\substack{m,m'\sim M_1\\ n_1q_2'd_2'\mp mn_2\equiv 0 \bmod d_2\\ n_1q_2d_2\pm m'n_2\equiv 0 \bmod d_2'}}\sum_{1\le|n_2|\ll N_2}|\lambda_f(m')|^2(m,n_1). \tag{8.3}$$

Fixing the parameters $(n_2,q_2,q_2',d_2,d_2',m')$, we count the number of m's as follows:

$$\begin{aligned}\sum_{\substack{m\sim M_1\\ n_1q_2'd_2'\mp mn_2\equiv 0 \bmod d_2}}(m,n_1)&=\sum_{\ell|n_1}\ell\sum_{\substack{m\sim M_1/\ell\\ n_1q_2'd_2'\bar{\ell}\mp mn_2\equiv 0 \bmod d_2}}1\\ &=\sum_{\ell|n_1}\ell\left((d_2,q_2'd_2',n_2)+\frac{M_1}{\ell d_2/(d_2,d_2'q_2',n_2)}\right)\\ &\ll (d_2,d_2'q_2',n_2)\left(n_1+\frac{M_1}{d_2}\right),\end{aligned} \tag{8.4}$$

where $\bar{\ell}$ is the inverse of ℓ modulo d_2, which follows from the fact $(d_2,n_1)=1$. On applying (8.4) with the bound $(d_2,n_2)(n_1+M_1/d_2)$ and then executing the sum over q_2' in (8.3), we arrive at

$$\begin{aligned}\frac{k^{\varepsilon}q_1^2rC^4}{n_1^3Q^2M_1N}\sup_{\tilde{N}\ll N_0}\tilde{N}\sum\sum_{d_2,d_2'\ll C/q_1}\frac{Cd_2}{q_1}\sum_{q_2\sim\frac{C}{d_2q_1}}\\ \times\sum_{\substack{1\le|n_2|\ll N_2\\ n_1q_2d_2\pm m'n_2\equiv 0 \bmod d_2'}}\sum_{m'\sim M_1}|\lambda_f(m')|^2(d_2,n_2)\left(n_1+\frac{M_1}{d_2}\right).\end{aligned} \tag{8.5}$$

We now count the number of (d_2,d_2',m') following the arguments in [22, Section 6.1].

Case 1: $n_1q_2d_2\pm m'n_2\equiv 0 \bmod d_2'$ but $n_1q_2d_2\pm m'n_2\neq 0$. On switching the order of summations over d_2' and m', we see that the d_2'-sum is bounded above by $d(|n_1q_2d_2\pm m'n_2|)\ll k^{\varepsilon}$, with $d(n)$ being the divisor function. Thus (8.5) is bounded above by

$$\begin{aligned}\frac{k^{\varepsilon}q_1^2rC^4}{n_1^3Q^2M_1N}\sup_{\tilde{N}\ll N_0}\tilde{N}\sum_{d_2\ll C/q_1}\frac{Cd_2}{q_1}\sum_{q_2\sim\frac{C}{d_2q_1}}\\ \times\sum_{1\le|n_2|\ll N_2}\sum_{m'\sim M_1}|\lambda_f(m')|^2(d_2,n_2)(n_1+M_1/d_2).\end{aligned}$$

On applying the Ramanujan bound on average to the m'-sum (see (2.1), (2.2)) and executing the n_2-sum, we arrive at

$$\frac{k^{\varepsilon}q_1^2rC^4}{n_1^3Q^2M_1N}\sup_{\tilde{N}\ll N_0}\tilde{N}N_2M_1\sum_{d_2\ll C/q_1}\frac{Cd_2}{q_1}\sum_{q_2\sim\frac{C}{d_2q_1}}\left(n_1+\frac{M_1}{d_2}\right).$$

Now executing the remaining sums, we get the expression

$$\frac{k^{\varepsilon}rC^6}{n_1^3Q^2M_1N}\sup_{\tilde{N}\ll N_0}\tilde{N}N_2\left(\frac{Cn_1M_1}{q_1}+M_1^2\right). \tag{8.6}$$

On applying the bounds $N_2=k^{\varepsilon}CN^{1/3}r^{2/3}n_1/(q_1\tilde{N}^{2/3})$ (see (6.10)) and $N_0\ll k^{\varepsilon}T^{3/2}\sqrt{N}\,r$ (see (5.7)), we note that

$$\sup_{\tilde{N}\ll N_0}\tilde{N}N_2\ll k^{\varepsilon}\frac{Cr^{2/3}n_1}{q_1}(N\tilde{N})^{1/3}\ll k^{\varepsilon}\frac{Cr^{2/3}n_1}{q_1}(NN_0)^{1/3}\ll\frac{k^{\varepsilon}rn_1}{q_1}(TN)^{1/2}C. \tag{8.7}$$

Thus, in Case 1, we get the following bound for $\Omega_{\pm}^{\neq 0}$:

$$\frac{k^{\varepsilon}r^2C^7(TN)^{1/2}}{n_1^2q_1Q^2M_1N}\left(\frac{Cn_1M_1}{q_1}+M_1^2\right). \tag{8.8}$$

Case 2: $n_1q_2d_2\pm m'n_2=0$. On applying (8.4) and switching some summations in (8.3), we arrive at

$$\frac{k^{\varepsilon}q_1^2rC^4}{n_1^3Q^2M_1N}\sup_{\tilde{N}\ll N_0}\tilde{N}\sum\sum_{d_2,d_2'\ll C/q_1}d_2d_2'\sum_{q_2'\sim\frac{C}{d_2'q_1}}\sum_{m'\sim M_1}|\lambda_f(m')|^2$$
$$\times\sum_{\substack{1\le|n_2|\ll N_2\\ n_1q_2d_2\pm m'n_2=0}}\sum_{q_2\sim\frac{C}{d_2q_1}}(d_2'q_2',n_2)\left(n_1+\frac{M_1}{d_2}\right). \tag{8.9}$$

Fixing the tuple (m',n_2,d_2), the number of q_2's turns out to be $O(k^{\varepsilon})$ (as $q_2\,|\,m'n_2$). Thus we arrive at

$$\frac{k^{\varepsilon}q_1^2rC^4}{n_1^3Q^2M_1N}\sup_{\tilde{N}\ll N_0}\tilde{N}\sum_{d_2'\ll C/q_1}d_2'\sum_{q_2'\sim\frac{C}{d_2'q_1}}\sum_{m'\sim M_1}|\lambda_f(m')|^2$$
$$\times\sum_{1\le|n_2|\ll N_2}(d_2'q_2',n_2)\sum_{\substack{d_2\ll C/q_1\\ d_2|m'n_2}}(n_1d_2+M_1).$$

Now executing the sum over d_2, followed by the sum over n_2, m', q_2' and d_2', we see that the above expression is bounded above by

$$\frac{k^{\varepsilon}rC^6}{n_1^3Q^2M_1N}\sup_{\tilde{N}\ll N_0}\tilde{N}N_2\left(\frac{Cn_1M_1}{q_1}+M_1^2\right).$$

Now estimating $\tilde{N}N_2$ as in Case 1, we get the first part of the lemma.

We will now prove (8.2). Consider the second term of the right hand side in (8.1). On substituting it in place of $\Omega_\pm$ in $S_r(N)$ in (5.4), we arrive at

$$\sup_{\substack{M\ll M_1\ll M_0\\ C\ll k^{1+\varepsilon}}} \frac{N^{5/3+\varepsilon}}{QTr^{2/3}C^3}\sum_{\pm}\sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2}\sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty}\left(\frac{r^2C^7(TN)^{1/2}M_1}{n_1^2q_1Q^2N}\right)^{1/2}$$

$$\ll \sup_{\substack{M\ll M_1\ll M_0\\ C\ll k^{1+\varepsilon}}} \frac{N^{5/3+\varepsilon}}{QTr^{2/3}C^3}\frac{r(TN)^{1/4}C^{7/2}M_1^{1/2}}{Q\sqrt{N}}\sum_{n_1\ll Cr} n_1^{-2/3}\Theta^{1/2}\sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty}\frac{1}{q_1^{1/2}}$$

$$\ll \sup_{\substack{M\ll M_1\ll M_0\\ C\ll k^{1+\varepsilon}}} \frac{N^{5/3+\varepsilon}}{QTr^{2/3}}\frac{r(TN)^{1/4}C^{1/2}M_1^{1/2}}{Q\sqrt{N}}\sum_{n_1\ll Cr}\frac{\sqrt{(n_1,r)}}{n_1^{7/6}}\Theta^{1/2}$$

$$\ll k^\varepsilon r^{1/2}k^{3-\eta/2},$$

where in the second last inequality we have used

$$\sum_{n_1\ll Cr}\frac{\sqrt{(n_1,r)}}{n_1^{7/6}}\Theta^{1/2}\ll_{\pi,\varepsilon} N_0^{1/6+\varepsilon}$$

from (7.3), $C\ll k^{1+\varepsilon}$, $N_0\ll k^\varepsilon r\sqrt{N}\,T^{3/2}$ and $M_0\ll k^{4+\varepsilon}/N$ as $C\ll k^{1+\varepsilon}$.

Now consider the first term on the right hand side of (8.1). We see that its contribution to $S_r(N)$ in (5.4) is given by

$$\sup_{\substack{M\ll M_1\ll M_0\\ C\ll k^{1+\varepsilon}}} \frac{N^{5/3+\varepsilon}}{QTr^{2/3}C^3}\sum_{\pm}\sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2}\sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty}\left(\frac{r^2C^7(TN)^{1/2}C}{n_1q_1^2Q^2N}\right)^{1/2}$$

$$\ll \sup_{\substack{M\ll M_1\ll M_0\\ C\ll k^{1+\varepsilon}}} \frac{N^{5/3+\varepsilon}}{QTr^{2/3}C^3}\frac{r(TN)^{1/4}C^{7/2}C^{1/2}}{Q\sqrt{N}}\sum_{n_1\ll Cr} n_1^{-1/6}\Theta^{1/2}\sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty}\frac{1}{q_1}$$

$$\ll \sup_{\substack{M\ll M_1\ll M_0\\ C\ll k^{1+\varepsilon}}} \frac{N^{5/3+\varepsilon}}{QTr^{2/3}}\frac{r(TN)^{1/4}C}{Q\sqrt{N}}\sum_{n_1\ll Cr}\frac{(n_1,r)}{n_1^{7/6}}\Theta^{1/2}$$

$$\ll k^{3-\eta/2}.$$

In the second last inequality, we have used the bound

$$\sum_{n_1\ll Cr}\frac{(n_1,r)}{n_1^{7/6}}\Theta^{1/2}\ll\left[\sum_{n_1\ll Cr}\frac{(n_1,r)^2}{n_1}\right]^{1/2}\left[\sum\sum_{n_1^2n_2\le N_0}\frac{|\lambda_\pi(n_1,n_2)|^2}{(n_1^2n_2)^{2/3}}\right]^{1/2}$$

$$\ll_{\pi,\varepsilon} r^{1/2}N_0^{1/6+\varepsilon}.$$

Thus we have the lemma. ∎

8.2. Estimates for generic q

Now we tackle the case when $C \gg k^{1+\varepsilon}$ and $n_2 \neq 0$. Let $S^{\neq 0}_{r,\text{generic}}(N)$ denote the part of $S^{\neq 0}_r(N)$ which is complementary to $S^{\neq 0}_{r,\text{small}}(N)$ (i.e., the contribution of $C \gg k^{1+\varepsilon}$) and $n_2 \neq 0$ to $S_r(N)$. We have the following lemma.

Lemma 19. *Let $S^{\neq 0}_{r,\text{generic}}(N)$ be as above. Then*

$$S^{\neq 0}_{r,\text{generic}}(N) \ll N^{1/2} k^{3/2-1/6+3\eta/4}. \tag{8.10}$$

Proof. Recall from the analysis of $\Omega^{\neq 0}_{\pm}$ in the proof of Lemma 18 (see (8.6)) that

$$\Omega^{\neq 0}_{\pm} \ll \frac{k^{\varepsilon} r C^6}{n_1^3 Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \tilde{N} N_2 \left(\frac{C n_1 M_1}{q_1} + M_1^2 \right). \tag{8.11}$$

To get this, we have used the bound $\mathcal{J}_{\pm} \ll k^{\varepsilon} C^4/(Q^2 M_1 N)$. For $C \gg k^{1+\varepsilon}$, we have a better bound for $\mathcal{J}_{\pm}$ (see Corollary 2). In fact,

$$\mathcal{J}_{\pm} \ll \frac{k^{\varepsilon} C^2}{Q^2} \frac{C r^{1/3} k^{2/3}}{k^2 (N\tilde{N})^{1/3}} \asymp \frac{k^{\varepsilon} C^4}{Q^2 M_1 N} \frac{C r^{1/3} k^{2/3}}{(N\tilde{N})^{1/3}}, \tag{8.12}$$

where we have used $\sqrt{M_1 N}/C \asymp k$ for $C \gg k^{1+\varepsilon}$. Thus, on applying the above bound, we see that

$$\Omega^{\neq 0}_{\pm} \ll \frac{k^{\varepsilon} r C^6}{n_1^3 Q^2 M_1 N} \times C r^{1/3} k^{2/3} \times \sup_{\tilde{N} \ll N_0} \frac{\tilde{N} N_2}{(N\tilde{N})^{1/3}} \left(\frac{C n_1 M_1}{q_1} + M_1^2 \right). \tag{8.13}$$

Recall from (8.7) that

$$\sup_{\tilde{N} \ll N_0} \frac{\tilde{N} N_2}{(N\tilde{N})^{1/3}} \ll k^{\varepsilon} \frac{C r^{2/3} n_1}{q_1}, \tag{8.14}$$

and

$$\sup_{\tilde{N} \ll N_0} \frac{\tilde{N} N_2}{(N\tilde{N})^{1/3}} = \frac{N_0 N_2}{(N N_0)^{1/3}}.$$

Thus we see that

$$\Omega^{\neq 0}_{\pm} \ll \frac{k^{\varepsilon} r C^6}{n_1^3 Q^2 M_1 N} \times C r^{1/3} k^{2/3} \times \frac{N_0 N_2}{(N N_0)^{1/3}} \left(\frac{C n_1 M_1}{q_1} + M_1^2 \right). \tag{8.15}$$

Comparing it with (8.8), we observe that we have an extra factor

$$\frac{C r^{1/3} k^{2/3}}{r^{1/3} (NT)^{1/2}} \ll \frac{Q k^{2/3}}{(NT)^{1/2}} = k^{\varepsilon+\eta-1/3}$$

in this case. Hence, taking it into account, we get

$$\Omega^{\neq 0}_{\pm} \ll \frac{k^{\varepsilon} r^2 C^7 (TN)^{1/2}}{n_1^2 q_1 Q^2 M_1 N} \times k^{\eta-1/3} \left(\frac{C n_1 M_1}{q_1} + M_1^2 \right). \tag{8.16}$$

Note that

$$\frac{Cn_1}{q_1} + M_1 \ll \frac{Qn_1}{q_1} + M_0 \ll \frac{n_1 k^\varepsilon}{q_1}\sqrt{\frac{N}{T}} + \frac{Q^2 k^{2+\varepsilon}}{N} \ll (n_1, r)k^\varepsilon \sqrt{\frac{N}{T}} + \frac{k^{2+\varepsilon}}{T} \\ \ll \frac{k^{2+\varepsilon}}{T},$$

where we have used $M_0 \ll Q^2 k^{2+\varepsilon}/N$, $Nr^2 \ll k^{3+\varepsilon}$, $Q = k^\varepsilon \sqrt{N/T}$, $T \ll k$ and $n_1/q_1 \le (n_1, r)$. Thus, on plugging the above bound into (8.16), we get

$$\Omega_\pm^{\neq 0} \ll \frac{k^\varepsilon r^2 C^7 (TN)^{1/2}}{n_1^2 q_1 Q^2 N} \times k^{\eta - 1/3} \times \frac{k^{2+\varepsilon}}{T}.$$

On substituting the above bound in place of $\Omega_\pm$ in (5.4), we see that $S^{\neq 0}_{r,\text{generic}}(N)$ is dominated by

$$\sup_{C \ll Q} \frac{N^{5/3+\varepsilon}}{QTr^{2/3}C^3} \sum_\pm \sum_{\frac{n_1}{(n_1,r)} \ll C} n_1^{1/3} \Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)} |q_1| (n_1 r)^\infty} \left(\frac{r^2 C^7 (TN)^{1/2}}{n_1^2 q_1 Q^2 N} \right)^{1/2} \times \frac{k^{5/6+\eta/2}}{\sqrt{T}}$$

$$\ll \sup_{C \ll Q} \frac{N^{5/3+\varepsilon}}{QTr^{2/3}C^3} \frac{r(TN)^{1/4} C^{7/2}}{Q\sqrt{N}} \sum_{n_1 \ll Cr} n_1^{-2/3} \Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)} |q_1| (n_1 r)^\infty} \frac{1}{q_1^{1/2}} \times \frac{k^{5/6+\eta/2}}{\sqrt{T}}$$

$$\ll \sup_{C \ll Q} \frac{N^{5/3+\varepsilon}}{QTr^{2/3}} \frac{r(TN)^{1/4} C^{1/2}}{Q\sqrt{N}} \sum_{n_1 \ll Cr} \frac{\sqrt{(n_1, r)}}{n_1^{7/6}} \Theta^{1/2} \times \frac{k^{5/6+\eta/2}}{\sqrt{T}}$$

$$\ll N^{1/2} k^{3/2 - 1/6 + 3\eta/4}.$$

Hence the lemma follows. ∎

8.3. *Estimates for the error term*

In this subsection we give estimates for $S_r(N)$ corresponding to the non-generic case $n_2^\star N \ll k^\varepsilon$ (see Lemma 12). Recall from (4.7) that if $n_2^\star N = n_1^2 n_2 N/(q^3 r) \ll k^\varepsilon$, then we have

$$\mathrm{S}_3 = q \sum_\pm \sum_{n_1 | qr} \sum_{n_2=1}^\infty \frac{\lambda_\pi(n_1, n_2)}{n_1 n_2} S(r\bar{a}, \pm n_2; qr/n_1) G_\pm(n_2^\star), \tag{8.17}$$

where $G_\pm(n_2^\star)$ is as defined in (4.5). On plugging (8.17) and (4.11) in place of S_3 and S_2 respectively into (3.4) we arrive at

$$\frac{2\pi i^k N^{1-it}}{QT} \sum_{1 \le q \le Q} \frac{1}{q} \sum_\pm \sum_{n_1 | qr} \sum_{n_2 \ll \frac{q^3 r k^\varepsilon}{n_1^2 N}} \frac{\lambda_\pi(n_1, n_2)}{n_1 n_2} \\ \times \sum_{M \le m \le M_0} \lambda_f(m) \mathcal{C}_\pm(\dots) \mathrm{I}_4(q, m, n_1^2 n_2) + O(k^{-2020}), \tag{8.18}$$

where

$$\mathscr{C}_\pm(\dots) := \sum_{a \bmod q}^\star S(r\bar{a}, \pm n_2; qr/n_1) e\left(\frac{\bar{a}m}{q}\right)$$
$$= \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sum_{\substack{\alpha \bmod qr/n_1 \\ n_1\alpha \equiv -m \bmod d}}^\star e\left(\pm\frac{\bar{\alpha}n_2}{qr/n_1}\right)$$
$$\ll (n_1, m, q)\left(q + \frac{qr}{n_1}\right) \ll \sqrt{(n_1, m)}\sqrt{(n_1, q)}\left(q + \frac{qr}{n_1}\right), \qquad (8.19)$$

and

$$\mathrm{I}_4(q, m, n_1^2 n_2) = \int_{\mathbb{R}} W(x/Q^\varepsilon) \int_{\mathbb{R}} V\left(\frac{t}{T}\right) g(q, x)\, \mathrm{I}_2(m, q, x) G_\pm(n_2^\star)\, \mathrm{d}t\, \mathrm{d}x,$$

with

$$\mathrm{I}_2(m, q, x) = \int_0^\infty U(y) y^{-it} e\left(\frac{-Nxy}{qQ}\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \mathrm{d}y,$$

and

$$G_\pm(n_2^\star) = \frac{1}{2\pi i}\int_{(\sigma)} (n_2^\star)^{-s} \gamma_\pm(s) \tilde{g}(-s)\, \mathrm{d}s$$
$$= \frac{N^{it}}{2\pi}\int_{-\infty}^\infty \frac{\gamma_\pm(\sigma + i\tau)}{(n_2^\star N)^{\sigma + i\tau}} \int_0^\infty e\left(\frac{z_1 N x}{qQ}\right) V(z_1) z_1^{-\sigma - i\tau + it} \frac{\mathrm{d}z_1}{z_1}\, \mathrm{d}\tau, \qquad (8.20)$$

where $\sigma > -1 + \max\{-\Re(\alpha_1), -\Re(\alpha_2), -\Re(\alpha_3)\}$. On analysing the x-integral and the t-integral following Lemma 14, we get the restriction

$$|z_1 - y| \ll k^\varepsilon q/(QT).$$

Thus, on replacing z_1 by $y + u$ with $u \ll k^\varepsilon q/(QT)$, we essentially arrive at

$$\mathrm{I}_4(q, m, n_1^2 n_2) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\gamma_\pm(\sigma + i\tau)}{(n_2^\star N)^{\sigma + i\tau}} \int_{\mathbb{R}} V\left(\frac{t}{T}\right) N^{it} \int_{u \ll \frac{k^\varepsilon q}{QT}} \mathrm{I}_u\, \mathrm{I}_5(m, q, u, \tau)\, \mathrm{d}u\, \mathrm{d}t\, \mathrm{d}\tau,$$

where

$$\mathrm{I}_u = \int_{\mathbb{R}} W(x/Q^\varepsilon) g(q, x) e\left(\frac{Nxu}{qQ}\right) \mathrm{d}x,$$
$$\mathrm{I}_5(m, q, u, \tau) = \int_0^\infty U_{t,u,\tau}(y) y^{-i\tau} J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \mathrm{d}y,$$

with $U_{t,u,\tau}(y) = U(y) y^{-\sigma}(1 + u/y)^{-\sigma - i\tau + it}$. Analysing $\mathrm{I}_5(m, q, u, \tau)$ like $\mathrm{I}_\pm(m, \tilde{N}w, q)$ (see Lemma 15), we get

$$\mathrm{I}_5(m, q, u, \tau) \ll \frac{k^\varepsilon q^{1/2}}{(mN)^{1/4}}.$$

We now move the contour σ in (8.20) to the left to $\sigma = -5/2$ passing through the poles given by

$$\frac{1 + \sigma + \Re(\alpha_i) + \ell}{2} = 0, \quad \text{i.e.,} \quad \sigma = -1 - \Re(\alpha_i) - \ell.$$

Thus, on treating the u- and t-integral trivially, we get

$$\mathrm{I}_4(q,m,n_1^2n_2) \ll (n_2^\star N)^{5/2}\frac{k^\varepsilon q^{3/2}}{Q(mN)^{1/4}}\int_{-\infty}^{\infty}|\gamma_\pm(-5/2+i\tau)|\,\mathrm{d}\tau + \frac{k^\varepsilon q^{3/2}}{Q(mN)^{1/4}} + \sum_{\ell=0,1}\sum_{i=1}^{3}(n_2^\star N)^{1+\ell+\Re(\alpha_i)}.$$

Now using the Stirling bound

$$|\gamma_\pm(-5/2+i\tau)| \ll (1+|\tau|)^{3(-5/2+1/2)} = (1+|\tau|)^{-6},$$

we arrive at

$$\mathrm{I}_4(q,m,n_1^2n_2) \ll \frac{k^\varepsilon q^{3/2}}{Q(mN)^{1/4}}\Big((n_2^\star N)^{5/2} + \sum_{\ell=0,1}\sum_{i=1}^{3}(n_2^\star N)^{1+\ell+\Re(\alpha_i)}\Big).$$

Note that $(n_2^\star N)^{5/2} = (n_2^\star N)^{1/4+9/4} \ll k^\varepsilon (n_2^\star N)^{1/4}$, and

$$\sum_{i=1}^{3}(n_2^\star N)^{1+\ell+\Re(\alpha_i)} = \sum_{i=1}^{3}(n_2^\star N)^{1/2+\beta_i} \ll k^\varepsilon (n_2^\star N)^{1/2} \ll k^\varepsilon (n_2^\star N)^{1/4}$$

as $1+\ell+\Re(\alpha_i) = 1/2+\beta_i$ for some $\beta_i > 0$. Thus we get

$$\mathrm{I}_4(q,m,n_1^2n_2) \ll \frac{k^\varepsilon q^{3/2}}{Q(mN)^{1/4}}(n_2^\star N)^{1/4} = \frac{k^\varepsilon q^{3/4}(n_1^2n_2)^{1/4}}{Qm^{1/4}r^{1/4}}. \tag{8.21}$$

Thus, on plugging the above bound and the bound (8.19) for $\mathcal{C}_\pm(\ldots)$ into (8.18) and then estimating the sum over m using the Ramanujan bound on average, we see that (8.18) is dominated by

$$\sum_{1\le q\le Q}\frac{NM_0^{3/4}}{Q^2Tr^{1/4}}\sum_{n_1|qr}\sum_{n_2\ll\frac{q^3rk^\varepsilon}{n_1^2N}}\frac{|\lambda_\pi(n_1,n_2)|}{n_1n_2}(n_1^2n_2)^{1/4}\sqrt{(n_1,q)}\Big(1+\frac{r}{n_1}\Big). \tag{8.22}$$

We estimate the sum over n_1 and n_2 as follows:

$$\begin{aligned}\sum_{n_1|qr}\sum_{n_2\ll\frac{q^3rk^\varepsilon}{n_1^2N}}&\frac{|\lambda_\pi(n_1,n_2)|}{n_1n_2}(n_1^2n_2)^{1/4}\sqrt{(n_1,q)}\Big(1+\frac{r}{n_1}\Big)\\ &\ll \sum_{n_1|qr}\sum_{n_2\ll\frac{q^3rk^\varepsilon}{n_1^2N}}|\lambda_\pi(n_1,n_2)|\frac{r}{\sqrt{n_2}}\\ &\ll \Big(\sum\sum_{n_1^2n_2\ll k^\varepsilon q^3r/N}|\lambda_\pi(n_1,n_2)|^2\Big)^{1/2}\Big(\sum_{n_1|qr}\sum_{n_2=1}^{\infty}\frac{r^2}{n_2}\Big)^{1/2} \ll \frac{q^{3/2}r^{3/2}}{\sqrt{N}}.\end{aligned}$$

Hence the contribution of the terms $n_1^2 n_2 N/(q^3 r) \ll k^\varepsilon$ to $S_r(N)$ is dominated by

$$k^\varepsilon \sum_{1\le q\le Q} \frac{NM_0^{3/4}}{Q^2 T r^{1/4}} \frac{q^{3/2} r^{3/2}}{\sqrt{N}} \ll \sqrt{N}\, k^{1+2\eta+3\eta/8+\varepsilon}, \tag{8.23}$$

where we have used $M_0 \ll k^{2+\varepsilon}/T$ and $Nr^2 \ll k^{3+\varepsilon}$.

9. Conclusion: Proof of Theorem 1

We now put together the bounds from Lemmas 16, 18, 19 and (8.23) to get

$$\frac{S_r(N)}{N^{1/2} k^{3/2+\varepsilon}} \ll k^{-1/2+2\eta+3\eta/8} + r^{1/2} k^{-\eta/2} + r^{1/2} \frac{k^{3/2-\eta/2}}{N^{1/2}} + k^{-1/6+3\eta/4}.$$

Using $k^{3-\theta} \ll Nr^2 \ll k^{3+\varepsilon}$ and $r \ll k^\theta$, we further get

$$\frac{S_r(N)}{N^{1/2} k^{3/2+\varepsilon}} \ll k^{-1/2+2\eta+3\eta/8} + k^{\theta/2-\eta/2} + k^{2\theta-\eta/2} + k^{-1/6+3\eta/4}.$$

Hence to get subconvexity, we need all of the above exponents to be negative. So the first and the third term give $4/19 > \eta > 4\theta$, and consequently the third and the fourth terms dominate the rest. Thus the above bound reduces to

$$\frac{S_r(N)}{N^{1/2} k^{3/2+\varepsilon}} \ll k^{2\theta-\eta/2} + k^{-1/6+3\eta/4}.$$

The optimal choice for η is $\eta = 8\theta/5 + 2/15$. On plugging this in Lemma 11, we get

$$L(1/2, \pi \times f) \ll k^{3/2+6\theta/5-1/15+\varepsilon} + k^{3/2-\theta/2+\varepsilon},$$

and with the optimal choice $\theta = 2/51$, we obtain the bound given in Theorem 1.

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