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Centro-affine differential geometry and the log-Minkowski problem

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Abstract. We interpret the log-Brunn–Minkowski conjecture of Böröczky–Lutwak–Yang–Zhang as a spectral problem in centro-affine differential geometry. In particular, we show that the Hilbert–Brunn–Minkowski operator coincides with the centro-affine Laplacian, thus obtaining a new avenue for tackling the conjecture using insights from affine differential geometry. As every strongly convex hypersurface in $\mathbb{R}^n$ is a centro-affine unit sphere, it has constant centro-affine Ricci curvature equal to $n-2$, in stark contrast to the standard weighted Ricci curvature of the associated metric-measure space, which will in general be negative. In particular, we may use the classical argument of Lichnerowicz and a centro-affine Bochner formula to give a new proof of the Brunn–Minkowski inequality. For origin-symmetric convex bodies enjoying fairly generous curvature pinching bounds (improving with dimension), we are able to show global uniqueness in the L^p- and log-Minkowski problems, as well as the corresponding global L^p- and log-Minkowski conjectured inequalities. As a consequence, we resolve the *isomorphic* version of the log-Minkowski problem: for any origin-symmetric convex body $\bar{K}$ in $\mathbb{R}^n$, there exists an origin-symmetric convex body K with $\bar{K} \subset K \subset 8\bar{K}$ such that K satisfies the log-Minkowski conjectured inequality, and such that K is uniquely determined by its cone-volume measure V_K. If $\bar{K}$ is not extremely far from a Euclidean ball to begin with, an analogous *isometric* result, where 8 is replaced by $1+\varepsilon$, is obtained as well.

Keywords. Uniqueness in L^p-Minkowski problem, log-Brunn–Minkowski inequality, centro-affine differential geometry, Hilbert–Brunn–Minkowski operator

1. Introduction

A central question in contemporary Brunn–Minkowski theory is that of existence and uniqueness in the L^p-Minkowski problem for $p \in (-\infty, 1)$: given a finite non-negative Borel measure μ on the Euclidean unit sphere $\mathbb{S}^* = S^{n-1}$, determine conditions on μ which ensure the existence and/or uniqueness of a convex body K in $\mathbb{R}^n$ such that

$$S_p K := h_K^{1-p} S_K = \mu. \tag{1.1}$$

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Here h_K and S_K denote the support function and surface-area measure of K, respectively – we refer to Section 2 for the standard definitions. When $h_K \in C^2(\mathbb{S}^*)$,

$$S_K = \det(D^2 h_K)\mathfrak{m},$$

where $\mathfrak{m}$ is the induced Lebesgue measure on $\mathbb{S}^*$, $D^2 h_K = {}^{\mathbb{S}^*}\nabla^2 h_K + h_K \delta^{\mathbb{S}^*}$ and ${}^{\mathbb{S}^*}\nabla$ is the Levi-Civita connection on $\mathbb{S}^*$ with its standard Riemannian metric $\delta^{\mathbb{S}^*}$. Consequently, (1.1) is a Monge–Ampère-type equation. It describes self-similar solutions to the (anisotropic) α-power-of-Gauss-curvature flow for $\alpha = \frac{1}{1-p}$ [4–6, 8, 22, 33, 35, 102, 103].

The case $p = 1$ above corresponds to the classical Minkowski problem of finding a convex body with prescribed surface-area measure; when μ is not concentrated on any hemisphere and its barycenter is at the origin, existence and uniqueness (up to translation) of K were established by Minkowski, Alexandrov and Fenchel–Jessen (see [96]), and regularity of K was studied by Lewy [64], Nirenberg [84], Cheng–Yau [31], Pogorelov [90], Caffarelli [23, 24] and many others. The extension to general p was put forth and publicized by E. Lutwak [73] as an L^p-analog of the Minkowski problem for the L^p surface-area measure $S_p K = h_K^{1-p} S_K$ which he introduced. Existence and uniqueness in the class of origin-symmetric convex bodies, when the measure μ is even and not concentrated in a hemisphere, was established for $n \neq p > 1$ by Lutwak [73] and for $p = n$ by Lutwak–Yang–Zhang [78]. A key tool in the range $p \geq 1$ is the prolific L^p-Brunn–Minkowski theory, initiated by Lutwak [73, 74] following Firey [41], and developed by Lutwak–Yang–Zhang (e.g. [76, 77, 79]) and others, which extends the classical $p = 1$ case. Further existence, uniqueness and regularity results in the range $p > 1$ under various assumptions on μ were obtained in [34, 44, 48, 51, 75, 108].

The case $p < 1$ turns out to be more challenging because of the lack of an appropriate L^p-Brunn–Minkowski theory. Existence, (non-)uniqueness and regularity under various conditions on μ were studied by numerous authors when $p < 1$ (from either side of the critical exponent $p = -n$), especially after the important work by Chou–Wang [34]; see e.g. [10, 11, 15, 16, 27–29, 45, 53, 70, 82, 98–100, 105–107, 109]. The case $p = 0$ is of particular importance as it corresponds to the *log-Minkowski problem* for the cone-volume measure

$$V_K := \frac{1}{n} h_K S_K = \frac{1}{n} S_0 K,$$

described next. Note that V_K is obtained as the push-forward of the cone measure on ∂K onto $\mathbb{S}^*$ via the Gauss map, and that the total mass of V_K is $V(K)$, the volume of K. Being a self-similar solution to the isotropic Gauss curvature flow, the case $p = 0$ and $\mu = \mathfrak{m}$ of (1.1) describes the ultimate fate of a worn stone in a model proposed by Firey [42] and further studied in [5, 8, 22, 33, 57].

Let $\mathcal{K}$ denote the collection of convex bodies in $\mathbb{R}^n$ containing the origin in their interior, and let $\mathcal{K}_e$ denote the subset of origin-symmetric elements. In [19], Böröczky–Lutwak–Yang–Zhang showed that an *even* measure μ is the cone-volume measure V_K of an *origin-symmetric* convex body $K \in \mathcal{K}_e$ if and only if it satisfies a certain subspace concentration condition, thereby completely resolving the existence part of the *even* log-Minkowski problem. As put forth by Böröczky–Lutwak–Yang–Zhang in their influential work [18, 19] and further developed in [61], the uniqueness question is intimately related to the validity of a conjectured L^0- (or log-)Brunn–Minkowski inequality for origin-symmetric convex bodies $K, L \in \mathcal{K}_e$, which would constitute a remarkable strengthening of the classical $p = 1$ case. The restriction to origin-symmetric bodies is natural, and necessitated by the fact that no L^p-Brunn–Minkowski inequality nor uniqueness in the L^p-Minkowski problem can hold for general convex bodies when $p < 1$ [7, 28, 29, 34, 45, 52, 61, 66, 67, 82, 99].

The following equivalence may be shown by following the arguments of [18, 19] (see Section 2.4 for a more general statement and further details). We denote by $\mathcal{K}^{2,\alpha}_{+,e}$ the subset of $\mathcal{K}_e$ having $C^{2,\alpha}$-smooth boundary and strictly positive curvature, and refer to Section 2 for the definition of the L^p-Minkowski sum $(1-\lambda)\cdot K +_p \lambda \cdot L$.

Theorem 1.1 (after Böröczky–Lutwak–Yang–Zhang). *The following statements are equivalent for any fixed $p \in (-n, 1)$:*

(1) *For any $q \in (p, 1)$, uniqueness holds in the even L^q-Minkowski problem for any $K \in \mathcal{K}^{2,\alpha}_{+,e}$:*
$$\forall L \in \mathcal{K}_e, \quad S_q L = S_q K \implies L = K. \tag{1.2}$$

(2) *The even L^p-Brunn–Minkowski inequality holds:*
$$\forall \lambda \in [0,1] \quad \forall K, L \in \mathcal{K}_e,$$
$$V((1-\lambda)\cdot K +_p \lambda \cdot L) \geq \left((1-\lambda)V(K)^{\frac{p}{n}} + \lambda V(L)^{\frac{p}{n}}\right)^{\frac{n}{p}}. \tag{1.3}$$

The case $p = 0$, called the even log-Brunn–Minkowski inequality, is interpreted in the limiting sense as
$$V((1-\lambda)\cdot K +_0 \lambda \cdot L) \geq V(K)^{1-\lambda} V(L)^{\lambda}.$$

(3) *The even L^p-Minkowski inequality holds:*
$$\forall K, L \in \mathcal{K}_e \quad \frac{1}{p}\int_{\mathbb{S}^*} h_L^p \, dS_p K \geq \frac{n}{p} V(K)^{1-\frac{p}{n}} V(L)^{\frac{p}{n}}. \tag{1.4}$$

The case $p = 0$, called the even log-Minkowski inequality, is interpreted in the limiting sense as:
$$\frac{1}{V(K)}\int_{\mathbb{S}^*} \log \frac{h_L}{h_K} \, dV_K \geq \frac{1}{n} \log \frac{V(L)}{V(K)}.$$

Using Jensen's inequality in formulation (1.4) (or (1.3)), it is immediate to check that the above (equivalent) statements become stronger as p decreases (i.e. that their validity for p_1 implies their validity for p_2 whenever $p_1 < p_2 < 1$).

Conjecture (Böröczky–Lutwak–Yang–Zhang, "Even log-Brunn–Minkowski Conjecture"). *Any* (*and hence all*) *of the above statements hold for origin-symmetric convex bodies in the "logarithmic case" $p = 0$* (*and hence for all $p \in [0, 1)$ as well*).

A confirmation of this conjecture would constitute a dramatic improvement over the classical Brunn–Minkowski theory for the subfamily of origin-symmetric convex bodies, which had gone unnoticed for over a century. The importance of this conjecture to the Brunn–Minkowski theory for general measures has been further expounded in several subsequent works [46, 57, 58, 69, 93, 94]; see below for additional information and partial results. The even log-Brunn–Minkowski conjecture (also called the log-Minkowski conjecture) is known to hold in the plane [18] (see also [43, 80, 91, 98, 104]), but remains open in general for $n \geq 3$.

It is easy to show that (1.3) or (1.4) are false for any $p < 0$ (see e.g. [61]). Moreover, uniqueness in (1.2) does not hold for general $K, L \in \mathcal{K}_e$ and $q = 0$, as may be verified by testing two different centered parallelepipeds with appropriately chosen parallel facets. The latter example is known to be the only exception to uniqueness in the log-Minkowski problem in the plane [18], but in higher dimension there are additional conjectured (non-smooth) cases of equality in (1.2) when $q = 0$ [14, 17]. In particular, when $K \in \mathcal{K}^{2,\alpha}_{+,e}$, the (somewhat stronger) conjecture is that (1.2) should hold not only for $q \in (0, 1)$ but for $q = 0$ as well – we will refer to this as the "uniqueness in the even log-Minkowski problem" conjecture.

1.1. Main results

We now turn to describe the main results of this work. As our first main result, we obtain the following uniqueness result for the even L^p-Minkowski problem, with corresponding even L^p-Minkowski inequality, under a fairly generous curvature bound assumption on K. We denote by $\mathrm{II}^{\partial K}$ the second fundamental form on $\partial K \subset \mathbb{R}^n$.

Theorem 1.2. *Let $K \in \mathcal{K}^{2,\alpha}_{+,e}$ have a centro-affine image $\tilde{K}$ such that the following curvature pinching bounds hold:*

$$\frac{1}{R}|X|^2 \leq \mathrm{II}^{\partial \tilde{K}}(X, X) \leq \frac{1}{r}|X|^2 \quad \forall X \in T\partial \tilde{K}, \tag{1.5}$$

for some $R \geq r > 0$; in other words, all radii of curvature of $\tilde{K}$ are bounded between r and R. Then for any p with

$$3 - \frac{n-1}{2}\frac{r^2}{R^2} < p < 1, \tag{1.6}$$

the even L^p-Minkowski problem for K has a unique solution:

$$\forall L \in \mathcal{K}_e, \quad S_p L = S_p K \implies L = K, \tag{1.7}$$

and the even L^p-Minkowski inequality holds:

$$\forall L \in \mathcal{K}_e, \quad \frac{1}{p}\int_{\mathbb{S}^*} h_L^p \, dS_p K \geq \frac{n}{p} V(K)^{1-\frac{p}{n}} V(L)^{\frac{p}{n}}, \tag{1.8}$$

with equality if and only if $L = cK$ for some $c > 0$.

The term “centro-affine image” above is synonymous with “(non-degenerate) linear image”, but we prefer to use the former to emphasize the centro-affine differential geometry underlying our results. Of course, the case $p = 0$ above is interpreted in the limiting sense as follows:

Corollary 1.3. *With the same conditions as above, whenever*

$$\frac{R^2}{r^2} < \frac{n-1}{6},$$

the even log-Minkowski problem for K has a unique solution:

$$\forall L \in \mathcal{K}_e, \quad V_L = V_K \implies L = K, \tag{1.9}$$

and the even log-Minkowski inequality holds:

$$\forall L \in \mathcal{K}_e, \quad \frac{1}{V(K)} \int_{\mathbb{S}^*} \log \frac{h_L}{h_K} \, dV_K \geq \frac{1}{n} \log \frac{V(L)}{V(K)}, \tag{1.10}$$

with equality if and only if $L = cK$ for some $c > 0$.

Remark 1.4. It is well-known that the measures $S_p K$ and in particular V_K are weakly continuous (i.e. in duality with $C(\mathbb{S}^*)$) with respect to convergence of K in the Hausdorff metric [96, pp. 212–215]. Consequently, for the purpose of deducing (1.8) or (1.10) without characterization of equality cases, it is enough to assume that $K \in \mathcal{K}_e$ can be approximated in the Hausdorff metric by $K_i \in \mathcal{K}^{2,\alpha}_{+,e}$ as above.

In fact, we will prove a strengthening of Theorem 1.2, involving two-sided bounds on $\mathrm{II}^{\partial \tilde{K}} / h_{\tilde{K}}(\mathrm{n}^{\partial \tilde{K}})$, and producing a linear dependence on these bounds (instead of quadratic as above) – see Theorem 6.4. Using this strengthened version, we can give a positive answer to the *isomorphic versions* of the uniqueness question for the even log-Minkowski problem and the even log-Minkowski inequality (curiously, we do not know how to establish these isomorphic results using the weaker formulation of Theorem 1.2 – see Remark 7.2). Our *isomorphic* nomenclature stems from Banach-space theory, where two Banach spaces are called isomorphic if, up to a linear bijection, their corresponding norms are equivalent up to constants. To better quantify this in the finite-dimensional geometric context, it is convenient to introduce the following distances for pairs of origin-symmetric convex bodies $K, L \in \mathcal{K}_e$ – *the geometric distance*

$$d_G(K, L) := \inf \left\{ ab > 0 \, ; \, \frac{1}{b} K \subset L \subset aK \right\},$$

and the *Banach–Mazur distance*

$$d_{BM}(K, L) := \inf \{ d_G(K, T(L)) \, ; \, T \in GL_n \}.$$

Clearly $d_G(K, L), d_{BM}(K, L) \geq 1$. Note that the classical John’s theorem [96, Section 10.12] asserts that $d_{BM}(K, B_2^n) \leq \sqrt{n}$ for any $K \in \mathcal{K}_e$, where B_2^n denotes the Euclidean unit ball in $\mathbb{R}^n$.

Theorem 1.5 (Isomorphic L^p-Minkowski). *Let $\bar{K} \in \mathcal{K}_e$, and denote $D := d_{BM}(\bar{K}, B_2^n)$. Given $\gamma > 0$, define*

$$p_{\gamma,D} := \frac{7}{3} - \frac{n-1}{24} \frac{\gamma^2}{D^2}. \tag{1.11}$$

Then for any $8 \leq \gamma \leq D/2$, there exists $\tilde{K} \in \mathcal{K}^{\infty}_{+,e}$ such that

$$d_G(\bar{K}, \tilde{K}) \leq \gamma,$$

and such that for any $p \in (p_{\gamma,D}, 1)$ and for any $T \in GL_n$, the even L^p-Minkowski problem for $K = T(\tilde{K})$ has a unique solution (1.7), and the even L^p-Minkowski inequality (1.8) holds for K, with equality if and only if $L = cK$ for some $c > 0$.

The above theorem agrees with the intuition that the smaller the Banach–Mazur distance D from $\bar{K}$ to B_2^n is, the smaller γ we can select (controlling the distance between the original $\bar{K}$ and the modified $\tilde{K}$) in order to hit a particular value of $p_{\gamma,D}$. Note that for $\gamma = D$, we can simply select $\tilde{K}$ to be the John ellipsoid $\mathcal{E}$ of $\bar{K}$ (so that $d_G(\bar{K}, \mathcal{E}) = D$), for which it is known (see the next subsection) that the above conclusion holds with $p_{\gamma,D} = -n$. At the other extreme, when $\bar{K} = [-1, 1]^n$ (so that $D = \sqrt{n}$), it follows from the results of [61] that for any fixed $p < 0$ there is no uniqueness in the even L^p-Minkowski problem for any $\tilde{K}$ with $\gamma = d_G(\bar{K}, \tilde{K})$ close enough to 1, so one cannot expect an estimate for $p_{1,\sqrt{n}}$ better than 0. In this sense, formula (1.11) for $p_{\gamma,D}$ captures the correct order of magnitude (proportional to $-n$ and to 1) when D/γ is of the order of 1 and $\sqrt{n}$, respectively.

Of particular interest is the logarithmic case. Specializing to $p = 0$ above, we resolve the isomorphic version of the conjecture regarding uniqueness in the even log-Minkowski problem:

Corollary 1.6 (Isomorphic log-Minkowski). *For any $\bar{K} \in \mathcal{K}_e$, there exists $\tilde{K} \in \mathcal{K}^{\infty}_{+,e}$ with*

$$d_G(\bar{K}, \tilde{K}) \leq 8,$$

such that for any $T \in GL_n$, the even log-Minkowski problem for $K = T(\tilde{K})$ has a unique solution (1.9), and the even log-Minkowski inequality (1.10) holds for K, with equality if and only if $L = cK$ for some $c > 0$.

This is an immediate corollary of Theorem 1.5 since $D \leq \sqrt{n}$ by John's theorem, and so when $n \leq 64$ we can simply use $\tilde{K} = \mathcal{E}$, John's ellipsoid (for which $D = d_G(\bar{K}, \mathcal{E}) \leq \sqrt{n} \leq 8$), and when $n \geq 65$ formula (1.11) ensures that $p_{8,\sqrt{n}} < 0$.

The constant 8 obtained in the isomorphic version above is the worst case behavior for a general $\bar{K} \in \mathcal{K}_e$, when $D = d_{BM}(\bar{K}, B_2^n)$ may be as large as John's upper bound $\sqrt{n}$. However, whenever $D \ll \sqrt{n}$, a slightly finer analysis yields an *isometric* version of the above results, where one only perturbs $\bar{K}$ by at most $\gamma = 1 + \varepsilon$. We only state the $p = 0$ case below:

Theorem 1.7 (Isometric log-Minkowski). *Let $\bar{K} \in \mathcal{K}_e$, and denote $D := d_{BM}(\bar{K}, B_2^n)$. There exists $\tilde{K} \in \mathcal{K}^{\infty}_{+,e}$ satisfying the conclusion of Corollary 1.6 such that*

$$d_G(\bar{K}, \tilde{K}) \leq 1 + C\frac{\sqrt{D}}{\sqrt[4]{n}},$$

where $C > 1$ is a universal constant.

1.2. Comparison with previous work

As already mentioned, the validity of the log-Minkowski inequality (1.10) for all $K \in \mathcal{K}_e$, including characterization of its equality cases, as well as the uniqueness in the even log-Minkowski problem (1.9) for $K \in \mathcal{K}_e$ which is not a parallelogram, was established when $n = 2$ by Böröczky–Lutwak–Yang–Zhang [18] (see also [80,91,104] for alternative derivations).

In our previous joint work with A. Kolesnikov [61], following the work of [37], we embarked on a systematic study of the validity of the *local* L^p-Brunn–Minkowski inequality for origin-symmetric convex bodies and $p < 1$; by "local" we mean on an infinitesimal scale, or equivalently, for pairs of bodies which are close enough to each other in an appropriate sense. To that end, we introduced an elliptic second-order differential operator on $C^2(\mathbb{S}^*)$, called the Hilbert–Brunn–Minkowski operator Δ_K, defined for $K \in \mathcal{K}^2_+$, which up to gauge transformations coincides with the operator introduced by Hilbert in his proof of the Brunn–Minkowski inequality (see [13]). Here $\mathcal{K}^m_+$ denotes the subset of $\mathcal{K}$ consisting of elements having C^m-smooth boundary and strictly positive curvature, and $\mathcal{K}^m_{+,e}$ denotes the subset of origin-symmetric elements. The operator $-\Delta_K$ is symmetric and positive semi-definite on $L^2(V_K)$, admitting a unique self-adjoint extension with compact resolvent. Its spectrum thus consists of a countable sequence of eigenvalues of finite multiplicity starting with 0 and tending to ∞. It was shown in [61] that Δ_K enjoys a remarkable centro-affine equivariance property, stating that for any $T \in GL_n$, $\Delta_{T(K)}$ and Δ_K are conjugates modulo an isometry of Hilbert spaces; in particular, the spectrum $\sigma(-\Delta_{T(K)})$ is the same for all T. One way to define $(\Delta_K + (n-p)\mathrm{Id})z$ is by linearizing $\log(h_K^{1-p}\det(D^2 h_K))$ appearing in the left-hand-side of (1.1) under a logarithmic variation $h_{K_\varepsilon} = h_K(1 + \varepsilon z)$. Consequently, understanding whether $n - p$ is in the spectrum of $-\Delta_K$ is of fundamental importance to the uniqueness question in the L^p-Minkowski problem.

It was Hilbert who realized that the classical Brunn–Minkowski inequality (the case $p = 1$) [96] is equivalent to the statement that $\sigma(-\Delta_K) \cap (0, n-1) = \emptyset$, and proved that indeed $\lambda_1(-\Delta_K) = n - 1$ where λ_1 denotes the first non-zero eigenvalue [13]. Similarly, given $K \in \mathcal{K}^2_{+,e}$, we denote the first non-zero *even* eigenvalue of $-\Delta_K$ (corresponding to an *even* eigenfunction) by $\lambda_{1,e}(-\Delta_K)$. It was shown in [61] that for any $p < 1$, the statement $\lambda_{1,e}(-\Delta_K) \geq n - p$ is equivalent to the *local* L^p-Brunn–Minkowski inequality for origin-symmetric perturbations of K, and implies the *local* uniqueness for the even L^q-Minkowski problem for any $q > p$. The fact that a *local* verification of these problems is enough to imply the *global* one was subsequently shown by Chen–Huang–Li–Liu [27] for the uniqueness of the L^p-Minkowski problem and by Putterman [91] for the L^p-Brunn–Minkowski inequality. We conjecture that $\lambda_{1,e}(-\Delta_K) > n$ for all $K \in \mathcal{K}^2_{+,e}$, which would confirm for all $p \in [0,1)$ the L^p-Brunn–Minkowski and L^p-Minkowski inequalities on $\mathcal{K}_e$ and the uniqueness in the L^p-Minkowski problem on $\mathcal{K}^2_{+,e}$.

Our main result in [61] was showing that $\lambda_{1,e}(-\Delta_K) \geq n - p_0$ for $p_0 := 1 - \frac{c}{n^{3/2}}$ and all $K \in \mathcal{K}^2_{+,e}$, where $c > 0$ is a universal constant, yielding local uniqueness in the even L^p-Minkowski problem for all $p \in (p_0, 1)$. In [27], Chen–Huang–Li–Liu established their local-to-global principle for the uniqueness question, and deduced (1.7) and (1.8) for all $K \in \mathcal{K}^{2,\alpha}_{+,e}$ and $p \in (p_0, 1)$. In fact, thanks to recent progress on the KLS conjecture due to Y. Chen [30], our estimate from [61, Corollary 6.8 and Theorem 6.9] immediately improves to $p_0 = 1 - \frac{c_\varepsilon}{n^{1+\varepsilon}}$ for any $\varepsilon > 0$, which together with the results of [27] yields the presently best known range of p's for which (1.7) and (1.8) are known to hold. Furthermore, Chen–Huang–Li–Liu established in [27] the validity of (1.9) and (1.10) for the class $H_{\varepsilon_n} := \{K \in \mathcal{K}^2_{+,e} \,;\, \|h_K - 1\|_{L^\infty} \leq \varepsilon_n\}$ and a sufficiently small $\varepsilon_n > 0$, employing a corresponding local uniqueness result for H_{ε_n} established in [61]; in fact, the same argument applies to any centro-affine image $K = T(\tilde{K})$, $T \in GL_n$ and $\tilde{K} \in H_{\varepsilon_n}$.

As already mentioned, it is known that for any $p < 0$ there exist $K \in \mathcal{K}^2_{+,e}$ for which (1.7) and (1.8) are false (see [82] for additional information, and [7, 34, 45, 52, 61, 66, 67] for previously known non-uniqueness results). Consequently, the logarithmic case $p = 0$ is precisely the conjectured threshold between the range $p \in [0,1)$ where (1.7) and (1.8) are expected to hold for all $K \in \mathcal{K}^2_{+,e}$, and the range $p < 0$ where it is known that they fail in general.

However, for a specific $K \in \mathcal{K}^2_{+,e}$, it is certainly possible for (1.7) and (1.8) to hold with $p < 0$. For example, it is possible to show that these statements hold for all centered ellipsoids K and for all $p \in (-n, 1)$. Even in the simplest case when $K = B_2^n$, the Euclidean unit ball, uniqueness in the L^p-Minkowski problem (1.7) was until recently a major open problem in the latter range of p's. As already eluded to above, this particular case is especially important because it describes self-similar solutions to the *isotropic* α-power-of-Gauss-curvature flow (for $\alpha = \frac{1}{1-p}$), a model proposed by Firey [42] for $\alpha = 1$ ($p = 0$). In the general anisotropic model, $x : \mathbb{S}^* \times [0,T) \to \mathbb{R}^n$ evolves according to

$$\frac{\partial x}{\partial t} = -(\rho(\mathfrak{n}_x^{\partial L_t})\kappa_x^{\partial L_t})^\alpha \mathfrak{n}_x^{\partial L_t}, \quad \rho := \frac{dS_p K}{d\mathfrak{m}},$$

where $\mathfrak{n}^{\partial L_t}$ is the outer unit normal to $\partial L_t := x(\mathbb{S}^*, t)$ and $\kappa^{\partial L_t}$ is the corresponding Gauss curvature. Following contributions in [5,8,33,35,42,47], uniqueness in (1.7) for the general isotropic case $K = B_2^n$ (without origin-symmetry, only assuming $L \in \mathcal{K}$) in the full range $p \in (-n, 1)$ was resolved by Brendle–Choi–Daskalopoulos [22]. In the origin-symmetric case, an extension of their uniqueness result from B_2^n to arbitrary centered ellipsoids $\mathcal{E}$ may be shown by following the arguments of [27, 61] – see Remark 2.7. Our uniqueness result of Theorem 1.2 thus extends the results of [22] in the origin-symmetric setting, from Euclidean balls (the isotropic case) to centro-affine images of convex bodies K enjoying a curvature pinching condition (pinched anisotropic case). Specializing to ellipsoids $\mathcal{E}$, while our general formula (1.6) does not recover the sharp exponent $p > -n$, we obtain the right order of magnitude ($p > 3 - \frac{n-1}{2}$). Note that uniqueness no longer holds below the critical exponent $p = -n$ due to the SL_n equivariance of the centro-affine Gauss curvature $h_K^{1+n} \det(D^2 h_K)$ [34, 101]; in particular, the centro-affine Gauss curvature of any centered ellipsoid $\mathcal{E}$ in $\mathbb{R}^n$ is constant and depends only on its volume: $S_{-n}\mathcal{E} = c_n V(\mathcal{E})^2 \mathfrak{m}$.

Strictly speaking, we are not aware of any other results establishing (1.7) or (1.8) for a given $K \in \mathcal{K}_e$, $p < 1$ and *all* $L \in \mathcal{K}_e$. Various additional results establish (1.7), (1.8) or the corresponding L^p-Brunn–Minkowski inequality for particular pairs of convex bodies $K, L \in \mathcal{K}_e$, which are typically perturbations of a well-understood example, or which enjoy certain symmetries [14,17,36,37,46,61,92–94]. Of particular historical significance was the case when $K, L \in \mathcal{K}^2_{+,e}$ are perturbations of B_2^n (C^2-perturbations of particular form in [36,37], and centro-affine images of general C^2- and even C^0-perturbations of B_2^n in [61]). However, the extent of these admissible C^2-perturbations was non-explicit and deteriorated with the dimension n. In contrast, note that the C^2-perturbations allowed by Theorem 1.2 and Corollary 1.3 are entirely explicit and in fact improve with the dimension – e.g. Corollary 1.3 applies to any $K \in \mathcal{K}^{2,\alpha}_{+,e}$ with $R^2/r^2 < \frac{n-1}{6}$.

As for the isomorphic and isometric results of Theorems 1.5 and 1.7, we are not aware of any prior results of this nature regarding the even L^p-Minkowski problem. The best comparison comes from a totally different yet equally fundamental problem posed by J. Bourgain [20] regarding a volumetric property of convex bodies – the Slicing Problem (see [21, 83]). The Slicing Problem has been confirmed for numerous families of convex bodies, and there has been recent dramatic advancement in the best known estimates for general convex bodies (obtained by combining the recent results of Chen [30] on the Kannan–Lovász–Simonovits conjecture [54] with the results of Eldan–Klartag [39]). While the Slicing Problem remains open in general, the *isomorphic version* of the Slicing Problem was fully resolved by B. Klartag [55]. Our results in Corollary 1.6 and Theorem 1.7 can be seen as the log-Minkowski analogues of Klartag's results for the Slicing Problem (despite the two problems being very different, and having no apparent relation between our corresponding proofs). Note that we are not aware of an analogous result for the (also spectral) KLS conjecture (apart from an isometric quantitative stability result established in [81]).

1.3. *Centro-affine differential geometry*

Perhaps more important than our main results described above is our rediscovery of the significance of affine differential geometry to the Brunn–Minkowski theory, and our apparently new observation about the crucial role played by the centro-affine normalization (we refer to [85, 96] for more background, and to Section 3 for an introduction to affine differential geometry). Historically, the Brunn–Minkowski theory of convex sets was initiated by Brunn and subsequently Minkowski towards the end of the 19th century, and further developed by Blaschke, Berwald, Kubota, Favard, Alexandrov, Bonnesen, Fenchel and others in the first third of the 20th century (a singular but especially relevant contribution to the theory was also made by Hilbert at the turn of the century). In parallel, the origins of affine differential geometry are often attributed to the works of Tzitzéica [101] circa 1908, following the axiomatization of affine geometry in Felix Klein's Erlangen program. A systematic study of affine differential geometry was subsequently undertaken between 1916 and 1923 by Blaschke in collaboration with Pick, Radon, Berwald and Thomsen, among others, and this was followed up in the work by Cartan, Kubota, Süss, Ślebodziński, Salkowski and others in the late 1920s and 1930s. It is apparent from the large overlap in mathematicians working on both theories during those formative years that these theories interacted quite significantly.

However, after this initial period, each theory developed along its own respective trajectory, with little to no overlap with the other. Some notable exceptions include the study of the affine surface area and affine isoperimetric inequality, initiated by Blaschke and further developed and extended by Deicke, Hug, Leichtweiss, Ludwig, Lutwak, Meyer, Petty, Reitzner, Santaló, Schütt, Werner, Ye and others (see [49, 50, 63, 71, 72, 74] and the references therein), the early work by R. Schneider in the 1960s on global affine differential geometry [95], and a more recent work of Klartag on convex affine hemispheres [56]; all of these works pertain to the affine differential geometry obtained by equipping a convex set with *Blaschke's equiaffine normal*, which is equivariant with respect to volume-preserving affine transformations. The equiaffine *normalization* is the most prevalent one used in affine differential geometry, and some of the highlights of the resulting theory include the works by Calabi (see [25, 26]) and Cheng–Yau [32] on classification of equiaffine spheres. However, we will make the case in this work that a much more natural normalization for studying the Brunn–Minkowski theory is the *centro-affine normalization*, which is equivariant with respect to centro-affine transformations (fixing the origin). In contrast to Blaschke's equiaffine normalization, where the classification of non-compact equiaffine spheres has proven to be a major challenge, the centro-affine normalization is in a sense trivial, since the boundary of every $K \in \mathcal{K}^2_+$ is a centro-affine sphere. However, it is precisely this property that makes the centro-affine normalization so useful for our purposes.

Before describing the relevance and usefulness of the centro-affine normalization to our setting, one should note that it has already been utilized in convex geometry through the notion of centro-affine surface area $\Omega_n(K)$, which coincides with the L^p-affine surface area for $p = n$ [49, 50, 74] – see Section 4.6. As is well-known, the centro-affine surface area is self-dual $\Omega_n(K) = \Omega_n(K^\circ)$, and furthermore, the centro-affine metric of a hypersurface is isometric to that of the polar (or dual) hypersurface – see Sections 4.2 and 4.6 for additional self-duality properties enjoyed by the centro-affine normalization, and for suggestions regarding further research in this direction. In addition, the critical case $p = -n$ of the L^p-Minkowski problem (1.1) was interpreted in [34] as the Minkowski problem for the centro-affine Gauss curvature. However, we are not aware of any other previously known connections between the centro-affine normalization and the Brunn–Minkowski inequality or any its variants – this appears to be a novel observation, which is the main insight we would like to put forth and emphasize in this work.

Our first observation is that the Hilbert–Brunn–Minkowski operator Δ_K precisely coincides with the centro-affine Laplacian associated with $K \in \mathcal{K}^2_+$. We provide all the relevant details in Sections 3 and 4, and for now only explain what the latter notion entails. Any selection of a normal vector field on ∂K defines a Riemannian metric g_K and a torsion-free affine connection ∇_K which in general is *not* the Levi-Civita connection for g_K. The Laplacian associated with the given normalization $\Delta^{\nabla_K, g_K} f$ is then defined as the connection divergence div^{∇_K} of the metric gradient $\mathrm{grad}_{g_K} f$ (the vector field obtained by identification with the covector df via the metric g_K). In addition, any (relative) normalization produces a volume measure ν_K, which in general does not coincide with the Riemannian volume measure ν_{g_K}, but is parallel with respect to ∇_K ($\nabla_K \nu_K = 0$); this allows us to integrate by parts:

$$\int (-\Delta^{\nabla_K, g_K} f) h \, d\nu_K = \int g_K(\mathrm{grad}_{g_K} f, \mathrm{grad}_{g_K} h) \, d\nu_K = \int f(-\Delta^{\nabla_K, g_K} h) \, d\nu_K. \tag{1.12}$$

It turns out that for the centro-affine normalization of ∂K, defining x itself to be the normal to ∂K at $x \in \partial K$, the above objects boil down to some familiar ones from the Brunn–Minkowski theory (after parametrizing ∂K on $\mathbb{S}^*$ via the Gauss map): the centro-affine volume measure ν_K coincides (up to constants) with the cone-volume measure V_K, the Riemannian volume measure ν_{g_K} for the centro-affine metric g_K coincides (up to constants) with the centro-affine surface area measure $\Omega_{n,K}$, and the centro-affine Laplacian Δ^{∇_K, g_K} coincides with the Hilbert–Brunn–Minkowski operator Δ_K.

In [61, Section 5.1], we had originally (implicitly) identified the metric $g_K = \frac{D^2 h_K}{h_K} > 0$ on $\mathbb{S}^*$ by starting with the Hilbert–Brunn–Minkowski operator Δ_K, performing intergration by parts in (1.12) with respect to V_K and computing the corresponding Dirichlet form, thereby interpreting Δ_K as the weighted Laplacian on $(\mathbb{S}^*, g_K, V_K)$. However, it was not entirely clear whether the choice of measure V_K and thus the construction of the metric g_K were canonical, or what was the direct relation between these two objects; we now finally have a satisfactory answer coming from the centro-affine normalization. In addition, this gives a satisfactory explanation for the centro-affine equivariance property of the Hilbert–Brunn–Minkowski operator, originally observed in [61, Section 5.2] following a lengthy computation. Furthermore, we deduce the centro-affine equivariance of all of the above differential objects (g_K, ∇_K, ν_K, etc.), as well as their behavior under duality. In particular, we deduce (the known fact) that $(\mathbb{S}^*, g_K)$ and $(\mathbb{S}^*, g_{K^\circ})$ are isometric, and so any quantity derived from g_K is the same for K and K° (for example, $\Omega_n(K) = \frac{1}{n}\|\nu_{g_K}\| = \frac{1}{n}\|\nu_{g_{K^\circ}}\| = \Omega_n(K^\circ)$).

One of the key takeaways of our work is that in the context of Brunn–Minkowski theory (and perhaps in other geometric problems), it is actually beneficial to use a calculus based on a well-suited non-Levi-Civita connection, instead of the usual weighted Levi-Civita calculus. As already mentioned, the boundary of any $K \in \mathcal{K}^2_+$ is a centro-affine ($(n-1)$-dimensional) unit sphere, and so in particular its centro-affine Ricci curvature is constant and equal to $n-2$. We stress that this is in stark contrast to the weighted Ricci curvature of $(\mathbb{S}^*, g_K, V_K)$, which will depend on third derivatives of h_K and so will not be positive in general. A classical theorem of Lichnerowicz [68] states that having a positive lower bound on the Ricci curvature of the Levi-Civita connection implies a lower bound on the first non-trivial eigenvalue of the associated Laplace–Beltrami operator. Lichnerowicz's proof is an immediate consequence of the L^2-method and an integrated Bochner formula for the Levi-Civita connection. It is possible to extend Bochner's formula to completely general affine connections, deriving an "asymmetric Bochner formula". Applying this to the centro-affine connection ∇_K, integrating with respect to ν_K, and using that the centro-affine Ricci curvature is $n-2$, we obtain in Section 5 the following centro-affine Bochner formula:

$$\int (\Delta_K f)^2 \, d\nu_K - \int \|\mathrm{Hess}^*_K f\|^2_{g_K} \, d\nu_K = (n-2) \int |\mathrm{grad}_{g_K} f|^2 \, d\nu_K$$

(here $\mathrm{Hess}^*_K f$ denotes the Hessian with respect to the conjugate connection to ∇_K – see Sections 3 and 4). As an immediate consequence, by verbatim repeating Lichnerowicz's argument, we obtain a new proof of the Brunn–Minkowski inequality (in its equivalent infinitesimal form) $\lambda_1(-\Delta_K) = n-1$, including the more delicate characterization of the corresponding n-dimensional eigenspace (originally due to Hilbert).

Using the centro-affine Bochner formula, it easily follows that the conjectured even log-Brunn–Minkowski / log-Minkowski inequalities for $K \in \mathcal{K}^2_{+,e}$ are equivalent to the following new inequality, which should hold for all *even* test functions f:

$$\int \|\mathrm{Hess}^*_K f\|^2_{g_K} \, d\nu_K \geq 2 \int |\mathrm{grad}_{g_K} f|^2 \, d\nu_K. \tag{1.13}$$

A particularly attractive feature of this new formulation is that the above inequality always holds for any $K \in \mathcal{K}^2_+$ and (not necessarily even) test function f with constant 1 instead of 2 above, in which case it becomes equivalent to the usual (infinitesimal) Brunn–Minkowski inequality:

$$\int z \, d\nu_K = 0 \implies \int (-\Delta_K z) z \, d\nu_K = \int |\mathrm{grad}_{g_K} z|^2 \, d\nu_K \geq (n-1) \int |z|^2 \, d\nu_K \tag{1.14}$$

(see Remark 5.6). Consequently, the challenge is to use the evenness of the data in (1.13) to get a two-fold increase in the "trivial" estimate, a factor which seems less mysterious than our previous local formulation from [61], where the goal was to pass from the known $\lambda_1(-\Delta_K) = n-1$ to the conjectured $\lambda_{1,e}(-\Delta_K) \geq n$. This is now very reminiscent of the challenge in the resolution of the B-conjecture by Cordero-Erausquin–Fradelizi–Maurey [38], where the evenness of the data was used to gain a factor of 2 in the corresponding even eigenvalue estimate. In some sense, the centro-affine normalization allows us to implement the strategy from [38], but we are still missing the final ingredient (1.13). Roughly speaking, the difficulty lies in the incompatibility between the centro-affine and Euclidean metrics, and so in contrast to the Euclidean setting of [38], applying (1.14) to $z_v = df(v)$ for some fixed vector $v \in \mathbb{S}$ and averaging over v does not yield the expressions appearing in (1.13). We are however able to verify (1.13) under the assumptions of Corollary 1.3 (and more generally, Theorem 6.3). To this end, the centro-affine geometric interpretation plays a crucial role.

The rest of this work is organized as follows. In Section 2 we begin with some notation and required preliminaries, establishing in particular Theorem 1.1. In Section 3, we provide the required background from affine differential geometry. In Section 4, we specialize the general theory to the centro-affine normalization for several useful parametrizations of ∂K, and compute various differential objects of interest. In Section 5, we derive the centro-affine Bochner formula, the equivalent local formulation (1.13), and a proof of the classical Brunn–Minkowski à la Lichnerowicz. In Section 6 we provide a proof of Theorem 1.2. In Section 7, we obtain our isomorphic and isometric Theorems 1.5 and 1.7.

2. Preliminaries

We begin with some preliminaries, referring to [61, 96] and the references therein for additional information.

2.1. Notation

Let $E = E^n$ denote an n-dimensional vector space over $\mathbb{R}$, which we will often identify with $\mathbb{R}^n$ via a fixed basis. The dual space to E is $E^* = (\mathbb{R}^n)^*$, which we identify with E via a fixed isomorphism $i : E \to E^*$. By abuse of notation, we also use $i : E^* \to E$ to denote the inverse isomorphism, and use $\langle\cdot,\cdot\rangle$ to denote both the natural pairing between E^* and E and the induced Euclidean scalar product on E and E^* via i (so that $\langle i(v), i(w)\rangle = \langle i(v), w\rangle = \langle v, w\rangle$ for all $v, w \in E$). We denote the Euclidean norm by $|x| = \sqrt{\langle x, x\rangle}$. The Euclidean unit spheres in $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, E and E^* are denoted by S^{n-1}, $\mathbb{S}$ and $\mathbb{S}^*$, respectively; they are equipped with their induced Lebesgue measures $\mathfrak{m}$, $\mathfrak{m}^{\mathbb{S}}$ and $\mathfrak{m}^{\mathbb{S}^*}$ (or simply $\mathfrak{m}$).

A *convex body* in $\mathbb{R}^n$ is a convex, compact set with non-empty interior. We denote by $\mathcal{K} = \mathcal{K}(E)$ the collection of convex bodies in E having the origin in their interior. The support function $h_K : E^* \to \mathbb{R}_+$ of $K \in \mathcal{K}(E)$ is defined as

$$h_K(x^*) := \max_{x \in K} \langle x^*, x\rangle, \quad x^* \in E^*.$$

It is easy to see that h_K is continuous, convex and positive outside the origin. Clearly, it is 1-homogeneous, so we will mostly consider its restriction to $\mathbb{S}^*$. Conversely, a convex 1-homogeneous function $h : E^* \to \mathbb{R}_+$ which is positive outside the origin is necessarily a support function of some $K \in \mathcal{K}$. The *dual body* $K^* \in \mathcal{K}(E^*)$ of $K \in \mathcal{K}(E)$ is defined as the convex body in E^* given by the level set $\{h_K \leq 1\}$; duality implies that $(K^*)^* = K$. The *Minkowski gauge function* of $K \in \mathcal{K}(E)$ is defined as

$$\|x\|_K := \inf\{t > 0 \,;\, x \in tK\}, \quad x \in E.$$

Note that $h_K = \|\cdot\|_{K^*}$ on E^* and $h_{K^*} = \|\cdot\|_K$ on E. Given $K \in \mathcal{K}(E)$, we define the *polar body* $K^\circ \in \mathcal{K}(E)$ by identifying it with $K^* \in \mathcal{K}(E^*)$ via i, i.e. $i(K^\circ) = K^*$. The

Minkowski sum $K_1 + K_2$ of two convex bodies is defined as $\{x_1 + x_2\,;\, x_i \in K_i\}$. Note that this operation is additive on the level of support functions: $h_{K_1+K_2} = h_{K_1} + h_{K_2}$.

We denote by $C^k(S^{n-1})$ and $C^{k,\alpha}(S^{n-1})$, $k = 0, 1, 2, \dots$ and $\alpha \in (0, 1)$, the space of k-times continuously and α-Hölder differentiable functions on S^{n-1}, respectively, equipped with their usual topologies. When $k = 0$, we simply write $C(S^{n-1})$ and $C^\alpha(S^{n-1})$. It is known [96, Section 1.8] that convergence of elements of $\mathcal{K}$ in the Hausdorff metric is equivalent to convergence of the corresponding support functions in the $C(\mathbb{S}^*)$ norm.

Given a smooth differentiable manifold M, the tangent and cotangent bundles are denoted by TM and T^*M, respectively, and $\Gamma^k(TM)$ and $\Gamma^k(T^*M)$ denote the collection of C^k-smooth vector and covector fields on M. We use X^i and ω_j to denote $X \in \Gamma^k(TM)$ and $\omega \in \Gamma^k(T^*M)$ in a local frame, and similarly for higher order contravariant and covariant tensors. A metric $(0, 2)$ tensor g is denoted by g_{ij}, and its inverse $(2, 0)$ tensor by g^{ij}, so that $g^{ij} g_{jk} = \delta^i_k$, the Kronecker delta. Given a C^1-smooth function f on M, we use f_j to denote the 1-form $(df)_j$ in a local frame.

The standard flat affine connection on $\mathbb{R}^n$ is denoted by $\bar{D}$. Given a Euclidean structure $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ and a closed smooth hypersurface H with outer unit normal $\mathfrak{n}^H$ in $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, we denote by ${}^H\nabla$ the induced Euclidean connection on H and by II^H the corresponding second fundamental form, given by the Gauss equation

$$\bar{D}_U V = {}^H\nabla_U V - \mathrm{II}^H(U, V)\mathfrak{n}^H, \quad U \in TH,\ V \in \Gamma^1(TH). \tag{2.1}$$

The induced Euclidean metric δ^H on H is given by $\delta^H_p(u, v) = \langle u, v\rangle$ for $u, v \in T_pH$. As usual, the (non-tensorial) Christoffel symbols associated to a local coordinate frame $\{e_1, \dots, e_{n-1}\}$ are defined via

$${}^H\nabla_{e_i} e_j = {}^H\Gamma^k_{ij} e_k,$$

and for any smooth function f on H we have

$${}^H\nabla^2_{ij} f = \partial^2_{ij} f - {}^H\Gamma^k_{ij}\partial_k f.$$

In addition, for any smooth extension of f to a neighborhood of H,

$$\bar{D}^2 f(u, v) = {}^H\nabla^2 f(u, v) + \mathrm{II}^H(u, v)\mathfrak{n}^H_p(f), \quad u, v \in T_pH. \tag{2.2}$$

Given $h \in C^2(\mathbb{S}^*)$, we extend h as a 1-homogeneous function on E^*, and define the symmetric 2-tensor D^2h on $\mathbb{S}^*$ as the restriction of $\bar{D}^2h$ onto $T\mathbb{S}^*$. Recalling (2.2) and using Euler's identity for 1-homogeneous functions $\mathfrak{n}^{\mathbb{S}^*}(h) = \langle\bar{D}h, \mathfrak{n}^{\mathbb{S}^*}\rangle = h$, it follows that in a local frame $\{e_1, \dots, e_{n-1}\}$ on $\mathbb{S}^*$,

$$D^2_{ij}h = \bar{D}^2h(e_i, e_j) = {}^{\mathbb{S}^*}\nabla^2_{ij}h + h\delta^{\mathbb{S}^*}_{ij}, \quad i, j = 1, \dots, n-1.$$

Denoting by $C^k_{>0}(\mathbb{S}^*)$ the subset of positive functions in $C^k(\mathbb{S}^*)$, note that $h \in C^2_{>0}(\mathbb{S}^*)$ is the support function of $K \in \mathcal{K}$ if and only if $D^2h_K \geq 0$.

We denote by $\mathcal{K}_+^m$ the subset of $\mathcal{K}$ of convex bodies with C^m-smooth boundary and strictly positive curvature. By [96, pp. 115–116, 120–121], for $m \geq 2$, $K \in \mathcal{K}_+^m$ if and only if $h_K \in C^m_{>0}(\mathbb{S}^*)$ and $D^2 h_K > 0$. Similarly, $\mathcal{K}_+^{m,\alpha}$ denotes the subset of $\mathcal{K}_+^m$ of convex bodies with $C^{m,\alpha}$-smooth boundary ($\alpha \in (0,1]$), and for $m \geq 2$, $K \in \mathcal{K}_+^{m,\alpha}$ if and only if $h_K \in C^{m,\alpha}_{>0}(\mathbb{S}^*)$ and $D^2 h_K > 0$. Consequently, by identifying elements of $\mathcal{K}_+^m$ and $\mathcal{K}_+^{m,\alpha}$ with their support functions whenever $m \geq 2$, we equip these spaces with their C^m and $C^{m,\alpha}$ topologies, respectively. It is well-known that $\mathcal{K}_+^\infty$ is dense in $\mathcal{K}$ with respect to the Hausdorff metric (e.g. [96, pp. 184–185]).

A convex body K is called *origin-symmetric* if $K = -K$. We will always use S_e to denote the origin-symmetric (or even) members of a set S, e.g. $\mathcal{K}_e$ and $\mathcal{K}^2_{+,e}$ denote the subsets of origin-symmetric bodies in $\mathcal{K}$ and $\mathcal{K}_+^2$, respectively, and $C^2_e(\mathbb{S}^*)$ denotes the subset of even functions in $C^2(\mathbb{S}^*)$.

We use $GL(E)$ to denote the group of non-singular linear (or centro-affine) transformations on E, and $SL(E)$ to denote the subgroup of volume and orientation preserving elements. When $E = \mathbb{R}^n$, we simply write GL_n.

2.2. Brunn–Minkowski theory

Given a convex body K in Euclidean space $(E^n, \langle\cdot,\cdot\rangle)$, its surface-area measure S_K is defined as the push-forward under the Gauss map $\mathfrak{n}^{\partial K} : \partial K \to \mathbb{S}^*$ of $\mathcal{H}^{n-1}|_{\partial K}$. Recall that $\mathfrak{n}^{\partial K}$ denotes the outer unit normal to K and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. When $K \in \mathcal{K}_+^2$, we have

$$S_K = \det(D^2 h_K)\mathfrak{m}.$$

More generally, Lutwak [73] introduced the L^p *surface-area measure* of K as

$$S_p K := h_K^{1-p} S_K.$$

The *cone-volume measure* V_K on $\mathbb{S}^*$ is defined as

$$V_K = V_K^{\mathbb{S}^*} := \frac{1}{n} h_K S_K;$$

it is obtained by first pushing forward the Lebesgue measure on K via the cone map $K \ni x \mapsto x/\|x\|_K \in \partial K$, and then pushing forward the resulting cone measure $V_K^{\partial K}$ on ∂K via the Gauss map $\mathfrak{n}^{\partial K} : \partial K \to \mathbb{S}^*$. For completeness, note that if we instead push forward the Lebesgue measure on K via the radial-projection map $K \ni x \mapsto x/|x| \in \mathbb{S}$, we obtain

$$V_K^{\mathbb{S}} := \frac{1}{n} \frac{1}{\|\theta\|_K^n} \mathfrak{m}(d\theta).$$

Given two convex bodies K_0, K_1 in E^n, the classical Brunn–Minkowski inequality states that

$$V(K_0 + K_1)^{\frac{1}{n}} \geq V(K_0)^{\frac{1}{n}} + V(K_1)^{\frac{1}{n}}, \tag{2.3}$$

where V denotes volume (Lebesgue measure) and $K_0 + K_1$ denotes the Minkowski sum of K_0 and K_1. The L^p*-Minkowski sum* $a \cdot K_0 +_p b \cdot K_1$ of $K_0, K_1 \in \mathcal{K}$ $(a, b \geq 0)$ was defined by Firey for $p \geq 1$ [41], and extended by Böröczky–Lutwak–Yang–Zhang [18,19] to all $p \in \mathbb{R}$, as the largest convex body L (with respect to inclusion) such that

$$h_L \leq (ah_{K_0}^p + bh_{K_1}^p)^{1/p}$$

(with the case $p = 0$ interpreted as $h_{K_0}^a h_{K_1}^b$ when $a + b = 1$). Note that for $p \geq 1$ one has equality above, that the case $p = 1$ coincides with the usual Minkowski sum, and that for $p < 1$ the resulting convex body $a \cdot K_0 +_p b \cdot K_1$ is the Alexandrov body associated to the continuous function on the right-hand side.

As a consequence of the Brunn–Minkowski and Jensen inequalities, Firey showed that for any $K_0, K_1 \in \mathcal{K}$ and $p \geq 1$,

$$\forall \lambda \in [0, 1], \quad V((1-\lambda) \cdot K_0 +_p \lambda \cdot K_1)^{\frac{p}{n}} \geq (1-\lambda)V(K_0)^{\frac{p}{n}} + \lambda V(K_1)^{\frac{p}{n}}. \tag{2.4}$$

It is not hard to show that the above statement for any $p<1$ is false for general $K_0, K_1 \in \mathcal{K}$. However, it was conjectured by Böröczky–Lutwak–Yang–Zhang [18] that for *origin-symmetric* $K_0, K_1 \in \mathcal{K}_e$, (2.4) does in fact hold for all $p \in [0, 1)$ – we refer to this as the (even) L^p*-Brunn–Minkowski conjecture*. The validity of (2.4) for all $K_0, K_1 \in \mathcal{K}_e$ and a given p implies the validity for all $K_0, K_1 \in \mathcal{K}_e$ and any $q > p$, and so the case $p = 0$, called the (even) *log-Brunn–Minkowski conjecture*, is the strongest in this hierarchy. As described in Theorem 1.1 from the Introduction, the even L^p-Brunn–Minkowski conjecture is intimately related to the even L^p-Minkowski inequality (1.4) and to the uniqueness question in the even L^p-Minkowski problem (1.2). It turns out that the conjecture is also related to a certain spectral problem, described next.

2.3. *Hilbert–Brunn–Minkowski operator*

Following the work of [37], the *local* version of the L^p-Brunn–Minkowski inequality (2.4) was studied in our previous work with Kolesnikov [61]. Given $K \in \mathcal{K}^2_+$, the *Hilbert–Brunn–Minkowski operator* $\Delta_K : C^2(\mathbb{S}^*) \to C(\mathbb{S}^*)$ was defined in [61] as

$$\begin{aligned}\Delta_K z = \Delta_K^{\mathbb{S}^*} z &:= ((D^2 h_K)^{-1})^{ij} D^2_{ij}(z h_K) - (n-1)z \\ &= ((D^2 h_K)^{-1})^{ij} (h_K{}^{\mathbb{S}^*}\nabla^2_{ij} z + (h_K)_i z_j + (h_K)_j z_i).\end{aligned}$$

Note that we are using a slightly different normalization than in [61], where the Hilbert–Brunn–Minkowski operator (denoted L_K) was defined as $L_K := \frac{1}{n-1}\Delta_K$. Introducing the following Riemannian (positive-definite) metric on $\mathbb{S}^*$:

$$g_K = g_K^{\mathbb{S}^*} := \frac{D^2 h_K}{h_K} > 0, \tag{2.5}$$

we may also write

$$\Delta_K z = g_K^{ij} ({}^{\mathbb{S}^*}\nabla^2_{ij} z + (\log h_K)_i z_j + (\log h_K)_j z_i). \tag{2.6}$$

Clearly, Δ_K is an elliptic second order differential operator with vanishing zeroth order term, and in particular $\Delta_K 1 = 0$. Up to gauge transformations, Δ_K coincides with the operator defined by Hilbert in his proof of the Brunn–Minkowski inequality [13].

It was shown in [61] that the following integration-by-parts formula holds:

$$\int_{\mathbb{S}^*} (-\Delta_K z) w \, dV_K = \int_{\mathbb{S}^*} g_K(\nabla z, \nabla w) \, dV_K = \int_{\mathbb{S}^*} (-\Delta_K w) z \, dV_K \quad \forall z, w \in C^2(\mathbb{S}^*).$$

We use the notation $g_K(\nabla z, \nabla w) = g_K^{ij} z_i w_j$ and $|\nabla z|^2_{g_K} = g_K(\nabla z, \nabla z)$. It follows that we may interpret Δ_K as the weighted Laplacian on the weighted Riemannian manifold $(\mathbb{S}^*, g_K, V_K)$ (see e.g. [59,60]). Consequently, $-\Delta_K$ is a symmetric positive semi-definite operator on $L^2(V_K)$. It uniquely extends to a self-adjoint positive semi-definite operator with Sobolev domain $H^2(\mathbb{S}^*)$ and compact resolvent, which we continue to denote by $-\Delta_K$. Its (discrete) spectrum is denoted by $\sigma(-\Delta_K)$, and its first non-zero eigenvalue is denoted by $\lambda_1(-\Delta_K)$.

As already known to Minkowski, the Brunn–Minkowski inequality (2.3) is equivalent to its local form (when K_1 is an infinitesimal perturbation of K_0). This local form was interpreted by Hilbert in a spectral language as

$$\lambda_1(-\Delta_K) \geq n - 1,$$

or equivalently

$$\int_{\mathbb{S}^*} z \, dV_K = 0 \implies \int_{\mathbb{S}^*} (-\Delta_K z) z \, dV_K \geq (n-1) \int_{\mathbb{S}^*} z^2 \, dV_K \quad \forall z \in C^2(\mathbb{S}^*).$$

Hilbert showed that in fact $\lambda_1(-\Delta_K) = n - 1$, characterizing in addition the corresponding eigenspace (see [61, Section 5] or Sections 4.5 and 5.3 for more information).

Now assume in addition that K is origin-symmetric, i.e. $K \in \mathcal{K}^2_{+,e}$. Denote by $H^2_e(\mathbb{S}^*)$ the even elements of the Sobolev space $H^2(\mathbb{S}^*)$ and by $\mathbf{1}^\perp$ those elements f for which $\int f \, dV_K = 0$. The first non-trivial *even* eigenvalue of $-\Delta_K$ is defined as

$$\begin{aligned} \lambda_{1,e}(-\Delta_K) &:= \min \sigma(-\Delta_K|_{H^2_e(\mathbb{S}^*) \cap \mathbf{1}^\perp}) \\ &= \inf \left\{ \frac{\int_{\mathbb{S}^*} |\nabla z|^2_{g_K} \, dV_K}{\int_{\mathbb{S}^*} z^2 \, dV_K} \, ; \, 0 \neq z \in C^2_e(\mathbb{S}^*), \int_{\mathbb{S}^*} z \, dV_K = 0 \right\}. \end{aligned} \tag{2.7}$$

It was shown in [61] that the validity of the *local* form of the even L^p-Brunn–Minkowski inequality (2.4) for $K \in \mathcal{K}^2_{+,e}$ is equivalent to the validity of the statement

$$\lambda_{1,e}(-\Delta_K) \geq n - p.$$

That the validity of the local form for all $K \in \mathcal{K}^2_{+,e}$ implies the validity of the global form (2.4) for all $K_0, K_1 \in \mathcal{K}_e$ is trivial for $p \geq 1$ but not obvious at all when $p < 1$. The latter was conjectured in [61] and proved by Putterman [91], after a prior local-to-global result for the uniqueness question in the even L^p-Minkowski problem by Chen–Huang–Li–Liu [27].

2.4. *Proof of Theorem 1.1*

We can now finally formulate an expanded version of Theorem 1.1 from the Introduction, utilizing the full array of results from [18,19,22,27,61], which we will require to establish our results.

Theorem 2.1. *For a fixed $p \in (-n, 1)$, statements (1)–(3) of Theorem 1.1 are equivalent to each other and to the following additional statements (with the usual interpretation when $q = 0$):*

(2b) *For all $q \in (p, 1)$ and $K, L \in \mathcal{K}_e$, the even L^q-Brunn–Minkowski inequality holds:*

$$\forall \lambda \in [0,1], \quad V((1-\lambda) \cdot K +_q \lambda \cdot L) \geq \left((1-\lambda)V(K)^{\frac{q}{n}} + \lambda V(L)^{\frac{q}{n}}\right)^{\frac{n}{q}}, \quad (2.8)$$

with equality for some $\lambda \in (0, 1)$ if and only if $L = cK$ for some $c > 0$.

(3b) *For all $q \in (p, 1)$ and $K \in \mathcal{K}^{2,\alpha}_{+,e}$, the even L^q-Minkowski inequality holds:*

$$\forall L \in \mathcal{K}_e, \quad \frac{1}{q}\int_{\mathbb{S}^*} h_L^q \, dS_q K \geq \frac{n}{q} V(K)^{1-\frac{q}{n}} V(L)^{\frac{q}{n}}, \quad (2.9)$$

with equality if and only if $L = cK$ for some $c > 0$.

(4) *For all $K \in \mathcal{K}^{2,\alpha}_{+,e}$, $\lambda_{1,e}(-\Delta_K) \geq n - p$.*

Moreover, let $\mathcal{F} \subset \mathcal{K}^{2,\alpha}_{+,e}$ be any subfamily containing B_2^n which is path-connected in the $C^{2,\alpha}$ topology. Namely, for any $K \in \mathcal{F}$, there exists $[0,1] \ni t \mapsto K_t \in \mathcal{F}$, a continuous path in the $C^{2,\alpha}$ topology such that $K_0 = B_2^n$ and $K_1 = K$. Then the implications (4)⇒(1)⇒(3b) remain valid if we replace $\mathcal{K}^{2,\alpha}_{+,e}$ in these statements by $\mathcal{F}$.

Before providing a proof of Theorem 2.1, we need to first collect several known ingredients from the literature. First, as explained in [82, Section 6], the standard regularity theory of the Monge–Ampère equation implies that any solution $L \in \mathcal{K}$ to

$$S_p L = f\mathfrak{m}, \quad f \in C^\alpha(\mathbb{S}^*), \ f > 0, \quad (2.10)$$

necessarily satisfies $L \in \mathcal{K}^{2,\alpha}_+$ (note that without *a priori* assuming that $L \in \mathcal{K}$, so that $h_L > 0$ on $\mathbb{S}^*$, the asserted regularity is false and L may not be C^2-smooth [34, Section 6]). In our context, this means that whenever $K \in \mathcal{K}^{2,\alpha}_{+,e}$, uniqueness in the L^p-Minkowski problem

$$S_p L = S_p K$$

is the same when considering solutions L in either of the classes $\mathcal{K}_e$ or $\mathcal{K}^{2,\alpha}_{+,e}$.

Next, we need the following local uniqueness statement from [61, Theorem 11.2]:

Theorem 2.2 (Kolesnikov–Milman). *Assume that the local even L^p-Brunn–Minkowski inequality (5.11) holds for $K \in \mathcal{K}^2_{+,e}$ and some $p_0 < 1$. Then for any $p \in (p_0, 1)$, the even L^p-Minkowski problem has a locally unique solution in a neighborhood of K in the following sense: there exists a C^2_e-neighborhood $N_{K,p}$ of K in $\mathcal{K}^2_{+,e}$ such that*

$$\forall L_1, L_2 \in N_{K,p}, \quad S_p L_1 = S_p L_2 \implies L_1 = L_2. \quad (2.11)$$

The next ingredient we need was obtained in [42] (for $p = 0$ and $L \in \mathcal{K}_e^\infty$), [8] (for $p \in [0, 1)$ and $L \in \mathcal{K}_e$) and finally completely resolved in [22] (see also [5,33,35,47] for additional contributions):

Theorem 2.3 (Firey, Andrews–Guan–Ni, Brendle–Choi–Daskalopoulos). *Let $-n < p < 1$ and $c > 0$. Then the L^p-Minkowski problem*

$$L \in \mathcal{K}, \quad S_p L = c \cdot \mathfrak{m}$$

has a unique solution L given by a centered Euclidean ball.

An additional crucial ingredient is the following theorem, which is the main new ingredient in the results of [27]; as it is not explicitly stated in the manner formulated below, we sketch its proof for completeness.

Theorem 2.4 (Chen–Huang–Li–Liu). *Let $p < 1$, and let $[0, 1] \ni t \mapsto K_t \in \mathcal{K}^{2,\alpha}_{+,e}$ be a continuous path in the $C^{2,\alpha}$ topology. Assume that the even L^p-Minkowski problem has a globally unique solution for K_0:*

$$\forall L \in \mathcal{K}^{2,\alpha}_{+,e}, \quad S_p L = S_p K_0 \implies L = K_0.$$

Assume that for all $t \in [0, 1]$, the even L^p-Minkowski problem has a locally unique solution in a $C^{2,\alpha}_e$-neighborhood $N_{K_t,p}$ of K_t in the sense of (2.11). Then the even L^p-Minkowski problem has a globally unique solution for K_1:

$$\forall L \in \mathcal{K}^{2,\alpha}_{+,e}, \quad S_p L = S_p K_1 \implies L = K_1.$$

Sketch of proof. Let $K \in \mathcal{K}^{2,\alpha}_{+,e}$, and assume that there exists a $C^{2,\alpha}_e$-neighborhood $N_{K,p}$ of K such that (2.11) holds. It was shown in [27, Lemma 3.1] that if the equation $S_p L = S_p K$ has a globally unique solution $L = K$ among all $L \in \mathcal{K}^{2,\alpha}_{+,e}$, then the equation $S_p L = S_p \tilde{K}$ has a globally unique solution $L = \tilde{K}$ for all $\tilde{K}$ in a $C^{2,\alpha}_e$ sub-neighborhood of K in $N_{K,p}$. On the other hand, as explained in the proof of [27, Theorem 1.4], it follows from [27, Lemmas 3.2–3.4] that if the equation $S_p L = S_p K$ has multiple distinct solutions $L \in \mathcal{K}^{2,\alpha}_{+,e}$, then the equation $S_p L = S_p \tilde{K}$ also has multiple distinct solutions for all $\tilde{K}$ in a $C^{2,\alpha}_e$ sub-neighborhood of K in $N_{K,p}$.

Now apply the method of continuity following [27]: Define $I \subset [0, 1]$ to be the subset of t's for which $S_p L = S_p K_t$ has a globally unique solution $L = K_t$ among all $L \in \mathcal{K}^{2,\alpha}_{+,e}$. The results above imply that I is both relatively open and closed in $[0, 1]$, and hence I is either empty or the entire $[0, 1]$. But our assumption was that $0 \in I$, and consequently $I = [0, 1]$. ■

Finally, we will use the existence of a global minimizer in the following optimization problem [34, Section 5]. Once it is shown that a global minimum is attained, a very general variational argument [73, Theorem 3.3], [19, Lemma 4.1] ensures that any local minimizer satisfies the corresponding Euler–Lagrange equation (2.12) (compare with the original argument of [34, Theorem D]):

Theorem 2.5 (Chou–Wang, Lutwak, Böröczky–Lutwak–Yang–Zhang). *Let $f \in C^{\alpha}_{e}(\mathbb{S}^*)$ be a strictly positive even density, and denote $\mu = f\,\mathfrak{m}$. Given $-n < p < 1$, consider the 0-homogeneous functional*

$$\mathcal{K}_e \in L \mapsto F_{\mu,p}(L) := \begin{cases} \dfrac{\frac{1}{p}\int h_L^p \, d\mu}{V(L)^{p/n}}, & p \neq 0, \\ \dfrac{\exp(\int \log h_L \, d\tilde{\mu})}{V(L)^{1/n}}, & p = 0, \end{cases}$$

where $\tilde{\mu}$ denotes the normalized measure $\mu/\|\mu\|$. Then $F_{\mu,p}$ attains a global minimum.

Moreover, any local minimizer L of $F_{\mu,p}$ (in the Hausdorff topology) satisfies

$$S_p L = c \cdot \mu \tag{2.12}$$

for some $c > 0$. In particular, by (2.10), necessarily $L \in \mathcal{K}^{2,\alpha}_{+,e}$.

Remark 2.6. Without the origin-symmetry assumption above, it is imperative to incorporate an additional maximization over all possible translations of L so that the origin remains in L, rendering the analysis much more delicate [11, 28, 29, 34]. Nevertheless, one can still guarantee the existence of a global minimizer under even more general conditions on μ than the ones stated above: this was shown in [34] for densities f satisfying $0 < c \leq f \leq C$, in [11] for densities $f \in L^{\frac{n}{n+p}}(\mathfrak{m})$ when $-n < p < 0$, and in [28,29] for finite Borel measures μ which are not concentrated on any hemisphere when $p \in (0, 1)$ or which satisfy the subspace concentration condition when $p = 0$.

We can now finally provide a proof of Theorem 2.1.

Proof of Theorem 2.1. Statement (1) implies (3b) for each individual $q \in (-n, 1)$ and $K \in \mathcal{K}^{2,\alpha}_{+,e}$. To see this, denote $\mu = S_q K$, and note that $\mu = f\,\mathfrak{m}$ with a positive density $f \in C^{\alpha}_{e}(\mathbb{S}^*)$. Recall from Theorem 2.5 that a global minimizer of $\mathcal{K}_e \ni L \mapsto F_{\mu,q}(L)$ always exists, and that any global minimizer L must satisfy $S_q L = c \cdot S_q K$ and is therefore in $\mathcal{K}^{2,\alpha}_{+,e}$. Consequently, if statement (3b) regarding the L^q-Minkowski inequality or its cases of equality were wrong, it would follow that there exists a global minimizer $L \in \mathcal{K}^{2,\alpha}_{+,e}$ which is different from K such that (after rescaling) $S_q L = S_q K$, in contradiction to the uniqueness in the even L^q-Minkowski problem asserted in (1).

The converse implication (3b)⇒(1) also holds for each individual q and *both* pairs (K, L) and (L, K), for any fixed $K, L \in \mathcal{K}_e$. While we do not require this here, we provide a quick proof for completeness following Lutwak [73]. Let $q \neq 0$; the case $q = 0$ is treated in an identical manner. Assume that (3b) holds for both pairs (K, L) and (L, K) and that $S_q K = S_q L$. Denote $V_q(K, L) := \frac{1}{q}\int_{\mathbb{S}^*} h_L^q \, dS_q K$. Then by (3b),

$$V_q(K, L) \geq V_q(K, K) = V_q(L, K) \geq V_q(L, L) = V_q(K, L).$$

Consequently, equality holds throughout, and since $\frac{n}{q}V(L) = V_q(L, L) = V_q(K, K) = \frac{n}{q}V(K)$, it follows that we have equality in (2.9), and hence $L = cK$ for some $c > 0$. But since $V(L) = V(K)$, we conclude that $L = K$, as asserted in (1).

Statement (3b) obviously implies (3) for general $K, L \in \mathcal{K}_e$ after recalling Remark 1.4 and taking the limit as $q \searrow p$. Similarly, statement (2b) trivially implies (2) by taking the limit $q \searrow p$. The equivalence of statements (3) and (2) was shown by Böröczky–Lutwak–Yang–Zhang [18]. That the global statement (2) implies the local one in (4) was shown in [61]. The local-to-global converse direction was established by Putterman [91], and also follows by the implications (4)⇒(1)⇒(3b)⇒(3)⇒(2).

That (3) implies (3b) for any $q > p$ is a simple consequence of Jensen's inequality (after rescaling K for convenience so that $\mathrm{Vol}(K) = 1$):

$$\left(\frac{1}{n}\int_{\mathbb{S}^*} h_L^q \, dS_q K\right)^{1/q} = \left(\int_{\mathbb{S}^*} \left(\frac{h_L}{h_K}\right)^q dV_K\right)^{1/q} \geq \left(\int_{\mathbb{S}^*} \left(\frac{h_L}{h_K}\right)^p dV_K\right)^{1/p} = \left(\frac{1}{n}\int_{\mathbb{S}^*} h_L^p \, dS_p K\right)^{1/p}. \tag{2.13}$$

To establish the characterization of equality in (3b), one may argue as in the proof of [18, Theorem 1.8]. Indeed, by (2.13) and (3) we have, for all $K \in \mathcal{K}^{2,\alpha}_{+,e}$ (say with $\mathrm{Vol}(K) = 1$) and $L \in \mathcal{K}_e$,

$$\left(\frac{1}{n}\int_{\mathbb{S}^*} h_L^q \, dS_q K\right)^{1/q} \geq \left(\frac{1}{n}\int_{\mathbb{S}^*} h_L^p \, dS_p K\right)^{1/p} \geq \mathrm{Vol}(L)^{1/n}.$$

Consequently, if equality holds between the leftmost and rightmost terms, we must have equality in Jensen's inequality (2.13), and hence h_L and h_K must be proportional V_K-a.e. But as $K \in \mathcal{K}^{2,\alpha}$, V_K is absolutely continuous with respect to $\mathfrak{m}$, and so by continuity $h_L(\theta) = c h_K(\theta)$ for some $c > 0$ and all $\theta \in \mathbb{S}^*$.

Similarly, it is well-known and easy to check (see [41, Theorem 2]) that (2) implies (2b), after rescaling K and L by homogeneity so that $\mathrm{Vol}(K) = \mathrm{Vol}(L) = 1$ and noting that by Jensen's inequality, whenever $q > p$,

$$(1-\lambda)\cdot K +_q \lambda \cdot L \supset (1-\lambda)\cdot K +_p \lambda \cdot L, \tag{2.14}$$

with equality for some $\lambda \in (0,1)$ if and only if K and L are dilates of each other. If equality holds in (2b) for some $\lambda_0 \in (0,1)$ and $K_0, L_0 \in \mathcal{K}_e$, then after rescaling K_0, L_0 into K, L such that $\mathrm{Vol}(K) = \mathrm{Vol}(L) = 1$, it follows by homogeneity that there exists $\lambda \in (0,1)$ so that equality holds in (2b) for λ, K and L. By (2.14) and (2) we know that

$$\mathrm{Vol}((1-\lambda)\cdot K +_q \lambda \cdot L) \geq \mathrm{Vol}((1-\lambda)\cdot K +_p \lambda \cdot L) \geq 1,$$

and as equality holds between the leftmost and righmost terms, we must have equality in (2.14) (up to null-sets, and as the corresponding compact sets have non-empty interior, pointwise equality). It follows that K and L must be dilates of each other, and hence so are K_0 and L_0.

It remains to show that statement (4) implies (1) for a path-connected $\mathcal{F} \subset \mathcal{K}^{2,\alpha}_{+,e}$ containing B_2^n. Given $K \in \mathcal{F}$, there exists a continuous path in $\mathcal{F}$ (equipped with the $C^{2,\alpha}$ topology), denoted $[0,1] \ni t \mapsto K_t$, such that $K_0 = B_2^n$ and $K_1 = K$. Fix $q \in (p,1)$. Statement (4) and Theorem 2.2 imply that for all $t \in [0,1]$, the even L^q-Minkowski problem has a locally unique solution in a neighborhood of K_t in the sense of (2.11). Consequently, as $K_0 = B_2^n$ satisfies the global uniqueness in the even L^q-Minkowski problem by Theorem 2.3, it follows by Theorem 2.4 that $K_1 = K$ also satisfies the global uniqueness in the even L^q-Minkowski problem in the class $\mathcal{K}^{2,\alpha}_{+,e}$. As explained at the beginning of this subsection, the regularity theory for (2.10) implies that the uniqueness extends to the entire $\mathcal{K}_e$, thereby establishing (1). ■

Remark 2.7. An immediate corollary of Theorem 2.1 is that uniqueness in the even L^p-Minkowski problem (1.7) holds for all $p \in (-n, 1)$ whenever K is a centered ellipsoid $\mathcal{E}$. Indeed, it is well-known that $\lambda_{1,e}(-\Delta_{B_2^n}) = 2n$, as $\Delta_{B_2^n}$ coincides with the usual Laplace–Beltrami operator on $\mathbb{S}^*$ (see e.g. [61]). As the spectrum of $-\Delta_K$ is invariant under centro-affine transformations [61, Section 5], it follows that $\lambda_{1,e}(-\Delta_{\mathcal{E}}) = 2n$ for all centered ellipsoids $\mathcal{E}$. Hence, applying Theorem 2.1 to the family $\mathcal{F} = \{\mathcal{E}\} \subset \mathcal{K}^{2,\alpha}_{+,e}$ of all centered ellipsoids, the implication (4)⇒(1) concludes the proof.

3. Affine differential geometry

In this section, we collect facts from affine differential geometry which we will need for this work; note that our sign choices in various places may be different from the standard ones. We refer to [12,85] for a detailed exposition and further information regarding affine differential geometry. For a development of the theory from the point of view of relative normalizations, we refer to [65,86,97], and from the point of view of statistical structures, we refer to [87,88].

3.1. Normalization and structure equations

Recall that $E = E^n$ denotes an n-dimensional linear vector space over $\mathbb{R}$. More general treatments assume that E is an n-dimensional affine space and distinguish between E and its tangent spaces, but for simplicity we will not require this here and identify $T_x E$ with E. The space E is equipped with its standard flat affine connection $\bar{D}$ and a determinant volume form Det (note that all determinant volume forms coincide up to a scalar multiple).

Let $M = M^{n-1}$ denote a smooth connected $(n-1)$-dimensional differentiable manifold. In our context, M will always be orientable and closed, i.e. compact without boundary. Let $x : M^{n-1} \to E^n$ be a smooth immersion, that is, a smooth map such that $d_p x$ is of maximal rank for all $p \in M$. In our context, $x : M \to E$ will *always be an embedding of a convex hypersurface with strictly positive curvature* (*"strongly convex"*). Let $\xi : M \to E$ denote a smooth transversal normal field to x, meaning that $\mathrm{rank}(d_p x, \xi(p)) = n$ for all $p \in M$. The transversal normal ξ induces a volume form ν_ξ on M:

$$\nu_\xi(e_1, \ldots, e_{n-1}) := \mathrm{Det}(dx(e_1), \ldots, dx(e_{n-1}), \xi), \quad e_i \in T_p M.$$

It also induces a connection $\nabla = \nabla^\xi$ and a bilinear form $g = g^\xi$ on M via the Gauss structure equation:

$$\bar{D}_U dx(V) = dx(\nabla^\xi_U V) - g^\xi(U, V)\xi, \quad U \in TM,\ V \in \Gamma^1(TM). \tag{3.1}$$

It turns out that ∇^ξ is always a torsion-free affine connection, and that g^ξ is a symmetric $(0,2)$ tensor, which is called the second fundamental form. In our context, since x is strongly convex, g^ξ is always definite, and so multiplying ξ by -1, we can always make sure that g^ξ is positive-definite, and hence defines a Riemannian metric on M.

In addition, ξ induces a $(1,1)$ tensor $S = S^\xi : TM \to TM$ called the shape operator and a 1-form θ^ξ on M via the Weingarten structure equation:

$$d\xi(V) = dx(S^\xi(V)) + \theta^\xi(V)\xi, \quad V \in TM. \tag{3.2}$$

3.2. *Conormalization and structure equations*

The dual space to E is denoted by E^*, and $\langle\cdot,\cdot\rangle : E^* \times E \to \mathbb{R}$ denotes the corresponding pairing. E^* is equipped with the same standard flat connection $\bar{D}$ and the dual volume form Det^*, uniquely defined by requiring that:

$$\mathrm{Det}^*(w^1,\ldots,w^n)\,\mathrm{Det}(v_1,\ldots,v_n) = \det((\langle w^i, v_j\rangle)_{ij}) \quad \forall w^i \in E^*,\ v_j \in E,$$

where det is the usual n by n determinant.

A *conormal field* $\xi^* : M \to E^*$ is a smooth vector field such that $\langle\xi^*, dx\rangle = 0$. We will always normalize ξ^* so that in addition $\langle\xi^*,\xi\rangle = 1$; since $\mathrm{rank}(dx,\xi) = n$, we see that ξ determines ξ^* uniquely.

Observe that

$$g^\xi(u,v) = \langle d\xi^*(u), dx(v)\rangle \quad \forall u,v \in T_pM;$$

in particular, the right-hand side is symmetric in u, v. Indeed, using that $\langle\xi^*, dx\rangle = 0$ twice and (3.1), we have, for $U \in TM$, $V \in \Gamma^1(TM)$,

$$\begin{aligned}\langle d\xi^*(U), dx(V)\rangle &= U(\langle\xi^*, dx(V)\rangle) - \langle\xi^*, \bar{D}_U dx(V)\rangle \\ &= -\langle\xi^*, dx(\nabla^\xi_U V) - g^\xi(U,V)\xi\rangle = \langle\xi^*,\xi\rangle g^\xi(U,V).\end{aligned}$$

The conormal field ξ^* induces a volume form ν^*_ξ on M:

$$\nu^*_\xi(e_1,\ldots,e_{n-1}) := \mathrm{Det}^*(d\xi^*(e_1),\ldots,d\xi^*(e_{n-1}),\xi^*), \quad e_i \in T_pM.$$

Since we assume that x is strongly convex, it follows that $\xi^* : M \to E^*$ is an immersion, and that ξ^* is transversal to $\xi^*(M)$; in particular, ν^*_ξ is non-trivial.

Repeating the same construction as before, ξ^* induces a torsion-free affine connection $\nabla^* = (\nabla^\xi)^*$ and a symmetric $(0,2)$ tensor $\hat{S} = \hat{S}^\xi$ called the *Weingarten form*, via the Gauss structure equation:

$$\bar{D}_U d\xi^*(V) = d\xi^*((\nabla^\xi)^*_U V) - \hat{S}^\xi(U,V)\xi^*, \quad U \in TM,\ V \in \Gamma^1(TM).$$

The Weingarten form and the shape operator are related by

$$\hat{S}(U,V) = g(S(U),V).$$

3.3. Affine and equiaffine invariance

(M, ξ, ξ^*) is called a *normalization* of the hypersurface $x : M \to E$.

Let $\alpha : E \to E$ be a regular affine transformation given by $\alpha z = Az + b$. Then the hypersurface x with normalization (ξ, ξ^*) and the hypersufrace αx with normalization $(A\xi, A^{-*}\xi^*)$ induce the following exact same structures on M: ∇, ∇^*, g, θ, S and $\hat{S}$. We will say that these structures are *affine-invariant*.

Note that the volume forms ν and ν^* are invariant under the above transformations only when $\det A = 1$, i.e. when α belongs to the unimodular (or equiaffine) group – we will say in this case that they are *equiaffine-invariant*.

3.4. Relative normalization

The transversal normal ξ is called a *relative normal*, and (M, ξ) is called a *relative normalization*, if the 1-form θ^ξ from the Weingarten equation (3.2) vanishes identically: $\theta^\xi = 0$. The following statements are easily shown to be equivalent:

(1) ξ is a relative normal:
$$d\xi(V) = dx(S^\xi(V)), \quad \forall V \in TM. \tag{3.3}$$

(2) ξ is equiaffine, meaning that
$$\nabla^\xi \nu_\xi = 0.$$
Note that in general one always has $\nabla^\xi \nu_\xi = \theta^\xi \nu_\xi$. We will not use the term equiaffine in this context, since it may be confused with Blaschke's notion of affine normal, which is a particular choice of relative normalization described below.

(3) The cubic form $A := -\frac{1}{2}\nabla^\xi g^\xi$ is a totally symmetric $(0,3)$ tensor. Since g^ξ is already symmetric, it is enough to verify the symmetry with respect to the first two variables:
$$(\nabla^\xi_X g^\xi)(Y, Z) = (\nabla^\xi_Y g^\xi)(X, Z) \quad \forall X, Y, Z \in T_pM. \tag{3.4}$$
A torsion-free affine connection ∇ on a Riemannian manifold (M, g) satisfying the Codazzi equation (3.4) is called a *statistical connection* for g, and (g, ∇) is called a *statistical structure* on M. This nomenclature is derived from the influential work of Amari (see e.g. [2, 3]) regarding applications of such structures in statistics, but is otherwise highly misleading, and so we will mostly avoid using it here.

For a strongly convex hypersurface there are inifinitely many different relative normalizations. As for the conormal ξ^*, it turns out that we always have
$$(\nabla^\xi)^* \nu^*_\xi = 0,$$
i.e. the conormal always gives rise to a relative (or equiaffine) conormalization (M, ξ^*).

Recall that $\langle \xi^*, dx \rangle = 0$. For a relative normalization, we also have the important property
$$\langle d\xi^*, \xi \rangle = 0.$$
Indeed, this follows by differentiating $\langle \xi^*, \xi \rangle = 1$ and using $d\xi = dx \circ S^\xi$.

From now on we assume that (M, ξ, ξ^*) is a relative normalization of the hypersurface x.

3.5. Conjugation via the metric

The connections $\nabla = \nabla^\xi$ and $\nabla^* = (\nabla^\xi)^*$ are conjugate connections with respect to $g = g^\xi$, i.e. they satisfy

$$Ug(V_1, V_2) = g(\nabla_U V_1, V_2) + g(V_1, \nabla^*_U V_2) \quad \forall U \in TM \ \forall V_1, V_2 \in \Gamma^1(TM).$$

In a local frame, it is straightforward to check that this is equivalent to

$$\nabla^*_i V^j = g^{ja} \nabla_i (g_{ab} V^b), \quad \nabla^*_i \omega_j = g_{ja} \nabla_i (g^{ab} \omega_b). \tag{3.5}$$

Denoting by ∇^g the Levi-Civita connection associated to the metric g, namely the unique torsion-free affine connection which is metric ($\nabla^g g = 0$), it easily follows that

$$\nabla^g = \tfrac{1}{2}(\nabla + \nabla^*).$$

In addition, the volume forms $\nu = \nu_\xi$ and $\nu^* = \nu^*_\xi$ are conjugate with respect to the Riemannian volume form ν_g:

$$\nu(e_1, \ldots, e_{n-1}) \nu^*(e_1, \ldots, e_{n-1}) = \det((g^\xi(e_i, e_j))_{ij}) = \nu_g^2(e_1, \ldots, e_{n-1}). \tag{3.6}$$

Here and elsewhere, ν_g denotes the Riemannian volume form associated to the metric g.

3.6. Differential calculus

Recall that the divergence of a vector field X on M relative to an affine connection ∇ is defined as

$$\operatorname{div}^\nabla X := \operatorname{tr}\{Y \mapsto \nabla_Y X\} = \nabla_i X^i,$$

and that the Hessian of a function $f \in C^2(M)$ is defined as

$$\operatorname{Hess}^\nabla f(X, Y) := \nabla^2_{X,Y} f = (\nabla_X df)(Y) = X(Y(f)) - (\nabla_X Y)(f).$$

The Hessian $\operatorname{Hess}^\nabla f$ is a $(0, 2)$ tensor, which is in addition symmetric if the connection ∇ is torsion-free. By definition $\nabla_X f := X(f) = df(X)$.

Assume that a volume form ν satisfies $\nabla \nu = 0$. The divergence theorem implies that $\int_M \operatorname{div}^\nabla X \, d\nu = 0$, and so we have the integration-by-parts formula

$$\int_M f \operatorname{div}^\nabla X \, d\nu = -\int_M \nabla_X f \, d\nu \quad \forall X \in \Gamma^1(TM) \ \forall f \in C^1(M).$$

While an affine connection is the only structure needed to define the above differential operators, this is not the case whenever a trace over two simultaneouesly covariant or contravariant coordinates is required; in particular, there is no intrinsic definition of the Laplacian of $f \in C^2(M)$ as the trace of its Hessian. To make sense of this, one needs an extra metric structure g on M. In that case, we denote by $\mathrm{grad}_g\, f \in \Gamma^1(TM)$ the unique vector field satisfying

$$g(\mathrm{grad}_g\, f, X) = X(f) \quad \forall X \in TM,$$

and define

$$\Delta^{\nabla,g} f := \mathrm{div}^{\nabla}\, \mathrm{grad}_g\, f.$$

In particular, we have the following integration-by-parts formula for all $f, h \in C^2(M)$:

$$\int_M (\Delta^{\nabla,g} f) h\, d\nu = -\int_M (\mathrm{grad}_g\, f)(h)\, d\nu = -\int_M g(\mathrm{grad}_g\, f, \mathrm{grad}_g\, h)\, d\nu$$
$$= \int_M f(\Delta^{\nabla,g} h)\, d\nu.$$

In our setting, all of the above applies to both pairs (∇^ξ, ν_ξ) and $((\nabla^\xi)^*, \nu_\xi^*)$.

In a local frame

$$(\mathrm{grad}_g\, f)^i = g^{ij} f_j, \quad \Delta^{\nabla,g} f = \nabla_i (g^{ij} f_j).$$

Using (3.5), we see that

$$\Delta^{\nabla,g} f = \nabla_i (g^{ij} f_j) = g^{ij} \nabla_i^* f_j = \mathrm{tr}_g\ \mathrm{Hess}^{\nabla^*} f,$$

where ∇^* is the connection g-conjugate to ∇. In our setting, this applies to our g^ξ-conjugate connection pair ∇^ξ and $(\nabla^\xi)^*$.

3.7. Curvature

Recall that the curvature R of an affine connection ∇ on M is defined as the following $(1, 3)$ tensor:

$$\mathrm{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and that the Ricci $(0, 2)$ tensor Ric is defined by tracing:

$$\mathrm{Ric}(Y, Z) = \mathrm{tr}\{X \mapsto \mathrm{R}(X, Y)Z\}.$$

We denote the curvature and Ricci tensors of $\nabla = \nabla^\xi$ and $\nabla^* = (\nabla^\xi)^*$ by R, Ric and $\mathrm{R}^*, \mathrm{Ric}^*$, respectively. We subsequently omit the superscripts ξ and ξ^* in our various differential structures.

The Gauss equations for R and R^* are

$$\mathrm{R}(X, Y)Z = g(Y, Z)SX - g(X, Z)SY,$$
$$\mathrm{R}^*(X, Y)Z = \hat{S}(Y, Z)X - \hat{S}(X, Z)Y.$$

In particular, ∇^* is always projectively flat, and we have

$$g(\mathrm{R}(X, Y)Z, W) = -g(\mathrm{R}^*(X, Y)W, Z).$$

Note that the usual symmetries of the Riemann curvature tensor need not hold for the curvature tensor R of a general torsion-free affine connection, and that in general the Ricci tensor will not be symmetric; however, in our context, $\mathrm{Ric}, \mathrm{Ric}^*$ turn out to be always symmetric:

$$\mathrm{Ric}(Y,Z) = \operatorname{tr} S \; g(Y,Z) - \hat{S}(Y,Z), \quad \mathrm{Ric}^*(Y,Z) = (n-2)\hat{S}(Y,Z).$$

It easily follows that $\mathrm{R} = \mathrm{R}^*$ iff $\mathrm{Ric} = \mathrm{Ric}^*$ iff $S = \lambda \mathrm{Id}$, i.e. the hypersurface $x : M \to E$ is a relative affine sphere; a particular instance of this is when x is a Blaschke (equi)affine sphere – see below. In the context of statistical structures, if $\mathrm{R} = \mathrm{R}^*$ then (g, ∇, ∇^*) is called a conjugate symmetric statistical structure.

3.8. *Structure equations in a local frame*

Recall that in a local frame $\{e_1, \ldots, e_{n-1}\}$ on M, we use x_i to denote the E-valued 1-form $(dx)_i$, and similarly for the E- and E^*-valued ξ_i and ξ_i^*. Let us stress that x_i should not be confused with the i-th coordinate of x in E (especially since E is not $(n-1)$-dimensional and since no coordinate system has been introduced on E). We summarize the (vector-valued) structure equations for a relative normalization in a local frame [65, p. 33]:

$$\begin{aligned} \mathrm{Hess}_{ij}^{\nabla^\xi} x &= \nabla_j^\xi x_i = -g_{ij}^\xi \xi && \text{(Gauss equation for } x\text{)},\\ \mathrm{Hess}_{ij}^{(\nabla^\xi)^*} \xi^* &= (\nabla^\xi)_j^* \xi_i^* = -\hat{S}_{ij}^\xi \xi^* && \text{(Gauss equation for } \xi^*\text{)},\\ \xi_i &= (S^\xi)_i^k \, x_k && \text{(Weingarten equation for } \xi\text{)}. \end{aligned}$$

We also have

$$\begin{aligned} g_{ij}^\xi &= \langle \xi_j^*, x_i \rangle = \langle d\xi^*(e_j), dx(e_i) \rangle,\\ \hat{S}_{ij}^\xi &= g_{ik}^\xi (S^\xi)_j^k. \end{aligned}$$

3.9. *Blaschke's equiaffine normalization*

The fundamental theorem of affine differential geometry states that there exists a unique relative normal ξ_0 such that the Riemannian volume measure $\nu_{g^{\xi_0}}$ associated to the metric g^{ξ_0} coincides with the induced volume measure ν_{ξ_0}. Equivalently (up to orientation), this is the same as requiring that $|\nu_{\xi_0}| = |\nu_{\xi_0}^*|$. This unique ξ_0 is called the *Blaschke affine normal*, g^{ξ_0} is called the *Blaschke metric* (or second fundamental form), and (M, ξ_0) or $(M, g^{\xi_0}, \nabla^{\xi_0})$ are called a *Blaschke hypersurface*. Clearly, the Blaschke normalization is equiaffine invariant, and is sometimes called the equiaffine normalization. There are various natural geometric and analytic ways to explicitly define the Blaschke affine normal [56, 65, 85, 86], and various related problems such as characterizing all Blaschke affine spheres have been an extremely active avenue of research [25, 26, 32]. However, in this work, we focus on a different natural relative normalization.

3.10. Centro-affine normalization

Recall that $x : M \to E$ is assumed to be strongly convex, and we assume further that the origin of E lies on the inside of $x(M)$ (i.e. in the interior of the bounded component of $E \setminus x(M)$). The centro-affine normalization of the hypersurface x is given by

$$\xi := x,$$

which is a transversal normal field thanks to our assumptions. The induced centro-affine metric is denoted by $g = g^x$. Clearly, this normalization is centro-affine invariant, namely, invariant under regular linear transformations (but not affine ones). Furthermore, inspecting (3.3), it is clearly a relative normalization with identity shape operator, and consequently

$$S = \mathrm{Id}, \quad \hat{S} = g.$$

This means that any strongly convex hypersurface $x : M \to E$ is always a centro-affine sphere. In particular, the centro-affine sectional and Ricci curvatures are always constant:

$$\begin{aligned} \mathrm{R}(X,Y)Z = \mathrm{R}^*(X,Y)Z &= g(Y,Z)X - g(X,Z)Y, \\ \mathrm{Ric} = \mathrm{Ric}^* &= (n-2)g. \end{aligned}$$

If ξ^* is the associated conormal field, we define

$$x^* := \xi^*.$$

Note that

$$\langle x^*, x\rangle = 1, \quad \langle x^*, dx\rangle = 0, \quad \langle dx^*, x\rangle = 0. \tag{3.7}$$

The symmetry between x and x^* immediately implies that the dual and primal centro-affine normalizations are related by conjugation (cf. [62], [86, Proposition 7.2.1]). By this we mean the following: Denote the metric and pairs of conjugate connections and volume forms on M for the hypersurface $x : M \to E$ equipped with the normalization (M, x, x^*) by g^x, ∇^x, $(\nabla^x)^*$, ν_x and ν_x^*, and for the hypersurface $x^* : M \to E^*$ equipped with the normalization (M, x^*, x) by g^{x^*}, ∇^{x^*}, $(\nabla^{x^*})^*$, ν_{x^*} and $\nu_{x^*}^*$, respectively. Then

$$g^x = g^{x^*}, \quad \nabla^x = (\nabla^{x^*})^*, \quad (\nabla^x)^* = \nabla^{x^*}, \quad \nu_x = \nu_{x^*}^*, \quad \nu_x^* = \nu_{x^*}. \tag{3.8}$$

Note that the induced metric remains invariant under duality.

The structure equations for the centro-affine normalization in a local frame $\{e_1, \dots, e_{n-1}\}$ are

$$\begin{aligned} \mathrm{Hess}_{ij}^{\nabla^x} x &= \nabla_j^x x_i = -g_{ij}^x x \quad \text{(Gauss equation for } x), \\ \mathrm{Hess}_{ij}^{\nabla^{x^*}} x^* &= \nabla_j^{x^*} x_i^* = -g_{ij}^{x^*} x^* \quad \text{(Gauss equation for } x^*), \end{aligned} \tag{3.9}$$

where

$$g_{ij}^x = g_{ij}^{x^*} = \langle x_j^*, x_i\rangle = \langle dx^*(e_j), dx(e_i)\rangle.$$

4. Centro-affine differential geometry of convex bodies

Fix a smooth convex body K with strictly positive curvature in E having the origin in its interior, $K \in \mathcal{K}^\infty_+$. Recall that we use a fixed isomorphism i to identify between E and E^*, and use $\langle \cdot, \cdot \rangle$ to denote both the natural pairing between E^* and E and the induced Euclidean scalar product on E and E^* via i. Having fixed i and thus the Euclidean structures on E and E^*, we uniquely select the determinant form Det on E so that $\mathrm{Det}(v_1, \ldots, v_n) = \sqrt{\det(\langle v_i, v_j \rangle)}$ and thus $\mathrm{Det}^*(w^1, \ldots, w^n) = \sqrt{\det(\langle w^i, w^j \rangle)}$ for all $v_i \in E$ and $w^j \in E^*$.

Recall that $K^* \subset E^*$ denotes the dual body to K, and that $K^\circ \subset E$ is the corresponding polar body given via $i(K^\circ) = K^*$. Also recall that $\bar{D}$ denotes the standard flat covariant derivative on E and E^*.

We equip the strongly convex hypersurface ∂K with the centro-affine normalization.

4.1. Parametrizations

It will be instructive to consider a parametrization $x^M_K : M \to \partial K \subset E$ for several natural manifolds M:

$$M \in \mathcal{M}_K := \{\mathbb{S}^*, \partial K, \mathbb{S}, \partial K^*\}.$$

We denote the induced metric and pairs of conjugate connections and volume forms on M by

$$g^M_K, \nabla^M_K, (\nabla^M_K)^*, \nu^M_K, (\nu^M_K)^*. \tag{4.1}$$

We will not distinguish between the above volume forms ν and the corresponding volume measures $|\nu|$, using ν to denote both. We will sometimes write $O^{M\,*}_K$ instead of $(O^M_K)^*$ for $O \in \{g, \nabla, \nu\}$, especially when concatenating with another operation.

The above parametrizations of ∂K are naturally obtained by appropriately composing the Gauss maps on ∂K, ∂K^* and their inverses with the radial spherical projection $v \mapsto v/|v|$ in E and E^*. Formally, for all $M_i, M_j \in \mathcal{M}_K$, we specify diffeomorphisms $T^{M_i \to M_j}_K : M_i \to M_j$ so that

$$T^{M_1 \to M_3}_K = T^{M_2 \to M_3}_K \circ T^{M_1 \to M_2}_K. \tag{4.2}$$

They are obtained by composing the following diffeomorphisms:

$$\begin{array}{ccccccccc} \mathbb{S}^* & \to & \partial K & \to & \mathbb{S} & \to & \partial K^* & \to & \mathbb{S}^* \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \theta^* & \mapsto & x = \bar{D} h_K(\theta^*) & \mapsto & \theta = \frac{x}{|x|} & \mapsto & x^* = \bar{D}\|\theta\|_K & \mapsto & \theta^* = \frac{x^*}{|x^*|}. \end{array}$$

It will be useful to also explicitly specify the inverse diffeomorphisms:

$$\begin{array}{ccccccccc} \mathbb{S}^* & \to & \partial K^* & \to & \mathbb{S} & \to & \partial K & \to & \mathbb{S}^* \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \theta^* & \mapsto & x^* = \frac{\theta^*}{h_K(\theta^*)} & \mapsto & \theta = \frac{\bar{D} h_K(x^*)}{|\bar{D} h_K(x^*)|} & \mapsto & x = \frac{\theta}{\|\theta\|_K} & \mapsto & \theta^* = \frac{\bar{D}\|x\|_K}{|\bar{D}\|x\|_K|}. \end{array}$$

It is well-known and straightforward to check that the above cycles close up, so that indeed $T_K^{M\to M} = \mathrm{Id}$ for all M and (4.2) holds. Note that $T_K^{\partial K\to\mathbb{S}^*}$ and $T_K^{\partial K^*\to\mathbb{S}}$ are the Gauss maps for ∂K and ∂K^*, respectively.

It should already be clear (and will be verified below) that our parametrizations are understood in the following natural sense: $x \in \partial K$ is both the hypersurface and the centro-affine normal, $\theta^* \in \mathbb{S}^*$ is the unit outer normal, $x^* \in \partial K^*$ is the centro-affine conormal (the dual point to x on ∂K^*), and $\theta \in \mathbb{S}$ is the unit outer normal to ∂K^* at x^*, pointing in the direction of x and thereby closing the cycle.

Setting
$$x_K^M := T_K^{M\to\partial K},$$
our definitions ensure that the following diagram commutes:

$$\begin{array}{ccc} M_1 & & \\ {\scriptstyle T_K^{M_1\to M_2}}\downarrow & \searrow{\scriptstyle x_K^{M_1}} & \\ M_2 & \xrightarrow{x_K^{M_2}} & \partial K \subset E \end{array}$$

Consequently, $T_K^{M_1\to M_2}$ induces an isomorphism between the objects $O_K^{M_i}$ defined on $M_i \in \mathcal{M}_K$ for each $O \in \{g, \nabla, \nabla^*, \nu, \nu^*\}$, and so for each of these, it is enough to calculate $O_K^{M_1}$ on a single convenient parametrization M_1, thereby obtaining $O_K^{M_2}$ for all other $M_2 \in \mathcal{M}_K$ by pushing forward:
$$O_K^{M_2} = (T_K^{M_1\to M_2})_* O_K^{M_1}.$$

Our main object of interest will be O_K, regardless of the parametrization M, and so we will often omit the superscript M.

4.2. Duality

Our parametrizations are compatible with the natural duality operation $\mathcal{D}_K$. For every $M \in \mathcal{M}_K$, denote by $M^* \in \mathcal{M}_K$ its obvious dual counterpart (e.g. $(\partial K)^* = \partial K^*$ and $(\mathbb{S}^*)^* = \mathbb{S}$). Note that for $M \in \{\mathbb{S}, \mathbb{S}^*\}$, $M^* = i(M)$ but not in general. By abuse of notation, we use the same notation $\mathcal{D}_K$ (omitting the reference to M) to denote the diffeomorphism
$$\mathcal{D}_K := T_K^{M\to M^*} : M \to M^*, \quad M^* = \mathcal{D}_K(M).$$
It is worthwhile to note that
$$\mathcal{D}_K : \partial K \ni x \mapsto x^* \in \partial K^*, \quad x^* = \mathcal{D}_K x = \bar{D}\|x\|_K.$$
To quickly see this, note that $\bar{D}\|x\|_K$ is clearly perpendicular to ∂K, and that
$$\langle x^*, x\rangle = \langle \bar{D}\|x\|_K, x\rangle = \|x\|_K = 1 \quad \forall x \in \partial K, \tag{4.3}$$

by Euler's identity for the 1-homogeneous function $\|x\|_K$. Hence

$$h_K(\bar{D}\|x\|_K) = \langle \bar{D}\|x\|_K, x\rangle = 1 \quad \forall x \in \partial K,$$

and we confirm that $\mathcal{D}_K$ maps ∂K onto ∂K^*. Similarly,

$$\mathcal{D}_K : \partial K^* \ni x^* \mapsto x \in \partial K, \quad x = \mathcal{D}_K x^* = \bar{D} h_K(x^*),$$
$$\mathcal{D}_K : \mathbb{S}^* \ni \theta^* \mapsto \theta \in \mathbb{S}, \quad \theta = \mathcal{D}_K \theta^* = \frac{\bar{D} h_K}{|\bar{D} h_K|}(\theta^*).$$

Next, we tautologically extend our construction to strongly convex bodies in E^* (and not just in E). Recall that $K^\circ \subset E$ and $K^* \subset E^*$ are related by $i(K^\circ) = K^*$, and hence $i(\partial K) = \partial(K^\circ)^* = (\partial K^\circ)^*$. We consequently define

$$\mathcal{M}_{K^*} := \mathcal{M}_K^* = i(\mathcal{M}_{K^\circ}),$$

and set

$$T_{K^*}^{M_1 \to M_2} = i \circ T_{K^\circ}^{i(M_1) \to i(M_2)} \circ i \quad \forall M_1, M_2 \in \mathcal{M}_{K^*}.$$

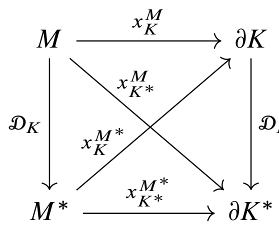
In particular,

$$x_{K^*}^M = i \circ x_{K^\circ}^{i(M)} \circ i.$$

As the identification between E and E^* via i is tautological and does not change any differential structure, we have, for every $M \in \mathcal{M}_K$,

$$g_{K^\circ}^{i(M)} = i_* g_{K^*}^M, \quad \nabla_{K^\circ}^{i(M)} = i_* \nabla_{K^*}^M, \quad (\nabla_{K^\circ}^{i(M)})^* = i_* (\nabla_{K^*}^M)^*,$$
$$\nu_{K^\circ}^{i(M)} = i_* \nu_{K^*}^M, \quad (\nu_{K^\circ}^{i(M)})^* = i_* (\nu_{K^*}^M)^*,$$

where i_* denote the push-forward via i. In addition, we see that our construction is compatible with the duality operator $\mathcal{D}_K$, in the sense that the following diagram commutes:



(4.4)

As discussed in Section 3.10, the duality operation is important in view of its conjugation role for the centro-affine normalization. Observe that the conormal $(x_K^M)^* : M \to E^*$ corresponding to the centro-affine normal $x_K^M : M \to E$ of the hypersurface ∂K is precisely given by

$$(x_K^M)^* := \mathcal{D}_K \circ x_K^M = x_{K^*}^M.$$

This follows immediately by verifying the validity of the defining equations (3.7); indeed, as already explained in (4.3), $x^* = \mathcal{D}_K x$ is perpendicular to ∂K and satisfies $\langle x^*, x\rangle = 1$.

Consequently, it follows from (3.8) that the induced structures on M via $x^M_K : M \to \partial K$ and via $x^M_{K^*} : M \to \partial K^*$ are conjugate to each other:

$$g^M_K = g^M_{K^*}, \nabla^M_K = (\nabla^M_{K^*})^*, (\nabla^M_K)^* = \nabla^M_{K^*}, \nu^M_K = (\nu^M_{K^*})^*, (\nu^M_K)^* = \nu^M_{K^*}.$$

Note that the induced centro-affine Riemannian metric g^M_K is self-dual, as is well-known [50,62,86]. We emphasize several equivalent alternative forms of the above duality, which follow immediately from the commutation in (4.4):

$$\begin{aligned}
g^{i(M^*)}_{K^\circ} &= i_* g^{M^*}_{K^*} &&= i_* g^{M^*}_K &&= (i \circ \mathcal{D}_K)_* g^M_K,\\
\nabla^{i(M^*)}_{K^\circ} &= i_* \nabla^{M^*}_{K^*} &&= i_* (\nabla^{M^*}_K)^* &&= (i \circ \mathcal{D}_K)_* (\nabla^M_K)^*,\\
(\nabla^{i(M^*)}_{K^\circ})^* &= i_* (\nabla^{M^*}_{K^*})^* &&= i_* \nabla^{M^*}_K &&= (i \circ \mathcal{D}_K)_* \nabla^M_K,\\
\nu^{i(M^*)}_{K^\circ} &= i_* \nu^{M^*}_{K^*} &&= i_* (\nu^{M^*}_K)^* &&= (i \circ \mathcal{D}_K)_* (\nu^M_K)^*,\\
(\nu^{i(M^*)}_{K^\circ})^* &= i_* (\nu^{M^*}_{K^*})^* &&= i_* \nu^{M^*}_K &&= (i \circ \mathcal{D}_K)_* \nu^M_K.
\end{aligned}$$

In other words, up to conjugation, $i : M^* \to i(M^*)$ pushes forward $O^{M^*}_K$ onto $O^{i(M^*)}_{K^\circ}$, and $i \circ \mathcal{D}_K : M \to i(M^*)$ pushes forward O^M_K onto $O^{i(M^*)}_{K^\circ}$, for $O \in \{g, \nabla, \nabla^*, \nu, \nu^*\}$. The latter is particularly useful when $M \in \{\mathbb{S}, \mathbb{S}^*\}$ since then $i(M^*) = M$.

It is a good exercise to verify directly that, e.g., $i \circ \mathcal{D}_K$ pushes forward $g^{\mathbb{S}^*}_K$ onto $g^{\mathbb{S}^*}_{K^\circ}$.

4.3. Centro-affine invariance

Let $A \in GL(E)$. The centro-affine invariance of the centro-affine normalization immediately implies that A pushes forward $O^{\partial K}_K$ onto $O^{\partial A(K)}_{A(K)}$ for all $O \in \{g, \nabla, \nabla^*\}$, as well as $O \in \{\nu, \nu^*\}$ whenever $A \in SL(E)$. As $A(K)^* = A^{-*}(K)$, it is also easy to see that A^{-*} pushes forward $O^{\partial K^*}_K$ onto $O^{\partial A(K)^*}_{A(K)}$ in the same manner as above.

The situation with our other parametrizations $M \in \{\mathbb{S}, \mathbb{S}^*\}$ requires a bit more thought. While these parametrizations are very natural from a geometric perspective, they are not centro-affine co- or contra-variant, e.g. it is not true that $x^{\mathbb{S}}_{A(K)} = A \circ x^{\mathbb{S}}_K$. Consequently, the centro-affine invariance from Section 3.3 only holds after a suitable change of variables:

Proposition 4.1. *Given $A \in GL(E)$, denote*

$$A^{(0)} : \mathbb{S} \to \mathbb{S}, \quad A^{(0)}(\theta) = \frac{A\theta}{|A\theta|},$$
$$(A^{-*})^{(0)} : \mathbb{S}^* \to \mathbb{S}^*, \quad (A^{-*})^{(0)}(\theta^*) = \frac{A^{-*}\theta^*}{|A^{-*}\theta^*|}.$$

Then

$$x^{\mathbb{S}}_{A(K)} = A \circ x^{\mathbb{S}}_K \circ (A^{(0)})^{-1}, \quad x^{\mathbb{S}^*}_{A(K)} = A^* \circ x^{\mathbb{S}^*}_K \circ ((A^{-*})^{(0)})^{-1}. \tag{4.5}$$

Consequently:

(1) *$A^{(0)}$ pushes forward $O^{\mathbb{S}}_K$ onto $O^{\mathbb{S}}_{A(K)}$ for all $O \in \{g, \nabla, \nabla^*\}$, and for $O \in \{\nu, \nu^*\}$ whenever $A \in SL(E)$.*

(2) $(A^{-*})^{(0)}$ *pushes forward* $O_K^{\mathbb{S}^*}$ *onto* $O_{A(K)}^{\mathbb{S}^*}$ *for all* $O \in \{g, \nabla, \nabla^*\}$*, and for* $O \in \{\nu, \nu^*\}$ *whenever* $A \in SL(E)$.

Proof. Recall that $x_K^{\mathbb{S}}(\theta) = \frac{\theta}{\|\theta\|_K}$, which is 0-homogeneous on E. Hence

$$x_{A(K)}^{\mathbb{S}}(\theta) = \frac{\theta}{\|\theta\|_{A(K)}} = \frac{\theta}{\|A^{-1}\theta\|_K} = A\frac{A^{-1}\theta}{\|A^{-1}\theta\|_K} = A \circ x_K^{\mathbb{S}}(A^{-1}\theta/|A^{-1}\theta|),$$

and the first identity in (4.5) follows. Also recall that $x_K^{\mathbb{S}^*}(\theta^*) = \bar{D}h_K(\theta^*)$, which is 0-homogeneous on E^*. Hence

$$\begin{aligned} x_{A(K)}^{\mathbb{S}^*}(\theta^*) &= \bar{D}h_{A(K)}(\theta^*) = \bar{D}(h_K(A^*\theta^*)) = A^*\bar{D}h_K(A^*\theta^*) \\ &= A^*\bar{D}h_K(A^*\theta^*/|A^*\theta^*|), \end{aligned}$$

and the second identity in (4.5) follows.

As for the second part of the proposition, let us only verify (2) (as the verification of (1) is identical). Denoting $x_{\text{aux}}^{\mathbb{S}^*} := x_K^{\mathbb{S}^*} \circ ((A^{-*})^{(0)})^{-1}$, it follows by usual centro-affine invariance (as in Section 3.3) that the two hypersurfaces $x_{\text{aux}}^{\mathbb{S}^*}, x_{A(K)}^{\mathbb{S}^*} : \mathbb{S}^* \to E$ (equipped with the centro-affine normalization) induce exactly the same metric, normal and conormal connections g, ∇, ∇^* on $\mathbb{S}^*$, and also the same volume measures ν, ν^* whenever $A \in SL(E)$. It remains to note that the two hypersurfaces $x_K^{\mathbb{S}^*}, x_{\text{aux}}^{\mathbb{S}^*} : \mathbb{S}^* \to E$ are identical, up to reparametrization of $\mathbb{S}^*$ via $(A^{-*})^{(0)}$; consequently, whenever these two hypersurfaces are equipped with the same normalization (as in context), their induced differential structures are isomorphic via $(A^{-*})^{(0)}$. ■

It is a good exercise to verify directly that, e.g., $A_*^{(0)}$ pushes forward $g_K^{\mathbb{S}^*}$ onto $g_{A(K)}^{\mathbb{S}^*}$.

4.4. *Explicit formulas*

We now calculate the centro-affine differential structures (4.1). As explained above, it is enough to perform the calculation on a convenient parametrization $M \in \mathcal{M}_K$. The most convenient choice for us is $M = \mathbb{S}^*$, but we also provide the corresponding expressions in other parametrizations. Recall from Section 2 the definitions of $V_K^{\mathbb{S}^*}$, $V_K^{\partial K}$ and $V_K^{\mathbb{S}}$, that D^2h_K denotes the restriction of $\bar{D}^2h_K$ onto $T\mathbb{S}^*$, and the discussion regarding induced Euclidean structures.

4.4.1. $M = \mathbb{S}^*$. We work on $M = \mathbb{S}^*$ and omit the superscript $\mathbb{S}^*$ (and often the subscript K as well) in our expressions. Recall that

$$\begin{aligned} x = x_K : \mathbb{S}^* \to \partial K, \quad & x(\theta^*) = \bar{D}h_K(\theta^*), \\ x^* = (x_K)^* = x_{K^*} : \mathbb{S}^* \to \partial K^*, \quad & x^*(\theta^*) = \frac{\theta^*}{h_K(\theta^*)}. \end{aligned}$$

We perform all calculations at a fixed $\theta^* \in \mathbb{S}^*$. Then

$$dx : T_{\theta^*}\mathbb{S}^* \to T_x\partial K, \quad dx(u) = \bar{D}_u\bar{D}h_K,$$

$$dx^* : T_{\theta^*}\mathbb{S}^* \to T_{x^*}\partial K^*, \quad dx^*(v) = \frac{1}{h_K}(v - v(\log h_K)\theta^*).$$

Since h_K is 1-homogeneous, it follows that $\bar{D}_{\theta^*}\bar{D}h_K = 0$. Consequently,

$$g_K^{\mathbb{S}^*}(u,v) = \langle dx^*(v), dx(u)\rangle = \frac{\bar{D}^2 h_K(u,v)}{h_K} = \frac{D^2 h_K(u,v)}{h_K}, \quad u, v \in T_{\theta^*}\mathbb{S}^*. \quad (4.6)$$

For a Euclidean orthonormal basis $e_1, \ldots, e_{n-1}$ in $T_{\theta^*}\mathbb{S}^*$, we have

$$\begin{aligned} \frac{d\nu_K}{d\mathfrak{m}^{\mathbb{S}^*}} &= \nu_K(e_1, \ldots, e_{n-1}) = \mathrm{Det}(dx(e_1), \ldots, dx(e_n), x) \\ &= h_K \,\mathrm{Det}(dx(e_1), \ldots, dx(e_{n-1}), \theta^*) = h_K \det(D^2 h_K) = n\frac{dV_K^{\mathbb{S}^*}}{d\mathfrak{m}^{\mathbb{S}^*}}, \end{aligned}$$

where we have used the fact that $P_{(T\partial K)^\perp} x = \langle \theta^*, x\rangle\theta^* = h_K\theta^*$. Similarly,

$$\begin{aligned} \frac{d\nu_K^*}{d\mathfrak{m}^{\mathbb{S}^*}} &= \nu_K^*(e_1, \ldots, e_{n-1}) = \mathrm{Det}^*(dx^*(e_1), \ldots, dx^*(e_n), x^*) \\ &= \frac{1}{h_K^n}\mathrm{Det}^*(e_1 - e_1(\log h_K)\theta^*, \ldots, e_{n-1} - e_{n-1}(\log h_K)\theta^*, \theta^*) \\ &= \frac{1}{h_K^n} = n\frac{d i_* V_{K^\circ}^{\mathbb{S}}}{d\mathfrak{m}^{\mathbb{S}^*}}. \end{aligned}$$

By the Gauss equation for x we have, for $U \in T\mathbb{S}^*$, $V \in \Gamma^1(T\mathbb{S}^*)$,

$$\bar{D}_U dx(V) = dx(\nabla_U V) - g_K(U,V)x.$$

Consequently,

$$\bar{D}^2_{U,V}\bar{D}h_K + \bar{D}_{\bar{D}_U V}\bar{D}h_K = \bar{D}_{\nabla_U V}\bar{D}h_K - g_K(U,V)x.$$

It follows that

$$\bar{D}^2 h_K(\nabla_U V - \bar{D}_U V, \xi^*) = \bar{D}^3 h_K(U, V, \xi^*) + g_K(U,V)\langle \xi^*, x\rangle \quad \forall \xi^* \in E^*.$$

Recalling that $g_K = \frac{\bar{D}^2 h_K}{h_K}$, $x = \bar{D}h_K$, $\bar{D}_U V = {}^{\mathbb{S}^*}\nabla_U V - \mathrm{II}^{\mathbb{S}^*}(U,V)\theta^*$ and $\bar{D}^2 h_K \cdot \theta^* = 0$, it follows that

$$g_K(\nabla_U V - {}^{\mathbb{S}^*}\nabla_U V, \xi^*) = \frac{\bar{D}^3 h_K(U,V,\xi^*)}{h_K} + g_K(U,V)\langle \xi^*, \bar{D}(\log h_K)\rangle \quad \forall \xi^* \in T_{\theta^*}\mathbb{S}^*.$$

Introducing a local frame $\{e_1, \ldots, e_{n-1}\}$ on $\mathbb{S}^*$, it follows that

$$(\nabla_U V - {}^{\mathbb{S}^*}\nabla_U V)^i = g_K^{ij}\left(\frac{\bar{D}^3 h_K(U,V,e_j)}{h_K} + g_K(U,V)(\log h_K)_j\right).$$

As expected, this expression depends on third derivatives of h_K.

A much more useful expression is obtained for the conjugate connection. By the Gauss equation for x^* (recall that $\hat{S} = g$ in the centro-affine normalization),

$$\bar{D}_U dx^*(V) = dx^*(\nabla^*_U V) - g_K(U, V)x^*,$$

i.e.

$$\bar{D}_U\left(\frac{V}{h_K} - \frac{V(h_K)}{h_K^2}\theta^*\right) = \frac{1}{h_K}(\nabla^*_U V - (\nabla^*_U V)(\log h_K)\theta^*) - g_K(U, V)\frac{\theta^*}{h_K(\theta^*)}.$$

Consequently, applying the Leibniz rule and multiplying by h_K, we have

$$-\frac{1}{h_K}U(h_K)V + \bar{D}_U V - h_K U(V(h_K)/h_K^2)\theta^* - V(\log h_K)\bar{D}_U\theta^* \\ = \nabla^*_U V - (\nabla^*_U V)(\log h_K)\theta^* - g_K(U, V)\theta^*.$$

Recall that $U \in T_{\theta^*}\mathbb{S}^*$ so that $\bar{D}_U\theta^* = U$. Orthogonally projecting onto $(\theta^*)^\perp$, we obtain

$$-U(\log h_K)V + {}^{\mathbb{S}^*}\nabla_U V - V(\log h_K)U = \nabla^*_U V.$$

In particular, we see that the conjugate connection only depends on first derivatives of h_K, which is already reassuring. By projecting onto θ^* and recalling (2.1), one rederives (4.6); for completeness, let us verify this:

$$\begin{aligned} g_K(u, v) &= \mathrm{II}^{\mathbb{S}^*}(U, V) + h_K U(V(h_K)/h_K^2) - (\nabla^*_U V)(\log h_K) \\ &= \mathrm{II}^{\mathbb{S}^*}(U, V) + \frac{{}^{\mathbb{S}^*}\nabla^2_{U,V}h_K}{h_K} + \frac{{}^{\mathbb{S}^*}\nabla_U V(h_K)}{h_K} - 2\frac{U(h_K)V(h_K)}{h_K^2} \\ &\quad + 2U(\log h_K)V(\log h_K) - {}^{\mathbb{S}^*}\nabla_U V(\log h_K) \\ &= \mathrm{II}^{\mathbb{S}^*}(U, V) + \frac{{}^{\mathbb{S}^*}\nabla^2_{U,V}h_K}{h_K} = \frac{\bar{D}^2 h_K(U, V)}{h_K}, \end{aligned}$$

where in the last transition we have used (2.2) and the fact that h_K is 1-homogeneous so that $\theta^*(h_K) = h_K$.

We summarize all of these computations in the following:

Proposition 4.2. *The differential structures* (4.1) *for the centro-affine normalization of K are given on $\mathbb{S}^*$ by*

$$\begin{aligned} g_K^{\mathbb{S}^*} &= \frac{D^2 h_K}{h_K}, \\ \nu_K^{\mathbb{S}^*} &= h_K S_K = h_K \det(D^2 h_K)\mathfrak{m}^{\mathbb{S}^*} = nV_K^{\mathbb{S}^*}, \\ (\nu_K^{\mathbb{S}^*})^* &= \frac{1}{h_K^n}\mathfrak{m}^{\mathbb{S}^*} = n i_* V_{K^\circ}^{\mathbb{S}}, \\ ((\nabla_K^{\mathbb{S}^*})_U V)^i &= ({}^{\mathbb{S}^*}\nabla_U V)^i + (g_K^{\mathbb{S}^*})^{ij}\left(\frac{\bar{D}^3 h_K(U, V, e_j)}{h_K} + g_K^{\mathbb{S}^*}(U, V)(\log h_K)_j\right), \\ (\nabla_K^{\mathbb{S}^*})^*_U V &= {}^{\mathbb{S}^*}\nabla_U V - U(\log h_K)V - V(\log h_K)U \end{aligned}$$

(for any $U \in T\mathbb{S}^*$, $V \in \Gamma^1(T\mathbb{S}^*)$ and local frame $\{e_1, \ldots, e_{n-1}\}$ on $\mathbb{S}^*$). In particular, the centro-affine metric $g_K^{\mathbb{S}^*}$ coincides with the metric (2.5) introduced in Section 2, and up to normalization, the centro-affine volume measure $\nu_K^{\mathbb{S}^*}$ coincides with the cone-volume measure $V_K^{\mathbb{S}^*}$.

In addition, recall that the following useful properties hold, regardless of parametrization:

- $\nabla_K \nu_K = \nabla_K^* \nu_K^* = 0$.
- K is a centro-affine unit sphere; in particular, the Ricci curvatures of ∇_K and ∇_K^* are constant and equal to $n-2$.

In our opinion, it is quite remarkable that the Ricci curvature turns out to even just be positive, let alone constant, for the centro-affine connection, in view of the fact that it involves three derivatives of h_K. It is a good but tedious exercise to verify this for both our connections directly from the associated Christoffel symbols on $\mathbb{S}^*$:

$$\begin{aligned} {}^{\nabla_K^*}\Gamma_{ij}^k &= {}^{\mathbb{S}^*}\Gamma_{ij}^k - \delta_i^k (\log h_K)_j - \delta_j^k (\log h_K)_i \\ {}^{\nabla_K}\Gamma_{ij}^k &= {}^{\mathbb{S}^*}\Gamma_{ij}^k + g_K^{kp}\left(\frac{\bar{D}^3_{ijp} h_K}{h_K} + (g_K)_{ij} (\log h_K)_p\right). \end{aligned} \tag{4.7}$$

Another good exercise is to verify that our connections are indeed g_K-conjugates using (3.5).

4.4.2. Other parametrizations. It will be convenient to also work on the parametrization $M = \partial K$ (we henceforth omit the corresponding superscript in our notation). Recall that

$$x = x_K : \partial K \to \partial K, \quad x = \mathrm{Id},$$
$$x^* = (x_K)^* = x_{K^*} : \partial K \to \partial K^*, \quad x^*(x) = \bar{D}\|x\|_K = \bar{D} h_{K^*}(x).$$

We fix a point on ∂K, which by abuse of notation we denote by x (there should be no confusion with the map x). Hence

$$dx : T_x \partial K \to T_x \partial K, \quad dx = \mathrm{Id},$$
$$dx^* : T_x \partial K \to T_{x^*} \partial K^*, \quad dx^*(v) = \bar{D}_v \bar{D} h_{K^*}.$$

Hence

$$g_K^{\partial K}(u, v) := \langle dx^*(v), dx(u) \rangle = \bar{D}_x^2 h_{K^*}(u, v), \quad u, v \in T_x \partial K.$$

Note that $\bar{D}_x^2 h_{K^*} \cdot x = 0$, whereas $u, v \perp \theta^*$; to emphasize this point, we write

$$g_K^{\partial K} = P_{(\theta^*)^\perp} \bar{D}_x^2 h_{K^*} P_{(\theta^*)^\perp}, \tag{4.8}$$

where P_H denotes orthogonal projection onto the corresponding subspace H. A more convenient expression is derived in

Lemma 4.3. *For all $x \in \partial K$,*

$$g_K^{\partial K}(x) = |x^*| \, \mathrm{II}_x^{\partial K} = \frac{\mathrm{II}_x^{\partial K}}{h_K(\theta^*)}.$$

Proof. It is well-known (e.g. [96, (2.48)]) that $\mathrm{II}_x^{\partial K} = (D^2_{\theta^*} h_K)^{-1} = \frac{1}{|x^*|}(D^2_{x^*} h_K)^{-1}$ (naturally identifying between the corresponding tangent spaces); using this and the duality between K and K^*, it is a nice exercise to derive the assertion from (4.8) by verifying that

$$P_{(\theta^*)^\perp} \bar{D}^2_x h_{K^*} P_{(\theta^*)^\perp} = (D^2_{x^*} h_K)^{-1}.$$

Alternatively, it is simpler to pull back $g_K^{\mathbb{S}^*} = \frac{D^2_{\theta^*} h_K}{h_K(\theta^*)}$ via the Gauss map $\partial K \ni x \mapsto \theta^* \in \mathbb{S}^*$ since $d_{\theta^*} x = D^2_{\theta^*} h_K$. Probably the simplest argument is to recall that the centro-affine conormalization of ∂K by x^* coincides with the Euclidean conormalization by θ^* up to a multiplicative factor of $|x^*|$, and hence the corresponding induced second fundamental forms $g_K^{\partial K}(x)$ and $\mathrm{II}_x^{\partial K}$ also coincide up to this factor [65, Proposition 1.23 (ii)]. Note that $|x^*| h_K(\theta^*) = |x^*| \langle \theta^*, x\rangle = \langle x^*, x\rangle = 1$. ∎

We leave the rest of the computations of our structures for the reader, as they will not be needed, and only state them. One can use all of the tools developed in the previous subsections to transfer the information from $\mathbb{S}^*$ to any other $M \in \mathcal{M}_K$: direct computation, pushing forward via our diffeomorphisms $T_K^{\mathbb{S}^* \to M}$, conjugation and duality. See also [65, Proposition 1.23].

Proposition 4.4. *The differential structures (4.1) for the centro-affine normalization of K are given on ∂K at $x \in \partial K$ by*

$$\begin{aligned}
g_K^{\partial K} &= P_{(\theta^*)^\perp} \bar{D}^2_x h_{K^\circ} P_{(\theta^*)^\perp} = |x^*|\, \mathrm{II}_x^{\partial K} = \frac{\mathrm{II}_x^{\partial K}}{h_K(\theta^*)},\\
\nu_K^{\partial K} &= \langle \theta^*, x\rangle \mathcal{H}^{n-1}|_{\partial K}(dx) = n V_K^{\partial K},\\
(\nu_K^{\partial K})^* &= \frac{\kappa_x^{\partial K}}{\langle \theta^*, x\rangle^n} \mathcal{H}^{n-1}|_{\partial K}(dx) =: n V_{K^\circ}^{\partial K},\\
(\nabla_K^{\partial K})_U V &= {}^{\partial K}\nabla_U V + g_K^{\partial K}(U, V) P_{(\theta^*)^\perp} x,\\
((\nabla_K^{\partial K})^*_U V)^i &= ({}^{\partial K}\nabla_U V)^i + (g_K^{\partial K})^{ij} \big(\bar{D}^3 h_{K^*}(U, V, e_j) + g_K^{\partial K}(U, V)(h_{K^*})_j\big)
\end{aligned}$$

(for any $U \in T_x\partial K$, $V \in \Gamma^1(T\partial K)$ and local frame $\{e_1, \ldots, e_{n-1}\}$ on ∂K). Here $\kappa_x^{\partial K} = \det \mathrm{II}_x^{\partial K} = 1/\det(D^2_{\theta^*} h_K)$ *denotes the Gauss curvature of ∂K at x.*

Proposition 4.5. *The differential structures (4.1) for the centro-affine normalization of K are given on $\mathbb{S}$ at $\theta \in \mathbb{S}$ by*

$$\begin{aligned}
g_K^{\mathbb{S}} &= g_{K^*}^{\mathbb{S}} = i_* g_{K^\circ}^{\mathbb{S}^*} = \frac{D^2 h_{K^*}}{h_{K^*}},\\
\nu_K^{\mathbb{S}} &= (\nu_{K^*}^{\mathbb{S}})^* = i_* (\nu_{K^\circ}^{\mathbb{S}^*})^* = \frac{1}{h_{K^*}^n} \mathfrak{m}^{\mathbb{S}} = \frac{1}{\|\cdot\|_K^n} \mathfrak{m}^{\mathbb{S}} = n V_K^{\mathbb{S}},\\
(\nu_K^{\mathbb{S}})^* &= \nu_{K^*}^{\mathbb{S}} = i_* \nu_{K^\circ}^{\mathbb{S}^*} = h_{K^*} S_{K^*} = h_{K^*} \det(D^2 h_{K^*}) \mathfrak{m}^{\mathbb{S}} = n i_* V_{K^\circ}^{\mathbb{S}^*},\\
(\nabla_K^{\mathbb{S}})_U V &= (\nabla_{K^*}^{\mathbb{S}\,*})_U V = (i_* \nabla_{K^\circ}^{\mathbb{S}^*\,*})_U V\\
&= {}^{\mathbb{S}}\nabla_U V - U(\log h_{K^*}) V - V(\log h_{K^*}) U,
\end{aligned}$$

$$((\nabla_K^{\mathbb{S}})_U^* V)^i = ((\nabla_{K^*}^{\mathbb{S}})_U V)^i = ((i_* \nabla_{K^\circ}^{\mathbb{S}^*})_U V)^i$$
$$= ({}^{\mathbb{S}}\nabla_U V)^i + (g_{K^*}^{\mathbb{S}})^{ij} \left(\frac{\bar{D}^3 h_{K^*}(U, V, e_j)}{h_{K^*}} + g_{K^*}(U, V)(\log h_{K^*})_j \right)$$

(*for any* $U \in T\mathbb{S}$, $V \in \Gamma^1(T\mathbb{S})$ *and local frame* $\{e_1, \ldots, e_{n-1}\}$ *on* $\mathbb{S}$).

Proposition 4.6. *The differential structures* (4.1) *for the centro-affine normalization of* K *are given on* ∂K^* *at* $x^* \in \partial K^*$ *by*

$$g_K^{\partial K^*} = g_{K^*}^{\partial K^*} = i_* g_{K^\circ}^{\partial K^\circ} = P_{\theta^\perp} \bar{D}_{x^*}^2 h_K P_{\theta^\perp} = |x| \, \mathrm{II}_{x^*}^{\partial K^*} = \frac{\mathrm{II}_{x^*}^{\partial K^*}}{h_{K^*}(\theta)},$$
$$\nu_K^{\partial K^*} = (\nu_{K^*}^{\partial K^*})^* = i_*(\nu_{K^\circ}^{\partial K^\circ})^* = \frac{\kappa_{x^*}^{\partial K^*}}{\langle x^*, \theta \rangle^n} \mathcal{H}^{n-1}|_{\partial K^*}(dx^*) = n i_* V_K^{\partial K^\circ},$$
$$(\nu_K^{\partial K^*})^* = \nu_{K^*}^{\partial K^*} = i_* \nu_{K^\circ}^{\partial K^\circ} = \langle x^*, \theta \rangle \mathcal{H}^{n-1}|_{\partial K^*}(dx^*) = n i_* V_{K^\circ}^{\partial K^\circ},$$
$$((\nabla_K^{\partial K^*})_U V)^i = ((\nabla_{K^*}^{\partial K^*})_U^* V)^i = ((i_* \nabla_{K^\circ}^{\partial K^\circ \, *})_U V)^i$$
$$= ({}^{\partial K^*}\nabla_U V)^i + (g_K^{\partial K^*})^{ij} \big(\bar{D}^3 h_K(U, V, e_j) + g_K^{\partial K^*}(U, V)(h_K)_j \big),$$
$$(\nabla_K^{\partial K^*})_U^* V = (\nabla_{K^*}^{\partial K^*})_U V = (i_* \nabla_{K^\circ}^{\partial K^\circ})_U V = {}^{\partial K^*}\nabla_U V + g_K^{\partial K^*}(U, V) P_{\theta^\perp} x^*$$

(*for any* $U \in T_{x^*}\partial K^*$, $V \in \Gamma^1(T\partial K^*)$ *and local frame* $\{e_1, \ldots, e_{n-1}\}$ *on* ∂K^*). *Here* $\kappa_{x^*}^{\partial K^*} = \det \mathrm{II}_{x^*}^{\partial K^*} = 1/\det(D_\theta^2 h_{K^*})$ *denotes the Gauss curvature of* ∂K^* *at* x^*.

4.5. Differential calculus and the Hilbert–Brunn–Minkowski operator

We denote the centro-affine divergence and Hessian operators by

$$\mathrm{div}_K^M := \mathrm{div}^{\nabla_K^M}, \qquad (\mathrm{div}_K^M)^* = \mathrm{div}^{(\nabla_K^M)^*}$$
$$\mathrm{Hess}_K^M := \mathrm{Hess}^{\nabla_K^M}, \quad (\mathrm{Hess}_K^M)^* = \mathrm{Hess}^{(\nabla_K^M)^*},$$

omitting the superscript M when the context is clear.

Lemma 4.7. *In a local frame on* $\mathbb{S}^*$ *we have, for any* $f \in C^2(\mathbb{S}^*)$,

$$(\mathrm{Hess}_K^*)_{ij} f = {}^{\mathbb{S}^*}\nabla_{ij}^2 f + (\log h_K)_i f_j + (\log h_K)_j f_i.$$

Proof. Recall that

$$(\mathrm{Hess}_K^*)_{ij} f = \partial_{ij}^2 f - {}^{\nabla_K^*}\Gamma_{ij}^k \partial_k f.$$

Plugging the expression for the Christoffel symbols on $\mathbb{S}^*$ derived in Proposition 4.2 and recorded in (4.7), the assertion immediately follows. ∎

Recalling the discussion in Section 3.6, we denote the corresponding centro-affine Laplacian operators by

$$\Delta_K^M := \mathrm{div}_K^M \ \mathrm{grad}_{g_K^M} = \mathrm{tr}_{g_K^M} (\mathrm{Hess}_K^M)^*, \quad (\Delta_K^M)^* := (\mathrm{div}_K^M)^* \ \mathrm{grad}_{g_K^M} = \mathrm{tr}_{g_K^M} \mathrm{Hess}_K^M.$$

Recall that the Hilbert–Brunn–Minkowski operator on $\mathbb{S}^*$ was introduced in Section 2 as the weighted Laplacian on $(\mathbb{S}^*, g_K^{\mathbb{S}^*}, V_K^{\mathbb{S}^*})$, and was denoted by $\Delta_K^{\mathbb{S}^*}$. While the reader may be concerned that there will be some ambiguity due to our identical notation for the Hilbert–Brunn–Minkowski and centro-affine Laplacian operators, our most important observation in this section is that there is no ambiguity. We omit the particular parametrization M, as it is irrelevant in the statement below.

Theorem 4.8. *The centro-affine Laplacian Δ_K coincides with the Hilbert–Brunn–Minkowski operator.*

Proof. We verify the claim on $\mathbb{S}^*$. By Lemma 4.7,

$$\Delta_K^{\mathbb{S}^*} f = g_K^{ij}(\mathrm{Hess}_K^* f)_{ij} = g_K^{ij}\big({}^{\mathbb{S}^*}\nabla_{ij}^2 f + (\log h_K)_i f_j + (\log h_K)_j f_i\big).$$

Recalling that centro-affine metric $g_K = g_K^{\mathbb{S}^*}$ coincides with the metric (2.5), we confirm that the right-hand side coincides with the Hilbert–Brunn–Minkowski operator (2.6). ∎

Theorem 4.8 finally gives a satisfactory explanation for the centro-affine equivariance property of the Hilbert–Brunn–Minkowski operator, originally observed in [61, Section 5.2] following a lengthy computation, but now an immediate consequence of Proposition 4.1.

Recall from Section 3.6 that (regardless of the parametrization M)

$$\int (-\Delta_K f) h \, d\nu_K = \int g_K(\mathrm{grad}_{g_K} f, \mathrm{grad}_{g_K} h) \, d\nu_K = \int f(-\Delta_K h) \, d\nu_K \tag{4.9}$$

for all $f, h \in C^2(M)$. Recall from Proposition 4.2 that on $\mathbb{S}^*$, $\nu_K^{\mathbb{S}^*}$ coincides (up to normalization) with the cone-volume measure $V_K^{\mathbb{S}^*}$. In [61, Section 5.1], we had originally (implicitly) identified the metric $g_K^{\mathbb{S}^*}$ by performing integration by parts in (4.9) with respect to V_K and computing the Dirichlet form, thereby interpreting the Hilbert–Brunn–Minkowski operator as the weighted Laplacian on $(\mathbb{S}^*, g_K^{\mathbb{S}^*}, V_K^{\mathbb{S}^*})$. However, it was not entirely clear whether the choice of measure $V_K^{\mathbb{S}^*}$ and thus the construction of the metric $g_K^{\mathbb{S}^*}$ were canonical, or what was the direct relation between these two objects (as in general V_K is not the Riemannian volume measure for g_K); we now finally have a satisfactory answer coming from the centro-affine normalization.

Consequently, regardless of the parametrization M, $-\Delta_K$ uniquely extends to a self-adjoint positive semi-definite operator on $L^2(\nu_K)$ with domain H^2 (the Sobolev space on M), as explained in [61, Section 5.1]. Its spectrum $\sigma(-\Delta_K)$ is thus inherently centro-affine invariant, and may be studied regardless of parametrization. The spectrum is discrete, consisting of a countable sequence of eigenvalues of finite multiplicity starting with 0 and tending to ∞. The first (trivial) eigenvalue $\lambda_0(-\Delta_K)$ is zero, corresponding to the constant eigenfunctions. As shown by Hilbert [13, 61], the next eigenvalue $\lambda_1(-\Delta_K)$ is $n-1$, and this fact is equivalent to the classical Brunn–Minkowski inequality; moreover, Hilbert showed that the multiplicity of the eigenvalue $n-1$ is precisely n. We will give a new proof of both of these statements in the next section using Lichnerowicz's method and Bochner's formula, utilizing the fact that ∂K is a centro-affine sphere having constant centro-affine Ricci curvature equal to $n-2$.

In Hilbert's original definition of his differential operator, the eigenfunctions corresponding to the first non-trivial eigenvalue $\lambda_1 = n-1$ on $\mathbb{S}^*$ were (the restriction to $\mathbb{S}^*$ of) linear functionals on E^*. However, with our definition of Δ_K, originating in [61] and further studied in [82], the corresponding eigenfunctions are the K-adapted linear functions

$$\mathrm{lin}^{\mathbb{S}^*}_{K,\xi} := \langle \cdot, \xi\rangle / h_K, \quad \xi \in E.$$

When K is a centered Euclidean ball (or ellipsoid), these coincide with the usual linear functionals, but not in general. While this originally appeared to us to be a caveat of our definition (compared to the one used by Hilbert), we now observe that this is in fact very natural. Indeed, the natural extension from a Euclidean ball to a general K should be to restrict the linear functionals on E^* to ∂K^* instead of $\mathbb{S}^*$. In a parametrization-free language this means using the conormal x^*_K (which on ∂K^* is just the identity map, and so $\langle x^*_K, \xi\rangle$ are just linear functions on ∂K^*):

Proposition 4.9.

$$\mathrm{lin}^{\mathbb{S}^*}_{K,\xi} = \langle (x^{\mathbb{S}^*}_K)^*, \xi\rangle \quad \forall \xi \in E, \tag{4.10}$$

and regardless of parametrization,

$$\mathrm{Hess}^*_K\, x^*_K = -x^*_K g_K, \tag{4.11}$$

$$-\Delta_K \langle x^*_K, \xi\rangle = (n-1)\langle x^*_K, \xi\rangle \quad \forall \xi \in E. \tag{4.12}$$

Proof. Recalling that $(x^{\mathbb{S}^*}_K)^*(\theta^*) = \theta^*/h_K(\theta^*)$ and the definition of $\mathrm{lin}^{\mathbb{S}^*}_{K,\xi}$, (4.10) is immediate. The property (4.11) is a direct consequence of our centro-affine structure equations (3.9). Tracing with respect to g_K immediately yields

$$\Delta_K x^*_K = \mathrm{tr}_{g_K} \mathrm{Hess}^*_K\, x^*_K = -g^{ij} g_{ij} x^*_K = -(n-1)x^*_K. \qquad \blacksquare$$

We will see in Theorem 5.4 that (4.12) is in fact equivalent to the a priori stronger property (4.11). One can also easily calculate expressions such as $\Delta_K (x^*_K)^p$ and $|\mathrm{grad}_{g_K} (x^*_K)^p|^2$ using the above calculus, which is what originally led us to the key calculation in [82] (the calculation there was derived without referring to the centro-affine normalization).

4.6. Self-duality

We conclude this section by reflecting a bit more on the remarkable self-duality property of the centro-affine metric g_K:

$$g^{i(M^*)}_{K^\circ} = i_* g^{M^*}_{K^*} = i_* g^{M^*}_K = (i \circ \mathcal{D}_K)_* g^M_K \quad \forall M \in \mathcal{M}_K.$$

Since $i(M^*) = M$ for $M \in \{\mathbb{S}, \mathbb{S}^*\}$, we see that $i \circ \mathcal{D}_K$ is an isometry between $(\mathbb{S}^*, g^{\mathbb{S}^*}_K)$ and $(\mathbb{S}^*, g^{\mathbb{S}^*}_{K^\circ})$. Consequently, any geometric quantity which is encoded by the centro-affine metric is the same for K and K°. Below are a few examples and many questions; we omit the reference to the parametrization M when it is irrelevant.

- The Riemannian volume measure $\nu^{\mathbb{S}^*}_{g_K}$ is (up to a constant) the well-known centro-affine surface-area measure $\Omega^{\mathbb{S}^*}_{n,K}$ [50, 62], which coincides with the L^p-affine surface-area measure of Lutwak [74] for $p = n$:

$$\frac{d\nu^{\mathbb{S}^*}_{g_K}}{d\mathfrak{m}^{\mathbb{S}^*}} = \sqrt{\det g_K} = \sqrt{\det(D^2 h_K / h_K)} = \frac{\sqrt{\det D^2 h_K}}{h_K^{(n-1)/2}} =: n \frac{d\Omega^{\mathbb{S}^*}_{n,K}}{d\mathfrak{m}^{\mathbb{S}^*}}.$$

This can also be seen from (3.6):

$$\begin{aligned}\left(\frac{d\nu^{\mathbb{S}^*}_{g_K}}{d\mathfrak{m}^{\mathbb{S}^*}}\right)^2 &= \frac{d\nu^{\mathbb{S}^*}_K}{d\mathfrak{m}^{\mathbb{S}^*}} \frac{d(\nu^{\mathbb{S}^*}_K)^*}{d\mathfrak{m}^{\mathbb{S}^*}} = \frac{n\,dV^{\mathbb{S}^*}_K}{d\mathfrak{m}^{\mathbb{S}^*}} \frac{n\,d i_* V^{\mathbb{S}}_{K^\circ}}{d\mathfrak{m}^{\mathbb{S}^*}} \\ &= \frac{h_K\, dS_K}{d\mathfrak{m}^{\mathbb{S}^*}} \frac{1}{h_K^n} = \frac{\det D^2 h_K}{h_K^{n-1}}. \end{aligned} \tag{4.13}$$

By Section 4.3, the centro-affine surface-area measure is indeed centro-affine invariant, i.e. $A_* \nu^{\partial K}_{g_K} = \nu^{\partial A(K)}_{g_{A(K)}}$ for all $A \in GL(E)$. Since $i \circ \mathcal{D}_K : \mathbb{S}^* \to \mathbb{S}^*$ pushes forward $\nu^{\mathbb{S}^*}_{g_K}$ onto $\nu^{\mathbb{S}^*}_{g_{K^\circ}}$, both measures have the same total mass, which up to normalization is the centro-affine surface area Ω_n:

$$\Omega_n(K) = \frac{1}{n}\|\nu_{g_K}\| = \frac{1}{n}\|\nu_{g_{K^\circ}}\| = \Omega_n(K^\circ).$$

Note that (4.13) and the Cauchy–Schwarz inequality immediately yield

$$\Omega_n(K)^2 \le V(K)V(K^\circ). \tag{4.14}$$

Unfortunately, $\Omega_n(K_i)$ converges to 0 whenever K_i converges in the Hausdorff metric to a polytope, and so this cannot be used to effectively lower bound the volume product on the right-hand side. All of this is of course well-known.

- Let d_{g_K} be the induced geodesic distance on $(\mathbb{S}^*, g_K)$. Given $p > -n+1$, define

$$W_p(K) := \left(\int_{\mathbb{S}^*}\int_{\mathbb{S}^*} d_{g_K}(x,y)^p\, \nu_{g_K}(dx)\, \nu_{g_K}(dy)\right)^{1/p}.$$

Then $W_p(K) = W_p(K^\circ)$ for all $p > -n+1$.

- $W_\infty(K) := \operatorname{diam}(\mathbb{S}^*, d_{g_K})$ also satisfies $W_\infty(K) = W_\infty(K^\circ)$. What is the geometric meaning of this quantity? Is this related to the resolution of Schäffer's conjecture by Álvarez-Paiva [1] (see also Faifman [40]), identifying the girths of K and K°? This seems unlikely, since the girth is a Finslerian notion. Note that it is not hard to show that $W_\infty(K+L) \le W_\infty(K) + W_\infty(L)$.
- Set $\Sigma(K) := \sigma(-\Delta_{g_K})$, the spectrum of the Laplace–Beltrami operator on $(\mathbb{S}^*, g_K)$. Then $\Sigma(K) = \Sigma(K^\circ)$. Is there a geometric meaning to (at least the first) eigenvalues?
- Let F_{g_K} denote any non-trivial real-valued function of the Riemann curvature tensor R_{g_K} and the metric tensor g_K – such as the scalar curvature $\mathfrak{s}_{g_K}$, the Hilbert–Schmidt norm of the Ricci tensor, or some other function of the sectional curvatures. Define

$\mathfrak{F}_p(K) = (\int |F_{g_K}|^p \, dv_{g_K})^{1/p}$. Then $\mathfrak{F}_p(K) = \mathfrak{F}_p(K^\circ)$. Of particular interest is the average scalar curvature

$$\mathfrak{S}(K) = \int \mathfrak{s}_{g_K} \, dv_{g_K}.$$

Does this quantity have a natural geometric meaning? One can show that $\mathfrak{s}_{g_K} \geq (n-1)(n-2)$ in the centro-affine normalization [87, formula (39)], and so $\mathfrak{S}(K) \geq (n-1)(n-2)\Omega_n(K)$. It would be interesting to lower bound the volume product $V(K)V(K^\circ)$ by a function of $\mathfrak{S}(K)^2 = \mathfrak{S}(K)\mathfrak{S}(K^\circ)$, in view of (4.14).

- Is it true that if $(\mathbb{S}^*, g_{K_1})$ and $(\mathbb{S}^*, g_{K_2})$ are isometric then K_1 and K_2 are congruent up to a centro-affine transformation and polarity? It is not hard to show that if $g_{K_1} = g_{K_2}$ then necessarily $K_2 = cK_1$ for some $c > 0$.

5. Bochner formula

5.1. Asymmetric Bochner formula

The classical Bochner formula [89, Chapter 9] states that if f is a smooth function on a Riemannian manifold (M, g) then

$$\frac{1}{2}\Delta_g |\mathrm{grad}_g f|^2 = g(\mathrm{grad}_g f, \mathrm{grad}_g \Delta_g f) + \mathrm{Ric}_g(\mathrm{grad}_g f, \mathrm{grad}_g f) + \|\mathrm{Hess}_g f\|_g^2.$$

In this formula, all higher order differential objects are computed with respect to the Levi-Civita connection ∇^g (which, recall, is the unique torsion-free connection which is in addition metric, i.e. $\nabla^g g = 0$); namely, $\Delta_g = \mathrm{div}^{\nabla^g} \mathrm{grad}_g$, $\mathrm{Hess}_g f = \nabla^g df$ and Ric_g denotes the Ricci curvature of ∇^g. We use the notation

$$|X|^2 = g(X, X), \quad \|A\|_g^2 = g(A, A) = \langle A, A\rangle_g = g^{ij} A_{jk} g^{kl} A_{li}$$

for the (squared) Riemannian length of a vector field X and Riemannian Hilbert–Schmidt norm of a symmetric $(0, 2)$ tensor A, respectively.

The Bochner formula is local, and applies to arbitrary vector fields X such that $\nabla^g X$ is a g-symmetric $(1, 1)$ tensor:

$$g(\nabla^g_Y X, Z) = g(\nabla^g_Z X, Y) \quad \forall Y, Z \in T_p M \;\; \forall X \in \Gamma^1(TM)$$

(and hence, locally, $X = \mathrm{grad}_g f$ for some function f). After polarization, it is equivalent to the following version, valid for any vector fields X, Y such that $\nabla^g X$ and $\nabla^g Y$ are g-symmetric $(1, 1)$ tensors:

$$\Delta_g \, g(X, Y) = X(\mathrm{div}^{\nabla^g} Y) + Y(\mathrm{div}^{\nabla^g} X) + 2\,\mathrm{Ric}_g(X, Y) + 2\,\mathrm{tr}(\nabla^g X \circ \nabla^g Y), \quad (5.1)$$

where $\nabla X \circ \nabla Y$ denotes the composition of $(1, 1)$ tensors, so that in local coordinates,

$$\mathrm{tr}(\nabla X \circ \nabla Y) = \nabla_j X^i \nabla_i Y^j.$$

An inspection of the proof of the Bochner formula confirms that it makes use of both the torsion-free and metric properties of the Levi-Civita connection.

However, it is possible to give a version of Bochner's formula which does not rely on any of these properties, holds for all vector fields, and in fact does not require a Riemannian metric g at all.

Proposition 5.1 (Asymmetric Bochner formula). *For any affine connection* ∇ *on* M *and* (*smooth*) *vector fields* X, Y,

$$\mathrm{div}^{\nabla}(\nabla_X Y) = X(\mathrm{div}^{\nabla} Y) + \mathrm{Ric}(Y, X) + \mathrm{tr}(\nabla X \circ \nabla Y). \tag{5.2}$$

A proof may be found in [87, Lemma 2.2]. For completeness, we provide a simple proof using local coordinates.

Proof. Denoting, as is customary, the components of the curvature tensors R and Ric by

$$\mathrm{R}(\partial_k, \partial_l)\partial_j = \mathrm{R}^i_{jkl}\partial_i, \quad \mathrm{Ric}_{jk} = \mathrm{Ric}(\partial_j, \partial_k),$$

so that $\mathrm{Ric}_{jk} = \mathrm{R}^i_{kij}$, we have in local coordinates

$$\begin{aligned}\nabla_j(X^i \nabla_i Y^j) &= \nabla_j X^i \nabla_i Y^j + X^i \nabla_j \nabla_i Y^j \\ &= \nabla_j X^i \nabla_i Y^j + X^i \nabla_i \nabla_j Y^j + X^i \mathrm{R}^j_{ijl} Y^l \\ &= \nabla_j X^i \nabla_i Y^j + X^i \nabla_i \nabla_j Y^j + \mathrm{Ric}_{li} X^i Y^l. \end{aligned}$$ ∎

Our "asymmetric" nomenclature stems from the fact that the formula is no longer symmetric in X, Y, and moreover does not assume any symmetry – neither of $\nabla X, \nabla Y$ (these are mixed tensors so in any case symmetry does not make any sense), nor of the affine connection ∇. The classical Bochner formula (5.1) is obtained from Proposition 5.1 by applying it to the Levi-Civita connection and symmetrizing (i.e. interchanging the roles of X, Y and summing). Indeed, note that by the metric property of ∇^g and as $\nabla^g X$ and $\nabla^g Y$ in (5.1) are assumed to be g-symmetric, we have

$$\nabla^g_Z\, g(X, Y) = g(\nabla^g_Z X, Y) + g(X, \nabla^g_Z Y) = g(\nabla^g_Y X, Z) + g(\nabla^g_X Y, Z),$$

and hence

$$\mathrm{grad}_g\, g(X, Y) = \nabla^g_X Y + \nabla^g_Y X.$$

The torsion-free property ensures that Ric_g is symmetric, and (5.1) immediately follows.

5.2. *Centro-affine Bochner formula*

Just as with the classical Bochner formula, integrating the asymmetric Bochner formula (5.2) with respect to an invariant measure ν yields an integrated Bochner identity [87, Theorem 9.9]. Applying this to our centro-affine differential structures, we immediately deduce

Theorem 5.2 (Centro-affine Bochner formula). *Let $K \in \mathcal{K}_+^\infty$ be a smooth convex body with strictly positive curvature which contains the origin in its interior. Let K be equipped with the centro-affine normalization. Then, regardless of a parametrization $M \in \mathcal{M}_K$, for any function $f \in C^2(M)$,*

$$\int (\Delta_K f)^2 \, d\nu_K - \int \|\mathrm{Hess}_K^* f\|_{g_K}^2 \, d\nu_K = (n-2) \int |\mathrm{grad}_{g_K} f|^2 \, d\nu_K. \tag{5.3}$$

Proof. By approximation, it is enough to prove the identity for smooth functions f. We abbreviate $\nabla = \nabla_K$. Recall that

$$\nabla_i (\mathrm{grad}_{g_K} f)^j = \nabla_i (g_K^{jk} f_k) = g_K^{jk} \nabla_i^* f_k = g_K^{jk} (\mathrm{Hess}_K^*)_{ki} f,$$

and hence

$$\mathrm{tr}(\nabla \, \mathrm{grad}_{g_K} f \circ \nabla \, \mathrm{grad}_{g_K} f) = \|\mathrm{Hess}_K^* f\|_{g_K}^2.$$

Let Ric_K denote the Ricci curvature of the centro-affine connection. Recall that ∂K is always a centro-affine ($(n-1)$-dimensional) unit sphere and thus

$$\mathrm{Ric}_K = (n-2) g_K.$$

Setting $X = Y = \mathrm{grad}_{g_K} f$ and applying the asymmetric Bochner formula (5.2) for $\nabla = \nabla_K$, we deduce that

$$\mathrm{div}^{\nabla_K} (\nabla_X X) = \mathrm{grad}_{g_K} f(\Delta_K f) + (n-2) |\mathrm{grad}_{g_K} f|^2 + \|\mathrm{Hess}_K^* f\|_{g_K}^2.$$

Finally, recall that $\nabla_K \nu_K = 0$ (since the centro-affine normalization is a relative normalization). Consequently, integrating with respect to ν_K as in Section 3.6, the integral on left-hand side above vanishes, and the first term on the right integrates by parts, yielding the asserted:

$$0 = -\int (\Delta_K f)^2 \, d\nu_K + (n-2) \int |\mathrm{grad}_{g_K} f|^2 \, d\nu_K + \int \|\mathrm{Hess}_K^* f\|_{g_K}^2 \, d\nu_K. \qquad \blacksquare$$

It may also be insightful to give an alternative proof of the centro-affine Bochner formula, not relying on centro-affine differential geometry, which is how we originally discovered it (thus realizing that there must be some underlying Bochner formula and constant Ricci curvature). We were informed by Bo'az Klartag and by Ramon van Handel that they also observed (unpublished argument) a similar, ultimately equivalent, formula, without any explicit reference to g_K, V_K, Hess_K^*, Ric_K, nor their centro-affine geometric interpretation.

Alternative proof of Theorem 5.2. Given a smooth f on $\mathbb{S}^*$, extend it as a 0-homogeneous function on $E^* \setminus \{0\}$ and define the following $(1,1)$ tensor in a local frame on $\mathbb{S}^*$:

$$A_k^i := ((D^2 h_K)^{-1})^{ij} D_{jk}^2 (f h_K) = g_K^{ij} \frac{D_{jk}^2 (f h_K)}{h_K}.$$

By the Leibniz rule,

$$\frac{D^2_{jk}(fh_K)}{h_K} = D^2_{jk}f + (\log h_K)_j f_k + (\log h_K)_k f_j + f(g_K)_{jk}, \tag{5.4}$$

and since $D^2_{jk}f = {}^{\mathbb{S}^*}\nabla^2_{jk}f$ as f is 0-homogeneous, it follows by Lemma 4.7 that

$$A^i_k = g^{ij}_K(\mathrm{Hess}^*_K f)_{jk} + f\delta^i_k. \tag{5.5}$$

From now on, we can use (5.5) as the definition of $\mathrm{Hess}^*_K f$ on $\mathbb{S}^*$.

Set $m = n-1$. Denoting by $\lambda = (\lambda_1, \dots, \lambda_m)$ the eigenvalues of the symmetric matrix

$$B := (D^2h_K)^{-1/2}D^2(fh_K)(D^2h_K)^{-1/2}$$

at a fixed point $\theta^* \in \mathbb{S}^*$, we clearly have

$$\mathrm{tr}(A^2) = \mathrm{tr}(B^2) = \sum_{i=1}^m \lambda_i^2 = m^2 \mathrm{e}_1^2(\lambda) - m(m-1)\mathrm{e}_2(\lambda), \tag{5.6}$$

where $\mathrm{e}_1(\lambda) := \frac{1}{m}\sum_{i=1}^m \lambda_i$ and $\mathrm{e}_2(\lambda) := \frac{1}{m(m-1)}\sum_{1\le i\ne j\le m}\lambda_i\lambda_j$ are the first two normalized symmetric polynomials in λ. Note that

$$\mathrm{e}_1(\lambda) = \frac{1}{m}\mathrm{tr}(B) = \frac{1}{m}\mathrm{tr}(A) = \frac{1}{m}(\Delta_K f + mf) =: \tilde{L}_K f.$$

In addition (see e.g. [9, formula (6)])

$$\begin{aligned}\mathrm{e}_2(\lambda) = D_m(B, B, \mathrm{Id}, \dots, \mathrm{Id}) &= \frac{D_m(D^2(fh_K), D^2(fh_K), D^2h_K, \dots, D^2h_K)}{\det D^2h_K} \\ &=: \frac{S_K(f,f)}{S_K};\end{aligned}$$

here D_m denotes the mixed discriminant of an m-tuple of m by m matrices, and we use the notation $S_K(f,f) = S_K(f;2)$, $S_K(f) = S_K(f;1)$ and $S_K = S_K(f;0) = \det D^2h_K$, where

$$S_K(f;p) = D_m(\underbrace{D^2(fh_K), \dots, D^2(fh_K)}_{p\text{ times}}, \underbrace{D^2h_K, \dots, D^2h_K}_{m-p\text{ times}}).$$

By [61, Section 5.1], $\tilde{L}_K f = S_K(f)/S_K$, and by [61, Lemma 4.1] we can integrate by parts:

$$\begin{aligned}\int_{\mathbb{S}^*}\frac{S_K(f,f)}{S_K}\,dV_K &= \frac{1}{n}\int_{\mathbb{S}^*} h_K S_K(f,f)\,d\mathfrak{m} = \frac{1}{n}\int_{\mathbb{S}^*} fh_K S_K(f)\,d\mathfrak{m} \\ &= \int_{\mathbb{S}^*} f\frac{S_K(f)}{S_K}\,dV_K = \int_{\mathbb{S}^*} f(\tilde{L}_K f)\,dV_K.\end{aligned}$$

We now compute $\int_{\mathbb{S}^*}\mathrm{tr}(A^2)\,dV_K$ in two different manners. On the one hand, by (5.5),

$$\int_{\mathbb{S}^*}\mathrm{tr}(A^2)\,dV_K = \int_{\mathbb{S}^*}\|\mathrm{Hess}^*_K f\|^2_{g_K}\,dV_K + 2\int_{\mathbb{S}^*} f(\Delta_K f)\,dV_K + m\int_{\mathbb{S}^*} f^2\,dV_K.$$

On the other hand, by (5.6) and the subsequent calculations,

$$\begin{aligned}\int_{\mathbb{S}^*} \mathrm{tr}(A^2)\, dV_K &= \int_{\mathbb{S}^*} (\Delta_K f + mf)^2\, dV_K - m(m-1)\int_{\mathbb{S}^*} f(\tilde{L}_K f)\, dV_K \\ &= \int_{\mathbb{S}^*} (\Delta_K f)^2\, dV_K + 2m\int_{\mathbb{S}^*} f(\Delta_K f)\, dV_K + m^2\int_{\mathbb{S}^*} f^2\, dV_K \\ &\quad - (m-1)\int_{\mathbb{S}^*} (f\Delta_K f + mf^2)\, dV_K.\end{aligned}$$

Comparing the two expressions, we obtain

$$\begin{aligned}\int_{\mathbb{S}^*} (\Delta_K f)^2\, dV_K - \int_{\mathbb{S}^*} \|\mathrm{Hess}^*_K f\|^2_{g_K}\, dV_K &= -(m-1)\int_{\mathbb{S}^*} f(\Delta_K f)\, dV_K \\ &= (m-1)\int_{\mathbb{S}^*} |\mathrm{grad}_{g_K} f|^2\, dV_K. \qquad \blacksquare\end{aligned}$$

5.3. *Proof of the Brunn–Minkowski inequality à la Lichnerowicz*

A classical theorem of Lichnerowicz [68] (see also [59]) states that the first non-zero eigenvalue $\lambda_1(-\Delta_g)$ of the Laplace–Beltrami operator $-\Delta_g$ on a closed connected $(n-1)$-dimensional Riemannian manifold (M^{n-1}, g) with $\mathrm{Ric}_g \geq \kappa g$ (as $(0,2)$ tensors) and $\kappa > 0$ satisfies

$$\lambda_1(-\Delta_g) \geq \frac{n-1}{n-2}\kappa.$$

The proof is immediate after applying the integrated Bochner formula to the first eigenfunction φ_1, and using Cauchy–Schwarz to relate $\|\mathrm{Hess}_g f\|_g^2$ and $(\Delta_g f)^2$.

Repeating the same argument verbatim for the centro-affine normalization on ∂K immediately gives a new proof of the Brunn–Minkowski inequality (note that in our case $\kappa = n-2$). By approximation and translation, it is enough to consider smooth strictly convex bodies in $\mathbb{R}^n$ having the origin in their interior $K \in \mathcal{K}^\infty_+$, and as observed by Minkowski and Hilbert, it is enough to derive the inequality in its infinitesimal (or local) form (see [61, Section 5]):

$$\begin{aligned}\int z\, d\nu_K = 0 \implies \int |\mathrm{grad}_{g_K} z|^2\, d\nu_K &= \int (-\Delta_K z) z\, d\nu_K \\ &\geq (n-1)\int z^2\, d\nu_K \quad \forall z \in C^2, \end{aligned} \tag{5.7}$$

which exactly means that

$$\lambda_1(-\Delta_K) \geq n-1. \tag{5.8}$$

It is well-known that (5.7) is equivalent to its dual form (via $z = \Delta_K f$)

$$\begin{aligned}\int (\Delta_K f)^2\, d\nu_K &\geq (n-1)\int |\mathrm{grad}_{g_K} f|^2\, d\nu_K \\ &= (n-1)\int (-\Delta_K f) f\, d\nu_K \quad \forall f \in C^2.\end{aligned} \tag{5.9}$$

This may be seen by either testing with the first eigenfunction φ_1 and using $-\Delta_K\varphi_1 = \lambda_1\varphi_1$, or by using the spectral theorem for $-\Delta_K \geq 0$ (which is self-adjoint on $L^2(\nu_K)$) and noting that (5.9) is equivalent to $P_2(-\Delta_K) \geq 0$ for the polynomial $P_2(t) = t^2 - (n-1)t$.

Theorem 5.3 (Local Brunn–Minkowski inequality). *For all $K \in \mathcal{K}^\infty_+$, the local Brunn–Minkowski inequality* (5.9) *holds.*

Proof à la Lichnerowicz. Given $f \in C^2$, by Cauchy–Schwarz we have

$$\|\mathrm{Hess}^*_K f\|^2_{g_K} \geq \frac{1}{n-1}(\mathrm{tr}_{g_K} \mathrm{Hess}^*_K f)^2 = \frac{1}{n-1}(\Delta_K f)^2. \tag{5.10}$$

Plugging this into the centro-affine Bochner formula (5.3), we obtain

$$\begin{aligned}\frac{n-2}{n-1}\int(\Delta_K f)^2\,d\nu_K &\geq \int(\Delta_K f)^2\,d\nu_K - \int\|\mathrm{Hess}^*_K f\|^2_{g_K}\,d\nu_K\\ &= (n-2)\int|\mathrm{grad}_{g_K} f|^2\,d\nu_K,\end{aligned}$$

yielding (5.9). ■

The above proof also immediately reveals the corresponding equality conditions, yielding a characterization of the eigenspace of $-\Delta_K$ corresponding to $\lambda_1 = n-1$, a result due to Hilbert.

Theorem 5.4 (Equality in local Brunn–Minkowski inequality). *For all $K \in \mathcal{K}^\infty_+$, the eigenspace of $-\Delta_K$ corresponding to the first non-zero eigenvalue $\lambda_1 = n-1$ is precisely n-dimensional, spanned by $\{\langle x^*_K, \xi\rangle\,;\,\xi \in E\}$.*

Proof. We have already seen in Proposition 4.9 that $\langle x^*_K, \xi\rangle$ are eigenfunctions corresponding to $\lambda_1 = n-1$. To show the converse, let $-\Delta_K\varphi_1 = (n-1)\varphi_1$, and hence equality holds in (5.9). Inspecting the proof of Theorem 5.3, it follows that we must have equality in the Cauchy–Schwarz inequality (5.10) ν_K-a.e., and hence by continuity (as ν_K is of full support) at every point. It follows that at every point p, $\mathrm{Hess}^*_K \varphi_1$ must be a multiple of g_K:

$$\begin{aligned}\mathrm{Hess}^*_K \varphi_1 &= c(p)g_K = \frac{\mathrm{tr}_{g_K}\mathrm{Hess}^*_K \varphi_1}{n-1}g_K = \frac{\Delta_K\varphi_1}{n-1}g_K\\ &= -\varphi_1 g_K.\end{aligned}$$

It is now convenient to use the parametrization on $M = \mathbb{S}^*$. It follows by (5.4) and Lemma 4.7 that if we extend φ_1 as a 0-homogeneous function on $E^* \setminus \{0\}$, we have

$$\frac{\bar{D}^2(\varphi_1 h_K)}{h_K} = \mathrm{Hess}^*_K \varphi_1 + \varphi_1 g_K = 0.$$

Consequently, $\varphi_1 h_K$ must be an affine function on E^*. But since h_K is 1-homogeneous, it vanishes at the origin, and we deduce that $\varphi_1 h_K$ must be a linear function $\langle\cdot,\xi\rangle$ for some $\xi \in E$. Hence $\varphi_1 = \mathrm{lin}_{K,\xi} = \langle x^*_K, \xi\rangle$, as asserted. ■

5.4. Equivalent formulations of the L^p-Brunn–Minkowski conjecture

Armed with the centro-affine Bochner formula, we can now give several equivalent formulations of the even local L^p-Brunn–Minkowski inequality, which is conjectured to hold for all $K \in \mathcal{K}^\infty_{+,e}$ and $p \in [0,1)$ (and the validity of which for all $K \in \mathcal{K}^\infty_{+,e}$ is equivalent to the validity of the global L^p-Brunn–Minkowski inequality (2.4) by [91]).

Theorem 5.5. *The following statements are equivalent for a given $K \in \mathcal{K}^\infty_{+,e}$ and $p \leq 1$:*

(1) *The even local L^p-Brunn–Minkowski conjectured inequality:*

$$\int f \, dv_K = 0 \implies \int |\mathrm{grad}_{g_K} f|^2 \, dv_K = \int (-\Delta_K f) f \, dv_K \geq (n-p) \int f^2 \, dv_K \quad \forall f \in C^2_e. \tag{5.11}$$

(2)
$$\int (\Delta_K f)^2 \, dv_K \geq (n-p) \int |\mathrm{grad}_{g_K} f|^2 \, dv_K = (n-p) \int (-\Delta_K f) f \, dv_K \quad \forall f \in C^2_e.$$

(3)
$$\int \|\mathrm{Hess}^*_K f\|^2_{g_K} \, dv_K \geq \frac{2-p}{n-p} \int (\Delta_K f)^2 \, dv_K \quad \forall f \in C^2_e.$$

(4)
$$\int \|\mathrm{Hess}^*_K f\|^2_{g_K} \, dv_K \geq (2-p) \int |\mathrm{grad}_{g_K} f|^2 \, dv_K \quad \forall f \in C^2_e.$$

When $p = 0$, these are equivalent to

(5)
$$\int \|\mathrm{Hess}^*_K f + f g_K\|^2_{g_K} \, dv_K \geq (n-1) \int f^2 \, dv_K \quad \forall f \in C^2_e.$$

Proof. The duality (say, via the spectral theorem) between (1) and (2) was already explained. The equivalence of (2)–(4) is then immediate by using the centro-affine Bochner formula (5.3). When $p = 0$, (4) is readily seen to be equivalent to (5) after noting that

$$\int \langle \mathrm{Hess}^*_K f, f g_K \rangle_{g_K} \, dv_K = \int f \Delta_K(f) \, dv_K = -\int |\mathrm{grad}_{g_K} f|^2 \, dv_K. \qquad \blacksquare$$

Remark 5.6. As we have already seen in (5.10), formulation (3) holds trivially for any $K \in \mathcal{K}^\infty_+$ with $p = 1$ by an application of the Cauchy–Schwarz inequality, regardless of evenness of f, and is actually equivalent to the local formulations of the Brunn–Minkowski inequality (5.7) and (5.9). It follows by the centro-affine Bochner formula as in Theorem 5.5 that the same applies to formulation (4).

Formulation (5) is rather appealing since by (4.11), we have $\mathrm{Hess}^*_K x^*_K = -x^*_K g_K$, and by Theorem 5.4, the left-hand side vanishes if and only if $f = \langle x^*_K, \xi \rangle$ for some $\xi \in E$, the (odd) eigenfunction of Δ_K corresponding to $\lambda_1(-\Delta_K) = n-1$; unfortunately, we do not know how to exploit this. Arranging the other conjectured inequalities above as follows:

$$\int \|\mathrm{Hess}^*_K f\|^2_{g_K}\, d\nu_K \geq_? \frac{2-p}{n-p} \int (\Delta_K f)^2\, d\nu_K$$
$$\geq_? (2-p) \int |\mathrm{grad}_{g_K} f|^2\, d\nu_K \quad \forall f \in C^2_e,$$

we see that formulation (4) relating the left most and right most terms is formally the weakest, and so we will concentrate on establishing it, at least for some rich class of convex bodies K.

5.5. *A possible strategy*

In view of Remark 5.6, we see that the remaining challenge is to exploit the evenness of f and origin-symmetry of K to turn the trivial constant 1 in (4) into the conjectured constant 2, corresponding to $p = 0$. This is very reminiscent of the challenge in the resolution of the B-conjecture in [38]. As in [38], a natural idea for deriving (4) is to apply (1) to some partial derivative z of f (which will necessarily be odd and hence integrate to zero as required in (1)), and then sum over all partial derivatives, thereby gaining one order of differentiation. However, in order to obtain $|\mathrm{grad}_{g_K} f|^2 = g^{ij}_K f_i f_j$ on the right-hand side of (4), we would need to incorporate the square-root of the metric g_K into the definition of z – an inherent complication in our non-Euclidean setting. Despite having the elegant representation $g_K = \langle dx^*, dx\rangle$, we do not know how to effectively take the square-root of g_K (without introducing additional parameters we have no control over). Although we will not use this here, we mention the following additional equivalent formulation, which gives us some more flexibility:

Theorem 5.7. *Given $K \in \mathcal{K}^\infty_{+,e}$ and $p \leq 1$, the even local L^p-Brunn–Minkowski conjectured inequality* (5.11) *holds if and only if for all $f \in C^2_e$, there exists $h \in C^2$ such that*

$$\begin{aligned} &\int f h\, d\nu_K > 0, \\ &\int \langle \mathrm{Hess}^*_K f, \mathrm{Hess}^*_K h\rangle_{g_K}\, d\nu_K \geq (2-p) \int g_K(\mathrm{grad}_{g_K} f, \mathrm{grad}_{g_K} h)\, d\nu_K. \end{aligned} \tag{5.12}$$

Proof. The "only if" direction follows by using $h = f$ in the implication (4)⇒(1) in Theorem 5.5. For the "if" direction, simply polarize the centro-affine Bochner formula (5.3):

$$\int (\Delta_K f)(\Delta_K h)\, d\nu_K - \int \langle \mathrm{Hess}^*_K f, \mathrm{Hess}^*_K h\rangle_{g_K}\, d\nu_K$$
$$= (n-2) \int g_K(\mathrm{grad}_{g_K} f, \mathrm{grad}_{g_K} h)\, d\nu_K \tag{5.13}$$

for all $f, h \in C^2$. By elliptic regularity, the first non-constant even eigenfunction f of $-\Delta_K$ corresponding to the eigenvalue $\lambda_{1,e}$ satisfies $f \in C^\infty_e$. Let $h \in C^2$ be the function from our assumption. Note that

$$\int g_K(\mathrm{grad}_{g_K} f, \mathrm{grad}_{g_K} h)\, d\nu_K = -\int (\Delta_K f) h\, d\nu_K = \lambda_{1,e} \int f h\, d\nu_K > 0. \tag{5.14}$$

Combining (5.12) and (5.13) and integrating by parts, we obtain

$$\lambda_{1,e} \int g_K(\mathrm{grad}_{g_K} f, \mathrm{grad}_{g_K} h)\, d\nu_K \geq (n-p) \int g_K(\mathrm{grad}_{g_K} f, \mathrm{grad}_{g_K} h)\, d\nu_K.$$

Using the positivity of (5.14), we deduce $\lambda_{1,e} \geq n - p$, which is equivalent to (5.11). ■

6. Curvature pinching implies even L^p-Minkowski uniqueness

We are now ready to reap the fruits of the insight gained from recasting our problem in the centro-affine differential language and establish the main results of this paper. First, it will be useful to record

Lemma 6.1. *Let $K \in \mathcal{K}^2_+$. Then for all $0 < A \leq B$:*
(1) $\frac{1}{B}\delta^{\partial K} \leq \mathrm{II}^{\partial K} \leq \frac{1}{A}\delta^{\partial K}$ *if and only if* $A\delta^{\mathbb{S}^*} \leq D^2 h_K \leq B\delta^{\mathbb{S}^*}$.
(2) $\frac{1}{B}\delta^{\partial K} \leq g_K^{\partial K} \leq \frac{1}{A}\delta^{\partial K}$ *if and only if* $A\delta^{\mathbb{S}^*} \leq h_K D^2 h_K \leq B\delta^{\mathbb{S}^*}$.

Recall that $\delta^{\partial K}$ denotes the induced Euclidean structure on ∂K (i.e. the first fundamental form), and that $g_K^{\partial K}(x) = \mathrm{II}_x^{\partial K}/h_K(\theta^*)$ by Proposition 4.4.

Proof. Let us pull back the assertions onto $\mathbb{S}^*$ using the inverse Gauss map $x : \mathbb{S}^* \ni \theta^* \mapsto \bar{D}h_K(\theta^*) \in \partial K$. Note that $dx(\theta^*) = D^2 h_K(\theta^*)$, and so the pull-back of $\delta_x^{\partial K}$ is $(D^2 h_K(\theta^*))^2$. Recalling that $\mathrm{II}_x^{\partial K} = (D^2 h_K(\theta^*))^{-1}$ under the natural identification between $T_x \partial K$ and $T_{\theta^*}\mathbb{S}^*$, the pull-back of $\mathrm{II}_x^{\partial K}$ is therefore $D^2 h_K(\theta^*)$. Consequently,
(1) $\frac{1}{B}\delta^{\partial K} \leq \mathrm{II}^{\partial K} \leq \frac{1}{A}\delta^{\partial K}$ if and only if

$$\frac{1}{B}(D^2 h_K(\theta^*))^2 \leq D^2 h_K(\theta^*) \leq \frac{1}{A}(D^2 h_K(\theta^*))^2.$$

(2) $\frac{1}{B}\delta_x^{\partial K} \leq \mathrm{II}_x^{\partial K}/h_K(\theta^*) \leq \frac{1}{A}\delta_x^{\partial K}$ if and only if

$$\frac{1}{B}(D^2 h_K(\theta^*))^2 \leq D^2 h_K(\theta^*)/h_K(\theta^*) \leq \frac{1}{A}(D^2 h_K(\theta^*))^2.$$ ■

Lemma 6.2. *Let $K \in \mathcal{K}^2_{+,e}$ and assume that*

$$\frac{1}{R}\delta^{\partial K} \leq \mathrm{II}^{\partial K} \leq \frac{1}{r}\delta^{\partial K}. \tag{6.1}$$

Then

$$rB_2^n \subset K \subset RB_2^n, \tag{6.2}$$

and consequently

$$\frac{1}{R^2}\delta^{\partial K} \leq g_K^{\partial K} \leq \frac{1}{r^2}\delta^{\partial K}.$$

Proof. By Blaschke's rolling ball theorem [96, Theorem 2.5.4/3.2.12], the assumption (6.1) implies that the Euclidean ball rB_2^n rolls freely inside K and that K moves freely inside the Euclidean ball RB_2^n. Convexity and origin-symmetry of K then imply (6.2) since $B_2^n \subset \mathrm{conv}((v + B_2^n) \cup (-v + B_2^n))$ and $B_2^n \supset (v + B_2^n) \cap (-v + B_2^n)$ for all $v \in E$. Recalling that $g_K^{\partial K}(x) = \mathrm{II}_x^{\partial K}/h_K(\theta^*)$, the final assertion follows. ∎

6.1. *Curvature pinching implies local even L^p-Brunn–Minkowski inequality*

Theorem 6.3. *Let $K \in \mathcal{K}^\infty_{+,e}$ have a centro-affine image $\tilde{K}$ such that*

$$\frac{1}{B}\delta^{\partial\tilde{K}} \le g_{\tilde{K}}^{\partial\tilde{K}} \le \frac{1}{A}\delta^{\partial\tilde{K}}, \quad \tilde{K} \subset RB_2^n, \tag{6.3}$$

for some $A, B, R > 0$. Then $\lambda_{1,e}(-\Delta_K) \ge n - p$, i.e. K satisfies the local even L^p-Brunn–Minkowski inequality (5.11) with

$$p = 2 - \frac{\frac{n-1}{2}A - R^2}{B}. \tag{6.4}$$

In particular, if K has a centro-affine image for which (6.1) holds, then K satisfies the local even L^p-Brunn–Minkowski inequality (5.11) with

$$p = 3 - \frac{n-1}{2}\frac{r^2}{R^2}.$$

In particular, whenever $(\frac{R}{r})^2 \le \frac{n-1}{6}$, K satisfies the local even log-Brunn–Minkowski inequality.

Proof. As the spectrum $\sigma(-\Delta_K)$ is centro-affine invariant, it is enough to prove the claim for $\tilde{K} = K$.

Let $\mathrm{e}^1, \dots, \mathrm{e}^n$ denote a fixed orthonormal basis of $(E, \langle\cdot,\cdot\rangle)$. Recall that $x_K : M \to \partial K$ is our centro-affine parametrization of ∂K on M. We abbreviate $g = g_K$, $x = x_K$, $\nabla = \nabla_K$, $\nabla^* = \nabla_K^*$ and $\mathrm{Hess}^* = \mathrm{Hess}_K^*$. We also denote

$$x^k := \langle \mathrm{e}^k, x\rangle : M \to \mathbb{R}, \quad x_i^k := (x^k)_i = dx^k(e_i)$$

(for some local frame $e_1, \dots, e_{n-1}$ on M). Define the following $(0,2)$ tensor on M:

$$\delta_{ij} = \langle dx(e_i), dx(e_j)\rangle = \sum_{k=1}^n x_i^k x_j^k. \tag{6.5}$$

Note that when $M = \partial K$, as $dx_K^{\partial K} = \mathrm{Id}$, δ is precisely the induced Euclidean metric $\delta^{\partial K}$ on ∂K, ensuring that our notation is consistent. We deduce that, regardless of parametrization, we have

$$\frac{1}{B}\delta \le g \le \frac{1}{A}\delta. \tag{6.6}$$

Given $f \in C^2_e(M)$ and $k = 1, \dots, n$, introduce the function $F^k \in C^1(M)$ given by

$$F^k := (\mathrm{grad}_g f)(x^k) = x^k_i f^i, \quad f^i := g^{ip} f_p.$$

Since K is origin-symmetric and f is even, F^k is clearly odd, and in particular $\int F^k \, d\nu_K = 0$. Consequently, applying the local (L^1-)Brunn–Minkowski inequality (5.7) to each F^k and summing (it is clearly enough that $F^k \in C^1$), we obtain

$$\sum_{k=1}^n \int |\mathrm{grad}_g F^k|^2 \, d\nu_K \geq (n-1) \sum_{k=1}^n \int (F^k)^2 \, d\nu_K.$$

Using (6.6), we first estimate from below:

$$\begin{aligned}\sum_{k=1}^n \int (F^k)^2 \, d\nu_K &= \int \sum_{k=1}^n x^k_i x^k_j f^i f^j \, d\nu_K = \int \delta_{ij} f^i f^j \, d\nu_K \\ &\geq A \int g_{ij} f^i f^j \, d\nu_K = A \int |\mathrm{grad}_g f|^2 \, d\nu_K.\end{aligned}$$

Next, note that by (3.5) and (3.9),

$$\begin{aligned}\partial_a(x^k_i f^i) &= \partial_a(x^k_i g^{ip} f_p) = (\nabla_a x^k_i) f^i + x^k_i g^{ip} \nabla^*_a f_p \\ &= -g_{ai} x^k f^i + x^k_i g^{ip} \, \mathrm{Hess}^*_{ap} f = -x^k f_a + x^k_i g^{ip} \, \mathrm{Hess}^*_{ap} f \\ &=: P^k_a + Q^k_a.\end{aligned} \tag{6.7}$$

Using (6.7), (6.5), (6.6) and (6.2), we now estimate from above:

$$\begin{aligned}&\sum_{k=1}^n \int |\mathrm{grad}_g F^k|^2 \, d\nu_K \\ &\quad = \sum_{k=1}^n \int g^{ab} \partial_a(x^k_i f^i) \partial_b(x^k_j f^j) \, d\nu_K \\ &\quad = \sum_{k=1}^n \int g^{ab} (P^k_a + Q^k_a)(P^k_b + Q^k_b) \, d\nu_K \\ &\quad \leq 2 \sum_{k=1}^n \int g^{ab} (P^k_a P^k_b + Q^k_a Q^k_b) \, d\nu_K \\ &\quad = 2 \int \Big(\sum_{k=1}^n (x^k)^2 g^{ab} f_a f_b + \Big(\sum_{k=1}^n x^k_i x^k_j \Big) g^{ab} g^{ip} g^{jq} \, \mathrm{Hess}^*_{ap} f \, \mathrm{Hess}^*_{bq} f \Big) \, d\nu_K \\ &\quad = 2 \int \big(|x|^2 |\mathrm{grad}_g f|^2 + \delta_{ij} g^{ab} g^{ip} g^{jq} \, \mathrm{Hess}^*_{ap} f \, \mathrm{Hess}^*_{bq} f \big) \, d\nu_K \\ &\quad \leq 2 \int (R^2 |\mathrm{grad}_g f|^2 + B \|\mathrm{Hess}^* f\|^2_g) \, d\nu_K,\end{aligned}$$

where in the last inequality we have used the well-known fact that $S_{ij}T^{ij} \geq 0$ whenever $S, T \geq 0$.

Combining the three inequalities derived above, we obtain

$$2\int (R^2|\mathrm{grad}_g f|^2 + B\|\mathrm{Hess}^* f\|_g^2)\, d\nu_K \geq (n-1)A\int |\mathrm{grad}_g f|^2\, d\nu_K.$$

Rearranging, we deduce that for every $f \in C_e^2$,

$$\int \|\mathrm{Hess}^* f\|_g^2\, d\nu_K \geq \frac{\frac{n-1}{2}A - R^2}{B}\int |\mathrm{grad}_g f|^2\, d\nu_K.$$

By Theorem 5.5, this is equivalent to the local L^p-Brunn–Minkowski inequality (5.11) with p given by (6.4).

Lastly, the "in particular" part of Theorem 6.3 follows immediately by Lemma 6.2. ■

6.2. *Uniqueness in the even L^p-Minkowski problem*

We can now translate Theorem 6.3 into a global uniqueness result for the even L^p-Minkowski problem and an even L^p-Minkowski inequality. We obtain the following strengthened version of Theorem 1.2, which we will require for the isomorphic results of the next section.

Theorem 6.4. *Let $K \in \mathcal{K}_{+,e}^{2,\alpha}$ have a centro-affine image $\tilde{K} \subset RB_2^n$ so that the following curvature pinching bounds hold:*

$$\frac{1}{B}\delta^{\partial\tilde{K}} \leq \frac{\mathrm{II}^{\partial\tilde{K}}}{h_{\tilde{K}}(\mathfrak{n}^{\partial\tilde{K}})} \leq \frac{1}{A}\delta^{\partial\tilde{K}} \tag{6.8}$$

for some $B \geq A > 0$ and $R > 0$. Then for any p with

$$2 - \frac{\frac{n-1}{2}A - R^2}{B} < p < 1, \tag{6.9}$$

the even L^p-Minkowski problem for K has a unique solution:

$$L \in \mathcal{K}_e, \quad S_pL = S_pK \implies L = K, \tag{6.10}$$

and the even L^p-Minkowski inequality holds:

$$\forall L \in \mathcal{K}_e, \quad \frac{\frac{1}{p}\int h_L^p\, dS_pK}{V(L)^{p/n}} \geq \frac{\frac{1}{p}\int h_K^p\, dS_pK}{V(K)^{p/n}} = \frac{n}{p}V(K)^{1-p/n}, \tag{6.11}$$

with equality if and only if $L = cK$ for some $c > 0$.

As usual, the case $p = 0$ above is interpreted in the limiting sense as in (1.10). Note that Theorem 6.4 immediately implies Theorem 1.2, since the assumption (1.5) implies (6.8) with $A = r^2$ and $B = R^2$ by Lemma 6.2.

Proof of Theorem 6.4. As Theorem 6.3 was formulated for $K \in \mathcal{K}^{\infty}_{+,e}$, let us first assume that this is the case. Denote by $T \in GL_n$ the linear map such that $K = T(\tilde{K})$ with

$$rB_2^n \subset \tilde{K} \subset RB_2^n \tag{6.12}$$

satisfying (6.8). Without loss of generality, we may assume that r, R are best possible in (6.12), and by scaling, we may assume that $rR = 1$, so that in particular $r \leq 1 \leq R$. By Lemma 6.2, since $\frac{r}{B}\delta^{\partial \tilde{K}} \leq \mathrm{II}^{\partial \tilde{K}} \leq \frac{R}{A}\delta^{\partial \tilde{K}}$, we have $R \leq B/r$ and $r \geq A/R$, and hence $A \leq 1 \leq B$.

Define, for $t \in [0, 1]$,

$$r_t := (1-t) + tr, \quad R_t := (1-t) + tR,$$

and let $\tilde{K}_t, K_t \in \mathcal{K}^{\infty}_{+,e}$ be defined via

$$h_{\tilde{K}_t} := (1-t) + th_{\tilde{K}}, \quad T_t := (1-t)\mathrm{Id} + tT, \; K_t := T_t(\tilde{K}_t),$$

so that $K_0 = B_2^n$ and $K_1 = K$.

By Lemma 6.1, the assumption (6.8) is equivalent to

$$A\delta^{\mathbb{S}^*} \leq h_{\tilde{K}} D^2 h_{\tilde{K}} \leq B\delta^{\mathbb{S}^*} \text{ on } \mathbb{S}^*.$$

Denote by $A_t, B_t > 0$ the best corresponding constants for $\tilde{K}_t$:

$$A_t\delta^{\mathbb{S}^*} \leq h_{\tilde{K}_t} D^2 h_{\tilde{K}_t} = ((1-t) + th_{\tilde{K}})((1-t)\delta^{\mathbb{S}^*} + tD^2 h_{\tilde{K}}) \leq B_t\delta^{\mathbb{S}^*} \text{ on } \mathbb{S}^*.$$

We claim that

$$A_t \geq A \quad \text{and} \quad B_t \leq B \quad \forall t \in [0, 1]. \tag{6.13}$$

Indeed, by fixing θ^* and diagonalizing $D^2 h_{\tilde{K}}(\theta^*) > 0$, it is enough to find the extremal values of the quadratic polynomial $Q(t) := ((1-t) + tp)((1-t) + tq)$ on the interval $[0, 1]$ when $p, q > 0$. A simple calculation verifies that this polynomial is always monotone (either increasing or decreasing) on $[0, 1]$, or equivalently, that the mid-point between its two roots lies outside the interval $(0, 1)$:

$$\frac{-\frac{1}{q-1} - \frac{1}{p-1}}{2} = \frac{1 - \frac{p+q}{2}}{1 + pq - (p+q)} \begin{Bmatrix} \geq \\ \leq \end{Bmatrix} \frac{1 - \frac{p+q}{2}}{1 + (\frac{p+q}{2})^2 - (p+q)}$$
$$= \frac{1}{1 - \frac{p+q}{2}} \begin{cases} \geq 1, & 0 < \frac{p+q}{2} < 1, \\ \leq 0, & \frac{p+q}{2} \geq 1. \end{cases}$$

Consequently, the minimum and maximum of Q on $[0, 1]$ are attained at the end points. Since $A \leq 1 \leq B$, (6.13) immediately follows.

Now, since clearly $R_t \leq R$, it follows that for all $t \in [0, 1]$,

$$2 - \frac{\frac{n-1}{2} A_t - R_t^2}{B_t} \leq 2 - \frac{\frac{n-1}{2} A - R^2}{B} =: p_0.$$

We may assume that $p_0 < 1$, otherwise there is nothing to prove. It follows by Theorem 6.3 that for all $t \in [0,1]$, K_t satisfies the local L^{p_0}-Brunn–Minkowski inequality (5.11). Note that it always holds that $p_0 > -n$ because $A \leq B$.

We now claim that the exact same conclusion also holds whenever $K \in \mathcal{K}^2_{+,e}$. Indeed, recall that K_t satisfies (5.11) if and only if $\lambda_{1,e}(-\Delta_{K_t}) \geq n - p_0$. It was shown in [61, Theorem 5.3] that the eigenvalues of $-\Delta_L$ are continuous in L with respect to the C^2 topology. Consequently, if we approximate K by $\{K^i\} \subset \mathcal{K}^\infty_{+,e}$ in C^2, since K^i_t obviously tends to K_t in C^2 as $i \to \infty$ as well, it follows that $\lambda_{1,e}(-\Delta_{K^i_t}) \to \lambda_{1,e}(-\Delta_{K_t})$. We thus deduce that $\lambda_{1,e}(-\Delta_{K_t}) \geq n - p_0$ for all $t \in [0,1]$ whenever $K \in \mathcal{K}^2_{+,e}$.

We now assume that $K \in \mathcal{K}^{2,\alpha}_{+,e}$, to ensure that $[0,1] \ni t \mapsto K_t \in \mathcal{K}^{2,\alpha}_{+,e}$ is a continuous deformation in the $C^{2,\alpha}$ topology. The implications (4)⇒(1)⇒(3b) of Theorem 2.1 for $\mathcal{F} := \{K_t\}_{t\in[0,1]} \subset \mathcal{K}^{2,\alpha}_{+,e}$ now immediately establish the asserted uniqueness in the even L^p-Minkowski problem for K (6.10), as well as the corresponding L^p-Minkowski inequality (6.11) along with its equality case, for any $p \in (p_0, 1)$. ∎

7. Isomorphic and isometric L^p-Minkowski problem

We conclude this work by deducing Theorems 1.5 and 1.7. To this end, we require the following:

Proposition 7.1. *Let $K \in \mathcal{K}_e$, and set $D = d_{BM}(K, B^n_2)$. Then for any $\alpha, \beta > 0$, there exists $\tilde{K} \in \mathcal{K}^\infty_{+,e}$ such that*

$$d_{BM}(K, \tilde{K}) \leq (1+\beta)\sqrt{1+\alpha^2}, \tag{7.1}$$

$$rB^n_2 \subset \tilde{K} \subset RB^n_2, \tag{7.2}$$

$$\frac{1}{B}\delta^{\partial\tilde{K}} \leq \frac{\mathrm{II}^{\partial\tilde{K}}}{h_{\tilde{K}}(\mathfrak{n}^{\partial\tilde{K}})} \leq \frac{1}{A}\delta^{\partial\tilde{K}}, \tag{7.3}$$

with $A, B, r, R > 0$ given by

$$r := \beta + \frac{1}{\sqrt{1+\alpha^2/D^2}}, \quad R := \frac{D}{\sqrt{1+\alpha^2}} + \beta, \tag{7.4}$$

$$A := \beta r, \quad B := \frac{D^2}{\alpha^2}\left(1 + \beta\sqrt{1+\alpha^2/D^2}\right) + \beta R. \tag{7.5}$$

Remark 7.2. It is natural to conjecture that with the above assumptions, for any $\gamma \in [1, D]$ there should exist $\tilde{K} \in \mathcal{K}^\infty_{+,e}$ such that

$$d_{BM}(K, \tilde{K}) \leq C\gamma$$

and

$$\delta^{\partial\tilde{K}} \leq \mathrm{II}^{\partial\tilde{K}} \leq \frac{CD}{F(\gamma)}\delta^{\partial\tilde{K}},$$

for some universal constant $C > 1$ and some increasing function $F : [1, D] \to [1, D]$ (probably $F(\gamma) = \gamma$). This would enable us to deduce the isomorphic results of this section directly from Theorem 1.2, without going through the stronger Theorem 6.4. Unfortunately, we do not see a simple argument for showing the above. The crux of the problem is that $\frac{1}{B}\delta^{\partial \tilde{K}} \leq \mathrm{II}^{\partial \tilde{K}} \leq \frac{1}{A}\delta^{\partial \tilde{K}}$ does not imply (in general) $A\delta^{\partial \tilde{K}^\circ} \leq \mathrm{II}^{\partial \tilde{K}^\circ} \leq B\delta^{\partial \tilde{K}^\circ}$, as the curvature of the polar body picks up additional factors depending on D.

Proof of Proposition 7.1. By applying a linear transformation, we may assume that K is in John's position, so that

$$B_2^n \subset K \subset DB_2^n. \tag{7.6}$$

Furthermore, by standard arguments (such as mollifying h_K, Minkowski adding a small Euclidean ball, and rescaling [96, pp. 184–185]), we may assume that $K \in \mathcal{K}^\infty_{+,e}$ without altering (7.6), and only incurring an extra (arbitrarily small) $\varepsilon > 0$ in the final estimate for $d_{BM}(K, \tilde{K})$, as described below.

We now construct the required body $\tilde{K} \in \mathcal{K}^\infty_{+,e}$ by defining

$$\tilde{K} := L + \beta B_2^n, \quad L := \left(K^\circ +_2 \left(\frac{\alpha}{D} B_2^n\right)\right)^\circ .$$

Here $A +_2 B$ denotes the L^2-Minkowski sum of two convex bodies $A, B \in \mathcal{K}$, defined via

$$h^2_{A+_2B} := h^2_A + h^2_B.$$

In other words,

$$\|x\|_L^2 = \|x\|_K^2 + \frac{\alpha^2}{D^2}|x|^2.$$

As

$$\frac{1}{D}|x| \leq \|x\|_K \leq |x|,$$

we have

$$\left(1 + \frac{\alpha^2}{D^2}\right)\|x\|_K^2, \frac{1+\alpha^2}{D^2}|x|^2 \leq \|x\|_L^2 \leq (1+\alpha^2)\|x\|_K^2, \left(1 + \frac{\alpha^2}{D^2}\right)|x|^2,$$

or equivalently

$$\frac{1}{\sqrt{1+\alpha^2}}K, \frac{1}{\sqrt{1+\alpha^2/D^2}}B_2^n \subset L \subset \frac{1}{\sqrt{1+\alpha^2/D^2}}K, \frac{D}{\sqrt{1+\alpha^2}}B_2^n. \tag{7.7}$$

Consequently,

$$\left(\frac{1}{\sqrt{1+\alpha^2}} + \frac{\beta}{D}\right)K, rB_2^n \subset \tilde{K} \subset \left(\frac{1}{\sqrt{1+\alpha^2/D^2}} + \beta\right)K, RB_2^n, \tag{7.8}$$

with $r, R > 0$ given by (7.4). Hence, reinserting the extra ε-error due to the smoothing of the original K, by choosing $\varepsilon > 0$ sufficiently small we may ensure

$$d_{BM}(K, \tilde{K}) \leq \frac{\frac{1}{\sqrt{1+\alpha^2/D^2}} + \beta}{\frac{1}{\sqrt{1+\alpha^2}} + \frac{\beta}{D}} + \varepsilon \leq (1+\beta)\sqrt{1+\alpha^2}.$$

Next, defining $f(x) := \|x\|_K^2 + \frac{\alpha^2}{D^2}|x|^2$, note that $\partial L = \{x\,;\, \|x\|_L = 1\} = \{f = 1\}$. Consequently, the unit outer normal to ∂L is $\mathfrak{n}^{\partial L}(x) = \frac{\bar{D} f(x)}{|\bar{D} f(x)|}$. A standard computation then verifies that

$$\mathrm{II}_x^{\partial L} = \frac{P_{(\mathfrak{n}^{\partial L})^\perp} \bar{D}^2 f \; P_{(\mathfrak{n}^{\partial L})^\perp}}{|\bar{D} f(x)|}.$$

As $\|x\|_K^2$ is convex on E, and as

$$|\bar{D} f(x)| h_L(\mathfrak{n}^{\partial L}(x)) = \langle \bar{D} f(x), x\rangle = 2 f(x) = 2 \quad \forall x \in \partial L,$$

by Euler's identity and the 2-homogeneity of f, it follows that

$$\mathrm{II}_x^{\partial L} \geq \frac{2\frac{\alpha^2}{D^2}\delta_x^{\partial L}}{2/h_L(\mathfrak{n}^{\partial L}(x))} = \frac{\alpha^2}{D^2} h_L(\mathfrak{n}^{\partial L}(x))\delta_x^{\partial L}.$$

Recalling that $D^2 h_L(\mathfrak{n}^{\partial L}(x)) = (\mathrm{II}_x^{\partial L})^{-1}$ and applying the same argument as in Lemma 6.1, we deduce

$$D^2 h_L \leq \frac{D^2}{\alpha^2}\frac{1}{h_L}\delta^{\mathbb{S}^*}.$$

As $h_{\tilde{K}} = h_L + \beta$ on $\mathbb{S}^*$, it follows that

$$\beta\delta^{\mathbb{S}^*} \leq D^2 h_{\tilde{K}} \leq \left(\frac{D^2}{\alpha^2}\frac{1}{h_L} + \beta\right)\delta^{\mathbb{S}^*},$$

and hence

$$\beta h_{\tilde{K}}\delta^{\mathbb{S}^*} \leq h_{\tilde{K}} D^2 h_{\tilde{K}} \leq \left(\frac{D^2}{\alpha^2}\left(1 + \frac{\beta}{h_L}\right) + \beta h_{\tilde{K}}\right)\delta^{\mathbb{S}^*}.$$

Recalling (7.7) and (7.8), we deduce that

$$A\delta^{\mathbb{S}^*} \leq h_{\tilde{K}} D^2 h_{\tilde{K}} \leq B\delta^{\mathbb{S}^*},$$

with $A, B > 0$ given by (7.5). A final application of Lemma 6.1 concludes the proof. ∎

We are now ready to complete the proofs of Theorems 1.5 and 1.7.

Proof of Theorems 1.5 and 1.7. Given $\bar{K} \in \mathcal{K}_e$ and $\gamma > 0$ where $D := d_{BM}(\bar{K}, B_2^n)$, we construct $\tilde{K} \in \mathcal{K}_{+,e}^\infty$ as in Proposition 7.1 with a certain choice of $\alpha, \beta > 0$ to be determined later. Recall that $d_{BM}(\bar{K}, \tilde{K}) \leq (1+\beta)\sqrt{1+\alpha^2}$ and that $\tilde{K}$ satisfies (7.2) and

(7.3), where $r, R, A, B > 0$ are parameters depending on α, β, D which are given by (7.4) and (7.5). By Theorem 6.4, for any p in the range

$$p_{\alpha,\beta,D} := 2 - \frac{\frac{n-1}{2}A - R^2}{B} < p < 1,$$

$\tilde{K}$ satisfies the conclusion of Theorem 1.5. Consequently, our task is to show that $p_{\alpha,\beta,D} \leq p_{\gamma,D}$ for some choice of $\alpha, \beta > 0$ with $(1+\beta)\sqrt{1+\alpha^2} = \gamma$, where $p_{\gamma,D}$ is given by (1.11).

Denote

$$b := \frac{\beta}{1+\beta}.$$

We will ensure that the parameters α, β, γ satisfy

$$b\gamma = \beta\sqrt{1+\alpha^2} \leq D(\sqrt{2}-1), \quad \alpha \leq D. \tag{7.9}$$

Plugging in the expressions for r, R, A, B, we have

$$r > \beta, \quad R \leq \frac{\sqrt{2}\,D}{\sqrt{1+\alpha^2}}, \quad \sqrt{1+\alpha^2/D^2} \leq \sqrt{2}.$$

We now estimate

$$\begin{aligned}\frac{\frac{n-1}{2}A - R^2}{B} &= \frac{\frac{n-1}{2}\beta r - R^2}{\beta R + \frac{D^2}{\alpha^2}(1+\beta\sqrt{1+\alpha^2/D^2})} > \frac{\frac{n-1}{2}\beta^2 - \frac{2D^2}{1+\alpha^2}}{\beta\frac{\sqrt{2}\,D}{\sqrt{1+\alpha^2}} + \frac{D^2}{\alpha^2}(1+\sqrt{2}\,\beta)} \\ &= \frac{\frac{n-1}{2}b^2\gamma^2 - 2D^2}{\sqrt{2}\,b\gamma D + D^2\frac{1+\alpha^2}{\alpha^2}(1+\sqrt{2}\,\beta)}. \end{aligned} \tag{7.10}$$

Let us start with the isomorphic result of Theorem 1.5. It is apparent that in order to maximize the latter expression given γ and D, it is preferable to choose β of the order of a constant. To obtain an aesthetically pleasing expression, we set

$$\beta = 1 + \sqrt{2} \iff b = \frac{1}{\sqrt{2}},$$

yielding

$$\frac{\frac{n-1}{2}A - R^2}{B} \geq \frac{\frac{n-1}{4}\gamma^2 - 2D^2}{\gamma D + D^2\frac{1+\alpha^2}{\alpha^2}(3+\sqrt{2})}.$$

Recalling our guarantee that

$$\gamma \leq \frac{\sqrt{2}-1}{b}D = (2-\sqrt{2})D,$$

it follows by also ensuring that $\alpha^2 \geq 3 + \sqrt{2}$ that

$$p_{\alpha,\beta,D} = 2 - \frac{\frac{n-1}{2}A - R^2}{B} < 2 - \frac{\frac{n-1}{4}\gamma^2 - 2D^2}{6D^2} = \frac{7}{3} - \frac{n-1}{24}\frac{\gamma^2}{D^2} = p_{\gamma,D}.$$

It remains to conveniently summarize our restrictions on $\gamma = (1+\beta)\sqrt{1+\alpha^2}$:

$$\gamma \geq (2+\sqrt{2})\sqrt{4+\sqrt{2}} \simeq 7.95, \quad \gamma \leq (2-\sqrt{2})D \simeq 0.58D.$$

This concludes the proof of Theorem 1.5.

As for the isometric result of Theorem 1.7, one just notes that in order to have $p_{\gamma,D} < 0$, it is enough to have the expression in (7.10) greater than or equal to 2, i.e.

$$\frac{n-1}{2}\beta^2(1+\alpha^2) \geq \sqrt{2}\,2\beta\sqrt{1+\alpha^2}\,D + 2D^2\frac{1+\alpha^2}{\alpha^2}(1+\sqrt{2}\,\beta) + 2D^2.$$

We set $\alpha = \sqrt{D}/\sqrt[4]{n}$ and $\beta = C_\beta \alpha$ for an appropriate universal constant $C_\beta > 1$ to be determined, and recall that $D \leq \sqrt{n}$ and hence $\alpha \leq 1$ and $\beta \leq C_\beta$. It is therefore enough to have

$$\frac{n-1}{2}C_\beta^2\frac{D}{\sqrt{n}} \geq 4C_\beta D + 4\sqrt{n}D(1+\sqrt{2}\,\beta) + 2D^2.$$

A sufficient condition for that is

$$\sqrt{n}D\left(\frac{1}{4}C_\beta^2 - 4\sqrt{2}\,C_\beta - 4\frac{C_\beta}{\sqrt{n}}\right) \geq 4\sqrt{n}D + 2D^2,$$

and in turn, a sufficient condition for that is

$$\frac{1}{4}C_\beta^2 - 4\sqrt{2}\,C_\beta - 4\frac{C_\beta}{\sqrt{n}} \geq 6.$$

Clearly, this holds for a sufficiently large constant $C_\beta > 1$. Finally, it follows that

$$\gamma = (1+\beta)\sqrt{1+\alpha^2} \leq 1 + C\frac{\sqrt{D}}{\sqrt[4]{n}}, \tag{7.11}$$

for another universal constant $C > 1$.

There is still one last point to take care of – we need to guarantee that the assumptions (7.9) hold. Since $\alpha \leq 1 \leq D$, we just need to make sure that $\sqrt{2}\,C_\beta \leq D(\sqrt{2}-1)$. Since we only ever used D as an upper bound on $d_{BM}(\bar{K}, B_2^n)$, it follows that we should use $\max(D, \frac{\sqrt{2}C_\beta}{\sqrt{2}-1})$ in place of D, which simply translates into using a different value for the constant $C > 1$ in (7.11). This concludes the proof of Theorem 1.7. ∎

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