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Symmetrized non-commutative tori revisited

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Abstract. For the flip action of $\mathbb{Z}_2$ on an n-dimensional noncommutative torus A_θ, using an exact sequence by Natsume, we compute the K-theory groups of $A_\theta \rtimes \mathbb{Z}_2$. The novelty of our method is that it also provides an explicit basis of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$, for any θ. As an application, for a simple n-dimensional torus A_θ, using classification techniques, we determine the isomorphism class of $A_\theta \rtimes \mathbb{Z}_2$ in terms of the isomorphism class of A_θ.

1. Introduction

For $n \geq 2$, let $\mathcal{T}_n$ denote the space of all $n \times n$ real skew-symmetric matrices. The n-dimensional noncommutative torus A_θ is the universal C^*-algebra generated by unitaries $U_1, U_2, U_3, \ldots, U_n$ subject to the relations

$$U_j U_k = e^{2\pi i \theta_{jk}} U_k U_j, \tag{1.1}$$

for $j, k = 1, 2, 3, \ldots, n$, where $\theta := (\theta_{jk}) \in \mathcal{T}_n$. For the two-dimensional noncommutative tori, since θ is determined by only one real number, θ_{12}, we will often denote the corresponding two-dimensional noncommutative torus by $A_{\theta_{12}}$. Recall that the action of $\mathbb{Z}_2$ on any n-dimensional A_θ—often called the *flip action*—is defined by sending U_i to U_i^{-1}, for all i. The study of the corresponding crossed product C^*-algebra $A_\theta \rtimes \mathbb{Z}_2$ for $n = 2$ goes back to the work [3]. Quickly this became one of the accessible examples of a noncommutative space. The algebra $A_\theta \rtimes \mathbb{Z}_2$, for a general n, also appears in M(atrix) theory and String theory; see [17, 18]. The Morita equivalence classes and the isomorphism classes of $A_\theta \rtimes \mathbb{Z}_2$ play an important role in [17, 18]. The K-theory of $A_\theta \rtimes \mathbb{Z}_2$ was computed by Kumjian [19] for the two-dimensional cases. Kumjian used an exact sequence of Natsume to compute such K-theory groups. Later, using the similar methods, Farsi and Watling, in [12], have computed the K-theory of $A_\theta \rtimes \mathbb{Z}_2$ for general n, and for a totally irrational θ (see Definition 3.4). However, in [8], a major gap was pointed out in the paper [12]. Recently this gap was rectified in [6] indirectly using a result of [1]. Using the tools developed by the authors in [8], we rectify the gap directly in this paper for general n. Note that this direct method also gives us a basis of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$, whereas the indirect method in [6] just computes the dimensions of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$. Note that $\mathrm{K}_1(A_\theta \rtimes \mathbb{Z}_2)$ is trivial.

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The authors in [12] used the following exact sequence of Natsume [22] to compute the K-theory of $A_\theta \rtimes \mathbb{Z}_2$, for a totally irrational θ. Since $A_\theta \rtimes \mathbb{Z}_2$ can be written as $A_{\theta'} \rtimes_\phi (\mathbb{Z}_2 * \mathbb{Z}_2)$, for some antisymmetric $(n-1) \times (n-1)$ matrix θ' and some action ϕ, in this case Natsume's exact sequence looks like

$$\begin{array}{ccccc}
\mathrm{K}_0(A_{\theta'}) & \xrightarrow{i_{1*}-i_{2*}} & \mathrm{K}_0(A_{\theta'} \rtimes \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta'} \rtimes \mathbb{Z}_2) & \xrightarrow{j_{1*}+j_{2*}} & \mathrm{K}_0(A_{\theta'} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2) \\
\uparrow & & & & \downarrow e_1 \\
\mathrm{K}_1(A_{\theta'} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2) & \xleftarrow[j_{1*}+j_{2*}]{} & \mathrm{K}_1(A_{\theta'} \rtimes \mathbb{Z}_2) \oplus \mathrm{K}_1(A_{\theta'} \rtimes \mathbb{Z}_2) & \xleftarrow[i_{1*}-i_{2*}]{} & \mathrm{K}_1(A_{\theta'}),
\end{array}$$

where i_1, i_2, j_1, j_2 are natural inclusions. In the proof of the main theorem [12, Theorem 7], it was stated that the map

$$i_{1*} - i_{2*} : \mathbb{Z}^{2^{n-2}} \longrightarrow \mathbb{Z}^{3.2^{n-1}},$$

in the K_0-level, is given by Diag(id, 0, −id, 0). (Note that Diag(id, 0, −id, 0) should be replaced by $(\mathrm{id}, 0, -\mathrm{id}, 0)^t$, as, for example, for $n = 2$, the map $i_{1*} - i_{2*}$ should be a 6×1 matrix and clearly the Diag(id, 0, −id, 0) is not.) However, the reference [26, Corollary 7.9], mentioned therein, does not clearly give the result. This was clarified in [8, Corollary 7.2, see also Remark 7.5] for $n = 3$ using the description of the Chern character map (from [29]), and using an explicit description of the K-theory classes of $A_\theta \rtimes \mathbb{Z}_2$, for two-dimensional A_θ. In [8, Corollary 7.2], the authors in fact show that $i_{1*} - i_{2*} : \mathbb{Z}^{2^{n-2}} \to \mathbb{Z}^{3.2^{n-1}}$ is given by $(\mathrm{id}, 0, -\mathrm{id}, 0)^t$, for $n = 3$.

Recently in a paper with Hua [7], the author of this paper has found a basis of $\mathrm{K}_0(A_\theta)$ consisting of projections inside A_θ, for an $n \times n$ strongly totally irrational θ (see Definition 3.15). Using this basis of $\mathrm{K}_0(A_\theta)$, for a strongly totally irrational θ, in this paper we prove that

$$i_{1*} - i_{2*} : \mathbb{Z}^{2^{n-2}} \longrightarrow \mathbb{Z}^{3.2^{n-1}}$$

is given by $(\mathrm{id}, 0, -\mathrm{id}, 0)^t$ for general n. This computes the K-theory of $A_\theta \rtimes \mathbb{Z}_2$, for all strongly totally irrational θ. We invoke Morita equivalence bi-modules for higher-dimensional noncommutative tori and an explicit description of the boundary map e_1 of the above exact sequence to conclude that $i_{1*} - i_{2*} = (\mathrm{id}, 0, -\mathrm{id}, 0)^t$. Our method gives an explicit basis of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$ for all strongly totally irrational θ, in terms of projections inside $A_\theta \rtimes \mathbb{Z}_2$. Let $\mathcal{P}_n$ be the set as in equation (4.6). Then results discussed above give the following.

Theorem 1.1 (Theorem 4.9). *Let θ be a strongly irrational $n \times n$ matrix. Then*

$$\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^{3 \cdot 2^{n-1}},$$

and a generating set of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$ *may be given by* $\{[1], [P] \mid P \in \mathcal{P}_n\}$.

For general θ, using a continuous field argument from [10], and ideas from [4], we construct an explicit basis of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$ in terms of projective modules over $A_\theta \rtimes \mathbb{Z}_2$. If Proj_n denotes the set as defined in Theorem 5.6, then we have the following result.

Theorem 1.2 (Theorem 5.6, Corollary 4.10). *Let $\theta \in \mathcal{T}_n$. Then* $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^{3\cdot 2^{n-1}}$, *and a generating set of* $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$ *is given by* $\{[1], [\mathcal{E}] \mid \mathcal{E} \in \mathrm{Proj}_n\}$.

It is known that θ is non-degenerate (see Definition 6.1) iff A_θ is simple. When $\theta \in \mathcal{T}_n$ is non-degenerate, $A_\theta \rtimes \mathbb{Z}_2$ is an AF algebra (see [10, Theorem 6.6], and also Corollary 6.4). For non-degenerate θ_1, θ_2, our construction of explicit bases of $\mathrm{K}_0(A_{\theta_i} \rtimes \mathbb{Z}_2)$, $i = 1, 2$, allows us to construct explicit isomorphisms between the Elliott invariants of $A_{\theta_1} \rtimes \mathbb{Z}_2$ and $A_{\theta_2} \rtimes \mathbb{Z}_2$ out of an isomorphism between the Elliott invariants of A_{θ_1} and A_{θ_2}, and the converse holds if, in addition, one of the θ_i is totally irrational. This results in the following theorem.

Theorem 1.3 (Theorem 6.2). *Let $\theta_1, \theta_2 \in \mathcal{T}_n$ be non-degenerate. Let $\mathbb{Z}_2$ act on A_{θ_1} and A_{θ_2} by the flip actions. Then $A_{\theta_1} \rtimes \mathbb{Z}_2$ is isomorphic to $A_{\theta_2} \rtimes \mathbb{Z}_2$ if A_{θ_1} is isomorphic to A_{θ_2}. Moreover, if any one of θ_1, θ_2 is totally irrational, the converse is true.*

The above theorem is a generalization of [10, Theorem 6.4] for general n. It is worth mentioning that the only canonical action (in the sense of [16]) of a finite cyclic group on a 3-dimensional torus A_θ, when θ is non-degenerate, is the flip action [16, Theorem 1.4].

It should be noted that, using a completely different approach, Davis and Lück [9] computed the K-theory of $A_\theta \rtimes \mathbb{Z}_2$ when θ is the $n \times n$ zero matrix. However, from their methods it is not clear how to extract a concrete basis for $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$, and hence a classification type result like Theorem 1.2.

This article is organized as follows. In Section 2, we define $A_\theta \rtimes \mathbb{Z}_2$ through twisted group C^*-algebras and study some basic properties of the crossed product. The K-theory of A_θ and a generating set of $\mathrm{K}_0(A_\theta)$, for a strongly totally irrational θ, are described in Section 3. Section 4 deals with descriptions of the maps that appear in Natsume's exact sequence, and the proof of Theorem 1.1. In Section 5, we use the continuous field approach of [10] to describe the explicit generators of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$ for general θ, and prove Theorem 1.2. The classification-type theorem, Theorem 1.3, is proved in Section 6. In Appendix A, we revisit the construction of the two-dimensional Rieffel projection which is used in the main construction of Sections 3 and 4, and in Appendix B, we give a class of examples of strongly totally irrational matrices. Finally, in Appendix C, we explicitly describe the continuous field which is used in Section 5.

Notation. $e(x)$ will always denote the number $e^{2\pi i x}$, and id_m (or without the "m" decoration if the context is clear) will be the $m \times m$ unit matrix.

2. $A_\theta \rtimes \mathbb{Z}_2$ – revisited

Let G be a discrete group. A map $\omega : G \times G \to \mathbb{T}$ is called a *2-cocycle* if

$$\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z) \quad \text{and} \quad \omega(x, 1) = 1 = \omega(1, x)$$

for all $x, y, z \in G$.

The *ω-twisted left regular representation* of the group G is given by the formula:

$$\big(L_\omega(x)f\big)(y) = \omega(x, x^{-1}y)f(x^{-1}y),$$

for $f \in l^2(G)$. The *reduced twisted group C^*-algebra* $C^*(G,\omega)$ is defined as the sub-C^*-algebra of $B(l^2(G))$ generated by the ω-twisted left regular representation of the group G. Since we do not talk about full group C^*-algebras in this paper, we simply call $C^*(G,\omega)$ the twisted group C^*-algebra of G with respect to ω. When $\omega = 1$, $C^*(G,\omega) =: C^*(G)$ is the usual reduced group C^*-algebra of G. We refer to [10, Section 1] for more on twisted group C^*-algebras and the details of the above construction.

Example 2.1. Let G be the group $\mathbb{Z}^n$. For each $\theta \in \mathcal{T}_n$, construct a 2-cocycle ω_θ on G by defining $\omega_\theta(x, y) = e(\langle -\theta x, y\rangle/2)$. The corresponding twisted group C^*-algebra $C^*(G, \omega_\theta)$ is isomorphic to the n-dimensional noncommutative torus A_θ, which was defined in the introduction. The isomorphism sends $\delta_{x_i} \in C^*(\mathbb{Z}^n, \omega_\theta)$ to U_i where $x_i = (0, \dots, 1, \dots, 0)$, with 1 at the ith position.

Example 2.2. Let $\mathbb{Z}_2$ act on $\mathbb{Z}^n$ by sending x to $-x$. Let us also take a $\theta \in \mathcal{T}_n$. Then we can define a 2-cocycle ω'_θ on $G := \mathbb{Z}^n \rtimes \mathbb{Z}_2$ by $\omega'_\theta((x,s),(y,t)) = \omega_\theta(x, s\cdot y)$. By [10, Lemma 2.1], we have

$$C^*(\mathbb{Z}^n \rtimes \mathbb{Z}_2, \omega'_\theta) = A_\theta \rtimes_\beta \mathbb{Z}_2,$$

where the action β of $\mathbb{Z}_2$ on A_θ is given by sending U_i to U_i^{-1} which is the *flip action*. For the crossed product with the flip action $A_\theta \rtimes_\beta \mathbb{Z}_2$, we shall often drop the decoration β from $A_\theta \rtimes_\beta \mathbb{Z}_2$, and denote it by $A_\theta \rtimes \mathbb{Z}_2$.

Let θ be as before and let θ' be the upper left $(n-1)\times(n-1)$ block of θ. In this case, A_θ can be written as a crossed product $A_{\theta'} \rtimes_\gamma \mathbb{Z}$, where the action γ of $\mathbb{Z}$ on $A_{\theta'}$ is determined on the positive generator of $\mathbb{Z}$ by mapping U_i to $e(-\theta_{in})U_i$, for $i = 1, \dots, n-1$. Now $A_\theta \rtimes \mathbb{Z}_2 = A_{\theta'} \rtimes_\phi (\mathbb{Z} \rtimes \mathbb{Z}_2) = A_{\theta'} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2$ since $\mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}_2$ as groups (cf. [12, Proposition 6]). Note that one copy of $\mathbb{Z}_2$ acts on $A_{\theta'}$ by the flip action β, and the other by $\alpha = \gamma \circ \beta$.

Lemma 2.3. *$A_{\theta'} \rtimes \mathbb{Z}_2 \cong A_{\theta'} \rtimes_\alpha \mathbb{Z}_2$.*

Proof. $A_{\theta'} \rtimes_\alpha \mathbb{Z}_2$ is generated by the unitaries $U_1, U_2, \dots, U_{n-1}$ and $W' = U_n W$ and we have the relations

$$W'^2 = 1, \quad U_j U_k = e(\theta_{jk}) U_k U_j, \quad W' U_i W' = e(\theta_{in}) U_i^{-1}.$$

Upon setting $\widetilde{U}_i = e(-\frac{1}{2}\theta_{in})U_i$ for $i = 1, 2, \dots, n-1$, we have that

$$\widetilde{U}_j \widetilde{U}_k = e(\theta_{jk}) \widetilde{U}_k \widetilde{U}_j, \quad W' \widetilde{U}_i W' = \widetilde{U}_i^{-1}.$$

So $A_{\theta'} \rtimes_\alpha \mathbb{Z}_2$ is isomorphic to $A_{\theta'} \rtimes \mathbb{Z}_2$. ∎

3. The construction of Rieffel-type projections and $K_0(A_\theta)$

For a formal definition of the pfaffian $\mathrm{pf}(A)$ of an $n \times n$ skew-symmetric matrix A with $n = 2m$ even, we refer to [4, Definition 3.1]. If $n = 2m$ for some integer $m \geq 1$, then for

$$A = \begin{pmatrix} 0 & \theta_{12} & \cdots & & \cdots & \theta_{1n} \\ -\theta_{12} & \ddots & \ddots & & & \theta_{2n} \\ \vdots & \ddots & & & & \\ & & & & \ddots & \vdots \\ -\theta_{1(n-1)} & & & \ddots & \ddots & \theta_{(n-1)n} \\ -\theta_{1n} & \cdots & & \cdots & -\theta_{(n-1)n} & 0 \end{pmatrix},$$

the pfaffian of A is given by $\sum_\xi (-1)^{|\xi|} \prod_{s=1}^m \theta_{\xi(2s-1)\xi(2s)}$, where the sum is taken over all elements ξ of the permutation group S_n such that $\xi(2s-1) < \xi(2s)$ for all $1 \leq s \leq m$ and $\xi(1) < \xi(3) < \cdots < \xi(2m-1)$.

Definition 3.1. Let $n \geq 2$ be an integer, and let p be an integer such that $1 \leq p \leq \frac{n}{2}$. A $2p$-*pfaffian minor* (or just *pfaffian minor*) of a skew-symmetric $n \times n$ matrix A is the pfaffian of a sub-matrix A_I of A consisting of rows and columns indexed by $i_1, i_2, \ldots, i_{2p}$ for some numbers $1 \leq i_1 < i_2 < \cdots < i_{2p} \leq n$, and $I = (i_1, i_2, \ldots, i_{2p})$. Define the length of I, $|I| := 2p$. We often use pf_I^A as the abbreviation of $\mathrm{pf}(A_I)$ without special emphasis.

For $p = 0$, define I to be the empty sequence $\emptyset$, and in this case, define $\mathrm{pf}(A_I) = \mathrm{pf}(A_\emptyset) := 1$. The length of $I = \emptyset$ is defined to be zero.

The set of all such I's, for a fixed n and varying $p, 0 \leq p \leq \frac{n}{2}$, is denoted by $\mathrm{Minor}(n)$. Of course, $\mathrm{Minor}(n) \subset \mathrm{Minor}(n+1)$, for all n. Note that the number of elements of $\mathrm{Minor}(n)$ is 2^{n-1}.

Let Tr denote the canonical tracial state on A_θ satisfying $\mathrm{Tr}(1) = 1$,

$$\mathrm{Tr}(U_1^{m_1} U_2^{m_2} \cdots U_n^{m_n}) = 0$$

unless $(m_1, m_2, \ldots, m_n) = 0 \in \mathbb{Z}^n$. We recall the following fact due to Elliott which will play a key role.

Theorem 3.2 (Elliott). *Let θ be a skew-symmetric real $n \times n$ matrix. Then there is an isomorphism $h : K_0(A_\theta) \to \Lambda^{\mathrm{even}}\mathbb{Z}^n$ such that $\exp_\wedge(\theta) \circ h = \mathrm{Tr}$, where $\exp_\wedge(\theta)$ is the exterior exponential map*

$$\exp_\wedge(\theta) : \Lambda^{\mathrm{even}}\mathbb{Z}^n \to \mathbb{R},$$

and such that $h([1])$ is the standard generator $1 \in \Lambda^0(\mathbb{Z}^n) = \mathbb{Z}$. In particular, $\mathrm{Tr}(K_0(A_\theta))$ is the range of the exterior exponential.

We refer to [11, Section 1.3, Theorems 2.2 and 3.1] for the definition of the exterior exponential and the proof of the above theorem. The range of the exterior exponential is well known and is given below as a corollary of the above theorem.

Corollary 3.3. *For $\theta \in \mathcal{T}_n$, $\mathrm{Tr}(\mathrm{K}_0(A_\theta))$ is the subgroup of $\mathbb{R}$ generated by 1 and the numbers*

$$\sum_{\xi}(-1)^{|\xi|}\Pi_{s=1}^{p}\theta_{j_{\xi(2s-1)}j_{\xi(2s)}}$$

for $1 \le j_1 < j_2 < \cdots < j_{2p} \le n$, where the sum is taken over all elements ξ of the permutation group S_{2p} such that $\xi(2s-1) < \xi(2s)$ for all $1 \le s \le p$ and $\xi(1) < \xi(3) < \cdots < \xi(2p-1)$.

Noting that $\sum_{\xi}(-1)^{|\xi|}\Pi_{s=1}^{p}\theta_{j_{\xi(2s-1)}j_{\xi(2s)}}$ is exactly the pfaffian of θ_I, where $I = (i_1, i_2, \ldots, i_{2p})$, we have

$$\mathrm{Tr}\left(\mathrm{K}_0(A_\theta)\right) = \sum_{I \in \mathrm{Minor}(n)} \mathrm{pf}(\theta_I)\mathbb{Z}. \tag{3.1}$$

Except for the case $I = \emptyset$, it is not clear whether the numbers $\mathrm{pf}(\theta_I)$ can be realized as traces of projections (which we call *Rieffel-type projections*, if exist) inside A_θ or not. For the case $I = \emptyset$, of course we can take the projection $1 \in A_\theta$. In this section, we will construct Rieffel-type projections for a large class of A_θ.

Definition 3.4 ([12, Definition 1]). We say that $\theta \in \mathcal{T}_n$ is *totally irrational* if $\exp_\wedge(\theta)$ is an injective map from $\Lambda^{\mathrm{even}}\mathbb{Z}^n$ to $\mathbb{R}$ (cf. [26, Sections 6 and 7]).

It is clear from Elliott's work (Theorem 3.2) that Tr is injective if and only if θ is totally irrational. Now the range of $\exp_\wedge(\theta)$ is given by

$$\sum_{I \in \mathrm{Minor}(n)} \mathrm{pf}(\theta_I)\mathbb{Z}.$$

Thus θ is totally irrational if and only if the numbers $\mathrm{pf}(\theta_I)$, $I \in \mathrm{Minor}(n)$, are rationally independent. Note that if θ is totally irrational, θ is also nondegenerate in the sense of [23] (see Definition 6.1), and A_θ is a simple C^*-algebra by [23, Theorem 1.9].

For

$$\theta = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{pmatrix} = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ -\theta_{1,2}^t & \theta_{2,2} \end{pmatrix} \in \mathcal{T}_n, \quad n = 2l \text{ for } l > 1,$$

such that

$$\theta_{1,1} = \begin{pmatrix} 0 & \theta_{12} \\ \theta_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \theta_{12} \\ -\theta_{12} & 0 \end{pmatrix} \in \mathcal{T}_2$$

is an invertible 2×2 matrix,

$$\theta_{2,2} = \begin{pmatrix} 0 & \theta_{34} & \cdots & \theta_{3n} \\ -\theta_{34} & 0 & \cdots & \theta_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ -\theta_{3n} & -\theta_{4n} & \cdots & 0 \end{pmatrix} \in \mathcal{T}_{n-2}, \quad \theta_{1,2} = \begin{pmatrix} \theta_{13} & \theta_{14} & \cdots & \theta_{1n} \\ \theta_{23} & \theta_{24} & \cdots & \theta_{2n} \end{pmatrix},$$

and

$$\theta_{2,1} = \begin{pmatrix} -\theta_{13} & -\theta_{23} \\ -\theta_{14} & -\theta_{24} \\ \vdots & \vdots \\ -\theta_{1n} & -\theta_{2n} \end{pmatrix};$$

we have

$$\theta_{1,1}^{-1} = \begin{pmatrix} 0 & -\frac{1}{\theta_{12}} \\ \frac{1}{\theta_{12}} & 0 \end{pmatrix} \in \mathcal{T}_2,$$

and

$$\theta_{2,2} - \theta_{2,1}\theta_{1,1}^{-1}\theta_{1,2}$$
$$= \begin{pmatrix} 0 & \theta_{34} - \frac{-\theta_{23}\theta_{14}+\theta_{13}\theta_{24}}{\theta_{12}} & \cdots & \theta_{3n} - \frac{-\theta_{23}\theta_{1n}+\theta_{13}\theta_{2n}}{\theta_{12}} \\ -\theta_{34} + \frac{-\theta_{23}\theta_{14}+\theta_{13}\theta_{24}}{\theta_{12}} & 0 & \cdots & \theta_{4n} - \frac{-\theta_{24}\theta_{1n}+\theta_{14}\theta_{2n}}{\theta_{12}} \\ \vdots & \vdots & \ddots & \vdots \\ -\theta_{3n} + \frac{-\theta_{23}\theta_{1n}+\theta_{13}\theta_{2n}}{\theta_{12}} & -\theta_{4n} + \frac{-\theta_{24}\theta_{1n}+\theta_{14}\theta_{2n}}{\theta_{12}} & \cdots & 0 \end{pmatrix}.$$

Hence we have

$$\theta_{2,2} - \theta_{2,1}\theta_{1,1}^{-1}\theta_{1,2} = \begin{pmatrix} 0 & \frac{\mathrm{pf}^{\theta}_{(1,2,3,4)}}{\theta_{12}} & \cdots & \frac{\mathrm{pf}^{\theta}_{(1,2,3,n)}}{\theta_{12}} \\ -\frac{\mathrm{pf}^{\theta}_{(1,2,3,4)}}{\theta_{12}} & 0 & \cdots & \frac{\mathrm{pf}^{\theta}_{(1,2,4,n)}}{\theta_{12}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\mathrm{pf}^{\theta}_{(1,2,3,n)}}{\theta_{12}} & -\frac{\mathrm{pf}^{\theta}_{(1,2,4,n)}}{\theta_{12}} & \cdots & 0 \end{pmatrix}.$$

We have the following lemma.

Lemma 3.5 ([7, Lemma 3.6]). *For any integer $n \geq 2$, let*

$$\theta = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{pmatrix} = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ -\theta_{1,2}^{t} & \theta_{2,2} \end{pmatrix} \in \mathcal{T}_n,$$

where $\theta_{1,1}$ is invertible 2×2 matrix, and one has

$$\mathrm{pf}(\theta_{1,1})\,\mathrm{pf}\left((\theta_{2,2} - \theta_{2,1}\theta_{1,1}^{-1}\theta_{1,2})_{I'}\right) = \mathrm{pf}(\theta_I), \tag{3.2}$$

where $I' \in \mathrm{Minor}(n-2) \setminus \{\emptyset\}$ and $I = (1, 2, i_1 + 2, i_2 + 2, \dots, i_{2l} + 2)$ for $I' = (i_1, i_2, \dots, i_{2l})$. In particular, when n is an even number, taking $I' = (1, 2, \dots, n-2)$, we have

$$\mathrm{pf}(\theta) = \mathrm{pf}(\theta_{1,1})\,\mathrm{pf}(\theta_{2,2} - \theta_{2,1}\theta_{1,1}^{-1}\theta_{1,2}). \tag{3.3}$$

Proof. See [7, proof of Lemma 3.6]. ∎

In order to explain our symbols, we give the following definition.

Definition 3.6. For

$$\theta = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{pmatrix} \in \mathcal{T}_n,$$

where θ_{11} is an invertible 2×2 matrix, and $n = 2l$, $l > 1$, define

$$F(\theta) = \theta_{2,2} - \theta_{2,1}\theta_{1,1}^{-1}\theta_{1,2} \in \mathcal{T}_{n-2}. \tag{3.4}$$

Hence from (3.3), we have

$$\mathrm{pf}(\theta) = \mathrm{pf}(\theta_{1,1})\,\mathrm{pf}\big(F(\theta)\big). \tag{3.5}$$

If m is an integer less than l, we denote by F^m the composition of F taken m times (when this makes sense) and $F^0(\theta) := \theta$. Note that F is defined for a θ such that θ_{11} is invertible, but still F^m may not make sense. The following lemma tells us when $F^m(\theta)$ is well defined.

Lemma 3.7 ([7, Lemmas 3.8 and 3.10]). *Let $\theta \in \mathcal{T}_n$ with $n = 2l$ for $l > 1$. If $\mathrm{pf}^{\theta}_{(1,2,\ldots,2s)} \neq 0$ for all $s = 1, 2, \ldots, l-1$, then $F^m(\theta)$ is well defined for $m = 1, 2, \ldots, l-1$ and is given by*

$$F^m(\theta) = \begin{pmatrix} 0 & \frac{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-1,n-p)}}{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-2)}} & \cdots & \frac{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-1,n)}}{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-2)}} \\ -\frac{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-1,n-p)}}{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-2)}} & 0 & \cdots & \frac{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p,n)}}{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-2)}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-1,n)}}{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-2)}} & -\frac{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p,n)}}{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-2)}} & \cdots & 0 \end{pmatrix}. \tag{3.6}$$

In particular,

$$F^m(\theta)_{jk} = \frac{\mathrm{pf}^{\theta}_{(1,2,\ldots,s',s'+j,s'+k)}}{\mathrm{pf}^{\theta}_{(1,2,\ldots,s')}}, \quad p = n-2m-2,\ s' = n-p-2 = 2m.$$

Proof. The lemma is exactly the content of [7, Lemmas 3.8 and 3.10]. See the proof of those. ∎

Let us use the above lemma to say more about $A_{F^m(\theta)}$ when $\theta \in \mathcal{T}_n$ is totally irrational with $n = 2l \geq 2$. When θ is totally irrational, the entries (above the diagonal) of $F^m(\theta)$ are all irrational and independent over $\mathbb{Q}$ for $m = 0, 1, \ldots, l-1$. This is because we have $\mathrm{pf}^{\theta}_{(1,2,\ldots,2s)} \neq 0$ for $s = 1, 2, \ldots, l-1$, by total irrationality of θ. By the above lemma, we have that $F^m(\theta)$ is well defined and

$$F^m(\theta)_{jk} = \frac{\mathrm{pf}^{\theta}_{(1,2,\ldots,s',s'+j,s'+k)}}{\mathrm{pf}^{\theta}_{(1,2,\ldots,s')}}, \quad p = n-2m-2,\ s' = n-p-2 = 2m,$$

for $m = 1, \dots, l-1$. Now since θ is totally irrational, the numbers $\mathrm{pf}^{\theta}_{(1,2,\dots,n-p-2,j,k)}$, $n-p-1 \le j < k \le n$ along with $\mathrm{pf}^{\theta}_{(1,2,\dots,n-p-2)}$, are irrational and independent over $\mathbb{Q}$. This means that the numbers

$$\frac{\mathrm{pf}^{\theta}_{(1,2,\dots,n-p-2,j,k)}}{\mathrm{pf}^{\theta}_{(1,2,\dots,n-p-2)}}$$

are irrational numbers. Next, to show that these numbers are independent over $\mathbb{Q}$, let us write

$$\sum_{n-p-1 \le j < k \le n} c_{j,k} \frac{\mathrm{pf}^{\theta}_{(1,2,\dots,n-p-2,j,k)}}{\mathrm{pf}^{\theta}_{(1,2,\dots,n-p-2)}} = 0, \quad c_{j,k} \in \mathbb{Q}.$$

This implies

$$\sum_{n-p-1 \le j < k \le n} c_{j,k}\, \mathrm{pf}^{\theta}_{(1,2,\dots,n-p-2,j,k)} = 0, \quad c_{j,k} \in \mathbb{Q}.$$

But since the numbers $\mathrm{pf}^{\theta}_{(1,2,\dots,n-p-2,j,k)}$, $n-p-1 \le j < k \le n$, are independent over $\mathbb{Q}$, $c_{j,k}$'s are all zero. Hence the numbers

$$\frac{\mathrm{pf}^{\theta}_{(1,2,\dots,n-p-2,j,k)}}{\mathrm{pf}^{\theta}_{(1,2,\dots,n-p-2)}}, \quad n-p-1 \le j < k \le n,$$

are rationally independent. So we have shown that the entries (above the diagonal) of $F^m(\theta)$ are irrational and rationally independent. Now, using [23, Lemma 1.7], it is easy to see that $F^m(\theta)$ is non-degenerate. Hence $A_{F^m(\theta)}$ is simple and has a unique tracial state for $m = 0, 1, \dots, l-1$.

In the following we shall construct Rieffel-type projections for the higher-dimensional noncommutative tori.

In [27], Rieffel and Schwarz defined (densely) an action of the group $\mathrm{SO}(n,n|\mathbb{Z})$ on $\mathcal{T}_n$. Recall that $\mathrm{SO}(n,n|\mathbb{Z})$ is the subgroup of $\mathrm{SL}(2n,\mathbb{Z})$, which consists of all matrices g with the following block form:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, and D are arbitrary $n \times n$ matrices over $\mathbb{Z}$ satisfying

$$A^t C + C^t A = 0, \quad B^t D + D^t B = 0, \quad \text{and} \quad A^t D + C^t B = \mathrm{id}_n.$$

The action of $\mathrm{SO}(n,n|\mathbb{Z})$ on $\mathcal{T}_n$ is defined as

$$g\theta := (A\theta + B)(C\theta + D)^{-1}$$

whenever $C\theta + D$ is invertible. The subset of $\mathcal{T}_n$ on which the action of every $g \in \mathrm{SO}(n,n|\mathbb{Z})$ is defined is dense in $\mathcal{T}_n$ (see [27, p. 291]). We have the following theorem due to Hanfeng Li.

Theorem 3.8 ([20, Theorem 1.1]). *For any $\theta \in \mathcal{T}_n$ and $g \in \mathrm{SO}(n, n|\mathbb{Z})$, if $g\theta$ is defined, then A_θ and $A_{g\theta}$ are strongly Morita equivalent.*

For any $R \in \mathrm{GL}(n, \mathbb{Z})$, let us denote by $\rho(R)$ the matrix $\left(\begin{smallmatrix} R & 0 \\ 0 & (R^{-1})^{\mathrm{t}} \end{smallmatrix}\right) \in \mathrm{SO}(n, n|\mathbb{Z})$, and for any $N \in \mathcal{T}_n \cap \mathrm{M}_n(\mathbb{Z})$, we denote by $\mu(N)$ the matrix $\left(\begin{smallmatrix} \mathrm{id}_n & N \\ 0 & \mathrm{id}_n \end{smallmatrix}\right) \in \mathrm{SO}(n, n|\mathbb{Z})$. Notice that the noncommutative tori corresponding to the matrices $\rho(R)\theta = R\theta R^t$ and $\mu(N)\theta = \theta + N$ are both isomorphic to A_θ. Also define

$$\mathrm{SO}(n, n|\mathbb{Z}) \ni \sigma_{2p} := \begin{pmatrix} 0 & 0 & \mathrm{id}_{2p} & 0 \\ 0 & \mathrm{id}_{n-2p} & 0 & 0 \\ \mathrm{id}_{2p} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{n-2p} \end{pmatrix}, \quad 1 \leqslant p \leqslant n/2.$$

We recall the approach of Rieffel [26] to find the $A_{\sigma_{2p}\theta} - A_\theta$ bimodule and follow the presentation in [20]. Let us fix a number p with $1 \leq p \leq n/2$ and let $q \geq 0$ be an integer such that $n = 2p + q$. Let us write $\theta \in \mathcal{T}_n$ as

$$\begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{pmatrix},$$

partitioned into four sub-matrices $\theta_{1,1}, \theta_{1,2}, \theta_{2,1}, \theta_{2,2}$, and assume $\theta_{1,1}$ to be an invertible $2p \times 2p$ matrix. Write σ_{2p} as σ. Then

$$\sigma(\theta) = \begin{pmatrix} \theta_{1,1}^{-1} & -\theta_{1,1}^{-1}\theta_{1,2} \\ \theta_{2,1}\theta_{1,1}^{-1} & \theta_{2,2} - \theta_{2,1}\theta_{1,1}^{-1}\theta_{1,2} \end{pmatrix}. \tag{3.7}$$

Set $A = A_\theta$ and $B = A_{\sigma(\theta)}$. Let $\mathcal{M}$ be the group $\mathbb{R}^p \times \mathbb{Z}^q$, $G = \mathcal{M} \times \widehat{\mathcal{M}}$, and $\langle \cdot, \cdot \rangle$ the natural pairing between $\mathcal{M}$ and its dual group $\widehat{\mathcal{M}}$ (our notation does not distinguish between the pairing of a group and its dual group, and the standard inner product on a linear space). Also, denote the linear dual of $\mathbb{R}^k$ by $(\mathbb{R}^k)^*$. Consider the Schwartz space $\mathcal{E}^\infty = \mathcal{S}(\mathcal{M})$ consisting of smooth and rapidly decreasing complex-valued functions on $\mathcal{M}$.

Denote by $\mathcal{A}^\infty = A_\theta^\infty$ and $\mathcal{B}^\infty = A_{\sigma(\theta)}^\infty$ the dense sub-algebras of A and B, respectively, consisting of formal series with rapidly decaying coefficients. Let us consider the following $(2p + 2q) \times (2p + q)$ real-valued matrix:

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & \mathrm{id}_q \\ T_{31} & T_{32} \end{pmatrix},$$

where T_{11} is an invertible matrix such that $T_{11}^t J_0 T_{11} = \theta_{1,1}$, $J_0 = \left(\begin{smallmatrix} 0 & \mathrm{id}_p \\ -\mathrm{id}_p & 0 \end{smallmatrix}\right)$, $T_{31} = \theta_{2,1}$, and T_{32} is any $q \times q$ matrix such that $\theta_{2,2} = T_{32} - T_{32}^t$. For our purposes, we take $T_{32} = \theta_{2,2}/2$.

We also define the following $(2p+2q) \times (2p+q)$ real-valued matrix:

$$S = \begin{pmatrix} J_0(T_{11}^t)^{-1} & -J_0(T_{11}^t)^{-1}T_{31}^t \\ 0 & \mathrm{id}_q \\ 0 & T_{32}^t \end{pmatrix}.$$

Let

$$J = \begin{pmatrix} J_0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_q \\ 0 & -\mathrm{id}_q & 0 \end{pmatrix}$$

and J' the matrix obtained from J by replacing the negative entries of it by zeroes. Note that T and S can be thought as maps from $(\mathbb{R}^n)^*$ to $\mathbb{R}^p \times (\mathbb{R}^p)^* \times \mathbb{R}^q \times (\mathbb{R}^q)^*$ (see the definition of an embedding map in [20, Definition 2.1]), and $S(\mathbb{Z}^n), T(\mathbb{Z}^n) \subset \mathbb{R}^p \times (\mathbb{R}^p)^* \times \mathbb{Z}^q \times (\mathbb{R}^q)^*$. Then we can think of $S(\mathbb{Z}^n), T(\mathbb{Z}^n)$ as in G via composing $S|_{\mathbb{Z}^n}, T|_{\mathbb{Z}^n}$ with the natural covering map

$$\mathbb{R}^p \times (\mathbb{R}^p)^* \times \mathbb{Z}^q \times (\mathbb{R}^q)^* \longrightarrow G.$$

Let P' and P'' be the canonical projections of G to $\mathcal{M}$ and $\widehat{\mathcal{M}}$, respectively, and let

$$T' = P' \circ T, \quad T'' = P'' \circ T, \quad S' = P' \circ S, \quad S'' = P'' \circ S.$$

Then the following set of formulas define a $\mathcal{B}^\infty$-$\mathcal{A}^\infty$ bimodule structure on $\mathcal{E}^\infty$:

$$(fU_l^\theta)(x) = e^{2\pi i \langle -T(l), J'T(l)/2\rangle} \big\langle x, T''(l)\big\rangle f\big(x - T'(l)\big), \tag{3.8}$$

$$\langle f, g\rangle_{\mathcal{A}^\infty}(l) = e^{2\pi i \langle -T(l), J'T(l)/2\rangle} \int_G \big\langle x, -T''(l)\big\rangle g\big(x + T'(l)\big)\overline{f(x)}dx, \tag{3.9}$$

$$(U_l^{\sigma(\theta)} f)(x) = e^{2\pi i \langle -S(l), J'S(l)/2\rangle} \big\langle x, -S''(l)\big\rangle f\big(x + S'(l)\big), \tag{3.10}$$

$${}_{\mathcal{B}^\infty}\langle f, g\rangle(l) = K e^{2\pi i \langle S(l), J'S(l)/2\rangle} \int_G \big\langle x, S''(l)\big\rangle \overline{g\big(x + S'(l)\big)} f(x)dx, \tag{3.11}$$

where $U_l^\theta, U_l^{\sigma(\theta)}$ denote the canonical unitaries with respect to the group element $l \in \mathbb{Z}^n$ in $\mathcal{A}^\infty$ and $\mathcal{B}^\infty$, respectively, and K is a positive constant. See [20, Proposition 2.2] for the following well-known result.

Theorem 3.9. *The smooth module $\mathcal{E}^\infty$, with above structures, is a $\mathcal{B}^\infty - \mathcal{A}^\infty$ Morita equivalence bi-module which can be completed to a strong $B - A$ Morita equivalence bi-module $\mathcal{E}$.*

The completion $\mathcal{E}$ of $\mathcal{E}^\infty$ of the above theorem becomes a finitely generated projective module over A (see the argument before [10, Proposition 4.6]). The resulting class $[\mathcal{E}] \in \mathrm{K}_0(A)$ is called the Bott class. We will soon see that it will contribute to a generating set of $\mathrm{K}_0(A)$.

Remark 3.10. The trace of the module $\mathcal{E}$, which was computed by Rieffel [26], is exactly the absolute value of the pfaffian of the upper left $2p \times 2p$ corner of the matrix θ, which is $\theta_{1,1}$. Indeed, [26, Proposition 4.3, p. 289] says that trace of $\mathcal{E}$ is $|\det \widetilde{T}|$, where

$$\widetilde{T} = \begin{pmatrix} T_{11} & 0 \\ 0 & \mathrm{id}_q \end{pmatrix}.$$

Thus the relation $T_{11}^t J_0 T_{11} = \theta_{1,1}$ and the fact $\det(J_0) = 1$ give the claim.

Let $\theta \in \mathcal{T}_n$. We will now see that for each non-zero pfaffian minor of θ, we can construct a projective module over A_θ such that the trace of this module is exactly the pfaffian minor. Fix $1 \leq p \leq \frac{n}{2}$. Choose $I := (i_1, i_2, \ldots, i_{2p})$ for $i_1 < i_2 < \cdots < i_{2p}$, and assume the pfaffian minor $\mathrm{pf}(\theta_I)$ is non-zero (so that θ_I is invertible). Choose a permutation $\Sigma \in \mathcal{S}_n$ such that $\Sigma(1) = i_1, \Sigma(2) = i_2, \ldots, \Sigma(2p) = i_{2p}$. If $U_1, U_2, \ldots, U_n$ are generators of A_θ, there exists an $n \times n$ skew-symmetric matrix, denoted by $\Sigma(\theta)$, such that $U_{\Sigma(1)}, U_{\Sigma(2)}, \ldots, U_{\Sigma(n)}$ are generators of $A_{\Sigma(\theta)}$ and $A_{\Sigma(\theta)} \cong A_\theta$. Note that the upper left $2p \times 2p$ block $\Sigma(\theta)$ is exactly θ_I, which is invertible. Now consider the projective module constructed as completion of $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^{n-2p})$ over $A_{\Sigma(\theta)}$ as in the previous subsection and denote it by $\mathcal{E}_I^\theta$. The trace of this module is the pfaffian of θ_I by the remark above, which is $\sum_{\xi \in \Pi} (-1)^{|\xi|} \prod_{s=1}^p \theta_{i_{\xi(2s-1)} i_{\xi(2s)}}$. Varying p, and assuming that all the pfaffian minors are non-zero, we get $2^{n-1} - 1$ projective modules. We call these $2^{n-1} - 1$ elements the *fundamental projective modules*.

So for a non-zero $\mathrm{pf}(\theta_I)$, $I = (i_1, i_2, \ldots, i_{2p})$, we have constructed a projective module $\mathcal{E}_I^\theta$ over A_θ, whose trace is $\mathrm{pf}(\theta_I)$. A quick thought shows that $\mathcal{E}_I^\theta$ is an equivalence bimodule between A_θ and $A_{g_{I,\Sigma}\theta}$ for some $g_{I,\Sigma} \in \mathrm{SO}(n, n \mid \mathbb{Z})$. Indeed, let R_I^Σ be the permutation matrix corresponding to the permutation Σ. Note that $\Sigma(\theta) = \rho(R_I^\Sigma)\theta$. Then clearly $g_{I,\Sigma} = \sigma_{2p}\rho(R_I^\Sigma)$. In Section 5, we will write down a basis of $\mathrm{K}_0(A_\theta)$ using these fundamental modules.

Next, we will construct specific (Rieffel-type) projections which represent the fundamental projective modules. The following theorem is a modification (according to our needs) of [7, Theorem 3.13]. From now on we shall often denote the canonical trace of A_θ by Tr_θ.

Theorem 3.11. *For any even number $n = 2l \geq 2$, let $\theta \in \mathcal{T}_n$ be totally irrational satisfying $\mathrm{pf}(F^j(\theta)_{1,1}) \in (\frac{1}{2}, 1)$ for $j = 0, 1, \ldots, l-1$. Then there exists a (Rieffel-type) projection p_m inside $A_{F^m(\theta)}$ such that*

$$\mathrm{Tr}_{F^m(\theta)}(p_m) = \mathrm{pf}\left(F^m(\theta)\right)$$

for $m = 0, 1, 2, \ldots, l-1$.

Proof. Since θ is totally irrational, it follows from the discussion after Lemma 3.7 that $\mathrm{pf}(F^m(\theta)_{1,1})$ is irrational and $A_{F^m(\theta)}$ is a simple C^*-algebra for $m = 0, 1, \ldots, l-1$. Now we do the proof using recursion on m. For $m = l-1$, $F^{l-1}(\theta)$ is a 2×2 matrix and

$\mathrm{pf}(F^{l-1}(\theta)_{1,1}) = \mathrm{pf}(F^{l-1}(\theta)) \in (\frac{1}{2}, 1)$ is irrational, the construction of such projection is well known (the projection is known as Rieffel projection), and the trace of this projection is $\mathrm{pf}(F^{l-1}(\theta))$ (see Appendix A.1). Now suppose that for some $m \in \{1, 2, \ldots, l-1\}$, there is such a projection in $A_{F^m(\theta)}$ such that

$$\mathrm{Tr}_{F^m(\theta)}(p_m) = \mathrm{pf}\big(F^m(\theta)\big). \tag{3.12}$$

Then we want to prove that there is a projection p_{m-1} in $A_{F^{m-1}(\theta)}$ with $\mathrm{Tr}_{F^{m-1}(\theta)}(p_{m-1}) = \mathrm{pf}(F^{m-1}(\theta))$. We follow the method of pp. 198–199 in [1] to construct such a projection. Write

$$F^{m-1}(\theta) = \begin{pmatrix} F^{m-1}(\theta)_{1,1} & F^{m-1}(\theta)_{1,2} \\ F^{m-1}(\theta)_{2,1} & F^{m-1}(\theta)_{2,2} \end{pmatrix} \in \mathcal{T}_{n-2(m-1)},$$

where $F^{m-1}(\theta)_{1,1}$ is a 2×2 block. From the previous discussion of this section, $A_{F^{m-1}(\theta)}$ is strong Morita equivalent to $A_{\sigma(F^{m-1}(\theta))}$, where

$$\begin{aligned} &\sigma\big(F^{m-1}(\theta)\big) \\ &= \begin{pmatrix} F^{m-1}(\theta)_{1,1}^{-1} & -F^{m-1}(\theta)_{1,1}^{-1} F^{m-1}(\theta)_{1,2} \\ F^{m-1}(\theta)_{2,1} F^{m-1}(\theta)_{1,1}^{-1} & F^{m-1}(\theta)_{2,2} - F^{m-1}(\theta)_{2,1} F^{m-1}(\theta)_{1,1}^{-1} F^{m-1}(\theta)_{1,2} \end{pmatrix} \\ &= \begin{pmatrix} F^{m-1}(\theta)_{1,1}^{-1} & -F^{m-1}(\theta)_{1,1}^{-1} F^{m-1}(\theta)_{1,2} \\ F^{m-1}(\theta)_{2,1} F^{m-1}(\theta)_{1,1}^{-1} & F^m(\theta) \end{pmatrix}. \end{aligned}$$

Denote the Rieffel projection in $A_{F^{m-1}(\theta)_{11}}$, by e and

$$\mathrm{Tr}_{F^{m-1}(\theta)}(e) = \mathrm{pf}(F^{m-1}(\theta)_{1,1})$$

(see Appendix A.1 for the construction of such e; here we use the assumption that $\mathrm{pf}(F^{m-1}(\theta)_{1,1})$ is in $(\frac{1}{2}, 1)$). It follows that $A_{\sigma(F^{m-1}(\theta))} \cong eA_{F^{m-1}(\theta)}e$, and we denote this isomorphism by ψ (see Appendix A.1 for the description of ψ). By the induction hypothesis and equation (3.12), there exists a projection $e' \in A_{F^m(\theta)} \subset A_{\sigma(F^{m-1}(\theta))}$ with $\mathrm{Tr}_{F^m(\theta)}(e') = \mathrm{pf}(F^m(\theta))$.

Now for the tracial state $\mathrm{Tr}_{F^{m-1}(\theta)}$ on $A_{F^{m-1}(\theta)}$, since $\psi(1_{A_{\sigma(F^{m-1}(\theta))}}) = e$ and

$$\mathrm{Tr}_{F^{m-1}(\theta)}(e) = \mathrm{pf}(F^{m-1}(\theta)_{1,1}),$$

we have

$$\frac{1}{\mathrm{pf}\big(F^{m-1}(\theta)_{1,1}\big)} \mathrm{Tr}_{F^{m-1}(\theta)} \circ \psi$$

is a tracial state on $A_{\sigma(F^{m-1}(\theta))}$. But this tracial state is $\mathrm{Tr}_{\sigma(F^{m-1}(\theta))}$ as $A_{\sigma(F^{m-1}(\theta))}$ has a unique tracial state (being Morita equivalent to $A_{F^{m-1}(\theta)}$, $F^{m-1}(\theta)$ is nondegenerate). Note $e' \in A_{F^m(\theta)} \subset A_{\sigma(F^{m-1}(\theta))}$, so $\mathrm{Tr}_{\sigma(F^{m-1}(\theta))}(e') = \mathrm{Tr}_{F^m(\theta)}(e')$. Then we get

$$\begin{aligned} &\frac{1}{\mathrm{pf}(F^{m-1}(\theta)_{1,1})} \mathrm{Tr}_{F^{m-1}(\theta)} \circ \psi(e') \\ &\quad = \mathrm{Tr}_{\sigma(F^{m-1}(\theta))}(e') = \mathrm{Tr}_{F^m(\theta)}(e') = \mathrm{pf}\big(F^m(\theta)\big). \end{aligned} \tag{3.13}$$

Let $p_{m-1} := \psi(e')$. From (3.13), we get

$$\begin{aligned}
\mathrm{Tr}_{F^{m-1}(\theta)}(p_{m-1}) &= \mathrm{Tr}_{F^{m-1}(\theta)}\left(\psi(e')\right) \\
&= \mathrm{pf}\left(F^{m-1}(\theta)_{1,1}\right)\left(\frac{1}{\mathrm{pf}\left(F^{m-1}(\theta)_{1,1}\right)}\mathrm{Tr}_{F^{m-1}(\theta)}\circ\psi(e')\right) \\
&= \mathrm{pf}\left(F^{m-1}(\theta)_{1,1}\right)\mathrm{pf}\left(F^{m}(\theta)\right) \\
&= \mathrm{pf}\left(F^{m-1}(\theta)\right),
\end{aligned}$$

using equations (3.2) and (3.3). ■

The above theorem tells us how to construct a higher-dimensional (Rieffel-type) projection with corresponding trace values from a low-dimensional projection under certain conditions. By Lemma 3.7, we know that

$$\mathrm{pf}\left(F^{j}(\theta)_{1,1}\right) = \frac{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-1,n-p)}}{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-2)}}, \quad p = n-2j-2,\ j = 1,\ldots,l-1.$$

Therefore, the conditions of the above theorem can also be described as

$$\frac{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-1,n-p)}}{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-2)}} \in \left(\frac{1}{2},1\right) \quad \text{for } p = n-2j-2,\ j = 1,\ldots,l-1$$

and $\theta_{ij} \in (\frac{1}{2},1)$ for $i < j$. (The last condition is stronger though.) We record this fact as a corollary below.

Corollary 3.12. *Let $\theta \in \mathcal{T}_n$ be totally irrational for $n = 2l \geq 2$. If θ satisfies $\theta_{ij} \in (\frac{1}{2},1)$ for $i < j$, and*

$$\frac{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-1,n-p)}}{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-2)}} \in \left(\frac{1}{2},1\right) \quad \textit{for } p = n-2j-2,\ j = 1,\ldots,l-1,$$

then there exists a (Rieffel-type) projection p_m inside $A_{F^m(\theta)}$ such that

$$\mathrm{Tr}_{F^m(\theta)}(p_m) = \mathrm{pf}\left(F^m(\theta)\right)$$

for $m = 0,1,2,\ldots,l-1$.

In particular, when $m = 0$, we have the following.

Corollary 3.13. *Let $\theta \in \mathcal{T}_n$ be totally irrational for $n = 2l \geq 2$. If θ satisfies $\theta_{ij} \in (\frac{1}{2},1)$ for $i < j$, and*

$$\frac{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-1,n-p)}}{\mathrm{pf}^{\theta}_{(1,2,\ldots,n-p-2)}} \in \left(\frac{1}{2},1\right) \quad \textit{for } p = n-2j-2,\ j = 1,\ldots,l-1,$$

then there exists a (Rieffel-type) projection $p = p_0$ inside A_θ such that $\mathrm{Tr}(p) = \mathrm{pf}(\theta)$.

Let us now show that under suitable conditions on θ, the numbers coming in the right-hand side of equation (3.1) may be realized as traces of the Rieffel-type projections in A_θ.

Theorem 3.14. *Let $\theta \in \mathcal{T}_n$ be totally irrational. If for some $I \in \mathrm{Minor}(n)$ with $|I| = 2m \geq 2$, $\mathrm{pf}(F^j(\theta_I)_{1,1}) \in (\frac{1}{2}, 1)$, for all $j = 0, 1, \ldots, m-1$, then there exist (Rieffel-type) projections P_I, such that*

$$\mathrm{Tr}(P_I) = \mathrm{pf}(\theta_I) \tag{3.14}$$

where Tr *is the canonical tracial state on A_θ.*

Proof. We use θ_I instead of θ in Corollary 3.13, noting that θ_I is totally irrational as well, to get $\mathrm{Tr}(P_I) = \mathrm{pf}(\theta_I)$. ■

We will now see that the above projections form a generating set of $\mathrm{K}_0(A_\theta)$, when θ is totally irrational, under the assumptions of the above theorem.

Definition 3.15. We say that θ is *strongly totally irrational* if $\theta = (\theta_{jk}) \in \mathcal{T}_n$ is a totally irrational matrix such that

$$\mathrm{pf}\left(F^j(\theta_I)_{1,1}\right) \in \left(\frac{1}{2}, 1\right) \tag{3.15}$$

for all $I \in \mathrm{Minor}(n)$ with $|I| = 2m \geq 2$, and for all $j = 0, 1, \ldots, m-1$.

We refer to Appendix B for more on strongly totally irrational matrices and for the construction of examples of such matrices.

Theorem 3.16. *For any integer $n \geq 2$, let $\theta \in \mathcal{T}_n$ be strongly totally irrational. Then there exist (Rieffel-type) projections P_I, for every $I \in \mathrm{Minor}(n)$ inside A_θ, such that $\mathrm{Tr}(P_I) = \mathrm{pf}(\theta_I)$, where* Tr *is the canonical tracial state on A_θ. Moreover, a generating set of $\mathrm{K}_0(A_\theta)$ is given by $\{[P_I] \mid I \in \mathrm{Minor}(n)\}$.*

Proof. For $I \neq \emptyset$, by Theorem 3.14 and from the definition of strong total irrationality, we know that those projections $\{P_I\}$ exist. For $I = \emptyset$, $P_I = 1$ by definition. Now since θ is totally irrational, Tr is injective. So $\{[P_I] \mid I \in \mathrm{Minor}(n)\}$ are the generators of $\mathrm{K}_0(A_\theta)$ by equation (3.1). ■

3.1. Pimsner–Voiculescu exact sequence and the Rieffel-type projections

Recall that for crossed products like $A \rtimes_\gamma \mathbb{Z}$, the Pimsner–Voiculescu sequence looks like

$$\begin{array}{ccccc}
\mathrm{K}_0(A) & \xrightarrow{\mathrm{id}-\gamma_*^{-1}} & \mathrm{K}_0(A) & \xrightarrow{i_*} & \mathrm{K}_0(A \rtimes \mathbb{Z}) \\
\uparrow & & & & \downarrow e_2 \\
\mathrm{K}_1(A \rtimes \mathbb{Z}) & \xleftarrow[i*]{} & \mathrm{K}_1(A) & \xleftarrow[\mathrm{id}-\gamma_*^{-1}]{} & \mathrm{K}_1(A)
\end{array}$$

where i is the inclusion.

Let $\theta \in \mathcal{T}_n$ and θ' be the upper left $(n-1)\times(n-1)$ block of θ. Let γ be the automorphism on $A_{\theta'}$ given by $\gamma(U_i) = e(-\theta_{in})U_i$ for $i = 1, \ldots, (n-1)$, as in Section 2. For the crossed product algebra $A_{\theta'} \rtimes_\gamma \mathbb{Z} \cong A_\theta$, the Pimsner–Voiculescu sequence becomes

$$\begin{array}{ccccc} \mathrm{K}_0(A_{\theta'}) & \xrightarrow{\mathrm{id}-\gamma_*^{-1}} & \mathrm{K}_0(A_{\theta'}) & \xrightarrow{i_*} & \mathrm{K}_0(A_\theta) \\ \uparrow & & & & \downarrow e_2 \\ \mathrm{K}_1(A_\theta) & \longleftarrow & \mathrm{K}_1(A_{\theta'}) & \xleftarrow[\mathrm{id}-\gamma_*^{-1}]{} & \mathrm{K}_1(A_{\theta'}) \end{array}$$

Since γ is homotopic to the identity map [23, Lemma 1.5], $\mathrm{id} - \gamma_*^{-1}$ is the zero map. Hence we get the following exact sequences:

$$0 \longrightarrow \mathrm{K}_0(A_{\theta'}) \xrightarrow{i_*} \mathrm{K}_0(A_\theta) \xrightarrow{e_2} \mathrm{K}_1(A_{\theta'}) \longrightarrow 0, \tag{3.16}$$

$$0 \longrightarrow \mathrm{K}_1(A_{\theta'}) \longrightarrow \mathrm{K}_1(A_\theta) \longrightarrow \mathrm{K}_0(A_{\theta'}) \longrightarrow 0. \tag{3.17}$$

From these two exact sequences (along with the fact $\mathrm{K}_0(C(\mathbb{T})) = \mathrm{K}_1(C(\mathbb{T})) = \mathbb{Z}$), by induction on n we get

$$\mathrm{K}_0(A_\theta) \cong \mathbb{Z}^{2^{n-1}} \cong \mathrm{K}_1(A_\theta).$$

Now we want to fit the Rieffel-type projections in equation (3.16). In order to achieve that, we again assume θ to be strongly totally irrational. Now we know from Theorem 3.16 that for different $I := (i_1, i_2, \ldots, i_{2p})$ such that $I \in \mathrm{Minor}(n)$, the K-theory classes of the Rieffel-type projections $P_I^\theta := P_I$ generate $\mathrm{K}_0(A_\theta)$. Now we claim that for $I \in \mathrm{Minor}(n-1)$,

$$i_*\big([P_I^{\theta'}]\big) = [P_I^\theta]. \tag{3.18}$$

This follows from the fact that they have the same trace in A_θ, which is $\mathrm{pf}(\theta_I)$. Now if $\widetilde{\mathrm{Minor}}(n)$ denotes the set of all $I \in \mathrm{Minor}(n)$ such that $i_{2p} = n$, i.e., $\widetilde{\mathrm{Minor}}(n) = \mathrm{Minor}(n) \setminus \mathrm{Minor}(n-1)$, the collection $\{[P_I^\theta]\}_{I \in \widetilde{\mathrm{Minor}}(n)}$ maps via e_2 to a generating set of $\mathrm{K}_1(A_{\theta'})$ which follows from the fact that equation (3.16) is exact. We record these observations in the following proposition.

Proposition 3.17. *For a strongly totally irrational θ, $i_*([P_I^{\theta'}]) = [P_I^\theta]$, for $I \in \mathrm{Minor}(n-1)$, and $\{e_2([P_I^\theta])\}_{I \in \widetilde{\mathrm{Minor}}(n)}$ form a generating set of $\mathrm{K}_1(A_{\theta'})$.*

4. K-theory of $A_\theta \rtimes \mathbb{Z}_2$

Before computing the K-theory groups of $A_\theta \rtimes \mathbb{Z}_2$ we will see how the Rieffel-type projections give rise to K_0-classes of $A_\theta \rtimes \mathbb{Z}_2$. We start with the following well known facts.

Proposition 4.1. *Suppose F is a finite group acting on a C^*-algebra A by the action α. Also suppose that $\mathcal{E}$ is a finitely generated projective (right) A-module with a right action $T : F \to \mathrm{Aut}(\mathcal{E})$, written $(\xi, g) \rightarrowtail \xi T_g$, such that $\xi(T_g)a = (\xi\alpha_g(a))T_g$ for all $\xi \in \mathcal{E}$,*

$a \in A$, and $g \in F$. Then $\mathcal{E}$ becomes a finitely generated projective $A \rtimes F$ module with action defined by

$$\xi \cdot \Big(\sum_{g \in F} a_g \delta_g\Big) = \sum_{g \in F} (\xi a_g) T_g.$$

Also, if we restrict the new module to A, we get the original A-module $\mathcal{E}$, with the action of F forgotten.

Proof. This is exactly the construction of the Green–Julg map, which is a map from $\mathrm{K}_0^F(A)$, the F-equivariant K-theory of A, to $\mathrm{K}_0(A \rtimes F)$. See [10, Proposition 4.5]. ∎

For a general crossed product $A \rtimes \mathbb{Z}_2$, for an action β of $\mathbb{Z}_2$ on A, the above Green–Julg map is easy to describe for $\mathbb{Z}_2$-invariant projections in A. If P is a $\mathbb{Z}_2$-invariant projection in A, the corresponding projection in $A \rtimes \mathbb{Z}_2$ is $\frac{P}{2}(1 + W)$, where W denotes the canonical non-trivial unitary of $\mathbb{Z}_2$ in $A \rtimes \mathbb{Z}_2$. Also define the natural map (regular representation) p which goes from $A \rtimes \mathbb{Z}_2$ to $\mathrm{M}_2(A)$ such that

$$p(a + bW) = \begin{pmatrix} a & b \\ \beta_g(b) & \beta_g(a) \end{pmatrix}, \quad a, b \in A, \tag{4.1}$$

where g is the non-trivial element of $\mathbb{Z}_2$. This induces a map $p_* : \mathrm{K}_0(A \rtimes \mathbb{Z}_2) \to \mathrm{K}_0(A)$, which is known to be the inverse of the Green–Julg map (see [14, p. 191]).

With the above facts in hand, we start with the projective modules over A_θ which were described in the previous section. Recall that the projective module $\mathcal{E}$ is a completion of the space $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q)$, for some $p, q \in \mathbb{Z}_{\geq 0}$ such that $n = 2p + q$. For the flip action of $\mathbb{Z}_2$ on A_θ, we define the following action, again called the flip action, of $\mathbb{Z}_2$ on the dense subspace $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q)$ of $\mathcal{E}$ by

$$T_g(f)(x,t) := f(-x,-t), \tag{4.2}$$

where g is the non-trivial element of $\mathbb{Z}_2$.

Using equations (3.8) and (3.9) it is quickly checked that $\mathbb{Z}_2$ defines an action on $\mathcal{E}$ which is compatible with the flip action of $\mathbb{Z}_2$ on A_θ in the sense of Proposition 4.1 (cf. [8, Section 7], [5, Section 3]). In particular, we have

$$\langle f T_g, f' T_g \rangle_{\mathcal{A}^\infty} = \beta_g\big(\langle f, f' \rangle_{\mathcal{A}^\infty}\big), \tag{4.3}$$

and

$$f(T_g)a = \big(f \beta_g(a)\big) T_g, \tag{4.4}$$

for f and $f' \in \mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q)$, $a \in A_\theta$ (see [5, Equation 3.9, Equation 3.11]). Similarly, the same set of equations holds true for the left $A_{\sigma(\theta)}$-module $\mathcal{E}$, where $\sigma(\theta)$ is as in equation (3.7). Hence $\mathcal{E}$ becomes a projective module over the crossed product $A_\theta \rtimes \mathbb{Z}_2$. We call this module $\widetilde{\mathcal{E}}$. Let $\mathrm{Tr}_\theta^{\mathbb{Z}_2}$ denote the canonical trace on $A_\theta \rtimes \mathbb{Z}_2$ defined by $\mathrm{Tr}_\theta^{\mathbb{Z}_2}(a + bW) = \mathrm{Tr}_\theta(a)$ for $a, b \in A_\theta$. From [5, Lemma 4.1] we can compute the trace of the K-theory class of $\widetilde{\mathcal{E}}$ as

$$\mathrm{Tr}_\theta^{\mathbb{Z}_2}\left([\widetilde{\mathcal{E}}]\right) = \frac{\mathrm{Tr}_\theta\left([\mathcal{E}]\right)}{2}. \tag{4.5}$$

Our next step is to understand how the Rieffel-type projections P_I^θ give various projections in $A_\theta \rtimes \mathbb{Z}_2$. First let us introduce some notations. For $I = (i_1, i_2, \ldots, i_{2p}) \in \mathrm{Minor}(n) \setminus \{\emptyset\}$, define $I^c := (i_{2p}+1, i_{2p}+2, \ldots, n)$, and for $I = \emptyset$, define $I^c := (1, 2, \ldots, n)$. Also regarding I^c as a finite sequence, by $J \subseteq I^c$ we mean a finite subsequence of I^c, with the understanding that J can be the empty sequence too. Finally for $J = (j_1, j_2, \ldots, j_q) \subseteq I^c$, define $U_J := U_{j_1} U_{j_2} \cdots U_{j_q} \in A_\theta$, and for $J = \emptyset$, $U_J := 1$. The length $|I^c|$ of I^c (or any sub-sequence of I^c) is defined as the number of elements in I^c.

From Appendix A.2, we know that all the P_I^θ's are $\mathbb{Z}_2$-invariant. (This follows from the fact that in Theorem 3.11, ψ is $\mathbb{Z}_2$-equivariant and e is $\mathbb{Z}_2$-invariant.) Now let us see how these give rise to different projections in $A_\theta \rtimes \mathbb{Z}_2$. For each $I \in \mathrm{Minor}(n)$, fix a $J \subseteq I^c$ as before. Let $\bar{J}$ denote $(j_q, j_{q-1}, \ldots, j_1)$ for $J = (j_1, j_2, \ldots, j_q)$. Let r_J be the number such that $U_J = e(-2r_J)U_{\bar{J}}$. Of course, r_J is a number involving $\theta_{j_l j_m}$, $1 \leq l < m \leq q$. Now one quickly checks that $e(r_J)U_J W =: W_J$ is a self-adjoint unitary in $A_\theta \rtimes \mathbb{Z}_2$. If $I = (i_1, i_2, \ldots, i_{2p})$, then $W_J U_{i_k} W_J = e(r_{J,i_k})U_{i_k}^{-1}$, for some number r_{J,i_k} involving $\theta_{i_k j_l}$, $1 \leq l \leq q$. As in Lemma 2.3, set

$$\widetilde{U}_{i_k} = e\Big(-\frac{r_{J,i_k}}{2}\Big)U_{i_k},$$

and we have $W_J \widetilde{U}_{i_k} W_J = \widetilde{U}_{i_k}^{-1}$ such that A_{θ_I} is generated by $\{\widetilde{U}_{i_k}\}_k$, and $A_{\theta_I} \rtimes \mathbb{Z}_2$ sits canonically inside $A_\theta \rtimes \mathbb{Z}_2$. Here in the crossed product $A_{\theta_I} \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2$ acts by the flip action and the generator of $\mathbb{Z}_2$ is identified with W_J inside $A_\theta \rtimes \mathbb{Z}_2$. Now we can construct $P_{I,J}^\theta := \frac{P_I^\theta}{2}(1 + W_J) \in A_{\theta_I} \rtimes \mathbb{Z}_2 \subseteq A_\theta \rtimes \mathbb{Z}_2$, since P_I^θ is flip invariant projection inside A_{θ_I}. For $I = \emptyset$, $P_{I,J}^\theta := \frac{1}{2}(1 + W_J)$. Hence for each $I \in \mathrm{Minor}(n)$, and each $J \subseteq I^c$, we have constructed projections inside $A_\theta \rtimes \mathbb{Z}_2$. Varying I, and varying J, we get a family of projections $\{P_{I,J}^\theta\}$ in $A_\theta \rtimes \mathbb{Z}_2$.

Now we claim that we have exactly $3 \cdot 2^{n-1} - 1$ projections $\{P_{I,J}^\theta\}$ if we restrict to $|J| \leq 2$; i.e., the set

$$\mathcal{P}_n := \bigcup_{I \in \mathrm{Minor}(n)} \left\{P_{I,J}^\theta \mid J \subseteq I^c, |J| \leq 2\right\} \tag{4.6}$$

has $3 \cdot 2^{n-1} - 1$ many elements. This can be shown using a simple induction argument. The statement holds for $n = 2, 3$ simply by counting. Now assume $|\mathcal{P}_n| = 3 \cdot 2^{n-1} - 1$, and we need to show that $|\mathcal{P}_{n+1}| = 3 \cdot 2^n - 1$. If $I \in \mathrm{Minor}(n)$, of course, $I \in \mathrm{Minor}(n+1)$, and if $J \subseteq I^c$, if we view I as an element of $\mathrm{Minor}(n)$, then we also have $J \subseteq I^c$ if we view I as an element of $\mathrm{Minor}(n+1)$. Hence we may regard $\mathcal{P}_n$ as a subset of $\mathcal{P}_{n+1}$. Thus we only need to show that there are $3 \cdot 2^n - 3 \cdot 2^{n-1} = 3 \cdot 2^{n-1} = 2^n + 2^{n-1}$ extra elements in $\mathcal{P}_{n+1} \backslash \mathcal{P}_n$. These extra elements are described below.

- For $I \in \text{Minor}(n)$, if we view I inside $\text{Minor}(n+1)$, we have elements $P^{\theta}_{I,J}$, where $J = (n+1)$; and for $I \in \widetilde{\text{Minor}}(n+1)$, we have elements $P^{\theta}_{I,J}$, where $J = \emptyset$. This way we get 2^n number of elements.
- For any $I \in \widetilde{\text{Minor}}(j)$ with $2 \leq j \leq n-1$, we have $(n-j)$ number of elements $P^{\theta}_{I,J}$, where $J = (j+1, n+1), (j+2, n+1), \ldots, (n, n+1)$. This way we get $2^{n-1} - n$ number of elements.
- Finally for $I = \emptyset$, we have n number of elements $P^{\theta}_{I,J}$, where $J = (1, n+1), (2, n+1), \ldots, (n, n+1)$.

Combining the above extra elements, we get our claim. Shortly we will see that the K-theory classes of the projections in $\mathcal{P}_n$ (along with the element [1]) generate $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$.

Let us recall the exact sequence due to Lance and Natsume. For $A_\theta \rtimes \mathbb{Z}_2 \cong A_{\theta'} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2$ (see Section 2), we have the following exact sequence [22]:

$$\begin{array}{ccccc}
\mathrm{K}_0(A_{\theta'}) & \xrightarrow{i_{1*}-i_{2*}} & \mathrm{K}_0(A_{\theta'} \rtimes_\alpha \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta'} \rtimes \mathbb{Z}_2) & \xrightarrow{j_{1*}+j_{2*}} & \mathrm{K}_0(A_{\theta'} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2) \\
\uparrow & & & & \downarrow e_1 \\
\mathrm{K}_1(A_{\theta'} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2) & \xleftarrow[j_{1*}+j_{2*}]{} & \mathrm{K}_1(A_{\theta'} \rtimes \mathbb{Z}_2) \oplus \mathrm{K}_1(A_{\theta'} \rtimes \mathbb{Z}_2) & \xleftarrow[i_{1*}-i_{2*}]{} & \mathrm{K}_1(A_{\theta'})
\end{array}$$

where i_1, i_2, j_1, j_2 are natural inclusions. Also $A_{\theta'} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to $(A_{\theta'} \rtimes_\gamma \mathbb{Z}) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $A_{\theta'}$ by the flip action and on the group $\mathbb{Z}$ by $x \to -x$. Before computing $\mathrm{K}_0(A_{\theta'} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2)$, we will first explicitly describe the maps in the above exact sequence.

The map e_1. As an immediate corollary of [8, Theorem 7.1], we have the following.

Proposition 4.2. *The diagram*

$$\begin{array}{ccc}
\mathrm{K}_0(A_{\theta'} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2) & \xrightarrow{e_1} & \mathrm{K}_1(A_{\theta'}) \\
& \searrow^{p_*} & \uparrow e_2 \\
& & \mathrm{K}_0(A_{\theta'} \rtimes_\gamma \mathbb{Z})
\end{array}$$

is commutative, where p_* is the map induced by the natural map (see equation (4.1)) $p : A_{\theta'} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2 \cong (A_{\theta'} \rtimes_\gamma \mathbb{Z}) \rtimes \mathbb{Z}_2 \to \mathrm{M}_2(A_{\theta'} \rtimes_\gamma \mathbb{Z})$.

Proof. Immediate from [8, Theorem 7.1]. ∎

Now from Proposition 3.17, $\{e_2([P^{\theta}_I])\}_{I \in \widetilde{\text{Minor}}(n)}$ form a generating set of $\mathrm{K}_1(A_{\theta'})$, for a strongly totally irrational θ. But for $I \in \widetilde{\text{Minor}}(n)$,

$$p_*([P^{\theta}_{I,\emptyset}]) = p_*\left(\left[\frac{P^{\theta}_I}{2}(1+W)\right]\right) = [P^{\theta}_I],$$

since p_* acts as the inverse of the Green–Julg map in Proposition 4.1. So we have the following.

Corollary 4.3. *For a strongly totally irrational* θ, $\{e_1([P^{\theta}_{I,\emptyset}])\}_{I\in\widetilde{\mathrm{Minor}(n)}}$ *form a generating set of* $\mathrm{K}_1(A_{\theta'})$.

Proof. Use Proposition 4.2. ∎

The map $j_{1*}+j_{2*}$. Next we want to understand the map

$$j_{1*}+j_{2*}:\mathrm{K}_0(A_{\theta'}\rtimes_\alpha\mathbb{Z}_2)\oplus\mathrm{K}_0(A_{\theta'}\rtimes\mathbb{Z}_2)\longrightarrow\mathrm{K}_0(A_{\theta'}\rtimes_\phi\mathbb{Z}_2*\mathbb{Z}_2),$$

for the elements in $\mathcal{P}_{n-1}$. Since both maps j_1, j_2 are inclusion maps, and $A_{\theta'}\rtimes_\alpha\mathbb{Z}_2$ is isomorphic to $A_{\theta'}\rtimes\mathbb{Z}_2$ by Lemma 2.3, it is enough to work with j_{2*}. Now j_{2*} is induced from the natural inclusion map

$$j_2:A_{\theta'}\rtimes\mathbb{Z}_2\longrightarrow A_\theta\rtimes\mathbb{Z}_2.$$

We now have the following proposition.

Proposition 4.4. *For a strongly totally irrational* θ,

$$j_{2*}([P^{\theta'}_{I,J}])=[P^{\theta}_{I,J}],$$

for $I\in\mathrm{Minor}(n-1)$, $J\subseteq I^c$.

Proof. This is clear from equation (3.18) as the map j_2 respects the inclusion map of $A_{\theta'}$ to A_θ, and respects the unitaries coming from $\mathbb{Z}_2$. ∎

The map $i_{1*}-i_{2*}$. We start with the following lemma.

Lemma 4.5. *The map*

$$i_{2*}:\mathrm{K}_0(A_\theta)\longrightarrow\mathrm{K}_0(A_\theta\rtimes\mathbb{Z}_2)$$

is injective when θ *is strongly totally irrational.*

Proof. If $\mathrm{Tr}^{\mathbb{Z}_2}_\theta$ denotes the canonical tracial state on $A_\theta\rtimes\mathbb{Z}_2$, we have $\mathrm{Tr}^{\mathbb{Z}_2}_\theta(i_{2*}([P^\theta_I]))=\mathrm{pf}(\theta_I)$. Recall that $\{[P^\theta_I]\mid I\in\mathrm{Minor}(n)\}$, and generate $\mathrm{K}_0(A_\theta)$. Now if $i_{2*}([X])=0$, writing $[X]=\sum r_I[P^\theta_I]$ for $r_I\in\mathbb{Z}$, we have $i_{2*}(\sum r_I[P^\theta_I])=0$. Since $\mathrm{pf}(\theta_I)$ are rationally independent, taking $\mathrm{Tr}^{\mathbb{Z}_2}_\theta$ of the expression $i_{2*}(\sum r_I[P^\theta_I])$ gives $r_I=0$. ∎

Corollary 4.6. $\mathrm{K}_1(A_\theta\rtimes\mathbb{Z}_2)=0$, *when* θ *is strongly totally irrational.*

Proof. From the left side of the diagram

$$\begin{array}{ccccc}
\mathrm{K}_0(A_{\theta'}) & \xrightarrow{i_{1*}-i_{2*}} & \mathrm{K}_0(A_{\theta'}\rtimes_\alpha\mathbb{Z}_2)\oplus\mathrm{K}_0(A_{\theta'}\rtimes\mathbb{Z}_2) & \xrightarrow{j_{1*}+j_{2*}} & \mathrm{K}_0(A_{\theta'}\rtimes_\phi\mathbb{Z}_2*\mathbb{Z}_2)\\
\uparrow & & & & \downarrow{\scriptstyle e_1}\\
\mathrm{K}_1(A_{\theta'}\rtimes_\phi\mathbb{Z}_2*\mathbb{Z}_2) & \xleftarrow[j_{1*}+j_{2*}]{} & \mathrm{K}_1(A_{\theta'}\rtimes\mathbb{Z}_2)\oplus\mathrm{K}_1(A_{\theta'}\rtimes\mathbb{Z}_2) & \xleftarrow[i_{1*}-i_{2*}]{} & \mathrm{K}_1(A_{\theta'})
\end{array}$$

we compute $\mathrm{K}_1(A_\theta\rtimes\mathbb{Z}_2)=\mathrm{K}_1(A_{\theta'}\rtimes_\phi\mathbb{Z}_2*\mathbb{Z}_2)$ by induction on n. Since $i_{1*}-i_{2*}$ is injective from Lemma 4.5, and we know $\mathrm{K}_1(A_\theta\rtimes\mathbb{Z}_2)=0$ for the low-dimensional cases, the result follows. ∎

Next we want to understand the map

$$i_{1*} - i_{2*} : \mathrm{K}_0(A_{\theta'}) \longrightarrow \mathrm{K}_0(A_{\theta'} \rtimes_\alpha \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta'} \rtimes \mathbb{Z}_2),$$

for a Rieffel-type projection P_I, $I \in \mathrm{Minor}(n-1)$. Since both maps i_{1*}, i_{2*} are inclusion maps, and $A_{\theta'} \rtimes_\alpha \mathbb{Z}_2$ is isomorphic to $A_{\theta'} \rtimes \mathbb{Z}_2$ by Lemma 2.3, it is enough to work with i_{2*}. Note that i_{2*} is induced from the natural inclusion map

$$i_2 : A_{\theta'} \longrightarrow A_{\theta'} \rtimes \mathbb{Z}_2.$$

We now have the following proposition.

Proposition 4.7. *Let θ be an $n \times n$ strongly totally irrational matrix and let $i_{2*} : \mathrm{K}_0(A_\theta) \to \mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$ be induced by the canonical inclusion map i_2. Then for $I \in \mathrm{Minor}(n) \setminus \{\emptyset\}$,*

$$i_{2*}([P_I^\theta]) = 2[P_{I,\emptyset}^\theta] - [P_{I'',\emptyset}^\theta] + [P_{I'',(i_{2p-1})}^\theta] - [P_{I'',(i_{2p})}^\theta] + [P_{I'',(i_{2p-1},i_{2p})}^\theta],$$

where $I = (i_1, i_2, \ldots, i_{2p})$ and I'' is obtained from I by deleting the last two numbers.

We would like to prove the above proposition by induction on the length $|I|$ of I. Before we go to the proof, we explain the two-dimensional case in the lemma below.

Lemma 4.8 (cf. the proof of [8, Corollary 7.2]). *Let θ_{12} be an irrational number in $(\frac{1}{2}, 1)$ and let $i_{2*} : \mathrm{K}_0(A_{\theta_{12}}) \to \mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z}_2)$ be induced by the canonical inclusion map i_2. Then*

$$i_{2*}([P_{(1,2)}^{\theta_{12}}]) = 2[P_{(1,2),\emptyset}^{\theta_{12}}] - [P_{\emptyset,\emptyset}^{\theta_{12}}] + [P_{\emptyset,(1)}^{\theta_{12}}] - [P_{\emptyset,(2)}^{\theta_{12}}] + [P_{\emptyset,(1,2)}^{\theta_{12}}].$$

Proof. In both cases, the two K-theory elements in RHS and in LHS have the same vector trace $(\theta; 0,0,0,0)$ [29, p. 597]. Then the result follows from [29, Corollary 5.6]. ∎

Proof of Proposition 4.7. As we already mentioned, we will prove this by induction on the length of I, $|I|$. For any $I \in \mathrm{Minor}(n) \setminus \{\emptyset\}$ we first have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{K}_0(A_{\theta_I}) & \xrightarrow{\ i_{2*}\ } & \mathrm{K}_0(A_{\theta_I} \rtimes \mathbb{Z}_2) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle j_{2*}} \\ \mathrm{K}_0(A_\theta) & \xrightarrow{\ i_{2*}\ } & \mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2) \end{array}$$

where all the maps are induced by inclusions. For $|I| = 2$, $I = (i_1, i_2)$, using Lemma 4.8 and using the above commutative diagram (along with Proposition 4.4), we indeed get

$$i_{2*}([P_{(i_1,i_2)}^\theta]) = 2[P_{(i_1,i_2),\emptyset}^\theta] - [P_{\emptyset,\emptyset}^\theta] + [P_{\emptyset,(i_1)}^\theta] - [P_{\emptyset,(i_2)}^\theta] + [P_{\emptyset,(i_1,i_2)}^\theta].$$

Now for the induction step, assume that the statement is true for any I with $|I| < 2p$. Then we will show that the statement is true for an I with $|I| = 2p$. Assume $I =$

$(i_1, i_2, \ldots, i_{2p})$. Due to the commutative diagram, it is enough to prove the statement viewing i_{2*} as a map from $\mathrm{K}_0(A_{\theta_I})$ to $\mathrm{K}_0(A_{\theta_I} \rtimes \mathbb{Z}_2)$. Hence we can assume $I = (1, 2, \ldots, 2p)$, and $\theta = \theta_I$, and we want to show

$$i_{2*}([P^{\theta}_{(1,2,\ldots,2p)}]) = 2[P^{\theta}_{I,\emptyset}] - [P^{\theta}_{I'',\emptyset}] + [P^{\theta}_{I'',(2p-1)}] - [P^{\theta}_{I'',(2p)}] + [P^{\theta}_{I'',(2p-1,2p)}].$$

From Appendix A.2, we have the following commutative diagrams:

$$\begin{array}{ccc} A_{\sigma(\theta)} & \xrightarrow{i_2} & A_{\sigma(\theta)} \rtimes \mathbb{Z}_2 \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ eA_\theta e & \xrightarrow{i_2} & eA_\theta e \rtimes \mathbb{Z}_2 \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ A_\theta & \xrightarrow{i_2} & A_\theta \rtimes \mathbb{Z}_2 \end{array} \qquad \begin{array}{ccc} \mathrm{K}_0(A_{\sigma(\theta)}) & \xrightarrow{i_{2*}} & \mathrm{K}_0(A_{\sigma(\theta)} \rtimes \mathbb{Z}_2) \\ \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \psi_*} \\ \mathrm{K}_0(eA_\theta e) & \xrightarrow{i_{2*}} & \mathrm{K}_0(eA_\theta e \rtimes \mathbb{Z}_2) \\ \downarrow{\scriptstyle i*} & & \downarrow{\scriptstyle i*} \\ \mathrm{K}_0(A_\theta) & \xrightarrow{i_{2*}} & \mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2) \end{array}$$

where

$$\sigma(\theta) := \begin{pmatrix} \theta_{1,1}^{-1} & -\theta_{1,1}^{-1}\theta_{1,2} \\ \theta_{2,1}\theta_{1,1}^{-1} & \theta_{2,2} - \theta_{2,1}\theta_{1,1}^{-1}\theta_{1,2} \end{pmatrix}$$

for

$$\theta = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{pmatrix}.$$

Here, as before $\theta_{1,1}$ is a 2×2 matrix and ψ as in the proof of Theorem 3.11. Let $V_1, V_2, \ldots, V_{2p}$ be the generators of $A_{\sigma(\theta)}$ as in Appendix A. As in the previous section, let $F(\theta) = \theta_{2,2} - \theta_{2,1}\theta_{1,1}^{-1}\theta_{1,2} \in \mathcal{T}_{n-2}$. Then the generators of $A_{F(\theta)} \subset A_{\sigma(\theta)}$ are $V_3, V_4, \ldots, V_{2p}$. Take $J = (1, 2, \ldots, 2p-2)$. Then by the induction step, we have

$$i_{2*}([P^{F(\theta)}_J]) = 2[P^{F(\theta)}_{J,\emptyset}] - [P^{F(\theta)}_{J'',\emptyset}] + [P^{F(\theta)}_{J'',(2p-3)}] - [P^{F(\theta)}_{J'',(2p-2)}] + [P^{F(\theta)}_{J'',(2p-3,2p-2)}].$$

From the proof of Theorem 3.11, we know that $\psi(P^{F(\theta)}_J) = P^{\theta}_I$. It is clear that

$$\psi(P^{F(\theta)}_{J,\emptyset}) = P^{\theta}_{I,\emptyset}.$$

Now we want to show that $\psi_*([P^{F(\theta)}_{J'',(2p-3)}]) = [P^{\theta}_{I'',(2p-1)}]$. By definition,

$$P^{F(\theta)}_{J'',(2p-3)} = \frac{P^{F(\theta)}_{J''}}{2}(1 + V_{2p-1}W).$$

Now $\psi(P^{F(\theta)}_{J''}) = P^{\theta}_{I''} \in eA_\theta e$. $\psi(P^{F(\theta)}_{J'',(2p-3)}) = \frac{P^{\theta}_{I''}}{2}(e + \psi(V_{2p-1})W)$. But with the computation of $\frac{1}{2}(e + \psi(V_{2p-1})W)$ in Appendix A.2, along with the arguments at the end of

Appendix A.2, it is shown that $\frac{P^{\theta}_{I''}}{2}(e+\psi(V_{2p-1})W)$ is homotopic to $P^{\theta}_{I''}\cdot\frac{e}{2}(1+U_{2p-1}W)$ in $A_\theta \rtimes \mathbb{Z}_2$. Since $P^{\theta}_{I''} \in eA_\theta e$, we indeed have

$$\psi_*([P^{F(\theta)}_{J'',(2p-3)}]) = [P^{\theta}_{I'',(2p-1)}].$$

A similar argument shows that $\psi_*([P^{F(\theta)}_{J'',(2p-2)}]) = [P^{\theta}_{I'',(2p)}]$. Now

$$\psi\big(P^{F(\theta)}_{J'',(2p-3,2p-2)}\big) = \psi\Bigg(\frac{P^{F(\theta)}_{J''}}{2}\bigg(1+e\bigg(-\frac{\mathrm{pf}^{\theta}_{(1,2,2p-1,2p)}}{2\theta_{12}}\bigg)V_{2p-1}V_{2p}W\bigg)\Bigg).$$

Like before, this element is homotopic to

$$\begin{aligned} &P^{\theta}_{I''}\cdot\frac{e}{2}\bigg(1+e\bigg(-\frac{\theta_{12}\theta_{2p-12p}}{2\theta_{12}}\bigg)U_{2p-1}U_{2p}W\bigg)\\ &\quad = P^{\theta}_{I''}\cdot\frac{e}{2}\bigg(1+e\bigg(-\frac{1}{2}\theta_{2p-12p}\bigg)U_{2p-1}U_{2p}W\bigg). \end{aligned}$$

Hence $\psi_*([P^{F(\theta)}_{J'',(2p-3,2p-2)}]) = [P^{\theta}_{I'',(2p-1,2p)}]$. Now the above commutative diagram (involving the K-theory) gives the result. ∎

The computation of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$. With the above results in hand, we now come to the computation of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$. Let us first see some lower-dimensional cases, i.e., the cases $n = 2, 3$.

Assume $\theta_{12} \in (\frac{1}{2}, 1)$ is an irrational number. We will compute the K-theory $\mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z}_2)$ and write down an explicit basis. Note that $A_{\theta_{12}} \rtimes \mathbb{Z}_2$ is generated by the unitaries U_1, U_2 and W such that $W^2 = 1$, $U_1U_2 = e(\theta_{12})U_2U_1$, $WUW = U^*$, $WVW = V^*$.

We now have

$$\begin{array}{ccccc} \mathrm{K}_0\big(C(\mathbb{T})\big) & \xrightarrow{i_{1*}-i_{2*}} & \mathrm{K}_0\big(C(\mathbb{T})\rtimes_\alpha \mathbb{Z}_2\big)\oplus \mathrm{K}_0\big(C(\mathbb{T})\rtimes\mathbb{Z}_2\big) & \xrightarrow{j_{1*}+j_{2*}} & \mathrm{K}_0(A_{\theta_{12}}\rtimes\mathbb{Z}_2)\\ \uparrow & & & & \downarrow e_1\\ 0 & \longleftarrow & 0 & \longleftarrow & \mathrm{K}_1\big(C(\mathbb{T})\big) \end{array}$$

In this case $i_{1*}-i_{2*}([1])=([1],-[1])$, and since we have already described the map e_1 (Corollary 4.3), we get a basis of $\mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z}_2)$ which is given by the K-theory classes of the following elements:

- 1;
- $P^{\theta_{12}}_{\emptyset,\emptyset} = \frac{1}{2}(1+W)$;
- $P^{\theta_{12}}_{\emptyset,(1)} = \frac{1}{2}(1+U_1W)$;
- $P^{\theta_{12}}_{\emptyset,(2)} = \frac{1}{2}(1+U_2W)$;
- $P^{\theta_{12}}_{\emptyset,(12)} = \frac{1}{2}(1+e(-\frac{1}{2}\theta_{12})U_1U_2W)$;
- $P^{\theta_{12}}_{I,\emptyset} = \frac{P^{\theta_{12}}_I}{2}(1+W)$, for $I = (1,2)$.

Hence a generating set of $\mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z}_2)$ is given by $\{[1],[P] \mid P \in \mathcal{P}_2\}$, using the notations introduced just after Corollary 4.3.

Let us now look at the case $n = 3$. Let

$$\theta = \begin{pmatrix} 0 & \theta_{12} & \theta_{13} \\ -\theta_{12} & 0 & \theta_{23} \\ -\theta_{13} & -\theta_{23} & 0 \end{pmatrix}$$

be a strongly irrational 3×3 matrix as in Definition 3.15. In this case it means that $\theta_{12}, \theta_{13}, \theta_{23} \in (\frac{1}{2}, 1)$, and they are rationally independent. We have the following exact sequence in this case:

$$\begin{array}{ccccc} \mathrm{K}_0(A_{\theta_{12}}) & \xrightarrow{i_{1*}-i_{2*}} & \mathrm{K}_0(A_{\theta_{12}} \rtimes_\alpha \mathbb{Z}_2) \oplus \mathrm{K}_0\,(A_{\theta_{12}} \rtimes \mathbb{Z}_2) & \xrightarrow{j_{1*}+j_{2*}} & \mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2) \\ \uparrow & & & & \downarrow e_1 \\ 0 & \longleftarrow & 0 & \longleftarrow & \mathrm{K}_1(A_{\theta_{12}}) \end{array}$$

Using the above two-dimensional computations, we first write down a basis for $A_{\theta_{12}} \rtimes_\alpha \mathbb{Z}_2$ and $A_{\theta_{12}} \rtimes \mathbb{Z}_2$. A basis of $\mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z}_2)$ is given by the K-theory classes of the following elements.

- 1;
- $P^{\theta_{12}}_{\emptyset,\emptyset} = \frac{1}{2}(1 + W)$;
- $P^{\theta_{12}}_{\emptyset,(1)} = \frac{1}{2}(1 + U_1 W)$;
- $P^{\theta_{12}}_{\emptyset,(2)} = \frac{1}{2}(1 + U_2 W)$;
- $P^{\theta_{12}}_{\emptyset,(12)} = \frac{1}{2}(1 + e(-\frac{1}{2}\theta_{12})U_1 U_2 W)$;
- $P^{\theta_{12}}_{I,\emptyset} = \frac{P^{\theta_{12}}_I}{2}(1 + W)$, for $I = (1, 2)$.

Now $A_{\theta_{12}} \rtimes_\alpha \mathbb{Z}_2$ is generated by the unitaries U_1, U_2, and $W' = U_3 W$ and we have the relations $W'^2 = 1$, $U_1 U_2 = e(\theta_{12}) U_2 U_1$, $W' U_1 W' = e(\theta_{13}) U_1^{-1}$, $W' U_2 W' = e(\theta_{23}) U_2^{-1}$. Using Lemma 2.3 and the above two-dimensional computation to get a basis $\{[1], [\widetilde{P}] \mid P \in \mathcal{P}_n\}$ of $\mathrm{K}_0(A_{\theta_{12}} \rtimes_\alpha \mathbb{Z}_2)$ by writing $\widetilde{P}$ we indicate that the class of P is taken inside $A_{\theta_{12}} \rtimes_\alpha \mathbb{Z}_2$. This is explicitly given by the K-theory classes of the following elements:

- 1;
- $\widetilde{P}^{\theta_{12}}_{\emptyset,\emptyset} = \frac{1}{2}(1 + U_3 W)$;
- $\widetilde{P}^{\theta_{12}}_{\emptyset,(1)} = \frac{1}{2}(1 + e(-\frac{1}{2}\theta_{13})U_1 U_3 W)$;
- $\widetilde{P}^{\theta_{12}}_{\emptyset,(2)} = \frac{1}{2}(1 + e(-\frac{1}{2}\theta_{23})U_2 U_3 W)$;
- $\widetilde{P}^{\theta_{12}}_{\emptyset,(12)} = \frac{1}{2}(1 + e(-\frac{1}{2}(\theta_{12} + \theta_{13} + \theta_{23}))U_1 U_2 U_3 W)$;
- $\widetilde{P}^{\theta_{12}}_{I,\emptyset} = \frac{P^{\theta_{12}}_I}{2}(1 + U_3 W)$, for $I = (1, 2)$.

Now in the above exact sequence,

$$i_{1*} - i_{2*}\big([1]\big) = \big([1], -[1]\big),$$

and by Proposition 4.7 (or by Lemma 4.8), we have

$$i_{2*}\big([P^{\theta_{12}}_{(1,2)}]\big) = 2\big[P^{\theta_{12}}_{(1,2),\emptyset}\big] - \big[P^{\theta_{12}}_{\emptyset,\emptyset}\big] + \big[P^{\theta_{12}}_{\emptyset,(1)}\big] - \big[P^{\theta_{12}}_{\emptyset,(2)}\big] + \big[P^{\theta_{12}}_{\emptyset,(1,2)}\big].$$

Similarly,

$$i_{1*}\big([P^{\theta_{12}}_{(1,2)}]\big) = 2\big[\widetilde{P}^{\theta_{12}}_{(1,2),\emptyset}\big] - \big[\widetilde{P}^{\theta_{12}}_{\emptyset,\emptyset}\big] + \big[\widetilde{P}^{\theta_{12}}_{\emptyset,(1)}\big] - \big[\widetilde{P}^{\theta_{12}}_{\emptyset,(2)}\big] + \big[\widetilde{P}^{\theta_{12}}_{\emptyset,(1,2)}\big].$$

Since $[1]$ and $[P^{\theta_{12}}_{(1,2)}]$ generate $\mathrm{K}_0(A_{\theta_{12}})$, looking at the above formulas of $i_{1*} - i_{2*}$ for $[1]$ and $[P^{\theta_{12}}_{(1,2)}]$, we can choose a basis of $\mathrm{K}_0(A_{\theta_{12}} \rtimes_\alpha \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z}_2)$ such that the map $i_{1*} - i_{2*}$ is exactly $(\mathrm{id}, 0, -\mathrm{id}, 0)^t$. Now we want to get the basis of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$. For this let us consider a basis of $\mathrm{K}_0(A_{\theta_{12}} \rtimes_\alpha \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z}_2)$ given by

$$\big\{[1] \oplus [0], [0] \oplus [1], [0] \oplus [P], [\widetilde{P}] \oplus [0] \mid P \in \mathcal{P}_2\big\}.$$

We can replace $[1] \oplus [0]$ by $i_{1*} - i_{2*}([1])$, and $[\widetilde{P}^{\theta_{12}}_{\emptyset,(1,2)}] \oplus [0]$ by $i_{1*} - i_{2*}([P^{\theta_{12}}_{(1,2)}])$ to get a new basis of $\mathrm{K}_0(A_{\theta_{12}} \rtimes_\alpha \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z}_2)$. Using this basis in the above sequence, with Corollary 4.3 and Proposition 4.4 in hand, we get the following basis of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$:

- 1;
- $P^\theta_{\emptyset,\emptyset} = \frac{1}{2}(1 + W)$;
- $P^\theta_{\emptyset,(1)} = \frac{1}{2}(1 + U_1 W)$;
- $P^\theta_{\emptyset,(2)} = \frac{1}{2}(1 + U_2 W)$;
- $P^\theta_{\emptyset,(1,2)} = \frac{1}{2}(1 + e(-\frac{1}{2}\theta_{12})U_1 U_2 W)$;
- $P^\theta_{I,\emptyset}, I = (1,2)$;
- $P^\theta_{\emptyset,(3)} = \frac{1}{2}(1 + U_3 W)$;
- $P^\theta_{\emptyset,(13)} = \frac{1}{2}(1 + e(-\frac{1}{2}\theta_{13})U_1 U_3 W)$;
- $P^\theta_{\emptyset,(23)} = \frac{1}{2}(1 + e(-\frac{1}{2}\theta_{23})U_2 U_3 W)$;
- $P^\theta_{I,(3)} = \frac{P^\theta_I}{2}(1 + U_3 W), I = (1,2)$;
- $P^\theta_{I,\emptyset} = \frac{P^\theta_I}{2}(1 + W), I = (1,3)$;
- $P^\theta_{I,\emptyset} = \frac{P^\theta_I}{2}(1 + W), I = (2,3)$.

So we have proved that $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^{12}$ and a basis of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$ may be given by $\{[1], [P] \mid P \in \mathcal{P}_3\}$.

We now have our main theorem

Theorem 4.9. *Let θ be a strongly irrational $n \times n$ matrix. Then $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^{3\cdot 2^{n-1}}$, and a generating set of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$ can be given by $\{[1], [P] \mid P \in \mathcal{P}_n\}$.*

Proof. We prove the theorem by induction on n. For $n = 2, 3$ we have already shown the result to be true. Assume that it holds for a number n. Then we must show that the result is true for $n + 1$. The proof is similar to the proof in the case for $n = 3$. Let θ be a strongly irrational $(n + 1) \times (n + 1)$ matrix. Let θ' be its upper left $n \times n$ corner. Then, by Natsume's theorem [22] we get the following six-term exact sequence:

$$\begin{array}{ccccc} \mathrm{K}_0(A_{\theta'}) & \xrightarrow{i_{1*}-i_{2*}} & \mathrm{K}_0(A_{\theta'} \rtimes_\alpha \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta'} \rtimes \mathbb{Z}_2) & \xrightarrow{j_{1*}+j_{2*}} & \mathrm{K}_0(A_{\theta'} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2) \\ \uparrow & & & & \downarrow e_1 \\ 0 & \longleftarrow & 0 & \longleftarrow & \mathrm{K}_1(A_{\theta'}) \end{array}$$

By induction hypothesis assume that a generating set of $\mathrm{K}_0(A_{\theta'} \rtimes \mathbb{Z}_2)$ is given by $\{[1], [P] \mid P \in \mathcal{P}_n\}$ and for $\mathrm{K}_0(A_{\theta'} \rtimes_\alpha \mathbb{Z}_2)$, the set is given by $\{[1], [\widetilde{P}] \mid P \in \mathcal{P}_n\}$ (as in the 3-dimensional case.)

Now $i_{1*} - i_{2*}([1]) = ([1], -[1])$. For each $I \in \mathrm{Minor}(n) \setminus \{\emptyset\}$, we have from Proposition 4.7

$$i_{2*}([P_I^{\theta'}]) = 2[P^{\theta'}_{I,\emptyset}] - [P^{\theta'}_{I'',\emptyset}] + [P^{\theta'}_{I'',(i_{2p-1})}] - [P^{\theta'}_{I'',(i_{2p})}] + [P^{\theta'}_{I'',(i_{2p-1},i_{2p})}],$$

where $I = (i_1, i_2, \ldots, i_{2p})$ and I'' is obtained from I by deleting the last two numbers. Similarly

$$i_{2*}([P_I^{\theta'}]) = 2[\widetilde{P}^{\theta'}_{I,\emptyset}] - [\widetilde{P}^{\theta'}_{I'',\emptyset}] + [\widetilde{P}^{\theta'}_{I'',(i_{2p-1})}] - [\widetilde{P}^{\theta'}_{I'',(i_{2p})}] + [\widetilde{P}^{\theta'}_{I'',(i_{2p-1},i_{2p})}].$$

Since [1] and $\{[P_I^{\theta'}]$ for $I \in \mathrm{Minor}(n) \setminus \{\emptyset\}\}$ generate $\mathrm{K}_0(A_{\theta'})$, looking at the above formulas of $i_{1*} - i_{2*}$ for [1] and $[P_I^{\theta'}]$, we can choose a basis of $\mathrm{K}_0(A_{\theta'} \rtimes_\alpha \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta'} \rtimes \mathbb{Z}_2)$ such that the map $i_{1*} - i_{2*}$ is exactly $(\mathrm{id}, 0, -\mathrm{id}, 0)^t$. This immediately gives that $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^{3 \cdot 2^n}$. To find a basis of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$, note that a basis of $\mathrm{K}_0(A_{\theta'} \rtimes_\alpha \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta'} \rtimes \mathbb{Z}_2)$ is given by

$$\{[1] \oplus [0], [0] \oplus [1], [0] \oplus [P], [\widetilde{P}] \oplus [0] \mid P \in \mathcal{P}_n\}.$$

But as in the 3-dimensional case, we can also replace $[1] \oplus [0]$ by $i_{1*} - i_{2*}([1])$, and $[\widetilde{P}^{\theta'}_{I'',(i_{2p-1},i_{2p-2})}] \oplus [0]$ by $i_{1*} - i_{2*}([P_I^{\theta'}])$, for each $I \in \mathrm{Minor}(n) \setminus \{\emptyset\}$, to get a new basis of $\mathrm{K}_0(A_{\theta'} \rtimes_\alpha \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta'} \rtimes \mathbb{Z}_2)$. Using this basis in the above exact sequence along with Corollary 4.3 and Proposition 4.4, we get our desired basis of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$. ■

We immediately get the following result, which was also obtained in [10] (see the proof of Theorem 6.6 there).

Corollary 4.10. *For any $\theta \in \mathcal{T}_n$,*

$$\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^{3 \cdot 2^{n-1}}, \quad \mathrm{K}_1(A_\theta \rtimes \mathbb{Z}_2) = 0.$$

Proof. From the proof of Theorem 6.6 in [10], it is enough to prove the statement for one $\theta \in \mathcal{T}_n$. Since Corollary B.6 of Appendix B gives a large class of examples of strongly totally irrational matrices, use Theorem 4.9. ■

5. Generators of $K_0(A_\theta \rtimes \mathbb{Z}_2)$ for a general θ

Using ideas from [10], in this section we construct a continuous field of projective modules, which will play the major role to compute the generators of $K_0(A_\theta \rtimes \mathbb{Z}_2)$ for a general θ. A similar idea was used in [4] for A_θ to compute the generators of $K_0(A_\theta)$ for a general θ; however there was a gap in the arguments therein. We fix the arguments and show that indeed such a construction of a field is possible for A_θ. We then extend these ideas to $A_\theta \rtimes \mathbb{Z}_2$.

Let G be a discrete group and let Ω be a $C([0,1],\mathbb{T})$-valued 2-cocycle on G (as in [10, Section 1]). One can then define the reduced twisted crossed product C^*-algebra $C([0,1]) \rtimes_\Omega G$ just like the twisted group C^*-algebras, where G acts trivially on $C([0,1])$. Here the underlying convolution algebra is the algebra of ℓ^1 functions on G with values in $C([0,1])$. Given any $t \in [0,1]$, the function

$$\omega_t := \Omega(\cdot,\cdot)(t)$$

is a $\mathbb{T}$-valued 2-cocycle on G. There is a canonical map (called the *evaluation map*)

$$\mathrm{ev}_t : C\big([0,1]\big) \rtimes_\Omega G \to C^*(G,\omega_t)$$

such that for each function $f \in \ell^1(G, C([0,1]))$ and each $x \in G$ we have $(\mathrm{ev}_t(f))(x) = (f(x))(t)$.

Theorem 5.1 ([10, Corollary 1.11]). *If G satisfies the Baum–Connes conjecture with coefficients, then the evaluation map* ev_t *induces an isomorphism on K-theory.*

Let Ω be a $C([0,1],\mathbb{T})$-valued 2-cocycle on $\mathbb{Z}^n$ such that

$$\Omega(\cdot,\cdot)(t) = \omega_{\theta_t}, \quad \theta_t \in \mathcal{T}_n,$$

is a 2-cocycle on $\mathbb{Z}^n$ as in Example 2.1 for all t. Also let $\widetilde{\Omega}$ be the $C([0,1],\mathbb{T})$-valued 2-cocycle on $\mathbb{Z}^n \rtimes \mathbb{Z}_2$ so that $\widetilde{\Omega}(\cdot,\cdot)(t) = \omega'_{\theta_t}$ (as in Example 2.2). Since the groups $\mathbb{Z}^n$ and $\mathbb{Z}^n \rtimes \mathbb{Z}_2$ satisfy the Baum–Connes conjecture with coefficients (see [15]), by Theorem 5.1 both evaluation maps

$$\begin{aligned}\mathrm{ev}_t &: C\big([0,1]\big) \rtimes_\Omega \mathbb{Z}^n \longrightarrow C^*(\mathbb{Z}^n, \omega_{\theta_t}),\\ \mathrm{ev}_t &: C\big([0,1]\big) \rtimes_{\widetilde{\Omega}} (\mathbb{Z}^n \rtimes \mathbb{Z}_2) \longrightarrow C^*(\mathbb{Z}^n \rtimes \mathbb{Z}_2, \omega'_{\theta_t})\end{aligned}$$

induce isomorphisms at the level of K_0 and K_1. As in the case of twisted group C^*-algebras, there is an identification

$$C\big([0,1]\big) \rtimes_{\widetilde{\Omega}} (\mathbb{Z}^n \rtimes \mathbb{Z}_2) \xrightarrow{\cong} \big(C\big([0,1]\big) \rtimes_\Omega \mathbb{Z}^n\big) \rtimes \mathbb{Z}_2$$

that respects the evaluation maps (see [10, Remark 2.3]).

As a result of the above discussions, using Theorem 5.1, we have (see also [10, Remark 2.3]) the following theorems.

Theorem 5.2. *Let* $[p_1], [p_2], \ldots, [p_m] \in \mathrm{K}_0(C([0,1]) \rtimes_\Omega \mathbb{Z}^n)$. *Then the following are equivalent:*

(1) $[p_1], [p_2], \ldots, [p_m]$ *form a basis of* $\mathrm{K}_0(C([0,1]) \rtimes_\Omega \mathbb{Z}^n)$.

(2) *For some* $t \in [0,1]$, *the evaluated classes* $[\mathrm{ev}_t(p_1)], [\mathrm{ev}_t(p_2)], \ldots, [\mathrm{ev}_t(p_m)]$ *form a basis of* $\mathrm{K}_0(C^*(\mathbb{Z}^n, \omega_{\theta_t}))$.

(3) *For every* $t \in [0,1]$, *the evaluated classes* $[\mathrm{ev}_t(p_1)], [\mathrm{ev}_t(p_2)], \ldots, [\mathrm{ev}_t(p_m)]$ *form a basis of* $\mathrm{K}_0(C^*(\mathbb{Z}^n, \omega_{\theta_t}))$.

Theorem 5.3. *Let* $[p_1], [p_2], \ldots, [p_m] \in \mathrm{K}_0((C([0,1]) \rtimes_\Omega \mathbb{Z}^n) \rtimes \mathbb{Z}_2)$. *Then the following are equivalent:*

(1) $[p_1], [p_2], \ldots, [p_m]$ *form a basis of* $\mathrm{K}_0((C([0,1]) \rtimes_\Omega \mathbb{Z}^n) \rtimes \mathbb{Z}_2)$.

(2) *For some* $t \in [0,1]$, *the evaluated classes* $[\mathrm{ev}_t(p_1)], [\mathrm{ev}_t(p_2)], \ldots, [\mathrm{ev}_t(p_m)]$ *form a basis of* $\mathrm{K}_0(C^*(\mathbb{Z}^n, \omega_{\theta_t}) \rtimes \mathbb{Z}_2)$.

(3) *For every* $t \in [0,1]$, *the evaluated classes* $[\mathrm{ev}_t(p_1)], [\mathrm{ev}_t(p_2)], \ldots, [\mathrm{ev}_t(p_m)]$ *form a basis of* $\mathrm{K}_0(C^*(\mathbb{Z}^n, \omega_{\theta_t}) \rtimes \mathbb{Z}_2)$.

Since $\mathbb{Z}^n \rtimes \mathbb{Z}_2$ is a discrete group, there is a canonical map from $\mathbb{Z}^n \rtimes \mathbb{Z}_2$ into the group of unitaries of $C([0,1]) \rtimes_{\tilde{\Omega}} (\mathbb{Z}^n \rtimes \mathbb{Z}_2)$. Let $\mathcal{U}_i \in C([0,1]) \rtimes_{\tilde{\Omega}} (\mathbb{Z}^n \rtimes \mathbb{Z}_2)$ be the images of the generators $x_i \in \mathbb{Z}^n \subset \mathbb{Z}^n \rtimes F$ (here x_i is as in Example 2.1) and $\mathcal{W}$ the image of the generator of $\mathbb{Z}_2 \subset \mathbb{Z}^n \rtimes \mathbb{Z}_2$ under the canonical map. Also, we denote by $\mathcal{U}_i$ the image of $x_i \in \mathbb{Z}^n$ in $C([0,1]) \rtimes_\Omega \mathbb{Z}^n$.

Next we will construct our required Ω. Consider the matrix $Z \in \mathcal{T}_n$ whose entries above the diagonal are all 1:

$$Z = \begin{pmatrix} 0 & 1 & \cdots & & \cdots & 1 \\ -1 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & & & & \\ & & & & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 1 \\ -1 & \cdots & & \cdots & -1 & 0 \end{pmatrix}. \tag{5.1}$$

If $\theta \in \mathcal{T}_n$, then by a translate of θ we understand any element $\theta^{\mathrm{tr}} \in \mathcal{T}_n$ such that $\theta - \theta^{\mathrm{tr}} \in M_n(\mathbb{Z})$. We then have $A_\theta = A_{\theta^{\mathrm{tr}}}$ since both matrices induce the same commutation relation on the generating unitaries (or, alternatively, since the corresponding circle-valued 2-cocycles ω_θ and $\omega_{\theta^{\mathrm{tr}}}$ coincide). In particular, this holds for $\theta^{\mathrm{tr}} = \theta + nZ$ for every $n \in \mathbb{Z}$. Now, for any $\theta \in \mathcal{T}_n$ there exists some $n \in \mathbb{Z}$ such that all pfaffian minors of $\theta^{\mathrm{tr}} = \theta + nZ$ are positive (see [4, Proposition 4.6]). Thus, replacing θ by θ^{tr} if necessary, we may assume without loss of generality that all the pfaffian minors of θ are positive.

Let us also fix a strongly totally irrational matrix $\psi \in \mathcal{T}_n$ as in Appendix B. Then, by definition (see Definition 3.15), all pfaffian minors of ψ are positive as well. Passing, again, to suitable translates θ^{tr} and ψ^{tr} of θ and ψ, if necessary, we may apply Proposition C.1 to obtain continuous paths $[0, \frac{1}{2}] \ni t \mapsto \theta(t) := (1-2t)\theta + 2tZ \in \mathcal{T}_n$ and $[\frac{1}{2}, 1] \ni t \mapsto \psi(t) := (2t-1)\psi + (2-2t)Z \in \mathcal{T}_n$ such that all pfaffian minors of $\theta(t)$ and $\psi(t)$, $t \in [0,1]$, are positive. Gluing both paths at $\theta(\frac{1}{2}) = Z = \psi(\frac{1}{2})$ we obtain a continuous path $[0,1] \ni t \to \nu(t) \in \mathcal{T}_n$ connecting θ with ψ such that all pfaffian minors of $\nu(t)$, $t \in [0,1]$, are positive. As a consequence, we can now formulate the following.

Proposition 5.4. *For each matrix $\theta \in \mathcal{T}_n$ there exists a strongly totally irrational matrix $\psi \in \mathcal{T}_n$ and a continuous path $[0,1] \ni t \mapsto \nu(t) \in \mathcal{T}_n$ with the following properties:*

(a) *All pfaffian minors of $\nu(t)$, $t \in [0,1]$, are positive.*

(b) *The endpoints $\nu(0) = \theta^{\mathrm{tr}}$ and $\nu(1) = \psi^{\mathrm{tr}}$ are translates of θ and ψ, respectively.*

The path in the above proposition now determines $C([0,1], \mathbb{T})$-valued 2-cocycles Ω_ν on $\mathbb{Z}^n$, and $\widetilde{\Omega}_\nu$ on $\mathbb{Z}^n \rtimes \mathbb{Z}_2$ by defining

$$\Omega_\nu(\cdot,\cdot)(t) = \omega_{\nu(t)}(\cdot,\cdot) \quad \text{and} \quad \widetilde{\Omega}_\nu(\cdot,\cdot)(t) = \omega'_{\nu(t)}(\cdot,\cdot), \qquad t \in [0,1].$$

In the following, we will construct a class of projective modules over

$$\left(C\left([0,1]\right) \rtimes_{\Omega_\nu} \mathbb{Z}^n\right) \rtimes \mathbb{Z}_2$$

such that the class restricted to each point t of $[0,1]$ gives a basis of $\mathrm{K}_0(C^*(\mathbb{Z}^n, \omega_{\nu(t)}) \rtimes \mathbb{Z}_2)$.

Now using [4, Theorem 3.3], for each $I \in \mathrm{Minor}(n) \setminus \{\emptyset\}$, we can construct a projective module $\mathcal{E}_I^{[0,1]}$ over $C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n$ such that $\mathcal{E}_I^{[0,1]}$ restricted to $t \in [0,1]$ is the module $\mathcal{E}_I^{\nu(t)}$. The idea of such a construction is as follows: using the notations of Section 3, for $I \in \mathrm{Minor}(n) \setminus \{\emptyset\}$, $|I| = 2p$, consider the algebra $C([0,1]) \rtimes_{\rho(R_I^\Sigma)\Omega_\nu} \mathbb{Z}^n$,

$$\rho(R_I^\Sigma)\Omega_\nu(\cdot,\cdot)(t) := \omega_{\rho(R_I^\Sigma)\nu(t)},$$

where $\rho(R_I^\Sigma)$ is as in the discussion which comes after Remark 3.10. Note that as in before, $C([0,1]) \rtimes_{\rho(R_I^\Sigma)\Omega_\nu} \mathbb{Z}^n$ is also canonically isomorphic to $C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n$. Also define $C([0,1]) \rtimes_{g_{I,\Sigma}(\Omega_\nu)} \mathbb{Z}^n$, $g_{I,\Sigma}(\Omega_\nu)(\cdot,\cdot)(t) := \omega_{g_{I,\Sigma}\nu(t)}$. For $M = \mathbb{R}^p \times \mathbb{Z}^{n-2p}$, define the space $\mathcal{S}(M,[0,1])$ consisting of all complex functions on $M \times [0,1]$ which are smooth and rapidly decreasing in the first variable and continuous in the second variable in each derivative of the first variable. Then using equations (3.8), (3.9), (3.10), (3.11) fibrewise, a suitable of completion of $\mathcal{S}(M,[0,1])$, $\mathcal{E}_I^{[0,1]}$, becomes a $C([0,1]) \rtimes_{g_{I,\Sigma}(\Omega_\nu)} \mathbb{Z}^n$ – $C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n$ strong Morita equivalence bi-module. Here in this construction we use the fact that $\mathrm{pf}((\nu(t))_I)$ is positive. Since $C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n$ is unital, $\mathcal{E}_I^{[0,1]}$ is also a projective module over $C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n$. The detailed proof of this construction can easily be deduced from the proof [4, Theorem 3.3] (see also the remark below).

Remark 5.5. In [4], in the construction of $C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n$ ($C^*(\mathbb{Z}^n \times I, \Omega)$ in the notations of [4]), Ω_ν was dependent on I; therefore, the proof of [4, Theorem 4.7] was incomplete. But we fix the proof in this paper by making Ω_ν independent of I and this results in Theorem 5.7.

Using Proposition 4.1, with the fibre-wise flip actions as in (4.2), it is easily checked that $\mathcal{E}_I^{[0,1]}$ becomes a projective module over the crossed product $(C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n) \rtimes \mathbb{Z}_2$.

We now construct a generating set of $\mathrm{K}_0((C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n) \rtimes \mathbb{Z}_2)$ using the projective modules over $(C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n) \rtimes \mathbb{Z}_2$ constructed above. Using the notations of Section 4, for each $I \in \mathrm{Minor}(n)$, fix a $J \subseteq I^c$. For $J = (j_1, j_2, \ldots, j_q)$, define $\mathcal{U}_J := \mathcal{U}_{j_1}\mathcal{U}_{j_2}\cdots\mathcal{U}_{j_q} \in C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n$, and for $J = \emptyset$, $\mathcal{U}_J := 1 \in C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n$. For a real function $f \in C([0,1],\mathbb{R})$, define $\exp(f) \in C([0,1],\mathbb{T})$ by $\exp(f)(t) = e(f(t))$. Let $r_J^{[0,1]} \in C([0,1],\mathbb{R})$ be the real function such that $\mathcal{U}_J = \exp(-2r_J^{[0,1]})\mathcal{U}_{\bar{J}}$. Now $\exp(r_J)\mathcal{U}_J\mathcal{W} =: \mathcal{W}_J$ is a self-adjoint unitary in $(C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n) \rtimes \mathbb{Z}_2$. One quickly checks that $\mathcal{W}_J\mathcal{U}_i\mathcal{W}_J = \exp(r_{J,i})\mathcal{U}_i^{-1}$, for some function $r_{J,i} \in C([0,1],\mathbb{R})$. A modified version of Lemma 2.3 shows that if we set $\widetilde{\mathcal{U}}_i = \exp(-\frac{r_{J,i}}{2})\mathcal{U}_i$ (so that we have $\mathcal{W}_J\widetilde{\mathcal{U}}_i\mathcal{W}_J = \widetilde{\mathcal{U}}_i^{-1}$), we have an isomorphism from $(C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n) \rtimes \mathbb{Z}_2$ to $(C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n) \rtimes \mathbb{Z}_2$ sending $\mathcal{U}_i$ to $\widetilde{\mathcal{U}}_i$ and $\mathcal{W}$ to $\mathcal{W}_J$. Here in the last crossed product $(C([0,1]) \rtimes_{\Omega_\nu})\mathbb{Z}^n \rtimes \mathbb{Z}_2$, the canonical unitary of $\mathbb{Z}_2$ is identified with $\mathcal{W}_J$. Using this identification, for each $I \in \mathrm{Minor}(n) \setminus \{\emptyset\}$, $J \subseteq I^c$. $\mathcal{E}_I^{[0,1]}$ becomes a module over $(C([0,1]) \rtimes_{\Omega_\nu})\mathbb{Z}^n \rtimes \mathbb{Z}_2$. We call this module $\mathcal{E}_{I,J}^{[0,1]}$. For $I = \emptyset$, and $J \subseteq I^c$, $\mathcal{E}_{I,J}^{[0,1]} := \frac{1}{2}(1 + \mathcal{W}_J)$. We consider the following family of elements in $\mathrm{K}_0((C([0,1]) \rtimes_{\Omega_\nu})\mathbb{Z}^n \rtimes \mathbb{Z}_2)$:

$$\bigcup_{I \in \mathrm{Minor}(n)} \{\mathcal{E}_{I,J}^{[0,1]} \mid J \subseteq I^c,\ |J| \le 2\}.$$

Our next step is to show that if we consider the K-theory classes $[\mathcal{E}_{I,J}^{[0,1]}]$ in the above family inside $(C([0,1]) \rtimes_{\Omega_\nu})\mathbb{Z}^n \rtimes \mathbb{Z}_2$, then the K-theory classes $[\mathrm{ev}_1(\mathcal{E}_{I,J}^{[0,1]})]$, along with $[1]$, provide a basis of $\mathrm{K}_0(A_{\nu(1)} \rtimes \mathbb{Z}_2) = \mathrm{K}_0(A_{\psi^{\mathrm{tr}}} \rtimes \mathbb{Z}_2) = \mathrm{K}_0(A_\psi \rtimes \mathbb{Z}_2)$. Let us write $\mathcal{E}_{I,J}^{\psi^{\mathrm{tr}}}$ for $\mathrm{ev}_1(\mathcal{E}_{I,J}^{[0,1]})$. Since ψ is totally irrational, the translate of ψ, ψ^{tr}, is also totally irrational. From the pfaffian summation formula (as in equation (C.1)) if we compute the pfaffian of ψ_I^{tr}, for a fixed $I \in \mathrm{Min}(n)$, we see that this is exactly $\mathrm{Tr}(P_I^\psi) + \sum_{|I'|<|I|} c_{I'}\,\mathrm{Tr}(P_{I'}^\psi)$, $c_{I'} \in \mathbb{Z}$. This also coincides with the trace of

$$\big[\mathcal{E}_I^{\psi^{\mathrm{tr}}}\big] = \big[\mathrm{ev}_1\big(\mathcal{E}_I^{[0,1]}\big)\big]$$

considering $\mathcal{E}_I^{[0,1]}$ as a projective module over $C([0,1]) \rtimes_{\Omega_\nu} \mathbb{Z}^n$. Hence

$$[P_I^\psi] + \sum_{|I'|<|I|} c_{I'}[P_{I'}^\psi] = [\mathcal{E}_I^{\psi^{\mathrm{tr}}}]$$

as the trace map is injective for $A_{\psi^{\mathrm{tr}}}$. Now for $J \subseteq I^c$, $[\mathcal{E}_{I,J}^{\psi^{\mathrm{tr}}}]$ and $[P_{I,J}^\psi] + \sum_{|I'|<|I|} c_{I'}[P_{I',J}^\psi]$ coincide inside $\mathrm{K}_0(A_{\psi^{\mathrm{tr}}} \rtimes \mathbb{Z}_2)$ as they are extended from the same element in $\mathrm{K}_0(A_{\psi^{\mathrm{tr}}})$ using the same method, i.e., the Green–Julg map. Now using Theorem 4.9, we know that the elements of the set

$$\bigcup_{I \in \mathrm{Minor}(n)} \Big\{\big[P_{I,J}^\psi\big] + \sum_{|I'|<|I|} c_{I'}\big[P_{I',J}^\psi\big] \mid J \subseteq I^c,\ |J| \le 2\Big\}$$

along with [1] form a basis of $\mathrm{K}_0(A_\psi \rtimes \mathbb{Z}_2) = \mathrm{K}_0(A_{\psi^{\mathrm{tr}}} \rtimes \mathbb{Z}_2)$. Considering $[\mathcal{E}_{I,J}^{[0,1]}]$ inside $(C([0,1]) \rtimes_{\Omega_\nu})\mathbb{Z}^n \rtimes \mathbb{Z}_2$, let us denote $\mathrm{ev}_0(\mathcal{E}_{I,J}^{[0,1]})$ by $\mathcal{E}_{I,J}^{\theta^{\mathrm{tr}}}$. Let

$$\mathrm{Proj}_n = \{\mathcal{E}_{I,J}^{\theta^{\mathrm{tr}}} : I \in \mathrm{Minor}(n),\ J \subseteq I^c,\ |J| \leq 2\}. \tag{5.2}$$

Now if we apply Theorem 5.3 to the cocycle Ω_ν constructed just after Proposition 5.4, as a result of the above discussion we immediately get the following theorem.

Theorem 5.6. *Let $\theta \in \mathcal{T}_n$. Then there exists a suitable translate $\theta^{\mathrm{tr}} \in \mathcal{T}_n$ for θ such that the set of classes $\{[1], [\mathcal{E}] \mid \mathcal{E} \in \mathrm{Proj}_n\}$, where Proj_n is as in equation (5.2), generate $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$.*

Using a similar idea as in the proof of the above theorem along with Theorem 5.2, we also get the following theorem.

Theorem 5.7 (cf. [4, Theorem 4.7]). *Let $\theta \in \mathcal{T}_n$. Then there exists a suitable translate $\theta^{\mathrm{tr}} \in \mathcal{T}_n$ for θ such that the set of classes*

$$\bigcup_{I \in \mathrm{Minor}(n) \setminus \{\emptyset\}} \{[1], [\mathcal{E}_I^{\theta^{\mathrm{tr}}}]\}$$

generate $\mathrm{K}_0(A_\theta)$.

From equation (4.5) it is clear that $\mathrm{Tr}_{\theta^{\mathrm{tr}}}^{\mathbb{Z}_2}([\mathcal{E}_{I,J}^{\theta^{\mathrm{tr}}}]) = \frac{\mathrm{pf}(\theta_I^{\mathrm{tr}})}{2}$. It follows that

$$\mathrm{Tr}_{\theta^{\mathrm{tr}}}^{\mathbb{Z}_2}\left(\mathrm{K}_0(A_{\theta^{\mathrm{tr}}} \rtimes \mathbb{Z}_2)\right) = \frac{\mathrm{Tr}_{\theta^{\mathrm{tr}}}\left(\mathrm{K}_0(A_{\theta^{\mathrm{tr}}})\right)}{2}, \tag{5.3}$$

which in turn gives that $\mathrm{Tr}_\theta^{\mathbb{Z}_2}(\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)) = \frac{\mathrm{Tr}_\theta(\mathrm{K}_0(A_\theta))}{2}$, from the computation of $\mathrm{pf}(\theta_I^{\mathrm{tr}})$ using the pfaffian summation formula (equation (C.1)). This result was already obtained in [5, Example 4.5].

6. Isomorphism classes of $A_\theta \rtimes \mathbb{Z}_2$

In this section we give an application of the explicit K-theory computations from the previous section. We start with the following definition.

Definition 6.1. A skew symmetric real $n \times n$ matrix θ is called *non-degenerate* if whenever $x \in \mathbb{Z}^n$ satisfies $e(\langle x, \theta y\rangle) = 1$ for all $y \in \mathbb{Z}^n$, then $x = 0$.

The theorem which we want to prove in this section is the following.

Theorem 6.2. *Let $\theta_1, \theta_2 \in \mathcal{T}_n$ be non-degenerate. Let $\mathbb{Z}_2$ act on A_{θ_1} and A_{θ_2} by the flip actions. Then $A_{\theta_1} \rtimes \mathbb{Z}_2$ is isomorphic to $A_{\theta_2} \rtimes \mathbb{Z}_2$ if A_{θ_1} is isomorphic to A_{θ_2}. Moreover, if any one of θ_1, θ_2 is totally irrational, the converse is true.*

We need some preparation before proving the above theorem.

Proposition 6.3 ([23, Proposition 3.7]). *Let A be a simple infinite dimensional separable unital nuclear C^*-algebra with tracial rank zero and which satisfies the Universal Coefficient Theorem. Then A is a simple AH algebra with real rank zero and no dimension growth. If $\mathrm{K}_*(A)$ is torsion-free, A is an AT algebra. If, in addition, $\mathrm{K}_1(A) = 0$, then A is an AF algebra.*

Let $\theta \in \mathcal{T}_n$ be non-degenerate. Then the following are known.

- A_θ is a simple C^*-algebra (even the converse is true: the simplicity of A_θ implies θ must be non-degenerate) with a unique tracial state [23, Theorem 1.9].
- A_θ is tracially AF [23, Theorem 3.6].
- If β is an action of a finite group on A_θ which has the tracial Rokhlin property (see [10, Section 5]), $A_\theta \rtimes_\beta F$ is a simple C^*-algebra with tracial rank zero [24, Corollary 1.6, Theorem 2.6]. Also, $A_\theta \rtimes_\beta F$ has a unique tracial state [10, Proposition 5.7].
- The flip action of $\mathbb{Z}_2$ on A_θ has the tracial Rokhlin property [10, Lemma 5.10 and Theorem 5.5].
- $A_\theta \rtimes \mathbb{Z}_2$ satisfies the Universal Coefficient Theorem (see [10, proof of Theorem 6.6]).

With the above list of results along with Proposition 6.3 and Theorem 4.10, we readily have the following corollary.

Corollary 6.4. *Let $\theta \in \mathcal{T}_n$ be non-degenerate. Then $A_\theta \rtimes \mathbb{Z}_2$ is an AF algebra.*

The above corollary was first obtained in [10, Theorem 6.6]. We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2. Since taking a translate of θ_i, $i = 1, 2$, does not change the isomorphism classes of A_{θ_i} or $A_{\theta_i} \rtimes \mathbb{Z}_2$, for this proof we may replace the θ_i by any of their translates.

To prove the last part, WLOG assume that θ_2 is totally irrational. Let there be a $*$-isomorphism f from $A_{\theta_1} \rtimes \mathbb{Z}_2$ to $A_{\theta_2} \rtimes \mathbb{Z}_2$. Then $\mathrm{Tr}_{\theta_1}^{\mathbb{Z}_2}$ and $\mathrm{Tr}_{\theta_2}^{\mathbb{Z}_2}$ have the same range, and hence using equation (5.3), Tr_{θ_1} and Tr_{θ_2} also have the same range. Now to show that A_{θ_1} and A_{θ_2} are $*$-isomorphic, it is enough to find an isomorphism $g : \mathrm{K}_0(A_{\theta_1}) \to \mathrm{K}_0(A_{\theta_2})$ such that $\mathrm{Tr}_{\theta_2} \circ g = \mathrm{Tr}_{\theta_1}$ and $g([1]) = [1]$. Indeed, g is then an order isomorphism by [2, Proposition 3.7], and using classification of tracially AF algebras by Lin [21, Theorem 5.2], we conclude that A_{θ_1} and A_{θ_2} are $*$-isomorphic. Let us now see the existence of the isomorphism g. Denote the ranges of Tr_{θ_1} and Tr_{θ_2} by R_1 and R_2, respectively. Since R_1 and R_2 are finitely generated subgroups of $\mathbb{R}$, they are free. Also $R_1 = R_2$ implies that they have the same rank. Now we have the following exact sequences:

$$0 \longrightarrow \ker(\mathrm{Tr}_{\theta_1}) \longrightarrow \mathrm{K}_0(A_{\theta_1}) \xrightarrow{\mathrm{Tr}_{\theta_1}} R_1 \longrightarrow 0,$$
$$0 \longrightarrow \ker(\mathrm{Tr}_{\theta_2}) \longrightarrow \mathrm{K}_0(A_{\theta_2}) \xrightarrow[\mathrm{Tr}_{\theta_2}]{} R_2 \longrightarrow 0.$$

Note that the above sequences split since the K-groups are torsion-free. Now $\ker(\mathrm{Tr}_{\theta_1})$ and $\ker(\mathrm{Tr}_{\theta_2})$ are finitely generated abelian groups of the same rank. So there exists an isomorphism ψ between them. Now g is defined as $\psi \oplus \phi$, where ϕ is the map between R_1 and R_2 given by multiplication with 1. Clearly $\mathrm{Tr}_{\theta_2} \circ g = \mathrm{Tr}_{\theta_1}$, since the following diagram commutes:

$$\begin{array}{ccc} \mathrm{K}_0(A_{\theta_1}) & \xrightarrow{\mathrm{Tr}_{\theta_1}} & R_1 \\ \downarrow g & & \downarrow \phi \\ \mathrm{K}_0(A_{\theta_2}) & \xrightarrow[\mathrm{Tr}_{\theta_2}]{} & R_2 \end{array}$$

Now $g([1]) = [1]$ follows from $\mathrm{Tr}_{\theta_2} \circ g = \mathrm{Tr}_{\theta_1}$, and total irrationality of θ_2. Indeed, using Theorem 5.7, let us write $g([1])$ as a linear combination of the basis elements. As we have $\mathrm{Tr}_{\theta_2}(g([1])) = 1$, then the total irrationality of θ_2 forces $g([1])$ to be $[1]$.

Now assume there is a $*$-isomorphism f from A_{θ_1} to A_{θ_2}. This gives $\mathrm{Tr}_{\theta_2} \circ f = \mathrm{Tr}_{\theta_1}$, since A_{θ_1} has a unique trace and $f(1) = 1$. So we have an order isomorphism, which we call by f again, from $\mathrm{K}_0(A_{\theta_1})$ to $\mathrm{K}_0(A_{\theta_2})$. Choosing the basis $\{[1], [\mathcal{E}_{I'}^{\theta_1}] \mid I' \in \mathrm{Minor}(n) \setminus \emptyset\}$ of $\mathrm{K}_0(A_{\theta_1})$ using Theorem 5.7, we have

$$f\big([\mathcal{E}_{I'}^{\theta_1}]\big) = \sum_{I'' \in \mathrm{Minor}(n)} C_{I''}^{I'} [\mathcal{E}_{I''}^{\theta_2}], \quad \text{for } I' \in \mathrm{Minor}(n),$$

where $C_{I''}^{I'} \in \mathbb{Z}$. We have that $A_{\theta_1} \rtimes \mathbb{Z}_2$ and $A_{\theta_2} \rtimes \mathbb{Z}_2$ are both AF algebras. Now to show that $A_{\theta_1} \rtimes \mathbb{Z}_2$ and $A_{\theta_2} \rtimes \mathbb{Z}_2$ are isomorphic, it is enough (just like before) to find an isomorphism $f' : \mathrm{K}_0(A_{\theta_1} \rtimes \mathbb{Z}_2) \to \mathrm{K}_0(A_{\theta_2} \rtimes \mathbb{Z}_2)$ such that $\mathrm{Tr}_{\theta_2}^{\mathbb{Z}_2} \circ f' = \mathrm{Tr}_{\theta_1}^{\mathbb{Z}_2}$ and $f'([1]) = [1]$. Now it is enough to define the map on a set of generators of $\mathrm{K}_0(A_{\theta_1} \rtimes \mathbb{Z}_2)$ which is given by $\{[1], [\mathcal{E}], \ \mathcal{E} \in \mathrm{Proj}_n\}$, where

$$\mathrm{Proj}_n = \bigcup_{I' \in \mathrm{Minor}(n)} \big\{\mathcal{E}_{I',J}^{\theta_1} \mid J \subseteq (I')^c, |J| \le 2\big\}$$

using Theorem 5.6. We define the map f' as follows: $f'([1]) = [1]$, for $I' \neq \emptyset$,

$$\begin{aligned} & f'\big([\mathcal{E}_{I',J}^{\theta_1}]\big) \\ & = \Big(\sum_{I'' \in \mathrm{Minor}(n) \setminus \{I', \emptyset\}} C_{I''}^{I'} [\mathcal{E}_{I'',\emptyset}^{\theta_2}]\Big) + [\mathcal{E}_{I',J}^{\theta_2}] + (C_{I'}^{I'} - 1)[\mathcal{E}_{I',\emptyset}^{\theta_2}] + C_{\emptyset}^{I'}[1] - C_{\emptyset}^{I'}[\mathcal{E}_{\emptyset,\emptyset}^{\theta_2}], \end{aligned}$$

where $C_{I''}^{I'}$ is as before, and for $I' = \emptyset$,

$$f'\big([\mathcal{E}_{\emptyset,J}^{\theta_1}]\big) = [\mathcal{E}_{\emptyset,J}^{\theta_2}].$$

Using $\mathrm{Tr}_{\theta_2} \circ f = \mathrm{Tr}_{\theta_1}$, we get the required tracial property of f'. Indeed,

$$\mathrm{Tr}_{\theta_2}^{\mathbb{Z}_2} \circ f'\big([1]\big) = 1 = \mathrm{Tr}_{\theta_1}^{\mathbb{Z}_2}\big([1]\big),$$

and for $I' = \emptyset$,

$$\mathrm{Tr}_{\theta_2}^{\mathbb{Z}_2} \circ f'\big([\mathcal{E}_{\emptyset,J}^{\theta_1}]\big) = \frac{1}{2} = \mathrm{Tr}_{\theta_1}^{\mathbb{Z}_2}\big([\mathcal{E}_{\emptyset,J}^{\theta_1}]\big),$$

and finally for $I' \neq \emptyset$,

$$\operatorname{Tr}^{\mathbb{Z}_2}_{\theta_2} \circ f'([\mathcal{E}^{\theta_1}_{I',J}]) = \frac{\operatorname{Tr}_{\theta_2}\left(\sum_{I'' \in \operatorname{Minor}(n)} C^{I'}_{I''}[\mathcal{E}^{\theta_2}_{I''}]\right)}{2} = \frac{\operatorname{Tr}_{\theta_2} \circ f\left([\mathcal{E}^{\theta_1}_{I'}]\right)}{2}$$
$$= \frac{\operatorname{Tr}_{\theta_1}\left([\mathcal{E}^{\theta_1}_{I'}]\right)}{2} = \operatorname{Tr}^{\mathbb{Z}_2}_{\theta_1}\left([\mathcal{E}^{\theta_1}_{I',J}]\right). \qquad \blacksquare$$

A. Rieffel-type projections and $\mathbb{Z}_2$-invariance

A.1. The Rieffel projection in n-dimensional tori

In this section we will give a description of the map ψ, and the construction of the Rieffel projection e used in Theorem 3.11.

Let us first write down the Morita equivalence construction of Section 3 explicitly for $p = 1$; i.e., when $\mathcal{M}$ is the group $\mathbb{R} \times \mathbb{Z}^q$, $q = n - 2$. As before, write

$$\theta = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{2,1} & \theta_{2,2} \end{pmatrix} = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} \\ -\theta^t_{1,2} & \theta_{2,2} \end{pmatrix} \in \mathcal{T}_n,$$

where

$$\theta_{1,1} = \begin{pmatrix} 0 & \theta_{12} \\ \theta_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \theta_{12} \\ -\theta_{12} & 0 \end{pmatrix} \in \mathcal{T}_2$$

is an invertible 2×2 matrix. Then consider the matrix

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & \mathrm{id}_q \\ T_{31} & T_{32} \end{pmatrix},$$

where $T_{11} = \begin{pmatrix} \theta_{12} & 0 \\ 0 & 1 \end{pmatrix}$, $T_{31} = \theta_{2,1}$, and $T_{32} = \frac{\theta_{2,2}}{2}$. Also consider

$$S = \begin{pmatrix} S_0 & -S_0 T^t_{31} \\ 0 & \mathrm{id}_q \\ 0 & T^t_{32} \end{pmatrix},$$

where $S_0 = \begin{pmatrix} 0 & 1 \\ -\theta^{-1}_{12} & 0 \end{pmatrix}$. Then the equations (3.8), (3.9), (3.10), (3.11) make $\mathcal{S}(\mathbb{R} \times \mathbb{Z}^q)$ an $A^\infty_{\sigma(\theta)} - A^\infty_\theta$ Morita equivalence bimodule, where

$$\sigma(\theta) = \begin{pmatrix} \theta^{-1}_{1,1} & -\theta^{-1}_{1,1}\theta_{1,2} \\ \theta_{2,1}\theta^{-1}_{1,1} & \theta_{2,2} - \theta_{2,1}\theta^{-1}_{1,1}\theta_{1,2} \end{pmatrix}.$$

Let us denote by $U_1, U_2, \ldots, U_n$ the canonical generators of A_θ. For $A_{\sigma(\theta)}$, we choose the generators $V_1, V_2, \ldots, V_n$ such that $\delta_{x_i} \in C^*(\mathbb{Z}^n, \omega_{\sigma(\theta)})$ is identified with V_i, for all i, where $x_i = (0, \ldots, -1, \ldots, 0)$, -1 is at the ith position.

According to [25, Proposition 2.1], if we can find an $f \in \mathcal{S}(\mathbb{R} \times \mathbb{Z}^q)$ such that

$$ {}_{A^\infty_{\sigma(\theta)}}\langle f, f \rangle = 1, $$

then $e = \langle f, f \rangle_{A^\infty_\theta}$ is a projection of trace θ_{12} in $A^\infty_\theta \subset A_\theta$, and we have an isomorphism $\psi : A_{\sigma(\theta)} \to eA_\theta e$, given by $\psi(a) = \langle f, af \rangle_{A_\theta}$.

Choose a smooth even function ϕ on $\mathbb{R}$ as in the definition of $f_{\theta_{12}}$ in [30, Section 2.2], assuming $\theta_{12} \in (1/2, 1)$. By using a standard regularization argument as in [1, Lemma 2.1], we can assume that ϕ is smooth and $\sqrt{\phi}$ is also smooth. We restrict θ_{12} to take values in $(\frac{1}{2}, 1)$ just to make sure that the projection e that we are going to construct is $\mathbb{Z}_2$-invariant. In general, one can find such ϕ with the required properties for a general $\theta_{12} \in (0, 1)$. Note that ϕ is a compactly supported function (with support in $[-\frac{1}{2}, \frac{1}{2}]$) satisfying $\phi(x_0 + \theta_{12}) = 1 - \phi(x_0)$ for $-\frac{1}{2} \le x_0 \le \frac{1}{2} - \theta_{12}$, $\phi = 1$ on $\frac{1}{2} - \theta_{12} \le x_0 \le \theta_{12} - \frac{1}{2}$, and $\phi(-x_0) = \phi(x_0)$ for $-\frac{1}{2} \le x \le \frac{1}{2}$.

Now define a function $f \in \mathcal{S}(\mathbb{R} \times \mathbb{Z}^q)$, given by $f(x_0, l) = c\sqrt{\phi}(x_0)$, $c = \frac{1}{\sqrt{K\theta_{12}}}$ (K is as in equation (3.11)), when $\mathbb{Z}^q \ni l = 0$ and $f(x, l) = 0$ otherwise. Let us first show that ${}_{A^\infty_{\sigma(\theta)}}\langle f, f \rangle = 1$. To this end, from equation (3.11), we have the following:

$$ {}_{A^\infty_{\sigma(\theta)}}\langle f, f \rangle(m) = Ke^{2\pi i \langle S(m), J'S(m)/2 \rangle} \int_{\mathbb{R}\times\mathbb{Z}^q} \langle x, S''(m) \rangle f\big(x + S'(m)\big) f(x) dx, $$

for $m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$, noting that f is real valued. Using the formula of S' and f, it is clear that the above expression is zero for any non-zero values of $(m_3, m_4, \dots, m_n) \in \mathbb{Z}^q$. Also when $m_2 \neq 0$, the above expression for $m = (m_1, m_2, 0, \dots, 0) \in \mathbb{Z}^n$ is zero since $x + S'(m) = (x_0 + m_2, x_1, \dots, x_q)$, where $x = (x_0, x_1, \dots, x_q)$, and ϕ is supported in an interval of length one and vanishing at the end points. Finally for an $m = (m_1, 0, \dots, 0)$ the expression becomes

$$ \begin{aligned} Kc^2 \int_{\mathbb{R}} e^{-\frac{2\pi i x_0 m_1}{\theta_{12}}} \phi(x_0) dx_0 &= c^2 K\theta_{12} \int_{\mathbb{R}} e^{-2\pi i x_0 m_1} \phi(x_0\theta_{12}) dx_0 \\ &= \int_{\mathbb{R}} e^{-2\pi i x_0 m_1} \widetilde{\phi}(x_0) dx_0, \end{aligned} $$

where $\widetilde{\phi}(x_0) = \phi(x_0\theta_{12})$. Now

$$ \int_{\mathbb{R}} e^{-2\pi i x_0 m_1} \widetilde{\phi}(x_0) dx_0 = \widehat{\widetilde{\phi}}(m_1) = \widehat{\Phi}(m_1), $$

where $\widehat{\ }$ denotes the Fourier transform and Φ is the periodic function defined by $\Phi(x_0) = \sum_{n\in\mathbb{Z}} \widetilde{\phi}(x_0 + n)$, $x_0 \in \mathbb{R}$. But $\sum_{n\in\mathbb{Z}} \widetilde{\phi}(x_0 + n) = \sum_{n\in\mathbb{Z}} \phi(\theta_{12}x_0 + \theta_{12}n) = 1$, using the defining properties of ϕ, for all $x_0 \in \mathbb{R}$. Hence we get ${}_{A^\infty_{\sigma(\theta)}}\langle f, f \rangle = 1$.

Now we want an explicit expression for the projection

$$ \langle f, f \rangle_{A^\infty_\theta}(m) = e^{2\pi i \langle -T(m), J'T(m)/2 \rangle} \int_{\mathbb{R}\times\mathbb{Z}^q} \langle x, -T''(m) \rangle f\big(x + T'(m)\big) f(x) dx. $$

Let us write $m = (m_1, m_2, \ldots, m_n)$, $x = (x_0, x_1, \ldots, x_q)$ as before. An easy observation shows that the above expression is zero exactly when $m_3 = m_4 = \cdots = m_n = 0$. This implies that $\langle f, f\rangle_{A_\theta^\infty} \in A_{\theta_{12}} \subseteq A_\theta$, and a direct computation yields

$$\begin{aligned}\langle f, f\rangle_{A_\theta^\infty}&(m_1, m_2, 0, 0, \ldots, 0)\\ &= e^{-\pi i\theta_{12}m_1m_2}\int_{\mathbb{R}} e^{-2\pi i x_o m_2}\sqrt{\phi}(x_0+\theta_{12}m_1)\sqrt{\phi}(x_0)dx_0. \end{aligned}\tag{A.1}$$

A.2. Rieffel-type projections in $A_\theta \rtimes \mathbb{Z}_2$

In this section we describe how to construct projections inside $A_\theta \rtimes \mathbb{Z}_2$ using the Rieffel projection constructed above, and some other technical results used in the proof of Proposition 4.7.

Let $f \in \mathcal{S}(\mathbb{R}\times\mathbb{Z}^q)$ be as above. Since, by construction, f is an even function, it follows from equation (4.3) that f and the projection $e = \langle f, f\rangle_{A_\theta^\infty}$ are $\mathbb{Z}_2$-invariant. Hence the flip-action is a well-defined action on $eA_\theta e$. Now let us show that the map $\psi : A_{\sigma(\theta)} \to eA_\theta e$ given by $\psi(a) = \langle f, af\rangle_{A_\theta}$ is flip equivariant. To this end, let us use equations (4.3) and (4.4) (for $A_{\sigma(\theta)}$) to check that $\psi(\beta(a)) = \beta(\psi(a))$. This is easy as $\beta(\psi(a)) = \beta(\langle f, af\rangle_{A_\theta}) = \langle fT_g, (af)T_g\rangle_{A_\theta} = \langle fT_g, \beta(a)(fT_g)\rangle_{A_\theta} = \langle f, \beta(a)f\rangle_{A_\theta} = \psi(\beta(a))$. Hence we get an isomorphism $\psi : A_{\sigma(\theta)} \rtimes \mathbb{Z}_2 \to eA_\theta e \rtimes \mathbb{Z}_2$, which we denoted by ψ again. Then we have the following commutative diagram:

$$\begin{array}{ccc} A_{\sigma_\theta} & \xrightarrow{\ i_2\ } & A_{\sigma_\theta} \rtimes \mathbb{Z}_2 \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ eA_\theta e & \xrightarrow{\ i_2\ } & eA_\theta e \rtimes \mathbb{Z}_2 \end{array}\tag{A.2}$$

where i_2 is the natural inclusion map.

Next let us understand the image of the projection $\frac{1}{2}(1 + V_kW) \in A_{\sigma_\theta} \rtimes \mathbb{Z}_2$, for $k = 3, 4, \ldots, n$, under the map ψ. Of course, we have $\psi(1) = e \in eA_\theta e \rtimes \mathbb{Z}_2$. It is then enough to compute $\psi(V_k)$.

Now from equation (3.10),

$$(V_k f)(x) = e^{2\pi i\langle -S(l), J'S(l)/2\rangle}\big\langle x, -S''(l)\big\rangle f\big(x + S'(l)\big),$$

where $l = (0, 0, \ldots, -1, \ldots, 0)$, -1 at the kth position. Now $(V_k f)(x)$ is only non-zero when $x_{k-2} = 1$, and $x_i = 0$, for $i \neq 0, k-2$. Then the value is

$$e^{\pi i\frac{\theta_{1k}\theta_{2k}}{\theta_{12}}}e^{2\pi i x_0\frac{-\theta_{1k}}{\theta_{12}}}f(x_0 - \theta_{2k}, 0, 0, \ldots, 0).$$

Now we want an explicit expression for

$$\langle f, V_k f\rangle_{A_\theta^\infty}(m) = e^{2\pi i\langle -T(m), J'T(m)/2\rangle}\int_{\mathbb{R}\times\mathbb{Z}^q}\big\langle x, -T''(m)\big\rangle(V_k f)\big(x + T'(m)\big)f(x)dx.$$

Now

$$(V_k f)\big(x + T'(m)\big) = (V_k f)(x_0 + \theta_{12}m_1, x_1 + m_3, x_2 + m_4, \dots, x_q + m_n).$$

Looking at the integral, it is only non-zero when $x_1 = x_2 = \cdots = x_q = 0$. So for $x = (x_0, 0, 0, \dots, 0)$,

$$(V_k f)\big(x + T'(m)\big) = (V_k f)(x_0 + \theta_{12}m_1, m_3, m_4, \dots, m_n).$$

But from the previous observation, $V_k f$ is only non-zero when $m_k = 1$ and $m_i = 0$ $(i \neq 1, k)$, and then the value is

$$e^{\pi i \frac{\theta_{1k}\theta_{2k}}{\theta_{12}}} e^{2\pi i (x_0 + \theta_{12}m_1)\frac{-\theta_{1k}}{\theta_{12}}} f(x_0 + \theta_{12}m_1 - \theta_{2k}, 0, 0).$$

Finally for $m = (m_1, m_2, 0, \dots, 1, \dots, 0)$ (1 at the kth position), we have

$$\begin{aligned}\langle f, V_k f\rangle_{A_\theta^\infty}(m) &= e^{2\pi i \langle -T(m), J'T(m)/2\rangle} \int_{\mathbb{R}\times\mathbb{Z}^q} \big\langle x, -T''(m)\big\rangle (V_k f)\big(x + T'(m)\big) f(x)dx \\ &= e^{\pi i(-\theta_{12}m_1m_2 + \theta_{1k}m_1 + \theta_{2k}m_2)} \int_{\mathbb{R}} e^{-2\pi i x_0 m_2} e^{\pi i \frac{\theta_{1k}\theta_{2k}}{\theta_{12}}} e^{-2\pi i (x_0 + \theta_{12}m_1)\frac{\theta_{1k}}{\theta_{12}}} \\ &\quad \times \sqrt{\phi}(x_0 + \theta_{12}m_1 - \theta_{2k})\sqrt{\phi}(x_0)dx_0.\end{aligned}$$

In particular, the above computation shows that the projection $\psi(\frac{1}{2}(1 + V_k W)) \in A_{\theta_{12}} \rtimes \mathbb{Z}_2 \subseteq A_\theta \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ in $A_{\theta_{12}} \rtimes \mathbb{Z}_2$ is generated by the self-adjoint unitary $U_k W =: W'$, acting on $A_{\theta_{12}}$ by the flip action (using Lemma 2.3). In the above formula of $\langle f, V_k f\rangle_{A_\theta^\infty}$ replacing θ_{1k} and θ_{2k} by $t\theta_{1k}$ and $t\theta_{2k}$, respectively, for $t \in [0, 1]$, we get a homotopy of projections in $A_{\theta_{12}} \rtimes \mathbb{Z}_2$ between $\psi(\frac{1}{2}(1 + V_k W))$ (at $t = 1$) and $\frac{e}{2}(1 + W')$ (at $t = 0$). (Note that we have used the fact that for $t = 0$, the above expression of $\langle f, V_k f\rangle_{A_\theta^\infty}$ matches with the same of $e = \langle f, f\rangle_{A_\theta^\infty}$ in equation (A.1).)

B. Strongly totally irrational matrices

This section is essentially [7, Appendix I], but with some modifications at the end. For completeness we repeat the whole construction.

Let $s = \{s_i\}_i$ be a sequence of integers such that $s_i > \sum_{j=1}^{i-1} s_j$, for all i, with $s_1 = 1$. We call such a sequence a super-increasing sequence. For $\alpha \in (0, 1)$ define the $n \times n$ antisymmetric matrix $\Theta(n)$ by induction: $\Theta(2) := \begin{pmatrix} 0 & \alpha^{s_1} \\ -\alpha^{s_1} & 0 \end{pmatrix}$; $\Theta(n)_{ij} = \Theta(n-1)_{ij}$, for $1 < i < j < n$, and $\Theta(n)_{in} := \alpha^{s_{p+i}}$, for $i = 1, \dots, n-1$, where $p = \frac{(n-1)(n-2)}{2}$. Hence

$$\Theta(3) = \begin{pmatrix} 0 & \alpha^{s_1} & \alpha^{s_2} \\ -\alpha^{s_1} & 0 & \alpha^{s_3} \\ -\alpha^{s_2} & -\alpha^{s_3} & 0 \end{pmatrix}, \quad \Theta(4) = \begin{pmatrix} 0 & \alpha^{s_1} & \alpha^{s_2} & \alpha^{s_4} \\ -\alpha^{s_1} & 0 & \alpha^{s_3} & \alpha^{s_5} \\ -\alpha^{s_2} & -\alpha^{s_3} & 0 & \alpha^{s_6} \\ -\alpha^{s_4} & -\alpha^{s_5} & -\alpha^{s_6} & 0 \end{pmatrix}.$$

Remark B.1. Note that all sub-matrices $\Theta(n)_I$ (as in Definition 3.1) of $\Theta(n)$ are like $\Theta(m)$, for some $m \leq n$, but for a different super-increasing sequence $s' = \{s'_i\}_i$, which is a subsequence of s.

We then have $\mathrm{pf}(\Theta(2)) = \alpha^{s_1}$, and

$$\mathrm{pf}\left(\Theta(4)\right) = \alpha^{s_1+s_6} - \alpha^{s_2+s_5} + \alpha^{s_4+s_3}.$$

Note that, using the super-increasing property of s, we have that

$$s_1 + s_6 > s_2 + s_5 > s_4 + s_3.$$

Denote by $s_{i,j}$ the exponent of α in the ijth entry of $\Theta(n)$. Let $s[n] := s_{(n-1),n} = s_{\frac{n(n-1)}{2}}$. Let us first recall the following recursive definition of pfaffian from [13, p. 116]. Let $\mathrm{pf}^{ij}(A)$ denote the pfaffian of the $(n-2) \times (n-2)$ skew-symmetric matrix, which we call A^{ij}, obtained from $A = (a_{jk})$ by removing the ith, jth row and the ith, jth column. Hence $\mathrm{pf}^{ij}(A) = \mathrm{pf}(A^{ij})$. Let n be even. Then for a fixed integer j, $1 \leq j \leq n$, one has the following recursive definition of pfaffian:

$$\mathrm{pf}(A) = \sum_{i<j} (-1)^{i+j-1} a_{ij}\, \mathrm{pf}^{ij}(A) + \sum_{i>j} (-1)^{i+j} a_{ij}\, \mathrm{pf}^{ij}(A).$$

In particular, when $j = 1$, we have

$$\mathrm{pf}(A) = \sum_{i>1} (-1)^{i+1} a_{i1}\, \mathrm{pf}^{i1}(A) = \sum_{i>1} (-1)^{i} a_{1i}\, \mathrm{pf}^{1i}(A), \tag{B.1}$$

and for $j = n$, we have

$$\mathrm{pf}(A) = \sum_{i=1}^{n-1} (-1)^{i+1} a_{in}\, \mathrm{pf}^{in}(A). \tag{B.2}$$

Lemma B.2. *For an even n we have*

$$\mathrm{pf}\left(\Theta(n)\right) = \alpha^{M(n)_1} - \alpha^{M(n)_2} + \alpha^{M(n)_3} - \alpha^{M(n)_4} + \cdots + \alpha^{M(n)_{R(n)}},$$

for a strictly decreasing sequence of numbers $M(n)_1, M(n)_2, \ldots, M(n)_{R(n)}$, where $R(n) = (n-1)!!$.[1]

Proof. We prove this by induction on n. As we have seen the statement is true for $n = 2, 4$ for all super-increasing sequences. Now assume that the statement is true for $n - 2$, for all super-increasing sequences. Then we must prove that the statement is true for n, for all super-increasing sequences. Fix a super-increasing sequence $s = \{s_i\}_i$. From

[1]Double factorial.

equation (B.2), we have

$$\begin{aligned}\operatorname{pf}\big(\Theta(n)\big) &= \sum_{i=1}^{n-1}(-1)^{i+1}\theta_{in}\operatorname{pf}^{in}(\Theta)\\ &= \theta_{(n-1)n}\operatorname{pf}^{(n-1)n}(\Theta) - \theta_{(n-2)n}\operatorname{pf}^{(n-2)n}(\Theta)\\ &\quad + \theta_{(n-3)n}\operatorname{pf}^{(n-3)n}(\Theta) - \cdots - \theta_{2n}\operatorname{pf}^{2n}(\Theta) + \theta_{1n}\operatorname{pf}^{1n}(\Theta)\\ &= \alpha^{s_{(n-1),n}}\operatorname{pf}^{(n-1)n}(\Theta) - \alpha^{s_{(n-2),n}}\operatorname{pf}^{(n-2)n}(\Theta)\\ &\quad + \alpha^{s_{(n-3),n}}\operatorname{pf}^{(n-3)n}(\Theta) - \cdots - \alpha^{s_{2,n}}\operatorname{pf}^{2n}(\Theta) + \alpha^{s_{1,n}}\operatorname{pf}^{1n}(\Theta).\end{aligned}$$

Now by the induction hypothesis we have that all $\operatorname{pf}^{in}(\Theta), i = 1, 2, \ldots, n-1$, are of the form as in the statement with $R(n-2) = (n-3)!!$. Now if we expand all $\operatorname{pf}^{in}(\Theta), i = 1, 2, \ldots, n-1$, keeping the form, we see that the expression

$$\begin{aligned}&\alpha^{s_{(n-1),n}}\operatorname{pf}^{(n-1)n}(\Theta) - \alpha^{s_{(n-2),n}}\operatorname{pf}^{(n-2)n}(\Theta)\\ &\quad + \alpha^{s_{(n-3),n}}\operatorname{pf}^{(n-3)n}(\Theta) - \cdots - \alpha^{s_{2,n}}\operatorname{pf}^{2n}(\Theta) + \alpha^{s_{1,n}}\operatorname{pf}^{1n}(\Theta)\end{aligned}$$

is already of the required form, using the super-increasing property of s, and noting that the exponents of α in the expressions of $\operatorname{pf}^{in}(\Theta), i = 1, 2, \ldots, n-1$, contain no term of the form $s_{*,n}$. Note that the total number of terms (after expanding) in the above expression is $(n-1)(n-3)!! = (n-1)!!$. ■

Remark B.3. It is also clear from above, again using the super-increasing property of s, that $M(n)_{R(n)} > M(n-2)_1$.

Lemma B.4. *For an even n, we have*

$$0 < \operatorname{pf}\big(\Theta(n)\big) < \operatorname{pf}\big(\Theta(n-2)\big) < 1.$$

Proof. Since

$$\operatorname{pf}\big(\Theta(n)\big) = \alpha^{M(n)_1} - \alpha^{M(n)_2} + \alpha^{M(n)_3} - \alpha^{M(n)_4} + \cdots + \alpha^{M(n)_{R(n)}},$$

we have that

$$\operatorname{pf}\big(\Theta(n)\big) > \alpha^{M(n)_1} > 0,$$

using $-\alpha^{M(n)_{2i}} + \alpha^{M(n)_{2i+1}} > 0$ (since $\alpha \in (0,1)$). To show $\operatorname{pf}(\Theta(n)) < \operatorname{pf}(\Theta(n-2))$, we look at the expression $\operatorname{pf}(\Theta(n)) - \operatorname{pf}(\Theta(n-2))$, which is

$$\begin{aligned}&\alpha^{M(n)_1} - \alpha^{M(n)_2} + \alpha^{M(n)_3} - \cdots + \alpha^{M(n)_{R(n)}} - \alpha^{M(n-2)_1}\\ &\quad + \alpha^{M(n-2)_2} - \alpha^{M(n-2)_3} + \cdots - \alpha^{M(n-2)_{R(n-2)}}.\end{aligned}$$

Note that $M(n)_1, M(n)_2, \ldots, M(n)_{R(n)}, M(n-2)_1, M(n-2)_2, \ldots, M(n-2)_{R(n-2)}$ is still a strictly decreasing finite sequence due to the above remark. Now using

$$\alpha^{M(n)_i} - \alpha^{M(n)_{i+1}} < 0, \quad \alpha^{M(n)_{R(n)}} - \alpha^{M(n-2)_1} < 0, \quad \alpha^{M(n-2)_{2i}} - \alpha^{M(n-2)_{2i+1}} < 0,$$

we get the above expression less than zero.

The last inequality in the statement of Lemma B.4 follows from an argument using induction on n. ∎

Now let us choose an $\alpha \in (0,1)$ such that $\alpha^{2s[n]} > \frac{1}{2}$. Then from the above, $M(n)_1 < 2s[n]$. But $\alpha^{2s[n]} < \alpha^{M(n)_1}$. So $\frac{1}{2} < \alpha^{M(n)_1}$. So we have, for such α,

$$\frac{1}{2} < \mathrm{pf}\big(\Theta(n)\big) < \mathrm{pf}\big(\Theta(n-2)\big) < 1, \tag{B.3}$$

for an even n.

Corollary B.5. *With the notations in Section 3, for an $\alpha \in (0,1)$ such that $\alpha^{2s[n]} > \frac{1}{2}$, we have*

$$\frac{1}{2} < \mathrm{pf}\big(F^j\big(\Theta(n)_I\big)_{11}\big) < 1,$$

for all $I \in \mathrm{Minor}(n)$ with $2 \leq |I| = 2m$, and for all $j = 0, 1, \ldots, m-1$.

Proof. For any $I = (i_1, i_2, \ldots, i_{2m})$, since $s^I[n] := s_{i_{2m-1}, i_{2m}} \leq s_{(n-1),n} = s[n]$, we still have $\alpha^{2s^I[n]} > \frac{1}{2}$. Hence from equation (B.3) (and using Remark B.1),

$$\frac{1}{2} < \mathrm{pf}\big(\Theta(n)_I\big) < \mathrm{pf}\big(\Theta(n)_{I''}\big) < 1,$$

where I'' is obtained from I by deleting the last two numbers. Now the corollary easily follows with the explicit expression (using Lemma 3.7) of $\mathrm{pf}(F^j(\Theta(n)_I)_{11})$ in hand. ∎

Choose the super-increasing sequence $\{s_i = 2^{i-1}\}_i$. When α is a transcendental number, it is well known that the numbers $\alpha, \alpha^2, \ldots, \alpha^{2^i}, \ldots$ as well as any products of these numbers are linear independent over $\mathbb{Q}$. So we have that $\Theta(n)$ is totally irrational. Then by using the above corollary, we get the following corollary.

Corollary B.6. *Let s be the super-increasing sequence $\{s_i = 2^{i-1}\}_i$ and let $\alpha \in (0,1)$ be a transcendental number such that $\alpha^{2s[n]} > \frac{1}{2}$. Let $\Theta(n)$ be the $n \times n$ antisymmetric matrix involving α and s. Then we have that $\Theta(n)$ is a strongly totally irrational matrix for $n \geq 2$.*

The above corollary gives a large class of examples of strongly totally irrational matrices.

C. Construction of $C([0,1]) \rtimes_{\Omega_A} \mathbb{Z}^n$

Let n be an even number. For $I \in \mathrm{Min}(n)$, define $\sigma(I) := i_1 + i_2 + \cdots + i_{2r}$ for $I = (i_1, i_2, \ldots, i_{2r})$ and $\sigma(I) := 0$ for $I = \emptyset$. Also for $I \in \mathrm{Min}(n)$, let I^{co} be the element in $\mathrm{Min}(n)$, $I^{\mathrm{co}} = (j_1, j_2, \ldots, j_{n-2r})$, such that $\{i_1, i_2, \ldots, i_{2r}, j_1, j_2, \ldots, j_{n-2r}\} = \{1, 2, \ldots, n\}$ and $\{i_1, i_2, \ldots, i_{2r}\} \cap \{j_1, j_2, \ldots, j_{n-2r}\} = \emptyset$. Then we have the *pfaffian* *summation formula* [28, Lemma 4.2]

$$\operatorname{pf}(A+B) = \sum_{|I|\le n} (-1)^{\sigma(I)-|I|/2} \operatorname{pf}(A_I) \operatorname{pf}(B_{I^{\mathrm{co}}}), \tag{C.1}$$

for $A, B \in \mathcal{T}_n$.

Recall that for a matrix $A \in \mathcal{T}_n$ a translate of A is any matrix A^{tr} such that $A - A^{\mathrm{tr}}$ has only integer entries. In this section we want to prove the following proposition.

Proposition C.1. *Assume that all the pfaffian minors of $A = (a_{jk}) \in \mathcal{T}_n$ are positive, where n is any number. Then there exists a translate of A, A^{tr}, such that all the pfaffian minors of the matrix $(1-t)A^{\mathrm{tr}} + tZ$ are positive for all $t \in [0,1]$, where Z is as in equation* (5.1).

Before proving the above proposition, let us try to understand why the above proposition should hold for $n = 2, 3, 4$. For $n = 2$ and 3, since all the pfaffian minors of A are positive, all pfaffian minors of the matrix $(1-t)A + tZ$ are also clearly positive for all $t \in [0,1]$. Hence we can just take $A^{\mathrm{tr}} = A$. Now for $n = 4$, if we compute the pfaffian of $(1-t)A + tZ$ using the summation formula above, we get

$$\begin{aligned}\operatorname{pf}\big((1-t)A + tZ\big)&\\ &= t^2 + (a_{12} + a_{34} - a_{13} - a_{24} + a_{14} + a_{23})t(1-t) + (1-t)^2 \operatorname{pf}(A).\end{aligned} \tag{C.2}$$

Now we can add a large positive integer n to a_{12} to make $a_{12} + a_{34} - a_{13} - a_{24} + a_{14} + a_{23}$ greater than zero. Now our required A^{tr} is the matrix obtained by adding

$$\begin{pmatrix} 0 & n & 0 & 0 \\ -n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

to A.

Let us now consider the matrix $(1-t)A + tZ$, for $A \in \mathcal{T}_n$ and n is an even number. Using the pfaffian summation formula we have

$$\operatorname{pf}\big((1-t)A + tZ\big) = \sum_{|I|=2r\le n} (-1)^{\sigma(I)-r} \operatorname{pf}(A_I) t^{n/2-r} (1-t)^r. \tag{C.3}$$

For all r such that $2r \le n$, set

$$c_{n,r} := \sum_{|I|=2r} (-1)^{\sigma(I)-r} \operatorname{pf}(A_I).$$

Then we claim that the coefficient of a_{12} in $c_{n,r}$ is the coefficient of $t^{n/2-r}(1-t)^r$ in the expression of $\operatorname{pf}^{12}((1-t)A + tZ)$ (refer to the paragraphs which come after Remark B.1 for the notation pf^{12}). To show this, first note that

$$\operatorname{pf}^{12}\big((1-t)A + tZ\big) = \sum_{|I'|=2r'} (-1)^{\sigma(I')-r'} \operatorname{pf}(A^{12}_{I'}) t^{(n-2)/2-r'} (1-t)^{r'}.$$

If we put $(n-2)/2 - r' = n/2 - r$, we get that $r' = r - 1$. Hence the coefficient of $t^{n/2-r}(1-t)^r$ in the expression of $\mathrm{pf}^{12}((1-t)A + tZ)$ is

$$\sum_{|I'|=2r-2} (-1)^{\sigma(I')-r+1}\, \mathrm{pf}(A^{12}_{I'}).$$

Now in the expression of $c_{n,r}$, only for the I's which are of the form $(1, 2, i_3, i_4, \ldots, i_{2r})$, for some $i_3, i_4, \ldots, i_{2r}$ so that $2 < i_3 < i_4 < \cdots < i_{2r} \le n$, $\mathrm{pf}(A_I)$ will contain a term involving a_{12}. The coefficient of a_{12} in such $\mathrm{pf}(A_I)$ is $\mathrm{pf}(A_{I''})$, for $I'' = (i_3, i_4, \ldots, i_{2r})$, using the recursive definition (equation (B.1)) of the pfaffian. But this I'' will correspond to $I' = (i_3 - 2, i_4 - 2, \ldots, i_{2r} - 2)$ in A^{12}. Now, it is clear that $(-1)^{\sigma(I)-r} = (-1)^{\sigma(I')-r+1}$. Since there are $2r-2$ many choices of I'' possible, our claim follows.

For two fixed numbers l_1, l_2, with $l_1 \le l_2$, let us denote by $(l_1, l_2, *, *, \ldots, *)$ any such $J \in \mathrm{Min}(n)$, whose first two components are l_1 and l_2, i.e. $J = (l_1, l_2, l_3, l_4, \ldots, l_{2r})$, for some r.

Proof of Proposition C.1. Our aim is to show that there exists a translate of A, A^{tr}, such that all pfaffian minors of the matrix $\mathrm{pf}(((1-t)A^{\mathrm{tr}} + tZ)_J) > 0$, for all $J \in \mathrm{Minor}(n)$. In fact we will show that there exists a A^{tr} such that each individual coefficient $c_{|J|,r}$, $2r \le |J|$, is greater than zero in the expression of $\mathrm{pf}(((1-t)A^{\mathrm{tr}} + tZ)_J)$ for all $J \in \mathrm{Minor}(n)$, coming from equation (C.3). We do this by induction on the length of J. Now clearly the statement is true for all J such that $|J| = 2$, since all the entries of A are positive. Assume that the statement is true for any J, such that $|J| \le 2m-2$. In this case we get a translate A^{tr}, which we call by A again, with required properties. We will now show that the statement is true for any J, such that $|J| \le 2m$. Now any such J so that $|J| = 2m$ will be of the form $(l_1, l_2, \ldots, l_{2m-2}, l_{2m})$, where $1 \le l_1 \le l_2 \le n - 2m + 2$. We perform the following algorithm to get our desired A^{tr}.

Set $l_1 = n - 2m + 1$, $l_2 = n - 2m + 2$.

Step 1. Consider any $J = (l_1, l_2, *, *, \ldots, *)$ so that $|J| = 2m$. Use the formula equation (C.3) for $\mathrm{pf}((1-t)A_J + tZ_J)$ and the observation made just before this proof to note that in the coefficients of $t^{m-r}(1-t)^r$, $r = 1, 2, \ldots, m$, the coefficients of $a_{l_1 l_2}$ are positive using the induction hypothesis. By adding a number to $a_{l_1 l_2}$ we can make sure that the coefficients of $t^{m-r}(1-t)^r$, $r = 1, 2, \ldots, m$, are positive. We can do this for each $J = (l_1, l_2, *, *, \ldots, *)$ and hence can add a sufficiently large number to $a_{l_1 l_2}$ such that the coefficients of $t^{m-r}(1-t)^r$, $r = 1, 2, \ldots, m$, are positive in all $\mathrm{pf}((1-t)A_J + tZ_J)$, for all $J = (l_1, l_2, *, *, \ldots, *)$. Thus by altering $a_{l_1 l_2}$ as above, we get a translate of A, which we call by A again. If $l_1 > 1$, then set $l_1 := l_1 - 1$, $l_2 := l_2$, and if $l_1 = 1$, then set $l_1 := l_2 - 2$, $l_2 := l_2 - 1$. Now if $l_1 = 1$, $l_2 = 2$, go to Step 3. Otherwise, go to Step 2.

Step 2. As in Step 1 add a sufficiently large number to $a_{l_1 l_2}$, such that the coefficients $t^{m-1-r}(1-t)^r$, $r = 1, 2, \ldots, m-1$, are positive in all $\mathrm{pf}((1-t)A_J + tZ_J)$, for all $J = (l_1, l_2, *, *, \ldots, *)$ such that $|J| = 2m-2$. Then repeat this for $J = (l_1, l_2, *, *, \ldots, *)$ such that $|J| = 2m-4$ and continue the process until we reach such J's so that $|J| = 4$. After the above alterations, we get a translate of A, which we call by A again. Go to Step 1.

Step 3. Again, as before, add a sufficiently large number to a_{12}, such that $t^{m-r}(1-t)^r$, $r = 1, 2, \dots, m$, are positive in all $\mathrm{pf}((1-t)A_J + tZ_J)$, for all $J = (1, 2, *, *, \dots, *)$ such that $|J| = 2m$.

After the above step, we end up with a translate A^{tr} of A. Now we claim that $\mathrm{pf}((1-t)A_J^{\mathrm{tr}} + tZ_J)$ is positive for all $|J| \leq 2m$. Let $J = (l_1, l_2, *, *, \dots, *)$ be fixed such that $|J| \leq 2m$. If $l_2 > n - 2m + 2$, $|J| \leq 2m - 2$. In this case, no entries of A_J have been altered in the above algorithm, and hence $\mathrm{pf}((1-t)A_J^{\mathrm{tr}} + tZ_J) = \mathrm{pf}((1-t)A_J + tZ_J)$ is positive by the induction hypothesis. If $l_2 \leq n - 2m + 2$, after one alteration of $a_{l_1 l_2}$ (let us call the translate A' so that we have $\mathrm{pf}((1-t)A'_J + tZ_J) > 0$), no alterations have been occurred for the entries of A'_J except possible alterations of $a_{l_1 l_2}$ which still keep $\mathrm{pf}((1-t)A'_J + tZ_J) > 0$. Hence $\mathrm{pf}((1-t)A_J^{\mathrm{tr}} + tZ_J) > 0$. ∎

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