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Atiyah sequences of braided Lie algebras and their splittings

Paolo Aschieri, Giovanni Landi, and Chiara Pagani

Abstract. Given an equivariant noncommutative principal bundle, we construct an Atiyah sequence of braided derivations whose splittings give connections on the bundle. Vertical braided derivations act as infinitesimal gauge transformations on connections. In the case of the principal SU(2)-bundle over the sphere S^4_θ an equivariant splitting of the Atiyah sequence recovers the instanton connection. An infinitesimal action of the braided conformal Lie algebra $\mathrm{so}_\theta(5,1)$ yields a five parameter family of splittings. On the principal $\mathrm{SO}_\theta(2n,\mathbb{R})$-bundle of orthonormal frames over the sphere S^{2n}_θ, a splitting of the sequence leads to the Levi-Civita connection for the 'round' metric on S^{2n}_θ. The corresponding Riemannian geometry of S^{2n}_θ is worked out.

1. Introduction

The Atiyah sequence of a principal bundle over a manifold M is an important tool in the study of the geometry of Yang–Mills theories [7]. It is given as a sequence of vector bundles over M with Lie algebra structures on the corresponding modules of sections, resulting in a short exact sequence of (infinite dimensional) Lie algebras,

$$0 \to \Gamma\,\mathrm{ad}(P) \to \mathfrak{X}_G(P) \to \mathfrak{X}(M) \to 0 \tag{1.1}$$

with principal G-bundle $P \to M$. Here $\mathfrak{X}(M)$ are the vector fields over M (the section of the tangent bundle TM), while $\mathfrak{X}_G(P)$ consists of G-invariant vector fields on P (the G-invariant sections of the tangent bundle TP along the fibres of P), and $\Gamma\,\mathrm{ad}(P)$ its Lie subalgebra made of vertical ones (with $\mathrm{ad}(P)$ the bundle associated to P via the adjoint representation of G on its Lie algebra).

A splitting of the sequence corresponds to a G-invariant direct sum decomposition of the tangent space TP in horizontal and vertical parts, that is, to a connection on $P \to M$. The space $\Gamma\,\mathrm{ad}(P)$ is the (infinite dimensional) Lie algebra of the gauge group of the principal bundle P. Its elements are infinitesimal gauge transformations, [7, §3].

Exact sequences of vector bundles (named after Atiyah) and their splittings were introduced in [6] in the complex analytic context, motivated by the study of connections and

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obstructions to their existence. Later versions are in [17, 20], with a braided case coming from a $\mathbb{Z}_2$-grading in [14]. The sequence can also be seen as a Lie algebroid [18].

In Section 3 of the present paper, we study a quantum version $0 \to \mathfrak{g} \to P \to T \to 0$ of the Lie algebroid (1.1) in the context of braided Lie algebras (of derivations) associated with a triangular Hopf algebra (K, R).[1] Each term in the sequence is a braided Lie–Rinehart pair for a braided commutative algebra B. Given any such an exact sequence, we define a connection to be a splitting of the underlying short exact sequence of vector spaces. The splitting does not need to be a braided Lie algebra morphism: the curvature is defined as the $\mathfrak{g}$-valued braided two-form on T that measures the extent to which the connection fails to be such. In Section 3.4, elements of $\mathfrak{g}$ are explicitly seen as infinitesimal gauge transformations which act on the set of connections (splittings) of the sequence. Having a connection, one can define the covariant derivative of $\mathfrak{g}$-valued braided forms on P. The connection and the curvature satisfy a structure equation and a Bianchi identity (Proposition 3.3). In this braided context, the covariant derivative of the curvature does not vanish in general (in agreement with the result in [12] and [11]), while it does for a K-equivariant connection.

In Section 4, we consider K-equivariant noncommutative principal bundles, that is K-equivariant Hopf–Galois extensions $B \subset A$, where (K, R) is a triangular Hopf algebra. We have then a corresponding sequence of braided Lie algebras of derivations as in (4.2),

$$0 \to \mathrm{aut}_B^{\mathsf{R}}(A) \to \mathrm{Der}_{\mathcal{M}^H}^{\mathsf{R}}(A) \to \mathrm{Der}^{\mathsf{R}}(B) \to 0$$

which, when exact, is a noncommutative version of the Atiyah sequence (1.1). The braided Lie algebra of infinitesimal gauge transformations $\mathrm{aut}_B^{\mathsf{R}}(A)$ of a Hopf–Galois extension consists of vertical braided derivations of the algebra A; it was introduced and studied in [4, 5]. The theory is exemplified with the construction in Section 4.1 of the Atiyah sequence of braided Lie algebras for the instanton bundle on the noncommutative sphere S^4_θ and a five parameter family of splittings (connections) of it. In a C^*-algebra context an Atiyah sequence for noncommutative principal bundles was given in [19].

The last part of the paper in Section 5 is dedicated to the example of the noncommutative principal $\mathrm{SO}_\theta(2n, \mathbb{R})$-bundle $\mathrm{SO}_\theta(2n+1, \mathbb{R}) \to S^{2n}_\theta$ on the noncommutative sphere S^{2n}_θ. This is the noncommutative orthogonal frame bundle through an identification of braided derivations of $\mathcal{O}(S^{2n}_\theta)$ as sections of the bundle associated to the principal bundle via the fundamental corepresentation of $\mathcal{O}(\mathrm{SO}_\theta(2n, \mathbb{R}))$ on the algebra $\mathcal{O}(\mathbb{R}^{2n}_\theta)$.

The corresponding Atiyah sequence is constructed and an equivariant splitting determined. This leads to a novel and very direct construction of the Levi-Civita connection on S^{2n}_θ; the connection is explicitly presented and globally defined using global coordinate functions on S^{2n}_θ. Indeed, the principal equivariant connection induces a covariant

[1] Indeed, in the present paper, we only consider triangular Hopf algebras (K, R) and the associated symmetric monoidal category of K-modules. Thus, rather than braided Lie algebras, a more appropriate term would be (K, R)-symmetric Lie algebras. The latter terminology is the one adopted in [4].

derivative on the associated tangent bundle which is torsion free and compatible with the 'round' metric. We then work out the corresponding Riemannian geometry of S^{2n}_θ. The latter is an Einstein space (the Ricci tensor being proportional to the metric, see (5.27)) and a space form (the scalar curvature being constant, see (5.28)).

The study of Levi-Civita connections in noncommutative geometry is a very active field of research. In the braided context, uniqueness and existence of Levi-Civita connections has been actively pursued, especially via Koszul formulas, see [1] and the different contributions there referred to. Our new contribution to this subject – in this paper on connections as splitting of Atiyah sequences – is the explicit and globally defined expression of the Levi-Civita connection on S^{2n}_θ given in (5.23).

2. Algebraic preliminaries

We work in the category of $\Bbbk$-modules with $\Bbbk$ a commutative field. All algebras are unital and associative and morphisms of algebras preserve the unit. All coalgebras satisfy the corresponding dual conditions. We use standard terminologies and notations in Hopf algebra theory. For H a bialgebra, we call H*-equivariant* a map of H-modules or H-comodules.

Recall that a bialgebra (or Hopf algebra) K is *quasitriangular* if there exists an invertible element $\mathsf{R} \in K \otimes K$ (the universal R-matrix of K) with respect to which the coproduct Δ of K is quasi-cocommutative

$$\Delta^{\mathrm{cop}}(k) = \mathsf{R}\Delta(k)\overline{\mathsf{R}} \tag{2.1}$$

for each $k \in K$, with $\Delta^{\mathrm{cop}} := \tau \circ \Delta$, τ the flip map, and $\overline{\mathsf{R}} \in K \otimes K$ the inverse of R, $\mathsf{R}\overline{\mathsf{R}} = \overline{\mathsf{R}}\mathsf{R} = 1 \otimes 1$. Moreover, R is required to satisfy,

$$(\Delta \otimes \mathrm{id})\mathsf{R} = \mathsf{R}_{13}\mathsf{R}_{23} \quad \text{and} \quad (\mathrm{id} \otimes \Delta)\mathsf{R} = \mathsf{R}_{13}\mathsf{R}_{12}. \tag{2.2}$$

We write $\mathsf{R} := \mathsf{R}^\alpha \otimes \mathsf{R}_\alpha$ with an implicit sum. Then $\mathsf{R}_{12} = \mathsf{R}^\alpha \otimes \mathsf{R}_\alpha \otimes 1$, and similarly for R_{23} and R_{13}. From conditions (2.1) and (2.2) it follows that R satisfies the quantum Yang–Baxter equation $\mathsf{R}_{12}\mathsf{R}_{13}\mathsf{R}_{23} = \mathsf{R}_{23}\mathsf{R}_{13}\mathsf{R}_{12}$. The R-matrix of a quasitriangular bialgebra (K, R) is unital: $(\varepsilon \otimes \mathrm{id})\mathsf{R} = 1 = (\mathrm{id} \otimes \varepsilon)\mathsf{R}$, with ε the counit of K. When K is a Hopf algebra, the quasitriangularity implies that its antipode S is invertible and satisfies

$$(S \otimes \mathrm{id})(\mathsf{R}) = \overline{\mathsf{R}}; \quad (\mathrm{id} \otimes S)(\overline{\mathsf{R}}) = \mathsf{R}; \quad (S \otimes S)(\mathsf{R}) = \mathsf{R}.$$

The Hopf algebra K is said to be *triangular* when $\overline{\mathsf{R}} = \mathsf{R}_{21}$, with $\mathsf{R}_{21} = \tau(\mathsf{R}) = \mathsf{R}_\alpha \otimes \mathsf{R}^\alpha$.

2.1. Braided Lie algebras of derivations

For the purpose of the present paper we only need to consider braided Lie algebras associated with a triangular Hopf algebra (K, R). A K-braided Lie algebra is a K-module $\mathfrak{g}$, with action $\triangleright\colon K \otimes \mathfrak{g} \to \mathfrak{g}$, which is endowed with a bilinear map (a bracket)

$$[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

which is equivariant, that is, $k \triangleright [u,v] = [k_{(1)} \triangleright u, k_{(2)} \triangleright v]$, for $\Delta(k) = k_{(1)} \otimes k_{(2)}$ the coproduct of K, and satisfies the conditions

(i) braided antisymmetry: $[u,v] = -[\mathsf{R}_\alpha \triangleright v, \mathsf{R}^\alpha \triangleright u]$,

(ii) braided Jacobi identity: $[u,[v,w]] = [[u,v],w] + [\mathsf{R}_\alpha \triangleright v, [\mathsf{R}^\alpha \triangleright u, w]]$,

for all $u, v, w \in \mathfrak{g}$, $k \in K$. A morphism of braided Lie algebras is a morphism of K-modules that in addition commutes with the brackets.

Any K-module algebra A is a K-braided Lie algebra for the braided commutator

$$[\ ,\ ] : A \otimes A \to A, \quad a \otimes b \mapsto [a,b] := ab - (\mathsf{R}_\alpha \triangleright b)(\mathsf{R}^\alpha \triangleright a). \tag{2.3}$$

Also, for A a K-module algebra, the K-module algebra $(\mathrm{Hom}(A,A), \triangleright_{\mathrm{Hom}(A,A)})$ of linear maps from A to A with action

$$\begin{aligned} \triangleright_{\mathrm{Hom}(A,A)} &: K \otimes \mathrm{Hom}(A,A) \to \mathrm{Hom}(A,A), \\ & k \otimes \psi \mapsto k \triangleright_{\mathrm{Hom}(A,A)} \psi : a \mapsto k_{(1)} \triangleright (\psi(S(k_{(2)}) \triangleright a)) \end{aligned} \tag{2.4}$$

is a braided Lie algebra with the braided commutator; here S is the antipode of K. An important example of a braided Lie algebra associated to a K-module algebra is that of braided derivations, [4, §5.2]. A braided derivation is any $Y \in \mathrm{Hom}(A,A)$ which satisfies

$$Y(aa') = Y(a)a' + (\mathsf{R}_\alpha \triangleright a)(\mathsf{R}^\alpha \triangleright_{\mathrm{Hom}(A,A)} Y)(a') \tag{2.5}$$

for any a, a' in A. We denote $\mathrm{Der}^{\mathsf{R}}(A)$ the $\Bbbk$-module of braided derivations of A (to lighten notation we often drop the label R). It is a K-submodule of $\mathrm{Hom}(A,A)$, with action given by the restriction of $\triangleright_{\mathrm{Hom}(A,A)}$

$$\begin{aligned} \triangleright_{\mathrm{Der}(A)} &: K \otimes \mathrm{Der}(A) \to \mathrm{Der}(A), \\ & k \otimes Y \mapsto k \triangleright_{\mathrm{Der}(A)} Y : a \mapsto k_{(1)} \triangleright Y(S(k_{(2)}) \triangleright a) \end{aligned} \tag{2.6}$$

and a braided Lie subalgebra of $\mathrm{Hom}(A,A)$ with braided commutator

$$\begin{aligned} [\ ,\ ] &: \mathrm{Der}(A) \otimes \mathrm{Der}(A) \to \mathrm{Der}(A), \\ & Y \otimes Y' \mapsto [Y,Y'] := Y \circ Y' - (\mathsf{R}_\alpha \triangleright_{\mathrm{Der}(A)} Y') \circ (\mathsf{R}^\alpha \triangleright_{\mathrm{Der}(A)} Y). \end{aligned} \tag{2.7}$$

Remark 2.1. When K is not finite dimensional over $\Bbbk$ the matrix R is in general not an element of $K \otimes K$ but rather belongs to a suitable topological completion of the tensor product algebra. In the examples of the present paper this fact does not constitute a problem since R acts diagonally in the fundamental representation and we consider representations that are algebraic direct sums of this one. Representation-wise we hence consider the braided monoidal category of K-modules that are algebraic direct sums of finite dimensional representations of K-modules.

2.2. Brackets of Lie algebra valued maps

Let N be a K-module and $\mathfrak{g}$ a braided Lie algebra. The space of $\mathfrak{g}$-valued multilinear maps $\eta : N \otimes \cdots \otimes N \to \mathfrak{g}$ which are braided antisymmetric in their arguments ($\mathfrak{g}$-valued forms on N), can be given the structure of a (super) braided Lie algebra. In the present paper we just need the bracket between one-forms and between a one-form and a two-form.

Given any two linear maps $\eta, \phi : N \to \mathfrak{g}$, their bracket is defined by

$$\begin{aligned} [\![\eta,\phi]\!](Y,Y') &:= \tfrac{1}{2}\big([\eta(\mathsf{R}_\gamma \triangleright Y), (\mathsf{R}^\gamma \triangleright \phi)(Y')] - (Y \otimes Y' \to \mathsf{R}_\alpha \triangleright Y' \otimes \mathsf{R}^\alpha \triangleright Y)\big) \\ &= \tfrac{1}{2}\big([\eta(\mathsf{R}_\gamma \triangleright Y), (\mathsf{R}^\gamma \triangleright \phi)(Y')] - [\eta(\mathsf{R}_\gamma \triangleright Y'), \mathsf{R}^\gamma \triangleright (\phi(Y))]\big) \end{aligned} \tag{2.8}$$

for $Y, Y' \in N$. The action on maps is the adjoint one in (2.4). By definition, the bracket is braided antisymmetric in the arguments, $[\![\eta,\phi]\!](Y,Y') = -[\![\eta,\phi]\!](\mathsf{R}_\alpha \triangleright Y', \mathsf{R}^\alpha \triangleright Y)$, while a short computation shows that

$$[\![\eta,\phi]\!] = [\![\mathsf{R}_\tau \triangleright \phi, \mathsf{R}^\tau \triangleright \eta]\!].$$

In particular, when $\phi = \eta$ the formula (2.8) can be written as

$$[\![\eta,\eta]\!](Y,Y') = \tfrac{1}{2}[\eta(\mathsf{R}_\gamma \triangleright Y) + \mathsf{R}_\gamma \triangleright (\eta(Y)), (\mathsf{R}^\gamma \triangleright \eta)(Y')]. \tag{2.9}$$

For a one-form η and a two-form Φ one defines

$$[\![\eta,\Phi]\!](X,Y,Z) := [\eta(\mathsf{R}_\gamma \triangleright X), (\mathsf{R}^\gamma \triangleright \Phi)(Y \otimes Z)] + \text{b.c.p.} \tag{2.10}$$

and

$$[\![\Phi,\eta]\!](X,Y,Z) := [\Phi(\mathsf{R}_\gamma \triangleright (X \otimes Y)), (\mathsf{R}^\gamma \triangleright \eta)(Z)] + \text{b.c.p.} \tag{2.11}$$

with b.c.p. standing for braided cyclic permutations of elements (X, Y, Z) in N. The braided antisymmetry of the two-form Φ implies the same property for both expressions above. A short computation shows that

$$[\![\eta,\Phi]\!] = -[\![\mathsf{R}_\lambda \triangleright \Phi, \mathsf{R}^\lambda \triangleright \eta]\!].$$

From this one also computes that

$$([\![\eta,\Phi]\!] - [\![\Phi,\eta]\!])(X,Y,Z) = [\eta(\mathsf{R}_\gamma \triangleright X) + \mathsf{R}_\gamma \triangleright (\eta(X)), (\mathsf{R}^\gamma \triangleright \Phi)(Y \otimes Z)] + \text{b.c.p.} \tag{2.12}$$

for $X, Y, Z \in N$, and in parallel with (2.9).

Finally, the K-equivariance of the bracket $[\ ,\ ]$ in $\mathfrak{g}$ leads to the K-equivariance of the bracket in (2.8),

$$k \triangleright [\![\eta,\phi]\!] = [\![k_{(1)} \triangleright \eta, k_{(2)} \triangleright \phi]\!],$$

and of those in (2.10) and (2.11) with similar expressions.

The bracket satisfies a braided Jacobi identity. For one-forms this reads as in the following proposition.

Proposition 2.2. *For one-forms ω, η, φ, we have the identity*

$$[\![\omega, [\![\eta, \varphi]\!]]\!] = [\![[\![\omega, \eta]\!], \varphi]\!] - [\![\mathsf{R}_\alpha \triangleright \eta, [\![\mathsf{R}^\alpha \triangleright \omega, \varphi]\!]]\!]. \tag{2.13}$$

Proof. We first compute separately the three terms in (2.13): from (2.10) and (2.8),

$$\begin{aligned} 2[\![\omega, [\![\eta, \varphi]\!]]\!](X, Y, Z) &= \big[\omega(\mathsf{R}_\lambda \mathsf{R}_\gamma \triangleright X), [(\mathsf{R}^\lambda \triangleright \eta)(\mathsf{R}_\alpha \triangleright Y), (\mathsf{R}^\alpha \mathsf{R}^\gamma \triangleright \varphi)(Z)]\big] \\ &\quad - \big[\omega(\mathsf{R}_\lambda \mathsf{R}_\gamma \triangleright X), [(\mathsf{R}^\lambda \triangleright \eta)(\mathsf{R}_\alpha \triangleright Z), \mathsf{R}^\alpha \triangleright ((\mathsf{R}^\gamma \triangleright \varphi)(Y))]\big] + \text{b.c.p.} \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &2[\![\mathsf{R}_\tau \triangleright \eta, [\![\mathsf{R}_\tau \triangleright \omega, \varphi]\!]]\!](X, Y, Z) \\ &\quad = \big[(\mathsf{R}_\tau \triangleright \eta)(\mathsf{R}_\lambda \mathsf{R}_\gamma \triangleright X), [(\mathsf{R}^\lambda \mathsf{R}^\tau \triangleright \omega)(\mathsf{R}_\alpha \triangleright Y), (\mathsf{R}^\alpha \mathsf{R}^\gamma \triangleright \varphi)(Z)]\big] \\ &\qquad - \big[(\mathsf{R}_\tau \triangleright \eta)(\mathsf{R}_\lambda \mathsf{R}_\gamma \triangleright X), [(\mathsf{R}^\lambda \mathsf{R}^\tau \triangleright \omega)(\mathsf{R}_\alpha \triangleright Z), \mathsf{R}^\alpha \triangleright ((\mathsf{R}^\gamma \triangleright \varphi)(Y))]\big] + \text{b.c.p..} \end{aligned} \tag{2.15}$$

While, from (2.11),

$$\begin{aligned} &2[\![[\![\omega, \eta]\!], \varphi]\!](X, Y, Z) \\ &\quad = \big[[\omega(\mathsf{R}_\lambda \mathsf{R}_\gamma \triangleright X), (\mathsf{R}^\lambda \triangleright \eta)(\mathsf{R}_\alpha \triangleright Y)], (\mathsf{R}^\alpha \mathsf{R}^\gamma \triangleright \varphi)(Z)\big] \\ &\qquad - \big[[\omega(\mathsf{R}_\lambda \mathsf{R}_\alpha \triangleright Y), \mathsf{R}^\lambda \triangleright (\eta(\mathsf{R}_\gamma \triangleright X))], (\mathsf{R}^\alpha \mathsf{R}^\gamma \triangleright \varphi)(Z)\big] + \text{b.c.p..} \end{aligned} \tag{2.16}$$

Using the braided cyclic permutation we substitute the second term in (2.15) with the following one:

$$\big[(\mathsf{R}_\tau \triangleright \eta)(\mathsf{R}_\lambda \mathsf{R}_\gamma \mathsf{R}_\mu \triangleright Y), [(\mathsf{R}^\lambda \mathsf{R}^\tau \triangleright \omega)(\mathsf{R}_\alpha \mathsf{R}^\nu \mathsf{R}^\mu \triangleright X), \mathsf{R}^\alpha \triangleright ((\mathsf{R}^\gamma \triangleright \varphi)(\mathsf{R}_\nu \triangleright Z))]\big]$$

which in turn, using (2.2), can be rewritten as

$$\big[\mathsf{R}_\tau \triangleright ((\mathsf{R}^\lambda \triangleright \eta)(\mathsf{R}^\nu \mathsf{R}_\gamma \mathsf{R}_\mu \triangleright Y)), [\mathsf{R}^\tau \triangleright (\omega(\mathsf{R}_\lambda \mathsf{R}_\nu \mathsf{R}_\alpha \mathsf{R}^\mu \triangleright X)), (\mathsf{R}^\alpha \mathsf{R}^\gamma \triangleright \varphi)(Z)]\big].$$

Next, using the Yang–Baxter equation on the indices ν, γ, α this reduces to

$$\begin{aligned} &\big[\mathsf{R}_\tau \triangleright ((\mathsf{R}^\lambda \triangleright \eta)(\mathsf{R}_\alpha \mathsf{R}^\nu \mathsf{R}_\mu \triangleright Y)), [\mathsf{R}^\tau \triangleright (\omega(\mathsf{R}_\lambda \mathsf{R}_\gamma \mathsf{R}_\nu \mathsf{R}^\mu \triangleright X)), (\mathsf{R}^\alpha \mathsf{R}^\gamma \triangleright \varphi)(Z)]\big] \\ &\quad = \big[\mathsf{R}_\tau \triangleright ((\mathsf{R}^\lambda \triangleright \eta)(\mathsf{R}_\alpha \triangleright Y)), [\mathsf{R}^\tau \triangleright (\omega(\mathsf{R}_\lambda \mathsf{R}_\gamma \triangleright X)), (\mathsf{R}^\alpha \mathsf{R}^\gamma \triangleright \varphi)(Z)]\big]. \end{aligned}$$

Finally, in the expression

$$\big([\![\omega, [\![\eta, \varphi]\!]]\!] - [\![[\![\omega, \eta]\!], \varphi]\!] + [\![\mathsf{R}_\tau \triangleright \eta, [\![\mathsf{R}^\tau \triangleright \omega, \varphi]\!]]\!]\big)(X, Y, Z)$$

one finds that this term and the two positive ones in (2.14) and (2.16) sum up to zero due to the Jacobi identity for derivations. A similar computation shows that the other three terms sum up to zero as well, thus establishing (2.13). ∎

2.3. Brackets of Lie algebra valued forms on bimodules

For a subalgebra $B \subseteq A$, the space $\mathrm{Hom}(A, A)$ is a left B-module via the left multiplication by elements in B. In general, this is not the case for $\mathrm{Der}(A)$. A sufficient condition for that is the quasi-centrality of B in A, that is

$$ba = (\mathsf{R}_\alpha \triangleright a)(\mathsf{R}^\alpha \triangleright b), \tag{2.17}$$

for all $b \in B$, $a \in A$. When the K-module algebra B is quasi-central in A, the braided Lie algebra $\mathrm{Der}(A)$ inherits the left B-module structure

$$(bY)(a) := bY(a), \tag{2.18}$$

for $Y \in \mathrm{Der}(A)$, $b \in B$ and $a \in A$. Then a right B-module structure is given by

$$Y \cdot b := (\mathsf{R}_\alpha \triangleright b)(\mathsf{R}^\alpha \triangleright Y). \tag{2.19}$$

We write $Y \cdot b$ to distinguish the right B-module structure from the evaluation of a derivation on an element in B. From the definition of the bimodule structure it follows compatibility with the K-action,

$$k \triangleright (bY \cdot b') = (k_{(1)} \triangleright b)(k_{(2)} \triangleright Y) \cdot (k_{(3)} \triangleright b')$$

for all $k \in K, b, b' \in B$ and $Y \in \mathrm{Der}(A)$. The Lie bracket of $\mathrm{Der}(A)$ then satisfies

$$\begin{aligned}[bY, Y' \cdot b'] &= b[Y, Y'] \cdot b' + b(\mathsf{R}_\alpha \triangleright Y') \cdot ((\mathsf{R}^\alpha \triangleright Y)(b')) \\ &\quad - ((\mathsf{R}_\alpha \mathsf{R}_\beta \triangleright_{\mathrm{Der}(A)} Y')(\mathsf{R}^\alpha \triangleright b))(\mathsf{R}^\beta \triangleright_{\mathrm{Der}(A)} Y) \cdot b'\end{aligned} \tag{2.20}$$

for all $b, b' \in B$ and $Y, Y' \in \mathrm{Der}(A)$.

Let N and N' be K-modules and B-bimodules as above. The space of right B-linear maps $\mathrm{Hom}_B(N, N')$ is a K-module with action

$$k \triangleright_{\mathrm{Hom}(N,N')} \eta : Y \mapsto k_{(1)} \triangleright (\eta(S(k_{(2)}) \triangleright Y)) \tag{2.21}$$

(cf. (2.4)) and a B-bimodule with actions $(b\eta)(Y) = b\eta(Y)$ and $\eta \cdot b = (\mathsf{R}_\alpha \triangleright b)(\mathsf{R}^\alpha \triangleright \eta)$, for $b \in B$, $\eta \in \mathrm{Hom}_B(N, N')$ and $Y \in N$.

With $\mathfrak{g}$ a braided Lie algebra over K, the definition of the bracket in Section 2.2 can be repeated for $\mathfrak{g}$-valued forms on N, i.e., $\mathfrak{g}$-valued right B-linear maps $\eta : N \otimes_B \cdots \otimes_B N \to \mathfrak{g}$ which are braided antisymmetric in their arguments. As we shall see this results into a (super) braided Lie–Rinehart algebra.

3. The Atiyah sequence of braided Lie algebras

The classical Atiyah sequence [6, 7] is a sequence of vector bundles over a base space M with Lie algebra structures on the corresponding modules of sections, this resulting in a short exact sequence of (infinite dimensional) Lie algebras. In this paper, we generalise the construction to a sequence of braided Lie algebras with braiding implemented by the triangular structure of a symmetry Hopf algebra K. In Section 4, we shall describe the basic example of this setting, the sequence of braided Lie algebras of a K-equivariant Hopf–Galois extension $B = A^{\mathrm{co}H} \subset A$, with structure Hopf algebra H.

3.1. The Atiyah sequence and the Lie–Rinehart structures

Let (K, R) be a triangular Hopf algebra and consider an exact sequence of K-braided Lie algebras,

$$0 \to \mathfrak{g} \xrightarrow{\iota} P \xrightarrow{\pi} T \to 0 \tag{3.1}$$

with ι and π braided Lie algebra morphisms. We identify $\mathfrak{g}$ with its image $\iota(\mathfrak{g})$ in P, while we write $\pi(Y) = Y^\pi$ for $Y \in P$, as the image in T via the projection π. By construction $\mathfrak{g}$ is an ideal in P for the braided commutator $[Y, V] \in \mathfrak{g}$, when $Y \in P$, $V \in \mathfrak{g}$.

We further take $\mathfrak{g}$, P and T to be B-bimodules with structures as in (2.18) and (2.19) for B a quasi-commutative algebra (that is, (2.17) holds for $A = B$) and ι, π to be right B-module morphisms (and hence left ones).

Finally, we take (B, T) to be a braided Lie–Rinehart pair: the algebra B is a T-module with elements of T acting as braided derivations of B,

$$X(bb') = X(b)b' + (\mathsf{R}_\alpha \triangleright b)(\mathsf{R}^\alpha \triangleright X)(b'), \quad b, b' \in B,\ X \in T,$$

and there is compatibility with the B-module structure of T,

$$[X, bX'] = X(b)X' + (\mathsf{R}_\alpha \triangleright b)[(\mathsf{R}^\alpha \triangleright X), X'], \quad b \in B,\ X, X' \in T, \tag{3.2}$$

(which also fixes $[bX, X' \cdot b']$, cf. (2.20)). The pair (B, P) is as well taken to be a braided Lie–Rinehart pair (and so is $(B, \mathfrak{g})$ for the trivial action): elements of P act as braided derivations of B via the map π, $Y(b) = Y^\pi(b)$, for $Y \in P$ and $b \in B$. The subalgebra $\mathfrak{g}$ acts trivially on B: elements of $\mathfrak{g}$ are 'vertical'. For simplicity we take $T = \mathrm{Der}^{\mathsf{R}}(B)$. The sequence (3.1) is a braided Lie algebroid with anchor π.

3.2. Splittings, connections and curvatures

A connection on the sequence (3.1) is a splitting of it, that is a right B-module map which is a section of π,

$$\rho : T \to P, \qquad \pi \circ \rho = \mathrm{id}_T. \tag{3.3}$$

In general, ρ is not K-equivariant, that is, it does not satisfy $k \triangleright \rho = \varepsilon(k)\rho$ for $k \in K$.

Using that $\rho(X)(b) = X(b)$ when $b \in B$ and recalling the adjoint action property $k \triangleright (\rho(X)) = (k_{(1)} \triangleright \rho)(k_{(2)} \triangleright X)$ for $X \in T$ and $k \in K$, one easily shows that

$$(k \triangleright \rho(X))(b) = (k \triangleright X)(b), \tag{3.4}$$

$$((k \triangleright \rho)(X))(b) = \varepsilon(k)X(b). \tag{3.5}$$

From the B-bimodule structure of T one also has $\rho(bX) = (\rho(\mathsf{R}_\alpha \triangleright X)) \cdot (\mathsf{R}^\alpha \triangleright b)$.

Remark 3.1. In accordance with the K-adjoint action $k \triangleright (\rho(X)) = (k_{(1)} \triangleright \rho)(k_{(2)} \triangleright X)$, the connection ρ 'acts from the left'. Therefore it is natural to take it to be right B-linear. When ρ is equivariant, right B-linearity also implies left B-linearity, $\rho(bX) = b\rho(X)$.

The corresponding vertical projection, the retract of $\imath$, is the right B-module map

$$\omega : P \to \mathfrak{g}, \quad \omega(Y) := Y - \rho(Y^{\pi}), \quad Y \in P. \tag{3.6}$$

Clearly $\omega(\rho(X)) = 0$, for each $X \in T$. We denote

$$h := \rho \circ \pi : P \to P \tag{3.7}$$

the horizontal projection. By definition $\omega + h = \mathrm{id}$.

In general, ρ (and so ω) is not a braided Lie algebra morphism. The extent to which it fails to be such is measured by the *(basic)* curvature

$$\begin{aligned} \Omega(X, X') &:= \rho([X, X']) - [\![\rho, \rho]\!](X, X') \\ &= \rho([X, X']) - \tfrac{1}{2}\big([\rho(\mathsf{R}_\alpha \triangleright X), (\mathsf{R}^\alpha \triangleright \rho)(X')] - [\rho(\mathsf{R}_\alpha \triangleright X'), \mathsf{R}^\alpha \triangleright (\rho(X))]\big), \end{aligned} \tag{3.8}$$

for $X, X' \in T$. From the K-equivariance of π and $\pi \circ \rho = \mathrm{id}_T$ it follows that $\pi \circ \Omega = 0$, and so Ω is $\mathfrak{g}$-valued. Also, from $\omega \circ \rho = 0$ it follows that

$$\Omega(X, X') = -(\omega \circ [\![\rho, \rho]\!])(X, X').$$

A braided antisymmetric right B-linear map from $T \otimes_B T$ to $\mathfrak{g}$ is called a $\mathfrak{g}$-valued braided two-form on T.

Proposition 3.2. *The curvature Ω is a $\mathfrak{g}$-valued braided two-form on T.*

Proof. We first show that Ω is well defined on $T \otimes_B T$, that is, $\Omega(X, bX') = \Omega(X \cdot b, X')$, for $b \in B$ and $X, X' \in T$. Indeed, using (3.2) and (2.19), we compute

$$\begin{aligned} [X, bX'] &= X(b)X' + (\mathsf{R}_\alpha \triangleright b)[\mathsf{R}^\alpha \triangleright X, X'] \\ &= (\mathsf{R}_\alpha \triangleright X') \cdot \mathsf{R}^\alpha \triangleright (X(b)) + \mathsf{R}_\beta \triangleright ([\mathsf{R}^\alpha \triangleright X, X']) \cdot (\mathsf{R}_\beta \mathsf{R}_\alpha \triangleright (b)) \\ &= (\mathsf{R}_\alpha \triangleright X') \cdot \mathsf{R}^\alpha \triangleright (X(b)) + [X, \mathsf{R}_\alpha \triangleright X'] \cdot (\mathsf{R}^\alpha \triangleright b) \end{aligned}$$

and

$$[X \cdot b, X'] = [X, \mathsf{R}_\alpha \triangleright X'] \cdot (\mathsf{R}^\alpha \triangleright b) + X \cdot (\mathsf{R}_\alpha \triangleright X')(\mathsf{R}^\alpha \triangleright b).$$

Then, the right B-linearity of ρ yields

$$\rho([X, bX']) - \rho([X \cdot b, X']) = \rho(\mathsf{R}_\alpha \triangleright X') \cdot \mathsf{R}^\alpha \triangleright (X(b)) - \rho(X) \cdot (\mathsf{R}_\alpha \triangleright X')(\mathsf{R}^\alpha \triangleright b). \tag{3.9}$$

Next, using the expression in (2.9) for the bracket,

$$\begin{aligned}
2[\![\rho,\rho]\!](X,bX') &= [\rho(\mathsf{R}_\alpha \triangleright X) + \mathsf{R}_\alpha \triangleright (\rho(X)), (\mathsf{R}^\alpha \triangleright \rho)(bX')] \\
&= [\rho(\mathsf{R}_\alpha \triangleright X) + \mathsf{R}_\alpha \triangleright (\rho(X)), (\mathsf{R}^\alpha \triangleright \rho)(\mathsf{R}_\tau \triangleright X')] \cdot (\mathsf{R}^\tau \triangleright b) \\
&\quad + \mathsf{R}_\sigma \triangleright ((\mathsf{R}^\alpha \triangleright \rho)(\mathsf{R}_\tau \triangleright X')) \\
&\quad \cdot (\mathsf{R}^\sigma \triangleright (\rho(\mathsf{R}_\alpha \triangleright X) + \mathsf{R}_\alpha \triangleright (\rho(X)))(\mathsf{R}^\tau \triangleright b)) \\
&= 2[\![\rho,\rho]\!](X,\mathsf{R}_\tau X') \cdot (\mathsf{R}^\tau \triangleright b) \\
&\quad + 2\mathsf{R}_\sigma \triangleright ((\mathsf{R}^\alpha \triangleright \rho)(\mathsf{R}_\tau \triangleright X')) \cdot ((\mathsf{R}^\sigma \mathsf{R}_\alpha \triangleright X)(\mathsf{R}^\tau \triangleright b)) \\
&= 2[\![\rho,\rho]\!](X,\mathsf{R}_\tau X') \cdot (\mathsf{R}^\tau \triangleright b) + 2\rho(\mathsf{R}_\tau \triangleright X') \cdot \mathsf{R}^\tau \triangleright (X(b)), \qquad (3.10)
\end{aligned}$$

using (3.4) in the last but one equality. While, using $(k \triangleright \rho)(X \cdot b) = (k \triangleright \rho)(X) \cdot b$,

$$\begin{aligned}
&2[\![\rho,\rho]\!](X \cdot b, X') \\
&\quad = [\rho(\mathsf{R}_\alpha \triangleright (X \cdot b)) + \mathsf{R}_\alpha \triangleright (\rho(X \cdot b)), (\mathsf{R}^\alpha \triangleright \rho)(X')] \\
&\quad = [(\rho(\mathsf{R}_\alpha \triangleright X) + \mathsf{R}_\alpha \triangleright (\rho(X))) \cdot (\mathsf{R}_\beta \triangleright b), (\mathsf{R}^\beta \mathsf{R}^\alpha \triangleright \rho)(X')] \\
&\quad = [(\rho(\mathsf{R}_\alpha \triangleright X) + \mathsf{R}_\alpha \triangleright (\rho(X))), \mathsf{R}_\lambda \triangleright ((\mathsf{R}^\beta \mathsf{R}^\alpha \triangleright \rho)(X'))] \cdot (\mathsf{R}^\lambda \mathsf{R}_\beta \triangleright b) \\
&\qquad + (\rho(\mathsf{R}_\alpha \triangleright X) + \mathsf{R}_\alpha \triangleright (\rho(X))) \cdot \mathsf{R}_\tau \triangleright ((\mathsf{R}^\beta \mathsf{R}^\alpha \triangleright \rho)(X'))(\mathsf{R}^\tau \mathsf{R}_\beta \triangleright b) \\
&\quad = [(\rho(\mathsf{R}_\alpha \triangleright X) + \mathsf{R}_\alpha \triangleright (\rho(X))), \mathsf{R}_\lambda \triangleright ((\mathsf{R}^\beta \mathsf{R}^\alpha \triangleright \rho)(X'))] \cdot (\mathsf{R}^\lambda \mathsf{R}_\beta \triangleright b) \\
&\qquad + 2\rho(X) \cdot (\mathsf{R}_\tau \triangleright X')(\mathsf{R}^\tau \triangleright b), \qquad (3.11)
\end{aligned}$$

using (3.5), in the last but one equality. Now, a direct computation shows that

$$\begin{aligned}
&[(\rho(\mathsf{R}_\alpha \triangleright X) + \mathsf{R}_\alpha \triangleright (\rho(X))), \mathsf{R}_\lambda \triangleright ((\mathsf{R}^\beta \mathsf{R}^\alpha \triangleright \rho)(X'))] \cdot (\mathsf{R}^\lambda \mathsf{R}_\beta \triangleright b) \\
&\quad = [\rho(\mathsf{R}_\alpha \triangleright X) + \mathsf{R}_\alpha \triangleright (\rho(X)), (\mathsf{R}^\alpha \triangleright \rho)(\mathsf{R}_\tau \triangleright X')] \cdot (\mathsf{R}^\tau \triangleright b) \\
&\quad = 2[\![\rho,\rho]\!](X,\mathsf{R}_\tau X') \cdot (\mathsf{R}^\tau \triangleright b).
\end{aligned}$$

With this, comparing (3.10) and (3.11), we obtain

$$\begin{aligned}
&[\![\rho,\rho]\!](X,bX') - [\![\rho,\rho]\!](X \cdot b, X') \\
&\quad = \rho(\mathsf{R}_\tau \triangleright X') \cdot \mathsf{R}^\tau \triangleright (X(b)) - \rho(X) \cdot (\mathsf{R}_\tau \triangleright X')(\mathsf{R}^\tau \triangleright b)
\end{aligned}$$

which, when compared with (3.9), amounts to $\Omega(X,bX') - \Omega(X \cdot b, X') = 0$.

Next we show that Ω is right B-linear, that is, $\Omega(X, X' \cdot b) = \Omega(X, X') \cdot b$. From $[X, X' \cdot b] = [X, X'] \cdot b + (\mathsf{R}_\alpha \triangleright X') \cdot (\mathsf{R}_\alpha \triangleright X)(b)$, the right B-linearity of ρ gives

$$\rho([X, X' \cdot b]) = \rho([X, X']) \cdot b + \rho(\mathsf{R}_\alpha \triangleright X') \cdot (\mathsf{R}^\alpha \triangleright X)(b). \qquad (3.12)$$

On the other hand, recalling that the K-action defined in (2.21) closes on right B-linear maps, $(k \triangleright \rho)(X \cdot b) = (k \triangleright \rho)(X) \cdot b$ and using (3.5), one computes

$$\begin{aligned} 2[\![\rho,\rho]\!](X, X'\cdot b) &= [\rho(\mathsf{R}_\alpha \triangleright X) + \mathsf{R}_\alpha \triangleright (\rho(X)), (\mathsf{R}^\alpha \triangleright \rho)(X'\cdot b)] \\ &= [\rho(\mathsf{R}_\alpha \triangleright X) + \mathsf{R}_\alpha \triangleright (\rho(X)), (\mathsf{R}^\alpha \triangleright \rho)(X')]\cdot b \\ &\quad + \mathsf{R}_\tau \triangleright ((\mathsf{R}^\alpha \triangleright \rho)(X'))\cdot(\mathsf{R}^\tau \triangleright (\rho(\mathsf{R}_\alpha \triangleright X) + \mathsf{R}_\alpha \triangleright (\rho(X))))(b) \\ &= 2[\![\rho,\rho]\!](X, X')\cdot b + 2\rho(\mathsf{R}_\tau \triangleright X')\cdot(\mathsf{R}^\tau \triangleright X)(b). \end{aligned}$$

A comparison with (3.12) yields $\Omega(X, X'\cdot b) - \Omega(X, X')\cdot b = 0$.

Finally, both terms in (3.8) are braided antisymmetric and hence so is the curvature: $\Omega(X, X') = -\Omega(\mathsf{R}_\alpha \triangleright X', \mathsf{R}^\alpha \triangleright X)$. ■

The curvature can be given as $\mathfrak{g}$-valued braided two-form on P (the *spatial* curvature),

$$\Omega_\omega(Y, Y') := \Omega(Y^\pi, Y'^\pi), \quad Y, Y' \in P. \tag{3.13}$$

This turns out to be *basic*, that is, $\Omega_\omega(Y, Y') = 0$ whenever Y or Y' is vertical (an element of $\mathfrak{g}$). Using (3.6) one also computes

$$\Omega_\omega(Y, Y') = [\omega(Y), Y'] + [\mathsf{R}_\alpha \triangleright Y, (\mathsf{R}^\alpha \triangleright \omega)(Y')] - \omega([Y, Y']) - [\![\omega,\omega]\!](Y, Y').$$

This expression can be read as a *structure equation*,

$$d\omega = \Omega_\omega + [\![\omega,\omega]\!]. \tag{3.14}$$

Here the $\mathfrak{g}$-valued two-form $[\![\omega,\omega]\!]$ on P is defined as in (2.8). Also, given a $\mathfrak{g}$-valued one-form η on P, its exterior derivative $d\eta$ is the $\mathfrak{g}$-valued two-form on P defined by

$$\begin{aligned} d\eta(Y, Y') &:= [\mathsf{R}_\alpha \triangleright Y, (\mathsf{R}^\alpha \triangleright \eta)(Y')] - [\mathsf{R}_\alpha \triangleright Y', \mathsf{R}^\alpha \triangleright (\eta(Y))] - \eta([Y, Y']) \\ &= 2[\![\mathrm{id}, \eta]\!](Y, Y') - \eta([Y, Y']) \end{aligned} \tag{3.15}$$

for $Y, Y' \in P$. The above is well defined since $\mathfrak{g}$, as mentioned, is an ideal in P for the braided commutator and braided antisymmetric by construction.

One could construct an exterior algebra of $\mathfrak{g}$-valued forms on P by extending the definition of the exterior derivative d to a form of any degree. For the sake of the present paper we just need it on one- and two-forms. For Φ a $\mathfrak{g}$-valued two-form on P, its exterior derivative is given as

$$d\Phi(X, Y, Z) := [\mathsf{R}_\lambda \triangleright X, (\mathsf{R}^\lambda \triangleright \Phi)(Y, Z)] + \Phi(X, [Y, Z]) + \text{b.c.p.} \tag{3.16}$$

for $X, Y, Z \in P$, and b.c.p. standing for braided cyclic permutations of (X, Y, Z). The result is a $\mathfrak{g}$-valued three-form on P. Using this definition and (3.15), a lengthy and intricate computation that uses the Jacobi identity and the Yang–Baxter equation shows $d^2 = 0$.

3.3. The covariant derivative

Having a connection, one can define the covariant derivative of (in particular) $\mathfrak{g}$-valued braided forms on P. This uses the horizontal projection h in (3.7). For a one-form η we define

$$\begin{aligned} D\eta(X,Y) &:= \big([h(\mathsf{R}_\tau \triangleright X), (\mathsf{R}^\tau \triangleright \eta)(Y)] - (X \otimes Y \to \mathsf{R}_\alpha \triangleright Y \otimes \mathsf{R}^\alpha \triangleright X)\big) \\ &\quad - \eta([X,Y]) \\ &= 2[\![h,\eta]\!](X,Y) - \eta([X,Y]), \end{aligned} \tag{3.17}$$

while for a two-form Φ its covariant derivative is defined as

$$\begin{aligned} D\Phi(X,Y,Z) &:= \tfrac{1}{2}[h(\mathsf{R}_\tau \triangleright X) + \mathsf{R}_\tau \triangleright (h(X)), (\mathsf{R}^\tau \triangleright \Phi)(Y,Z)] + \Phi(X,[Y,Z]) \\ &\quad + \text{b.c.p.}. \\ &= \tfrac{1}{2}([\![h,\Phi]\!] - [\![\Phi,h]\!])(X,Y,Z) + (\Phi(X,[Y,Z]) + \text{b.c.p.}) \end{aligned} \tag{3.18}$$

using (2.12) for the second equality. These definitions can be generalised to forms of any degree, obtaining braided versions of the classical formulas [8, eq. (I.1)], [13, eq. (19)].

From (3.13) and (3.8) the curvature is written in terms of the horizontal projection as

$$\Omega_\omega(Y,Y') = h([Y,Y']) - [\![h,h]\!](Y,Y'). \tag{3.19}$$

Then one shows the following proposition.

Proposition 3.3. *For the connection and the curvature there is a structure equation*

$$\Omega_\omega = D\omega + [\![\omega,\omega]\!]$$

and a Bianchi identity

$$D\Omega_\omega = \tfrac{1}{2}[\![\mathsf{R}_\alpha \triangleright h, [\![\mathsf{R}^\alpha \triangleright h, h]\!]]\!]. \tag{3.20}$$

The covariant derivative $D\Omega_\omega$ vanishes for an equivariant connection.

Proof. From the structure equation (3.14) and (3.15), and from the definition (3.17) of the covariant derivative we have

$$\Omega_\omega = 2[\![\mathrm{id},\omega]\!] - \omega\circ[\ ,\ ] - [\![\omega,\omega]\!] = 2[\![h,\omega]\!] - \omega\circ[\ ,\ ] + [\![\omega,\omega]\!] = D\omega + [\![\omega,\omega]\!].$$

Then, for $X,Y,Z \in P$, from definitions (3.18) and (3.19) one computes

$$\begin{aligned} D\Omega_\omega(X,Y,Z) &= \tfrac{1}{2}[h(\mathsf{R}_\gamma \triangleright X) + \mathsf{R}_\gamma \triangleright (h(X)), (\mathsf{R}^\gamma \triangleright h)([Y,Z])] \\ &\quad - \tfrac{1}{2}[h(\mathsf{R}_\gamma \triangleright X) + \mathsf{R}_\gamma \triangleright (h(X)), (\mathsf{R}^\gamma \triangleright [\![h,h]\!])(Y\otimes Z)] \\ &\quad + h([X,[Y,Z]]) - [\![h,h]\!](X\otimes[Y,Z]) + \text{b.c.p.}. \end{aligned}$$

The term $h([X,[Y,Z]]) + \text{b.c.p.}$ vanishes by the Jacobi identity while the first and fourth terms cancel each other. The remaining one, the second term, then gives

$$D\Omega_\omega = -\tfrac{1}{2}([\![h,[\![h,h]\!]]\!] - [\![[\![h,h]\!],h]\!]) = \tfrac{1}{2}[\![\mathsf{R}_\alpha \triangleright h, [\![\mathsf{R}^\alpha \triangleright h, h]\!]]\!].$$

The second equality follows from the Jacobi identity (2.13). When the connection is equivariant, $k \triangleright h = \varepsilon(k)h$ and the right-hand side of (3.20) vanishes by the Jacobi identity,

$$([\![h, [\![h, h]\!]]\!])(X, Y, Z) = [h(X), [h(Y), h(Z)]] + \text{b.c.p.} = 0.$$

Then the relation in (3.20) is the usual Bianchi identity. ∎

An easy computation also gives that the covariant derivative of the curvature as in (3.18) can be written as

$$D\Omega_\omega = d\Omega_\omega - \tfrac{1}{2}([\![\omega, \Omega_\omega]\!] - [\![\Omega_\omega, \omega]\!]).$$

Remark 3.4. Our result in (3.20) compares with a similar one in [11, §5.4]. In the (dual) notation of [12] it says that our connection is regular but not necessarily multiplicative.

3.4. The space of connections and the gauge transformations

The space $C(T, \mathfrak{g})$ of connections on the sequence (3.1) is an affine space modelled on the linear space of right B-module maps $\eta : T \to \mathfrak{g}$ (the $\mathfrak{g}$-valued one-forms on B). Indeed, given a connection $\rho : T \to P$ and such a map $\eta : T \to \mathfrak{g}$, one has

$$\pi \circ (\rho + \eta) = \pi \circ \rho = \mathrm{id}_T$$

and thus $\rho' = \rho + \eta$ is a connection as well, with $\omega_{\rho'} = \omega_\rho - \eta \circ \pi$. In the examples of the present paper we shall use this decomposition with ρ a K-equivariant connection.

The space of connections $C(T, \mathfrak{g})$ is a subset of the K-module $\mathrm{Hom}_B(T, P)$. The latter carries an action of the braided Lie algebra P (cf. [4, §5.3]).

Proposition 3.5. *The map* $\delta : P \otimes \mathrm{Hom}_B(T, P) \to \mathrm{Hom}_B(T, P)$ *given by*

$$(\delta_Y \rho)(X) := [Y, \rho(X)] - (\mathsf{R}_\alpha \triangleright \rho)([\mathsf{R}^\alpha \triangleright Y^\pi, X]) \tag{3.21}$$

is an action of the K*-braided Lie algebra* P*, that is,*

$$k \triangleright (\delta_Y \rho) = \delta_{k_{(1)} \triangleright Y}(k_{(2)} \triangleright \rho) \tag{3.22}$$

and for $Y, Y' \in P$*,*

$$\delta_{[Y,Y']} = \delta_Y \circ \delta_{Y'} - \delta_{\mathsf{R}_\alpha \triangleright Y'} \circ \delta_{\mathsf{R}^\alpha \triangleright Y}. \tag{3.23}$$

Proof. Formula (3.22) follows from the quasi-cocommutativity (2.1) and the explicit expression (2.21) for the K-action on morphisms. Then, using formula (3.22), we compute

$$\begin{aligned}(\delta_Y(\delta_{Y'}\rho))(X) &= [Y, \delta_{Y'}\rho(X)] - (\mathsf{R}_\alpha \triangleright \delta_{Y'}\rho)([\mathsf{R}^\alpha \triangleright Y^\pi, X]) \\ &= [Y, [Y', \rho(X)]] - [Y, (\mathsf{R}_\gamma \triangleright \rho)([\mathsf{R}^\gamma \triangleright Y'^\pi, X])] \\ &\quad - [\mathsf{R}_\alpha \triangleright Y', (\mathsf{R}_\beta \triangleright \rho)([\mathsf{R}^\beta \mathsf{R}^\alpha \triangleright Y^\pi, X])] \\ &\quad + (\mathsf{R}_\gamma \mathsf{R}_\beta \triangleright \rho)([\mathsf{R}^\gamma \mathsf{R}_\alpha \triangleright Y'^\pi, [\mathsf{R}^\beta \mathsf{R}^\alpha \triangleright Y^\pi, X]]).\end{aligned}$$

This also yields

$$
\begin{aligned}
&(\delta_{\mathsf{R}_\alpha \triangleright Y'}(\delta_{\mathsf{R}^\alpha \triangleright Y}\rho))(X) \\
&\quad = [\mathsf{R}_\alpha \triangleright Y', [\mathsf{R}^\alpha \triangleright Y, \rho(X)]] - [\mathsf{R}_\alpha \triangleright Y', (\mathsf{R}_\gamma \triangleright \rho)([\mathsf{R}^\gamma \mathsf{R}^\alpha \triangleright Y^\pi, X])] \\
&\qquad - [Y, (\mathsf{R}_\beta \triangleright \rho)([\mathsf{R}^\beta \triangleright Y'^\pi, X])] + (\mathsf{R}_\gamma \mathsf{R}_\beta \triangleright \rho)([\mathsf{R}^\gamma \triangleright Y^\pi, [\mathsf{R}^\beta \triangleright Y'^\pi, X]]).
\end{aligned}
$$

These two expressions have two terms in common so their difference is given by

$$
\begin{aligned}
&\delta_Y(\delta_{Y'}\rho)(X) - \delta_{\mathsf{R}_\alpha \triangleright Y'}(\delta_{\mathsf{R}^\alpha \triangleright Y}\rho)(X) \\
&\quad = [Y, [Y', \rho(X)]] - [\mathsf{R}_\alpha \triangleright Y', [\mathsf{R}^\alpha \triangleright Y, \rho(X)]] \\
&\qquad + (\mathsf{R}_\gamma \mathsf{R}_\beta \triangleright \rho)\big([\mathsf{R}^\gamma \mathsf{R}_\alpha \triangleright Y'^\pi, [\mathsf{R}^\beta \mathsf{R}^\alpha \triangleright Y^\pi, X]] - [\mathsf{R}^\gamma \triangleright Y^\pi, [\mathsf{R}^\beta \triangleright Y'^\pi, X]]\big) \\
&\quad = [Y, [Y', \rho(X)]] - [\mathsf{R}_\alpha \triangleright Y', [\mathsf{R}^\alpha \triangleright Y, \rho(X)]] \\
&\qquad + (\mathsf{R}_\gamma \mathsf{R}_\beta \triangleright \rho)\big([\mathsf{R}_\alpha \mathsf{R}^\beta \triangleright Y'^\pi, [\mathsf{R}^\alpha \mathsf{R}^\gamma \triangleright Y^\pi, X]] - [\mathsf{R}^\gamma \triangleright Y^\pi, [\mathsf{R}^\beta \triangleright Y'^\pi, X]]\big)
\end{aligned}
$$

using the Yang–Baxter equation for the second summand. Finally, the Jacobi identity gives

$$
\begin{aligned}
&\delta_Y(\delta_{Y'}\rho)(X) - \delta_{\mathsf{R}_\alpha \triangleright Y'}(\delta_{\mathsf{R}^\alpha \triangleright Y}\rho)(X) \\
&\quad = [[Y, Y'], \rho(X)] - (\mathsf{R}_\beta \triangleright \rho)([\mathsf{R}^\beta \triangleright [Y^\pi, Y'^\pi], X])
\end{aligned}
$$

which is just $(\delta_{[Y,Y']}\rho)(X)$ and coincides with the right-hand side of (3.23). ■

The braided Lie algebra action of P on $\mathrm{Hom}_B(T, P)$ gives rise to a map

$$P \times C(T, \mathfrak{g}) \to C(T, \mathfrak{g}), \quad (Y, \rho) \mapsto \rho' = \rho + \delta_Y \rho.$$

Indeed, from $\pi \circ \rho = \mathrm{id}_T$, it follows that $\pi \circ (\delta_Y \rho) = 0$ and so $\delta_Y \rho : T \to \mathfrak{g}$.

Proposition 3.5 can be generalised. Let M be a right B-module with a compatible P-action $\triangleright_P$. Then there is a P-action $\delta : P \otimes \mathrm{Hom}_B(M, P) \to \mathrm{Hom}_B(M, P)$,

$$(\delta_Y \Phi)(m) := [Y, \Phi(m)] - (\mathsf{R}_\alpha \triangleright \Phi)((\mathsf{R}^\alpha \triangleright Y^\pi) \triangleright_P m). \tag{3.24}$$

Since the P-action on $T \otimes T$ is by braided derivations, when Φ is a P-valued two-form on T, the above becomes

$$
\begin{aligned}
(\delta_Y \Phi)(X, X') = {}& [Y, \Phi(X, X')] - (\mathsf{R}_\alpha \triangleright \Phi)([\mathsf{R}^\alpha \triangleright Y^\pi, X], X') \\
& - (\mathsf{R}_\alpha \triangleright \Phi)([\mathsf{R}_\lambda \triangleright X, [\mathsf{R}^\lambda \mathsf{R}^\alpha \triangleright Y^\pi, X']]).
\end{aligned} \tag{3.25}
$$

Proposition 3.6. *The variation $\delta_Y \Omega$ of the curvature of the connection ρ for the action of an element $Y \in P$, as defined in (3.25), explicitly reads*

$$\delta_Y \Omega = (\delta_Y \rho) \circ [\ ,\ ] - [\![\delta_Y \rho, \rho]\!] - [\![\mathsf{R}_\alpha \triangleright \rho, \delta_{\mathsf{R}^\alpha \triangleright Y}\rho]\!].$$

Proof. This follows from linearity of the action δ_Y and from its braided derivation rule. ■

In analogy with the classical case, an element $V \in \mathfrak{g}$ acts on a connection and on the corresponding curvature as an infinitesimal gauge transformation.

Corollary 3.7. *The variation* (3.21) *of a connection* ρ *for the action of a vertical element* $V \in \mathfrak{g}$ *is given by*

$$(\delta_V \rho)(X) = [V, \rho(X)],$$

while the variation (3.25) *of the curvature is*

$$(\delta_V \Omega)(X, X') = [V, \Omega(X, X')].$$

Then in view of (3.23), $\mathfrak{g}$ is the braided Lie algebra of infinitesimal gauge transformations. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is the braided Hopf algebra of such transformations [4, §5.3].

Remark 3.8. Let Ω' be the curvature of the transformed connection $\rho' = \rho + \delta_Y \rho$, for $Y \in P$. Since the action δ_Y is a braided derivation, the variation $\delta_Y \Omega$ differs from the first order term in the difference $\Omega' - \Omega$ (which by construction is a derivation),

$$\Omega' - \Omega = \delta_Y \Omega + [\![\mathsf{R}_\beta \triangleright \rho, \delta_{\mathsf{R}^\beta \triangleright Y} \rho]\!] - [\![\rho, \delta_Y \rho]\!] + [\![\delta_Y \rho, \delta_Y \rho]\!]. \tag{3.26}$$

When the splitting ρ is K-equivariant, $k \triangleright \rho = \varepsilon(k)\rho$, the variations of the connection and of the curvature reduce to

$$\begin{aligned}(\delta_Y \rho)(X) &= [Y, \rho(X)] - \rho([Y^\pi, X]),\\ \delta_Y \Omega(X, X') &= [Y, \Omega(X, X')] - \Omega([Y^\pi, X], X') - \Omega(\mathsf{R}_\alpha \triangleright X, [\mathsf{R}^\alpha \triangleright Y^\pi, X']).\end{aligned} \tag{3.27}$$

Moreover, the extra term in the right-hand side of (3.26) vanishes.

4. Atiyah sequences for Hopf–Galois extensions

Recall that an algebra A is a right H-comodule algebra for a Hopf algebra H if it carries a right coaction $\delta : A \to A \otimes H$ which is a morphism of algebras. We write $\delta(a) = a_{(0)} \otimes a_{(1)}$ in Sweedler notation with an implicit sum. The subspace of coinvariants $B := A^{\mathrm{co}H} = \{b \in A \mid \delta(b) = b \otimes 1_H\}$ is a subalgebra of A. There is a canonical map

$$\chi := (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes_B \delta) : A \otimes_B A \to A \otimes H, \quad a' \otimes_B a \mapsto a' a_{(0)} \otimes a_{(1)}.$$

The algebra extension $B \subseteq A$ is called an H-Galois extension if this map is bijective.

In the present paper, we deal with H-Galois extensions which are K-equivariant. That is, A also carries a left K-action $\triangleright : K \otimes A \to A$, for K a Hopf algebra, that commutes with the right H-coaction, $\delta \circ \triangleright = (\triangleright \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta)$ (the coaction δ is a K-module map where H has trivial K-action). On elements $k \in K, a \in A$,

$$(k \triangleright a)_{(0)} \otimes (k \triangleright a)_{(1)} = (k \triangleright a_{(0)}) \otimes a_{(1)}.$$

Given a K-equivariant Hopf–Galois extension $B = A^{\mathrm{co}H} \subseteq A$, with triangular Hopf algebra (K, R), the Lie algebra $\mathrm{Der}(A)$ of K-braided derivations of A has two distinguished Lie subalgebras. Firstly, the Lie subalgebra of braided derivations that are H-equivariant (that is, H-comodule maps),

$$\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) = \{X \in \mathrm{Der}(A) \mid \delta(X(a)) = X(a_{(0)}) \otimes a_{(1)},\ a \in A\} \tag{4.1}$$

and then its Lie subalgebra of vertical derivations

$$\mathrm{aut}^{\mathsf{R}}_B(A) := \{X \in \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \mid X(b) = 0,\ b \in B\}.$$

The linear spaces $\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A)$ and $\mathrm{aut}^{\mathsf{R}}_B(A)$ are K-braided Lie subalgebras of $\mathrm{Der}(A)$, [4, Prop. 7.2]. Elements of $\mathrm{aut}^{\mathsf{R}}_B(A)$ are regarded as infinitesimal gauge transformations of the K-equivariant Hopf–Galois extension $B = A^{\mathrm{co}H} \subseteq A$, [4, Def. 7.1].

Each derivation in $\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A)$, being H-equivariant, restricts to a derivation on the subalgebra of coinvariant elements $B = A^{\mathrm{co}H}$. Thus, associated to $B = A^{\mathrm{co}H} \subseteq A$, there is the sequence of braided Lie algebras $\mathrm{aut}^{\mathsf{R}}_B(A) \to \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \to \mathrm{Der}^{\mathsf{R}}(B)$. When exact,

$$0 \to \mathrm{aut}^{\mathsf{R}}_B(A) \to \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \to \mathrm{Der}^{\mathsf{R}}(B) \to 0 \tag{4.2}$$

is a version of the Atiyah sequence of a (commutative) principal fibre bundle.

When the K-module algebra B is quasi-central in A, see (2.17), the braided Lie algebras in the sequence are also B-bimodules with module structures as in (2.18) and (2.19). The above is then a sequence of braided Lie–Rinehart pairs as in Section 3.1.

As studied in the previous section, a connection on the bundle can be given as an H-equivariant splitting of the sequence, associating to a derivation X in $\mathrm{Der}^{\mathsf{R}}(B)$ a unique horizontal derivation in $\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A)$ projecting onto X. The curvature of the connection measures the extend to which this map fails to be a braided Lie algebra morphism.

An example of this construction with the study of moduli spaces of connections is given in the next two subsections. In Section 5 we present a sequence for the orthogonal frame bundle over the sphere S^{2n}_θ with a splitting which leads to the Levi-Civita connection.

4.1. The sequence for the SU(2)-bundle over the sphere S^4_θ

This section is devoted to the Atiyah sequence of braided Lie algebras associated with the $\mathcal{O}(\mathrm{SU}(2))$ Hopf–Galois extension $\mathcal{O}(S^4_\theta) \subset \mathcal{O}(S^7_\theta)$ constructed in [15] and related connections.

Let $\theta \in \mathbb{R}$. The $*$-algebra $\mathcal{O}(S^7_\theta)$ has generators z_r, z_r^*, $r = 1, 2, 3, 4$ with commutation relations determined by the action of a 2-torus. For K the Hopf algebra generated by two commuting elements H_1, H_2, the generators z_r are eigenfunctions for the action of H_1, H_2 of eigenvalues $\mu^r = (\mu^r_1, \mu^r_2) = \frac{1}{2}(1, -1)$, $\frac{1}{2}(-1, 1)$, $\frac{1}{2}(-1, -1)$, $\frac{1}{2}(1, 1)$, for $r = 1, 2, 3, 4$, and so are their $*$-conjugated z_r^* with eigenvalues $-\mu^r$. Then, the commutation relations among the z_r read

$$z_r \bullet_\theta z_s = \lambda^{2 r \wedge s} z_s \bullet_\theta z_r, \quad \lambda = e^{-\pi i \theta}, \quad r \wedge s := \mu^r_1 \mu^s_2 - \mu^r_2 \mu^s_1, \tag{4.3}$$

for $r, s = \pm1, \pm2, \pm3, \pm4$ and $z_{-r} := z_r^*$. In addition, the generators satisfy a sphere relation $z_1 \bullet_\theta z_1^* + z_2 \bullet_\theta z_2^* + z_3 \bullet_\theta z_3^* + z_4 \bullet_\theta z_4^* = 1$.

The algebra $\mathcal{O}(S^7_\theta)$ is a right $\mathcal{O}(\mathrm{SU}(2))$-comodule algebra with right coaction which is defined on the algebra generators as

$$\delta : \mathcal{O}(S^7_\theta) \to \mathcal{O}(S^7_\theta) \otimes \mathcal{O}(\mathrm{SU}(2)),$$
$$\mathsf{u} \mapsto \mathsf{u} \dot{\otimes} \mathsf{w}, \quad \mathsf{u} := \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ -z_2^* & z_1^* & -z_4^* & z_3^* \end{pmatrix}^t, \quad \mathsf{w} := \begin{pmatrix} w_1 & -w_2^* \\ w_2 & w_1^* \end{pmatrix}. \tag{4.4}$$

Here $\dot{\otimes}$ denotes the composition of the tensor product $\otimes$ with matrix multiplication. As usual, the coaction is extended to the whole $\mathcal{O}(S^7_\theta)$ as a $*$-algebra morphism.

The subalgebra $B = \mathcal{O}(S^7_\theta)^{\mathrm{co}\mathcal{O}(\mathrm{SU}(2))}$ of coinvariant elements for the coaction δ is generated by the entries of the matrix $p := \mathsf{u} \bullet_\theta \mathsf{u}^\dagger$ and is identified with the algebra $\mathcal{O}(S^4_\theta)$ of coordinate functions on the 4-sphere S^4_θ in [10]. The K-action on $\mathcal{O}(S^7_\theta)$ commutes with the $\mathcal{O}(\mathrm{SU}(2))$-coaction (4.4) and the sphere $\mathcal{O}(S^4_\theta)$ carries the induced action of K. We denote the generators of $\mathcal{O}(S^4_\theta)$ by b_μ, with eigenvalues $\mu = (\mu_1, \mu_2) = (0,0), (\pm1, 0), (0, \pm1)$ of the action of H_1, H_2. Explicitly,

$$b_{10} := \sqrt{2}(z_1 \bullet_\theta z_3^* + z_2^* \bullet_\theta z_4), \quad b_{01} := \sqrt{2}(z_2 \bullet_\theta z_3^* - z_1^* \bullet_\theta z_4),$$
$$b_{00} := z_1 \bullet_\theta z_1^* + z_2 \bullet_\theta z_2^* - z_3 \bullet_\theta z_3^* - z_4 \bullet_\theta z_4^*$$

and $b_{-\mu} = b_\mu^*$. They have commutation relations

$$b_\mu \bullet_\theta b_\nu = \lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta b_\mu, \quad \lambda = e^{-\pi i\theta}, \quad \mu \wedge \nu := \mu_1\nu_2 - \mu_2\nu_1 \tag{4.5}$$

and satisfy the sphere relation $\sum_\mu b_\mu^* \bullet_\theta b_\mu = 1$.

From the above, one works out [5, eq. (4.27)] the following mixed relations:

$$\begin{aligned} (1 - b_{00}) \bullet_\theta z_1 &= \sqrt{2}(b_{10} \bullet_\theta z_3 - z_4 \bullet_\theta b_{0-1}), \\ (1 - b_{00}) \bullet_\theta z_2 &= \sqrt{2}(z_4 \bullet_\theta b_{-10} + b_{01} \bullet_\theta z_3), \\ (1 + b_{00}) \bullet_\theta z_3 &= \sqrt{2}(b_{-10} \bullet_\theta z_1 + b_{0-1} \bullet_\theta z_2), \\ (1 + b_{00}) \bullet_\theta z_4 &= \sqrt{2}(z_2 \bullet_\theta b_{10} - z_1 \bullet_\theta b_{01}). \end{aligned} \tag{4.6}$$

4.1.1. The braided Lie algebra $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))$. The commutation relations (4.5) among the generators b_μ of $\mathcal{O}(S^4_\theta)$ can be written as

$$b_\mu \bullet_\theta b_\nu = (\mathsf{R}_\alpha \triangleright b_\nu) \bullet_\theta (\mathsf{R}^\alpha \triangleright b_\mu),$$

with braiding implemented by the triangular structure of K,

$$\mathsf{R} := e^{-2\pi i\theta(H_1 \otimes H_2 - H_2 \otimes H_1)},$$

which, as in Remark 2.1, belongs to a topological completion of $K \otimes K$. Similarly for the commutation relations (4.3) of the generators z_r of $\mathcal{O}(S^7_\theta)$. In particular, both of the algebras $\mathcal{O}(S^7_\theta)$ and $\mathcal{O}(S^4_\theta)$ are quasi-commutative. Then the braided Lie algebra $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))$

of braided derivations is a $\mathcal{O}(S^4_\theta)$-bimodule. As a left $\mathcal{O}(S^4_\theta)$-module, it is generated by operators T_μ defined on the algebra generators as

$$T_\mu(b_\nu) := \delta_{\mu\nu^*} - b_\mu \bullet_\theta b_\nu, \quad \mu = (0,0), (\pm 1, 0), (0, \pm 1). \tag{4.7}$$

These are extended to the whole algebra $\mathcal{O}(S^4_\theta)$ as braided derivations

$$T_\mu(b_\nu \bullet_\theta b_\tau) = T_\mu(b_\nu) \bullet_\theta b_\tau + \lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta T_\mu(b_\tau).$$

It is easy to verify that

$$T_\mu\Big(\sum_\nu b^*_\nu \bullet_\theta b_\nu\Big) = 0,$$

showing that the T_μ's are well defined as derivations of $\mathcal{O}(S^4_\theta)$. Moreover, using the sphere relation, one also sees that they are not independent but rather satisfy the relation

$$\sum_\mu b^*_\mu T_\mu = 0 \tag{4.8}$$

for the left module structure as in (2.18). The action of H_1, H_2, the generators of K, on the generators b_μ lifts to the adjoint action (2.6) on the derivations T_μ,

$$H_j \triangleright T_\mu = [H_j, T_\mu] = \mu_j T_\mu \tag{4.9}$$

being $[H_j, T_\mu](b_\nu) = H_j \triangleright (T_\mu(b_\nu)) - T_\mu(H_j \triangleright b_\nu)$.

The derivations T_μ are the simplest combinations of the five basis derivations of $\mathcal{O}(\mathbb{R}^5_\theta)$ that preserve the sphere. In the classical limit $\theta = 0$, the derivations T_μ reduce to $T_\mu = \partial_{\mu^*} - b_\mu D$, for $D = \sum_\mu b_\mu \partial_\mu$ the Liouville vector field. The five weights μ are those of the five-dimensional representation [5] of so(5). Indeed, the vector fields H_μ carry such a representation. The bracket in $\mathrm{Der}(\mathcal{O}(S^4_\theta))$ is the braided commutator in (2.3). Using (2.20), it is determined by its computation on the generators.

Proposition 4.1. *The braided Lie algebra structure of* $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))$ *is given by*

$$[T_\mu, T_\nu] = b_\mu T_\nu - \lambda^{2\mu\wedge\nu} b_\nu T_\mu. \tag{4.10}$$

Proof. We compute the braided commutator of two generators T_μ, T_ν of $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))$,

$$\begin{aligned}
[T_\mu, T_\nu](b_\tau) &= (T_\mu \circ T_\nu - \lambda^{2\mu\wedge\nu} T_\nu \circ T_\mu)(b_\tau) \\
&= -T_\mu(b_\nu \bullet_\theta b_\tau) + \lambda^{2\mu\wedge\nu} T_\nu(b_\mu \bullet_\theta b_\tau) \\
&= -(T_\mu(b_\nu) \bullet_\theta b_\tau + \lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta T_\mu(b_\tau)) \\
&\quad + \lambda^{2\mu\wedge\nu}(T_\nu(b_\mu) \bullet_\theta b_\tau + \lambda^{2\nu\wedge\mu} b_\mu \bullet_\theta T_\nu(b_\tau)) \\
&= -T_\mu(b_\nu) \bullet_\theta b_\tau - \lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta T_\mu(b_\tau) \\
&\quad + \lambda^{2\mu\wedge\nu} T_\nu(b_\mu) \bullet_\theta b_\tau + b_\mu \bullet_\theta T_\nu(b_\tau) \\
&= -\lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta T_\mu(b_\tau) + b_\mu \bullet_\theta T_\nu(b_\tau),
\end{aligned}$$

for $\mu, \nu, \tau = (0,0), (\pm 1, 0), (0, \pm 1)$. For the last equality we have used that $\nu^* \wedge \nu = 0$ implies $T_\mu(b_\nu) = \lambda^{2\mu\wedge\nu} T_\nu(b_\mu)$ for each μ, ν. Then, identity (4.10) is verified. ∎

We conclude this subsection with two lemmas that we use later on. We first observe that formula (4.10) allows us to write the derivations T_μ in terms of their commutators.

Lemma 4.2. *The generators T_μ can be expressed in terms of their commutators as*

$$T_\nu = \sum_\mu b^*_\mu [T_\mu, T_\nu].$$

Proof. From expression (4.10) of the commutators, we compute

$$\begin{aligned}\sum_\mu b^*_\mu [T_\mu, T_\nu] &= \sum_\mu b^*_\mu \bullet_\theta b_\mu T_\nu - \sum_\mu \lambda^{2\mu\wedge\nu} b^*_\mu \bullet_\theta b_\nu T_\mu \\ &= T_\nu - b_\nu \sum_\mu b^*_\mu T_\mu = T_\nu,\end{aligned}$$

where we used the sphere relation for the second equality and (4.8) for the last one. ∎

For the braided antisymmetric commutators we introduce the notation

$$L^\pi_{\mu,\nu} := [T_\mu, T_\nu] = -\lambda^{2\mu\wedge\nu} L^\pi_{\nu,\mu}. \tag{4.11}$$

The action (4.9) of H_1 and H_2 on the derivations T_μ implies

$$H_j \triangleright L^\pi_{\mu,\nu} = (\mu_j + \nu_j) L^\pi_{\mu,\nu}.$$

From (4.10), on the generators of $\mathcal{O}(S^4_\theta)$ these derivations are given by

$$L^\pi_{\mu,\nu}(b_\sigma) = b_\mu \delta_{\nu^*\sigma} - \lambda^{2\mu\wedge\nu} b_\nu \delta_{\mu^*\sigma}. \tag{4.12}$$

Proposition 4.3. *The derivations $L^\pi_{\mu,\nu}$ give a faithful representation of the braided Lie algebra* $\mathrm{so}_\theta(5)$*, that is,*

$$[L^\pi_{\mu,\nu}, L^\pi_{\tau,\sigma}] = \delta_{\nu^*\tau} L^\pi_{\mu,\sigma} - \lambda^{2\mu\wedge\nu} \delta_{\mu^*\tau} L^\pi_{\nu,\sigma} - \lambda^{2\tau\wedge\sigma} (\delta_{\nu^*\sigma} L^\pi_{\mu,\tau} - \lambda^{2\mu\wedge\nu} \delta_{\mu^*\sigma} L^\pi_{\nu,\tau}).$$

Proof. From (4.12) a direct computation yields

$$L^\pi_{\mu,\nu} \circ L^\pi_{\tau,\sigma}(b_\alpha) = (b_\mu \delta_{\nu^*\tau} - \lambda^{2\mu\wedge\nu} b_\nu \delta_{\mu^*\tau}) \delta_{\sigma^*\alpha} - \lambda^{2\tau\wedge\sigma} (b_\mu \delta_{\nu^*\sigma} - \lambda^{2\mu\wedge\nu} b_\nu \delta_{\mu^*\sigma}) \delta_{\tau^*\alpha} \tag{4.13}$$

and the one with indices exchanged

$$L^\pi_{\tau,\sigma} \circ L^\pi_{\mu,\nu}(b_\alpha) = (b_\tau \delta_{\sigma^*\mu} - \lambda^{2\tau\wedge\sigma} b_\sigma \delta_{\tau^*\mu}) \delta_{\nu^*\alpha} - \lambda^{2\mu\wedge\nu} (b_\tau \delta_{\sigma^*\nu} - \lambda^{2\tau\wedge\sigma} b_\sigma \delta_{\tau^*\nu}) \delta_{\mu^*\alpha}. \tag{4.14}$$

The braided commutator is just given by the terms in (4.13) minus the terms in (4.14) with an extra factor coming from the braiding, that is, $\lambda^{2(\mu+\nu)\wedge(\tau+\sigma)}$. The result is obtained by pairing the terms with the same δ's. For instance, the one coming from $\delta_{\nu^*\tau} = \delta_{\tau^*\nu}$ is

$$b_\mu \delta_{\sigma^*\alpha} - \lambda^{2(\mu+\nu)\wedge(\tau+\sigma)+2(\mu\wedge\nu+\tau\wedge\sigma)} b_\sigma \delta_{\mu^*\alpha} = b_\mu \delta_{\sigma^*\alpha} - \lambda^{2(\mu\wedge\sigma)} b_\sigma \delta_{\mu^*\alpha} = L^\pi_{\mu,\sigma}(b_\alpha),$$

using that $\tau = -\nu$ to simplify the exponent of λ. The other terms behave similarly. ∎

In the classical limit, the derivations $L^\pi_{\mu,\nu}$ give the representation [10] of so(5), thus the name $\mathrm{so}_\theta(5)$ for the braided Lie algebra generated by the deformed generators $L^\pi_{\mu,\nu}$.

4.1.2. The sequence and the connection. For the SU(2) bundle on the sphere S^4_θ we have the short sequence of braided Lie algebras

$$0 \to \mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta)) \xrightarrow{\iota} \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta)) \xrightarrow{\pi} \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta)) \to 0. \tag{4.15}$$

The algebra $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))$ was given in Section 4.1.1. The braided Lie algebra $\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))$ of derivations of $\mathcal{O}(S^7_\theta)$ which are equivariant in the sense of (4.1) is also given as an $\mathcal{O}(S^4_\theta)$-module generated by a set of elements. We use 'partial derivatives' ∂_r, $\partial_r^* = \partial_{-r}$ defined by $\partial_r(z_s) = \delta_{rs}$, for $r, s = \pm 1, \pm 2, \pm 3, \pm 4$, on the algebra generators and extended as braided derivations. There are two commuting elements (the Cartan subalgebra)

$$\begin{aligned} L_1 = L_{10,-10} &:= \tfrac{1}{2}(z_1\partial_1 - z_1^*\partial_1^* - z_2\partial_2 + z_2^*\partial_2^* - z_3\partial_3 + z_3^*\partial_3^* + z_4\partial_4 - z_4^*\partial_4^*), \\ L_2 = L_{01,0-1} &:= \tfrac{1}{2}(-z_1\partial_1 + z_1^*\partial_1^* + z_2\partial_2 - z_2^*\partial_2^* - z_3\partial_3 + z_3^*\partial_3^* + z_4\partial_4 - z_4^*\partial_4^*). \end{aligned} \tag{4.16}$$

The remaining generators can be given, for $\mu, \nu = (0,0), (\pm 1, 0), (0, \pm 1)$, as

$$L_{\mu+\nu} = L_{\mu,\nu}, \quad \mu \neq \nu^*,\ \mu < \nu,$$

with the lexicographic order $(-1,0) < (0,-1) < (0,0) < (0,1) < (1,0)$. Then, the vectors $\mathsf{r} = (\mu + \nu) \in \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\}$ are the 'root vectors' of the braided Lie algebra $\mathrm{so}_\theta(5)$. Explicitly,

$$\begin{aligned} L_{10} = L_{00,10} &:= \tfrac{1}{\sqrt{2}}(z_1\partial_3 - \lambda z_3^*\partial_1^* - \lambda z_4\partial_2 + z_2^*\partial_4^*), \\ L_{-10} = L_{-10,00} &:= \tfrac{1}{\sqrt{2}}(z_3\partial_1 - \lambda^{-1} z_1^*\partial_3^* - \lambda^{-1} z_2\partial_4 + z_4^*\partial_2^*), \\ L_{01} = L_{00,01} &:= \tfrac{1}{\sqrt{2}}(z_2\partial_3 - \lambda^{-1} z_3^*\partial_2^* + \lambda^{-1} z_4\partial_1 - z_1^*\partial_4^*), \\ L_{0-1} = L_{0-1,00} &:= \tfrac{1}{\sqrt{2}}(z_3\partial_2 - \lambda z_2^*\partial_3^* + \lambda z_1\partial_4 - z_4^*\partial_1^*), \\ L_{11} = L_{01,10} &:= -\lambda^{-1} z_4\partial_3 + \lambda^{-1} z_3^*\partial_4^*, \\ L_{-1-1} = L_{-10,0-1} &:= \lambda z_4^*\partial_3^* - \lambda z_3\partial_4, \\ L_{1-1} = L_{0-1,10} &:= -\lambda^2 z_1\partial_2 + \lambda^2 z_2^*\partial_1^*, \\ L_{-11} = L_{-10,01} &:= -\lambda^{-2} z_2\partial_1 + \lambda^{-2} z_1^*\partial_2^*. \end{aligned} \tag{4.17}$$

For $\mu > \nu$ we set $L_{\mu,\nu} = -\lambda^{2\mu\wedge\nu} L_{\nu,\mu}$.

These derivations give a faithful representation of the braided Lie algebra $\mathrm{so}_\theta(5)$,

$$[L_{\mu,\nu}, L_{\tau,\sigma}] = \delta_{\nu^*\tau} L_{\mu,\sigma} - \lambda^{2\mu\wedge\nu}\delta_{\mu^*\tau} L_{\nu,\sigma} - \lambda^{2\tau\wedge\sigma}(\delta_{\nu^*\sigma} L_{\mu,\tau} - \lambda^{2\mu\wedge\nu}\delta_{\mu^*\sigma} L_{\nu,\tau}). \tag{4.18}$$

In particular, with the Cartan generators (4.16), one finds that

$$[L_j, L_{\mu+\nu}] = [L_j, L_{\mu+\nu}] = (\mu+\nu)_j L_{\mu+\nu}, \quad j = 1, 2.$$

In the classical limit $\theta = 0$, the derivations $L_{\mu,\nu}$ or $L_{\mu+\nu}$ are a representation of the Lie algebra so(5) as vector fields on the sphere S^7.

The braided Lie subalgebra $\mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta)) \subset \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))$ consists of vertical derivations. It turns out it is generated, as an $\mathcal{O}(S^4_\theta)$-module, by the ten derivations

$$\begin{aligned} Y_{\mu,\nu} &:= \sum_\gamma b^*_\gamma \bullet_\theta (b_\gamma L_{\mu,\nu} - \lambda^{-2\mu\wedge\gamma} b_\mu L_{\gamma,\nu} + \lambda^{-2\nu\wedge(\gamma+\mu)} b_\nu L_{\gamma,\mu}) \\ &= L_{\mu,\nu} - \sum_{\gamma,\tau} (b_\mu \bullet_\theta b^*_\gamma \delta_{\nu\tau} - \lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta b^*_\gamma \delta_{\mu\tau}) L_{\gamma,\tau} \end{aligned} \tag{4.19}$$

for μ, ν and sums on γ, τ in $\{(0,0), (\pm 1, 0), (0, \pm 1)\}$. It is direct to see that these derivations are vertical, that is $Y_{\mu,\nu}(b) = 0$, for $b \in \mathcal{O}(S^4_\theta)$. The proof that they generate $\mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$ follows the corresponding classical results in [5, Prop. 4.1].

The sequence (4.15) is exact. Any derivation $X \in \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))$, being H-equivariant, restricts to a derivation X^π of the subalgebra $\mathcal{O}(S^4_\theta)$. This determines the map $\pi : \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta)) \to \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))$. By construction, $\ker \pi = \imath(\mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta)))$. By a direct computation, one verifies that the restrictions $\pi(L_{\mu,\nu}) = L^\pi_{\mu,\nu}$ of the derivations $L_{\mu,\nu}$ in (4.16) and (4.17) to $\mathcal{O}(S^4_\theta)$ coincide with the derivations $L^\pi_{\mu,\nu} = [T_\mu, T_\nu]$ defined in (4.11) (thus the use of the same symbol). Moreover, from Lemma 4.2, the derivations T_ν can be written as $T_\nu = \sum_\mu b^*_\mu L^\pi_{\mu,\nu}$, also giving that

$$T_\nu = \pi\Big(\sum_\mu b^*_\mu L_{\mu,\nu}\Big). \tag{4.20}$$

This shows the surjectivity of $\pi : \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta)) \to \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))$, and thus the exactness of the sequence in (4.15) (since the elements of $\mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$ are vertical). We know from the general theory in Section 3.4 that the braided Lie algebra $\mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$ is that of infinitesimal gauge transformations.

In the limit $\theta = 0$, the sequence of Lie algebras in (4.15) is the Atiyah sequence of the SU(2)-principal Hopf bundle $S^7 \to S^4$, with $\mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ the (infinite dimensional classical) Lie algebra of infinitesimal gauge transformations.

Remark 4.4. The braided Lie algebras in the sequence (4.15) were obtained in [5] from a twist deformation quantization of the corresponding Lie algebras of the SU(2)-classical fibration $S^7 \to S^4$. The braided Lie algebra $\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))$ was generated by derivations $\widetilde{H}_j$, $j = 1, 2$, and $\widetilde{E}_{\mathsf{r}}$ for r the roots of the Lie algebra so(5). These are related to the generators $L_{\mu,\nu}$ by

$$L_j = \widetilde{H}_j, \quad j = 1, 2, \qquad L_{\mu,\nu} = \lambda^{\mu\wedge\nu} \widetilde{E}_{\mu+\nu}, \quad \mu \neq \nu^*, \ \mu < \nu,$$

with the previous lexicographic order on the μ's. Moreover,

$$L^\pi_j = \widetilde{H}^\pi_j, \quad j = 1, 2, \qquad L^\pi_{\mu,\nu} = \lambda^{\mu\wedge\nu} \varphi_\mu \varphi_\nu \widetilde{E}^\pi_{\mu+\nu}, \quad \mu \neq \nu^*, \ \mu < \nu,$$

for their restrictions, where $\varphi_{00} := 1$, $\varphi_{\pm 10} := \lambda^{\mp\frac{1}{2}}$, $\varphi_{0\pm 1} := \lambda^{\pm\frac{1}{2}}$ (see [5, Rem. 4.7]).

Moreover, each of the non vanishing terms in parentheses in (4.19) is just one of the derivations $\widetilde{K}$'s and $\widetilde{W}$'s in [5, Prop. 4.14]. Then, the ten vertical derivations $Y_{\mu,\nu}$ are those associated to the quantization $Y_{(11)} \propto L_{(0,1)+(1,0)}$ of the operator Y_{11}, the highest weight vector of the representation [10].

A connection on the SU(2) Hopf–Galois extension $\mathcal{O}(S^4_\theta) \subset \mathcal{O}(S^7_\theta)$ is a splitting of its Atiyah sequence (4.15). This results into an equivariant direct sum decomposition of $\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))$ in horizontal and vertical components.

Proposition 4.5. *The right $\mathcal{O}(S^4_\theta)$-module map $\rho : \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta)) \to \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))$, defined on the generators T_ν of $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))$ as*

$$\rho(T_\nu) := \sum_\mu b^*_\mu L_{\mu,\nu} \tag{4.21}$$

is a right splitting of the sequence (4.15).

Proof. From (4.20), it is immediate to see that $\pi \circ \rho = \mathrm{id}$. ∎

From Remark 3.1, this connection is left $\mathcal{O}(S^4_\theta)$-linear as well since it is equivariant.

Proposition 4.6. *The connection ρ is invariant under the action (by braided commutators) of the braided Lie algebra $\mathrm{so}_\theta(5)$ of $\mathcal{O}(\mathrm{SO}_\theta(4))$-equivariant derivations on $\mathcal{O}(\mathrm{SO}_\theta(5))$: for every $L_{\mu,\nu} \in \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^7_\theta))$ we have $(\delta_{L_{\mu,\nu}}\rho)(X) = 0$, for $X \in \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))$, that is,*

$$[L_{\mu,\nu}, \rho(X)] - \rho([L^\pi_{\mu,\nu}, X]) = 0.$$

Proof. From right $\mathcal{O}(S^4_\theta)$-linearity it is enough to show that

$$[L_{\mu,\nu}, \rho(T_\sigma)] - \rho([L^\pi_{\mu,\nu}, T_\sigma]) = 0,$$

for all generators T_σ of $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))$.

From (2.20) a simple computation gives $[b_\mu T_\nu, T_\sigma] = b_\mu \bullet_\theta b_\nu T_\sigma - \lambda^{2\nu\wedge\sigma}\delta_{\sigma^*\mu} T_\nu$. Thus, using $L^\pi_{\mu,\nu} = [T_\mu, T_\nu] = b_\mu T_\nu - \lambda^{2\mu\wedge\nu} b_\nu T_\mu$ from (4.10), we have

$$\begin{aligned}
[L^\pi_{\mu,\nu}, T_\sigma] &= [b_\mu T_\nu, T_\sigma] - \lambda^{2\mu\wedge\nu}[b_\nu T_\mu, T_\sigma] \\
&= b_\mu \bullet_\theta b_\nu T_\sigma - \lambda^{2\nu\wedge\sigma}\delta_{\sigma^*\mu} T_\nu - \lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta b_\mu T_\sigma + \lambda^{2\mu\wedge(\nu+\sigma)}\delta_{\sigma^*\nu} T_\mu \\
&= \delta_{\sigma^*\nu} T_\mu - \lambda^{2\mu\wedge\nu}\delta_{\sigma^*\mu} T_\nu .
\end{aligned}$$

Then

$$\rho([L^\pi_{\mu,\nu}, T_\sigma]) = \sum_\tau b^*_\tau (\delta_{\sigma^*\nu} L_{\tau,\mu} - \lambda^{2\mu\wedge\nu}\delta_{\sigma^*\mu} L_{\tau,\nu}).$$

On the other hand, using (4.12) and the braided commutator in (4.18), we compute

$$\begin{aligned}
[L_{\mu,\nu}, \rho(T_\sigma)] &= \sum_\tau [L_{\mu,\nu}, b^*_\tau L_{\tau,\sigma}] \\
&= \sum_\tau (L_{\mu,\nu}(b^*_\tau) L_{\tau,\sigma} + \lambda^{-2(\mu+\nu)\wedge\tau} b^*_\tau [L_{\mu,\nu}, L_{\tau,\sigma}])
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\tau}(b_\mu \delta_{\nu\tau} - \lambda^{2\mu\wedge\nu} b_\nu \delta_{\mu\tau}) L_{\tau,\sigma} \\
&\quad + \sum_{\tau} b_\tau^*(\lambda^{-2\mu\wedge\tau}\delta_{\nu^*\tau} L_{\mu,\sigma} - \lambda^{2(\mu+\tau)\wedge\nu}\delta_{\mu^*\tau} L_{\nu,\sigma} \\
&\quad - \lambda^{-2\mu\wedge\tau}\delta_{\nu^*\sigma} L_{\mu,\tau} + \lambda^{2(\mu+\tau)\wedge\nu}\delta_{\sigma^*\mu} L_{\nu,\tau}) \\
&= b_\mu L_{\nu,\sigma} - \lambda^{2\mu\wedge\nu} b_\nu L_{\mu,\sigma} + \lambda^{2\mu\wedge\nu} b_\nu L_{\mu,\sigma} - b_\mu L_{\nu,\sigma} \\
&\quad + \sum_{\tau} b_\tau^*(-\lambda^{-2\mu\wedge\tau}\delta_{\nu^*\sigma} L_{\mu,\tau} + \lambda^{2(\mu+\tau)\wedge\nu}\delta_{\sigma^*\mu} L_{\nu,\tau}) \\
&= \sum_{\tau} b_\tau^*(-\lambda^{-2\mu\wedge\tau}\delta_{\nu^*\sigma} L_{\mu,\tau} + \lambda^{2(\mu+\tau)\wedge\nu}\delta_{\sigma^*\mu} L_{\nu,\tau}).
\end{aligned}
$$

This expression coincides with $\rho([L^\pi_{\mu,\nu}, T_\sigma])$ due to the braided antisymmetry property $L_{\mu,\nu} = -\lambda^{2\mu\wedge\nu} L_{\nu,\mu}$. ∎

As mentioned, the splitting gives an equivariant direct sum decomposition of

$$
\begin{aligned}
\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta)) &= \rho(\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))) \oplus \imath(\mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))) \\
&\simeq \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta)) \oplus \mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))
\end{aligned} \tag{4.22}
$$

in horizontal and vertical parts. For the corresponding vertical projection, we have the following.

Corollary 4.7. *The $\mathcal{O}(S^4_\theta)$-module map $\omega : \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta)) \to \mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$,*

$$
\omega(L_{\mu,\nu}) := L_{\mu,\nu} - \rho(L^\pi_{\mu,\nu}) = L_{\mu,\nu} - (b_\mu \rho(T_\nu) - \lambda^{2\mu\wedge\nu} b_\nu \rho(T_\mu))
$$

is a left splitting of the sequence (4.15); ω is the vertical projection.

The decomposition (4.22) is then given, equivalently, by the idempotent $(\imath\circ\omega)^2 = \imath\circ\omega$. By construction, the horizontal derivations are the kernel of the vertical projection ω. A direct check uses the sphere relation and (4.8),

$$
\begin{aligned}
\omega\circ\rho(T_\nu) &= \sum_\mu b_\mu^* \omega(L_{\mu,\nu}) \\
&= \sum_\mu b_\mu^* L_{\mu,\nu} - \sum_\mu b_\mu^* \bullet_\theta b_\mu \rho(T_\nu) + \sum_\mu \lambda^{2\mu\wedge\nu} b_\mu^* \bullet_\theta b_\nu \rho(T_\mu) \\
&= \rho(T_\nu) - \rho(T_\nu) + b_\nu \rho(\sum_\mu b_\mu^* T_\mu) = 0.
\end{aligned}
$$

The derivations $\omega(L_{\mu,\nu})$ generate the module $\mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$. More explicitly,

$$
\begin{aligned}
\omega(L_{\mu,\nu}) &= L_{\mu,\nu} - b_\mu \rho(T_\nu) + \lambda^{2\mu\wedge\nu} b_\nu \rho(T_\mu) \\
&= \sum_\tau b_\tau^* \bullet_\theta b_\tau L_{\mu,\nu} - b_\mu \sum_\tau b_\tau^* L_{\tau,\nu} + \lambda^{\mu\wedge\nu} b_\nu \sum_\tau b_\tau^* L_{\tau,\mu} \\
&= \sum_\tau b_\tau^* (b_\tau L_{\mu,\nu} - \lambda^{-2\mu\wedge\tau} b_\mu L_{\tau,\nu} + \lambda^{-\nu\wedge(\tau+\mu)} b_\nu L_{\tau,\mu}).
\end{aligned}
$$

The last sum is actually limited to $\tau \neq \mu, \nu$ since for these two choices for the index τ the term in parentheses vanishes. Then, the ten vertical derivations $\omega(L_{\mu,\nu})$ coincide with the generators $Y_{\mu,\nu}$ in (4.19) of the braided Lie algebra $\mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$.

The connection ρ assigns to every derivation X on $\mathcal{O}(S^4_\theta)$ a unique horizontal derivation on $\mathcal{O}(S^7_\theta)$ that projects on X via π. As mentioned, this needs not be a braided Lie algebra morphism in general: the commutators of horizontal vector fields need not be horizontal. A measure of this failure is the curvature (3.8).

Proposition 4.8. *On the generators T_μ of the braided Lie algebra $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))$ the curvature of the connection in (4.21) is given explicitly by*

$$-\Omega(T_\mu, T_\nu) = \omega(L_{\mu,\nu}) = L_{\mu,\nu} - (b_\mu \rho(T_\nu) - \lambda^{2\mu\wedge\nu} b_\nu \rho(T_\mu)) = Y_{\mu,\nu}, \tag{4.23}$$

where the $Y_{\mu,\nu}$ are the derivations defined in (4.19).

Proof. We show that

$$[\rho(T_\mu), \rho(T_\nu)] = L_{\mu,\nu}. \tag{4.24}$$

Using the module structure, we compute

$$\begin{aligned}[\rho(T_\mu), \rho(T_\nu)] &= \Big[\sum_\sigma b^*_\sigma L_{\sigma,\mu}, \sum_\tau b^*_\tau L_{\tau,\nu}\Big] \\ &= \sum_{\sigma,\tau} \big(b^*_\sigma \bullet_\theta L_{\sigma,\mu}(b^*_\tau) L_{\tau,\nu} - \lambda^{2\mu\wedge\nu} b^*_\tau \bullet_\theta L_{\tau,\nu}(b^*_\sigma) L_{\sigma,\mu} \\ &\qquad + \lambda^{-2\mu\wedge\tau} b^*_\tau \bullet_\theta b^*_\sigma [L_{\sigma,\mu}, L_{\tau,\nu}]\big).\end{aligned}$$

Now, $L_{\mu,\nu}(b_\sigma) = L^\pi_{\mu,\nu}(b_\sigma)$ on the generators of $\mathcal{O}(S^4_\theta)$, and thus from (4.12) the first two terms in the above expression are equal to

$$\begin{aligned}&\sum_{\sigma,\tau}\big(b^*_\sigma \bullet_\theta (\delta_{\mu\tau} b_\sigma - \delta_{\tau\sigma}\lambda^{2\sigma\wedge\mu} b_\mu) L_{\tau,\nu} - \lambda^{2\mu\wedge\nu} b^*_\tau \bullet_\theta (\delta_{\nu\sigma} b_\tau - \delta_{\tau\sigma}\lambda^{2\tau\wedge\nu} b_\nu) L_{\sigma,\mu}\big) \\ &\quad = L_{\mu,\nu} - \sum_\tau b^*_\tau \bullet_\theta \lambda^{2\tau\wedge\mu} b_\mu L_{\tau,\nu} - \lambda^{2\mu\wedge\nu} L_{\nu,\mu} + \sum_\tau \lambda^{2\mu\wedge\nu} b^*_\tau \bullet_\theta \lambda^{2\tau\wedge\nu} b_\nu L_{\tau,\mu} \\ &\quad = 2L_{\mu,\nu} - b_\mu \bullet_\theta \sum_\tau b^*_\tau L_{\tau,\nu} + \lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta \sum_\tau b^*_\tau L_{\tau,\mu}.\end{aligned}$$

As for the third term, using the braided commutator in (4.18), one gets

$$\begin{aligned}&\lambda^{-2\mu\wedge\tau} b^*_\tau \bullet_\theta b^*_\sigma [L_{\sigma,\mu}, L_{\tau,\nu}] \\ &\quad = \sum_\sigma \lambda^{2\sigma\wedge\mu} b^*_\sigma \bullet_\theta b_\mu L_{\sigma,\nu} - L_{\mu,\nu} - \delta_{\mu *\nu} \sum_{\sigma,\tau} \lambda^{-2\sigma\wedge\tau} b^*_\sigma \bullet_\theta b^*_\tau L_{\sigma,\tau} \\ &\qquad + \sum_\tau \lambda^{-2(\mu\wedge\tau+\nu\wedge\mu)} b_\nu \bullet_\theta b^*_\tau L_{\mu,\tau}.\end{aligned}$$

Putting the three terms together, one then arrives at

$$[\rho(T_\mu), \rho(T_\nu)] = L_{\mu,\nu} - \delta_{\mu *\nu} \sum_{\sigma,\tau} b^*_\tau \bullet_\theta b^*_\sigma L_{\sigma,\tau}.$$

For the last term, the $\mathcal{O}(S^4_\theta)$-linearity of the horizontal lift and the identity (4.8) give

$$\sum_{\sigma,\tau} b^*_\tau \bullet_\theta b^*_\sigma L_{\sigma,\tau} = \sum_\tau b^*_\tau \rho(T_\tau) = \rho\Big(\sum_\tau b^*_\tau T_\tau\Big) = 0.$$

Thus, $[\rho(T_\mu), \rho(T_\nu)] = L_{\mu,\nu}$ as claimed. ∎

4.1.3. The connection one-form. The connection on the SU(2)-bundle $\mathcal{O}(S^4_\theta) \subset \mathcal{O}(S^7_\theta)$ given by splitting the sequence (4.15) corresponds to a Lie algebra valued one-form on the bundle. Indeed, the partial isometry u in (4.4) gives a projection $p = \mathsf{u} \bullet_\theta \mathsf{u}^*$ and a canonical covariant derivative $\nabla = p \circ d$ on the module of sections $\Gamma = p(\mathcal{O}(S^4_\theta)^4)$ of the vector bundle associated to the fundamental representation of SU(2) [15]. When translated to the principal bundle this corresponds to an su(2)-valued one-form $\omega : \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^7_\theta)) \to$ su(2). Explicitly,

$$\omega = -d\mathsf{u}^* \bullet_\theta \mathsf{u} = \omega_{22}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \omega_{21}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \omega^*_{21}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \tag{4.25}$$

with one-forms

$$\begin{aligned}\omega_{22} &= dz_1 \bullet_\theta z^*_1 + dz_2 \bullet_\theta z^*_2 + dz_3 \bullet_\theta z^*_3 + dz_4 \bullet_\theta z^*_4,\\ \omega_{21} &= -dz_1 \bullet_\theta z_2 + dz_2 \bullet_\theta z_1 - dz_3 \bullet_\theta z_4 + dz_4 \bullet_\theta z_3.\end{aligned}$$

The elements $dz_a, dz^*_a, a = 1, \dots, 4$, are the degree 1 generators of the graded differential algebra $\Omega(S^7_\theta)$ of the canonical differential calculus $(\Omega(S^7_\theta), d)$ on the algebra $\mathcal{O}(S^7_\theta)$ [9]. The commutation relations of the dz's with the z's are the same as those of the z's. The pairing between derivations and one-forms is defined by

$$\langle\ ,\ \rangle : \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^7_\theta)) \otimes \Omega^1(S^7_\theta) \to \mathcal{O}(S^7_\theta), \quad \langle X, da \bullet_\theta a'\rangle := X(a) \bullet_\theta a'.$$

By construction ω transforms under the adjoint coaction of $\mathcal{O}(\mathrm{SU}(2))$,

$$\delta\omega_{jk} = \sum_{s,t} \omega_{st} \otimes \mathsf{w}^*_{js}\mathsf{w}_{tk},$$

for $\mathsf{w} = (\mathsf{w}_{jk})$ as in (4.4), with $\mathsf{w}_{jk} \in \mathcal{O}(\mathrm{SU}(2))$. The form ω is the identity on the fundamental derivations of $\mathcal{O}(S^7_\theta)$ defined by the $\mathcal{O}(\mathrm{SU}(2))$-coaction and is vertical, that is, it vanishes on any horizontal derivation in (4.21),

$$\langle \rho(T_\nu), \omega\rangle = -(\rho(T_\nu)(\mathsf{u}^*)) \bullet_\theta \mathsf{u} = 0.$$

This can also be seen by a direct computation using the expressions for the horizontal derivations in Table 1 below (with slightly different notation: $\alpha = \sqrt{2}b_{10}$, $\beta = \sqrt{2}b_{01}$ and their conjugated) and using relations (4.6).

Then, for instance,

$$\begin{aligned} 2\langle \rho(T_{00}), \omega_{22} \rangle &= (1-x) \bullet_\theta (z_1 \bullet_\theta z_1^* + z_2 \bullet_\theta z_2^*) - (1+x) \bullet_\theta (z_3 \bullet_\theta z_3^* + z_4 \bullet_\theta z_4^*) \\ &= x - x \bullet_\theta \sum z_a \bullet_\theta z_a^* = 0, \\ 2\langle \rho(T_{00}), \omega_{21} \rangle &= (1-x) \bullet_\theta (-z_1 \bullet_\theta z_2 + z_2 \bullet_\theta z_1) - (1+x) \bullet_\theta (-z_3 \bullet_\theta z_4 + z_4 \bullet_\theta z_3) \\ &= 0. \end{aligned}$$

	z_1	z_2
$2\rho(T_{00})$	$\alpha \bullet_\theta z_3 - z_4 \bullet_\theta \beta^*$ $= (1-x) \bullet_\theta z_1$	$\beta \bullet_\theta z_3 + z_4 \bullet_\theta \alpha^*$ $= (1-x) \bullet_\theta z_2$
$2\sqrt{2}\rho(T_{01})$	$\lambda^{-1}(z_1 \bullet_\theta \beta + 2x \bullet_\theta z_4 - 2z_2 \bullet_\theta \alpha)$ $= -\lambda^{-1}((1-x) \bullet_\theta z_4 + z_2 \bullet_\theta \alpha)$	$-\beta \bullet_\theta z_2$
$2\sqrt{2}\rho(T_{0-1})$	$-\beta^* \bullet_\theta z_1$	$\beta^* \bullet_\theta z_2 - 2x \bullet_\theta z_3 + 2\alpha^* \bullet_\theta z_1$ $= (1-x) \bullet_\theta z_3 + \alpha^* \bullet_\theta z_1$
$2\sqrt{2}\rho(T_{10})$	$-\alpha \bullet_\theta z_1$	$\lambda(z_2 \bullet_\theta \alpha - 2x \bullet_\theta z_4 - 2z_1 \bullet_\theta \beta)$ $= \lambda((1-x) \bullet_\theta z_4 - z_1 \bullet_\theta \beta)$
$2\sqrt{2}\rho(T_{-10})$	$\alpha^* \bullet_\theta z_1 - 2x \bullet_\theta z_3 + 2\beta^* \bullet_\theta z_2$ $= (1-x) \bullet_\theta z_3 + \beta^* \bullet_\theta z_2$	$-\alpha^* \bullet_\theta z_2$

	z_3	z_4
$2\rho(T_{00})$	$-\beta^* \bullet_\theta z_2 - \alpha^* \bullet_\theta z_1$ $= -(1+x) \bullet_\theta z_3$	$z_1 \bullet_\theta \beta - z_2 \bullet_\theta \alpha$ $= -(1+x) \bullet_\theta z_4$
$2\sqrt{2}\rho(T_{01})$	$\beta \bullet_\theta z_3 + 2x \bullet_\theta z_2 + 2z_4 \bullet_\theta \alpha^*$ $= (1+x) \bullet_\theta z_2 + z_4 \bullet_\theta \alpha^*$	$-\beta \bullet_\theta z_4$
$2\sqrt{2}\rho(T_{0-1})$	$-\beta^* \bullet_\theta z_3$	$\lambda(z_4 \bullet_\theta \beta^* - 2x \bullet_\theta z_1 - 2\alpha \bullet_\theta z_3)$ $= -\lambda((1+x) \bullet_\theta z_1 + \alpha \bullet_\theta z_3)$
$2\sqrt{2}\rho(T_{10})$	$(\alpha \bullet_\theta z_3 + 2x \bullet_\theta z_1 - 2z_4 \bullet_\theta \beta^*)$ $= ((1+x) \bullet_\theta z_1 - z_4 \bullet_\theta \beta^*)$	$-\alpha \bullet_\theta z_4$
$2\sqrt{2}\rho(T_{-10})$	$-\alpha^* \bullet_\theta z_3$	$\lambda^{-1}(z_4 \bullet_\theta \alpha^* + 2x \bullet_\theta z_2 + 2\beta \bullet_\theta z_3)$ $= \lambda^{-1}((1+x) \bullet_\theta z_2 + \beta \bullet_\theta z_3)$

Table 1. Horizontal vector fields on $\mathcal{O}(S^7_\theta)$

Likewise, using also the commutation relations between the z_a's and the b_μ's, we compute

$$
\begin{aligned}
&2\sqrt{2}\langle\rho(T_{10}),\omega_{22}\rangle\\
&\quad= -\alpha \bullet_\theta z_1 \bullet_\theta z_1^* + \lambda(1-x)\bullet_\theta z_4 \bullet_\theta z_2^* - \lambda z_1 \bullet_\theta \beta \bullet_\theta z_2^*\\
&\qquad + (1+x)\bullet_\theta z_1 \bullet_\theta z_3^* - z_4 \bullet_\theta \beta^* \bullet_\theta z_3^* - \lambda z_4 \bullet_\theta \alpha \bullet_\theta z_4^*\\
&\quad= -[\alpha \bullet_\theta z_1 \bullet_\theta z_1^* - (1-x)\bullet_\theta z_2^* \bullet_\theta z_4 + z_1 \bullet_\theta (z_2^* \bullet_\theta \beta)\\
&\qquad - (1+x)\bullet_\theta z_1 \bullet_\theta z_3^* + (z_4 \bullet_\theta \beta^*)\bullet_\theta z_3^* + \alpha \bullet_\theta z_4 \bullet_\theta z_4^*]\\
&\quad= -[\alpha \bullet_\theta z_1 \bullet_\theta z_1^* - (1-x)\bullet_\theta z_2^* \bullet_\theta z_4 + z_1 \bullet_\theta ((x+1)\bullet_\theta z_3^* - z_1^* \bullet_\theta \alpha)\\
&\qquad - (1+x)\bullet_\theta z_1 \bullet_\theta z_3^* + (-(1-x)\bullet_\theta z_1 + \alpha \bullet_\theta z_3)\bullet_\theta z_3^* + \alpha \bullet_\theta z_4 \bullet_\theta z_4^*]\\
&\quad= -[(x-1)\bullet_\theta (z_2^* \bullet_\theta z_4 + z_1 \bullet_\theta z_3^*) + \alpha \bullet_\theta (z_3 \bullet_\theta z_3^* + z_4 \bullet_\theta z_4^*)]\\
&\quad= -\tfrac{1}{2}[(x-1)\bullet_\theta \alpha + \alpha \bullet_\theta (1-x)] = 0
\end{aligned}
$$

and

$$
\begin{aligned}
2\sqrt{2}\langle\rho(T_{10}),\omega_{21}\rangle &= \alpha \bullet_\theta z_1 \bullet_\theta z_2 + \lambda(1-x)\bullet_\theta z_4 \bullet_\theta z_1 - \lambda z_1 \bullet_\theta \beta \bullet_\theta z_1\\
&\quad - (1+x)\bullet_\theta z_1 \bullet_\theta z_4 + z_4 \bullet_\theta \beta^* \bullet_\theta z_4 - \lambda z_4 \bullet_\theta \alpha \bullet_\theta z_3\\
&= \alpha \bullet_\theta z_1 \bullet_\theta z_2 + (1-x)\bullet_\theta z_1 \bullet_\theta z_4 - z_1 \bullet_\theta z_1 \bullet_\theta \beta\\
&\quad - (1+x)\bullet_\theta z_1 \bullet_\theta z_4 + (z_4 \bullet_\theta \beta^*)\bullet_\theta z_4 - \alpha \bullet_\theta z_4 \bullet_\theta z_3\\
&= \alpha \bullet_\theta z_1 \bullet_\theta z_2 - 2x \bullet_\theta z_1 \bullet_\theta z_4 - z_1 \bullet_\theta z_1 \bullet_\theta \beta\\
&\quad + (\alpha \bullet_\theta z_3 - (1-x)\bullet_\theta z_1)\bullet_\theta z_4 - \alpha \bullet_\theta z_4 \bullet_\theta z_3\\
&= z_1 \bullet_\theta [z_2 \bullet_\theta \alpha - 2x \bullet_\theta z_4 - z_1 \bullet_\theta \beta - (1-x)\bullet_\theta z_4]\\
&= z_1 \bullet_\theta [(x+1)\bullet_\theta z_4 - 2x \bullet_\theta z_4 - (1-x)\bullet_\theta z_4] = 0.
\end{aligned}
$$

The computations for the other horizontal derivations are similar.

4.2. The conformal algebra and its action on connections

The 'basic' connection ρ we have described in the previous sections is an instanton in the sense that the curvature of the connection one-form in (4.25) is anti-selfdual [9, 10]. An infinitesimal action of twisted conformal transformations yields a five parameter family of instantons [16]. We obtain here a five parameter family of splittings of the Atiyah sequence (4.15) associated with the braided conformal Lie algebra $\mathrm{so}_\theta(5,1)$.

The braided Lie algebra $\mathrm{so}_\theta(5,1)$ as a linear space is spanned by generators $\mathcal{T}_\mu$ and $\mathcal{L}_{\mu,\nu} = -\lambda^{2\mu\wedge\nu}\mathcal{L}_{\nu,\mu}$, for $\mu,\nu = (0,0),(\pm1,0),(0,\pm1)$. It is a K-module with action

$$
H_j \triangleright \mathcal{T}_\mu = \mu_j \mathcal{T}_\mu, \quad H_j \triangleright \mathcal{L}_{\mu,\nu} = (\mu_j + \nu_j)\mathcal{L}_{\mu,\nu} \tag{4.26}
$$

and has braided brackets

$$
\begin{aligned}
[\mathcal{T}_\mu, \mathcal{T}_\nu] &= \mathcal{L}_{\mu,\nu},\\
[\mathcal{L}_{\mu,\nu}, \mathcal{T}_\sigma] &= \delta_{\sigma^*\nu}\mathcal{T}_\mu - \lambda^{2\mu\wedge\nu}\delta_{\sigma^*\mu}\mathcal{T}_\nu,\\
[\mathcal{L}_{\mu,\nu}, \mathcal{L}_{\tau,\sigma}] &= \delta_{\nu^*\tau}\mathcal{L}_{\mu,\sigma} - \lambda^{2\mu\wedge\nu}\delta_{\mu^*\tau}\mathcal{L}_{\nu,\sigma} - \lambda^{2\tau\wedge\sigma}(\delta_{\nu^*\sigma}\mathcal{L}_{\mu,\tau} - \lambda^{2\mu\wedge\nu}\delta_{\mu^*\sigma}\mathcal{L}_{\nu,\tau}).
\end{aligned}
$$

A representation as braided derivations on $\mathcal{O}(S^4_\theta)$ is given by the operators $L^\pi_{\mu,\nu}$ and T_σ in (4.12) and (4.7) respectively. On the other hand, a representation as braided derivations on $\mathcal{O}(S^7_\theta)$ is given by the operators $L_{\mu,\nu}$ in (4.16) and (4.17) together with the following additional generators

$$\begin{aligned}
\mathcal{T}_{00} &:= -\tfrac{1}{2}b_{00}\Delta + \tfrac{1}{2}(z_1\partial_1 + z_2\partial_2 + z_1^*\partial_1^* + z_2^*\partial_2^* - z_3\partial_3 - z_4\partial_4 - z_3^*\partial_3^* - z_4^*\partial_4^*),\\
\mathcal{T}_{10} &:= -\tfrac{1}{2}b_{10}\Delta + \tfrac{\sqrt{2}}{2}(\lambda z_4\partial_2 + z_1\partial_3 + \lambda z_3^*\partial_1^* + z_2^*\partial_4^*),\\
\mathcal{T}_{01} &:= -\tfrac{1}{2}b_{01}\Delta + \tfrac{\sqrt{2}}{2}(-\lambda^{-1}z_4\partial_1 + z_2\partial_3 + \lambda^{-1}z_3^*\partial_2^* - z_1^*\partial_4^*),\\
\mathcal{T}_{-10} &:= -\tfrac{1}{2}b_{-10}\Delta + \tfrac{\sqrt{2}}{2}(z_3\partial_1 + \lambda^{-1}z_2\partial_4 + z_4^*\partial_2^* + \lambda^{-1}z_1^*\partial_3^*),\\
\mathcal{T}_{0-1} &:= -\tfrac{1}{2}b_{0-1}\Delta + \tfrac{\sqrt{2}}{2}(z_3\partial_2 - \lambda z_1\partial_4 - z_4^*\partial_1^* + \lambda z_2^*\partial_3^*),
\end{aligned}$$

with $\Delta = \sum_r (z_r\partial_r + z_r^*\partial_r^*)$ the Liouville derivation on $\mathcal{O}(S^7_\theta)$.

A direct computation shows that they project to the derivations T_μ on $S^4_\theta : \mathcal{T}^\pi_\mu = T_\mu$. Moreover, using the mixed relations (4.6), one also computes that $\sum_\tau b_\tau^*\mathcal{T}_\tau = 0$.

We know from Proposition 4.6 that the basic connection in (4.21) is invariant under the action of the braided Lie algebra generated by the elements $\mathcal{L}_{\mu,\nu}$. The remaining five generators $\mathcal{T}_\mu$ will give new, not gauge equivalent connections $\rho_\mu = \rho + \delta_\mu\rho$, with

$$(\delta_\mu\rho)(X) := (\delta_{\mathcal{T}_\mu}\rho)(X) = [\mathcal{T}_\mu, \rho(X)] - \rho([\mathcal{T}^\pi_\mu, X]), \quad X \in \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta)).$$

While the connection ρ is K-equivariant the connections ρ_μ are not: due to (4.26) one finds $H_j \triangleright \rho_\mu = \mu_j\delta_\mu\rho$. On the generators T_ν one computes, using (4.21) and (4.24),

$$\begin{aligned}
(\delta_\mu\rho)(T_\nu) &= -\rho([T_\mu, T_\nu]) + [\mathcal{T}_\mu, \rho(T_\nu)]\\
&= -\rho([T_\mu, T_\nu]) + \sum_\tau \mathcal{T}_\mu(b_\tau^*)L_{\tau,\nu} + \sum_\tau \lambda^{-2\mu\wedge\tau}b_\tau^*[\mathcal{T}_\mu, L_{\tau,\nu}]\\
&= -\rho([T_\mu, T_\nu]) + L_{\mu,\nu} - b_\mu\rho(T_\nu) - \sum_\tau \lambda^{2\mu\wedge\nu}b_\tau^*(\delta_{\mu^*\nu}\mathcal{T}_\tau - \lambda^{2\tau\wedge\nu}\delta_{\mu^*\tau}\mathcal{T}_\nu)\\
&= -\rho([T_\mu, T_\nu]) + [\rho(T_\mu), \rho(T_\nu)] - b_\mu\rho(T_\nu) + b_\mu\mathcal{T}_\nu - \delta_{\mu^*\nu}\sum_\tau b_\tau^*\mathcal{T}_\tau\\
&= -\Omega(T_\mu, T_\nu) + b_\mu(\mathcal{T}_\nu - \rho(T_\nu))\\
&= -\Omega(T_\mu, T_\nu) + b_\mu\omega(\mathcal{T}_\nu).
\end{aligned}$$

The last equality follows from $\omega(\mathcal{T}_\nu) = \mathcal{T}_\mu - \rho(\mathcal{T}^\pi_\mu) = \mathcal{T}_\mu - \rho(T_\mu)$ and $\sum_\tau b_\tau^*\mathcal{T}_\tau = 0$.

Lemma 4.9. *On generators of* $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^4_\theta))$ *the variation of the curvature is given by*

$$\begin{aligned}
(\delta_\sigma\Omega)(T_\mu, T_\nu) &= -2b_\sigma\Omega(T_\mu, T_\nu) - T_\sigma(b_\mu)\omega(\mathcal{T}_\nu) + \lambda^{2\mu\wedge\nu}T_\sigma(b_\nu)\omega(\mathcal{T}_\mu)\\
&= -2b_\sigma\Omega(T_\mu, T_\nu) - \omega\big(T_\sigma(b_\mu)\mathcal{T}_\nu - \lambda^{2\mu\wedge\nu}T_\sigma(b_\nu)\mathcal{T}_\mu\big).
\end{aligned}$$

Proof. From the general theory, see (3.27), the variation of the curvature is

$$(\delta_\sigma \Omega)(T_\mu, T_\nu) = [\mathcal{T}_\sigma, \Omega(T_\mu, T_\nu)] - \Omega([T_\sigma, T_\mu], T_\nu) + \lambda^{2\mu\wedge\nu}\Omega([T_\sigma, T_\nu], T_\mu).$$

For the last two summands, using formula (4.10) for the commutator of the T_μ, one gets

$$\begin{aligned}-\Omega([T_\sigma, T_\mu], T_\nu) &= -\Omega(b_\sigma T_\mu - \lambda^{2\sigma\wedge\mu} b_\mu T_\sigma, T_\nu)\\ &= -b_\sigma \Omega(T_\mu, T_\nu) + \lambda^{2\sigma\wedge\mu} b_\mu \Omega(T_\sigma, T_\nu)\end{aligned}$$

and similarly

$$\lambda^{2\mu\wedge\nu}\Omega([T_\sigma, T_\nu], T_\mu) = \lambda^{2\mu\wedge\nu} b_\sigma \Omega(T_\nu, T_\mu) - \lambda^{2(\mu+\sigma)\wedge\nu} b_\nu \Omega(T_\sigma, T_\mu).$$

For the first summand, from $\Omega(T_\mu, T_\nu) = -\omega(L_{\mu,\nu}) = -Y_{\mu,\nu}$ and their explicit expression in (4.19), we compute

$$\begin{aligned}
&[\mathcal{T}_\sigma, \Omega(T_\mu, T_\nu)]\\
&\quad = -[\mathcal{T}_\sigma, Y_{\mu,\nu}]\\
&\quad = -[\mathcal{T}_\sigma, L_{\mu,\nu}] + \sum_{\gamma,\tau} \mathcal{T}_\sigma(b_\mu \bullet_\theta b^*_\gamma \delta_{\nu\tau} - \lambda^{2\mu\wedge\nu} b_\nu \bullet_\theta b^*_\gamma \delta_{\mu\tau}) L_{\gamma,\tau}\\
&\qquad + \sum_{\gamma,\tau} (\lambda^{2\sigma\wedge(\mu-\gamma)} b_\mu \bullet_\theta b^*_\gamma \delta_{\nu\tau} - \lambda^{2\sigma\wedge(\nu-\gamma)+2\mu\wedge\nu} b_\nu \bullet_\theta b^*_\gamma \delta_{\mu\tau})[\mathcal{T}_\sigma, L_{\gamma,\tau}]\\
&\quad = -\delta_{\sigma^*\mu}\mathcal{T}_\nu + \lambda^{2\sigma\wedge\mu}\delta_{\sigma^*\nu}\mathcal{T}_\mu\\
&\qquad + \sum_{\gamma,\tau} \big((\delta_{\sigma^*\mu} - b_\sigma \bullet_\theta b_\mu) \bullet_\theta b^*_\gamma + \lambda^{2\sigma\wedge\mu} b_\mu \bullet_\theta (\delta_{\sigma\gamma} - b_\sigma \bullet_\theta b^*_\gamma)\big)\delta_{\nu\tau} L_{\gamma,\tau}\\
&\qquad - \lambda^{2\mu\wedge\nu} \sum_{\gamma,\tau} \big((\delta_{\sigma^*\nu} - b_\sigma \bullet_\theta b_\nu) \bullet_\theta b^*_\gamma + \lambda^{2\sigma\wedge\nu} b_\nu \bullet_\theta (\delta_{\sigma\gamma} - b_\sigma \bullet_\theta b^*_\gamma)\big)\delta_{\mu\tau} L_{\gamma,\tau}\\
&\qquad + \sum_{\gamma,\tau} (\lambda^{2\sigma\wedge(\mu-\gamma)} b_\mu \bullet_\theta b^*_\gamma \delta_{\nu\tau} - \lambda^{2\sigma\wedge(\nu-\gamma)+2\mu\wedge\nu} b_\nu \bullet_\theta b^*_\gamma \delta_{\mu\tau})\delta_{\sigma^*\gamma}\mathcal{T}_\tau\\
&\qquad - \sum_{\gamma,\tau} (\lambda^{2\sigma\wedge(\mu-\gamma)} b_\mu \bullet_\theta b^*_\gamma \delta_{\nu\tau} - \lambda^{2\sigma\wedge(\nu-\gamma)+2\mu\wedge\nu} b_\nu \bullet_\theta b^*_\gamma \delta_{\mu\tau})\lambda^{2\sigma\wedge\gamma}\delta_{\sigma^*\tau}\mathcal{T}_\gamma\\
&\quad = -\delta_{\sigma^*\mu}\mathcal{T}_\nu + \lambda^{2\sigma\wedge\mu}\delta_{\sigma^*\nu}\mathcal{T}_\mu\\
&\qquad + (\delta_{\sigma^*\mu} - b_\sigma \bullet_\theta b_\mu)\rho(T_\nu) + \lambda^{2\sigma\wedge\mu} b_\mu L_{\sigma,\nu} - \lambda^{2\sigma\wedge\mu} b_\mu \bullet_\theta b_\sigma \rho(T_\nu)\\
&\qquad - \lambda^{2\mu\wedge\nu}(\delta_{\sigma^*\nu} - b_\sigma \bullet_\theta b_\nu)\rho(T_\mu) - \lambda^{2(\mu+\sigma)\wedge\nu} b_\nu L_{\sigma,\mu}\\
&\qquad + \lambda^{2(\mu+\sigma)\wedge\nu} b_\nu \bullet_\theta b_\sigma \rho(T_\mu) + \lambda^{2\sigma\wedge\mu} b_\mu \bullet_\theta b_\sigma \mathcal{T}_\nu - \lambda^{2\sigma\wedge\nu+2\mu\wedge\nu} b_\nu \bullet_\theta b_\sigma \mathcal{T}_\mu\\
&\qquad - (\lambda^{2\sigma\wedge\mu}\delta_{\nu\sigma^*} b_\mu - \lambda^{2\sigma\wedge\nu+2\mu\wedge\nu}\delta_{\mu\sigma^*} b_\nu) \sum_\gamma b^*_\gamma \mathcal{T}_\gamma
\end{aligned}$$

where we used formulas $[\mathcal{T}_\sigma, L_{\mu,\nu}] = \delta_{\sigma^*\mu}\mathcal{T}_\nu - \lambda^{2\sigma\wedge\mu}\delta_{\sigma^*\nu}\mathcal{T}_\mu$ for the braided commutators and $\mathcal{T}_\sigma(b_\mu \bullet_\theta b^*_\gamma) = (\delta_{\sigma^*\mu} - b_\sigma \bullet_\theta b_\mu) \bullet_\theta b^*_\gamma + \lambda^{2\sigma\wedge\mu} b_\mu \bullet_\theta (\delta_{\sigma\gamma} - b_\sigma \bullet_\theta b^*_\gamma)$ for the third equality, and (4.21) for the fourth equality. Finally, recalling that $\sum_\gamma b_\gamma^* \mathcal{T}_\gamma = 0$, we have

$$
\begin{aligned}
&[\mathcal{T}_\sigma, \Omega(T_\mu, T_\nu)] \\
&\quad = -\delta_{\sigma^*\mu}(\mathcal{T}_\nu - \rho(T_\nu)) + \lambda^{2\sigma\wedge\mu}\delta_{\sigma^*\nu}(\mathcal{T}_\mu - \rho(T_\mu)) \\
&\qquad - b_\sigma \bullet_\theta b_\mu \rho(T_\nu) + b_\sigma \bullet_\theta b_\mu(\mathcal{T}_\nu - \rho(T_\nu)) \\
&\qquad + \lambda^{2\mu\wedge\nu} b_\sigma \bullet_\theta b_\nu \rho(T_\mu) - \lambda^{2\mu\wedge\nu} b_\sigma \bullet_\theta b_\nu(\mathcal{T}_\mu - \rho(T_\mu)) \\
&\qquad + \lambda^{2\sigma\wedge\mu} b_\mu L_{\sigma,\nu} - \lambda^{2(\mu+\sigma)\wedge\nu} b_\nu L_{\sigma,\mu} \\
&\quad = (b_\sigma \bullet_\theta b_\mu - \delta_{\sigma^*\mu})\omega(\mathcal{T}_\nu) - \lambda^{2\mu\wedge\nu}(b_\sigma \bullet_\theta b_\nu - \delta_{\sigma^*\nu})\omega(\mathcal{T}_\mu) \\
&\qquad + \lambda^{2\sigma\wedge\mu} b_\mu \bullet_\theta (L_{\sigma,\nu} - b_\sigma \rho(T_\nu)) - \lambda^{2(\mu+\sigma)\wedge\nu} b_\nu \bullet_\theta (L_{\sigma,\mu} - b_\sigma \rho(T_\mu)) \\
&\quad = -T_\sigma(b_\mu)\omega(\mathcal{T}_\nu) + \lambda^{2\mu\wedge\nu} T_\sigma(b_\nu)\omega(\mathcal{T}_\mu) \\
&\qquad + \lambda^{2\sigma\wedge\mu} b_\mu \bullet_\theta (L_{\sigma,\nu} - b_\sigma \rho(T_\nu)) - \lambda^{2(\mu+\sigma)\wedge\nu} b_\nu \bullet_\theta (L_{\sigma,\mu} - b_\sigma \rho(T_\mu)).
\end{aligned}
$$

Finally, using (4.23), we obtain

$$
\begin{aligned}
(\delta_\sigma \Omega)(T_\mu, T_\nu) &= -2b_\sigma \Omega(T_\mu, T_\nu) - T_\sigma(b_\mu)\omega(\mathcal{T}_\nu) + \lambda^{2\mu\wedge\nu} T_\sigma(b_\nu)\omega(\mathcal{T}_\mu) \\
&\quad + \lambda^{2\sigma\wedge\mu} b_\mu \bullet_\theta (\Omega(T_\sigma, T_\nu) + L_{\sigma,\nu} - b_\sigma \rho(T_\nu)) \\
&\quad - \lambda^{2(\mu+\sigma)\wedge\nu} b_\nu \bullet_\theta (\Omega(T_\sigma, T_\mu) + L_{\sigma,\mu} - b_\sigma \rho(T_\mu)) \\
&= -2b_\sigma \Omega(T_\mu, T_\nu) - T_\sigma(b_\mu)\omega(\mathcal{T}_\nu) + \lambda^{2\mu\wedge\nu} T_\sigma(b_\nu)\omega(\mathcal{T}_\mu) \\
&\quad + \lambda^{2\sigma\wedge\mu} b_\mu \bullet_\theta (-\lambda^{2\sigma\wedge\nu} b_\nu \rho(T_\sigma)) \\
&\quad - \lambda^{2(\mu+\sigma)\wedge\nu} b_\nu \bullet_\theta (-\lambda^{2\sigma\wedge\mu} b_\mu \rho(T_\sigma)) \\
&= -2b_\sigma \Omega(T_\mu, T_\nu) - T_\sigma(b_\mu)\omega(\mathcal{T}_\nu) + \lambda^{2\mu\wedge\nu} T_\sigma(b_\nu)\omega(\mathcal{T}_\mu)
\end{aligned}
$$

hence concluding the proof. ■

5. The Riemannian geometry of the noncommutative sphere S_θ^{2n}

We consider the $\mathcal{O}(\mathrm{SO}_\theta(2n, \mathbb{R}))$-Hopf–Galois extension $\mathcal{O}(S_\theta^{2n}) \subset \mathcal{O}(\mathrm{SO}_\theta(2n+1, \mathbb{R}))$ on the noncommutative θ-spheres S_θ^{2n}. In analogy with the classical case this is thought of as the bundle of orthonormal frames on S_θ^{2n} via the identification of derivations $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S_\theta^{2n}))$ with sections of the associated bundle for the fundamental corepresentation of the Hopf algebra $\mathcal{O}(\mathrm{SO}_\theta(2n, \mathbb{R}))$ on the algebra $\mathcal{O}(\mathbb{R}_\theta^{2n})$. An equivariant splitting of the Atiyah sequence of the frame bundle then leads to an explicit and globally defined expression for the Levi-Civita connection of the 'round' metric on $\mathcal{O}(S_\theta^{2n})$. The corresponding curvature and Ricci tensors and the scalar curvature are then computed.

The Hopf algebra $\mathcal{O}(\mathrm{SO}_\theta(2n+1, \mathbb{R}))$ is the noncommutative algebra generated by the entries of a matrix $N = (n_{IJ})$, $I, J = 1, \dots 2n+1$, modulo the orthogonality conditions

$$N^t \bullet_\theta Q \bullet_\theta N = Q, \quad N \bullet_\theta Q \bullet_\theta N^t = Q \tag{5.1}$$

and $\det(N) = 1$ for the quantum determinant. There the matrix Q is given as

$$Q := \begin{pmatrix} 0 & \mathbb{1}_n & 0 \\ \mathbb{1}_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.2}$$

In slight more generality, we may assume the matrix Q to be symmetric and have a single entry equal to 1 in each row. For fixed index J, we set J' to be the unique index such that $Q_{JJ'} = 1$. Clearly $(J')' = J$. We denote by 0 the unique index $J \in \{1, \dots, 2n+1\}$ such that $Q_{JJ} = 1$ and by $\mathcal{I}$ the subset consisting of the n indices J such that $J < J'$. For the choice of Q in (5.2) one has $J' = n + J$ for $J \leq n$, $0 = 2n + 1$ and $\mathcal{I} = \{1, \dots, n\}$. In components, the orthogonality conditions give

$$\sum_K n_{K'I'} \bullet_\theta n_{KJ} = \delta_{IJ}, \quad \sum_K n_{IK} \bullet_\theta n_{J'K'} = \delta_{IJ}. \tag{5.3}$$

The commutation relations among the generators n_{IJ} are given by

$$n_{IJ} \bullet_\theta n_{KL} = \lambda_{IK}\lambda_{LJ} n_{KL} \bullet_\theta n_{IJ}, \quad \lambda_{IJ} = \exp(-2i\pi\theta_{IJ}) \tag{5.4}$$

and depend on an antisymmetric matrix of real deformation parameters

$$\theta_{IJ} = -\theta_{JI} = -\theta_{IJ'}, \quad I, J = 1, \dots, 2n+1. \tag{5.5}$$

Thus $\lambda_{IJ} = 1$ if I or J is equal to 0.

The algebra $\mathcal{O}(\mathrm{SO}_\theta(2n+1, \mathbb{R}))$ becomes a Hopf algebra with (in matrix notation) coproduct and counit $\Delta(N) = N \otimes N$, $\varepsilon(N) = \mathbb{1}_{2n+1}$, and antipode $S(N) = Q \bullet_\theta N^t \bullet_\theta Q$, $S(n_{JK}) = n_{K'J'}$. It is a $*$-Hopf algebra with $*$-structure $*(N) = QNQ$ or

$$(n_{JK})^* = S(n_{KJ}) = n_{J'K'}.$$

This corresponds to Euclidean signature $2n + 1$ (see [2, §4.1.1] for details).

Consider the n functionals $t_j : \mathcal{O}(\mathrm{SO}_\theta(2n+1, \mathbb{R})) \to \mathbb{C}$, $j \in \mathcal{I}$, defined by

$$t_j(n_{KL}) = \delta_{jK}\delta_{jL} - \delta_{jK'}\delta_{jL'} \quad \text{and} \quad t_j(aa') = t_j(a)\varepsilon(a') + \varepsilon(a)t_j(a'),$$

for any $a, a' \in \mathcal{O}(\mathrm{SO}_\theta(2n+1, \mathbb{R}))$. They commute under the convolution product and give the cotriangular structure of $\mathcal{O}(\mathrm{SO}_\theta(2n+1, \mathbb{R}))$ as $R = e^{2\pi i \theta_{jk} t_j \otimes t_k}$ (sum over $j, k \in \mathcal{I}$ is understood). They generate the Hopf algebra $U(\mathfrak{t}^n)$, the universal enveloping algebra of the Lie algebra $\mathfrak{t}^n$ of the (commutative) n-torus. This Hopf algebra has triangular structure $R = e^{2\pi i \theta_{jk} t_j \otimes t_k}$ and therefore the Hopf algebra $K = U(\mathfrak{t}^n)^{\mathrm{op}} \otimes U(\mathfrak{t}^n)$ has triangular structure

$$\mathsf{R} = (\mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id})(R^{-1} \otimes R).$$

The functionals t_j are lifted to the derivations $H_j = (t_j \otimes \mathrm{id}) \circ \Delta$ and $\widetilde{H}_j = (\mathrm{id} \otimes t_j) \circ \Delta$ of $\mathcal{O}(\mathrm{SO}_\theta(2n+1, \mathbb{R}))$, which are right and left $\mathcal{O}(\mathrm{SO}_\theta(2n+1, \mathbb{R}))$-invariant, respectively. They define an action of the Hopf algebra $K = U(\mathfrak{t}^n)^{\mathrm{op}} \otimes U(\mathfrak{t}^n)$ on $\mathcal{O}(\mathrm{SO}_\theta(2n+1, \mathbb{R}))$ given explicitly by

$$H_j \triangleright n_{KL} = (\delta_{jK} - \delta_{j'K}) n_{KL}, \quad \widetilde{H}_j \triangleright n_{KL} = (\delta_{jL} - \delta_{j'L}) n_{KL}. \tag{5.6}$$

It is not difficult to see that cotriangularity of $\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$ is then equivalent to quasi-commutativity of the (K,R)-module algebra $\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$ (cf. also [4, Ex. 5.10]). Explicitly,

$$a \bullet_\theta a' = m_\theta(e^{2\pi i\theta_{jk}(H_j\otimes H_k - \widetilde{H}_j\otimes\widetilde{H}_k)}(a'\otimes a)),$$

for $a, a' \in \mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$ and m_θ the algebra multiplication in $\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$.

Similarly one defines $\mathcal{O}(\mathrm{SO}_\theta(2n,\mathbb{R}))$. This Hopf $*$-algebra is a quantum subgroup of $\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$, with Hopf $*$-algebra surjection $\eta : \mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R})) \to \mathcal{O}(\mathrm{SO}_\theta(2n,\mathbb{R}))$. It is the quotient of $\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$ by the Hopf ideal

$$\langle n_{00}-1, n_{J0}, n_{0J}, J = 1,\dots,2n+1, J \neq 0\rangle.$$

Hence there is a natural right coaction of $\mathcal{O}(\mathrm{SO}_\theta(2n,\mathbb{R}))$ on $\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$,

$$\delta := (\mathrm{id}\otimes\eta)\Delta, \quad N \mapsto N \dot{\otimes} \eta(N), \quad \delta(n_{JK}) = \sum_L n_{JL}\otimes m_{LK},$$

having denoted $\eta(N) = M = (m_{IJ})$. The corresponding subalgebra of coinvariants is the quantum homogeneous space $\mathcal{O}(S^{2n}_\theta)$. It is generated by elements $u_I := n_{I0}$, $I = 1,\dots,2n+1$, with induced $*$-structure $u_I^* = u_{I'}$ and commutation relations

$$u_I \bullet_\theta u_J = \lambda_{IJ} u_J \bullet_\theta u_I.$$

The orthogonality relations (5.3) imply the sphere relation $\sum_I u_{I'} \bullet_\theta u_I = \sum_I u_I^* \bullet_\theta u_I = 1$.

The algebra inclusion $\mathcal{O}(S^{2n}_\theta) \subset \mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$ is a K-equivariant Hopf–Galois extension, with $\mathcal{O}(S^{2n}_\theta)$ quasi-central in $\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$. We study the corresponding braided Lie algebra of derivations.

The braided Lie algebra $\mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$ of $\mathcal{O}(\mathrm{SO}_\theta(2n))$-equivariant derivations on $\mathcal{O}(\mathrm{SO}_\theta(2n+1))$ is generated, as an $\mathcal{O}(S^{2n}_\theta)$-module, by elements L_{IJ} which are defined on the algebra generators as

$$L_{IJ}(n_{ST}) := \delta_{JS'} n_{IT} - \lambda_{IJ}\delta_{I'S} n_{JT} \tag{5.7}$$

and are extended to the whole algebra as braided derivations. They are braided antisymmetric, $L_{IJ} = -\lambda_{IJ} L_{JI}$, and in particular $L_{jj'} = H_j$. The K-action on the derivations L_{IJ} is the lift of the action on $\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$ given in (5.6). Since these derivations are right $\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$-invariant they commute with the left invariant ones $\widetilde{H}_j$ and the action of the latter is trivial. As for the action of H_j one finds

$$H_j \triangleright L_{ST} = (\delta_{jS} + \delta_{jT} - \delta_{j'S} - \delta_{j'T}) L_{ST}.$$

It follows that for these generators the braided commutator in (2.7) explicitly reads

$$[L_{IJ}, L_{ST}] = L_{IJ}L_{ST} - \lambda_{IS}\lambda_{IT}\lambda_{JS}\lambda_{JT} L_{ST}L_{IJ}.$$

Proposition 5.1. *The derivations L_{IJ} close the braided Lie algebra* $\mathrm{so}_\theta(2n+1)$*:*

$$[L_{IJ}, L_{ST}] = \delta_{J'S} L_{IT} - \lambda_{IJ}\delta_{I'S} L_{JT} - \lambda_{ST}(\delta_{J'T} L_{IS} - \lambda_{IJ}\delta_{I'T} L_{JS}). \tag{5.8}$$

Proof. From (5.7) one computes

$$L_{IJ} L_{ST}(n_{KL}) = (n_{IL}\delta_{S'J} - \lambda_{IJ} n_{JL}\delta_{S'I})\delta_{T'K} - \lambda_{ST}(n_{IL}\delta_{T'J} - \lambda_{IJ} n_{JL}\delta_{T'I})\delta_{S'K}$$

and similarly

$$L_{ST} L_{IJ}(n_{KL}) = (n_{SL}\delta_{I'T} - \lambda_{ST} n_{TL}\delta_{I'S})\delta_{J'K} - \lambda_{IJ}(n_{SL}\delta_{J'T} - \lambda_{ST} n_{TL}\delta_{J'S})\delta_{I'K}.$$

Then, using (5.5) to simplify products of λ's, one gets

$$\begin{aligned}
&[L_{IJ}, L_{ST}](n_{KL}) \\
&\quad = \delta_{S'J}(n_{IL}\delta_{T'K} - \lambda_{IT} n_{TL}\delta_{I'K}) - \delta_{S'I}(\lambda_{IJ} n_{JL}\delta_{T'K} - \lambda_{JS}\lambda_{JT} n_{TL}\delta_{J'K}) \\
&\qquad - \delta_{T'J}(\lambda_{ST} n_{IL}\delta_{S'K} - \lambda_{IS}\lambda_{JS} n_{SL}\delta_{I'K}) \\
&\qquad + \delta_{T'I}(\lambda_{ST}\lambda_{IJ} n_{JL}\delta_{S'K} - \lambda_{IS}\lambda_{JS}\lambda_{JT} n_{SL}\delta_{J'K}) \\
&\quad = \delta_{S'J} L_{IT}(n_{KL}) - \delta_{S'I}\lambda_{IJ} L_{JT}(n_{KL}) - \delta_{T'J}\lambda_{ST} L_{IS}(n_{KL}) \\
&\qquad + \delta_{T'I}\lambda_{ST}\lambda_{IJ} L_{JS}(n_{KL})
\end{aligned}$$

and equation (5.8) is verified. ∎

It is clear that any derivation in (5.7) restricts to a derivation of the subalgebra of $\mathcal{O}(\mathrm{SO}_\theta(2n+1))$ generated by the entries of any column of N, thus in particular to a derivation of $\mathcal{O}(S^{2n}_\theta)$. We denote $\pi : \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))) \to \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$ the map which associates to $X \in \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$ its restriction X^π to $\mathcal{O}(S^{2n}_\theta)$, $\pi(X) := X^\pi$. For the L_{IJ} in (5.7), one easily computes

$$L^\pi_{IJ}(u_K) = u_I\delta_{J'K} - \lambda_{IJ} u_J \delta_{I'K}. \tag{5.9}$$

The restrictions L^π_{IJ} close the braided Lie algebra $\mathrm{so}_\theta(2n+1)$, as in (5.8), too.

Lemma 5.2. *The elements*

$$T_J := \sum_I u_{I'} L^\pi_{IJ} \tag{5.10}$$

generate the $\mathcal{O}(S^{2n}_\theta)$*-module* $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$ *of derivations of* $\mathcal{O}(S^{2n}_\theta)$*.*

Proof. We establish the lemma by showing that

$$L^\pi_{IJ} = u_I T_J - \lambda_{IJ} u_J T_I.$$

Since both sides are braided derivations it is enough to show the equality on the generators of $\mathcal{O}(S^{2n}_\theta)$. From (5.9), and using the relation $\sum_I u_{I'} \bullet_\theta u_I = 1$, one computes

$$T_J(u_K) = \delta_{JK'} - \lambda_{K'J} u_K \bullet_\theta u_J = \delta_{JK'} - u_J \bullet_\theta u_K. \tag{5.11}$$

Then,

$$\begin{aligned}(u_I T_J - \lambda_{IJ} u_J T_I)(u_K) \\ &= (u_I \delta_{J'K} - u_I \bullet_\theta u_J \bullet_\theta u_K) - \lambda_{IJ}(u_J \delta_{I'K} - u_J \bullet_\theta u_I \bullet_\theta u_K) \\ &= u_I \delta_{J'K} - \lambda_{IJ} u_J \delta_{I'K}.\end{aligned}$$

A comparison with (5.9) shows that this coincides with the evaluation $L^\pi_{IJ}(u_K)$. ∎

Notice that the sphere relation implies the generators are constrained as $\sum_J u_{J'} T_J = 0$.

Proposition 5.3. *The braided Lie algebra structure of* $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$ *is given by*

$$[T_I, T_J] = u_I T_J - \lambda_{IJ} u_J T_I.$$

Proof. From (5.11) we compute

$$\begin{aligned}T_I T_J(u_K) &= -T_I(u_J \bullet_\theta u_K) \\ &= -(\delta_{I'J} - u_I \bullet_\theta u_J) \bullet_\theta u_K - \lambda_{IJ} u_J \bullet_\theta (\delta_{I'K} - u_I \bullet_\theta u_K).\end{aligned}$$

Then,

$$\begin{aligned}[T_I, T_J](u_K) &= (T_I T_J - \lambda_{IJ} T_J T_I)(u_K) \\ &= -\delta_{I'J} u_K + 2u_I \bullet_\theta u_J \bullet_\theta u_K - \lambda_{IJ} \delta_{I'K} u_J + \lambda_{IJ} \delta_{J'I} u_K \\ &\quad - 2\lambda_{IJ} u_J \bullet_\theta u_I \bullet_\theta u_K + \delta_{J'K} u_I \\ &= -\lambda_{IJ} \delta_{I'K} u_J + \delta_{J'K} u_I \\ &= (u_I T_J - \lambda_{IJ} u_J T_I)(u_K).\end{aligned}$$

∎

In the limit $\theta_{IJ} = 0$, the derivations L^π_{IJ} give a representation of the Lie algebra $\mathrm{so}(2n+1)$ as vector fields on the sphere S^{2n}.

5.1. The sequence and the equivariant connection

From the definition of the generators $T_J = \sum_I u_{I'} L^\pi_{IJ}$ of $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$ in (5.10), the right $\mathcal{O}(S^{2n}_\theta)$-module morphism

$$\rho : \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta)) \to \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(\mathrm{SO}_\theta(2n+1)), \quad T_J \mapsto \rho(T_J) := \sum_I u_{I'} L_{IJ} \tag{5.12}$$

is a section of the projection π, that is, $\pi \circ \rho = \mathrm{id}_{\mathcal{O}(S^{2n}_\theta)}$. Explicitly,

$$\begin{aligned}\rho(T_J)(n_{ST}) &= \sum_K u_{K'} \bullet_\theta (n_{KT}\delta_{J'S} - \lambda_{KJ} n_{JT} \delta_{K'S}) = \delta_{0T}\delta_{JS'} - \lambda_{S'J} u_S \bullet_\theta n_{JT} \\ &= \delta_{0T}\delta_{JS'} - n_{JT} \bullet_\theta u_S,\end{aligned} \tag{5.13}$$

where we used (5.7) and the orthogonality condition (5.3) for the third equality.

Proposition 5.4. *The connection ρ is invariant under the action of the braided Lie algebra* $\mathrm{so}_\theta(2n+1)$*: for every* $L_{IJ} \in \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(\mathrm{SO}_\theta(2n+1)))$ *we have* $(\delta_{L_{IJ}}\rho)(X) = 0$*, that is,*

$$[L_{IJ}, \rho(X)] - \rho([L^{\pi}_{IJ}, X]) = 0$$

for all $X \in \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$.

The proof is analogous to that of Proposition 4.6 and we omit it. Due to this proposition the connection ρ is left $\mathcal{O}(S^{2n}_\theta)$-linear as well. Using (5.7), the map ρ satisfies $\rho(\sum_J u_{J'}T_J) = \sum_{I,J} u_{J'} \bullet_\theta u_{I'}L_{IJ} = 0$ as it should be due to $\sum_J u_{J'}T_J = 0$.

The kernel of the projection π is generated, as an $\mathcal{O}(S^{2n}_\theta)$-module, by the derivations

$$Y_{IJ} := L_{IJ} - \rho(L^{\pi}_{IJ}), \tag{5.14}$$

where

$$\rho(L^{\pi}_{IJ}) = u_I\rho(T_J) - \lambda_{IJ}u_J\rho(T_I) = \sum_K u_I \bullet_\theta u_{K'}L_{KJ} - \lambda_{IJ}u_J \bullet_\theta u_{K'}L_{KI}.$$

We set

$$\mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))) = \ker\pi,$$

the braided Lie subalgebra of vertical derivations. From Section 3.4 this is the braided Lie algebra of infinitesimal gauge transformations of the $\mathcal{O}(\mathrm{SO}_\theta(2n,\mathbb{R}))$ Hopf–Galois extension $\mathcal{O}(S^{2n}_\theta) \subset \mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$ of the noncommutative θ-spheres S^{2n}_θ. For $n = 2$, the braided Lie algebra $\mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(\mathrm{SO}_\theta(5,\mathbb{R})))$ was studied in [5] in the context of twist deformation quantization.

The braided Lie algebras above give rise to a short exact sequence,

$$\begin{aligned} 0 \to \mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^{2n}_\theta)}(\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))) &\xrightarrow{\iota} \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))) \\ &\xrightarrow{\pi} \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta)) \to 0. \end{aligned} \tag{5.15}$$

The sequence is split by the map ρ in (5.12), the horizontal lift. The corresponding vertical projection is then (5.14),

$$\begin{aligned} \omega : \mathrm{Der}^{\mathsf{R}}_{\mathcal{M}^H}(\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))) &\to \mathrm{aut}^{\mathsf{R}}_{\mathcal{O}(S^{2n}_\theta)}(\mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))), \\ L_{IJ} &\mapsto \omega(L_{IJ}) := L_{IJ} - \rho(L^{\pi}_{IJ}) = Y_{IJ}. \end{aligned}$$

As for the curvature one has the following.

Proposition 5.5. *On the generators* $T_\nu \in \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$*, the curvature is given by*

$$\Omega(T_I, T_J) = -\omega(L_{IJ}) = -Y_{IJ}.$$

Proof. We need to show that $[\rho(T_I), \rho(T_J)] = L_{IJ}$. For this we use the expression for $\rho(T_J)(n_{ST})$ in (5.13) and that $\rho(T_J)(u_S) = T_J(u_S) = \delta_{J'S} - u_J \bullet_\theta u_S$. Then

$$\begin{aligned}\rho(T_I)(\rho(T_J)(n_{ST})) &= -\rho(T_I)(n_{JT}) \bullet_\theta u_S - \lambda_{IJ} n_{JT} \bullet_\theta \rho(T_I)(u_S) \\ &= -(\delta_{0T}\delta_{I'J} - n_{IT} \bullet_\theta u_J) \bullet_\theta u_S - \lambda_{IJ} n_{JT} \bullet_\theta (\delta_{I'S} - u_I \bullet_\theta u_S).\end{aligned}$$

Next for the braided commutator,

$$\begin{aligned}&(\rho(T_I)\rho(T_J) - \lambda_{IJ}\rho(T_J)\rho(T_I))(n_{ST}) \\ &\quad = -\delta_{0T}\delta_{I'J}u_S + n_{IT} \bullet_\theta u_J \bullet_\theta u_S - \lambda_{IJ}\delta_{I'S}n_{JT} + \lambda_{IJ}n_{JT} \bullet_\theta u_I \bullet_\theta u_S \\ &\qquad + \delta_{0T}\delta_{J'I}\lambda_{IJ}u_S - \lambda_{IJ}n_{JT} \bullet_\theta u_I \bullet_\theta u_S + \delta_{J'S}n_{IT} - n_{IT} \bullet_\theta u_J \bullet_\theta u_S \\ &\quad = -\lambda_{IJ}\delta_{I'S}n_{JT} + \delta_{J'S}n_{IT} = L_{IJ}(n_{ST})\end{aligned}$$

where the last equality follows from (5.7). ■

The $so_\theta(2n)$-valued connection 1-form on the bundle $\mathcal{O}(S^{2n}_\theta) \subseteq \mathcal{O}(SO_\theta(2n+1,\mathbb{R}))$, corresponding to the splitting of the Atiyah sequence (5.15), is the projection $\omega_{|_{so_\theta(2n)}}$ to $so_\theta(2n)$ of the Maurer–Cartan form

$$\omega = -dN^\dagger \bullet_\theta N$$

on $\mathcal{O}(SO_\theta(2n+1,\mathbb{R}))$. Here $(N^\dagger)_{JK} = n_{K'J'}$.

The differential calculus on the algebras $\mathcal{O}(SO_\theta(2n+1,\mathbb{R}))$ was constructed in [3,9]. The commutation relations among degree-zero and degree-one generators of the differential algebra $\Omega(SO_\theta(2n+1,\mathbb{R}))$ are given by

$$n_{IJ} \bullet_\theta dn_{KL} = \lambda_{IK}\lambda_{LJ} dn_{KL} \bullet_\theta n_{IJ}, \quad dn_{IJ} \bullet_\theta dn_{KL} = -\lambda_{IK}\lambda_{LJ} dn_{KL} \bullet_\theta dn_{IJ}.$$

Moreover, from $N^\dagger N = \mathbb{1}_{n+1}$ one has $\sum_L (dn_{L'K'} \bullet_\theta n_{LJ} + \lambda_{JK'} dn_{LJ} \bullet_\theta n_{L'K'}) = 0$. Then,

$$\omega_{K'J'} = -\sum_L dn_{L'K} \bullet_\theta n_{LJ'} = \lambda_{J'K} \sum_L dn_{LJ'} \bullet_\theta n_{L'K} = -\lambda_{KJ}\omega_{JK}.$$

With E_{JK} the elementary matrices (with component 1 in position JK and zero otherwise) one computes

$$\omega = \sum_{J,L} \omega_{JL} E_{JL} = \tfrac{1}{2} \sum_{J,L} \omega_{JL}(E_{JL} - \lambda_{LJ} E_{L'J'}) = \tfrac{1}{2} \sum_{J,L} \omega_{JL} K_{JL'}$$

where in the last equality we have defined the $d(d-1)/2$ matrices K_{JL} with $d = 2n+1$. More in general, for any even or odd d we have the following.

Lemma 5.6. *The matrices* $K_{JL} = E_{JL'} - \lambda_{JL} E_{LJ'}$, *for* $J, L = 1, \dots, d$, *satisfy the equality* $K_{JL} = -\lambda_{JL} K_{LJ}$. *The entries of each matrix* K_{JL} *satisfy* $((K_{JL})^t Q)_{AB} = -\lambda_{AB}(QK_{JL})_{AB}$. *The matrices* K_{JL} *close the braided Lie algebra* $so_\theta(d)$,

$$[K_{IJ}, K_{ST}] = \delta_{JS'} K_{IT} - \lambda_{IJ}\delta_{IS'} K_{JT} - \lambda_{ST}(\delta_{JT'} K_{IS} - \lambda_{IJ}\delta_{IT'} K_{JS}). \qquad (5.16)$$

Proof. The first equality is clear, the second follows from a direct computation. For the last statement we compute

$$K_{IJ}K_{ST} = \delta_{J'S}E_{IT'} - \delta_{J'T}\lambda_{ST}E_{IS'} - \delta_{I'S}\lambda_{IJ}E_{JT'} + \delta_{I'T}\lambda_{ST}\lambda_{IJ}E_{JS'}$$

and renaming the indices,

$$K_{ST}K_{IJ} = \delta_{T'I}E_{SJ'} - \delta_{T'J}\lambda_{IJ}E_{SI'} - \delta_{S'I}\lambda_{ST}E_{TJ'} + \delta_{S'J}\lambda_{IJ}\lambda_{ST}E_{TI'}.$$

Then

$$\begin{aligned}[K_{IJ}, K_{ST}] &= K_{IJ}K_{ST} - \lambda_{IS}\lambda_{IT}\lambda_{JS}\lambda_{JT}K_{ST}K_{IJ} \\ &= \delta_{J'S}(E_{IT'} - \lambda_{IS}\lambda_{IT}\lambda_{JS}\lambda_{JT}\lambda_{IJ}\lambda_{ST}E_{TI'}) \\ &\quad - \delta_{I'S}(\lambda_{IJ}E_{JT'} - \lambda_{IS}\lambda_{IT}\lambda_{JS}\lambda_{JT}\lambda_{ST}E_{TJ'}) \\ &\quad - \delta_{J'T}(\lambda_{ST}E_{IS'} - \lambda_{IS}\lambda_{IT}\lambda_{JS}\lambda_{JT}\lambda_{IJ}E_{SI'}) \\ &\quad + \delta_{I'T}(\lambda_{ST}\lambda_{IJ}E_{JS'} - \lambda_{IS}\lambda_{IT}\lambda_{JS}\lambda_{JT}E_{SJ'}) \\ &= \delta_{J'S}(E_{IT'} - \lambda_{IT}E_{TI'}) - \delta_{I'S}\lambda_{IJ}(E_{JT'} - \lambda_{JT}E_{TJ'}) \\ &\quad - \delta_{J'T}\lambda_{ST}(E_{IS'} - \lambda_{IS}E_{SI'}) + \delta_{I'T}\lambda_{ST}\lambda_{IJ}(E_{JS'} - \lambda_{JS}E_{SJ'}),\end{aligned}$$

proving (5.16). ■

The projection $\omega_{|_{\mathrm{so}_\theta(2n)}}$ of ω to the braided Lie subalgebra $\mathrm{so}_\theta(2n)$ of $\mathrm{so}_\theta(2n+1)$ is just

$$\omega_{|_{\mathrm{so}(2n)}} = \sum_{J,K\neq 0} \omega_{JK}K_{JK'}. \tag{5.17}$$

As a consistency check we show that $\omega_{|_{\mathrm{so}_\theta(2n)}}$ is zero on horizontal fields:

$$\langle \rho(T_I), \omega_{|_{\mathrm{so}(2n)}}\rangle = -(\rho(T_I)(N^\dagger)) \bullet_\theta N_{|_{\mathrm{so}(2n)}} = 0$$

for each horizontal field $\rho(T_I)$ in (5.12). Firstly, we compute

$$\begin{aligned}\langle \rho(T_I), \omega_{JK}\rangle &= -\sum_L \rho(T_I)(n_{L'J'}) \bullet_\theta n_{LK} = -\sum_L (\delta_{0J}\delta_{IL} - n_{IJ'} \bullet_\theta u_{L'}) \bullet_\theta n_{LK} \\ &= -\delta_{0J}n_{IK} + \delta_{0K}n_{IJ'}\end{aligned}$$

where we used the expression for $\rho(T_I)(n_{ST})$ computed in (5.13) and the orthogonality condition (5.3). This then gives $\langle \rho(T_I), \omega_{|_{\mathrm{so}(2n)}}\rangle = 0$ when considering the projection $\omega_{|_{\mathrm{so}_\theta(2n)}}$ of ω to $\mathrm{so}_\theta(2n)$ in (5.17).

5.2. The derivations as sections of an associated bundle

Given the $\mathcal{O}(\mathrm{SO}_\theta(2n,\mathbb{R}))$-Hopf–Galois extension $\mathcal{O}(S^{2n}_\theta) \subset \mathcal{O}(\mathrm{SO}_\theta(2n+1,\mathbb{R}))$ of orthonormal frames on S^{2n}_θ, we identify the derivations $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$ as the sections of the associated bundle for the fundamental corepresentation of the Hopf algebra $\mathcal{O}(\mathrm{SO}_\theta(2n,\mathbb{R}))$ on the algebra $\mathcal{O}(\mathbb{R}^{2n}_\theta)$.

Given a right H-comodule algebra A with coaction $\delta : A \to A \otimes H$, $\delta(a) = a_{(0)} \otimes a_{(1)}$ and a left H-comodule V with coaction $\gamma : V \to H \otimes V$, $\gamma(v) = v_{(-1)} \otimes v_{(0)}$, sections of the vector bundle associated with the corepresentation γ can be identified with linear maps $\phi : V \to A$ which are H-equivariant

$$\phi(v)_{(0)} \otimes \phi(v)_{(1)} = \phi(v_{(0)}) \otimes S(v_{(-1)}). \tag{5.18}$$

The collection $\mathcal{E}$ of such maps is a left B-module for $B \subseteq A$ the subalgebra of coinvariants for the H-coaction.

For $H = \mathcal{O}(\mathrm{SO}_\theta(2n, \mathbb{R}))$ consider the fundamental corepresentation

$$\gamma : \mathbb{R}^{2n} \to \mathcal{O}(\mathrm{SO}_\theta(2n, \mathbb{R})) \otimes \mathbb{R}^{2n}, \quad e_\alpha \mapsto \sum_\beta m_{\alpha\beta} \otimes e_\beta$$

on the vector space $\mathbb{R}^{2n}$ with the $2n$ basis elements e_α, $\alpha \neq \alpha'$. We denote by $\mathcal{E}_T$ the $\mathcal{O}(S^{2n}_\theta)$-module of equivariant maps, defined as in (5.18), associated via this corepresentation to the $\mathcal{O}(\mathrm{SO}_\theta(2n, \mathbb{R}))$-Hopf–Galois extension $\mathcal{O}(S^{2n}_\theta) \subset \mathcal{O}(\mathrm{SO}_\theta(2n+1, \mathbb{R}))$.

Proposition 5.7. *The $\mathcal{O}(S^{2n}_\theta)$-module $\mathcal{E}_T$ of equivariant maps is generated by the $2n+1$ linear maps*

$$\begin{aligned} \phi_{(J)} &: \mathbb{R}^{2n} \to \mathcal{O}(\mathrm{SO}_\theta(2n+1, \mathbb{R})), \\ \phi_{(J)}(e_\alpha) &:= (NQ)_{J\alpha} = n_{J\alpha'}, \quad J = 1, \dots, 2n+1. \end{aligned} \tag{5.19}$$

Proof. With the coactions $\delta(n_{JK}) = \sum_L n_{JL} \otimes m_{LK}$ and $\gamma(e_\alpha) = \sum_\beta m_{\alpha\beta} \otimes e_\beta$, the condition in (5.18) implies that an equivariant map is linear in the generators n_{KL},

$$\phi(e_\alpha) = \sum_{J,K} b^{JK}_\alpha \bullet_\theta n_{JK}, \quad b^{JK}_\alpha \in \mathcal{O}(S^{2n}_\theta).$$

Then (5.18) yields

$$\sum_{J,K,L} b^{JK}_\alpha \bullet_\theta n_{JL} \otimes m_{LK} = \sum_{\beta,J,K} b^{JK}_\beta \bullet_\theta n_{JK} \otimes m_{\beta'\alpha'}.$$

Thus $b^{JK}_\alpha = 0$ when $K \neq \alpha'$ and $\phi(e_\alpha)$ reduces to $\phi(e_\alpha) = \sum_J b^J_\alpha \bullet_\theta n_{J\alpha'}$. The equivariance implies $b^J_\alpha = b^J_\beta$, for all β, that is, the coefficients do not depend on the basis element e_α. The general equivariant map is thus written as $\phi(e_\alpha) = \sum_J \beta^J \bullet_\theta n_{J\alpha'}$ with $\beta^J \in \mathcal{O}(S^{2n}_\theta)$. This concludes the proof. ∎

The module generators above are not independent,

$$\sum_J u_{J'} \phi_{(J)} = 0$$

for $u_J = n_{J0}$ the generators of the sphere $\mathcal{O}(S^{2n}_\theta)$ and module structure written as $(b\phi_{(J)})(e_\alpha) = b \bullet_\theta (\phi_{(J)}(e_\alpha))$, for $b \in \mathcal{O}(S^{2n}_\theta)$. Indeed, from (5.3),

$$\sum_J u_{J'} \bullet_\theta \phi_{(J)}(e_\alpha) = \sum_J u_{J'} \bullet_\theta n_{J\alpha'} = \sum_J n_{J'0} \bullet_\theta n_{J\alpha'} = \delta_{0\alpha'} = 0.$$

The $\mathcal{O}(S^{2n}_\theta)$-module of equivariant maps $\mathcal{E}_T$ can be realised as the image of the free module via a suitable projection with entries in $\mathcal{O}(S^{2n}_\theta)$. Define $2n$ vectors $|\varphi_\alpha\rangle$, $\alpha \neq \alpha'$, with components

$$|\varphi_\alpha\rangle_J := n_{J\alpha'}, \quad J = 1, \dots, 2n+1.$$

From the orthogonality condition of the matrix N in (5.3), these are orthonormal, $\langle \varphi_\alpha, \varphi_\beta \rangle = \sum_J n_{J'\alpha} n_{J\beta'} = \delta_{\alpha\beta}$, and we get a matrix projection

$$p := \sum_{\alpha \neq \alpha'} |\varphi_\alpha\rangle\langle\varphi_\alpha|.$$

The entries of p are in $\mathcal{O}(S^{2n}_\theta)$: using again the orthogonality condition one computes

$$p_{IJ} = \sum_\alpha n_{I\alpha} \bullet_\theta n_{J'\alpha'} = \delta_{IJ} - n_{I0} \bullet_\theta n_{J'0} = \delta_{IJ} - u_I \bullet_\theta u_{J'}. \tag{5.20}$$

This projector has rank $2n$, its trace is

$$\operatorname{tr}(p) = \sum_{J=1}^{2n+1} (\delta_{JJ} - u_J \bullet_\theta u_J^*) = 2n \cdot 1,$$

with 1 the constant function. We can then identify $\mathcal{E}_T = (\mathcal{O}(S^{2n}_\theta)^{2n+1})p$. In this way, the rows of p are a set of generators for the module $\mathcal{E}_T$.

The module $\mathcal{E}_T$ gives the 'tangent module', that is, the derivations $\mathrm{Der}^{\mathrm{R}}(\mathcal{O}(S^{2n}_\theta))$. This can be seen in two different ways. On the one hand, from expression (5.20), the rows of p, that is, the generators of $\mathcal{E}_T$, are the components of the generators T_J in (5.11).

On the other hand, let $|\Delta\rangle$ be the unit vector with components $(u_J, J = 1, \dots, 2n+1)$, and let $p_N = |\Delta\rangle\langle\Delta|$ be the 'normal' projection with corresponding 'normal' bundle $\mathcal{E}_N$. From $p \oplus p_N = \mathrm{id}$ we read the direct sum decomposition $\mathcal{O}(S^{2n}_\theta)^{2n+1} = \mathcal{E}_T \oplus \mathcal{E}_N$. We have as a consequence a module isomorphism between the $\mathcal{O}(S^{2n}_\theta)$-module $\mathrm{Der}^{\mathrm{R}}(\mathcal{O}(S^{2n}_\theta))$ of derivations of $\mathcal{O}(S^{2n}_\theta)$ and the $\mathcal{O}(S^{2n}_\theta)$-module $\mathcal{E}_T$.

We then identify the generators T_J of $\mathrm{Der}^{\mathrm{R}}(\mathcal{O}(S^{2n}_\theta))$ in (5.10) with the generators $\phi_{(J)}$ of the module $\mathcal{E}_T$ in (5.19) via the module map

$$\Gamma : \mathrm{Der}^{\mathrm{R}}(\mathcal{O}(S^{2n}_\theta)) \to \mathcal{E}_T, \quad T_J \mapsto \phi_{(J)}, \quad J = 1, \dots, 2n+1.$$

The matrix Q defining the orthogonality condition (5.1) is used for a metric on $\mathcal{E}_T$,

$$g : \mathrm{Der}^{\mathrm{R}}(\mathcal{O}(S^{2n}_\theta)) \times \mathrm{Der}^{\mathrm{R}}(\mathcal{O}(S^{2n}_\theta)) \to \mathcal{O}(S^{2n}_\theta).$$

This is the restriction of the standard metric on the free module $\mathcal{O}(S^{2n}_\theta)^{2n+1}$ to the module $\mathcal{E}_T = (\mathcal{O}(S^{2n}_\theta)^{2n+1})p \simeq \mathrm{Der}^{\mathrm{R}}(\mathcal{O}(S^{2n}_\theta))$. Thinking of the generators T_J as the rows of the projection (5.20), on these the metric is defined by

$$g(T_J, T_K) := \sum_{L,I} (T_J)_L \bullet_\theta Q_{LI} \bullet_\theta (T_K)_I$$

and computed to be

$$\begin{aligned} g(T_J, T_K) &= \sum_{L,I} \delta_{LI'} (T_J)_L \bullet_\theta (T_K)_I = \sum_L (T_J)_L \bullet_\theta (T_K)_{L'} \\ &= \sum_L p_{JL} \bullet_\theta p_{KL'} = \sum_L p_{JL} \bullet_\theta p_{LK'} = p_{JK'} \\ &= \delta_{JK'} - u_J \bullet_\theta u_K = T_J(u_K). \end{aligned} \tag{5.21}$$

Here we used the properties $p^\dagger = p$ and $p^2 = p$ read as $p_{K'L'} = p_{LK}$ and $\sum_L p_{KL} \bullet_\theta p_{LJ} = p_{KJ}$. Expression (5.21) is extended by $\mathcal{O}(S^{2n}_\theta)$-bilinearity. Clearly the metric g is braided-symmetric: $g(T_K, T_J) = \lambda_{KJ} g(T_J, T_K)$.

5.3. The Levi-Civita connection

With the identification above, the connection (the splitting) ρ of the Atiyah sequence in (5.12) induces an affine connection on $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$.

Definition 5.8. For $X, Y \in \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$, the covariant derivative of Y along X is defined to be

$$\nabla_X Y := \Gamma^{-1}(\rho(X)(\Gamma(Y))) \tag{5.22}$$

for ρ the splitting and the derivation $\rho(X)$ acting on the function $\Gamma(Y)$.

Using that ρ is equivariant and thus right and left $\mathcal{O}(S^{2n}_\theta)$-linear, one proves the following covariant derivative properties.

Proposition 5.9. *The covariant derivative has the properties:*

(i) $\nabla_{bX} Y = b \nabla_X Y$;

(ii) $\nabla_X (bY) = X(b) Y + (\mathsf{R}_\alpha \triangleright b) \nabla_{\mathsf{R}^\alpha \triangleright X} Y$

for $X, Y \in \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$ *and* $b \in \mathcal{O}(S^{2n}_\theta)$.

The Riemannian curvature of the covariant derivative is naturally defined as

$$\mathsf{R}(X, Y) Z := \nabla_X \nabla_Y Z - \nabla_{\mathsf{R}_\alpha \triangleright Y} \nabla_{\mathsf{R}^\alpha \triangleright X} Z - \nabla_{[X,Y]} Z,$$

for $X, Y, Z \in \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$. The properties in Proposition 5.9 imply that the curvature is left $\mathcal{O}(S^{2n}_\theta)$-linear while the equivariance of ρ implies that it is also right $\mathcal{O}(S^{2n}_\theta)$-linear.

Proposition 5.10. *Let* $\Omega(X, Y) = \rho([X, Y]) - [\rho(X), \rho(Y)]$ *be the curvature of the equivariant splitting (the connection)* ρ. *Then*

$$\mathsf{R}(X, Y) Z = -\Gamma^{-1}(\Omega(X, Y)(\Gamma(Z))).$$

Proof. From $\Gamma(\nabla_Y Z) = \rho(Y)(\Gamma(Z))$ one gets $\Gamma(\nabla_X \nabla_Y Z) = \rho(X)\rho(Y)(\Gamma(Z))$. Thus,

$$\Gamma(\mathsf{R}(X, Y) Z) = \big(\rho(X)\rho(Y) - \rho(\mathsf{R}_\alpha \triangleright Y)\rho(\mathsf{R}^\alpha \triangleright X) - \rho([X, Y])\big)(\Gamma(Z))$$

from which the stated equality follows. ∎

Furthermore, the torsion tensor of the covariant derivative is defined as

$$\mathsf{T}(X,Y) := \nabla_X Y - \nabla_{\mathsf{R}_\alpha \triangleright Y}(\mathsf{R}^\alpha \triangleright X) - [X,Y],$$

for $X, Y \in \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$. Again, it is $\mathcal{O}(S^{2n}_\theta)$-bilinear.

As we see below, the covariant derivative in (5.22) is the Levi-Civita one. On the generators T_J of the module $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$, formula (5.22) gives

$$\nabla_{T_J} T_K = -T_J \cdot u_K. \tag{5.23}$$

(As before we write $X \cdot b$ to distinguish the right $\mathcal{O}(S^{2n}_\theta)$-module structure from the evaluation of a derivation on an element in $\mathcal{O}(S^{2n}_\theta)$.) Indeed, using (5.13), one computes

$$\Gamma(\nabla_{T_J} T_K)(e_\alpha) = \rho(T_J)(n_{K\alpha'}) = -n_{J\alpha'} \bullet_\theta u_K = -(\Gamma(T_J)(e_\alpha)) \bullet_\theta u_K.$$

Proposition 5.11. *The covariant derivative in* (5.22) *is torsion free.*

Proof. Using (5.23), we compute

$$\nabla_{T_I} T_J - \lambda^{IJ} \nabla_{T_J} T_I = -T_I \cdot u_J + \lambda_{IJ} T_J \cdot u_I = -\lambda_{IJ} u_J T_I + u_I T_J = [T_I, T_J]$$

where we used the module structure (2.19) for the second equality and Proposition 5.3 for the last one. Then $\mathsf{T}(T_I, T_J) = 0$ as claimed. ∎

Proposition 5.12. *The covariant derivative is compatible with the metric* (5.21)*,*

$$g(\nabla_X Y, Z) + g(\mathsf{R}_\alpha \triangleright Y, \nabla_{\mathsf{R}^\alpha \triangleright X} Z) = X(g(Y,Z)),$$

for $X, Y, Z \in \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$.

Proof. On generators the above becomes

$$g(\nabla_{T_L} T_J, T_K) + \lambda_{JL} g(T_J, \nabla_{T_L} T_K) = T_L(g(T_J, T_K)). \tag{5.24}$$

Then, from the explicit expression (5.23) and (5.21) one computes,

$$\begin{aligned} g(\nabla_{T_L} T_J, T_K) + \lambda_{JL} g(T_J, \nabla_{T_L} T_K) &= -g(T_L \cdot u_J, T_K) - \lambda_{JL} g(T_J, T_L \cdot u_K) \\ &= -\lambda_{JL} u_J \bullet_\theta g(T_L, T_K) - g(T_L, T_J) \bullet_\theta u_K \\ &= -\lambda_{JL} u_J \bullet_\theta T_L(u_K) - T_L(u_J) \bullet_\theta u_K \end{aligned}$$

which coincides with the right-hand side of (5.24) by the last line in (5.21). ∎

5.4. The Riemannian geometry

For the covariant derivative ∇ in (5.22), the Riemannian curvature on the generators $T_J \in \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$ is given by

$$\begin{aligned}\mathsf{R}(T_I, T_J)T_K &= T_I \cdot T_J(u_K) - \lambda_{IJ} T_J \cdot T_I(u_K) \\ &= T_I \cdot g(T_J, T_K) - \lambda_{IJ} T_J \cdot g(T_I, T_K).\end{aligned} \tag{5.25}$$

First, using Proposition 5.9, we have

$$\begin{aligned}\nabla_{T_I} \nabla_{T_J} T_K &= \nabla_{T_I}(-T_J \cdot u_K) = -\lambda_{JK} \nabla_{T_I}(u_K T_J) \\ &= -\lambda_{IJ} T_J \cdot T_I(u_K) + T_I \cdot u_J \bullet_\theta u_K\end{aligned}$$

from (5.23), and

$$\nabla_{[T_I, T_J]} T_K = \nabla_{-\lambda_{IJ} u_J T_I + u_I T_J} T_K = -u_I T_J \cdot u_K + \lambda_{IJ} u_J T_I \cdot u_K.$$

Then,

$$\begin{aligned}\mathsf{R}(T_I, T_J)T_K &= \nabla_{T_I} \nabla_{T_J} T_K - \lambda_{IJ} \nabla_{T_J} \nabla_{T_I} T_K - \nabla_{[T_I, T_J]} T_K \\ &= -\lambda_{IJ} T_J \cdot T_I(u_K) + T_I \cdot u_J \bullet_\theta u_K + T_I \cdot T_J(u_K) \\ &\quad - \lambda_{IJ} T_J \cdot u_I \bullet_\theta u_K + u_I T_J \cdot u_K - \lambda_{IJ} u_J T_I \cdot u_K \\ &= -\lambda_{IJ} T_J \cdot T_I(u_K) + T_I \cdot T_J(u_K).\end{aligned}$$

The second equality in (5.25) follows from the last line in definition (5.21).

We next consider the dual $(\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta)))'$ of $\mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta))$ with respect to the metric g in (5.21). That is, we have a map still denoted g and defined on generators by

$$\theta_J := g(T_J) : \mathrm{Der}^{\mathsf{R}}(\mathcal{O}(S^{2n}_\theta)) \to \mathcal{O}(S^{2n}_\theta), \quad T_K \mapsto \theta_J(T_K) = g(T_J, T_K) = T_J(u_K). \tag{5.26}$$

Proposition 5.13. *The Ricci tensor, defined by*

$$\mathsf{R}(T_J, T_K) := \sum_I \theta_{I'}(\mathsf{R}(T_I, T_J)T_K)$$

is computed to be

$$\mathsf{R}(T_J, T_K) = (2n-1) T_j(u_K) = (2n-1) g(T_J, T_K). \tag{5.27}$$

The scalar curvature defined by

$$\mathsf{r} = \sum_J \mathsf{R}(g^{-1}(\theta_{J'}), T_J)$$

is computed to be

$$\mathsf{r} = 2n(2n-1). \tag{5.28}$$

Proof. For the Ricci tensor, from (5.25) and (5.26),

$$\mathsf{R}(T_J, T_K) := \sum_I (\theta_{I'}(T_I)T_J(u_K) - \lambda_{IJ}\theta_{I'}(T_J)T_I(u_K)).$$

The first term is given by $\sum_I (\delta_{I'I'} - u_{I'} \bullet_\theta u_I)T_J(u_K) = 2nT_J(u_K)$ while the second one is

$$-\sum_I \lambda_{IJ}(\delta_{I'J'} - u_{I'} \bullet_\theta u_J)T_I(u_K) = -T_J(u_K) + u_J \bullet_\theta \sum_I u_{I'}(T_I(u_K)) = -T_J(u_K),$$

being $\sum_I u_{I'}T_I = 0$. When added they give (5.27) using that $T_J(u_K) = g(T_J, T_K)$.

For the scalar curvature one has then

$$r = (2n-1)\sum_J T_J(u_{J'}) = (2n-1)\sum_J(\delta_{JJ} - u_{J'} \bullet_\theta u_J) = (2n-1)2n$$

as stated. ■

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Paolo Aschieri
Dipartimento di Scienze e Innovazione Tecnologica, Università del Piemonte Orientale, Viale T. Michel 11, 15121 Alessandria; INFN Torino, Via P. Giuria 1, 10125 Torino; Arnold-Regge centre, Via P. Giuria 1, 10125 Torino, Italy; paolo.aschieri@uniupo.it

Giovanni Landi
Matematica, Università di Trieste, Via A. Valerio 12/1, 34127 Trieste; INFN Sezione di Trieste, Trieste, Italy; landi@units.it

Chiara Pagani
Dipartimento di Matematica, Alma Mater Studiorum Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna; current address: Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli Federico II, Via Cintia 21, 80126 Napoli, Italy; cpagani@units.it