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Self-similar quantum groups

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Abstract. We introduce the notion of self-similarity for compact quantum groups. For a finite set X, we introduce a C^*-algebra $\mathbb{A}_X$, which is the quantum automorphism group of the infinite homogeneous rooted tree X^*. Self-similar quantum groups are then certain quantum subgroups of $\mathbb{A}_X$. Our main class of examples are called finitely constrained self-similar quantum groups, and we find a class of these examples that can be described as quantum wreath products by subgroups of the quantum permutation group.

1. Introduction

Self-similar groups are a class of groups acting faithfully on an infinite rooted homogeneous tree X^*. In particular, given an automorphism $g \in \mathrm{Aut}(X^*)$ and a vertex $w \in X^*$, by identifying wX^* with $g(w)X^*$, we get an automorphism $g|_w \in \mathrm{Aut}(X^*)$ which is uniquely determined by the identity

$$g \cdot (wv) = (g \cdot w)g|_w \cdot v \quad \text{for all } v \in X^*.$$

The automorphism $g|_w$ is called the *restriction* of g by w, and a subgroup $G \leq \mathrm{Aut}(X^*)$ is *self-similar* if it is closed under restrictions. Self-similar groups are a significant class of groups that play an important role in geometric group theory, and have been a rich source of groups displaying interesting phenomena. Most notably, the Grigorchuk group [6] is a self-similar group which is an infinite, finitely generated periodic group and provided the first example of a group with intermediate growth, as well as the first known amenable group to not be elementary amenable.

When the group of automorphisms $\mathrm{Aut}(X^*)$ is equipped with the permutation topology, the closed self-similar groups are examples of compact, totally disconnected groups, and hence are profinite groups. A particular class of examples of interest are the self-similar groups of *finite type*, which are subgroups of automorphisms of X^* that act like elements of a given finite group locally around every vertex. Grigorchuk introduced this concept in [5], where he also showed that the closure of the Grigorchuk group is a self-similar group of finite type. Note that these groups are called finitely constrained self-similar groups in [9], and we will use that terminology.

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The theory of compact quantum groups is by now a very substantial part of the wider field of quantum groups, and one which sits in the framework of operator algebras. The theory started with Woronowicz's introduction of the quantum SU(2) group in [14]. Woronowicz then defined compact matrix quantum groups in [13], before developing a general theory of compact quantum groups in [15]. An important class of compact matrix quantum groups was identified and studied by Wang through his quantum permutation groups in [12]. Wang was motivated by one of Connes' questions from his noncommutative geometry program: what is the *quantum* automorphism group of a space? Wang's work in [12] provided an answer for finite spaces; in particular, Wang formally defined the notion of a quantum automorphism group, and then showed that his quantum permutation group $A_s(n)$ is the quantum automorphism group of the space with n points. For three or fewer points this algebra is commutative, and hence indicating no quantum permutations; but for four or more points, remarkably the algebra is noncommutative and infinite-dimensional.

Since the appearance of [12], follow-up work progressed in multiple directions, including the results of Bichon in [2] in which he introduced quantum automorphisms of finite graphs. These algebras are quantum subgroups of the quantum permutation groups. Bichon used this construction to define the quantum dihedral group D_4. Later still in [1], Banica and Bichon classified all the compact quantum groups acting on four points; that is, all the compact quantum subgroups of $A_s(4)$. Quantum automorphisms of infinite graphs have recently been considered by Rollier and Vaes in [8], and by Voigt in [10].

Our current work is the result of us asking the question: is there a reasonable notion of self-similarity for *quantum* groups? We answer this question in the affirmative for compact quantum groups. We do this by first constructing the quantum automorphism group $\mathbb{A}_X$ of the homogeneous rooted tree X^*, and then identifying the quantum analogue of the restriction maps $g \mapsto g|_w$ for $g \in \mathrm{Aut}(X^*)$, $w \in X^*$. We then define a self-similar quantum group to be any quantum subgroup A of $\mathbb{A}_X$ for which the restriction maps factor through the quotient map $\mathbb{A}_X \to A$. We characterise self-similar quantum groups in terms of a certain homomorphism $A \otimes C(X) \to C(X) \otimes A$, which can be thought of as quantum state-transition function. The main class of examples we examine are quantum analogues of finitely constrained self-similar groups. In our main theorem about these examples we describe a class of finitely constrained self-similar groups as free wreath products by quantum subgroups of quantum permutation groups.

We start with a small preliminaries section in which we collect all the required definitions from the literature on compact quantum groups. In Section 3, we then identify a compact quantum group $\mathbb{A}_X$ which we prove is the quantum automorphism group of the homogeneous rooted tree X^*. The C^*-algebra $\mathbb{A}_X$ is a noncommutative, infinite-dimensional C^*-algebra whose abelianisation is the algebra of continuous functions on the automorphism group of the tree X^*. In Section 4, we introduce the notion of self-similarity for compact quantum groups, and we characterise self-similar quantum groups A in terms of morphisms $A \otimes C(X) \to C(X) \otimes A$, mimicking the fact that classical self-similar actions are governed by the maps $G \times X \to X \times G\colon (g, x) \mapsto (g \cdot x, g|_x)$.

In Section 5, we define finitely constrained self-similar quantum groups, which are the quantum analogues of the classical finitely constrained self-similar groups studied in [4, 9]. In particular, we consider subalgebras $\mathbb{A}_d$ of $\mathbb{A}_X$, which are the quantum automorphism groups of the finite subtrees $X^{[d]}$ of X^* of depth d. To each quantum subgroup $\mathbb{P}$ of $\mathbb{A}_d$, we construct a quantum subgroup $A_{\mathbb{P}}$, which we prove is a self-similar quantum group. We then build on the work of Bichon in [3] by constructing free wreath products of compact quantum groups by quantum subgroups of the quantum permutation group (which corresponds to the subalgebra $\mathbb{A}_1$ of $\mathbb{A}_X$), and we prove that every $A_{\mathbb{P}}$ coming from a quantum subgroup $\mathbb{P}$ of $\mathbb{A}_1$ is canonically isomorphic to the free wreath product $A_{\mathbb{P}} *_w \mathbb{P}$.

2. Preliminaries

In this section, we collect some basics on compact quantum groups. We start with Woronowicz's definition of a compact quantum group [15].

Definition 2.1. A *compact quantum group* is a pair (A, Φ) where A is a unital C^*-algebra and $\Phi : A \to A \otimes A$ is a unital $*$-homomorphism such that

(1) $(\Phi \otimes \mathrm{id})\Phi = (\mathrm{id} \otimes \Phi)\Phi$,

(2) $\overline{(A \otimes 1)\Phi(A)} = A \otimes A = \overline{(1 \otimes A)\Phi(A)}$.

We call Φ the *comultiplication* and (1) is called *coassociativity*.

Remark 2.2. It is proved in [15] that (A, Φ) is a compact quantum group if and only if there is a family of matrices $\{a^\lambda = (a^\lambda_{i,j}) \in M_{d_\lambda}(A) : \lambda \in \Lambda\}$ for some indexing set Λ such that

(1) $\Phi(a^\lambda_{i,j}) = \sum_{k=1}^{d_\lambda} a^\lambda_{i,k} \otimes a^\lambda_{k,j}$ for all $\lambda \in \Lambda$ and $1 \le i, j \le d_\lambda$,

(2) a^λ and its transpose $(a^\lambda)^T$ are invertible elements of $M_{d_\lambda}(A)$ for every $\lambda \in \Lambda$,

(3) the $*$-subalgebra $\mathcal{A}$ of A generated by the entries $\{a^\lambda_{i,j} : 1 \le i, j \le d_\lambda,\ \lambda \in \Lambda\}$ is dense in A.

Example 2.3. A key example for us are Wang's *quantum permutation groups* $(A_s(n), \Phi)$ from [12]. Here, n is a positive integer, and $A_s(n)$ is the universal C^*-algebra generated by elements a_{ij}, $1 \le i, j \le n$, satisfying

$$a_{ij}^2 = a_{ij} = a_{ij}^* \quad \text{for all } 1 \le i, j \le n,$$

$$\sum_{j=1}^{n} a_{ij} = 1 \quad \text{for all } 1 \le i \le n,$$

$$\sum_{i=1}^{n} a_{ij} = 1 \quad \text{for all } 1 \le j \le n.$$

The comultiplication Φ satisfies $\Phi(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj}$ for all $1 \le i, j \le n$.

Definition 2.4. If (A_1, Φ_1) and (A_2, Φ_2) are compact quantum groups, then a *morphism* π from (A_1, Φ_1) to (A_2, Φ_2) is a homomorphism of C^*-algebras $\pi\colon A_1 \to A_2$ satisfying $(\pi \otimes \pi) \circ \Phi_1 = \Phi_2 \circ \pi$.

Definition 2.5. Let (A, Φ) be a compact quantum group. A *Woronowicz ideal* is an ideal I of A such that $\Phi(I) \subseteq \ker(q \otimes q)$, where q is the quotient map $A \to A/I$. Then $(A/I, \Phi')$, where $\Phi'\colon A/I \to A/I \otimes A/I$ satisfies $\Phi' \circ q = (q \otimes q) \circ \Phi$ is a compact quantum group called a *quantum subgroup* of (A, Φ).

Definition 2.6. A *(left) coaction* of a compact quantum group (A, Φ) on a unital C^*-algebra B is a unital $*$-homomorphism $\alpha\colon B \to A \otimes B$ satisfying

(1) $(\mathrm{id} \otimes \alpha)\alpha = (\Phi \otimes \mathrm{id})\alpha$,

(2) $\overline{\alpha(B)(A \otimes 1)} = A \otimes B$.

We refer to (1) as the *coaction identity* and (2) is known as the *Podleś condition*.

3. Quantum automorphisms of a homogeneous rooted tree

In this section, we introduce a compact quantum group $\mathbb{A}_X$ which we prove is the quantum automorphism group of the infinite homogeneous rooted tree X^*. We start with the notion of an action of a compact quantum group on X^*. Note that for $n \geq 0$ we write X^n for all the words in X of length n, and then the tree X^* can be identified with $\bigcup_{n\geq} X^n$, where $X^0 = \{\varnothing\}$ and $\varnothing$ is the root of the tree.

Definition 3.1. Let X be a finite set and let (A, Φ) be a compact quantum group. An action of A on the homogeneous rooted tree X^* is a system

$$\alpha = (\alpha_n\colon C(X^n) \to A \otimes C(X^n))$$

of left coactions, such that for any $m < n$ the diagram

$$\begin{array}{ccc} C(X^m) & \xrightarrow{\ i_{m,n}\ } & C(X^n) \\ \big\downarrow{\scriptstyle \alpha_m} & & \big\downarrow{\scriptstyle \alpha_n} \\ A \otimes C(X^m) & \xrightarrow{\mathrm{id} \otimes i_{m,n}} & A \otimes C(X^n) \end{array}$$

commutes, where $i_{m,n} : C(X^m) \to C(X^n)$ is the injective homomorphism satisfying

$$i_{m,n}(p_w) = \sum_{w' \in X^{n-m}} p_{ww'}.$$

We now define the main object of interest in this section, the C^*-algebra $\mathbb{A}_X$, before proving that it is indeed a compact quantum group in Theorem 3.4. At some point in the later stages of this project we became aware of [8], and their notion of the quantum automorphism group QAut Π of a locally finite connected graph Π. A straightforward argument shows that $\mathbb{A}_X$ is QAut Π for Π the homogeneous rooted tree, but we include the proof of Theorem 3.4 for completeness.

Definition 3.2. Let X be a finite set. Define $\mathbb{A}_X$ to be the universal C^*-algebra generated by elements $\{a_{u,v} : u, v \in X^n, n \geq 0\}$ subject to the following relations:

(1) $a_{\varnothing,\varnothing} = 1$,

(2) for any $n \geq 0$, $u, v \in X^n$, $a^*_{u,v} = a^2_{u,v} = a_{u,v}$,

(3) for any $n \geq 0$, $u, v \in X^n$ and $x \in X$,

$$a_{u,v} = \sum_{y \in X} a_{ux,vy} = \sum_{z \in X} a_{uz,vx}.$$

Remarks 3.3. (i) For each $d \in \mathbb{N}$ we denote by $\mathbb{A}_d$ the subalgebra of $\mathbb{A}_X$ generated by $\{a_{u,v} : u, v \in X^d\}$. Note that $\mathbb{A}_1$ is Wang's quantum permutation group $A_s(|X|)$ from Example 2.3.

(ii) We can interpret (3) as follows: each projection $a_{u,v}$ decomposes as an $|X| \times |X|$ square of projections $\{a_{ux,vy} : x, y \in X\}$ with a magic square type property where every row and column sums to $a_{u,v}$. For example, if $X = \{0, 1, 2\}$ we have the following structure:

$$a_{u,v} \mapsto \begin{array}{|c|c|c|}\hline a_{u0,v0} & a_{u0,v1} & a_{u0,v2} \\ \hline a_{u1,v0} & a_{u1,v1} & a_{u1,v2} \\ \hline a_{u2,v0} & a_{u2,v1} & a_{u2,v2} \\ \hline \end{array}.$$

(iii) Repeated applications of (3) from Definition 3.2 show that for all $u, u', v, v', w \in X^n, n \in \mathbb{N}$, we have

$$u \neq u',\ v \neq v' \implies a_{u,w}a_{u',w} = 0 = a_{w,v}a_{w,v'},$$

and that for all $u = u_1 \cdots u_n$, $v = v_1 \cdots v_n \in X^n$, $n \in \mathbb{N}$, and $x, y \in X$ we have

$$a_{x,y}a_{u,v} = a_{u,v}a_{x,y} = \begin{cases} a_{u,v} & \text{if } u_1 = x,\ v_1 = y, \\ 0 & \text{otherwise.} \end{cases}$$

We will freely use these two identities without comment throughout the rest of the paper.

Theorem 3.4. *The C^*-algebra $\mathbb{A}_X$ is a compact quantum group with comultiplication $\Delta\colon \mathbb{A}_X \to \mathbb{A}_X \otimes \mathbb{A}_X$ satisfying*

$$\Delta(a_{u,v}) = \sum_{w \in X^n} a_{u,w} \otimes a_{w,v},$$

for all $u, v \in X^n$ and $n \geq 1$.

Proof. To see that Δ exists, it is enough to show that the elements

$$b_{u,v} := \sum_{w \in X^n} a_{u,w} \otimes a_{w,v}$$

for $u, v \in X^n$ and $n \geq 1$ satisfy Definition 3.2.

Firstly, $b_{\varnothing,\varnothing} = \Delta(a_{\varnothing,\varnothing}) = a_{\varnothing,\varnothing} \otimes a_{\varnothing,\varnothing} = 1 \otimes 1$. For (2), we have

$$b_{u,v}^* = \sum_{w \in X^n} a_{u,w}^* \otimes a_{w,v}^* = \sum_{w \in X^n} a_{u,w} \otimes a_{w,v} = b_{u,v}$$

and

$$\begin{aligned} b_{u,v}^2 &= \Big(\sum_{w \in X^n} a_{u,w} \otimes a_{w,v} \Big)^2 \\ &= \sum_{w,z \in X^n} a_{u,w} a_{u,z} \otimes a_{w,v} a_{z,v} \\ &= \sum_{w \in X^n} a_{u,w}^2 \otimes a_{w,v}^2 \\ &= \sum_{w \in X^n} a_{u,w} \otimes a_{w,v} \\ &= b_{u,v}. \end{aligned}$$

For (3), fix $u, v \in X^n$ and $x \in X$. Then

$$\begin{aligned} b_{u,v} &= \sum_{w \in X^n} a_{u,w} \otimes a_{w,v} \\ &= \sum_{w \in X^n} \sum_{z \in X} a_{ux,wz} \otimes a_{w,v} \\ &= \sum_{w \in X^n} \sum_{z \in X} a_{ux,wz} \otimes \sum_{y \in X} a_{wz,vy} \\ &= \sum_{y \in X} \sum_{w \in X^{n+1}} a_{ux,w} \otimes a_{w,vy} \\ &= \sum_{y \in X} b_{ux,vy}. \end{aligned}$$

So by the universal property of $\mathbb{A}_X$ there is a homomorphism $\Delta \colon \mathbb{A}_X \to \mathbb{A}_X \otimes \mathbb{A}_X$ such that

$$\Delta(a_{u,v}) = \sum_{w \in X^n} a_{u,w} \otimes a_{w,v}.$$

For coassociativity, we have

$$\begin{aligned} (\mathrm{id} \otimes \Delta) \circ \Delta(a_{u,v}) &= \sum_{w \in X^n} a_{u,w} \otimes \Delta(a_{w,v}) \\ &= \sum_{w \in X^n} a_{u,w} \otimes \Big(\sum_{z \in X^n} a_{w,z} \otimes a_{z,v} \Big) \end{aligned}$$

$$= \sum_{z \in X^n} \Big(\sum_{w \in X^n} a_{u,w} \otimes a_{w,z} \Big) \otimes a_{z,v}$$
$$= \sum_{z \in X^n} \Delta(a_{u,z}) \otimes a_{z,v}$$
$$= (\Delta \otimes \mathrm{id}) \circ \Delta(a_{u,v}).$$

Finally, we show that the set of matrices

$$\{a_n = (a_{u,v})_{u,v \in X^n} \in M_{X^n}(\mathbb{A}_X) : n \geq 1\}$$

satisfies the conditions of Remark 2.2. Conditions (1) and (3) are clear. For (2) we show that given any $n \geq 1$ the matrix a_n is invertible with inverse given by $(a_n)^T$. Given $u, v \in X^n$ we have

$$(a_n(a_n)^T)_{u,v} = \sum_{w \in X^n} a_{u,w} a_{v,w} = \delta_{u,v} \sum_{w \in X^n} a_{u,w} = \delta_{u,v} 1_A.$$

Likewise, we can show $((a_n)^T a_n)_{u,v} = \delta_{u,v} 1_A$ and hence $(a_n)^T = a_n^{-1}$ as required. ■

Remark 3.5. The canonical dense $*$-subalgebra of $\mathbb{A}_X$ is the $*$-subalgebra generated by the projections $\{a_{u,v} : u, v \in X^n, n \geq 0\}$. This is a Hopf $*$-algebra with counit $\varepsilon\colon \mathbb{A}_X \to \mathbb{C}$ and coinverse $\kappa\colon \mathbb{A}_X \to \mathbb{A}_X$ satisfying $\varepsilon(a_{u,v}) = \delta_{u,v}$ and $\kappa(a_{u,v}) = a_{v,u}$, for $u, v \in X^n$, $n \in \mathbb{N}$.

We now show that $(\mathbb{A}_X, \Delta)$ is the quantum automorphism group (in the sense of [12, Definition 2.3]) of the homogeneous rooted tree.

Proposition 3.6. *There is an action $\gamma = (\gamma_n)_{n=1}^\infty$ of $\mathbb{A}_X$ on X^*. Moreover, if $\alpha = (\alpha_n)_{n=1}^\infty$ is an action of a compact quantum group (A, Φ) on X^* then there is a quantum group homomorphism $\pi\colon \mathbb{A}_X \to A$ such that $(\pi \otimes \mathrm{id}) \circ \gamma_n = \alpha_n$ for any $n \geq 1$.*

Proof. For any $n \geq 1$, the elements

$$q_w := \sum_{w' \in X^n} a_{w,w'} \otimes p_{w'} \in \mathbb{A}_X \otimes C(X^n)$$

for each $w \in X^n$ are mutually orthogonal projections and satisfy

$$\sum_{w \in X^n} q_w = \sum_{w,w' \in X^n} a_{w,w'} \otimes p_{w'} = 1 \otimes 1.$$

Therefore, there is a unital $*$-homomorphism $\gamma_n\colon C(X^n) \to \mathbb{A}_X \otimes C(X^n)$ satisfying $\gamma_n(p_w) = q_w$. We have

$$(\Delta \otimes \mathrm{id})\gamma_n(p_w) = \sum_{w' \in X^n} \Delta(a_{w,w'}) \otimes p_{w'}$$
$$= \sum_{w',z \in X^n} a_{w,z} \otimes a_{z,w'} \otimes p'_w$$

$$= \sum_{z \in X^n} a_{w,z} \otimes \alpha_n(p_z)$$
$$= (\mathrm{id} \otimes \gamma_n)\gamma_n(p_w),$$

and so each γ_n satisfies the coaction identity.

For a fixed $v \in X^n$ we have

$$\sum_{u \in X^n} \gamma_n(p_u)(a_{u,v} \otimes 1) = \sum_{u,w \in X^n} a_{u,w}a_{u,v} \otimes p_w = \sum_{u \in X^n} a_{u,v} \otimes p_v = 1 \otimes p_v.$$

Multiplying by any element $a \otimes 1 \in \mathbb{A}_X \otimes 1$ shows that $\gamma_n(C(X^n))(\mathbb{A}_X \otimes 1)$ contains the elements $a \otimes p_v$ of $\mathbb{A}_X \otimes C(X^n)$ and hence the required density is satisfied.

Finally, fix $m < n$ and $w \in X^m$. Then

$$\begin{aligned}(\mathrm{id} \otimes i_{m,n})\gamma_m(p_w) &= \sum_{z \in X^m} a_{w,z} \otimes i_{m,n}(p_z) \\ &= \sum_{z \in X^m} \sum_{z' \in X^{n-m}} a_{w,z} \otimes p_{zz'} \\ &= \sum_{z \in X^m} \sum_{z' \in X^{n-m}} \sum_{w' \in X^{n-m}} a_{ww',zz'} \otimes p_{zz'} \\ &= \sum_{w' \in X^{n-m}} \alpha_n(p_{ww'}) \\ &= \gamma_n(i_{m,n}(p_w)),\end{aligned}$$

and so the collection $\gamma = (\gamma_n)_{n=1}^{\infty}$ defines an action of $(\mathbb{A}_X, \Delta)$ on the homogeneous rooted tree X^*.

Now suppose $(\alpha_n)_{n=1}^{\infty}$ is an action of a compact quantum group (A, Φ) on X^*. Let $b_{\varnothing,\varnothing} := 1 \in A$ and for $n \geq 1$ and $u, v \in X^n$ define $b_{u,v} \in A$ to be the unique elements satisfying

$$\alpha_n(p_u) = \sum_{v \in X^n} b_{u,v} \otimes p_v.$$

The coaction identity for α_n says that

$$\Phi(b_{u,v}) = \sum_{w \in X^n} b_{u,w} \otimes b_{w,v} \tag{3.1}$$

for any $u, v \in X^n$.

We claim that the collection $\{b_{u,v} : u, v \in X^n,\ n \geq 0\} \subseteq A$ satisfies Definition 3.2. Condition (1) is by definition. For (2) and (3), we appeal to the universal property of the quantum permutation groups $A_s(|X|^n)$ for $n \geq 1$. Since for any $n \geq 1$, α_n defines a coaction of (A, Φ) on $C(X^n)$, [12, Theorem 3.1] says that the elements $\{b_{u,v} : u, v \in X^n\}$ satisfy conditions (3.1)–(3.3) of [12, Section 3]. Condition (3.1) is precisely (2). Conditions (3.1) and (3.2) say that for any $v \in X^n$ we have

$$\sum_{u \in X^n} b_{u,v} = 1_A = \sum_{w \in X^n} b_{v,w}.$$

For any $u \in X^n$ and $x \in X$ we have

$$p_{ux} \leq \sum_{y \in X} p_{uy} = i_{n,n+1}(p_u),$$

and hence

$$\begin{aligned}\sum_{v \in X^n} \sum_{y \in X} b_{ux,vy} \otimes p_{vy} &= \alpha_{n+1}(p_{ux}) \\ &\leq \alpha_{n+1}(i_{n,n+1}(p_u)) \\ &= (\mathrm{id}_A \otimes i_{n,n+1})\alpha_n(p_u) \\ &= \sum_{v \in X^n} \sum_{y \in X} b_{u,v} \otimes p_{vy}.\end{aligned}$$

It follows that $b_{ux,vy} \leq b_{u,v}$ for any $x, y \in X$. Therefore, for any $u, v \in X^n$ and $x \in X$ we have

$$b_{u,v} = b_{u,v}\Big(\sum_{w \in X^n} \sum_{y \in X} b_{ux,wy}\Big) = \sum_{y \in X} b_{ux,vy}.$$

Likewise for any $y \in X$ we have $b_{u,v} = \sum_{x \in X} b_{ux,vy}$ and (3) holds.

Therefore, the universal property of $\mathbb{A}_X$ provides a homomorphism $\pi \colon \mathbb{A}_X \to A$ satisfying $\pi(a_{u,v}) = b_{u,v}$. It follows from (3.1) that $(\pi \otimes \pi) \circ \Delta = \Phi \otimes \pi$ and so π is a compact quantum group homomorphism. The identity $(\pi \otimes \mathrm{id}) \circ \gamma_n = \alpha_n$ is immediate. ∎

Proposition 3.7. *For $|X| \geq 2$ the C^*-algebra $\mathbb{A}_X$ is noncommutative and infinite-dimensional.*

Proof. Without loss of generality, assume $X = \{0, 1\}$. Let B be the universal unital C^*-algebra generated by two (noncommuting) projections p and q. It is known from [7] that $B \cong C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$, which is noncommutative and infinite-dimensional. Define the matrix

$$(b_{u,v})_{u,v \in X^2} = \begin{pmatrix} p & 1_B - p & 0 & 0 \\ 1_B - p & p & 0 & 0 \\ 0 & 0 & q & 1_B - q \\ 0 & 0 & 1_B - q & q \end{pmatrix} \in M_4(B).$$

Define $b_{\varnothing,\varnothing} = b_{0,0} = b_{1,1} = 1_B$, $b_{0,1} = b_{1,0} = 0$ and for $u, v \in X^2$ and $w, w' \in X^*$ define $b_{uw,vw'} := \delta_{w,w'} b_{u,v}$. Then these elements satisfy the relations in Definition 3.2 and hence there is a surjective homomorphism $\mathbb{A}_X \to B$. Since B is noncommutative and infinite-dimensional so is $\mathbb{A}_X$. ∎

Remark 3.8. The group $\mathrm{Aut}(X^*)$ of automorphisms of a homogeneous rooted tree X^* is a compact totally disconnected Hausdorff group under the permutation topology. A neighbourhood basis of the identity is given by the family of subgroups

$$\big\{G_u := \{g \in \mathrm{Aut}(X^*) : g \cdot u = u\} : u \in X^*\big\},$$

and since the orbit of any $u \in X^*$ is finite, each of these open subgroups is closed and hence compact. Cosets of these subgroups are of the form $G_{u,v} := \{g \in G : g \cdot v = u\}$. Then $\{G_{u,v} : u, v \in X^*\}$ is a basis of compact open sets for the topology on $\mathrm{Aut}(X^*)$. It follows that the indicator functions $f_{u,v} := 1_{G_{u,v}}$ span a dense subset of $C(\mathrm{Aut}(X^*))$. It is easily checked that the elements $f_{u,v}$ satisfy (1)–(3) of Definition 3.2 and the universal property of $C(\mathrm{Aut}(X^*))$ then implies that it is the abelianisation of $\mathbb{A}_X$.

4. Self-similarity

If $g \in \mathrm{Aut}(X^*)$ and $x \in X$, the *restriction* $g|_x$ is the unique element of $\mathrm{Aut}(X^*)$ satisfying

$$g \cdot (xw) = (g \cdot x) g|_x \cdot w \quad \text{for all } w \in X^*.$$

A subgroup $G \leq \mathrm{Aut}(X^*)$ is called *self-similar* if G is closed under taking restrictions. That is, whenever $g \in G$ and $x \in X$, the restriction $g|_x$ is an element of G. With the topology inherited from $\mathrm{Aut}(X^*)$, the restriction map $G \to G\colon g \mapsto g|_x$ is continuous. If G is any group acting on X^* by automorphisms, we call the action *self-similar* if the image of G in $\mathrm{Aut}(X^*)$ is self-similar.

To have a reasonable notion of self-similarity for quantum subgroups of $\mathbb{A}_X$, we need to understand how restriction manifests itself in the function algebra $C(\mathrm{Aut}(X^*))$. Given $x \in X$ and $u, v \in X^n$ we have

$$\begin{aligned}\{g : g|_x \cdot u = v\} &= \Big(\bigcup_{y \in X} \{g : g \cdot x = y\}\Big) \cap \{g : g|_x \cdot u = v\}\\ &= \bigcup_{y\in X} \big(\{g : g \cdot x = y\} \cap \{g : g|_x \cdot u = v\}\big)\\ &= \bigcup_{y \in X} \{g : g \cdot (xu) = yv\},\end{aligned}$$

and hence the corresponding indicator functions satisfy

$$1_{\{g : g|_x \cdot u = v\}} = \sum_{y \in X} 1_{\{g : g \cdot (xu) = yv\}}.$$

This formula motivates the following result.

Proposition 4.1. *For each $x \in X$ there is a homomorphism $\rho_x\colon \mathbb{A}_X \to \mathbb{A}_X$ satisfying*

$$\rho_x(a_{u,v}) = \sum_{y \in X} a_{yu,xv},$$

for all $u, v \in X^n$.

We illustrate the formula for a restriction map in Figure 1 by considering $X = \{0, 1, 2\}$ and looking at what the restriction map ρ_1 does to the projection $a_{1,2}$.

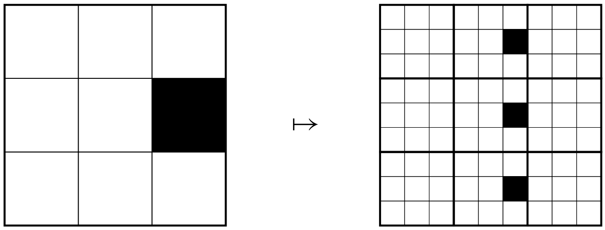

Figure 1. $\rho_1(a_{1,2}) = a_{01,12} + a_{11,12} + a_{21,12}$

Proof of Proposition 4.1. Fix $x \in X$. We show that the elements

$$\{b_{u,v} := \rho_x(a_{u,v}) : u, v \in X^n,\ n \geq 1\}$$

satisfy the conditions of Definition 3.2. For (1) we have

$$b_{\varnothing,\varnothing} = \rho_x(a_{\varnothing,\varnothing}) = \sum_{y \in X} a_{y,x} = 1.$$

For (2), we have

$$b_{u,v}^* = \Big(\sum_{y \in X} a_{yu,xv}\Big)^* = \sum_{y \in X} a_{yu,xv}^* = \sum_{y \in X} a_{yu,xv} = b_{u,v}$$

and

$$b_{u,v}^2 = \Big(\sum_{y \in X} a_{yu,xv}\Big)^2 = \sum_{y,z \in X} a_{yu,xv} a_{zu,xv} = \Big(\sum_{y \in X} a_{yu,xv}\Big) = b_{u,v}.$$

For (3), fix $y \in X$. Then

$$\sum_{z \in X} b_{uy,vz} = \sum_{z \in X} \sum_{w \in X} a_{wuy,xvz} = \sum_{w \in X} a_{wu,xv} = b_{u,v}.$$

A similar calculation shows $\sum_{z \in X} b_{uz,vy} = b_{u,v}$. Hence there is a homomorphism ρ_x with the desired formula. ∎

Remark 4.2. We define $\rho_\varnothing$ to be the identity homomorphism $\mathbb{A}_X \to \mathbb{A}_X$, and for $w = w_1 \cdots w_n \in X^n$ we define ρ_w to be the composition $\rho_{w_1} \circ \cdots \circ \rho_{w_n}$. A routine calculation shows that for all $u, v \in X^n$ we have

$$\rho_w(a_{u,v}) = \sum_{z \in X^n} a_{zu,wv}.$$

Remark 4.3. A similar argument to the one in the proof of Proposition 4.1 shows that for each $x \in X$ there is a homomorphism $\sigma_x \colon \mathbb{A}_X \to \mathbb{A}_X$ satisfying

$$\sigma_x(a_{u,v}) = \sum_{y \in X} a_{xu,yv}$$

for all $u, v \in X^n$, $n \in \mathbb{N}$. It is straightforward to see that $\sigma_x = \kappa \circ \rho_x \circ \kappa$, where κ is the coinverse.

We can now state the main definition of the paper.

Definition 4.4. We call ρ_w the *restriction by* w. A quantum subgroup A of $\mathbb{A}_X$ is *self-similar* if for each $x \in X$ the restriction ρ_x factors through the quotient map $q\colon \mathbb{A}_X \to A$; that is, if there exists a homomorphism $\widetilde{\rho_x}\colon A \to A$ such that the diagram

$$\begin{array}{ccc} \mathbb{A}_X & \xrightarrow{\rho_x} & \mathbb{A}_X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{\widetilde{\rho_x}} & A \end{array}$$

commutes.

To motivate the main result of this section, let G be a group. To construct a self-similar action of G on X^*, it suffices to have a function $f\colon G \times X \to X \times G$ such that $f(e, x) = (x, e)$ for all $x \in X$, and such that the following diagram commutes:

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m_G \times \mathrm{id}_X} & G \times X \\ \downarrow{\scriptstyle \mathrm{id}_G \times f} & & \\ G \times X \times G & & \downarrow{\scriptstyle f} \\ \downarrow{\scriptstyle f \times \mathrm{id}_G} & & \\ X \times G \times G & \xrightarrow{\mathrm{id}_X \times m_G} & X \times G. \end{array}$$

This data allows us to define an action of G on X^*, which is self-similar with $g \cdot x$ and $g|_x$ the unique elements of X and G satisfying $(g \cdot x, g|_x) := f(g, x)$.

Our next result is a compact quantum group analogue of the above result. We will be working with multiple different identity homomorphisms and units. For clarity we adopt the following notational conventions: we write id_A for the identity homomorphism on a C^*-algebra A, and for $n \geq 1$ write id_n for the identity homomorphism on the commutative C^*-algebra $C(X^n)$. Likewise, 1_A will denote the unit of A, 1 and 1_n will denote the units of $C(X)$ and $C(X^n)$ respectively.

Theorem 4.5. *Suppose* (A, Φ) *is a compact quantum group equipped with a unital* $*$-*homomorphism* $\psi\colon C(X) \otimes A \to A \otimes C(X)$ *satisfying*

$$(\Phi \otimes \mathrm{id}_1)\psi = (\mathrm{id}_A \otimes \psi)(\psi \otimes \mathrm{id}_A)(\mathrm{id}_1 \otimes \Phi) \tag{4.1}$$

and

$$\overline{\psi(C(X) \otimes 1_A)(A \otimes 1)} = A \otimes C(X). \tag{4.2}$$

Then (A, Φ) *acts on the homogeneous rooted tree* X^* *and, moreover, the image of* $\mathbb{A}_X$*, under the homomorphism* $\pi\colon \mathbb{A}_X \to A$ *from Proposition 3.6, is a self-similar compact quantum group.*

Proof. We begin by defining an action of (A, Φ) on X^*. Identify $C(X)$ with $C(X) \otimes 1_A \subseteq C(X) \otimes A$ and let $\alpha_1 := \psi|_{C(X) \otimes 1_A}$. Then α_1 is clearly unital and the coaction identity and Podleś condition for α_1 follow from (4.1) and (4.2). Now inductively define $\alpha_{n+1} := (\psi \otimes \mathrm{id}_n)(\mathrm{id}_1 \otimes \alpha_n)\colon C(X^{n+1}) \to A \otimes C(X^{n+1})$ for $n \geq 1$, where we are suppressing the canonical isomorphism $C(X^{n+1}) \cong C(X) \otimes C(X^n)$. Again, α_{n+1} is clearly unital whenever α_n is. If we assume α_n satisfies the coaction identity, then

$$\begin{aligned}(\Phi \otimes \mathrm{id}_{n+1})\alpha_{n+1} &= (\Phi \otimes \mathrm{id}_{n+1})(\psi \otimes \mathrm{id}_n)(\mathrm{id}_1 \otimes \alpha_n)\\ &= (\mathrm{id}_A \otimes \psi \otimes \mathrm{id}_n)(\psi \otimes \mathrm{id}_A \otimes \mathrm{id}_n)(\mathrm{id}_1 \otimes \Phi \otimes \mathrm{id}_n)(\mathrm{id}_1 \otimes \alpha_n)\\ &= (\mathrm{id}_A \otimes \psi \otimes \mathrm{id}_n)(\psi \otimes \mathrm{id}_A \otimes \mathrm{id}_n)(\mathrm{id}_1 \otimes \mathrm{id}_A \otimes \alpha_n)(\mathrm{id}_1 \otimes \alpha_n)\\ &= (\mathrm{id}_A \otimes \psi \otimes \mathrm{id}_n)(\mathrm{id}_A \otimes \mathrm{id}_1 \otimes \alpha_n)(\psi \otimes \mathrm{id}_n)(\mathrm{id}_1 \otimes \alpha_n)\\ &= (\mathrm{id}_A \otimes \alpha_{n+1})\alpha_{n+1},\end{aligned}$$

and so α_{n+1} also satisfies the coaction identity. Since α_1 is a coaction, we see that α_n satisfies the coaction identity for any $n \geq 1$.

To see that each α_n satisfies the Podleś condition, we argue by induction. We know it is satisfied for $n = 1$. Suppose for some $n \geq 1$ that

$$\overline{\alpha_n(C(X^n))(A \otimes 1_n)} = A \otimes C(X^n).$$

Fix a spanning element $a \otimes p_u \otimes p_x \in A \otimes C(X^{n+1})$ where $u \in X^n$ and $x \in X$. By the inductive hypothesis we can approximate

$$a \otimes p_u \sim \sum_i \alpha_n(f_i)(a_i \otimes 1_n),$$

where $f_i \in C(X^n)$ and $a_i \in A$. Then

$$a \otimes p_u \otimes p_x \sim \sum_i (\alpha_n(f_i) \otimes 1)(1_A \otimes 1_n \otimes p_x)(a_i \otimes 1_{n+1}). \tag{4.3}$$

By definition of α_n, for any $f \in C(X^n)$ we have

$$\begin{aligned}\alpha_n(f) \otimes 1 &= \big((\psi \otimes \mathrm{id}_{n-1}) \cdots (\mathrm{id}_{n-1} \otimes \psi)(f \otimes 1_A)\big) \otimes 1\\ &= (\psi \otimes \mathrm{id}_n) \cdots (\mathrm{id}_{n-1} \otimes \psi \otimes \mathrm{id}_1)(f \otimes 1_A \otimes 1)\\ &= (\psi \otimes \mathrm{id}_n) \cdots (\mathrm{id}_{n-1} \otimes \psi \otimes \mathrm{id}_1)(\mathrm{id}_n \otimes \psi)(f \otimes 1 \otimes 1_A)\\ &= \alpha_{n+1}(f \otimes 1).\end{aligned}$$

So we can write (4.3) as

$$\sum_i \alpha_{n+1}(f_i \otimes 1)(1_A \otimes 1_n \otimes p_x)(a_i \otimes 1_{n+1}).$$

Since ψ is unital, we have

$$1_A \otimes 1_n \otimes p_x = (\psi \otimes \mathrm{id}_n)(1 \otimes 1_A \otimes 1_{n-1} \otimes p_x),$$

which can be approximated using the induction hypothesis by

$$\begin{aligned}(\psi \otimes \mathrm{id}_n)(1 \otimes 1_A \otimes 1_{n-1} \otimes p_x) &\sim (\psi \otimes \mathrm{id}_n)\Big(1 \otimes \sum_j \alpha_n(g_j)(b_j \otimes 1_n)\Big)\\ &= (\psi \otimes \mathrm{id}_n)\Big(\sum_j (\mathrm{id}_1 \otimes \alpha_n)(1 \otimes g_j)(1 \otimes b_j \otimes 1_n)\Big)\\ &= \sum_j \alpha_{n+1}(1 \otimes g_j)(\psi \otimes \mathrm{id}_n)(1 \otimes b_j \otimes 1_n).\end{aligned}$$

Finally, applying the Podleś condition for α_1, we can approximate

$$\psi(1 \otimes b_j) \sim \sum_k \alpha_1(h_k)(c_k \otimes 1) = \sum_k \psi(h_k \otimes 1_A)(c_k \otimes 1),$$

so

$$\begin{aligned}(\psi \otimes \mathrm{id}_n)(1 \otimes b_j \otimes 1_n) &\sim \sum_k (\psi(h_k \otimes 1_A) \otimes 1_n)(c_k \otimes 1_{n+1})\\ &= \sum_k \alpha_{n+1}(h_k \otimes 1_n)(c_k \otimes 1_{n+1}).\end{aligned}$$

Combining these approximations we can write

$$a \otimes p_u \otimes p_x \sim \sum_{i,j,k} \alpha_{n+1}((f_i \otimes 1)(h_k \otimes g_j))(c_k a_i \otimes 1_{n+1}),$$

where $f_i, g_j \in C(X^n)$, $h_k \in C(X)$ and $a_i, c_k \in A$. Thus α_{n+1} satisfies the Podleś condition and so by induction α_n satisfies the Podleś condition for every $n \geq 1$.

It remains to show that $\alpha_n \circ i_{m,n} = (\mathrm{id}_A \otimes i_{m,n}) \circ \alpha_m$ for any $m < n$. As in the proof of Proposition 3.6 , for any $n \geq 1$ and $u, v \in X^n$ we will let $b_{u,v} \in A$ be the unique elements satisfying

$$\alpha_n(p_u) = \sum_{v \in X^n} b_{u,v} \otimes p_v.$$

We know from the same proof that for any $n \geq 1$ and $v \in X^n$ we have

$$\sum_{u \in X^n} b_{u,v} = 1_A.$$

If $m < n$, for any $u \in X^m$ we have

$$\begin{aligned}\alpha_n \circ i_{m,n}(p_u) &= (\psi \otimes \mathrm{id}_{n-1}) \cdots (\mathrm{id}_{m-1} \otimes \psi \otimes \mathrm{id}_{n-m})(\mathrm{id}_m \otimes \alpha_{n-m})(i_{m,n}(p_u))\\ &= \sum_{w \in X^{n-m}} (\psi \otimes \mathrm{id}_{n-1}) \cdots (\mathrm{id}_{m-1} \otimes \psi \otimes \mathrm{id}_{n-m})(\mathrm{id}_m \otimes \alpha_{n-m})(p_u \otimes p_w)\\ &= \sum_{w,z \in X^{n-m}} (\psi \otimes \mathrm{id}_{n-1}) \cdots (\mathrm{id}_{m-1} \otimes \psi \otimes \mathrm{id}_{n-m})(p_u \otimes b_{w,z} \otimes p_z)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{z\in X^{n-m}} (\psi\otimes \mathrm{id}_{m-1})\cdots(\mathrm{id}_{m-1}\otimes\psi)\Big(p_u\otimes \sum_{w\in X^{n-m}} b_{w,z}\Big)\otimes p_z\\
&= \sum_{z\in X^{n-m}} (\psi\otimes \mathrm{id}_{m-1})\cdots(\mathrm{id}_{m-1}\otimes\psi)(p_u\otimes 1_A)\otimes p_z\\
&= \sum_{z\in X^{n-m}} \alpha_m(p_u)\otimes p_z\\
&= \sum_{v\in X^m} b_{u,v}\otimes \sum_{z\in X^{n-m}} p_v\otimes p_z\\
&= \sum_{v\in X^m} b_{u,v}\otimes i_{m,n}(p_v)\\
&= (\mathrm{id}_A\otimes i_{m,n})\circ\alpha_m(p_u).
\end{aligned}
$$

So we have that $(\alpha_n)_{n=1}^{\infty}$ defines an action of (A,Φ) on X^*.

Finally, let $\pi\colon \mathbb{A}_X\to A$ be the homomorphism from Proposition 3.6. We have

$$\pi(a_{u,v}) = b_{u,v}$$

for any $u,v\in X^n$ and $n\geq 1$. For each $x\in X$, define a homomorphism $\widetilde{\rho_x}\colon A\to A$ by

$$\psi(1\otimes a) = \sum_{x\in X}\widetilde{\rho_x}(a)\otimes p_x,$$

where $a\in A$. For any $u\in X^n$ we have

$$\alpha_{n+1}(1\otimes p_u) = \sum_{y\in X}\alpha_{n+1}(p_{yu}) = \sum_{v\in X^n}\sum_{x,y\in X} b_{yu,xv}\otimes p_x\otimes p_v.$$

On the other hand, we know $\alpha_{n+1} = (\psi\otimes\mathrm{id}_n)(\mathrm{id}_1\otimes\alpha_n)$ and

$$(\psi\otimes\mathrm{id}_n)(\mathrm{id}_1\otimes\alpha_n)(1\otimes p_u) = \sum_{v\in X^n}\psi(1\otimes b_{u,v})\otimes p_v = \sum_{v\in X^n}\sum_{x\in X}\widetilde{\rho_x}(b_{u,v})\otimes p_x\otimes p_v,$$

and by comparing tensor factors, we see that $\widetilde{\rho_x}(b_{u,v}) = \sum_{y\in X} b_{yu,xv}$. Hence, the diagram

$$
\begin{array}{ccc}
\mathbb{A}_X & \xrightarrow{\ \rho_x\ } & \mathbb{A}_X\\
\big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi}\\
\pi(\mathbb{A}_X) & \xrightarrow{\ \widetilde{\rho_x}\ } & \pi(\mathbb{A}_X)
\end{array}
$$

commutes, and so $\pi(\mathbb{A}_X)\subseteq A$ is a self-similar quantum group. ∎

Proposition 4.6. *The following are equivalent:*

(1) (A,Φ) *is a quantum self-similar group, and*

(2) (A,Φ) *is a quantum subgroup of* $(\mathbb{A}_X,\Delta)$ *and there is a homomorphism* $\psi\colon C(X)\otimes A\to A\otimes C(X)$ *satisfying the hypotheses of Theorem 4.5.*

Proof. Theorem 4.5 is the implication (2) ⇒ (1). To see (1) ⇒ (2) suppose (A, Φ) is a quantum self-similar group. By definition there is a surjective quantum group morphism $q\colon \mathbb{A}_X \to A$. It is routine to check that there is a homomorphism $\psi\colon C(X) \otimes A \to A \otimes C(X)$ satisfying

$$\psi(p_x \otimes q(a_{u,v})) = \sum_{y \in X} q(a_{xu,yv}) \otimes p_y.$$

Given $u, v \in X^n$ we have

$$\begin{aligned}
(\Phi \otimes \mathrm{id}_1)\psi(p_x \otimes q(a_{u,v})) &= \sum_{y \in X} \Phi(q(a_{xu,yv})) \otimes p_y \\
&= \sum_{w \in X^n} \sum_{y,z \in X} q(a_{xu,zw}) \otimes q(a_{zw,yv}) \otimes p_y \\
&= (\mathrm{id}_A \otimes \psi)\Big(\sum_{w \in X^n} \sum_{z \in X} q(a_{xu,zw}) \otimes p_z \otimes q(a_{w,v}) \Big) \\
&= (\mathrm{id}_A \otimes \psi)(\psi \otimes \mathrm{id}_A)\Big(\sum_{w \in X^n} p_x \otimes q(a_{u,w}) \otimes q(a_{w,v}) \Big) \\
&= (\mathrm{id}_A \otimes \psi)(\psi \otimes \mathrm{id}_A)(\mathrm{id}_1 \otimes \Phi)(p_x \otimes q(a_{u,v})),
\end{aligned}$$

and so ψ satisfies (4.1). For (4.2) notice that for any $q(a) \in A$ and $z \in X$ we have

$$\begin{aligned}
q(a) \otimes p_z &= (1 \otimes p_z)(q(a) \otimes 1) \\
&= \Big(\sum_{x \in X} q(a_{x,z}) \otimes p_z \Big)(q(a) \otimes 1) \\
&= \Big(\sum_{x,w \in X} (q(a_{x,w}) \otimes p_w)(q(a_{x,z}) \otimes 1) \Big)(q(a) \otimes 1) \\
&= \sum_{x \in X} \psi(p_x \otimes 1)(q(a_{x,z}a) \otimes 1).
\end{aligned}$$ ∎

Example 4.7. If G is a closed subgroup of $\mathrm{Aut}(X^*)$ which is self-similar, then $C(G)$ is a commutative self-similar quantum group. The quotient map $\mathbb{A}_X \to C(\mathrm{Aut}(X^*))$ takes a generator $a_{u,v}$ to the indicator function $f_{u,v}$ defined in Remark 3.8. For a function $f \in C(\mathrm{Aut}(X^*))$ and $x \in X$ the restriction homomorphism $\widetilde{\rho_x}$ satisfies $\widetilde{\rho_x}(f)(g) = f(g|_x)$, for any $g \in G$.

5. Finitely constrained self-similar quantum groups

5.1. Classical finitely constrained self-similar groups

Fix $d \geq 1$, and let $X^{[d]} = \bigcup_{k \leq d} X^k$ be the finite subtree of X^* of depth d. The group of automorphisms $\mathrm{Aut}(X^{[d]})$ is a quotient of $\mathrm{Aut}(X^*)$, and the quotient map is given by restriction to the finite subtree. We write $r_d\colon \mathrm{Aut}(X^*) \to \mathrm{Aut}(X^{[d]})$ for this restriction map.

Fix a subgroup $P \leq \mathrm{Aut}(X^{[d]})$. Define

$$G_P := \{g \in \mathrm{Aut}(X^*) : r_d(g|_w) \in P \text{ for all } w \in X^*\}.$$

By the properties of restriction, if $g, h \in G_P$, then for any $w \in X^*$

$$r_d((gh)|_w) = r_d(g|_{h\cdot w} h|_w) = r_d(g|_{h\cdot w}) r_d(h|_w) \in P.$$

Likewise, $r_d(g^{-1}|_w) = r_d(g|_{g^{-1}\cdot w})^{-1} \in P$. Hence G_P is a self-similar group, called a *finitely constrained self-similar group*. More details for these groups can be found in [9].

5.2. Finitely constrained self-similar quantum groups

Consider the subalgebra $\mathbb{A}_d \subseteq \mathbb{A}_X$ generated by the elements $\{a_{u,v} : |u| = |v| \leq d\}$. Since $\Delta\colon \mathbb{A}_d \to \mathbb{A}_d \otimes \mathbb{A}_d$, the subalgebra $\mathbb{A}_d$ is a quotient quantum group. The abelianisation of $\mathbb{A}_d$ is the algebra $C(\mathrm{Aut}(X^{[d]})$ of continuous functions on the finite group $\mathrm{Aut}(X^{[d]})$.

Definition 5.1. Suppose $\mathbb{P}$ is a quantum subgroup of $\mathbb{A}_d$, where $\mathbb{P} = \mathbb{A}_d/I$. Denote by $q_I\colon \mathbb{A}_d \to \mathbb{P}$ the quotient map; so $I = \ker(q_I)$. We denote by J the smallest closed 2-sided ideal of $\mathbb{A}_X$ generated by $\{\rho_w(I) : w \in X^*\}$, and by $A_{\mathbb{P}}$ the quotient $A_{\mathbb{P}} := \mathbb{A}_X/J$. In the next result we prove that $A_{\mathbb{P}}$ is a self-similar quantum group, and we call it a *finitely constrained self-similar quantum group*.

Proposition 5.2. *Each $A_{\mathbb{P}}$ is a self-similar quantum group.*

To prove Proposition 5.2 we need two lemmas. Recall that for $g, h \in \mathrm{Aut}(X^*)$, $w \in X^*$ we have

$$(gh)|_w = g|_{h\cdot w} h|_w.$$

In the first lemma, we establish an analogous relationship between the comultiplication Δ on $\mathbb{A}_X$ and the restriction maps ρ_w.

Lemma 5.3. *For any $n \geq 1$, $w \in X^n$ and $a \in \mathbb{A}_X$ we have*

$$(\Delta \circ \rho_w)(a) = \sum_{y \in X^n} (1 \otimes a_{y,w})(\rho_y \otimes \rho_w)(\Delta(a)).$$

Proof. Let $a_{u,v}$ be a generator of $\mathbb{A}_X$, with $|u| = |v| = k \geq 0$. Then

$$\begin{aligned}
(\Delta \circ \rho_w)(a_{u,v}) &= \Delta\Big(\sum_{\alpha \in X^n} a_{\alpha u, wv}\Big) \\
&= \sum_{y \in X^n} \sum_{\beta \in X^k} \sum_{\alpha \in X^n} a_{\alpha u, y\beta} \otimes a_{y\beta, wv} \\
&= \sum_{y \in X^n} \sum_{\beta \in X^k} \rho_y(a_{u,\beta}) \otimes a_{y\beta, wv} \\
&= \sum_{y \in X^n} \sum_{\beta \in X^k} \rho_y(a_{u,\beta}) \otimes a_{y,w} \rho_w(a_{\beta,v})
\end{aligned}$$

$$= \sum_{y\in X^n} (1 \otimes a_{y,w})(\rho_y \otimes \rho_w)\Big(\sum_{\beta\in X^k} a_{u,\beta} \otimes a_{\beta,v} \Big)$$
$$= \sum_{y\in X^n} (1 \otimes a_{y,w})(\rho_y \otimes \rho_w)(\Delta(a_{u,v})).$$

To see that this formula extends to $\mathbb{A}_X$, it is enough to show that for any $w \in X^*$ the map

$$a \mapsto \sum_{y\in X^n} (1 \otimes a_{y,w})(\rho_y \otimes \rho_w)(\Delta(a))$$

is linear and multiplicative. Linearity is clear, and multiplicativity follows from the orthogonality of the projections $1 \otimes a_{y,w}$ and $1 \otimes a_{z,w}$ for $y \neq z$. ∎

Lemma 5.4. *Consider the quotient maps $q_I\colon \mathbb{A}_d \to \mathbb{A}_d/I$ and $q_J\colon \mathbb{A}_X \to \mathbb{A}_X/J$. Then for any $n \geq 1$ and $y, w \in X^n$*

$$\ker(q_I \otimes q_I) \subseteq \ker((q_J \circ \rho_y) \otimes (q_J \circ \rho_w)).$$

Proof. By definition of J we have $I \subseteq J \circ \rho_w$ for any $w \in X^*$. Therefore there is a commuting diagram

$$\begin{array}{ccc} \mathbb{A}_d & \hookrightarrow & \mathbb{A}_X \\ \downarrow{\scriptstyle q_I} & & \downarrow{\scriptstyle q_J\circ\rho_w} \\ \mathbb{A}_d/I & \xrightarrow{\pi_w} & \mathbb{A}_X/\ker(q_J \circ \rho_w). \end{array}$$

Then if $c \in \ker(q_I \otimes q_I)$ we have

$$(q_J \circ \rho_y) \otimes (q_J \circ \rho_w)(c) = (\pi_y \circ q_I) \otimes (\pi_w \circ q_I)(c) = (\pi_y \otimes \pi_w) \circ (q_I \otimes q_I)(c) = 0,$$

as required. ∎

Proof of Proposition 5.2. To see that $A_{\mathbb{P}}$ is a compact quantum group, it suffices to show that J is a Woronowicz ideal. In other words, we need to show that $\Delta(J) \subset \ker(q_J \otimes q_J)$ where $q_J\colon \mathbb{A}_X \to \mathbb{A}_X/J =: A_{\mathbb{P}}$ is the quotient map. Since J is generated as an ideal by $\bigcup_{w\in X^*} \rho_w(I)$, it is enough to show that

$$(q_J \otimes q_J)(\Delta \circ \rho_w(i)) = 0$$

for any $i \in I$ and $w \in X^*$. Since I is a Woronowicz ideal we know that $\Delta(i) \in \ker(q_I \otimes q_I)$. Then by Lemmas 5.3 and 5.4 we have

$$(q_J \otimes q_J)(\Delta \circ \rho_w(i)) = (q_J \otimes q_J)\Big(\sum_{y\in X^n} (1 \otimes a_{y,w})(\rho_y \otimes \rho_w)(\Delta(i)) \Big)$$
$$= \sum_{y\in X^n} (1 \otimes q_J(a_{y,w}))(q_J \circ \rho_y \otimes q_J \circ \rho_w)(\Delta(i))$$
$$= 0.$$

Finally, $A_{\mathbb{P}}$ is self-similar since by definition of J we have $\rho_w(J) \subset J$ for any $w \in X^*$. ∎

5.3. Free wreath products

It is well known that for any $d \geq 1$ the group $\mathrm{Aut}(X^{[d+1]})$ is isomorphic to the wreath product $\mathrm{Aut}(X^{[d]}) \wr \mathrm{Sym}(X)$. Since $\mathrm{Aut}(X^*)$ is the inverse limit over d of the groups $\mathrm{Aut}(X^{[d]})$, it can be thought as the infinitely iterated wreath product $\ldots \wr \mathrm{Sym}(X) \wr \mathrm{Sym}(X)$. It follows that $\mathrm{Aut}(X^*) \cong \mathrm{Aut}(X^*) \wr \mathrm{Sym}(X)$. More generally, it is shown in [4] that if $P \leq \mathrm{Sym}(X) = \mathrm{Aut}(X^{[1]})$, then the finitely constrained self-similar group G_P is the infinitely iterated wreath product $\ldots \wr P \wr P$. In this section, we prove in Theorem 5.7 an analogue of this result for finitely constrained self-similar quantum groups.

In [3], Bichon constructs a free wreath product of a compact quantum group by the quantum permutation group $\mathbb{A}_s(n)$. Bichon also comments in [3, Remark 2.4] that there is a natural analogue of this construction for free wreath products by quantum subgroups of $\mathbb{A}_s(n)$. In this section, we formally extend this definition to take free wreath products by any quantum subgroup of $\mathbb{A}_s(n)$, and we prove that the finitely constrained self-similar quantum group $A_{\mathbb{P}}$ induced from a quantum subgroup $\mathbb{P}$ of $A_s(n)$ is a free wreath product by $\mathbb{P}$. We begin by recalling the definition of the free wreath product from [3]; note that we use our notation $\mathbb{A}_1$ instead of $A_s(|X|)$.

Definition 5.5. Let X be a set of at least two elements. Let (A, Φ) be a compact quantum group, and $\mathbb{P}$ a quantum subgroup of $\mathbb{A}_1$. For each $x \in X$, we denote by ν_x the inclusion of A in the free product C^*-algebra $(*_{x\in X} A) * \mathbb{P}$. The *free wreath product* of A by $\mathbb{P}$ is the quotient of $(*_{x\in X} A) * \mathbb{P}$ by the two-sided ideal generated by the elements

$$\nu_x(a)q_I(a_{x,y}) - q_I(a_{x,y})\nu_x(a), \quad x, y \in X,\ a \in A.$$

The resulting C^*-algebra is denoted by $A *_{X,w} \mathbb{P}$, and the quotient map is denoted by q_w. If X is understood, we typically just write $A *_w \mathbb{P}$.

Theorem 5.6. *Let (A, Φ) be a compact quantum group, and $\mathbb{P}$ a quantum subgroup of $\mathbb{A}_1$. The free wreath product $A *_w \mathbb{P}$ from Definition 5.5 is a compact quantum group with comultiplication Φ_w satisfying*

$$\Phi_w(q_w(q_I(a_{x,y}))) = \sum_{z\in X} q_w(q_I(a_{x,z})) \otimes q_w(q_I(a_{z,y})), \tag{5.1}$$

$$\Phi_w(q_w(\nu_x(a))) = \sum_{z\in X} (q_w \otimes q_w)\big((\nu_x \otimes \nu_z)(\Phi(a))(q_I(a_{x,z}) \otimes 1)\big), \tag{5.2}$$

for each $x, y \in X$ and $a \in A$.

Proof. Since I is a Woronowicz ideal, we have $\Delta|_I \subseteq \ker(q_I \otimes q_I)$, and so the map $(q_I \otimes q_I) \circ \Delta_{\mathbb{A}_1}$ descends to a map

$$\phi\colon \mathbb{P} \to \mathbb{P} \otimes \mathbb{P} \subseteq ((*_{x\in X} A) * \mathbb{P})^{\otimes 2}.$$

Then $(q_w \otimes q_w) \circ \phi\colon \mathbb{P} \to (A *_w \mathbb{P})^{\otimes 2}$ satisfies

$$(q_w \otimes q_w) \circ \phi(q_I(a_{x,y})) = \sum_{z\in X} q_w(q_I(a_{x,z})) \otimes q_w(q_I(a_{z,y})) \quad \text{for all } x, y \in X.$$

For each $x \in X$, consider the continuous linear map $\phi_x \colon A \to (A *_w \mathbb{P})^{\otimes 2}$ given by

$$\phi_x(a) = (q_w \otimes q_w)\Big(\sum_{z\in X}(\nu_x \otimes \nu_z)(\Phi(a))(q_I(a_{x,z}) \otimes 1)\Big).$$

We claim that ϕ_x is a homomorphism. To see this, let $\{a^\lambda = (a^\lambda_{i,j}) \in M_{d_\lambda}(A) : \lambda \in \Lambda\}$ be a family of matrices satisfying (1)–(3) of Remark 2.2, and $\mathcal{A}$ be the $*$-subalgebra of A spanned by the entries $a^\lambda_{i,j}$. Let $a, b \in \mathcal{A}$ and use Sweedler's notation to write $\Phi(a) = a_{(1)} \otimes a_{(2)}$ and $\Phi(b) = b_{(1)} \otimes b_{(2)}$. We have

$$\phi_x(a)\phi_x(b) = \sum_{z,z'\in X} q_w\big(\nu_x(a_{(1)})q_I(a_{x,z})\nu_x(b_{(1)})q_I(a_{x,z'})\big) \otimes q_w\big(\nu_z(a_{(2)})\nu_{z'}(b_{(2)})\big),$$

and then since

$$\begin{aligned} q_w\big(\nu_x(a_{(1)})q_I(a_{x,z})\nu_x(b_{(1)})q_I(a_{x,z'})\big) &= q_w\big(\nu_x(a_{(1)}b_{(1)})q_I(a_{x,z}a_{x,z'})\big) \\ &= \delta_{z,z'}q_w\big(\nu_x(a_{(1)}b_{(1)})q_I(a_{x,z})\big), \end{aligned}$$

we have

$$\begin{aligned} \phi_x(a)\phi_x(b) &= \sum_{z\in X} q_w\big(\nu_x(a_{(1)}b_{(1)})q_I(a_{x,z})\big) \otimes q_w\big(\nu_z(a_{(2)})\nu_z(b_{(2)})\big) \\ &= (q_w \otimes q_w)\Big(\sum_{z\in X}(\nu_x \otimes \nu_z)(a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)})(q_I(a_{x,z}) \otimes 1)\Big) \\ &= (q_w \otimes q_w)\Big(\sum_{z\in X}(\nu_x \otimes \nu_z)(\Phi(ab))(q_I(a_{x,z}) \otimes 1)\Big) \\ &= \phi_x(ab). \end{aligned}$$

Since $\mathcal{A}$ is dense in A, it follows that ϕ_x is a homomorphism on A.

The universal property of $(*_{x\in X} A) * \mathbb{P}$ now gives a homomorphism $\widetilde{\Phi}\colon (*_{x\in X} A) * \mathbb{P} \to (A *_w \mathbb{P})^{\otimes 2}$ satisfying

$$\begin{aligned} \widetilde{\Phi}(q_I(a_{x,y})) &= \sum_{z\in X} q_w(q_I(a_{x,z})) \otimes q_w(q_{\mathbb{P}}(a_{z,y})), \\ \widetilde{\Phi}(\nu_x(a)) &= \sum_{z\in X} (q_w \otimes q_w)\big((\nu_x \otimes \nu_z)(\Phi(a))(q_I(a_{x,z}) \otimes 1)\big). \end{aligned}$$

For each $a \in \mathcal{A}$, $x, y \in X$ we have

$$\begin{aligned} \widetilde{\Phi}(\nu_x(a)q_I(a_{x,y})) &= \sum_{z,z'\in X} q_w\big(\nu_x(a_{(1)})q_I(a_{x,z})q_I(a_{x,z'})\big) \otimes q_w\big(\nu_z(a_{(2)})q_I(a_{z',y})\big) \\ &= \sum_{z,z'\in X} q_w\big(q_I(a_{x,z})\nu_x(a_{(1)})q_I(a_{x,z'})\big) \otimes q_w\big(q_I(a_{z',y})\nu_z(a_{(2)})\big) \\ &= \widetilde{\Phi}(q_I(a_{x,y})\nu_x(a)). \end{aligned}$$

It follows that $\widetilde{\Phi}(\nu_x(a)q_I(a_{x,y})) = \widetilde{\Phi}(q_I(a_{x,y})\nu_x(a))$ for each $a \in A$, $x, y \in X$, and hence $\widetilde{\Phi}$ descends to the desired $\Phi_w \colon A *_w \mathbb{P} \to (A *_w \mathbb{P})^{\otimes 2}$.

We now claim that $(\mathrm{id} \otimes \Phi_w) \circ \Phi_w = (\Phi_w \otimes \mathrm{id}) \circ \Phi_w$. Since $\mathcal{A}$ is dense in A, to see that $(\mathrm{id} \otimes \Phi_w) \circ \Phi_w$ and $(\Phi_w \otimes \mathrm{id}) \circ \Phi_w$ agree on each $q_w(\nu_x(A))$, it suffices to show that

$$\begin{aligned}(\mathrm{id} \otimes \Phi_w) \circ \Phi_w(q_w(\nu_x(a^\lambda_{i,j})))\\ = (\Phi_w \otimes \mathrm{id}) \circ \Phi_w(q_w(\nu_x(a^\lambda_{i,j}))), \quad \text{for all } \lambda \in \Lambda,\ 1 \le i, j \le d_\lambda.\end{aligned}$$

Routine calculations using (5.1) and (5.2) show that both sides of this equation are equal to

$$\sum_{z,z' \in X} \sum_{1 \le k,l \le d_\lambda} q_w^{\otimes 3}\big(\nu_x(a^\lambda_{i,k})q_I(a_{x,z}) \otimes \nu_z(a^\lambda_{k,l})q_I(a_{z,z'}) \otimes \nu_{z'}(a^\lambda_{l,j})\big),$$

and hence are equal. So we have

$$(\mathrm{id} \otimes \Phi_w) \circ \Phi_w(q_q(\nu_x(A))) = (\Phi_w \otimes \mathrm{id}) \circ \Phi_w(q_q(\nu_x(A)))$$

for each $x \in X$. It is straightforward to check that evaluating both $(\mathrm{id} \otimes \Phi_w) \circ \Phi_w$ and $(\Phi_w \otimes \mathrm{id}) \circ \Phi_w$ at $q_I(a_{x,y})$ gives

$$\sum_{z,z' \in X} q_w^{\otimes 3}\big(q_I^{\otimes 3}(a_{x,z} \otimes a_{z,z'} \otimes a_{z',y})\big).$$

Hence we have $(\mathrm{id} \otimes \Phi_w) \circ \Phi_w = (\Phi_w \otimes \mathrm{id}) \circ \Phi_w$.

We now define the matrix a^X by $a^X_{x,y} := q_w(q_I(a_{x,y}))$, for $x, y \in X$; and for each $\lambda \in \Lambda$, $1 \le i, j \le d_\lambda$, $x, y \in X$, the elements

$$a^{(\lambda,X)}_{(i,x),(j,y)} := q_w(\nu_x(a^\lambda_{i,j})q_I(a_{x,y})) \in A *_w \mathbb{P},$$

define matrices $a^{(\lambda,X)} = (a^{(\lambda,X)}_{(i,x),(j,y)})$. To finish the proof we have to show that these matrices satisfy (1)–(3) of Remark 2.2.

We have

$$\begin{aligned}\Phi_w(a^{(\lambda,X)}_{(i,x),(j,y)}) = \Bigg(\sum_{z \in X}(q_w \otimes q_w)\big((\nu_x \otimes \nu_z)(\Phi(a^\lambda_{i,j}))(q_I(a_{x,z}) \otimes 1)\big)\Bigg)\\ \cdot (q_w \circ q_I)^{\otimes 2}(\Delta(a_{x,y})).\end{aligned}$$

We know that (1) is satisfied for the matrix a^X. For each $z \in X$ we have

$$\begin{aligned}&(q_w \otimes q_w)\big((\nu_x \otimes \nu_z)(\Phi(a^\lambda_{i,j}))(q_I(a_{x,z}) \otimes 1)\big)(q_w \circ q_I)^{\otimes 2}(\Delta(a_{x,y}))\\ &\quad = (q_w \otimes q_w)\Bigg((\nu_x \otimes \nu_z)\Bigg(\sum_{1 \le k \le d_\lambda} a^\lambda_{i,k} \otimes a^\lambda_{k,j}\Bigg)(q_I(a_{x,z}) \otimes 1)\Bigg)\\ &\quad\quad \cdot (q_w \circ q_I)^{\otimes 2}(\Delta(a_{x,y}))\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\le k\le d_\lambda} \big(q_w(\nu_x(a^\lambda_{i,k})) \otimes q_w(\nu_z(a^\lambda_{k,j}))\big) \\
&\quad \cdot (q_w(q_I(a_{x,z})) \otimes 1)(q_w \circ q_I)^{\otimes 2}(\Delta(a_{x,y})) \\
&= \sum_{1\le k\le d_\lambda} \big(q_w(\nu_x(a^\lambda_{i,k})) \otimes q_w(\nu_z(a^\lambda_{k,j}))\big) \sum_{z'\in X} q_w(q_I(a_{x,z}a_{x,z'})) \otimes q_w(q_I(a_{z',y})) \\
&= \sum_{1\le k\le d_\lambda} \big(q_w(\nu_x(a^\lambda_{i,k})) \otimes q_w(\nu_z(a^\lambda_{k,j}))\big)\big(q_w(q_I(a_{x,z})) \otimes q_w(q_I(a_{z,y}))\big) \\
&= \sum_{1\le k\le d_\lambda} q_w(\nu_x(a^\lambda_{i,k})q_I(a_{x,z})) \otimes q_w(\nu_z(a^\lambda_{k,j})q_I(a_{z,y})).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\Phi_w(a^{(\lambda,X)}_{(i,x),(j,y)}) &= \sum_{z\in X}\sum_{1\le k\le d_\lambda} q_w(\nu_x(a^\lambda_{i,k})q_I(a_{x,z})) \otimes q_w(\nu_z(a^\lambda_{k,j})q_I(a_{z,y})) \\
&= \sum_{z\in X}\sum_{1\le k\le d_\lambda} a^{(\lambda,X)}_{(i,x),(k,z)} \otimes a^{(\lambda,X)}_{(k,z),(j,y)},
\end{aligned}
$$

and so (1) holds for all matrices $a^{(\lambda,X)}$. To see that $a^{(\lambda,X)}$ is invertible, we define $b^{(\lambda,X)}$ by

$$
b^{(\lambda,X)}_{(i,x),(j,y)} := q_w(q_I(a_{y,x})\nu_y((a^\lambda)^{-1}_{i,j})).
$$

Then we have

$$
\begin{aligned}
(a^{(\lambda,X)}b^{(\lambda,X)})_{(i,x),(j,y)} &= \sum_{z\in X}\sum_{1\le k\le d_\lambda} a^{(\lambda,X)}_{(i,x),(k,z)} b^{(\lambda,X)}_{(k,z),(j,y)} \\
&= q_w\Bigg(\sum_{z\in X}\sum_{1\le k\le d_\lambda} \nu_x(a^\lambda_{i,k})q_I(a_{x,z})q_I(a_{y,z})\nu_y((a^\lambda)^{-1}_{k,j})\Bigg) \\
&= \delta_{x,y} q_w\Bigg(\sum_{1\le k\le d_\lambda} \nu_x(a^\lambda_{i,k})\bigg(\sum_{z\in X} q_I(a_{x,z})\bigg)\nu_x((a^\lambda)^{-1}_{k,j})\Bigg) \\
&= \delta_{x,y} q_w\Bigg(\nu_x\bigg(\sum_{1\le k\le d_\lambda} a^\lambda_{i,k}(a^\lambda)^{-1}_{k,j}\bigg)\Bigg) \\
&= \delta_{x,y} q_w(\nu_x((a^\lambda(a^\lambda)^{-1})_{i,j})) \\
&= \delta_{x,y}\delta_{i,j} 1.
\end{aligned}
$$

A similar calculation shows that $(b^{(\lambda,X)}a^{(\lambda,X)})_{(i,x),(j,y)} = \delta_{x,y}\delta_{i,j}1$, and so $a^{(\lambda,X)}$ is invertible. Similar calculations also show that $c^{(\lambda,X)}$ with entries

$$
c^{(\lambda,X)}_{(i,x),(j,y)} := q_w(q_I(a_{x,y})\nu_x(((a^\lambda)^T)^{-1}_{i,j}))
$$

is the inverse of $(a^{(\lambda,X)})^T$.

We also have

$$(a^X(a^X)^T)_{x,y} = \sum_{z\in X} a^X_{x,z}a^X_{y,z} = q_w\bigg(q_I\bigg(\sum_{z\in X} a_{x,z}a_{y,z}\bigg)\bigg) = \delta_{x,y}1.$$

Similarly, $(a^X)^T a^X$ is the identity. So a^X and $(a^X)^T$ are mutually inverse, and (2) is satisfied.

We now claim that the entries of the matrices $\{a^{(\lambda,X)} : \lambda \in \Lambda\} \cup \{a^X\}$ span a dense subset of $A *_{X,w} \mathbb{P}$. For each $x, y \in X$ we obviously have $q_w(q_I(a_{x,y}))$ in this span since they are the entries of a^X. For each $x \in X$, $\lambda \in \Lambda$ and $1 \le i, j \le d_\lambda$ we have

$$\sum_{y\in X} a^{(\lambda,X)}_{(i,x),(j,y)} = q_w\bigg(\nu_x(a^\lambda_{i,j})q_I\bigg(\sum_{y\in X} a_{x,y}\bigg)\bigg) = q_w(\nu_x(a^\lambda_{i,j})),$$

and so each $q_w(\nu_x(a^\lambda_{i,j}))$ is in the span of the entries. The claim follows, and so (3) holds. ∎

Theorem 5.7. *Let $A_\mathbb{P}$ be a finitely constrained self-similar quantum group in the sense of Definition 5.1. There is a unital quantum group isomorphism $\pi\colon A_\mathbb{P} \to A_\mathbb{P} *_w \mathbb{P}$ satisfying*

$$\pi(q_J(a_{xu,yv})) = q_w(q_I(a_{x,y})\nu_x(q_J(a_{u,v})))$$

for all $x, y \in X$, $u, v \in X^m$, $m \ge 0$.

Proof. We define $b_{\varnothing,\varnothing}$ to be the identity of $A_\mathbb{P} *_w \mathbb{P}$, and for each $x, y \in X$, $u, v \in X^m$, $m \ge 0$,

$$b_{xu,yv} := q_w(q_I(a_{x,y})\nu_x(q_J(a_{u,v}))).$$

We claim that this gives a family of projections satisfying (1)–(3) of Definition 3.2. Condition (1) holds by definition. We have

$$\begin{aligned} b^*_{xu,yv} &= q_w(\nu_x(q_J(a^*_{u,v}))q_I(a^*_{x,y})) \\ &= q_w(\nu_x(q_J(a_{u,v}))q_I(a_{x,y})) \\ &= q_w(q_I(a_{x,y})\nu_x(q_J(a_{u,v}))) \\ &= b_{xu,yv} \end{aligned}$$

and

$$\begin{aligned} b^2_{xu,yv} &= q_w\big(q_I(a_{x,y})\nu_x(q_J(a_{u,v}))q_I(a_{x,y})\nu_x(q_J(a_{u,v}))\big) \\ &= q_w\big(q_I(a^2_{x,y})\nu_x(q_J(a^2_{u,v}))\big) = b_{xu,yv}. \end{aligned}$$

So (2) holds. For each $w \in X$ we have

$$\begin{aligned} \sum_{z\in X} b_{xuw,yvz} &= \sum_{z\in X} q_w(q_I(a_{x,y})\nu_x(q_J(a_{uw,vz}))) \\ &= q_w\bigg(q_I(a_{x,y})\nu_x\bigg(q_J\bigg(\sum_{z\in X} a_{uw,vz}\bigg)\bigg)\bigg) \end{aligned}$$

$$= q_w(q_I(a_{x,y})\nu_x(q_J(a_{u,v})))$$
$$= b_{xu,yv},$$

and

$$\begin{aligned}
\sum_{z\in X} b_{xuz,yvw} &= \sum_{z\in X} q_w(q_I(a_{x,y})\nu_x(q_J(a_{uz,vw}))) \\
&= q_w\Big(q_I(a_{x,y})\nu_x\Big(q_J\Big(\sum_{z\in X} a_{uz,vw}\Big)\Big)\Big) \\
&= q_w(q_I(a_{x,y})\nu_x(q_J(a_{u,v}))) \\
&= b_{xu,yv},
\end{aligned}$$

and hence (3) holds. This proves the claim, and hence the universal property of $\mathbb{A}_X$ now gives a homomorphism $\widetilde{\pi}: \mathbb{A}_X \to A_{\mathbb{P}} *_w \mathbb{P}$ satisfying

$$\widetilde{\pi}(a_{xu,yv}) = q_w(q_I(a_{x,y})\nu_x(q_J(a_{u,v}))),$$

for all $x, y \in X$, $u, v \in X^m$, $m \geq 0$.

We now claim that J is contained in $\ker\widetilde{\pi}$. To see this, fix $w \in X^n$, with $w = w_1 w'$ for $w_1 \in X$, $w' \in X^{n-1}$. We first prove the claim that for each $x_k := a_{u_1,v_1}\cdots a_{u_k,v_k}$, where $k \geq 1$ and each pair $u_i, v_i \in X^{m_i}$ for some $m_i \geq 0$, we have

$$\widetilde{\pi}(\rho_w(x_k)) = \sum_{y\in X} q_w(q_I(a_{y,w_1})\nu_y(q_J(\rho_{w'}(x_k)))). \tag{5.3}$$

Let $k = 1$. Then

$$\begin{aligned}
\widetilde{\pi}(\rho_w(a_{u_1,v_1})) &= \sum_{y\in X}\sum_{\alpha\in X^{n-1}} \widetilde{\pi}(a_{y\alpha u_1,wv_1}) \\
&= \sum_{y\in X}\sum_{\alpha\in X^{n-1}} q_w(q_I(a_{y,w_1})\nu_y(q_J(a_{\alpha u_1,w'v_1}))) \\
&= \sum_{y\in X} q_w\Big(q_I(a_{y,w_1})\nu_y\Big(q_J\Big(\sum_{\alpha\in X^{n-1}} a_{\alpha u_1,w'v_1}\Big)\Big)\Big) \\
&= \sum_{y\in X} q_w(q_I(a_{y,w_1})\nu_y(q_J(\rho_{w'}(a_{u_1,v_1})))),
\end{aligned}$$

and so (5.3) holds for $k = 1$. We now assume true for x_k, and prove for x_{k+1}. Note that for $y, y' \in X$ we have $q_I(a_{y,w_1})q_I(a_{y',w_1}) = \delta_{y,y'}q_I(a_{y,w_1})$, and hence

$$\begin{aligned}
&q_w\big(q_I(a_{y,w_1})\nu_y(q_J(\rho_{w'}(x_k)))q_I(a_{y',w_1})\nu_{y'}(q_J(\rho_{w'}(a_{u_{k+1},v_{k+1}})))\big) \\
&\quad = q_w\big(\nu_y(q_J(\rho_{w'}(x_k)))q_I(a_{y,w_1})q_I(a_{y',w_1})\nu_{y'}(q_J(\rho_{w'}(a_{u_{k+1},v_{k+1}})))\big) \\
&\quad = \delta_{y,y'}q_w\big(\nu_y(q_J(\rho_{w'}(x_k)))q_I(a_{y,w_1})\nu_y(q_J(\rho_{w'}(a_{u_{k+1},v_{k+1}})))\big) \\
&\quad = \delta_{y,y'}q_w\big(q_I(a_{y',w_1})\nu_y(q_J(\rho_{w'}(x_k)))\nu_y(q_J(\rho_{w'}(a_{u_{k+1},v_{k+1}})))\big) \\
&\quad = \delta_{y,y'}q_w\big(q_I(a_{y',w_1})\nu_y(q_J(\rho_{w'}(x_k a_{u_{k+1},v_{k+1}})))\big).
\end{aligned}$$

It follows that

$$\begin{aligned}\widetilde{\pi}(\rho_w(x_{k+1})) &= \widetilde{\pi}(\rho_w(x_k))\widetilde{\pi}(\rho_w(a_{u_{k+1},v_{k+1}}))\\ &= \sum_{y\in X} q_w\big(q_I(a_{y',w_1})\nu_y(q_J(\rho_{w'}(x_k a_{u_{k+1},v_{k+1}})))\big),\end{aligned}$$

and it follows that (5.3) holds for all k. Since linear combinations of products of the form x_k are a dense subalgebra of $\mathbb{A}_X$, it follows that

$$\widetilde{\pi}(\rho_w(a)) = \sum_{y\in X} q_w\big(q_I(a_{y,w_1})\nu_y(q_J(\rho_{w'}(a)))\big)$$

for all $a \in \mathbb{A}_X$. Now, if $a \in I$, then $\rho_{w'}(a) \in J = \ker q_J$, and hence the above equations show that $\widetilde{\pi}(\rho_w(a)) = 0$. Hence $\rho_w(a) \in \ker\widetilde{\pi}$ for all $w \in X^n$ and $a \in I$, and hence $J \subseteq \ker\widetilde{\pi}$. This means $\widetilde{\pi}$ descends to a homomorphism $\pi\colon A_{\mathbb{P}} \to A_{\mathbb{P}} *_w \mathbb{P}$ satisfying

$$\pi(q_J(a_{xu,yv})) = q_w(q_I(a_{x,y})\nu_x(q_J(a_{u,v})))$$

for all $x, y \in X$, $u, v \in X^m$, $m \geq 0$.

We now show that π is an isomorphism by finding an inverse. For each $x \in X$ consider the homomorphism $q_J \circ \sigma_x\colon \mathbb{A}_X \to A_{\mathbb{P}}$, where σ_x is the homomorphism from Remark 4.3. Since $\sigma_x = \kappa \circ \rho_x \circ \kappa$, and we know from [11, Remark 2.10] that $\kappa(J) \subseteq J$, it follows that $q_J \circ \sigma_x$ descends to a homomorphism $\phi_x\colon A_{\mathbb{P}} \to A_{\mathbb{P}}$ satisfying

$$\phi_x(q_J(a_{u,v})) = q_J(\sigma_x(a_{u,v})) = \sum_{y\in X} q_J(a_{xu,yv}),$$

for all $u, v \in X^m$, $m \geq 0$.

Each ϕ_x, and the map $q_I(a) \mapsto q_J(a)$ from $\mathbb{P}$ to $A_{\mathbb{P}}$, now allow us to apply the universal property of the free product $(*_{x\in X} A_{\mathbb{P}}) * \mathbb{P}$ to get a homomorphism $\widetilde{\phi}\colon (*_{x\in X} A_{\mathbb{P}}) * \mathbb{P} \to A_{\mathbb{P}}$ satisfying $\widetilde{\phi} \circ \nu_x = \phi_x$ for each $x \in X$, and $\widetilde{\phi}(q_I(a)) = q_J(a)$ for all $a \in \mathbb{A}_1 \subseteq \mathbb{A}_X$. We claim that

$$\widetilde{\phi}\big(\nu_x(q_J(a_{u,v}))q_I(a_{x,y}) - q_I(a_{x,y})\nu_x(q_J(a_{u,v}))\big) = 0,$$

for each $x \in X$, $u, v \in X^m$, $m \in \mathbb{N}$. We have

$$\begin{aligned}&\widetilde{\phi}\big(\nu_x(q_J(a_{u,v}))q_I(a_{x,y}) - q_I(a_{x,y})\nu_x(q_J(a_{u,v}))\big)\\ &\quad= \phi_x(q_J(a_{u,v}))q_J(a_{x,y}) - q_G(a_{x,y})\phi_x(q_J(a_{u,v}))\\ &\quad= \sum_{y\in X} q_J(a_{xu,yv})q_J(a_{x,y}) - \sum_{y'\in X} q_J(a_{x,y})q_J(a_{xu,y'v})\\ &\quad= q_J(a_{xu,yv}) - q_J(a_{xu,yv})\\ &\quad= 0.\end{aligned}$$

It follows that $\widetilde{\phi}$ descends to a homomorphism $\phi\colon A_{\mathbb{P}} *_w \mathbb{P} \to A_{\mathbb{P}}$ satisfying

$$\phi(q_w(\nu_x(q_J(a_{u,v})))) = q_J(\sigma_x(a_{u,v})) = \sum_{y\in X} q_J(a_{xu,yv})$$

for all $x \in X$, $u, v \in X^m$, $m \geq 0$, and

$$\phi(q_w(q_I(a_{x,y}))) = q_J(a_{x,y})$$

for all $x, y \in X$.

We claim that π and ϕ are mutually inverse. For $x, y \in X$, $u, v \in X^m$, $m \geq 0$, we have

$$\begin{aligned}\phi(\pi(q_J(a_{xu,yv}))) &= \phi\big(q_w(q_I(a_{x,y})\nu_x(q_J(a_{u,v})))\big)\\ &= q_J(a_{x,y}) \sum_{y\in X} q_J(a_{xu,yv}) = q_J(a_{xu,yv}),\end{aligned}$$

and it follows that $\phi \circ \pi$ is the identity on $A_{\mathbb{P}}$. For $x \in X$, $u, v \in X^m$, $m \geq 0$, we have

$$\begin{aligned}\pi(\phi(q_w(\nu_x(q_J(a_{u,v}))))) &= \pi\Big(\sum_{y\in X} q_J(a_{xu,yv})\Big)\\ &= \sum_{y\in X} q_w(q_I(a_{x,y})\nu_x(q_J(a_{u,v})))\\ &= q_w\Big(q_I\Big(\sum_{y\in X} a_{x,y}\Big)\nu_x(q_J(a_{u,v}))\Big)\\ &= q_w(\nu_x(q_J(a_{u,v}))),\end{aligned}$$

and for all $x, w \in X$ we have

$$\pi(\phi(q_w(q_I(a_{x,y})))) = \pi(q_J(a_{x,y})) = q_w(q_I(a_{x,y})).$$

Hence $\pi \circ \phi$ is the identity on $A_{\mathbb{P}} *_w \mathbb{P}$, and so π is an isomorphism.

We now need to show that π is a homomorphism of compact quantum groups, which means that $\Delta_w \circ \pi = (\pi \otimes \pi) \circ \Delta_J$, where Δ_J is the comultiplication on $A_{\mathbb{P}}$. For $x, y \in X$, $u, v \in X^m$, $m \geq 0$, we have

$$\begin{aligned}&(\pi \otimes \pi) \circ \Delta_J(q_J(a_{xu,yv}))\\ &\quad= \sum_{\alpha^{m+1}} \pi(q_J(a_{xu,\alpha})) \otimes \pi(q_J(a_{\alpha,yv}))\\ &\quad= \sum_{z\in X}\sum_{\beta\in X^m} \pi(q_J(a_{xu,z\beta})) \otimes \pi(q_J(a_{z\beta,yv}))\\ &\quad= \sum_{z\in X}\sum_{\beta\in X^m} q_w(q_I(a_{x,z})\nu_x(q_J(a_{u,\beta}))) \otimes q_w(q_I(a_{z,y})\nu_x(q_J(a_{\beta,v}))).\end{aligned}$$

We have

$$\Delta_w \circ \pi(q_J(a_{xu,yv})) = \Delta_w(q_w(q_I(a_{x,y})))\Delta_w(q_w(\nu_x(q_J(a_{u,v})))),$$

where

$$\Delta_w(q_w(q_I(a_{x,y}))) = \sum_{z\in X} q_w(q_I(a_{x,z})) \otimes q_w(q_{\mathbb{P}}(a_{z,y})), \tag{5.4}$$

and

$$\begin{aligned}\Delta_w(q_w(\nu_x(q_J(a_{u,v})))) &= \sum_{z'\in X}(q_w\otimes q_w)\big((\nu_x\otimes\nu_{z'})(\Delta_J(q_J(a_{u,v})))(q_I(a_{x,z'})\otimes 1)\big)\\ &= \sum_{z'\in X}\sum_{\beta\in X^m} q_w(\nu_x(q_J(a_{u,\beta}))q_I(a_{x,z'}))\otimes q_w(\nu_{z'}(q_J(a_{\beta,v}))).\end{aligned} \tag{5.5}$$

A typical summand in the product of the expressions in (5.4) and (5.5) is

$$\begin{aligned}&q_w\big(q_I(a_{x,z})\nu_x(q_J(a_{u,\beta}))q_I(a_{x,z'})\big)\otimes q_w\big(q_{\mathbb{P}}(a_{z,y})\nu_{z'}(q_J(a_{\beta,v}))\big)\\ &\quad= q_w\big(\nu_x(q_J(a_{u,\beta}))q_I(a_{x,z})q_I(a_{x,z'})\big)\otimes q_w\big(q_{\mathbb{P}}(a_{z,y})\nu_{z'}(q_J(a_{\beta,v}))\big)\\ &\quad= \delta_{z,z'}q_w\big(\nu_x(q_J(a_{u,\beta}))q_I(a_{x,z})\big)\otimes q_w\big(q_{\mathbb{P}}(a_{z,y})\nu_z(q_J(a_{\beta,v}))\big)\\ &\quad= \delta_{z,z'}q_w\big(q_I(a_{x,z})\nu_x(q_J(a_{u,\beta}))\big)\otimes q_w\big(q_{\mathbb{P}}(a_{z,y})\nu_z(q_J(a_{\beta,v}))\big).\end{aligned}$$

Hence

$$\begin{aligned}&\Delta_w\circ\pi(q_J(a_{xu,yv}))\\ &\quad= \sum_{z\in X} q_w(q_I(a_{x,z}))\otimes q_w(q_{\mathbb{P}}(a_{z,y}))\\ &\quad= \sum_{z\in X}\sum_{\beta\in X^m} q_w\big(q_I(a_{x,z})\nu_x(q_J(a_{u,\beta}))\big)\otimes q_w\big(q_I(a_{z,y})\nu_x(q_J(a_{\beta,v}))\big)\\ &\quad= (\pi\otimes\pi)\circ\Delta_J(q_J(a_{xu,yv})),\end{aligned}$$

and it follows that $\Delta_w \circ \pi = (\pi \otimes \pi) \circ \Delta_J$. ■

Example 5.8. An immediate consequence of Theorem 5.7 is that $A_{\mathbb{P}}$ is noncommutative whenever $\mathbb{P}$ is a noncommutative quantum subgroup of $\mathbb{A}_1$. A class of such examples comes from Banica and Bichon's [1, Theorem 1.1], in which they classify all the quantum subgroups $\mathbb{P}$ of $\mathbb{A}_1$ for $|X| = 4$; the corresponding list of quantum groups $A_{\mathbb{P}}$ gives us a list of potentially interesting self-similar quantum groups for further study.

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