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Lifting of fractional Sobolev mappings to noncompact covering spaces

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Abstract. Given compact Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$, a Riemannian covering $\pi: \widetilde{\mathcal{N}} \to \mathcal{N}$ by a noncompact covering space $\widetilde{\mathcal{N}}$, $1 < p < \infty$ and $0 < s < 1$, the space of liftings of fractional Sobolev maps in $\dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$ is characterized when $sp > 1$ and an optimal nonlinear fractional Sobolev estimate is obtained when moreover $sp \geq \dim \mathcal{M}$. A nonlinear characterization of the sum of spaces $\dot{W}^{s,p}(\mathcal{M}, \mathbb{R}) + \dot{W}^{1,sp}(\mathcal{M}, \mathbb{R})$ is also provided.

1. Introduction

Given a covering map $\pi: \widetilde{\mathcal{N}} \to \mathcal{N}$, that is, a map π such that for every $y \in \mathcal{N}$ there exists some open set $U \subseteq \mathcal{N}$ such that $y \in U$ and $\pi^{-1}(U)$ is a disjoint union of open subsets of $\widetilde{\mathcal{N}}$ on which π is a homeomorphism, the classical topological lifting theory states that if $\mathcal{M}$ is a simply connected topological manifold and if π is surjective, then every mapping $u \in C(\mathcal{M}, \mathcal{N})$ can be written as $u = \pi \circ \tilde{u}$ for some map $\tilde{u} \in C(\mathcal{M}, \widetilde{\mathcal{N}})$ (see for example [15, Prop. 1.33]). For instance, the universal covering of the circle $\pi: \mathbb{R} \to \mathbb{S}^1$ defined for each $\tilde{y} \in \mathbb{R}$ by $\pi(\tilde{y}) := e^{i\tilde{y}} \in \mathbb{S}^1 \subseteq \mathbb{R}^2 \simeq \mathbb{C}$ allows one to classify the homotopy classes of maps from the circle $\mathbb{S}^1$ to itself (see for example [15, Thm. 1.7]).

When the manifolds $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are both endowed with a Riemannian metric, we say that $\pi: \widetilde{\mathcal{N}} \to \mathcal{N}$ is a *Riemannian covering* whenever it is a covering and it is a local isometry, that is, it preserves the metric tensor. In fact, if $\mathcal{N}$ is a Riemannian manifold and π is a topological covering map, there exists a unique Riemannian metric on $\widetilde{\mathcal{N}}$ such that $\pi: \widetilde{\mathcal{N}} \to \mathcal{N}$ is a Riemannian covering (see [16, Prop. 2.31], [13, §2.A.4]).

Given a Riemannian covering $\pi: \widetilde{\mathcal{N}} \to \mathcal{N}$, a Riemannian manifold $\mathcal{M}$, $s \in (0, 1]$ and $p \in [1, \infty)$, the *lifting problem in Sobolev spaces* amounts to determining whether each mapping $u \in \dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$ can be written as $u = \pi \circ \tilde{u}$ on $\mathcal{M}$, for some map $\tilde{u} \in \dot{W}^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}})$ [2, 5].

When $s = 1$, the space $\dot{W}^{1,p}(\mathcal{M}, \mathcal{N})$ is the *homogeneous first-order Sobolev space* defined – if the Riemannian manifold $\mathcal{N}$ is assumed without loss of generality in view of Nash's embedding theorem [26] to be isometrically embedded into some Euclidean space

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$\mathbb{R}^\nu$ – as

$$\dot{W}^{1,p}(\mathcal{M},\mathcal{N}) := \bigl\{u\colon \mathcal{M} \to \mathcal{N} \bigm| u \text{ is weakly differentiable and } \textstyle\int_{\mathcal{M}} |Du|^p < \infty\bigr\}.$$

If the domain manifold $\mathcal{M}$ is simply connected, then the first-order Sobolev spaces in which each map admits a lifting have been characterized for the universal covering of the circle $\pi\colon \mathbb{R} \to \mathbb{S}^1$ by Bourgain, Brezis and Mironescu [5, Thm. 3] and for a general Riemannian covering map $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ by Bethuel and Chiron [2, Thm. 1] (see also [11, Thm. 1.1]): if the covering π is surjective and not injective, every map $u \in \dot{W}^{1,p}(\mathcal{M},\mathcal{N})$ can be written as $u = \pi \circ \tilde{u}$ for some mapping $\tilde{u} \in \dot{W}^{1,p}(\mathcal{M},\widetilde{\mathcal{N}})$ if and only if $p \geq \min\{2, \dim \mathcal{M}\}$; moreover one has then

$$\int_{\mathcal{M}} |D\tilde{u}|^p = \int_{\mathcal{M}} |Du|^p; \tag{1}$$

once the existence of the lifting $\tilde{u}$ is known, the identity (1) follows directly from the chain rule for the Sobolev functions since the covering map π is a local Riemannian isometry so that $|D\tilde{u}| = |Du|$ almost everywhere on $\mathcal{M}$.

In the fractional case $0 < s < 1$, the corresponding *homogeneous fractional Sobolev–Slobodeckiĭ space* $\dot{W}^{s,p}(\mathcal{M},\mathcal{N})$ can be defined through the finiteness of the Gagliardo fractional energy as

$$\dot{W}^{s,p}(\mathcal{M},\mathcal{N}) := \Bigl\{u\colon \mathcal{M} \to \mathcal{N} \Bigm| \|u\|^p_{\dot{W}^{s,p}(\mathcal{M})} := \textstyle\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y),u(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x < \infty\Bigr\},$$

where $d_{\mathcal{M}}$ and $d_{\mathcal{N}}$ respectively denote the geodesic distances on the connected Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$ and where $m := \dim \mathcal{M}$.

When $sp < 1$, by the works of Bourgain, Brezis and Mironescu for the universal covering of the circle $\pi\colon \mathbb{R} \to \mathbb{S}^1$ [5, Thm. 2], [11, Thms. 5.1 & 5.2] and of Bethuel and Chiron [2, Thm. 3], every map $u \in \dot{W}^{s,p}(\mathcal{M},\mathcal{N})$ can be written as $u = \pi \circ \tilde{u}$ with $\tilde{u} \in \dot{W}^{s,p}(\mathcal{M},\widetilde{\mathcal{N}})$ and one then has the lifting estimate

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \leq C \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y),u(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x. \tag{2}$$

In this régime $sp < 1$, fractional Sobolev maps are quite rough mappings, and the possibility of jumps leaves much room for the construction of the lifting, which is quite challenging because of the highly nonunique character of the lifting.

When the covering space $\widetilde{\mathcal{N}}$ is *compact*, fractional Sobolev spaces $\dot{W}^{s,p}(\mathcal{M},\mathcal{N})$ for which any map admits a lifting have been characterized in the works of Bethuel and Chiron [2, Thm. 3], and of Mironescu and Van Schaftingen [24]: if the covering π is surjective and not injective, every map $u \in \dot{W}^{s,p}(\mathcal{M},\mathcal{N})$ can be written as $u = \pi \circ \tilde{u}$ for some $\tilde{u} \in \dot{W}^{s,p}(\mathcal{M},\widetilde{\mathcal{N}})$ if and only if $sp \geq \min\{2, \dim \mathcal{M}\}$. Moreover, if $sp > 1$, estimate (2) holds for any $u \in \dot{W}^{s,p}(\mathcal{M},\mathcal{N})$ that can be written as $u = \pi \circ \tilde{u}$ with $\tilde{u} \in \dot{W}^{s,p}(\mathcal{M},\widetilde{\mathcal{N}})$; this crucial estimate is a consequence of a reverse oscillation estimate, combined with the observation that the diameter of the covering space $\widetilde{\mathcal{N}}$ can be bounded by a multiple of $\operatorname{inj}(\mathcal{N})$, the injectivity radius of the manifold $\mathcal{N}$.

When the covering space $\widetilde{\mathcal{N}}$ is *not compact*, one encounters an *analytical obstruction* for $1 \le sp < m$: there exist maps in $\dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$ that are smooth except at a single point and that cannot be written as $u = \pi \circ \tilde{u}$ for some map $\tilde{u} \in \dot{W}^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}})$ [5, Thm. 2], [2, Thm. 3]. This does not end the story, as one can still try to describe the functional space of liftings.

In the case of the universal covering of the circle $\pi: \mathbb{R} \to \mathbb{S}^1$, the liftings have been characterized in a sequence of works by Bourgain, Brezis, Mironescu and Nguyen [4, 11, 19–22, 27]:

Theorem 1.1. *Let $\mathcal{M}$ be a compact Riemannian manifold, let $m := \dim \mathcal{M}$, let $s \in (0, 1)$ and let $p \in (1, \infty)$. If $\mathcal{M}$ is simply connected and if $sp \ge 2$, then there exists a constant $C \in (0, \infty)$ such that every map $u \in \dot{W}^{s,p}(\mathcal{M}, \mathbb{S}^1)$ can be written as $u = \pi \circ \tilde{u}$ on $\mathcal{M}$ with $\tilde{u} = \tilde{v} + \widetilde{w}$, where the functions $\tilde{v} \in \dot{W}^{s,p}(\mathcal{M}, \mathbb{R})$ and $\widetilde{w} \in \dot{W}^{1,sp}(\mathcal{M}, \mathbb{R})$ satisfy the estimate*

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{|\tilde{v}(y) - \tilde{v}(x)|^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x + \int_{\mathcal{M}} |D\widetilde{w}|^{sp} \le C \iint_{\mathcal{M}\times\mathcal{M}} \frac{|u(y) - u(x)|^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x. \quad (3)$$

In other words, Theorem 1.1 states that any map $u \in \dot{W}^{s,p}(\mathcal{M}, \mathbb{S}^1)$ has a lifting $\tilde{u} \in \dot{W}^{s,p}(\mathcal{M}, \mathbb{R}) + \dot{W}^{1,sp}(\mathcal{M}, \mathbb{R})$.

Conversely to Theorem 1.1, in view of the fractional Gagliardo–Nirenberg interpolation inequality (see for example [8, Cor. 3.2], [33, Lem. 2.1], [10]), if $\tilde{u} = \tilde{v} + \widetilde{w}$ with $\tilde{v} \in \dot{W}^{s,p}(\mathcal{M}, \mathbb{R})$ and $\widetilde{w} \in \dot{W}^{1,sp}(\mathcal{M}, \mathbb{R})$, then $u := \pi \circ \tilde{u} \in \dot{W}^{s,p}(\mathcal{M}, \mathbb{S}^1)$, with inequality (3) reversed. Theorem 1.1 characterizes thus completely the lifting space of $\dot{W}^{s,p}(\mathcal{M}, \mathbb{S}^1)$ for $sp \ge 2$ as the sum of linear spaces $\dot{W}^{s,p}(\mathcal{M}, \mathbb{R}) + \dot{W}^{1,sp}(\mathcal{M}, \mathbb{R})$.

The first goal of the present work is to obtain a counterpart of Theorem 1.1 for a general covering map $\pi: \widetilde{\mathcal{N}} \to \mathcal{N}$ when the covering space $\widetilde{\mathcal{N}}$ is not compact. This endeavour is delicate from its very beginning, since $\dot{W}^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}}) + \dot{W}^{1,sp}(\mathcal{M}, \widetilde{\mathcal{N}})$ has no straightforward definition or generalization when the covering space $\widetilde{\mathcal{N}}$ is not a linear space. We characterize the lifting space as follows.

Theorem 1.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be compact Riemannian manifolds, let $m := \dim \mathcal{M}$, let $\pi: \widetilde{\mathcal{N}} \to \mathcal{N}$ be a surjective Riemannian covering map, let $s \in (0, 1)$ and let $p \in (1, \infty)$. If $\mathcal{M}$ is simply connected and if $sp \ge 2$, then there exists a constant $C \in (0, \infty)$ such that for every map $u \in \dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$ there exists a measurable map $\tilde{u}: \mathcal{M} \to \widetilde{\mathcal{N}}$ satisfying $\pi \circ \tilde{u} = u$ almost everywhere on $\mathcal{M}$ and*

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p \wedge 1}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \le C \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x. \quad (4)$$

The integrand in the left-hand side of (4) only differs from the classical Gagliardo energy in the right-hand side by truncating, through the minimum $\wedge$ operation, the value of the distance at 1; in terms of metric space, this can be interpreted as taking, on the covering space $\widetilde{\mathcal{N}}$, a bounded distance for which the covering map π is an isometry at small scales.

The characterization of liftings of Theorem 1.2 is sharp, in the sense that if for some mapping $\tilde{u}\colon \mathcal{M} \to \widetilde{\mathcal{N}}$ the left-hand side of (4) is finite, then by the local isometry property of liftings one has $u = \pi \circ \tilde{u} \in \dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$ together with estimate (4) reversed.

The core of the proof of Theorem 1.2 is the reverse oscillation estimate of Mironescu and Van Schaftingen [24, Lem. 3.1] (see Proposition 2.1 below), combined with the approximation of maps that are smooth outside a finite union of manifolds of dimension $\lceil m - sp - 1\rceil$ by Brezis and Mironescu [9], a suitable variant of the fractional Rellich compactness theorem under a boundedness assumption on the left-hand side of (4) (see Proposition 2.8 below) and, at a more technical level, the equivalent characterization of the lifting space (see Proposition 1.5 below).

The lifting in the space of functions such that the left-hand side of (4) is finite enjoys a uniqueness property. In order to state this, we define the space

$$X(\mathcal{M}, \widetilde{\mathcal{N}}) := \Bigl\{\tilde{u}\colon \mathcal{M} \to \widetilde{\mathcal{N}} \Bigm| \tilde{u} \text{ is measurable and } \iint\limits_{\substack{x,y\in\mathcal{M}\\ d_{\widetilde{\mathcal{N}}}(\tilde{u}(x),\tilde{u}(y))\geq \operatorname{inj}(\mathcal{N})/2}} \frac{1}{d_{\mathcal{M}}(x,y)^{m+1}} \,\mathrm{d}y \,\mathrm{d}x < \infty\Bigr\};$$

the latter space contains mappings for which the left-hand side of (4) is finite (see Proposition 2.15 below) and the uniqueness of the lifting then follows from the next proposition.

Proposition 1.3. *Let $\mathcal{M}$ be a compact Riemannian manifold and let $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ be a Riemannian covering. If $\mathcal{M}$ is connected, if $\tilde{u}_0, \tilde{u}_1 \in X(\mathcal{M}, \widetilde{\mathcal{N}})$ and if $\pi \circ \tilde{u}_0 = \pi \circ \tilde{u}_1$ almost everywhere on $\mathcal{M}$, then either $\tilde{u}_0 = \tilde{u}_1$ almost everywhere on $\mathcal{M}$ or $\tilde{u}_0 \neq \tilde{u}_1$ almost everywhere on $\mathcal{M}$.*

When $1 < sp \leq 2$ or when the manifold $\mathcal{M}$ is not simply connected, topological obstructions can exclude the existence of a lifting; it turns out however that when a lifting exists in $X(\mathcal{M}, \widetilde{\mathcal{N}})$, then such a lifting has to satisfy the estimate of Theorem 1.2.

Theorem 1.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be compact Riemannian manifolds, let $m := \dim \mathcal{M}$, let $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ be a Riemannian covering map, let $s \in (0, 1)$ and let $p \in (1, \infty)$. If $sp > 1$, then there exists a constant $C \in (0, \infty)$ such that if $\tilde{u} \in X(\mathcal{M}, \widetilde{\mathcal{N}})$ and if $u := \pi \circ \tilde{u} \in \dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$, then*

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p \wedge 1}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \leq C \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x.$$

The restriction to $sp > 1$ is essential in Theorem 1.4 in both the compact and noncompact cases: if $sp = 1$ and if π is surjective and not injective, then there is no estimate on the lifting [24, Lem. 5.1].

Theorems 1.2 and 1.4 motivate studying the quantity on the left-hand side of (4), which turns out be equivalent to a wide family of similar quantities.

Proposition 1.5. *Let $\mathcal{M}$ be a compact Riemannian manifold with $m := \dim \mathcal{M}$, let $\widetilde{\mathcal{N}}$ be a Riemannian manifold, let $s \in (0, 1)$, let $p \in (1, \infty)$ and let $q_0, q_1 \in [0, \infty)$. If $q_0 \vee q_1 \vee 1 < sp$, then there exists a constant $C \in (0, \infty)$ such that every measurable mapping $\tilde{u} : \mathcal{M} \to \widetilde{\mathcal{N}}$ satisfies*

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p \wedge d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^{q_0}}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \\ \le C \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p \wedge d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^{q_1}}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x.$$

In Proposition 1.5, the case $q_0 \le q_1$ is trivial. The estimate of Proposition 1.5 generalizes similar estimates obtained in the context of estimates of homotopy classes with $sp = \dim \mathcal{M}$ [36, §5].

As a consequence of Theorems 1.1, 1.2 and Proposition 1.5, we obtain the following nonlinear characterization of the linear sum of Sobolev spaces.

Theorem 1.6. *Let $\mathcal{M}$ be a compact Riemannian manifold, let $m := \dim \mathcal{M}$, let $s \in (0, 1)$ and let $p \in (1, \infty)$. If $sp > 1$ and if $0 < q < sp$, then*

$$\Big\{f : \mathcal{M} \to \mathbb{R} \;\Big|\; \iint_{\mathcal{M}\times\mathcal{M}} \tfrac{|f(y)-f(x)|^p \wedge |f(y)-f(x)|^q}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x < \infty\Big\} \\ = \dot{W}^{s,p}(\mathcal{M}, \mathbb{R}) + \dot{W}^{1,sp}(\mathcal{M}, \mathbb{R}). \tag{5}$$

Moreover, the quantities

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{|f(y) - f(x)|^p \wedge |f(y) - f(x)|^q}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x$$

and

$$\inf_{\substack{g \in \dot{W}^{s,p}(\mathcal{M},\mathbb{R})\\ h \in \dot{W}^{1,sp}(\mathcal{M},\mathbb{R})\\ f = g+h}} \iint_{\mathcal{M}\times\mathcal{M}} \frac{|g(y) - g(x)|^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x + \int_{\mathcal{M}} |Dh|^{sp}$$

are equivalent in the sense that each of them is bounded by a constant multiple of the other.

Theorem 1.6 complements the characterization of sums of fractional Sobolev spaces by Rodiac and Van Schaftingen [32], which states that if $q > sp$,

$$\Big\{f : \mathcal{M} \to \mathbb{R} \;\Big|\; \iint_{\mathcal{M}\times\mathcal{M}} \tfrac{|f(y)-f(x)|^p \wedge |f(y)-f(x)|^q}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x < \infty\Big\} \\ = \dot{W}^{s,p}(\mathcal{M}, \mathbb{R}) + \dot{W}^{\frac{sp}{q},q}(\mathcal{M}, \mathbb{R}). \tag{6}$$

We give a proof of Theorem 1.6 relying on the characterization of liftings of mappings into the circle Theorem 1.1; it would be enlightening to have a direct proof.

Open Problem 1. Give a direct proof of Theorem 1.6.

In the case $q = sp$, it turns out that the identifications (5) and (6) fail (see Proposition 3.2), leading to the following question.

Open Problem 2. Given a compact Riemannian manifold $\mathcal{M}$ with $m = \dim \mathcal{M}$, $p \in [1, \infty)$ and $s \in (0, 1)$ such that $sp > 1$, characterize the set

$$X^{s,p}(\mathcal{M}, \mathbb{R}) := \Big\{f : \mathcal{M} \to \mathbb{R} \;\Big|\; \iint_{\mathcal{M}\times\mathcal{M}} \tfrac{|f(y)-f(x)|^p \wedge |f(y)-f(x)|^{sp}}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x < \infty\Big\}.$$

We have some information about what the space $X^{s,p}(\mathcal{M}, \mathbb{R})$ could be: by Theorem 1.6 and by (6), we have

$$\begin{aligned}\bigcup_{s<\theta<1} \big(\dot{W}^{s,p}(\mathcal{M}, \mathbb{R}) + \dot{W}^{s/\theta,\theta p}(\mathcal{M}, \mathbb{R})\big) &\subseteq X^{s,p}(\mathcal{M}, \mathbb{R}) \\ &\subseteq \dot{W}^{s,p}(\mathcal{M}, \mathbb{R}) + \dot{W}^{1,sp}(\mathcal{M}, \mathbb{R}),\end{aligned} \tag{7}$$

whereas by Proposition 3.2 below, we have

$$\dot{W}^{1,sp}(\mathcal{M}, \mathbb{R}) \not\subseteq X^{s,p}(\mathcal{M}, \mathbb{R}), \tag{8}$$

so that the second inclusion in (7) cannot be an equality.

The second goal of the present work is to investigate *estimates* for the lifting when $sp \ge m$. In this case every map $u \in \dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$ can be written as $\pi \circ \tilde{u}$ with $\tilde{u} \in \dot{W}^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}})$ [5, Thm. 2], [2, Thm. 3] (see also [11, Thms. 5.1 & 5.2]). When $sp = 1 = m$, it is known that there is no estimate on the lifting when the covering map π is surjective and not injective [5, Rem. 3], [24, Lem. 5.1] (see also [11, Prop. 9.2]). If the covering space $\widetilde{\mathcal{N}}$ is not compact, then it is also known that there cannot be any estimate of the form (2) (see [23, Prop. 5.7] for the universal covering of the circle $\pi : \mathbb{R} \to \mathbb{S}^1$).

For the universal covering of the circle $\pi : \mathbb{R} \to \mathbb{S}^1$, Merlet and Mironescu and Molnar have obtained the following nonlinear estimate [18, Thm. 1.1], [23, Thm. 5.4] (see also [11, Thm. 9.6]).

Theorem 1.7. *Let $\mathcal{M}$ be a compact Riemannian manifold, let $m := \dim \mathcal{M}$, let $s \in (0, 1)$ and let $p \in (1, \infty)$. If $sp \ge m$ and $sp > 1$, then there exists a constant $C \in (0, \infty)$ such that if $\tilde{u} \in \dot{W}^{s,p}(\mathcal{M}, \mathbb{R})$ and if $u := e^{i\tilde{u}}$, we have*

$$\begin{aligned}&\iint_{\mathcal{M}\times\mathcal{M}} \frac{|\tilde{u}(y) - \tilde{u}(x)|^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \\ &\quad\le C\bigg(\iint_{\mathcal{M}\times\mathcal{M}} \frac{|u(y) - u(x)|^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x + \Big(\iint_{\mathcal{M}\times\mathcal{M}} \frac{|u(y) - u(x)|^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x\Big)^{\frac{1}{s}}\bigg).\end{aligned}$$

We generalize Theorem 1.7 to a general covering $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$.

Theorem 1.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be compact Riemannian manifolds, let $m := \dim \mathcal{M}$, let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be a Riemannian covering map, let $s \in (0, 1)$ and let $p \in (1, \infty)$. If $sp \ge m$ and $sp > 1$, then there exists a constant $C \in (0, \infty)$ such that if $\tilde{u} \in X(\mathcal{M}, \widetilde{\mathcal{N}})$ and if*

$u := \pi \circ \tilde{u} \in \dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$, *we have* $\tilde{u} \in \dot{W}^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}})$ *and*

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x \le C\biggl(\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x + \biggl(\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x\biggr)^{\frac{1}{s}}\biggr). \tag{9}$$

Theorem 1.7 can be proved by combining estimate (3) on the linear decomposition of the lifting with a fractional Sobolev embedding [23]; the latter embedding turns out to be a consequence of Theorem 1.8 (see Remark 4.2 below). Since the decomposition of the lifting into a sum (3) does not subsist for a general covering space $\widetilde{\mathcal{N}}$, we give a direct proof of Theorem 1.8; the structure of the proof with weak-type estimates on some level sets of differences is akin to the proof of Marcinkiewicz's real interpolation theorem and Sobolev's embedding theorem by interpolation (see for example [35, Chap. I, Thm. 5]).

As a consequence of Theorem 1.7 and of the classical extension of traces in the fractional space $\dot{W}^{1-1/p,p}(\mathcal{M}, \mathbb{R})$ into $\dot{W}^{1,p}(\mathcal{M}\times(0,1), \mathbb{R})$, one gets the following extension estimate: if $p \ge \dim \mathcal{M} + 1$, then there exists a constant $C \in (0,\infty)$ such that every map $u \in \dot{W}^{1-1/p,p}(\mathcal{M}, \mathbb{S}^1)$ is the trace on $\mathcal{M}\times\{0\}$ of a mapping $U \in \dot{W}^{1,p}(\mathcal{M}\times(0,1), \mathbb{S}^1)$ satisfying the estimate

$$\int_{\mathcal{M}\times(0,1)} |DU|^p \le C\biggl(\iint_{\mathcal{M}\times\mathcal{M}} \frac{|u(y)-u(x)|^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x + \biggl(\iint_{\mathcal{M}\times\mathcal{M}} \frac{|u(y)-u(x)|^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x\biggr)^{\frac{p}{p-1}}\biggr). \tag{10}$$

For a general target manifold $\mathcal{N}$, it is known that if $p \ge \dim \mathcal{M} + 1$, every map $u \in \dot{W}^{1-1/p,p}(\mathcal{M}, \mathcal{N})$ is the trace of a mapping $U \in \dot{W}^{1,p}(\mathcal{M}\times(0,1), \mathcal{N})$ [3, Thm. 1]. When $p > \dim \mathcal{M} + 1$, a compactness argument shows that the extension U can be taken to remain in a bounded set of $\dot{W}^{1,p}(\mathcal{M}\times(0,1), \mathcal{N})$ when the trace u remains bounded in $\dot{W}^{1-1/p,p}(\mathcal{M}, \mathcal{N})$ (see for example [30, Thm. 4]). When $p = \dim \mathcal{M} + 1$ and $\pi_{p-1}(\mathcal{N}) \not\simeq \{0\}$, such a boundedness cannot hold [29, Prop. 2.8], [25, Thm. 1.10]; one still gets estimates when the mapping u has a small fractional Sobolev energy and weak-type estimates in general [29, 30].

In the particular case where $\pi_1(\mathcal{N}) \simeq \dots \simeq \pi_{\lfloor p-1\rfloor}(\mathcal{N}) \simeq \{0\}$, where $\lfloor r\rfloor \in \mathbb{Z}$ denotes the integer part of $r \in \mathbb{R}$, Hardt and Lin [14, Thm. 6.2] have proved that there exists a constant $C \in (0,\infty)$ such that every map $u \in \dot{W}^{1-1/p,p}(\mathcal{M}, \mathcal{N})$ is the trace of a mapping $U \in \dot{W}^{1,p}(\mathcal{M}\times(0,1), \mathcal{N})$ satisfying the estimate

$$\int_{\mathcal{M}\times(0,1)} |DU|^p \le C \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x. \tag{11}$$

The surjectivity of the trace with the linear estimate (11) fails when the homotopy group $\pi_{\lfloor p-1\rfloor}(\mathcal{N})$ is nontrivial or when one of the homotopy groups $\pi_1(\mathcal{N}), \dots, \pi_{\lfloor p-2\rfloor}(\mathcal{N})$ is infinite [14, §6.3], [3, Thm. 4], [1, Prop. 1.13], [25, Thm. 1.10].

Estimates (10) and (11) naturally raise the following question.

Open Problem 3. Given compact Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$ and $p \geq \dim \mathcal{M} + 1$, is there a constant $C \in (0, \infty)$ such that every map $u \in \dot{W}^{1-1/p,p}(\mathcal{M}, \mathcal{N})$ is the trace on $\mathcal{M} \times \{0\}$ of a mapping $U \in \dot{W}^{1,p}(\mathcal{M} \times (0,1), \mathcal{N})$ satisfying the estimate

$$\int_{\mathcal{M}\times(0,1)} |DU|^p \leq C\biggl(\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y),u(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x + \Bigl(\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y),u(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x\Bigr)^{\frac{p}{p-1}}\biggr)?$$

In the case where the fundamental group $\pi_1(\mathcal{N})$ is infinite and where $\pi_2(\mathcal{N}) \simeq \cdots \simeq \pi_{\lfloor p-1 \rfloor}(\mathcal{N}) \simeq \{0\}$, although Theorem 1.8 provides a lifting in $\dot{W}^{1-1/p,p}(\mathcal{M}, \widetilde{\mathcal{N}})$, a universal covering space $\widetilde{\mathcal{N}}$ fails to be compact, so that Hardt and Lin's theorem on the extension of traces [14, Thm. 6.2] is not applicable.

2. Characterizations of the lifting space and related estimates

2.1. A priori estimate for regular liftings

We begin by proving an priori estimate on the lifting that will be the main analytical tool for the construction and estimate of liftings in Theorems 1.2 and 1.4. Given a convex open set $\Omega \subseteq \mathbb{R}^m$, we define the space of mappings that are essentially continuous on almost every segment of Ω:

$$\begin{aligned} Y(\Omega, \widetilde{\mathcal{N}}) := \bigl\{\tilde{u}\colon \Omega \to \widetilde{\mathcal{N}} \bigm| & \text{ for almost every } x, y \in \Omega, \text{ there exists } \tilde{u}_{x,y} \in C([x,y], \widetilde{\mathcal{N}}) \\ & \text{such that } \tilde{u}_{x,y}(x) = \tilde{u}(x),\ \tilde{u}_{x,y}(y) = \tilde{u}(y) \\ & \text{and } \tilde{u}_{x,y} = \tilde{u}|_{[x,y]} \text{ almost everywhere on } [x,y]\bigr\}, \end{aligned} \tag{12}$$

for which we prove the following a priori estimate.

Proposition 2.1. *Let $m \in \mathbb{N} \setminus \{0\}$, let $s \in (0,1)$ and let $p \in (1, \infty)$. If $sp > 1$, then there exists a constant $C \in (0, \infty)$ such that if $\Omega \subseteq \mathbb{R}^m$ is open and convex, if $\tilde{u} \in Y(\Omega, \widetilde{\mathcal{N}})$ and if $u := \pi \circ \tilde{u} \in \dot{W}^{s,p}(\Omega, \mathcal{N})$, then*

$$\iint_{\Omega\times\Omega} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(x))^p \wedge 1}{|y-x|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x \leq C \iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y),u(x))^p}{|y-x|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x. \tag{13}$$

Proposition 2.1 was initially stated and proved in the case where the covering space $\widetilde{\mathcal{N}}$ is compact [24], where it was an essential tool in the construction of liftings; the same argument also yields reverse superposition estimates in fractional Sobolev spaces [37]. We perform here a straightforward adaptation of the proof to the case where the covering space $\widetilde{\mathcal{N}}$ is not compact.

As in the proof in the compact case [24], the main analytic ingredient of the proof of Proposition 2.1 is the following estimate on Gagliardo seminorms on segments.

Lemma 2.2. *Let $m \in \mathbb{N} \setminus \{0\}$, let $s, \sigma \in (0,1)$ and let $p \in (1,\infty)$. If the set $\Omega \subseteq \mathbb{R}^m$ is open and convex, if $0 < \sigma < s$ and if the mapping $u \colon \Omega \to \mathcal{N}$ is measurable, then*

$$\iint_{\Omega\times\Omega} \biggl(\iint_{[0,1]\times[0,1]} \frac{d_{\mathcal{N}}\bigl(u((1-t)x+ty), u((1-r)x+ry)\bigr)^p}{|t-r|^{1+\sigma p}|y-x|^{m+sp}} \,\mathrm{d}r \,\mathrm{d}t \biggr) \mathrm{d}y \,\mathrm{d}x \\ \le \frac{8}{(2(s-\sigma)p+1)^2-1} \iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y),u(x))^p}{|x-y|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x. \tag{14}$$

It will appear in the proof of Lemma 2.2 that the constant in inequality (14) is sharp: equality holds in (14) if $\Omega = \mathbb{R}^m$. The left-hand side of (14) cannot be bounded for $\sigma = s$.

Proof of Lemma 2.2. We apply the change of variable $(z,w) = ((1-t)x+ty, (1-r)x+ry)$ in the integral on the left-hand side of (14), and we obtain, since $z - w = (t-r)(y-x)$ and $\det(\left(\begin{smallmatrix} 1-t & t \\ 1-r & r \end{smallmatrix}\right)) = -(t-r)$,

$$\iint_{\Omega\times\Omega} \biggl(\iint_{[0,1]\times[0,1]} \frac{d_{\mathcal{N}}\bigl(u((1-t)x+ty), u((1-r)x+ry)\bigr)^p}{|t-r|^{1+\sigma p}|y-x|^{m+sp}} \,\mathrm{d}t \,\mathrm{d}r \biggr) \mathrm{d}y \,\mathrm{d}x \\ = \iint_{\Omega\times\Omega} \iint_{\Sigma_{z,w}} \frac{d_{\mathcal{N}}(u(z),u(w))^p}{|t-r|^{1-(s-\sigma)p}|z-w|^{m+sp}} \,\mathrm{d}t \,\mathrm{d}r \,\mathrm{d}z \,\mathrm{d}w, \tag{15}$$

where we have defined for each $z, w \in \Omega$ the set

$$\Sigma_{z,w} := \Bigl\{ (r,t) \in [0,1]\times[0,1] \Bigm| \tfrac{rz-tw}{r-t} \in \Omega \text{ and } \tfrac{(1-r)z-(1-t)w}{t-r} \in \Omega \Bigr\}.$$

We observe that, since $s > \sigma$, we have by domain-monotonicity of the integral and by direct computation for each $z, w \in \Omega$,

$$\iint_{\Sigma_{z,w}} \frac{1}{|t-r|^{1-(s-\sigma)p}} \,\mathrm{d}t \,\mathrm{d}r \le \int_0^1 \int_0^1 \frac{1}{|t-r|^{1-(s-\sigma)p}} \,\mathrm{d}t \,\mathrm{d}r \\ = \frac{1}{(s-\sigma)p} \int_0^1 |1-r|^{(s-\sigma)p} + |r|^{(s-\sigma)p} \,\mathrm{d}r \\ = \frac{8}{(2(s-\sigma)p+1)^2-1} < \infty, \tag{16}$$

and conclusion (14) follows from identity (15) and estimate (16). ∎

Our second tool is the following elementary geometric result on covering space.

Lemma 2.3. *Let $\pi \colon \widetilde{\mathcal{N}} \to \mathcal{N}$ be a Riemannian covering map. If the manifold $\mathcal{N}$ has a positive injectivity radius $\operatorname{inj}(\mathcal{N}) > 0$, then for every $\tilde{x}, \tilde{y} \in \widetilde{\mathcal{N}}$ such that $d_{\widetilde{\mathcal{N}}}(\tilde{x},\tilde{y}) \le \operatorname{inj}(\mathcal{N})$, one has $d_{\widetilde{\mathcal{N}}}(\tilde{x},\tilde{y}) = d_{\mathcal{N}}(\pi(\tilde{x}),\pi(\tilde{y}))$.*

The proof of Lemma 2.3 follows from the definition of injectivity radius $\operatorname{inj}(\mathcal{N})$ and from the lifting of geodesics (see for example [24, Lem. 2.1]).

Proof of Proposition 2.1. We first assume that the set $\Omega \subseteq \mathbb{R}^m$ is open and convex. By the convexity of Ω and by the definition of $Y(\Omega, \widetilde{\mathcal{N}})$ in (12), for almost every $x, y \in \Omega$, we have $[x, y] \subset \Omega$, the restriction $\tilde{u} \restriction_{[x,y]}$ of $\tilde{u}$ to the segment $[x, y]$ satisfies $\tilde{u} \restriction_{[x,y]} = \tilde{u}_{x,y}$ almost everywhere on $[x, y]$, $\tilde{u}_{x,y}(x) = \tilde{u}(x)$ and $\tilde{u}_{x,y}(y) = \tilde{u}(y)$, with $\tilde{u}_{x,y} \in C([x, y], \widetilde{\mathcal{N}})$. By the intermediate value theorem, there exists $z \in [x, y]$ such that

$$\operatorname{inj}(\mathcal{N}) \wedge d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x)) = \operatorname{inj}(\mathcal{N}) \wedge d_{\widetilde{\mathcal{N}}}(\tilde{u}_{x,y}(y), \tilde{u}_{x,y}(x)) = d_{\widetilde{\mathcal{N}}}(\tilde{u}_{x,y}(z), \tilde{u}_{x,y}(y));$$

by Lemma 2.3, we have thus

$$\operatorname{inj}(\mathcal{N}) \wedge d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x)) = d_{\mathcal{N}}(u_{x,y}(z), u_{x,y}(y)),$$

with $u_{x,y} := \pi \circ \tilde{u}_{x,y} \in C([x, y], \mathcal{N})$. We have thus proved that

$$\operatorname{inj}(\mathcal{N}) \wedge d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x)) \le \sup_{z \in [x,y]} d_{\mathcal{N}}(u_{x,y}(z), u_{x,y}(y)). \tag{17}$$

Fixing $\sigma \in (0, 1)$ such that $1/p < \sigma < s$, we deduce from the one-dimensional fractional Morrey–Sobolev embedding (see for example [17, Thm. 2.8]) and from (17) that

$$\begin{aligned} &\operatorname{inj}(\mathcal{N})^p \wedge d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p \\ &\quad \le C_1 \iint_{[0,1]\times[0,1]} \frac{d_{\mathcal{N}}\big(u_{x,y}((1-t)x + ty), u_{x,y}((1-r)x + ry)\big)^p}{|t - r|^{1+\sigma p}} \,\mathrm{d}t \,\mathrm{d}r \\ &\quad = C_1 \iint_{[0,1]\times[0,1]} \frac{d_{\mathcal{N}}\big(u((1-t)x + ty), u((1-r)x + ry)\big)^p}{|t - r|^{1+\sigma p}} \,\mathrm{d}t \,\mathrm{d}r, \end{aligned} \tag{18}$$

since $\sigma p > 1$ and $u_{x,y} = u \restriction_{[x,y]}$ almost everywhere on $[x, y]$. The conclusion follows then by integration of (18) thanks to Lemma 2.2. ∎

2.2. Variations on the lower exponent

We exhibit a whole family of characterizations of the space appearing in the description of liftings of Theorem 1.2; our analysis follows and extends the results obtained for $m = sp$ in the context of homotopy estimates [36, §5]. The results of the present section are valid under the quite general assumption that the target $\mathcal{E}$ is any metric space.

Proposition 2.4 (Exponent improvement). *Let $\mathcal{M}$ be a Riemannian manifold, let $\mathcal{E}$ be a metric space, let $s \in (0, 1)$, let $p \in (1, \infty)$ and let $q_0, q_1 \in (0, \infty)$. If $sp > 1 \vee q_0 \vee q_1$, then there exists a constant $C \in (0, \infty)$ such that for every measurable map $f : \mathcal{M} \to \mathcal{E}$ one has*

$$\begin{aligned} &\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{E}}(f(y), f(x))^p \wedge d_{\mathcal{E}}(f(y), f(x))^{q_1}}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \\ &\qquad \le C \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{E}}(f(y), f(x))^p \wedge d_{\mathcal{E}}(f(y), f(x))^{q_0}}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x, \end{aligned} \tag{19}$$

with $m := \dim \mathcal{M}$.

The main tool to prove Proposition 2.4 is the following estimate which was already known in the special case $\gamma = m$ [36, Prop. 5.5].

Proposition 2.5. *Let $q_0, q_1 \in [0, +\infty)$, let $\eta \in (0, 1)$ and let $\gamma \in (0, \infty)$. If $q_1 < \gamma$ and if either $q_0 \geq 1$ or $\gamma > 1$, then there exists a constant $C \in (0, \infty)$ such that for every $m \in \mathbb{N} \setminus \{0\}$, for every convex open set $\Omega \subseteq \mathbb{R}^m$ and for every measurable map $f : \Omega \to \mathcal{E}$, one has*

$$\iint_{\substack{(x,y)\in\Omega\times\Omega \\ d_{\mathcal{E}}(f(y),f(x))\geq\lambda}} \frac{\bigl(d_{\mathcal{E}}(f(y), f(x)) - \lambda\bigr)^{q_1}}{|y-x|^{m+\gamma}} \,\mathrm{d}y \,\mathrm{d}x \leq C\lambda^{q_1-q_0} \iint_{\substack{(x,y)\in\Omega\times\Omega \\ d_{\mathcal{E}}(f(y),f(x))\geq\eta\lambda}} \frac{\bigl(d_{\mathcal{E}}(f(y), f(x)) - \eta\lambda\bigr)^{q_0}}{|y-x|^{m+\gamma}} \,\mathrm{d}y \,\mathrm{d}x. \tag{20}$$

In the particular case $q_1 \leq q_0$, one has the pointwise estimate

$$(t-\lambda)^{q_1} \leq (t-\eta\lambda)^{q_0}/((1-\eta)\lambda)^{q_0-q_1},$$

and (20) follows immediately by integration.

Proposition 2.5 is reminiscent of an estimate of Nguyen that appears in characterizations of first-order Sobolev spaces [28, Thm. 1 (a)].

Our first tool to prove Proposition 2.5 in general, is the following scaling inequality (when $\gamma = m$ see [36, Prop. 5.1]).

Lemma 2.6. *For every $m \in \mathbb{N} \setminus \{0\}$, for every convex open set $\Omega \subset \mathbb{R}^m$, for every measurable map $f : \Omega \to \mathcal{E}$, for every $q \in [0, \infty)$ and for every $\gamma \in \mathbb{R}$, if $\lambda_0 < \lambda_1$ one has*

$$\begin{aligned}\iint_{\substack{(x,y)\in\Omega\times\Omega \\ d_{\mathcal{E}}(f(y),f(x))\geq\lambda_1}} &\frac{\bigl(d_{\mathcal{E}}(f(y), f(x)) - \lambda_1\bigr)^{q}}{|y-x|^{m+\gamma}} \,\mathrm{d}y \,\mathrm{d}x \\ &\leq 2^{(\gamma-1-(q-1)_+)_+}\Bigl(\frac{\lambda_1}{\lambda_0}\Bigr)^{(q-1)_+-\gamma+1} \\ &\quad\times \iint_{\substack{(x,y)\in\Omega\times\Omega \\ d_{\mathcal{E}}(f(y),f(x))\geq\lambda_0}} \frac{\bigl(d_{\mathcal{E}}(f(y), f(x)) - \lambda_0\bigr)^{q}}{|y-x|^{m+\gamma}} \,\mathrm{d}y \,\mathrm{d}x.\end{aligned} \tag{21}$$

Proof of Lemma 2.6. Since the set Ω is convex, for every $x, y \in \Omega$, we have $\frac{x+y}{2} \in \Omega$ and thus by the triangle inequality,

$$d_{\mathcal{E}}(f(y), f(x)) - \lambda_1 \leq d_{\mathcal{E}}(f(y), f(\tfrac{x+y}{2})) - \tfrac{\lambda_1}{2} + d_{\mathcal{E}}(f(\tfrac{x+y}{2}), f(x)) - \tfrac{\lambda_1}{2},$$

so that

$$\begin{aligned}\iint_{\substack{(x,y)\in\Omega\times\Omega \\ d_{\mathcal{E}}(f(y),f(x))\geq\lambda_1}} &\frac{\bigl(d_{\mathcal{E}}(f(y), f(x)) - \lambda_1\bigr)^{q}}{|y-x|^{m+\gamma}} \,\mathrm{d}y \,\mathrm{d}x \\ &\leq 2^{(q-1)_+} \iint_{\substack{(x,y)\in\Omega\times\Omega \\ d_{\mathcal{E}}(f(y),f(\frac{x+y}{2}))\geq\frac{\lambda_1}{2}}} \frac{\bigl(d_{\mathcal{E}}(f(y), f(\frac{x+y}{2})) - \frac{\lambda_1}{2}\bigr)^{q}}{|y-x|^{m+\gamma}} \,\mathrm{d}y \,\mathrm{d}x \\ &\quad + 2^{(q-1)_+} \iint_{\substack{(x,y)\in\Omega\times\Omega \\ d_{\mathcal{E}}(f(\frac{x+y}{2}),f(x))\geq\frac{\lambda_1}{2}}} \frac{\bigl(d_{\mathcal{E}}(f(\frac{x+y}{2}), f(x)) - \frac{\lambda_1}{2}\bigr)^{q}}{|y-x|^{m+\gamma}} \,\mathrm{d}y \,\mathrm{d}x.\end{aligned} \tag{22}$$

Therefore, by symmetry between both terms in the right-hand side of (22) under exchange of the variables x and y in the integral, we have

$$\begin{aligned}\iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\geq\lambda_1}} \frac{\big(d_{\mathcal{E}}(f(y), f(x)) - \lambda_1\big)^q}{|y-x|^{m+\gamma}}\,\mathrm{d}y\,\mathrm{d}x\\ = 2^{(q-1)_+ + 1} \iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(\frac{x+y}{2}),f(x))\geq\frac{\lambda_1}{2}}} \frac{\big(d_{\mathcal{E}}(f(\frac{x+y}{2}), f(x)) - \frac{\lambda_1}{2}\big)^q}{|y-x|^{m+\gamma}}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}\tag{23}$$

By the change of variable $y = 2z - x$, we have $|y - x| = 2|z - x|$ and thus

$$\begin{aligned}\iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(\frac{x+y}{2}),f(x))\geq\frac{\lambda_1}{2}}} \frac{\big(d_{\mathcal{E}}(f(\frac{x+y}{2}), f(x)) - \frac{\lambda_1}{2}\big)^q}{|y-x|^{m+\gamma}}\,\mathrm{d}y\,\mathrm{d}x\\ = \frac{1}{2^\gamma}\int_\Omega\bigg(\int_{\Sigma_x} \frac{\big(d_{\mathcal{E}}(f(z), f(x)) - \frac{\lambda_1}{2}\big)^q}{|z-x|^{m+\gamma}}\,\mathrm{d}z\bigg)\,\mathrm{d}x\\ \leq \frac{1}{2^\gamma}\iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\geq\frac{\lambda_1}{2}}} \frac{\big(d_{\mathcal{E}}(f(y), f(x)) - \frac{\lambda_1}{2}\big)^q}{|y-x|^{m+\gamma}}\,\mathrm{d}y\,\mathrm{d}x, \end{aligned}\tag{24}$$

where for every $x \in \Omega$, the set Σ_x is defined as

$$\Sigma_x := \big\{z \in \Omega \;\big|\; 2z - x \in \Omega \text{ and } d_{\mathcal{E}}(f(z), f(x)) \geq \tfrac{\lambda_1}{2}\big\}.$$

By (23) and (24), we deduce that for every $\lambda_1 > 0$,

$$\begin{aligned}\iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\geq\lambda_1}} \frac{\big(d_{\mathcal{E}}(f(y), f(x)) - \lambda_1\big)^q}{|y-x|^{m+\gamma}}\,\mathrm{d}y\,\mathrm{d}x\\ \leq 2^{(q-1)_+ - (\gamma-1)} \iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\geq\frac{\lambda_1}{2}}} \frac{\big(d_{\mathcal{E}}(f(y), f(x)) - \frac{\lambda_1}{2}\big)^q}{|y-x|^{m+\gamma}}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}\tag{25}$$

Iterating estimate (25), we deduce that for every nonnegative integer $\ell \in \mathbb{N}$,

$$\begin{aligned}\iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\geq\lambda_1}} \frac{\big(d_{\mathcal{E}}(f(y), f(x)) - \lambda_1\big)^q}{|y-x|^{m+\gamma}}\,\mathrm{d}y\,\mathrm{d}x\\ \leq 2^{\ell((q-1)_+ - (\gamma-1))} \iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\geq\frac{\lambda_1}{2^\ell}}} \frac{\big(d_{\mathcal{E}}(f(y), f(x)) - \frac{\lambda_1}{2^\ell}\big)^q}{|y-x|^{m+\gamma}}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}\tag{26}$$

If $\lambda_0 \in (0, \lambda_1)$, we let $\ell \in \mathbb{N}$ in (26) be defined by the condition $2^{-(\ell+1)}\lambda_1 \leq \lambda_0 < 2^{-\ell}\lambda_1$ and we conclude that (21) holds. ∎

Our second tool for the proof of Proposition 2.5 is the next elementary integral inequality [36, Lem. 5.6].

Lemma 2.7 (Integral estimate of truncated powers). *For every $q_0, q_1 \in [0, \infty)$ and every $\eta \in (0, 1)$, there exists a constant $C > 0$ such that for every $t \in [1, \infty)$,*

$$(t-1)^{q_1} \le C \int_\eta^t \frac{(t-r)^{q_0}}{r^{1+q_0-q_1}} \,\mathrm{d}r.$$

Proof of Proposition 2.5. Applying Lemma 2.7 with $t := d_{\mathcal{E}}(f(y), f(x))/\lambda$ at each $x, y \in \Omega$, integrating the result and interchanging the integrals, we have

$$\begin{aligned}\iint\limits_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\ge\lambda}} &\frac{\big(d_{\mathcal{E}}(f(y), f(x)) - \lambda\big)^{q_1}}{|y-x|^{m+\gamma}} \,\mathrm{d}y\,\mathrm{d}x\\ &\le C_1 \lambda^{q_1-q_0} \int_\eta^\infty \iint\limits_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\ge r\lambda}} \frac{\big(d_{\mathcal{E}}(f(y), f(x)) - r\lambda\big)^{q_0}}{r^{1+q_0-q_1}|y-x|^{m+\gamma}} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}r. \end{aligned} \tag{27}$$

Since the set $\Omega \subseteq \mathbb{R}^m$ is convex, by Lemma 2.6, we have for every $r \in (\eta, \infty)$,

$$\begin{aligned}\iint\limits_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\ge r\lambda}} &\frac{\big(d_{\mathcal{E}}(f(y), f(x)) - r\lambda\big)^{q_0}}{|y-x|^{m+\gamma}} \,\mathrm{d}y\,\mathrm{d}x\\ &\le C_2 \frac{1}{r^{(\gamma-1)-(q_0-1)_+}} \iint\limits_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\ge\eta\lambda}} \frac{\big(d_{\mathcal{E}}(f(y), f(x)) - \eta\lambda\big)^{q_0}}{|y-x|^{m+\gamma}} \,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{28}$$

Combining estimates (27) and (28), we deduce that

$$\begin{aligned}\iint\limits_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\ge\lambda}} &\frac{\big(d_{\mathcal{E}}(f(y), f(x)) - \lambda\big)^{q_1}}{|y-x|^{m+\gamma}} \,\mathrm{d}y\,\mathrm{d}x\\ &\le C_3 \lambda^{q_1-q_0} \int_\eta^\infty \frac{1}{r^{\gamma+1-(1-q_0)_+-q_1}} \,\mathrm{d}r\\ &\quad\times \iint\limits_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\ge\eta\lambda}} \frac{\big(d_{\mathcal{E}}(f(y), f(x)) - \eta\lambda\big)^{q_1}}{|y-x|^{m+\gamma}} \,\mathrm{d}y\,\mathrm{d}x, \end{aligned} \tag{29}$$

since $q_0 - 1 - (q_0-1)_+ = -(1-q_0)_+$. If $q_1 < \gamma - (1-q_0)_+$, then

$$\int_\eta^\infty \frac{1}{r^{\gamma+1-(1-q_0)_+-q_1}} \,\mathrm{d}r = \frac{1}{(\gamma-(1-q_0)_+-q_1)\eta^{\gamma-(1-q_0)_+-q_1}} < \infty,$$

and estimate (20) follows from (29).

If $q_0 \ge 1$, then we have proved the estimate for $q_1 < \gamma$. Otherwise, $q_0 < 1$, and we have proved estimate (20) for $q_1 < q_0 + (\gamma - 1)$. Iterating the estimate finitely many times we reach the interval $q_1 \in [0, 1]$ and conclusion (20) then follows for $q_1 < \gamma$. ■

We are now in a position to prove Proposition 2.4.

Proof of Proposition 2.4. Since the case $q_1 \le q_0$ follows from the fact that for every $t \in (0, \infty)$, we have $t^p \wedge t^{q_0} \le t^p \wedge t^{q_1}$, we consider the case $q_1 > q_0$. Letting $\Omega \subseteq \mathbb{R}^m$ be a convex open set and the mapping $f\colon \Omega \to \mathcal{E}$ be measurable, and defining the set

$$A := \bigl\{(x, y) \in \Omega \times \Omega \bigm| d_{\mathcal{E}}(f(y), f(x)) \ge 1\bigr\},$$

we decompose, since $q_1 < sp < p$, the integral in the left-hand side of (19) as

$$\begin{aligned}
&\iint_{\Omega\times\Omega} \frac{d_{\mathcal{E}}(f(y), f(x))^p \wedge d_{\mathcal{E}}(f(y), f(x))^{q_1}}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad= \iint_{\Omega\times\Omega\setminus A} \frac{d_{\mathcal{E}}(f(y), f(x))^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x + \iint_{A} \frac{d_{\mathcal{E}}(f(y), f(x))^{q_1}}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned} \tag{30}$$

On the one hand, we immediately have

$$\begin{aligned}
&\iint_{\Omega\times\Omega\setminus A} \frac{d_{\mathcal{E}}(f(y), f(x))^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad\le \iint_{\Omega\times\Omega} \frac{d_{\mathcal{E}}(f(y), f(x))^p \wedge d_{\mathcal{E}}(f(y), f(x))^{q_0}}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned} \tag{31}$$

On the other hand, by Proposition 2.5, since $sp > 1$ and $q_1 < sp$, we have

$$\begin{aligned}
&\iint_{A} \frac{d_{\mathcal{E}}(f(y), f(x))^{q_1}}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad\le 2^{q_1} \iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\ge\frac{1}{2}}} \frac{\bigl(d_{\mathcal{E}}(f(y), f(x)) - \frac{1}{2}\bigr)^{q_1}}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad\le C_1 \iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\mathcal{E}}(f(y),f(x))\ge\frac{1}{3}}} \frac{\bigl(d_{\mathcal{E}}(f(y), f(x)) - \frac{1}{3}\bigr)^{q_0}}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned} \tag{32}$$

and it follows thus from (30), (31) and (32) that

$$\begin{aligned}
&\iint_{\Omega\times\Omega} \frac{d_{\mathcal{E}}(f(y), f(x))^p \wedge d_{\mathcal{E}}(f(y), f(x))^{q_1}}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad\le C_2 \iint_{\Omega\times\Omega} \frac{d_{\mathcal{E}}(f(y), f(x))^p \wedge d_{\mathcal{E}}(f(y), f(x))^{q_0}}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned} \tag{33}$$

since $q_0 < p$. The announced conclusion (19) then follows from (33) and the covering of Lemma 2.11. ∎

Thanks to Proposition 2.4, we can now prove Proposition 1.5.

Proof of Proposition 1.5. This follows from Proposition 2.4, with $\mathcal{E} = \widetilde{\mathcal{N}}$. ∎

2.3. Compactness in the space of liftings

Given the estimate on the lifting of Proposition 2.1 on a set which is dense in the fractional Sobolev space $\dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$ [9], a classical approach to prove the existence of a lifting would be to consider the limit of the liftings of an approximating sequence. In order to perform this, we need a compactness result on sets for which the left-hand side of (13) is uniformly bounded.

Proposition 2.8. *Let $\mathcal{M}$ be a Riemannian manifold with finite volume, let $\mathcal{E}$ be a metric space, let $0 \le q \le p$ and let $0 < s < 1$. Assume that every bounded subset of $\mathcal{E}$ is totally bounded. If $\mathcal{S}$ is a set of measurable functions from $\mathcal{M}$ to $\mathcal{E}$ such that*

$$\sup_{f \in \mathcal{S}} \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{E}}(f(y), f(x))^p \wedge d_{\mathcal{E}}(f(y), f(x))^q}{d_{\mathcal{M}}(y,x)^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x < \infty, \tag{34}$$

with $m := \dim \mathcal{M}$, and such that

$$\inf_{f,g\in\mathcal{S}} \int_{\mathcal{M}} \frac{1}{1 + d_{\mathcal{E}}(g, f)} > 0, \tag{35}$$

then the set $\mathcal{S}$ is totally bounded for the distance

$$d_\mu(f, g) := \int_{\mathcal{M}} \frac{d_{\mathcal{E}}(f, g)}{1 + d_{\mathcal{E}}(f, g)}. \tag{36}$$

Although the case $p = q = 0$ is covered in Proposition 2.8, it is not particulary interesting since in view of Lemma 2.14, the mapping f should be constant on every connected component of $\mathcal{M}$.

If the metric space $\mathcal{E}$ is complete, the assumption that any of its subsets is totally bounded is equivalent to $\mathcal{E}$ having the Bolzano–Weierstraß property or to $\mathcal{E}$ being a proper space.

Convergence with respect to the distance d_μ defined in (36) is convergence in measure. We first remark that this distance can be controlled on finite-measure sets by a quantity reminiscent of the integrand in (34).

Lemma 2.9. *Let μ be a measure on Ω and let $\mathcal{E}$ be a metric space. If $0 \le q \le p$ and if the mappings $f, g \colon \Omega \to \mathcal{E}$ are measurable, then*

$$\int_{\mathcal{M}} \frac{d_{\mathcal{E}}(f, g)}{1 + d_{\mathcal{E}}(f, g)}\,\mathrm{d}\mu \le \mu(\Omega)^{(1-1/p)_+} \left(\int_{\Omega} d_{\mathcal{E}}(f, g)^p \wedge d_{\mathcal{E}}(f, g)^q \,\mathrm{d}\mu\right)^{\frac{1}{p}\wedge 1}. \tag{37}$$

Proof. When $0 \le p \le 1$, (37) follows from the fact that for every $t \in [0, \infty)$ one has

$$t/(1 + t) \le t \wedge 1 \le t^p \wedge t^q,$$

whereas when $p > 1$, (37) follows from the fact that

$$t/(1 + t) \le t \wedge 1 \le t \wedge t^{q/p}$$

and Hölder's inequality. ∎

The proof of Proposition 2.8 will rely on the following inequality.

Lemma 2.10. *If $p, q \in [0, \infty)$, then for every $\ell \in \mathbb{N}$ and $a_1, \dots, a_\ell \in [0, \infty)$, we have*

$$\Big(\sum_{i=1}^{\ell} a_i\Big)^p \wedge \Big(\sum_{i=1}^{\ell} a_i\Big)^q \le \max_{i \in \{1,\dots,\ell\}} (\ell a_i)^p \wedge (\ell a_i)^q .$$

Proof. Without loss of generality, we can assume that for each $i \in \{1, \dots, \ell\}$ one has $a_i \le a_1$, so that $\sum_{i=1}^{\ell} a_i \le \ell a_1$ and

$$\Big(\sum_{i=1}^{\ell} a_i\Big)^p \wedge \Big(\sum_{i=1}^{\ell} a_i\Big)^q \le (\ell a_1)^p \wedge (\ell a_1)^q = \max_{i \in \{1,\dots,\ell\}} (\ell a_i)^p \wedge (\ell a_i)^q . \qquad \blacksquare$$

Proof of Proposition 2.8. By the finiteness of the volume and a local charts argument, we assume that $\Omega = \mathbb{Q}^m := [0, 1]^m$. For every $k \in \mathbb{N} \setminus \{0\}$, we subdivide the cube $\mathbb{Q}^m$ in a set $\mathcal{Q}^k$ of k^m cubes of edge length $1/k$. Given $f \in \mathcal{S}$, we define the map $f_k : \mathbb{Q}^m \to \mathcal{E}$ in such a way that f_k is constant on each cube $Q \in \mathcal{Q}^k$ and for every $x \in Q \in \mathcal{Q}^k$,

$$\begin{aligned} &\int_Q d_{\mathcal{E}}(f(x), f_k(x))^p \wedge d_{\mathcal{E}}(f(x), f_k(x))^q \,\mathrm{d}x \\ &\qquad \le k^m \iint_{Q \times Q} d_{\mathcal{E}}(f(y), f(x))^p \wedge d_{\mathcal{E}}(f(y), f(x))^q \,\mathrm{d}y \,\mathrm{d}x. \end{aligned} \tag{38}$$

It follows immediately from (38) that

$$\begin{aligned} &\int_{\mathbb{Q}^m} d_{\mathcal{E}}(f(x), f_k(x))^p \wedge d_{\mathcal{E}}(f(x), f_k(x))^q \,\mathrm{d}x \\ &\qquad \le \frac{m^{\frac{m+sp}{2}}}{k^{sp}} \iint_{\mathbb{Q}^m \times \mathbb{Q}^m} \frac{d_{\mathcal{E}}(f(y), f(x))^p \wedge d_{\mathcal{E}}(f(y), f(x))^q}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x, \end{aligned} \tag{39}$$

and thus by Lemma 2.9 that

$$\begin{aligned} &\int_{\mathbb{Q}^m} \frac{d_{\mathcal{E}}(f(x), f_k(x))}{1 + d_{\mathcal{E}}(f(x), f_k(x))} \,\mathrm{d}x \\ &\qquad \le \Big(\frac{m^{\frac{m+sp}{2}}}{k^{sp}} \iint_{\mathbb{Q}^m \times \mathbb{Q}^m} \frac{d_{\mathcal{E}}(f(x), f(y))^p \wedge d_{\mathcal{E}}(f(x), f(y))^q}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x\Big)^{1 \wedge \frac{1}{p}}. \end{aligned} \tag{40}$$

Assumption (34) and estimate (40) imply that for $k \in \mathbb{N} \setminus \{0\}$ large enough, the set $\mathcal{S}$ is contained in an arbitrarily small neighbourhood of the set of mappings f_k. Since for every $k \in \mathbb{N}$, the set of mappings taking constant value on each $Q \in \mathcal{Q}^k$ is bi-Lipschitz equivalent to the manifold $\mathcal{E}^{k^m}$, and since bounded subsets of $\mathcal{E}$ are totally bounded, it remains to prove that for any $k \in \mathbb{N}$, the mappings f_k are contained in a bounded set.

For every $\lambda \in (0, \infty)$ and $f, g : \mathcal{M} \to \mathcal{E}$, we have

$$\int_{\mathbb{Q}} \frac{1}{1 + d_{\mathcal{E}}(g(x), f(x))} \,\mathrm{d}x \le \big|\{x \in \mathbb{Q}^m \mid d_{\mathcal{E}}(g(x), f(x)) \le \lambda\}\big| + \frac{1}{1 + \lambda},$$

and therefore by our assumption (35), there exist $\lambda \in (0, \infty)$ and $\eta \in (0, \infty)$ such that for every $f, g \in \mathcal{S}$,

$$\left|\left\{x \in \mathbb{Q}^m \;\middle|\; d_{\mathcal{E}}(g(x), f(x)) \le \lambda\right\}\right| \ge \eta. \tag{41}$$

For every $x, y \in \mathbb{Q}^m$, we have by the triangle inequality,

$$\begin{aligned} d_{\mathcal{E}}(g_k(x), f_k(x)) \le d_{\mathcal{E}}(g_k(x), g(x)) + d_{\mathcal{E}}(g(x), g(y)) \\ + d_{\mathcal{E}}(g(y), f(y)) + d_{\mathcal{E}}(f(y), f(x)) + d_{\mathcal{E}}(f(x), f_k(x)), \end{aligned}$$

and thus by Lemma 2.10,

$$\begin{aligned} &\int_{\mathbb{Q}^m} d_{\mathcal{E}}(g_k(x), f_k(x))^p \wedge d_{\mathcal{E}}(g_k(x), f_k(x))^q \,\mathrm{d}x \\ &\quad\le \int_{\mathbb{Q}^m} \bigl(5d_{\mathcal{E}}(g_k(x), g(x))\bigr)^p \wedge \bigl(5d_{\mathcal{E}}(g_k(x), g(x))\bigr)^q \,\mathrm{d}x \\ &\qquad + \int_{\mathbb{Q}^m} \fint_A \bigl(5d_{\mathcal{E}}(g(y), g(x))\bigr)^p \wedge \bigl(5d_{\mathcal{E}}(g(y), g(x))\bigr)^q \,\mathrm{d}y\,\mathrm{d}x \\ &\qquad + \fint_A \bigl(5d_{\mathcal{E}}(g(y), f(y))\bigr)^p \wedge \bigl(5d_{\mathcal{E}}(g(y), f(y))\bigr)^q \,\mathrm{d}y \\ &\qquad + \int_{\mathbb{Q}^m} \fint_A \bigl(5d_{\mathcal{E}}(f(y), f(x))\bigr)^p \wedge \bigl(5d_{\mathcal{E}}(f(y), f(x))\bigr)^q \,\mathrm{d}y\,\mathrm{d}x \\ &\qquad + \int_{\mathbb{Q}^m} \bigl(5d_{\mathcal{E}}(f_k(x), f(x))\bigr)^p \wedge \bigl(5d_{\mathcal{E}}(f_k(x), f(x))\bigr)^q \,\mathrm{d}x, \end{aligned} \tag{42}$$

with

$$A := \left\{x \in \mathbb{Q}^m \;\middle|\; d_{\mathcal{E}}(g(x), f(x)) \le \lambda\right\}. \tag{43}$$

Inserting (34), (39) and (41) in (42) combined with (43) and with Lemma 2.9, we get

$$\int_{\mathbb{Q}^m} \frac{d_{\mathcal{E}}(g_k(x), f_k(x))}{1 + d_{\mathcal{E}}(g_k(x), f_k(x))} \,\mathrm{d}x \le C_1 \Bigl(\frac{1}{k^{sp}} + \frac{1}{\eta} + \lambda^p \wedge \lambda^q\Bigr)^{\frac{1}{p} \wedge 1},$$

and the announced boundedness follows. ∎

2.4. Existence of a lifting

The last tool we will use to prove Theorem 1.2 is the existence of local charts that cover the product.

Lemma 2.11. *If $\mathcal{M}$ is a connected compact manifold with $m := \dim \mathcal{M}$, then there exist open sets $V_1, \dots, V_\ell \subset \mathcal{M}$ such that for each $i \in \{1, \dots, \ell\}$, the set $\overline{V_i}$ is diffeomorphic to the closed ball $\overline{\mathbb{B}}_1 \subset \mathbb{R}^m$ and such that*

$$\mathcal{M} \times \mathcal{M} \subseteq \bigcup_{i=1}^{\ell} V_i \times V_i.$$

Proof. Since the manifold $\mathcal{M}$ is connected, every doubleton $\{x, y\} \subset \mathcal{M}$ is contained in an open set $V \subseteq \mathcal{M}$ such that $\overline{V}$ is diffeomorphic to the closed ball $\overline{B}_1 \subset \mathbb{R}^m$. In particular, $(x, y) \in V \times V$. We conclude by compactness of $\mathcal{M} \times \mathcal{M}$. ∎

The proof of Theorem 1.2 will rely on the notion of normal covering. A covering map $\pi: \widetilde{\mathcal{N}} \to \mathcal{N}$ is *normal* (or regular) whenever, for every $\tilde{y} \in \widetilde{\mathcal{N}}$, we have

$$\pi^{-1}(\{\pi(\tilde{y})\}) = \{\tau(\tilde{y}) \mid \tau \in \operatorname{Aut}(\pi)\}, \tag{44}$$

where the *group of deck transformations* (or group of covering transformations or Galois group) of the covering π is the group

$$\operatorname{Aut}(\pi) = \{\tau: \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}} \mid \tau \text{ is a homeomorphism and } \pi \circ \tau = \pi\}$$

endowed with the composition operation [15, §1.3], [34, Chap. 2, §6]. When π is a Riemannian covering, π is a local isometry and any $\tau \in \operatorname{Aut}(\pi)$ is a global isometry of $\widetilde{\mathcal{N}}$.

If $\pi: \widetilde{\mathcal{N}} \to \mathcal{N}$ is a universal covering, that is when π is surjective and $\widetilde{\mathcal{N}}$ is simply connected, then π is normal.

We proceed to the proof of existence of a lifting.

Proof of Theorem 1.2. We first assume that $\pi: \widetilde{\mathcal{N}} \to \mathcal{N}$ is a normal covering of $\mathcal{N}$.

Given a map $u \in \dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$, by Brezis and Mironescu's approximation result for fractional Sobolev mappings [9], there exists a sequence of mappings $(u_j)_{j \in \mathbb{N}}$ in the set $\mathcal{R}^0_{m-2}(\mathcal{M}, \mathcal{N}) \cap \dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$ that converges strongly to the mapping u in $\dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$, where $\mathcal{R}^0_k(\mathcal{M}, \mathcal{N})$ denotes for $k \in \{0, \dots, m-1\}$ the set of maps from a manifold $\mathcal{M}$ to a manifold $\mathcal{N}$ that are continuous outside a finite union of k-dimensional submanifolds with boundary of $\mathcal{M}$.

For every $j \in \mathbb{N}$, the mapping u_j is continuous outside an $(m-2)$-dimensional subset $\Sigma_j \subset \mathcal{M}$. Since the manifold $\mathcal{M}$ is simply connected, the set $\mathcal{M} \setminus \Sigma_j$ is also simply connected and there exists $\tilde{u}_j \in C(\mathcal{M} \setminus \Sigma_j, \widetilde{\mathcal{N}})$ such that $\pi \circ \tilde{u}_j = u_j \restriction_{\mathcal{M} \setminus \Sigma_j}$, where $u_j \restriction_{\mathcal{M} \setminus \Sigma_j}$ is the restriction of u_j to the set $\mathcal{M} \setminus \Sigma_j$. In particular, we have $\tilde{u}_j \in \mathcal{R}^0_{m-2}(\mathcal{M}, \widetilde{\mathcal{N}})$. Since for every convex open set $\Omega \subseteq \mathbb{R}^m$ we have $\mathcal{R}^0_{m-2}(\Omega, \widetilde{\mathcal{N}}) \subseteq Y(\Omega, \widetilde{\mathcal{N}})$ and since $sp > 1$, by the a priori estimate on the lifting (Proposition 2.1), by the diagonal covering (Lemma 2.11) and by Proposition 2.4, we have

$$\sup_{j \in \mathbb{N}} \iint_{\mathcal{M} \times \mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}_j(x), \tilde{u}_j(y))^p \wedge d_{\widetilde{\mathcal{N}}}(\tilde{u}_j(x), \tilde{u}_j(y))}{d_{\mathcal{M}}(y, x)^{m+sp}} \, \mathrm{d}y \, \mathrm{d}x < \infty. \tag{45}$$

By (45), there exists thus $\lambda \in (0, \infty)$ such that for every $j \in \mathbb{N}$, there exists $x_j \in \mathcal{M}$ for which if we set

$$A_j := \{x \in \mathcal{M} \mid d_{\widetilde{\mathcal{N}}}(\tilde{u}_j(x), \tilde{u}_j(x_j)) \le \lambda\} \tag{46}$$

we have then

$$|A_j| \ge \tfrac{2}{3} |\mathcal{M}|. \tag{47}$$

Since the manifold $\mathcal{N}$ is compact and since the covering π is normal, there exists an open bounded set $\widetilde{W} \subseteq \widetilde{\mathcal{N}}$ such that $\pi(\widetilde{W}) = \mathcal{N}$ and

$$\widetilde{\mathcal{N}} = \bigcup_{\tau \in \operatorname{Aut}(\pi)} \tau^{-1}(\widetilde{W}), \tag{48}$$

in view of (44). By (48), for every $j \in \mathbb{N}$, there exists thus $\tau_j \in \operatorname{Aut}(\pi)$ such that $\tau_j(\tilde{u}_j(x_j)) \in \widetilde{W}$. Without loss of generality we assume that for each $j \in \mathbb{N}$ we have $\tau_j = \mathrm{id}_{\widetilde{\mathcal{N}}}$, so that $\tilde{u}_j(x_j) \in \widetilde{W}$.

We deduce from (46) that for every $i, j \in \mathbb{N}$ and $x \in A_{i,j} := A_i \cap A_j$,

$$\begin{aligned} d_{\widetilde{\mathcal{N}}}(\tilde{u}_j(x), \tilde{u}_i(x)) &\leq d_{\widetilde{\mathcal{N}}}(\tilde{u}_j(x), \tilde{u}_j(x_j)) + d_{\widetilde{\mathcal{N}}}(\tilde{u}_j(x_j), \tilde{u}_i(x_i)) + d_{\widetilde{\mathcal{N}}}(\tilde{u}_i(x_i), \tilde{u}_i(x)) \\ &\leq 2\lambda + \operatorname{diam}(\widetilde{W}); \end{aligned} \tag{49}$$

by (47), we have

$$|A_{i,j}| = |A_i \cap A_j| = |A_i| + |A_j| - |A_i \cup A_j| \geq \tfrac{2}{3}|\mathcal{M}| + \tfrac{2}{3}|\mathcal{M}| - |\mathcal{M}| = \tfrac{1}{3}|\mathcal{M}|. \tag{50}$$

Therefore, we have by (49) and (50),

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{1}{1 + d_{\widetilde{\mathcal{N}}}(\tilde{u}_j, \tilde{u}_i)} \geq \frac{|A_{i,j}|}{1 + 2\lambda + \operatorname{diam}(\widetilde{W})} \geq \frac{|\mathcal{M}|}{3(1 + 2\lambda + \operatorname{diam}(\widetilde{W}))},$$

and it follows from Propositions 2.1 and 2.8 and from the completeness of the manifold $\widetilde{\mathcal{N}}$ that, up to a subsequence, the sequence $(\tilde{u}_j)_{j\in\mathbb{N}}$ converges almost everywhere on $\mathcal{M}$ to some mapping $\tilde{u}\colon \mathcal{M} \to \widetilde{\mathcal{N}}$; we also have $\pi \circ \tilde{u} = \lim_{j\to\infty} \pi \circ \tilde{u}_j = \lim_{j\to\infty} u_j = u$ almost everywhere; by Fatou's lemma, by the a priori estimate on the lifting (Proposition 2.1) and by the diagonal covering (Lemma 2.11) we have

$$\begin{aligned} \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p \wedge 1}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x &\leq \liminf_{j\to\infty} \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}_j(y), \tilde{u}_j(x))^p \wedge 1}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x \\ &\leq C_1 \liminf_{j\to\infty} \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u_j(y), u_j(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x \\ &= C_1 \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x, \end{aligned}$$

which proves the statement and estimate (4) when $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ is a normal covering.

If $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ is not a normal covering, we choose $\pi_*\colon \widetilde{\mathcal{N}}_* \to \widetilde{\mathcal{N}}$ to be a universal covering of $\widetilde{\mathcal{N}}$, so that in particular $\pi \circ \pi_*\colon \widetilde{\mathcal{N}}_* \to \mathcal{N}$ is a universal covering of $\mathcal{N}$ and thus also a normal covering. Applying the first part of the proof, we get a mapping $\tilde{u}_* \in \dot{W}^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}}_*)$ such that $\pi \circ \pi_* \circ \tilde{u}_* = u$ on $\mathcal{M}$; setting $\tilde{u} := \pi_* \circ \tilde{u}_*$, we reach the conclusion in the general case. ∎

As a by-product of the proof of Theorem 1.2, we get under the weaker condition $sp > 1$ the existence of a lifting with an estimate for maps that are continuous outside a submanifold of codimension 2.

Theorem 2.12. *Let $\mathcal{M}$ and $\mathcal{N}$ be compact Riemannian manifolds, let $m := \dim \mathcal{M}$, let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be a surjective Riemannian covering map, let $s \in (0, 1)$ and let $p \in (1, \infty)$. If $\mathcal{M}$ is simply connected and if $sp > 1$, then there exists a constant $C \in (0, \infty)$ such that for every map $u \in \mathcal{R}^0_{m-2}(\mathcal{M}, \mathcal{N}) \cap \dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$ there exists a measurable map $\tilde{u} : \mathcal{M} \to \widetilde{\mathcal{N}}$ such that $\pi \circ \tilde{u} = u$ almost everywhere on $\mathcal{M}$ and (4) holds.*

As a consequence of Theorem 2.12 and Proposition 2.8, any map which is the almost everywhere limit of a sequence of maps $(u_j)_{j\in\mathbb{N}}$ in $\mathcal{R}^0_{m-2}(\mathcal{M}, \mathcal{N}) \cap \dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$ that is bounded in $\dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$ has a lifting $\tilde{u} : \mathcal{M} \to \widetilde{\mathcal{N}}$ satisfying

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p \wedge 1}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \le \liminf_{j\to\infty} C \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u_j(y), u_j(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x.$$

When $1 < sp < 2$, the assumption that the domain $\mathcal{M}$ is simply connected in Theorems 1.2 and 2.12 can be replaced by a smallness assumption on the map to be lifted.

Theorem 2.13. *Let $\mathcal{M}$ and $\mathcal{N}$ be compact Riemannian manifolds, let $m := \dim \mathcal{M}$, let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be a surjective Riemannian covering map, let $s \in (0, 1)$ and let $p \in (1, \infty)$. If $sp > 1$, then there exist constants $\varepsilon, C \in (0, \infty)$ such for every map $u \in \dot{W}^{s,p}(\mathcal{M}, \mathcal{N})$ satisfying*

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \le \varepsilon, \tag{51}$$

and also satisfying $u \in \mathcal{R}^0_{m-2}(\mathcal{M}, \mathcal{N})$ when $1 < sp < 2$, there exists a measurable map $\tilde{u} : \mathcal{M} \to \widetilde{\mathcal{N}}$ such that $\pi \circ \tilde{u} = u$ almost everywhere on $\mathcal{M}$ and (4) holds.

When $\pi : \mathbb{R} \to \mathbb{S}^1$ is the universal covering of the circle, Theorem 2.13 is a reformulation of a result of Brezis and Mironescu [11, Thm. 14.5 & §14.6.2].

Proof of Theorem 2.13. We follow the proof of Theorem 1.2, noting that $\pi_1(\mathcal{M})$ has finitely many generators, so that if $\varepsilon \in (0, \infty)$ is taken small enough, the smallness assumption (51) implies that u_j has a lifting on a finite set of loops generating $\pi_1(\mathcal{N})$ and not intersecting the singular set of u_j, and hence u_j has a lifting outside its singular set. ∎

2.5. Uniqueness of the lifting

The lifting given by Theorem 1.2 turns out to be essentially unique, as it is well established for the lifting in fractional Sobolev spaces [5, App. B], [2, Lem. A.4].

The main analytical tool is the following result of Bourgain, Brezis and Mironescu [7, App. B], [6, Cor. 6.4] (see also [11, 12, 31]).

Lemma 2.14. *Let $\mathcal{M}$ be a connected Riemannian manifold with $m := \dim \mathcal{M}$. If the set $A \subseteq \mathcal{M}$ is measurable and if*

$$\int_A \int_{\mathcal{M}\setminus A} \frac{1}{\operatorname{dist}_{\mathcal{M}}(y, x)^{m+1}} \,\mathrm{d}y \,\mathrm{d}x < \infty,$$

then either $|A| = 0$ or $|\mathcal{M} \setminus A| = 0$.

Proof of Proposition 1.3. We define the set

$$A := \{x \in \mathcal{M} \mid \tilde{u}_1(x) = \tilde{u}_0(x)\}. \tag{52}$$

We observe that if $x \in A$ and $y \in \mathcal{M} \setminus A$, then by Lemma 2.3 and by the triangle inequality,

$$\begin{aligned} \operatorname{inj}(\mathcal{N}) &\le d_{\tilde{\mathcal{N}}}(\tilde{u}_1(y), \tilde{u}_0(y)) \\ &\le d_{\tilde{\mathcal{N}}}(\tilde{u}_1(y), \tilde{u}_1(x)) + d_{\tilde{\mathcal{N}}}(\tilde{u}_1(x), \tilde{u}_0(x)) + d_{\tilde{\mathcal{N}}}(\tilde{u}_0(x), \tilde{u}_0(y)) \\ &= d_{\tilde{\mathcal{N}}}(\tilde{u}_1(y), \tilde{u}_1(x)) + d_{\tilde{\mathcal{N}}}(\tilde{u}_0(x), \tilde{u}_0(y)), \end{aligned}$$

and thus either $d_{\tilde{\mathcal{N}}}(\tilde{u}_0(x), \tilde{u}_0(y)) \ge \operatorname{inj}(\mathcal{N})/2$ or $d_{\tilde{\mathcal{N}}}(\tilde{u}_1(x), \tilde{u}_1(y)) \ge \operatorname{inj}(\mathcal{N})/2$, and thus

$$\begin{aligned} &\int_A \int_{\mathcal{M}\setminus A} \frac{1}{d_{\mathcal{M}}(x,y)^{m+1}} \,\mathrm{d}y\,\mathrm{d}x \\ &\qquad \le \sum_{j\in\{0,1\}} \iint_{\substack{x,y\in\mathcal{M} \\ d_{\tilde{\mathcal{N}}}(\tilde{u}_j(x),\tilde{u}_j(y))\ge \operatorname{inj}(\mathcal{N})/2}} \frac{1}{d_{\mathcal{M}}(x,y)^{m+1}} \,\mathrm{d}y\,\mathrm{d}x < \infty. \end{aligned}$$

It follows then from Lemma 2.14 that either $|A| = 0$ or $|\mathcal{M} \setminus A| = 0$ and the conclusion follows from the definition of A in (52). ∎

The space $X(\mathcal{M}, \tilde{\mathcal{N}})$ contains all functions such that the left-hand side of (4) in Theorem 1.2 is finite.

Proposition 2.15. *Let $\mathcal{M}$ be a compact Riemannian manifold, let $\pi\colon \tilde{\mathcal{N}} \to \mathcal{N}$ be a Riemannian covering, let $s \in (0,1)$, let $p \in (1,\infty)$ and let $q \in [0,\infty)$. If $sp > 1$ and if the mapping $u\colon \mathcal{M} \to \tilde{\mathcal{N}}$ is measurable and satisfies*

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\tilde{\mathcal{N}}}(\tilde{u}(x), \tilde{u}(y))^p \wedge d_{\tilde{\mathcal{N}}}(\tilde{u}(x), \tilde{u}(y))^q}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x < \infty,$$

with $m := \dim \mathcal{M}$, then $\tilde{u} \in X(\mathcal{M}, \tilde{\mathcal{N}})$.

Proof. We have

$$\begin{aligned} &\iint_{\substack{x,y\in\mathcal{M} \\ d_{\tilde{\mathcal{N}}}(\tilde{u}(x),\tilde{u}(y))\ge \operatorname{inj}(\mathcal{N})/2}} \frac{(\operatorname{inj}(\mathcal{N})/2)^p \wedge (\operatorname{inj}(\mathcal{N})/2)^q}{d_{\mathcal{M}}(x,y)^{m+1}} \,\mathrm{d}y\,\mathrm{d}x \\ &\qquad \le \operatorname{diam}(\mathcal{M})^{sp-1} \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\tilde{\mathcal{N}}}(\tilde{u}(x), \tilde{u}(y))^p \wedge d_{\tilde{\mathcal{N}}}(\tilde{u}(x), \tilde{u}(y))^q}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x < \infty. \end{aligned}$$ ∎

Classical fractional uniqueness results for $u_0, u_1 \in \dot{W}^{s,p}(\mathcal{M}, \tilde{\mathcal{N}})$ with $0 < s < 1$ [2,6] can be recovered from Propositions 1.3 and 2.15.

The uniqueness property of the lifting also allows one to write any lifting in terms of a fixed lifting over a normal covering.

Proposition 2.16. *Let $\mathcal{M}$ be a Riemannian manifold, let $\pi_\sharp\colon \widetilde{\mathcal{N}}_\sharp \to \widetilde{\mathcal{N}}_\flat$ and $\pi_\flat\colon \widetilde{\mathcal{N}}_\flat \to \mathcal{N}$ be Riemannian coverings and let $\tilde{u}_\sharp \in X(\mathcal{M}, \widetilde{\mathcal{N}}_\sharp)$ and $\tilde{u}_\flat \in X(\mathcal{M}, \widetilde{\mathcal{N}}_\flat)$. If $\mathcal{M}$ is connected, if the covering $\pi_\sharp$ is surjective, if the covering $\pi_\flat \circ \pi_\sharp$ is normal and if $\pi_\flat \circ \tilde{u}_\flat = \pi_\flat \circ \pi_\sharp \circ \tilde{u}_\sharp$ almost everywhere on $\mathcal{M}$, then there exists $\tau \in \operatorname{Aut}(\pi_\flat \circ \pi_\sharp)$ such that $\tilde{u}_\flat = \pi_\sharp \circ \tau \circ \tilde{u}_\sharp$ almost everywhere on $\mathcal{M}$.*

Proof. Since the covering $\pi_\sharp$ is surjective, for every $x \in \mathcal{M}$, there exists $\tilde{y}_\sharp \in \widetilde{\mathcal{N}}_\sharp$ such that $\pi_\sharp(\tilde{y}_\sharp) = \tilde{u}_\flat(x)$. For almost every $x \in \mathcal{M}$, since $\pi_\flat(\pi_\sharp(\tilde{y}_\sharp)) = \pi_\flat(\tilde{u}_\flat(x)) = \pi_\flat(\pi_\sharp(\tilde{u}_\sharp(x)))$ and since the covering $\pi_\flat \circ \pi_\sharp$ is normal, there exists $\tau \in \operatorname{Aut}(\pi_\flat \circ \pi_\sharp)$ such that $\tilde{y}_\sharp = \tau(\tilde{u}_\sharp(x))$ and thus $\tilde{u}_\flat(x) = \pi_\sharp(\tau(\tilde{u}_\sharp(x)))$. Hence we have

$$\mathcal{M} = \bigcup_{\tau \in \operatorname{Aut}(\pi_\flat \circ \pi_\sharp)} \bigl\{x \in \mathcal{M} \bigm| \tilde{u}_\flat(x) = \pi_\sharp(\tau(\tilde{u}_\sharp(x)))\bigr\} \cup E,$$

where $E \subseteq \mathcal{M}$ satisfies $|E| = 0$. Since the set $\operatorname{Aut}(\pi_\flat \circ \pi_\sharp)$ is countable, there exists $\tau \in \operatorname{Aut}(\pi_\flat \circ \pi_\sharp)$ such that $\tilde{u}_\flat = \pi_\sharp \circ \tau \circ \tilde{u}_\sharp$ on a set of positive measure of $\mathcal{M}$ and the identity then holds outside a null set by the uniqueness of lifting (Proposition 1.3) since $\mathcal{M}$ is connected. ■

As a consequence of Proposition 2.16, we get that a lifting in $X(\mathcal{M}, \widetilde{\mathcal{N}})$ of a continuous map is necessarily essentially continuous.

Proposition 2.17. *Let $\mathcal{M}$ be a Riemannian manifold and let $\pi\colon \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$ be a Riemannian covering. If $\tilde{u} \in X(\mathcal{M}, \widetilde{\mathcal{N}})$ and if $u = \pi \circ \tilde{u}$ is continuous, then there exists $\tilde{v} \in C(\mathcal{M}, \widetilde{\mathcal{N}})$ such that $\tilde{v} = \tilde{u}$ almost everywhere on $\mathcal{M}$.*

Proof. We first assume that the manifold $\mathcal{M}$ is simply connected. We then apply Proposition 2.16 with $\pi_\flat = \pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$, $\pi_\sharp\colon \widetilde{\mathcal{N}}_* \to \widetilde{\mathcal{N}}$ a universal covering and $\tilde{v} \in C(\mathcal{M}, \widetilde{\mathcal{N}}_*)$ such that $\pi \circ \tilde{v} = \pi \circ \tilde{u}$. The conclusion then follows from Proposition 2.16.

In the general case, we cover the manifold $\mathcal{M}$ by simply connected open sets $U_j \subseteq \mathcal{M}$, with $j \in J$. By the first part of the proof, for every $j \in J$, there exists a mapping $\tilde{v}_j \in C(U_j, \widetilde{\mathcal{N}})$ such that $\tilde{u} = \tilde{v}_j$ almost everywhere in U_j. For every $j, \ell \in J$, it follows in view of the continuity of the mappings $\tilde{v}_j$ and $\tilde{v}_\ell$ that $\tilde{v}_j = \tilde{v}_\ell$ everywhere in $U_j \cap U_\ell$. Therefore the map $\tilde{v}$ can be defined in such a way that for every $j \in J$ its restriction $\tilde{v} \restriction_{U_j}$ to the set U_j satisfies $\tilde{v} \restriction_{U_j} = \tilde{v}_j$ and that $\tilde{v}$ is continuous on $\mathcal{M}$. ■

2.6. A priori estimate on the lifting

Theorem 1.4 will be proved as a consequence of Proposition 2.18, once one notices that liftings in $X(\Omega, \widetilde{\mathcal{N}})$ of maps in $\dot{W}^{s,p}(\Omega, \mathcal{N})$ with $sp > 1$ turn out to be in $Y(\Omega, \widetilde{\mathcal{N}})$.

Proposition 2.18. *Let $m \in \mathbb{N} \setminus \{0\}$, let $\Omega \subseteq \mathbb{R}^m$ be open and convex, let $\mathcal{N}$ be a compact Riemannian manifold, let $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ be a Riemannian covering map, let $s \in (0, 1)$ and let $p \in (1, \infty)$. If $sp > 1$, if $\tilde{u} \in X(\Omega, \widetilde{\mathcal{N}})$ and if $u := \pi \circ \tilde{u} \in \dot{W}^{s,p}(\Omega, \mathcal{N})$, then $\tilde{u} \in Y(\Omega, \widetilde{\mathcal{N}})$.*

In order to prove Proposition 2.18, we will use the following consequence of Fubini's theorem, which implements the rotation method on the space $X(\Omega, \widetilde{\mathcal{N}})$.

Lemma 2.19. *For every $m \in \mathbb{N} \setminus \{0\}$, there exists a constant $C \in (0, \infty)$ such that for every convex open set $\Omega \subset \mathbb{R}^m$, every metric space $\mathcal{E}$, every $\delta \in (0, \infty)$ and every measurable function $f : \Omega \to \mathcal{E}$, we have*

$$\iint_{\substack{x,y\in\Omega \\ d_{\mathcal{E}}(f(y),f(x))\geq\delta}} \frac{1}{|y-x|^{m+1}} \,\mathrm{d}y\,\mathrm{d}x = C \int_{\mathbb{S}^{m-1}} \int_{w^\perp} \iint_{\substack{x,y\in\Omega\cap(z+\mathbb{R}w) \\ d_{\mathcal{E}}(f(y),f(x))\geq\delta}} \frac{1}{|y-x|^2} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}w.$$

Proof of Proposition 2.18. Since $sp > 1$, by Fubini's theorem and the fractional Morrey–Sobolev embedding, for every straight line $L \subseteq \mathbb{R}^m$, there exists a mapping $u_L \in C(\Omega \cap L, \widetilde{\mathcal{N}})$ such that $u \restriction_{\Omega\cap L} = u_L = \pi \circ \tilde{u} \restriction_{\Omega\cap L}$ almost everywhere in $\Omega \cap L$. Similarly, by Lemma 2.19, we have $\tilde{u} \restriction_{\Omega\cap L} \in X(\Omega \cap L, \mathcal{N})$. By Proposition 2.17, there exists a mapping $\tilde{u}_L \in C(\Omega \cap L, \widetilde{\mathcal{N}})$ such that $\tilde{u} \restriction_{\Omega\cap L} = \tilde{u}_L$ almost everywhere on $\Omega \cap L$. It follows thus by definition (12) that $\tilde{u} \in Y(\Omega, \widetilde{\mathcal{N}})$. ■

Proof of Theorem 1.4. By Proposition 2.18, the a priori estimate Proposition 2.1 holds on any local chart. We reach the conclusion by the covering of Lemma 2.11. ■

3. Relationship to linear Sobolev spaces

3.1. Characterization as a sum of Sobolev spaces

Our proof of Theorem 1.6 that characterizes the space of liftings appearing in Theorem 1.2 and Proposition 1.5 will use the following density result.

Proposition 3.1. *Let $m \in \mathbb{N} \setminus \{0\}$, let $s \in (0, 1)$ and let $p \in [1, \infty)$. If $U \subseteq \mathbb{R}^m$ is open and if $f : U \to \mathbb{R}$ is a measurable function satisfying*

$$\iint_{U\times U} \frac{|f(y)-f(x)|^p \wedge |f(y)-f(x)|}{|y-x|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x < \infty, \tag{53}$$

then for every set $\Omega \subseteq U$ such that $\operatorname{dist}(\Omega, \mathbb{R}^m \setminus U) > 0$, there exists a sequence $(f_j)_{j\in\mathbb{N}}$ in $C^\infty(\overline{\Omega}, \mathbb{R})$ such that $f_j \to f$ almost everywhere in Ω as $j \to \infty$ and

$$\sup_{j\in\mathbb{N}} \iint_{\Omega\times\Omega} \frac{|f_j(y)-f_j(x)|^p \wedge |f_j(y)-f_j(x)|}{|y-x|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x < \infty. \tag{54}$$

Proof. We define the function $\Phi : \mathbb{R} \to \mathbb{R}$ for each $t \in \mathbb{R}$ by

$$\Phi(t) := \begin{cases} |t|^p & \text{if } |t| \leq 1, \\ 1 + p(|t|-1) & \text{if } |t| \geq 1. \end{cases} \tag{55}$$

We observe that the function Φ is convex and that it satisfies for every $t \in \mathbb{R}$,

$$|t|^p \wedge |t| \le \Phi(t) \le |t|^p \wedge (p|t|) \le p(|t|^p \wedge |t|). \tag{56}$$

We fix a function $\eta \in C_c^\infty(\mathbb{R}^m, \mathbb{R})$ such that $\eta \ge 0$ and $\int_{\mathbb{R}^m} \eta = 1$. Since condition (53) implies the local integrability of the function f, there exists a sequence $(\delta_j)_{j\in\mathbb{N}}$ in $(0,\infty)$ such that the function $f_j : \bar{\Omega} \to \mathbb{R}$ defined for each $x \in \Omega$ by

$$f_j(x) := \int_{\mathbb{R}^m} \eta(z) f(x - \delta_j z)\,\mathrm{d}z$$

is well defined and $f_j \to f$ almost everywhere in Ω as $j \to \infty$. Moreover, since the function Φ is convex, we have for every $j \in \mathbb{N}$,

$$\iint_{\Omega\times\Omega} \frac{\Phi(f_j(y) - f_j(x))}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x \le \iint_{U\times U} \frac{\Phi(f(y) - f(x))}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x, \tag{57}$$

and (54) follows from (56) and (57). ∎

We now prove the characterization of the sum $W^{s,p}(\mathcal{M}, \mathbb{R}) + W^{1,sp}(\mathcal{M}, \mathbb{R})$.

Proof of Theorem 1.6. In order to prove the inclusion $\subseteq$ in (5), we first assume that the set $\Omega \subseteq \mathbb{R}^m$ is bounded and open with a smooth boundary $\partial\Omega$, that $\Omega \subseteq U$, with $\operatorname{dist}(\Omega, \mathbb{R}^m \setminus U) > 0$ for some open set $U \supseteq \bar{\Omega}$, and that, in view of Proposition 1.5,

$$\begin{aligned}\iint_{U\times U} &\frac{|f(x) - f(y)|^p \wedge |f(x) - f(x)|}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x \\ &\le \iint_{U\times U} \frac{|f(x) - f(y)|^p \wedge |f(x) - f(x)|^q}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x < \infty.\end{aligned} \tag{58}$$

By Proposition 3.1 and by (58), there exists a sequence $(f_j)_{j\in\mathbb{N}}$ in $C^\infty(\bar{\Omega}, \mathbb{R})$ such that $f_j \to f$ almost everywhere in Ω and

$$\begin{aligned}\sup_{j\in\mathbb{N}} \iint_{\Omega\times\Omega} &\frac{|f_j(y) - f_j(x)|^p \wedge 1}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x \\ &\le C_1 \iint_{U\times U} \frac{|f(y) - f(x)|^p \wedge |f(y) - f(x)|}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x < \infty.\end{aligned} \tag{59}$$

For every $j \in \mathbb{N}$, setting $u_j := e^{if_j}$, we have

$$\iint_{\Omega\times\Omega} \frac{|u_j(y) - u_j(x)|^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x \le C_2 \iint_{\Omega\times\Omega} \frac{|f_j(x) - f_j(y)|^p \wedge 1}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x < \infty. \tag{60}$$

Since $sp > 1$, by the lifting in the sum of Sobolev spaces [11, Thm. 8.8], [22, Thm. 2] (see also [6, 19, 20, 27]), we can write $f_j = g_j + h_j$ with the functions $g_j \in \dot{W}^{s,p}(\Omega, \mathbb{R})$ and

$h_j \in \dot{W}^{1,sp}(\Omega, \mathbb{R})$ satisfying the estimates

$$\iint_{\Omega\times\Omega} \frac{|g_j(y) - g_j(x)|^p}{|y-x|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x \le C_3 \iint_{\Omega\times\Omega} \frac{|u_j(x) - u_j(y)|^p}{|y-x|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x, \tag{61}$$

$$\int_\Omega |Dh_j|^{sp} \le C_4 \iint_{\Omega\times\Omega} \frac{|u_j(x) - u_j(y)|^p}{|y-x|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x \tag{62}$$

and the conditions

$$\int_\Omega g_j = \int_\Omega h_j = \frac{1}{2}\int_\Omega f_j. \tag{63}$$

Up to a subsequence, we can assume that $g_j \to g$ and $h_j \to h$ almost everywhere in Ω as $j \to \infty$, with the functions $g \in \dot{W}^{s,p}(\Omega, \mathbb{R})$ and $h \in \dot{W}^{1,sp}(\Omega, \mathbb{R})$ satisfying in view of (59), (60), (61) and (62),

$$\iint_{\Omega\times\Omega} \frac{|g(y) - g(x)|^p}{|y-x|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x \le C_5 \iint_{U\times U} \frac{|f(x) - f(y)|^p \wedge |f(x) - f(x)|}{|y-x|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x, \tag{64}$$

$$\int_\Omega |Dh|^{sp} \le C_6 \iint_{U\times U} \frac{|f(x) - f(y)|^p \wedge |f(x) - f(x)|}{|y-x|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x, \tag{65}$$

and in view of (63),

$$\int_\Omega g = \int_\Omega h = \frac{1}{2}\int_\Omega f. \tag{66}$$

In the general case we follow Rodiac and Van Schaftingen [32, proof of Prop. 4.1]. Since $\mathcal{M}$ is a compact manifold with boundary, there exist $N \in \mathbb{N}$ and, for $k \in \{1, \dots, N\}$, a diffeomorphism $\psi_k : U_k \to \mathbb{R}^m$ such that either $\psi_k(U_k) = \mathbb{B}^m \subset \mathbb{R}^m$ or $\psi_k(U_k) = \mathbb{B}^m \cap \mathbb{R}^{m-1} \times [0, \infty)$ and such that $\mathcal{M} = \bigcup_{k=1}^N U_k$. We take a partition of unity $(\varphi_k)_{1\le k\le N}$ associated to the sets U_k, that is, for every $k \in \{1, \dots, N\}$, $\varphi_k \in C^1(\mathcal{M}, \mathbb{R})$ and $\varphi_k = 0$ in $\mathcal{M} \setminus U_k$, and $\sum_{i=1}^N \varphi_k = 1$ on $\mathcal{M}$. Given a measurable function $f : \mathcal{M} \to \mathbb{R}$, for each $k \in \{1, \dots, N\}$, we define the function $f_k := f \circ \psi_k^{-1} : \psi_k(U_k) \to \mathbb{R}$ to which we apply the first part of the proof which yields functions $g_k \in \dot{W}^{s,p}(\psi_k(U_k), \mathbb{R})$ and $h_k \in \dot{W}^{1,sp}(\psi_k(U_k), \mathbb{R})$ satisfying (64), (65) and (66) with $\Omega = \psi_k(U_k)$. Defining the functions

$$g_* := \sum_{k=1}^N \varphi_k \Big(g_k \circ \psi_k - \fint_{U_k} g_k \circ \psi_k \Big),$$

$$h_* := \sum_{k=1}^N \varphi_k \Big(h_k \circ \psi_k - \fint_{U_k} h_k \circ \psi_k \Big),$$

and the low-frequency component

$$f_0 := \sum_{k=1}^N \varphi_k \fint_{U_k} f,$$

we have $f = f_0 + g_* + h_*$ on $\mathcal{M}$. Moreover, since

$$f_0 := \sum_{k=1}^{N} \varphi_k \Bigl(\fint_{U_k} f - \fint_{\mathcal{M}} f \Bigr) + \fint_{\mathcal{M}} f,$$

where the last term is constant, we have

$$\|Df_0\|_{L^\infty(\mathcal{M})} \le C_7 \iint_{\mathcal{M}\times\mathcal{M}} |f(y) - f(x)| \,\mathrm{d}y \,\mathrm{d}x$$

and

$$\Phi(\|Df_0\|_{L^\infty(\mathcal{M})}) \le C_8 \iint_{\mathcal{M}\times\mathcal{M}} \Phi(|f(y) - f(x)|) \,\mathrm{d}y \,\mathrm{d}x,$$

with the convex function Φ defined as in (55). Since $1 < sp < p$, by (56), we have

$$\begin{aligned}\|Df_0\|^p_{L^\infty(\mathcal{M})} \wedge \|Df_0\|^{sp}_{L^\infty(\mathcal{M})} &\le \|Df_0\|^p_{L^\infty(\mathcal{M})} \wedge \|Df_0\|_{L^\infty(\mathcal{M})}\\ &\le C_9 \iint_{\mathcal{M}\times\mathcal{M}} \frac{|f(y) - f(x)|^p \wedge |f(y) - f(x)|}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x,\end{aligned}$$

so that

$$\begin{aligned}&\iint_{\mathcal{M}\times\mathcal{M}} \frac{|f_0(x) - f_0(y)|^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \wedge \int_{\mathcal{M}} |Df_0|^{sp}\\ &\qquad\le C_{10} \iint_{\mathcal{M}\times\mathcal{M}} \frac{|f(x) - f(y)|^p \wedge |f(x) - f(x)|}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x.\end{aligned}$$

By either taking $g := g_*$ and $h := h_* + f_0$ or $g := g_* + f_0$ and $h := h_*$, we finally get, in view of Proposition 1.5,

$$\begin{aligned}&\iint_{\mathcal{M}\times\mathcal{M}} \frac{|g(y) - g(x)|^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x + \int_{\mathcal{M}} |Dh|^{sp}\\ &\qquad\le C_{11} \iint_{\mathcal{M}\times\mathcal{M}} \frac{|f(y) - f(x)|^p \wedge |f(y) - f(x)|^q}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x,\end{aligned}$$

which gives the first estimate and inclusion.

We now prove the reverse inclusion $\supseteq$ in (5). If $f = g + h$ with $g \in \dot{W}^{s,p}(\mathcal{M},\mathbb{R})$ and $h \in \dot{W}^{1,sp}(\mathcal{M},\mathbb{R})$, there exist sequences of smooth maps $(g_j)_{j\in\mathbb{N}}$ and $(h_j)_{j\in\mathbb{N}}$ in $C^\infty(\mathcal{M},\mathbb{R})$, such that, as $j \to \infty$, $g_j \to g$ in $\dot{W}^{s,p}(\mathcal{M},\mathbb{R})$ and $h_j \to h$ in $\dot{W}^{1,sp}(\mathcal{M},\mathbb{R})$. For every $j \in \mathbb{N}$, defining $f_j := g_j + h_j$ and $u_j := e^{if_j}$, we have by the fractional Gagliardo–Nirenberg interpolation inequality (see for example [8, Cor. 3.2], [33, Lem. 2.1], [10]), since $sp > 1$ and $|e^{ih_j}| \le 1$,

$$\begin{aligned}&\iint_{\mathcal{M}\times\mathcal{M}} \frac{|u_j(y) - u_j(x)|^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x\\ &\quad\le \iint_{\mathcal{M}\times\mathcal{M}} \frac{|e^{ig_j(y)} - e^{ig_j(x)}|^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x + \iint_{\mathcal{M}\times\mathcal{M}} \frac{|e^{ih_j(y)} - e^{ih_j(x)}|^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x\\ &\quad\le C_{12} \Bigl(\iint_{\mathcal{M}\times\mathcal{M}} \frac{|g_j(y) - g_j(x)|^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x + \int_{\mathcal{M}} |Dh_j|^{sp} \Bigr). \qquad (67)\end{aligned}$$

By Theorem 1.4, we have for every $j \in \mathbb{N}$,

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{|f_j(y) - f_j(x)|^p \wedge 1}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x \le \iint_{\mathcal{M}\times\mathcal{M}} \frac{|u_j(y) - u_j(x)|^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x. \tag{68}$$

Letting $j \to \infty$ in (67) and (68) and applying Proposition 1.5, we get

$$\begin{aligned}\iint_{\mathcal{M}\times\mathcal{M}} &\frac{|f(y) - f(x)|^p \wedge |f(y) - f(x)|^q}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x\\ &\le C_{13}\Big(\iint_{\mathcal{M}\times\mathcal{M}} \frac{|g(y) - g(x)|^p}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x + \int_{\mathcal{M}} |Dh|^{sp}\Big),\end{aligned}$$

which proves the announced reverse inclusion and estimate. ∎

3.2. About the critical lower exponent

If the function $f\colon \mathbb{B}^m \to \mathbb{R}$ is measurable and if $q \in [1,\infty)$, then it is known that

$$\iint_{\mathbb{B}^m\times\mathbb{B}^m} \frac{|f(y) - f(x)|^q}{|y-x|^{m+q}} \,\mathrm{d}y\,\mathrm{d}x = \infty$$

unless the function f is constant [7, Prop. 2]. Although the integral restricted to a region of large oscillation

$$\iint_{\substack{(x,y)\in\mathbb{B}^m\times\mathbb{B}^m\\ |f(y)-f(x)|\ge 1}} \frac{|f(y) - f(x)|^q}{|y-x|^{m+q}} \,\mathrm{d}y\,\mathrm{d}x = \infty \tag{69}$$

might be finite for a function of small oscillation, there are still Sobolev functions for which the large oscillation part of the integral (69) blows up.

Proposition 3.2. *Let $m \in \mathbb{N} \setminus \{0,1\}$. If $1 \le q < m$, then there is a function $f \in \dot{W}^{1,q}_0(\mathbb{B}^m, \mathbb{R})$ such that*

$$\iint_{\substack{(x,y)\in\mathbb{B}^m\times\mathbb{B}^m\\ |f(y)-f(x)|\ge 1}} \frac{|f(y) - f(x)|^q}{|y-x|^{m+q}} \,\mathrm{d}y\,\mathrm{d}x = \infty.$$

As a consequence of Proposition 3.2, if $1 \le sp < m$, there exists some function $f \in \dot{W}^{1,sp}_0(\mathbb{B}^m_1, \mathbb{R})$ such that

$$\iint_{\substack{(x,y)\in\mathcal{M}\times\mathcal{M}\\ |f(y)-f(x)|\ge 1}} \frac{|f(y) - f(x)|^p \wedge |f(y) - f(x)|^{sp}}{d_{\mathcal{M}}(y,x)^{m+sp}} \,\mathrm{d}y\,\mathrm{d}x = \infty$$

and thus the noninclusion (8) holds.

Proof of Proposition 3.2. We choose a function $\psi \in C^\infty_c(\mathbb{R}^m, \mathbb{R})$ such that $\psi(x) = x_1$ when $x \in B_{1/2}$ and $\operatorname{supp}\psi \subset B_1$. For every $\lambda \in (2,\infty)$, we have

$$\begin{aligned}\iint_{\substack{(x,y)\in\mathbb{B}^m\times\mathbb{B}^m\\ |\lambda\psi(y)-\lambda\psi(x)|\ge 1}} \frac{|\lambda\psi(y) - \lambda\psi(x)|^q}{|y-x|^{m+q}} \,\mathrm{d}y\,\mathrm{d}x &\ge \lambda^q \iint_{\substack{(x,y)\in B_{1/2}\times B_{1/2}\\ \lambda|y_1-x_1|\ge 1}} \frac{1}{|y-x|^m} \,\mathrm{d}y\,\mathrm{d}x\\ &\ge C_1 \lambda^q \ln(\tfrac{\lambda}{2} - 1),\end{aligned} \tag{70}$$

for some constant $C_1 \in (0, \infty)$. We now define for each $j \in \mathbb{N}$ the numbers

$$\lambda_j := \lambda_0 2^{j^2} \quad \text{and} \quad \rho_j := (\lambda_j^q \ln(\tfrac{\lambda_j}{2} - 1))^{-1/(m-q)}, \tag{71}$$

with $\lambda_0 \in (2, \infty)$ large enough so that there exists a sequence of points $(a_j)_{j \in \mathbb{N}}$ for which the closed balls $\bar{B}_{\rho_j}(a_j)$ are pairwise disjoint and all contained in $\mathbb{B}^m$ (this is possible since $q < m$). We define the function $f : \mathbb{B}^m \to \mathbb{R}$ for every $x \in \mathbb{B}^m$ by

$$f(x) := \begin{cases} \lambda_j \psi\Big(\dfrac{x - a_j}{\rho_j}\Big) & \text{if } x \in B_{\rho_j}(a_j), \\ 0 & \text{otherwise.} \end{cases}$$

By the disjointness of the balls, by scaling and by (71), we have

$$\begin{aligned} \int_{\mathbb{B}^m} |Df|^q = \sum_{j \in \mathbb{N}} \int_{B_{\rho_j}(a_j)} |Df|^q &= \sum_{j \in \mathbb{N}} \lambda_j^q \rho_j^{m-q} \int_{\mathbb{B}^m} |D\psi|^q \\ &= \sum_{j \in \mathbb{N}} \frac{1}{\ln(\lambda_0 2^{j^2 - 1} - 1)} \int_{\mathbb{B}^m} |D\psi|^q < \infty, \end{aligned}$$

so that $f \in W_0^{1,q}(\mathbb{B}^m, \mathbb{R})$. On the other hand, by the disjointness of the balls, by scaling, by (70) and by (71), we have

$$\begin{aligned} &\iint_{\substack{(x,y) \in \mathbb{B}^m \times \mathbb{B}^m \\ |f(y) - f(x)| \geq 1}} \frac{|f(y) - f(x)|^q}{|y - x|^{m+q}} \,\mathrm{d}y \,\mathrm{d}x \\ &\quad \geq \sum_{j \in \mathbb{N}} \rho_j^{m-q} \iint_{\substack{(x,y) \in \mathbb{B}^m \times \mathbb{B}^m \\ |\lambda_j \psi(y) - \lambda_j \psi(x)| \geq 1}} \frac{|\lambda_j \psi(y) - \lambda_j \psi(x)|^q}{|y - x|^{m+q}} \,\mathrm{d}y \,\mathrm{d}x \\ &\quad \geq \sum_{j \in \mathbb{N}} C_1 = \infty. \end{aligned}$$

■

4. Estimate of the lifting in subcritical dimension

This section is devoted to the proof of Theorem 1.8. We first observe that by Theorem 1.4, for every $\delta \in (0, \infty)$, the map $\tilde{u} : \mathcal{M} \to \widetilde{\mathcal{N}}$ immediately satisfies the *small-scale estimate*

$$\iint_{\substack{(x,y) \in \mathcal{M} \times \mathcal{M} \\ d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x)) \leq \delta}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \leq C \iint_{\mathcal{M} \times \mathcal{M}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x,$$

so that it will be sufficient to estimate the *large-scale integral*

$$\iint_{\substack{(x,y) \in \mathcal{M} \times \mathcal{M} \\ d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x)) > \delta}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x.$$

We will prove the following counterpart of Proposition 2.1 for *large-scale oscillations*.

Proposition 4.1. *Let $m \in \mathbb{N} \setminus \{0\}$, let $s, s_* \in (0,1)$ and let $p, p_* \in [1, \infty)$. If $sp > 1$, then there exists a constant $C \in (0, \infty)$ such that if $\pi \colon \widetilde{\mathcal{N}} \to \mathcal{N}$ is a Riemannian covering, if $\Omega \subseteq \mathbb{R}^m$ is open and convex, if $\tilde{u} \in Y(\Omega, \widetilde{\mathcal{N}})$, if $u := \pi \circ \tilde{u}$, if $\delta \leq \operatorname{inj}(\mathcal{N})$ and if*

$$\frac{1 - s_*}{m} = \frac{1}{sp} - \frac{1}{p_*}, \tag{72}$$

then

$$\begin{aligned} &\iint_{\substack{(x,y)\in\Omega\times\Omega \\ d_{\widetilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(x))\geq\delta}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^{p_*}}{|y-x|^{m+s_* p_*}} \,\mathrm{d}y \,\mathrm{d}x \\ &\qquad \leq C \left(\frac{1}{\delta^{(1-s)p}} \iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{|y-x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \right)^{\frac{p_*}{sp}}. \end{aligned} \tag{73}$$

We recall that the space $Y(\Omega, \widetilde{\mathcal{N}})$ was defined in (12) as the set of maps whose restriction on almost every segment coincides almost everywhere with a continuous function taking the same value at the extremities.

Remark 4.2. Proposition 4.1 implies a fractional Sobolev embedding: for $s_* \in (0,1)$ and $p, p_* \in (1, \infty)$ such that $1/p_* = 1/p - (1 - s_*)/m$, letting $\pi \colon \mathbb{R} \to \mathbb{S}^1$ be the universal covering of the circle and choosing $\tilde{u} := tf$ for $t > 0$ in (73) with $\delta = 1$, one gets by the fractional Gagliardo–Nirenberg interpolation inequality, since $|e^{itf}| \leq 1$ in Ω,

$$\begin{aligned} &\iint_{\substack{(x,y)\in\Omega\times\Omega \\ |f(y)-f(x)|\geq t}} \frac{|f(y)-f(x)|^{p_*}}{|y-x|^{m+s_* p_*}} \,\mathrm{d}y \,\mathrm{d}x \\ &\qquad \leq \frac{C_2}{t^{p_*}} \left(\iint_{\Omega\times\Omega} \frac{|e^{itf(y)} - e^{itf(x)}|^{p/s_*}}{|y-x|^{m+p}} \,\mathrm{d}y \,\mathrm{d}x \right)^{p_*/p} \\ &\qquad \leq \frac{C_3}{t^{p_*}} \left(\int_\Omega |De^{itf}|^p \right)^{p_*/p} = C_3 \left(\int_\Omega |Df|^p \right)^{p_*/p}; \end{aligned} \tag{74}$$

letting $t \to 0$ in (74), one gets the fractional Sobolev embedding

$$\iint_{\Omega\times\Omega} \frac{|f(y)-f(x)|^{p_*}}{|y-x|^{m+s_* p_*}} \,\mathrm{d}y \,\mathrm{d}x \leq C_3 \left(\int_\Omega |Df|^p \right)^{p_*/p}.$$

4.1. One-dimensional estimates

Our first tool towards the proof of Proposition 4.1 is the following truncated fractional Morrey–Sobolev inequality.

Lemma 4.3. *Let $s \in (0,1)$ and let $p \in [1, \infty)$. If $sp > 1$, then there exists a constant $C \in (0, \infty)$ such that if $I \subseteq \mathbb{R}$ is an interval, if $\mathcal{N}$ is a Riemannian manifold, if the mapping*

$u\colon I \to \mathcal{N}$ is measurable and if $\mu \in [0, \infty)$, then for almost every $x, y \in I$, we have

$$d_{\mathcal{N}}(u(y), u(x)) \le C\biggl(\biggl(\iint_{[x,y]\times[x,y]} \Bigl(\frac{d_{\mathcal{N}}(u(w), u(v))}{|w - v|^s} - \mu\Bigr)_+^p \frac{\mathrm{d}w\,\mathrm{d}v}{|w - v|}\biggr)^{\frac{1}{p}} |y - x|^{s-\frac{1}{p}} + \mu|y - x|^s\biggr). \tag{75}$$

When $\mu = 0$, estimate (75) reduces to the fractional Morrey–Sobolev inequality; when $\mu > 0$, (75) shows that when small values of the difference quotient are removed, one still gets some truncated uniform bound.

The proof will use the following Minkowski inequality for mean oscillations.

Lemma 4.4. *Let $m \in \mathbb{N} \setminus \{0\}$, let $p \in [1, +\infty)$, let $\Omega \subseteq \mathbb{R}^m$ be measurable and let the mapping $u\colon \Omega \to \mathcal{N}$ be measurable. For every $k \in \mathbb{N}$ and measurable sets $A_0, \dots, A_k \subseteq \Omega$ such that for every $j \in \{0, \dots, k\}$, $\mathcal{L}^d(A_j) > 0$, one has*

$$\Bigl(\fint_{A_0}\fint_{A_k} d_{\mathcal{N}}(u(y), u(x))^p \,\mathrm{d}y\,\mathrm{d}x\Bigr)^{\frac{1}{p}} \le \sum_{j=0}^{k-1} \Bigl(\fint_{A_j}\fint_{A_{j+1}} d_{\mathcal{N}}(u(y), u(x))^p \,\mathrm{d}y\,\mathrm{d}x\Bigr)^{\frac{1}{p}}. \tag{76}$$

Proof. We have by the triangle inequality and by Minkowski's inequality,

$$\begin{aligned}
\Bigl(\fint_{A_0}\fint_{A_k} & d_{\mathcal{N}}(u(y), u(x))^p \,\mathrm{d}y\,\mathrm{d}x\Bigr)^{\frac{1}{p}} \\
&= \Bigl(\fint_{A_0}\cdots\fint_{A_k} d_{\mathcal{N}}(u(x_k), u(x_0))^p \,\mathrm{d}x_k \cdots \mathrm{d}x_0\Bigr)^{\frac{1}{p}} \\
&\le \sum_{j=0}^{k-1} \Bigl(\fint_{A_0}\cdots\fint_{A_k} d_{\mathcal{N}}(u(x_{j+1}), u(x_j))^p \,\mathrm{d}x_k \cdots \mathrm{d}x_0\Bigr)^{\frac{1}{p}} \\
&= \sum_{j=0}^{k-1} \Bigl(\fint_{A_j}\fint_{A_{j+1}} d_{\mathcal{N}}(u(y), u(x))^p \,\mathrm{d}y\,\mathrm{d}x\Bigr)^{\frac{1}{p}},
\end{aligned}$$

which proves (76). ■

We now prove the truncated fractional Morrey–Sobolev embedding.

Proof of Lemma 4.3. Since the mapping u is measurable, we can assume without loss of generality that x and y are Lebesgue points of u and that $I = (x, y)$. We define for each $j \in \mathbb{N}$ the set $I_j^x := x + 2^{-j}(I - x) \subseteq I$. Since x is a Lebesgue point of u, we have

$$\lim_{j\to\infty} \fint_{I_j} d_{\mathcal{N}}(u(x), u(z))^p \,\mathrm{d}z = 0, \tag{77}$$

and then by (77), by Lemma 4.4 and by Minkowski's inequality,

$$\begin{aligned}\Big(\fint_I d_{\mathcal{N}}(u(x),u(z))^p\,\mathrm{d}z\Big)^{\frac{1}{p}} &\le \sum_{j=0}^{\infty}\Big(\fint_{I_j^x}\fint_{I_{j+1}^x} d_{\mathcal{N}}(u(w),u(v))^p\,\mathrm{d}w\,\mathrm{d}v\Big)^{\frac{1}{p}}\\ &\le \sum_{j=0}^{\infty}\Big(\fint_{I_j^x}\fint_{I_{j+1}^x} \big(d_{\mathcal{N}}(u(w),u(v))-\mu|w-v|^s\big)_+^p\,\mathrm{d}w\,\mathrm{d}v\Big)^{\frac{1}{p}}\\ &\quad+\sum_{j=0}^{\infty}\Big(\fint_{I_j^x}\fint_{I_{j+1}^x} \mu^p|w-v|^{sp}\,\mathrm{d}w\,\mathrm{d}v\Big)^{\frac{1}{p}}.\end{aligned}\tag{78}$$

For the first term in the right-hand side of (78), we have for every $j\in\mathbb{N}$, since $sp\ge 1$,

$$\begin{aligned}&\fint_{I_j^x}\fint_{I_{j+1}^x}\big(d_{\mathcal{N}}(u(w),u(v))-\mu|w-v|^s\big)_+^p\,\mathrm{d}w\,\mathrm{d}v\\ &\qquad\le \frac{2\operatorname{diam}(I)^{sp-1}}{2^{j(sp-1)}}\iint_{I\times I}\Big(\frac{d_{\mathcal{N}}(u(w),u(v))}{|w-v|^s}-\mu\Big)_+^p\frac{\mathrm{d}w\,\mathrm{d}v}{|w-v|},\end{aligned}\tag{79}$$

while for the second term in the right-hand side of (78), we have for every $j\in\mathbb{N}$,

$$\fint_{I_j^x}\fint_{I_{j+1}^x}|w-v|^{sp}\,\mathrm{d}w\,\mathrm{d}v\le\frac{C_1\operatorname{diam}(I)^{sp}}{2^{jsp}}.\tag{80}$$

Inserting (79) and (80) into (78), we get, since $sp>1$,

$$\begin{aligned}&\Big(\fint_I d_{\mathcal{N}}(u(x),u(z))^p\,\mathrm{d}z\Big)^{\frac{1}{p}}\\ &\le C_2\Big(\operatorname{diam}(I)^{sp-1}\iint_{I\times I}\Big(\frac{d_{\mathcal{N}}(u(w),u(v))}{|w-v|^s}-\mu\Big)_+^p\frac{\mathrm{d}w\,\mathrm{d}v}{|w-v|}+\mu^p\operatorname{diam}(I)^{sp}\Big)^{\frac{1}{p}},\end{aligned}\tag{81}$$

and conclusion (75) follows from (81) and the triangle inequality. ∎

Next we use Lemma 4.3 to estimate the large-scale oscillations of a lifting by a truncated fractional Sobolev seminorm.

Lemma 4.5. *Let $s\in(0,1)$ and let $p\in[1,\infty)$. If $sp>1$, then there exists a constant $C\in(0,\infty)$ such that if $I\subseteq\mathbb{R}$ is an interval, if $\pi\colon\widetilde{\mathcal{N}}\to\mathcal{N}$ is a Riemannian covering, if $\tilde u\in C(I,\widetilde{\mathcal{N}})$ and if $u:=\pi\circ\tilde u$, then for almost every $x,y\in I$, every $\mu\in[0,\infty)$ and every $\delta\in[0,\operatorname{inj}(\mathcal{N}))$, one has*

$$\begin{aligned}&\big(d_{\widetilde{\mathcal{N}}}(\tilde u(y),\tilde u(x))-\delta\big)_+^{sp}\\ &\le\frac{C}{\delta^{p(1-s)}}\Big(\iint_{[x,y]\times[x,y]}\Big(\frac{d_{\mathcal{N}}(u(w),u(v))}{|w-v|^s}-\mu\Big)_+^p\frac{\mathrm{d}w\,\mathrm{d}v}{|w-v|}|y-x|^{sp-1}+\mu^p|y-x|^{sp}\Big).\end{aligned}\tag{82}$$

Lemma 4.5 gives a growth estimate corresponding to what the Morrey–Sobolev embedding would give if one had $\tilde{u} \in \dot{W}^{1,sp}(I, \widetilde{\mathcal{N}})$.

When $\mu = 0$, Lemma 4.5 shows that on large scales the lifting $\tilde{u}$ behaves like a Hölder-continuous mapping of exponent $1 - 1/sp$, which is not as good as the exponent $s - 1/p$ that the fractional Morrey–Sobolev embedding gives on the original function u; this generalizes the results obtained for the universal covering of the circle by Merlet [18, Lem. 8.25] and Mironescu and Molnar [23].

Proof of Lemma 4.5. Let $\ell := \lfloor d_{\widetilde{\mathcal{N}}}(\tilde{u}(x), \tilde{u}(y))/\delta \rfloor$, so that

$$\big(d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x)) - \delta\big)_+ \le \ell\delta. \tag{83}$$

Since the mapping $\tilde{u}$ is continuous, by the intermediate value theorem, there exist points $z_0 = x \le z_1 \le z_2 \le \dots \le z_\ell \le y$ such that for every $i \in \{1, \dots, \ell\}$, one has $d_{\widetilde{\mathcal{N}}}(\tilde{u}(z_i), \tilde{u}(z_{i-1})) = \delta$. Since $\delta \le \operatorname{inj}(\mathcal{N})$, by Lemma 2.3, we also have $d_{\mathcal{N}}(u(z_i), u(z_{i-1})) = \delta$. Therefore, since $sp > 1$, it follows from Lemma 4.3 that for each $i \in \{1, \dots, \ell\}$,

$$\begin{aligned}\delta \le C_1\Bigg(&\bigg(\iint_{[z_{i-1},z_i]^2}\Big(\frac{d_{\mathcal{N}}(u(z),u(w))}{|z-w|^s} - \mu\Big)_+^p \frac{\mathrm{d}z\,\mathrm{d}w}{|z-w|}|z_i - z_{i-1}|^{sp-1}\bigg)^{\frac{1}{p}}\\ &+ \mu|z_i - z_{i-1}|^s\Bigg).\end{aligned} \tag{84}$$

Summing (84) we have

$$\begin{aligned}\ell \le \Big(\frac{C_1}{\delta}\Big)^{\frac{1}{s}}\sum_{i=1}^{\ell}\Bigg(&\bigg(\iint_{[z_{i-1},z_i]^2}\Big(\frac{d_{\mathcal{N}}(u(z),u(w))}{|z-w|^s} - \mu\Big)_+^p \frac{\mathrm{d}z\,\mathrm{d}w}{|z-w|}\bigg)^{\frac{1}{sp}}|z_i - z_{i-1}|^{1-\frac{1}{sp}}\\ &+ \mu^{\frac{1}{s}}|z_i - z_{i-1}|\Bigg).\end{aligned} \tag{85}$$

Applying the discrete Hölder inequality to the right-hand side of (85), we get

$$\begin{aligned}\ell \le \Big(\frac{C_1}{\delta}\Big)^{\frac{1}{s}}\Bigg(&\bigg(\sum_{i=1}^{\ell}\iint_{[z_{i-1},z_i]^2}\Big(\frac{d_{\mathcal{N}}(u(z),u(w))}{|z-w|^s} - \mu\Big)_+^p \frac{\mathrm{d}z\,\mathrm{d}w}{|z-w|}\bigg)^{\frac{1}{sp}}\bigg(\sum_{i=1}^{\ell}|z_i - z_{i-1}|\bigg)^{1-\frac{1}{sp}}\\ &+ \mu^{\frac{1}{s}}\sum_{i=1}^{\ell}|z_i - z_{i-1}|\Bigg).\end{aligned} \tag{86}$$

Since $x \le z_0 \le z_1 \le \dots \le z_\ell \le y$, the sets $(z_{i-1}, z_i)^2$ are disjoint subsets of $[x, y]^2$ and we deduce from (86) that

$$\begin{aligned}\ell \le \Big(\frac{C_1}{\delta}\Big)^{\frac{1}{s}}\Bigg(&\bigg(\iint_{[x,y]^2}\Big(\frac{d_{\mathcal{N}}(u(z),u(w))}{|z-w|^s} - \mu\Big)_+^p \frac{\mathrm{d}z\,\mathrm{d}w}{|z-w|}\bigg)^{\frac{1}{sp}}|y-x|^{1-\frac{1}{sp}}\\ &+ \mu^{\frac{1}{s}}|y-x|\Bigg).\end{aligned} \tag{87}$$

Recalling (83), conclusion (82) follows from (87). ∎

4.2. Mean integral oscillation estimates

Integrating the estimate of Lemma 4.5, we will obtain the following estimate on truncated mean oscillation by a truncated fractional Sobolev norm.

Lemma 4.6. *Let $m \in \mathbb{N} \setminus \{0\}$, let $s \in (0,1)$ and let $p \in (1,\infty)$. If $sp > 1$, then there exists a constant $C \in (0,\infty)$ such that if the set $\Omega \subset \mathbb{R}^m$ is bounded and convex, if $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ is a Riemannian covering map, if $\tilde{u} \in Y(\Omega, \widetilde{\mathcal{N}})$, if $u := \pi \circ \tilde{u}$, if $\delta \in (0, \operatorname{inj}(\mathcal{N})]$ and if $\mu \in [0,\infty)$, then*

$$\begin{aligned}\iint_{\Omega\times\Omega} \big(d_{\widetilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(z)) - \delta\big)_+^{sp} \,\mathrm{d}y\,\mathrm{d}z \\ \le \frac{C_2}{\delta^{(1-s)p}}\bigg(\frac{\operatorname{diam}(\Omega)^{m+sp}}{m+sp}\iint_{\Omega\times\Omega}\Big(\frac{d_{\mathcal{N}}(u(y),u(x))}{|y-x|^s} - \mu\Big)_+^p \frac{\mathrm{d}y\,\mathrm{d}x}{|y-x|^m} \\ + \mu^p \operatorname{diam}(\Omega)^{2m+sp}\bigg). \qquad (88)\end{aligned}$$

Lemma 4.6 will be deduced from Lemma 4.5 and the next integral estimate.

Lemma 4.7. *Let $m \in \mathbb{N} \setminus \{0\}$. If the set $\Omega \subseteq \mathbb{R}^m$ is open and convex, if the function $F\colon \Omega\times\Omega \to [0,\infty)$ is measurable and if $\gamma > -m$, then*

$$\begin{aligned}\iint_{\Omega\times\Omega}\bigg(\iint_{[x,y]\times[x,y]} F(w,v)\,\mathrm{d}w\,\mathrm{d}v\bigg)\frac{\mathrm{d}y\,\mathrm{d}x}{|y-x|^{1-\gamma}} \\ \le \frac{2\operatorname{diam}(\Omega)^{m+\gamma}}{m+\gamma}\iint_{\Omega\times\Omega}\frac{F(x,y)}{|y-x|^{m-1}}\,\mathrm{d}y\,\mathrm{d}x. \qquad (89)\end{aligned}$$

Proof. We have by definition of an integral on a segment,

$$\begin{aligned}\iint_{\Omega\times\Omega}\bigg(\iint_{[x,y]\times[x,y]} F(w,v)\,\mathrm{d}w\,\mathrm{d}v\bigg)|y-x|^{\gamma-1}\,\mathrm{d}y\,\mathrm{d}x \\ = \iint_{\Omega\times\Omega}\iint_{[0,1]\times[0,1]} F((1-t)x+ty,(1-r)x+ry)|y-x|^{\gamma+1}\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}x. \qquad (90)\end{aligned}$$

By the change of variables $v = (1-r)x + ry$, $w = (1-t)x + ty$ in the right-hand side of (90), we obtain, since $|v-w| = |t-r|\,|y-x|$,

$$\begin{aligned}\iint_{\Omega\times\Omega}\bigg(\iint_{[x,y]\times[x,y]} F(w,v)\,\mathrm{d}w\,\mathrm{d}v\bigg)\frac{\mathrm{d}y\,\mathrm{d}x}{|y-x|^{1-\gamma}} \\ = \iint_{\Omega\times\Omega}\iint_{\Sigma_{v,w}}\frac{F(w,v)|w-v|^{\gamma+1}}{|t-r|^{m+\gamma+1}}\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}w\,\mathrm{d}v, \qquad (91)\end{aligned}$$

where we have defined for every $v, w \in \Omega$ the set

$$\Sigma_{v,w} := \Big\{(t,r) \in [0,1]\times[0,1] \;\Big|\; \tfrac{rv-tw}{r-t} \in \Omega \text{ and } \tfrac{(1-r)v-(1-t)w}{t-r} \in \Omega\Big\}.$$

Since

$$\Sigma_{v,w} \subseteq \bigl\{(t,r) \in [0,1]\times[0,1] \bigm| |t-r| \ge \tfrac{|w-v|}{\operatorname{diam}\Omega}\bigr\},$$

we have

$$\iint_{\Sigma_{v,w}} \frac{1}{|t-r|^{m+\gamma+1}}\,\mathrm{d}t\,\mathrm{d}r \le \int_{|s|\ge \frac{|w-v|}{\operatorname{diam}\Omega}} \frac{\mathrm{d}s}{|s|^{m+\gamma+1}} = \frac{2\operatorname{diam}(\Omega)^{m+\gamma}}{(m+\gamma)|w-v|^{m+\gamma}}, \tag{92}$$

and we deduce from (91) and (92) that (89) holds. ■

We proceed now to the proof of Lemma 4.6.

Proof of Lemma 4.6. We have by Lemma 4.5, since $sp > 1$,

$$\begin{aligned}\iint_{\Omega\times\Omega} &\bigl(d_{\tilde{\mathcal{N}}}(\tilde u(y),\tilde u(x)) - \delta\bigr)_+^{sp}\,\mathrm{d}y\,\mathrm{d}x\\ &\le \frac{C_1}{\delta^{(1-s)p}}\biggl(\iint_{\Omega\times\Omega}\iint_{[x,y]\times[x,y]}\Bigl(\frac{d_{\mathcal{N}}(u(w),u(v))}{|w-v|^s} - \mu\Bigr)_+^p \frac{\mathrm{d}w\,\mathrm{d}v}{|w-v|}|y-x|^{sp-1}\,\mathrm{d}y\,\mathrm{d}x\\ &\qquad + \mu^p\iint_{\Omega\times\Omega}|y-x|^{sp}\,\mathrm{d}y\,\mathrm{d}x\biggr).\end{aligned} \tag{93}$$

For the first term in the right-hand side of (93), we proceed by Lemma 4.7 to infer from (93), since $sp > -m$, that

$$\begin{aligned}\iint_{\Omega\times\Omega}\iint_{[x,y]\times[x,y]}&\Bigl(\frac{d_{\mathcal{N}}(u(w),u(v))}{|w-v|^s} - \mu\Bigr)_+^p \frac{\mathrm{d}w\,\mathrm{d}v}{|w-v|}|y-x|^{sp-1}\,\mathrm{d}y\,\mathrm{d}x\\ &\le \frac{2\operatorname{diam}(\Omega)^{m+sp}}{m+sp}\iint_{\Omega\times\Omega}\Bigl(\frac{d_{\mathcal{N}}(u(y),u(x))}{|y-x|^s} - \mu\Bigr)_+^p \frac{\mathrm{d}y\,\mathrm{d}x}{|y-x|^m},\end{aligned} \tag{94}$$

whereas for the second term in the right-hand side of (93) we have

$$\iint_{\Omega\times\Omega}|y-x|^{sp}\,\mathrm{d}y\,\mathrm{d}x \le C_2\operatorname{diam}(\Omega)^{2m+sp}. \tag{95}$$

Estimate (88) then follows from inequalities (93), (94) and (95). ■

4.3. Integral truncated mean oscillation estimate

We now obtain an interpolation estimate similar to Proposition 4.1 on an integral of truncated mean oscillations.

Proposition 4.8. *Let $\mathcal{M}$ be a compact Riemannian manifold, let $s, s_* \in (0,1)$, let $p, p_* \in [1,\infty)$ and let $m := \dim\mathcal{M}$. If $sp > 1$, then there exists a constant $C \in (0,\infty)$ such that if $\pi\colon\tilde{\mathcal{N}}\to\mathcal{N}$ is a Riemannian covering, if $\tilde u \in Y(\mathcal{M},\tilde{\mathcal{N}})$, if $u := \pi\circ\tilde u$, if $\delta \le \operatorname{inj}(\mathcal{N})$ and if*

$$\frac{1-s_*}{m} = \frac{1}{sp} - \frac{1}{p_*}, \tag{96}$$

then

$$\int_{\Omega}\int_0^{\operatorname{diam}\Omega}\Big(\frac{1}{r^{2m+s_*}}\iint_{(\Omega\cap B_r(x))^2}\big(d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(z))-\delta\big)_+\,\mathrm{d}y\,\mathrm{d}z\Big)^{p_*}\frac{\mathrm{d}r}{r}\,\mathrm{d}x$$
$$\leq C\Big(\frac{1}{\delta^{(1-s)p}}\iint_{\Omega\times\Omega}\frac{d_{\mathcal{N}}(u(y),u(z))^p}{|y-z|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}z\Big)^{\frac{p_*}{sp}}.$$

The proof of Lemma 4.8 is reminiscent of the proof of the Marcinkiewicz real interpolation theorem, although the framework here is much more nonlinear.

Proof of Lemma 4.8. We have, by the layer-cake representation of integrals (Cavalieri's principle),

$$\int_{\Omega}\Big(\frac{1}{r^{2m}}\iint_{(\Omega\cap B_r(x))^2}\big(d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(z))-\delta\big)_+\,\mathrm{d}y\,\mathrm{d}z\Big)^{p_*}\,\mathrm{d}x$$
$$=(p_*-1)\int_0^{\infty}\mathcal{L}^m(E_\lambda^r)\lambda^{p^*-1}\,\mathrm{d}\lambda, \tag{97}$$

where for each $\lambda\in(0,\infty)$ and $r\in(0,\infty)$ we have defined the set

$$E_\lambda^r:=\big\{x\in\Omega\;\big|\;\textstyle\iint_{(\Omega\cap B_r(x))^2}\big(d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(z))-\delta\big)_+\,\mathrm{d}y\,\mathrm{d}z\geq\lambda r^{2m}\big\}.$$

On the one hand, fixing $q\in(\frac{1}{s},p)$ – which is possible since $sp>1$ – for each $x\in E_\lambda^r$ and $\mu\in[0,\infty)$, we have by Jensen's inequality and by Lemma 4.6, since $sq>1$,

$$\begin{aligned}\lambda^{sq}&\leq\Big(\frac{1}{r^{2m}}\iint_{(\Omega\cap B_r(x))^2}\big(d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(z))-\delta\big)_+\,\mathrm{d}y\,\mathrm{d}w\Big)^{sq}\\&\leq C_1\frac{1}{r^{2m}}\iint_{(\Omega\cap B_r(x))^2}\big(d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(z))-\delta\big)_+^{sq}\,\mathrm{d}y\,\mathrm{d}z\\&\leq C_2\frac{r^{sq-m}}{\delta^{(1-s)q}}\iint_{(\Omega\cap B_r(x))^2}\Big(\frac{d_{\mathcal{N}}(u(y),u(z))}{|y-z|^s}-\mu\Big)_+^q\frac{\mathrm{d}y\,\mathrm{d}z}{|y-z|^m}+C_3\frac{\mu^q r^{sq}}{\delta^{(1-s)q}}.\end{aligned}\tag{98}$$

If we take now μ to be given by

$$\mu_\lambda^r:=C_4\frac{\lambda^s\delta^{1-s}}{r^s},$$

with $C_4^qC_3=\frac{1}{2}$, for each $x\in E_\lambda^r$, we have by (98),

$$\lambda^{sq}\leq C_5\frac{r^{sq-m}}{\delta^{(1-s)q}}\iint_{(\Omega\cap B_r(x))^2}\Big(\frac{d_{\mathcal{N}}(u(y),u(z))}{|y-z|^s}-\mu_\lambda^r\Big)_+^q\frac{\mathrm{d}y\,\mathrm{d}z}{|y-z|^m}. \tag{99}$$

Hence, we have by (99),

$$\mathcal{L}^m(E_\lambda^r)$$
$$\leq\frac{C_5r^{sq-m}}{\lambda^{sq}\delta^{(1-s)q}}\int_{\Omega}\iint_{(\Omega\cap B_r(x))^2}\Big(\frac{d_{\mathcal{N}}(u(y),u(z))}{|y-z|^s}-\mu_\lambda^r\Big)_+^q\frac{\mathrm{d}y\,\mathrm{d}z}{|y-z|^m}\,\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{C_5 r^{sq-m}}{\lambda^{sq}\delta^{(1-s)q}} \iint_{\Omega\times\Omega} \Big(\frac{d_{\mathcal{N}}(u(y),u(z))}{|y-z|^s} - \mu^r_\lambda\Big)^q_+ \mathcal{L}^m(\Omega\cap B_r(y)\cap B_r(z)) \frac{\mathrm{d}y\,\mathrm{d}z}{|y-z|^m}\\
&\le \frac{C_6 r^{sq}}{\lambda^{sq}\delta^{(1-s)q}} \iint_{\Omega\times\Omega} \Big(\frac{d_{\mathcal{N}}(u(y),u(z))}{|y-z|^s} - \mu^r_\lambda\Big)^q_+ \frac{\mathrm{d}y\,\mathrm{d}z}{|y-z|^m}. \qquad (100)
\end{aligned}
$$

On the other hand, since $sp > 1$, if $x \in E^r_\lambda$, we have by Jensen's inequality and by Lemma 4.6 with $\mu = 0$,

$$
\begin{aligned}
\lambda^{sp} &\le \Big(\frac{1}{r^{2m}} \iint_{(\Omega\cap B_r(x))^2} \big(d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(z)) - \delta\big)_+ \,\mathrm{d}y\,\mathrm{d}z\Big)^{sp}\\
&\le C_7 \frac{1}{r^{2m}} \iint_{(\Omega\cap B_r(x))^2} \big(d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(z)) - \delta\big)^{sp}_+ \,\mathrm{d}y\,\mathrm{d}z\\
&\le C_8 \frac{r^{sp-m}}{\delta^{(1-s)p}} \iint_{(\Omega\cap B_r(x))^2} \frac{d_{\mathcal{N}}(u(y),u(z))^p}{|y-z|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}z; \qquad (101)
\end{aligned}
$$

it follows then from (101) that

$$
\begin{aligned}
&\big\{(r,\lambda) \in (0,\infty)^2 \;\big|\; E^\lambda_r \neq \emptyset\big\}\\
&\subseteq H := \Big\{(r,\lambda) \in (0,\infty)^2 \;\Big|\; \lambda^{sp} r^{m-sp} \le \tfrac{C_8^{sp}}{\delta^{(1-s)p}} \iint_{\Omega\times\Omega} \tfrac{d_{\mathcal{N}}(u(y),u(z))^p}{|y-z|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}z\Big\}. \qquad (102)
\end{aligned}
$$

By (97), (100) and (102), we have

$$
\begin{aligned}
&\int_\Omega \int_0^{\operatorname{diam}\Omega} \Big(\frac{1}{r^{2m}} \iint_{(\Omega\cap B_r(x))^2} \big(d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(z)) - \delta\big)_+ \,\mathrm{d}y\,\mathrm{d}z\Big)^{p_*} \frac{\mathrm{d}r}{r^{1+s_* p_*}} \,\mathrm{d}x\\
&\quad\le \frac{C_9}{\delta^{(1-s)q}} \iint_H \iint_{\Omega\times\Omega} \Big(\frac{d_{\mathcal{N}}(u(y),u(z))}{|y-z|^s} - \mu^r_\lambda\Big)^q_+ \frac{\mathrm{d}y\,\mathrm{d}z}{|y-z|^m}\\
&\qquad\qquad \times \frac{r^{sq}\lambda^{p_*}}{r^{1+s_* p_*}\lambda^{1+sq}} \,\mathrm{d}\lambda\,\mathrm{d}r. \qquad (103)
\end{aligned}
$$

Applying the change of variable

$$
\mu = C_4 \frac{\lambda^s \delta^{1-s}}{r^s} \quad \text{and} \quad t = \lambda^{sp} r^{m-sp}
$$

in (103), we infer from (96) that

$$
\begin{aligned}
&\int_\Omega \int_0^{\operatorname{diam}(\Omega)} \Big(\frac{1}{r^{2m}} \iint_{(\Omega\cap B_r(x))^2} \big(d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(z)) - \delta\big)_+ \,\mathrm{d}y\,\mathrm{d}z\Big)^{p_*} \frac{\mathrm{d}r}{r^{1+s_* p_*}} \,\mathrm{d}x\\
&\quad\le \frac{C_{10}}{\delta^{(1-s)p}} \iint_{\Omega\times\Omega} \int_0^{\bar{t}} \int_0^\infty \Big(\frac{d_{\mathcal{N}}(u(y),u(z))}{|y-z|^s} - \mu\Big)^q_+ \mu^{p-q-1} t^{\frac{p^*}{sp}-2} \,\mathrm{d}\mu\,\mathrm{d}t\\
&\qquad\qquad \times \frac{\mathrm{d}y\,\mathrm{d}z}{|y-z|^m}, \qquad (104)
\end{aligned}
$$

with

$$\bar{t} := \frac{C_8^{sp}}{\delta^{(1-s)p}} \iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y), u(z))^p}{|y-z|^{m+sp}} \,\mathrm{d}y\,\mathrm{d}z.$$

The conclusion follows by the integration in μ and t of the right-hand side of inequality (104), since $q < p$ and $p^* > sp$. ■

4.4. Proof of the large-scale estimate

We now use Lemma 4.8 to prove Proposition 4.1. The main idea consists in applying Lemma 4.8 with the triangle inequality; because of the truncation in the left-hand side we need to rely on Lemma 4.8 with values of δ arbitrarily close to 0.

Proof of Proposition 4.1. By a comparison argument, we have

$$\begin{aligned}\iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(x))\ge\delta}} &\frac{d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(x))^{p_*}}{|y-x|^{m+s_*p_*}}\,\mathrm{d}y\,\mathrm{d}x\\ &\le 2^{p_*}\iint_{\Omega\times\Omega}\frac{\big(d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(x))-\frac{\delta}{2}\big)_+^{p_*}}{|y-x|^{m+s_*p_*}}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{105}$$

By the triangle inequality and by symmetry, we then have

$$\begin{aligned}\iint_{\Omega\times\Omega}&\frac{\big(d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(x))-\frac{\delta}{2}\big)_+^{p_*}}{|y-x|^{m+s_*p_*}}\,\mathrm{d}y\,\mathrm{d}x\\ &\le 2^{p_*-1}\iint_{\Omega\times\Omega}\bigg(\fint_{\Omega\cap B_{|y-x|/2}(\frac{x+y}{2})}\big(d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(z))-\tfrac{\delta}{4}\big)_+\,\mathrm{d}z\bigg)^{p_*}\\ &\quad+\bigg(\fint_{\Omega\cap B_{|y-x|/2}(\frac{x+y}{2})}\big(d_{\tilde{\mathcal{N}}}(\tilde{u}(z),\tilde{u}(x))-\tfrac{\delta}{4}\big)_+\,\mathrm{d}z\bigg)^{p_*}\frac{\mathrm{d}y\,\mathrm{d}x}{|y-x|^{m+s_*p_*}}\\ &= 2^{p_*}\iint_{\Omega\times\Omega}\bigg(\fint_{\Omega\cap B_{|y-x|/2}(\frac{x+y}{2})}\big(d_{\tilde{\mathcal{N}}}(\tilde{u}(z),\tilde{u}(x))-\tfrac{\delta}{4}\big)_+\,\mathrm{d}z\bigg)^{p_*}\\ &\qquad\times\frac{\mathrm{d}y\,\mathrm{d}x}{|y-x|^{m+s_*p_*}}. \end{aligned}\tag{106}$$

By (105) and (106) we have by integration in spherical coordinates,

$$\begin{aligned}\iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(x))\ge\delta}} &\frac{d_{\tilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(x))^{p_*}}{|y-x|^{m+s_*p_*}}\,\mathrm{d}y\,\mathrm{d}x\\ &\le C_1\iint_{\Omega\times\Omega}\bigg(\fint_{\Omega\cap B_{|y-x|}(x)}\big(d_{\tilde{\mathcal{N}}}(\tilde{u}(z),\tilde{u}(x))-\tfrac{\delta}{4}\big)_+\,\mathrm{d}z\bigg)^{p_*}\frac{\mathrm{d}y\,\mathrm{d}x}{|y-x|^{m+s_*p_*}}\\ &\le C_2\int_\Omega\int_0^{\operatorname{diam}(\Omega)}\bigg(\fint_{\Omega\cap B_r(x)}\big(d_{\tilde{\mathcal{N}}}(\tilde{u}(z),\tilde{u}(x))-\tfrac{\delta}{4}\big)_+\,\mathrm{d}z\bigg)^{p_*}\frac{\mathrm{d}r}{r^{1+s_*p_*}}\,\mathrm{d}x. \end{aligned}$$

By the triangle inequality, similarly to the proof of Lemma 4.4, we have for almost every $x \in \Omega$ and every $r \in (0, \operatorname{diam}(\Omega))$,

$$\begin{aligned}
&\fint_{\Omega \cap B_r(x)} \bigl(d_{\widetilde{\mathcal{N}}}(\tilde{u}(z), \tilde{u}(x)) - \tfrac{\delta}{4}\bigr)_+ \,\mathrm{d}z \\
&\quad\le \sum_{j \in \mathbb{N}} \fint_{\Omega \cap B_{2^{-j}r}(x)} \fint_{\Omega \cap B_{2^{-j-1}r}(x)} \bigl(d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(z)) - \delta_j\bigr)_+ \,\mathrm{d}y \,\mathrm{d}z \\
&\quad\le C_3 \frac{r^{2m}}{2^{2mj}} \sum_{j \in \mathbb{N}} \int_{\Omega \cap B_{2^{-j}r}(x)} \int_{\Omega \cap B_{2^{-j}r}(x)} \bigl(d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(z)) - \delta_j\bigr)_+ \,\mathrm{d}y \,\mathrm{d}z, \qquad (107)
\end{aligned}$$

where we have set for each $j \in \mathbb{N}$,

$$\delta_j := \frac{\delta \kappa^j}{4(1-\kappa)}, \qquad (108)$$

with a constant $\kappa \in (0, 1)$ to be determined later, since $\sum_{j \in \mathbb{N}} \delta_j = \frac{\delta}{4}$. We have then by (107) and by Minkowski's inequality,

$$\begin{aligned}
&\biggl(\int_\Omega \int_0^{\operatorname{diam}\Omega} \Bigl(\fint_{\Omega \cap B_r(x)} \bigl(d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x)) - \tfrac{\delta}{4}\bigr)_+ \,\mathrm{d}y\Bigr)^{p_*} \frac{\mathrm{d}r}{r^{1+s_* p_*}} \,\mathrm{d}x\biggr)^{\frac{1}{p_*}} \\
&\quad\le C_3 \sum_{j \in \mathbb{N}} \biggl(\int_\Omega \int_0^{\operatorname{diam}\Omega} \Bigl(\frac{2^{2mj}}{r^{2m}} \iint_{(\Omega \cap B_{2^{-j}r}(x))^2} \bigl(d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(z)) - \delta_j\bigr)_+ \,\mathrm{d}y \,\mathrm{d}z\Bigr)^{p_*} \\
&\qquad\qquad \times \frac{\mathrm{d}r}{r^{1+s_* p_*}} \,\mathrm{d}x\biggr)^{\frac{1}{p_*}}. \qquad (109)
\end{aligned}$$

For every $j \in \mathbb{N}$, we have by a change of variable in the outer integral,

$$\begin{aligned}
&\int_0^{\operatorname{diam}\Omega} \Bigl(\frac{2^{2mj}}{r^{2m}} \iint_{(\Omega \cap B_{2^{-j}r}(x))^2} \bigl(d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(z)) - \delta_j\bigr)_+ \,\mathrm{d}y \,\mathrm{d}z\Bigr)^{p_*} \frac{\mathrm{d}r}{r^{1+s_* p_*}} \\
&\quad= \frac{1}{2^{s_* p_* j}} \int_0^{2^{-j}\operatorname{diam}\Omega} \Bigl(\frac{1}{r^{2m}} \iint_{(\Omega \cap B_r(x))^2} \bigl(d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(z)) - \delta_j\bigr)_+ \,\mathrm{d}y \,\mathrm{d}z\Bigr)^{p_*} \\
&\qquad\qquad \times \frac{\mathrm{d}r}{r^{1+s_* p_*}}, \qquad (110)
\end{aligned}$$

whereas by Lemma 4.8,

$$\begin{aligned}
&\int_0^{\operatorname{diam}\Omega} \Bigl(\frac{1}{r^{2m}} \iint_{(\Omega \cap B_r(x))^2} \bigl(d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(z)) - \delta_j\bigr)_+ \,\mathrm{d}y \,\mathrm{d}z\Bigr)^{p_*} \frac{\mathrm{d}r}{r^{1+s_* p_*}} \\
&\quad\le C_4 \Bigl(\frac{1}{\delta_j^{(1-s)p}} \iint_{\Omega \times \Omega} \frac{d_{\mathcal{N}}(u(y), u(z))^p}{|y - z|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}z\Bigr). \qquad (111)
\end{aligned}$$

Combining (109), (110) and (111), we obtain in view of (108),

$$\Big(\int_\Omega \int_0^{\operatorname{diam}\Omega} \Big(\fint_{\Omega\cap B_r(x)} \big(d_{\tilde{\mathcal{N}}}(\tilde u(y),\tilde u(x)) - \tfrac{\delta}{4}\big)_+ \,\mathrm{d}y\Big)^{p_*} \frac{\mathrm{d}r}{r^{1+s_* p_*}}\,\mathrm{d}x\Big)^{\frac{1}{p_*}} \\ \le C_5 \sum_{j\in\mathbb{N}} \frac{1}{(2^{s_*}\kappa^{\frac{1-s}{s}})^j} \Big(\frac{1}{\delta^{(1-s)p}} \iint_{\Omega^2} \frac{d_{\mathcal{N}}(u(y),u(z))^p}{|y-z|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}z\Big)^{\frac{1}{sp}},$$

and the conclusion follows provided $\kappa \in (0,1)$ is chosen in such a way that $\kappa > 2^{-\frac{s_* s}{1-s}}$. ∎

4.5. Conclusion and further estimate

We now deduce Theorem 1.8 from Proposition 4.1.

Proof of Theorem 1.8. We first assume that $\mathcal{M} = \Omega$, where the set $\Omega \subset \mathbb{R}^m$ is open, bounded and convex. By Proposition 2.18, we have $\tilde u \in Y(\Omega, \tilde{\mathcal{N}})$. Letting $p_* := p$, we have $s_* = s + (1-s)(1 - \frac{m}{sp}) \ge s$. We get, since Ω is bounded and $s_* p_* \ge sp$,

$$\iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\tilde{\mathcal{N}}}(\tilde u(y),\tilde u(x))\ge \operatorname{inj}(\mathcal{N})}} \frac{d_{\tilde{\mathcal{N}}}(\tilde u(y),\tilde u(x))^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x \\ \le C_1 \iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\tilde{\mathcal{N}}}(\tilde u(y),\tilde u(x))\ge \operatorname{inj}\mathcal{N}}} \frac{d_{\tilde{\mathcal{N}}}(\tilde u(y),\tilde u(x))^{p_*}}{|y-x|^{m+s_* p_*}}\,\mathrm{d}y\,\mathrm{d}x \\ \le C_2 \Big(\iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y),u(x))^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x\Big)^{\frac{1}{s}}, \tag{112}$$

by Proposition 4.1 with $\delta = \operatorname{inj}(\mathcal{N})$. Combining estimate (112) with Proposition 2.1, we get

$$\iint_{\Omega\times\Omega} \frac{d_{\tilde{\mathcal{N}}}(\tilde u(y),\tilde u(x))^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x \le C_3 \Big(\iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y),u(x))^p}{|y-x|^{m+sp}}\,\mathrm{d}z\,\mathrm{d}y \\ + \Big(\iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y),u(x))^p}{|y-x|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x\Big)^{\frac{1}{s}}\Big). \tag{113}$$

We reach conclusion (9) on a general compact manifold $\mathcal{M}$ thanks to estimate (113) and the covering of Lemma 2.11. ∎

Remark 4.9. The exponent $\frac{1}{s}$ in (9) is optimal. Indeed, assuming that

$$\|\tilde u\|_{\dot W^{s,p}} \le C_1(\|u\|_{\dot W^{s,p}} + \|u\|^{\gamma}_{\dot W^{s,p}})$$

holds and taking $\pi\colon \mathbb{R} \to \mathbb{S}^1$ to be the universal covering of the unit circle and $\tilde u = t\varphi$, for some $\varphi \in C^\infty(\mathcal{M},\mathbb{R})$ and every $t \in \mathbb{R}$, one gets from (9) that $|t| \le C_2(|t|^s + |t|^{\gamma s})$, which can only hold if $\gamma \ge \frac{1}{s}$.

Proposition 4.1 can also be applied to obtain a result in which the nonlinear part in the estimate contains a critical fractional Sobolev energy.

Theorem 4.10. *Let $\mathcal{M}$ and $\mathcal{N}$ be compact Riemannian manifolds, let $m := \dim \mathcal{M}$, let $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be a Riemannian covering map, let $s \in (0, 1)$ and let $p \in (1, \infty)$. If $sp > 1$, then there exists a constant $C \in (0, \infty)$ such that for every $\tilde{u} \in X(\mathcal{M}, \widetilde{\mathcal{N}})$, we have $\tilde{u} \in \dot{W}^{s,p}(\mathcal{M}, \widetilde{\mathcal{N}})$ and*

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \leq C_3 \Bigl(1 + \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y, x)^{2m}} \,\mathrm{d}y \,\mathrm{d}x\Bigr)^{\frac{(1-s)p}{m}} \iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x.$$

Although no restriction is put on the exponent, in practice the first integral in the right-hand side will be finite for some nonconstant function u if and only if $p > m$.

Proof of Theorem 4.10. We proceed as in the proof of Theorem 1.8, now applying Proposition 4.1 with s being given by $s_0 = \frac{1}{1+(1-s)p/m}$, $p_* = p$ and so that s_* is then given by s in (72). Since $sp > 1$, we have $s_0 p = \frac{p}{1+(1-s)p/m} > 1$, and thus by Proposition 4.1,

$$\iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\widetilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(x))\geq \operatorname{inj}(\mathcal{N})}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \leq C_1 \Bigl(\iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{|y - x|^{m+\frac{mp}{m+(1-s)p}}} \,\mathrm{d}y \,\mathrm{d}x\Bigr)^{\frac{m+(1-s)p}{m}}. \tag{114}$$

If $sp \geq m$, we have $m \leq \frac{mp}{m+(1-s)p} \leq sp$, whereas if $sp \leq m$ we have $sp \leq \frac{mp}{m+(1-s)p} \leq m$, and thus by Hölder's inequality and (114) we get

$$\begin{aligned}\iint_{\substack{(x,y)\in\Omega\times\Omega\\ d_{\widetilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(x))\geq \operatorname{inj}(\mathcal{N})}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x &\leq C_2 \Bigl(1 + \iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{|y - x|^{2m}} \,\mathrm{d}y \,\mathrm{d}x\Bigr)^{\frac{(1-s)p}{m}} \\ &\quad\times \iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x.\end{aligned} \tag{115}$$

Hence, combining (115) with Proposition 2.1, we get

$$\begin{aligned}\iint_{\Omega\times\Omega} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x &\leq C_3 \Bigl(1 + \iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{|y - x|^{2m}} \,\mathrm{d}y \,\mathrm{d}x\Bigr)^{\frac{(1-s)p}{m}} \\ &\quad\times \iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x.\end{aligned} \tag{116}$$

Combining (116) with the covering of Lemma 2.11, we conclude. ∎

Remark 4.11. Again, the exponent $\frac{1}{s}$ in (9) is optimal. Indeed, assuming that we have

$$\|\tilde{u}\|_{\dot{W}^{s,p}} \le C_1(1 + \|u\|^{\gamma}_{\dot{W}^{m/p,p}})\|u\|_{\dot{W}^{s,p}} \tag{117}$$

and taking $\pi\colon \mathbb{R} \to \mathbb{S}^1$ to be the universal covering of the unit circle and $\tilde{u} = t\varphi$, for some $\varphi \in C^\infty(\mathcal{M}, \mathbb{R})$ and every $t \in \mathbb{R}$, one gets from (117) that $|t| \le C_2(1 + |t|^{\gamma m/p})|t|^s$, which can only hold if $\gamma \ge \frac{(1-s)p}{m}$.

Finally, the same methods can be used to get an estimate on a lower-order fractional Sobolev energy when the dimension is supercritical.

Theorem 4.12. *Let $\mathcal{M}$ and $\mathcal{N}$ be compact Riemannian manifolds, let $m := \dim \mathcal{M}$, let $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ be a Riemannian covering map, let $s \in (0, 1)$ and let $p \in (1, \infty)$. If*

$$1 - s < \frac{sp}{m} < 1 \tag{118}$$

and if $\tilde{u} \in X(\mathcal{M}, \widetilde{\mathcal{N}})$, then $\tilde{u} \in \dot{W}^{s_\flat,p}(\mathcal{M}, \widetilde{\mathcal{N}})$ and

$$\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p}{d_{\mathcal{M}}(y, x)^{m+s_\flat p}} \,\mathrm{d}y \,\mathrm{d}x \le C_3\biggl(\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x + \Bigl(\iint_{\mathcal{M}\times\mathcal{M}} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{d_{\mathcal{M}}(y, x)^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x\Bigr)^{\frac{1}{s}}\biggr),$$

with

$$s_\flat := s - (1 - s)\Bigl(\frac{m}{sp} - 1\Bigr). \tag{119}$$

Proof. We follow the structure of the proof of Theorem 1.8. Considering $\tilde{u} \in Y(\Omega, \widetilde{\mathcal{N}})$, we apply Proposition 4.1 with $p_* = p$ so that $s_* = s_\flat$ in view of (119) since by (118),

$$s_\flat p = p + m\Bigl(\frac{1}{s} - 1\Bigr) > 1$$

and we get

$$\iint_{\substack{(x,y)\in\Omega\times\Omega \\ d_{\widetilde{\mathcal{N}}}(\tilde{u}(y),\tilde{u}(x))\ge \mathrm{inj}(\mathcal{N})}} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p}{|y - x|^{m+s_\flat p}} \,\mathrm{d}y \,\mathrm{d}x \le C_1\Bigl(\iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x\Bigr)^{\frac{1}{s}}.$$

On the other hand, by Proposition 2.1, since $s_\flat < s$ and since the set Ω is bounded, we get

$$\iint_{\Omega\times\Omega} \frac{d_{\widetilde{\mathcal{N}}}(\tilde{u}(y), \tilde{u}(x))^p}{|y - x|^{m+s_\flat p}} \,\mathrm{d}y \,\mathrm{d}x \le C_2\biggl(\iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x + \Bigl(\iint_{\Omega\times\Omega} \frac{d_{\mathcal{N}}(u(y), u(x))^p}{|y - x|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x\Bigr)^{\frac{1}{s}}\biggr). \tag{120}$$

The conclusion follows from (120) and Lemma 2.11. ∎

Remark 4.13. The value $s_\flat$ in the statement of Theorem 4.12 is optimal: taking $\pi\colon \mathbb{R} \to \mathbb{S}^1$ to be the universal covering of the unit circle and defining $\tilde{u}(x) := |x|^{-\alpha}$, then $u \in \dot{W}^{1,sp}(\mathbb{B}^m, \mathbb{R})$ if and only if $(\alpha + 1)sp < m$. By the fractional Gagliardo–Nirenberg interpolation inequality, one then has $\pi \circ \tilde{u} \in \dot{W}^{s,p}(\mathbb{B}^m, \mathbb{S}^1)$. We also have $u \in \dot{W}^{s_*,p}(\mathbb{B}^m, \mathbb{R})$ if and only if $(\alpha + s_*)p < m$. This implies that we can have $\tilde{u} \notin \dot{W}^{s_*,p}(\mathbb{B}^m, \mathbb{R})$ and $u \in \dot{W}^{s,p}(\mathbb{B}^m, \mathbb{S}^1)$, when $\frac{m}{p} - s_* < \frac{m}{sp} - 1$, or equivalently $s_* > s_\flat$.

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