Ann. Inst. H. Poincaré
Anal. Non Linéaire 42 (2025), 163–207
DOI 10.4171/AIHPC/108

Regularity for a geometrically nonlinear flat Cosserat micropolar membrane shell with curvature

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Abstract. We consider the rigorously derived thin shell membrane Γ-limit of a three-dimensional isotropic geometrically nonlinear Cosserat micropolar model and deduce full interior regularity of both the midsurface deformation $m\colon \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ and the orthogonal microrotation tensor field $R\colon \omega \subset \mathbb{R}^2 \to \mathrm{SO}(3)$. The only further structural assumption is that the curvature energy depends solely on the uni-constant isotropic Dirichlet-type energy term $|\mathrm{D}R|^2$. We use Rivière's regularity techniques of harmonic-map-type systems for our system which couples harmonic maps to SO(3) with a linear equation for m. The additional coupling term in the harmonic map equation is of critical integrability and can only be handled because of its special structure.

1. Introduction

1.1. Regularity background and setting of the problem

This paper contributes to the wide field of regularity theory of harmonic-map-type equations. Driven by the application to a geometrically nonlinear flat Cosserat shell model, we extend known regularity results to a system that couples a harmonic map equation with another uniformly elliptic equation. The system we consider is of the form

$$\operatorname{Div} S(\mathrm{D}m, R) = 0, \tag{1.1}$$

$$\Delta R - \Omega_R \cdot \mathrm{D}R - \operatorname{skew}(\mathrm{D}m \circ S(\mathrm{D}m, R))R = 0, \tag{1.2}$$

where here

$$S(\mathrm{D}m, R) := \pi_{12}\big(2R\mathbb{P}^2(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big), \tag{1.3}$$

$$\Omega_R := -\begin{pmatrix}\operatorname{skew}(R\partial_x R^\intercal)\\ \operatorname{skew}(R\partial_y R^\intercal)\end{pmatrix}, \tag{1.4}$$

with some notation to be explained in Section 5. The unknown functions here are the midsurface deformation $m \in W^{1,2}(\omega, \mathbb{R}^3)$ and the microrotation $R \in W^{1,2}(\omega, \mathrm{SO}(3))$,

Mathematics Subject Classification 2020: 58E20 (primary); 74G40, 74B20, 35J35 (secondary).
Keywords: flat shell, membrane, Γ-limit, regularity, Cosserat surface, Cosserat shell, micropolar shell, harmonic maps, graphene, generalized continua.

while $\omega \subset \mathbb{R}^2$ is a smooth domain. The $\mathfrak{so}(3)$-valued 1-form Ω_R is the one that makes

$$\Delta R - \Omega_R \cdot \mathrm{D}R = 0 \tag{1.5}$$

the harmonic map equation for harmonic mappings to $\mathrm{SO}(3) \subset \mathbb{R}^{3\times 3}$. The theory of harmonic map equations of two-dimensional domains (to any sufficiently smooth compact target manifold, here SO(3)) has a long history. It was proven in 1948 by Morrey [44] that minimizing weakly harmonic maps are smooth. In 1981, Grüter [29] generalized that to conformal weakly harmonic maps, and then in 1984 Schoen [70] to stationary ones. The regularity proof for general weakly harmonic maps was then found in 1990 by Hélein [31, 32]. (Note that in our case, the target manifold SO(3) is a Lie group, and in this case the harmonic map problem has a lot of interesting extra structure, many aspects of which are covered in Helein's book [33].) Later, in 2007, Rivière [63] revisited harmonic-map-type equations and asked for which Ω_R all weak solutions of (1.5) on a two-dimensional domain are smooth. It turned out that Ω_R need not come from the harmonic map equation (in which case it can be seen as the anti-symmetrized tensor derived from the second fundamental form of the target manifold), but for the regularity result only the skew-symmetry of Ω_R is needed. This gave deeper insight into the structures necessary to have regularity results, and it is Rivière's philosophy that we rely upon.

Before we comment on the structure of our equations, and hence on the regularity theory methods required, we state our main result.

Theorem 1.1. *Every weak solution* $(m, R) \in W^{1,2}(\omega, \mathbb{R}^3 \times \mathrm{SO}(3))$ *of* (1.1)–(1.2)*, with* $S(\mathrm{D}m, R)$ *and* Ω_R *given by* (1.3)–(1.4)*, is smooth on the interior of* ω.

In order to understand the structure of (1.1)–(1.2), we first look at (1.5). With Ω_R and $\mathrm{D}R$ being in L^2, the nonlinear term $\Omega_R \cdot \mathrm{D}R$ in (1.5) is only in L^1, and if it did not have any further structure, it would be difficult to start with any regularity theory, due to the lack of an L^p-theory working for $p = 1$. But it turns out that the product $\Omega_R \cdot \mathrm{D}R$, after a suitable gauge transformation, is the sum of products of divergence-free vector fields and gradients in L^2, which is known to be in the Hardy space $\mathscr{H}^1$ rather than L^1. This little bit of extra regularity is enough to perform regularity theory.

Now let us have a look at our equation (1.2). Compared with (1.5), it has an extra term $\mathrm{skew}(\mathrm{D}m \circ S(\mathrm{D}m, R))R$, and again, with $\mathrm{D}R \in L^2$, $S(\mathrm{D}m, R) \in L^2$, and $R \in L^\infty$, this has only L^1-integrability. But once more, $\mathrm{D}R$ is a gradient, and $S(\mathrm{D}m, R)$ is divergence-free due to equation (1.1). This time, we have the product of a gradient $\mathrm{D}m$, a divergence-free vector field $S(\mathrm{D}m, R)$, and a bounded function R. Based on a crucial estimate by Coifman, Lions, Meyer, and Semmes [14], Rivière and Struwe [64] were able to handle such products in their work on partial regularity in dimensions ≥ 3. They encountered such products in the course of their proof for equation (1.5) without any extra terms, and we can modify their arguments to handle our extra term from the coupling. The handling of the first equation, which is linear in m with some right-hand side, is easier, in principle. But we have to do the iteration procedure for both equations simultaneously in the proof of Hölder continuity, resulting in some technicalities.

Once we have that, we still only have Hölder continuity of m and R, as proven in Section 6.1. Note that for Rivière's equation (1.5), in general, one can only expect Hölder regularity of the solutions. For the important special case of the harmonic map equation, however, Ω_R depends on $\mathrm{D}R$ only linearly, allowing one to bootstrap and achieve C^∞ regularity once Hölder regularity *of the gradient* has been proven as a start. The same applies here, since our Ω_R is that of the harmonic map equation. Due to the additional nonlinearity in (1.2), however, deriving the Hölder continuity of gradients from that of solutions requires one to modify arguments from Moser's book [45] to make them fit for our coupled system. Combining such methods with standard Schauder estimates for (1.1), we succeed in proving $C^{1,\alpha}$ and then C^∞ regularity, which is the content of Section 6.2.

1.2. Engineering background and application

The Cosserat model is one of the best known generalized continuum models [13]. It assumes that material points can undergo translation, described by the standard deformation map $\varphi\colon \mathcal{U} \to \mathbb{R}^3$ and independent microrotations described by the orthogonal tensor field $R\colon \mathcal{U} \to \mathrm{SO}(3)$, where $\mathcal{U} \subset \mathbb{R}^3$ describes the smooth reference configuration of the material. Therefore, the geometrically nonlinear Cosserat model immediately induces the Lie-group structure on the configuration space $\mathbb{R}^3 \times \mathrm{SO}(3)$.

Both fields are coupled in the assumed elastic energy $W = W(\mathrm{D}\varphi, R, \mathrm{D}R)$ and the static Cosserat model appears as a two-field minimization problem which is automatically geometrically nonlinear due to the presence of the nonabelian rotation group SO(3). Material frame indifference (objectivity) dictates left invariance of the Lagrangian W under the action of SO(3) and material symmetry (here isotropy) implies right invariance under action of SO(3).

In the early 20th century the Cosserat brothers E. and F. Cosserat introduced this model in its full geometrically nonlinear splendor [17] in a bold attempt to unify field theories embracing mechanics, optics, and electrodynamics through a common principle of least action. They used the invariance of the energy under Euclidean transformations [4, 16] to deduce the correct form of the energy $W = W(R^{\mathsf{T}}\mathrm{D}\varphi, R^{\mathsf{T}}\partial_x R, R^{\mathsf{T}}\partial_y R, R^{\mathsf{T}}\partial_z R)$ and to derive the equations of balance of forces (variations with respect to the deformation φ, the force–stress tensor may lose symmetry [57]) and balance of angular momentum (variations with respect to rotations R). The Cosserat brothers, however, did not provide any specific constitutive form of the energy since they were not interested in applications.

While the appearance of an additional rotational field R for describing the elastic response of bulk material requires getting used to, such an appearance is most natural in the case of shell theory. There, the Frenet–Darboux trièdre [15] (trièdre caché in the terminology of the Cosserats, trihedron) naturally plays a role and it is no big step to assume that this orthogonal field is supposed to be kinematically independent of the former (trièdre mobile). Hence the Cosserat approach [15]; the independent rotation field R describes the rotations of the cross-sections of the shell (including in-plane drill rotations about the

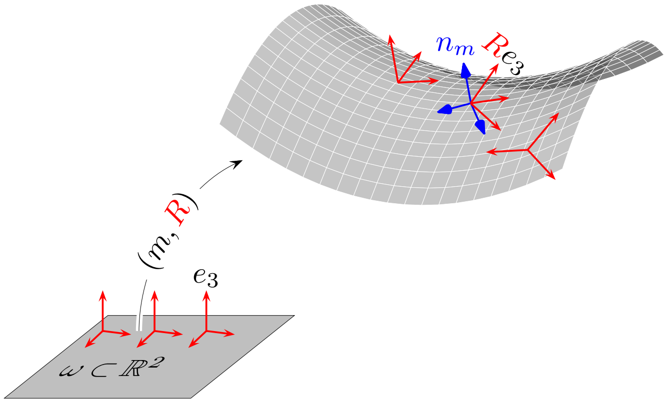


Figure 1. The mapping $m\colon \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ describes the deformation of the flat midsurface $\omega \subset \mathbb{R}^2$. The Frenet–Darboux frame (in blue, trièdre caché) is tangent to the midsurface. The independent frame mapped by $R \in \mathrm{SO}(3)$ is the trièdre mobile (in red, not necessary tangent to the midsurface). Both fields m and R are coupled in the variational problem.

normal n_m to the midsurface m) and these cross-sections are all allowed to shear with respect to the normal of the midsurface ($Re_3 \neq n_m$). See Figure 1.

On this basis, very efficient ad hoc Cosserat shell models have been introduced; see e.g. [2, 3]. A special case of these shell models is the family of Reissner–Mindlin shells in which the in-plane rotations are discarded (no drill energy) [36] and one is left with a one director theory [38].[1] Upon identifying/constraining the trièdre mobile with the trièdre caché (microrotation equals continuum rotation, Cosserat couple modulus $\mu_c \to \infty$), canonical shell models of Kirchhoff–Love type emerge [43]. However, engineers would often prefer the Cosserat shell models since these yield nonlinear balance equations of second order [7, 34, 62, 68, 73, 74].

The precise derivation of Cosserat shell models may proceed in several different ways: integration of equilibrium equations through the thickness [18, 62], direct modeling as a two-dimensional directed surface [2, 3, 28], or the derivation approach, which starts from a three-dimensional variational problem and introduces certain assumptions for the deformation behavior through the thickness. The second author has introduced this derivation procedure based on the geometrically nonlinear Cosserat model in his habilitation thesis [49, 51]. Lastly, there is the "ansatz-free" method of Γ-convergence [8, 11, 12] (while letting the thickness h tend to zero) to perform the dimensional descent.

In this method, one needs to choose an energy scaling regime, and typically one obtains either membrane or bending-like theories [22, 38–40] when starting from clas-

[1] One-director geometrically nonlinear, physically linear Reissner–Mindlin shells are typically not well posed, since the membrane stretch energy part depends quadratically on $\mathrm{D}m^\top \mathrm{D}m - \mathbb{1}_2$, which is not rank-one elliptic in the compression regime. For a more detailed exposition, see the appendix.

sical finite strain elasticity [21–23]. However, the Γ-limit membrane model [38, 39] has a serious shortcoming which is connected to the necessary relaxation step: it does not predict any resistance against compression and averages out the expected fine-scale wrinkling response. The situation is strikingly different when starting from a three-dimensional Cosserat model, as done in [52]. This is true since the bulk-Cosserat model already features a curvature term (derivatives of R) which "survives" the membrane scaling.

The Cosserat membrane Γ-limit with remaining curvature effects can be used as an effective surrogate model to describe ultra-thin graphene mono-layers. Graphene is the name given to a single atomic layer of carbon atoms tightly packed into a two-dimensional honeycomb lattice (see Figure 2). It can be wrapped up to form fullerenes, rolled into nanotubes [75], or stacked into graphite. Its stiffness properties are extreme. Such a graphene layer has resistance against in-plane stretch and curvature changes but its thickness is so small that a classical membrane-bending model (where the bending terms scale with h^3 while the membrane terms scale with h) is clearly insufficient. It is simply impossible to speak about the "thickness" of graphene in a classical continuum framework. Researchers then usually resort to introducing an "effective bending rigidity" in order to apply concepts from classical shell theory. This can be completely avoided in the Cosserat membrane model.

In this paper we will consider, for the first time, the challenging regularity questions for the flat shell Cosserat membrane Γ-limit. To the best of the authors' knowledge, such a regularity investigation for the flat Cosserat membrane shell has never been undertaken. Two recent previous contributions consider the regularity issue for the geometrically isotropic nonlinear Cosserat bulk equations [24, 41], both times restricting attention to the uni-constant Dirichlet curvature energy $|\mathrm{D}R|^2$, leading to a ΔR-term in the Euler–Lagrange equations and allowing the sophisticated techniques for harmonic-map-type systems to be used.

This paper is structured as follows. After this introduction and the introduction of our notation, in Section 3 we will introduce the three-dimensional isotropic Cosserat model, together with a short discussion of suitable representations for the curvature term. Following, in Section 4, we briefly describe the dimensional descent towards a membrane shell, juxtaposing the result of the Γ-limit procedure and a formal engineering approach. In Section 5 we introduce the final two-dimensional Cosserat membrane shell model, together with some pertinent notation and simplifications. The remainder of the paper is devoted to showing the interior Hölder regularity of these weak solutions. In the appendix we gather further useful calculations, like the three-dimensional Euler–Lagrange equations in dislocation tensor format. We present a more engineering-oriented derivation of the two-dimensional Euler–Lagrange equations and give a glimpse of a related Reissner–Mindlin model. Finally, we show some numerical experiments on the flat Cosserat membrane shell model in compression.

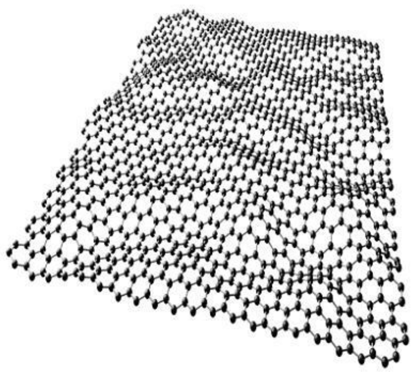


Figure 2. A deformed graphene mono-atomic layer resisting in-plane stretches (membrane effects) and curvature. Classical continuum models are no longer suitable, since there is no tangible thickness, cf. [75]. Graphene is thought to be the strongest among all known materials. Nevertheless, it is soft in the sense that it can be easily bent due to its one-atom-thin nature.

2. Notation

Let $a, b \in \mathbb{R}^3$. We denote the scalar product on $\mathbb{R}^3$ with $\langle a, b\rangle_{\mathbb{R}^3}$ and the associated vector norm by $|a|^2_{\mathbb{R}^3} = \langle a, a\rangle_{\mathbb{R}^3}$. The set of real-valued 3×3 second-order tensors is denoted by $\mathbb{R}^{3\times 3}$.

The standard Euclidean scalar product on $\mathbb{R}^{3\times 3}$ is given by $\langle X, Y\rangle_{\mathbb{R}^{3\times 3}} = \operatorname{tr}(XY^\intercal)$, and the associated norm is $|X|^2 = \langle X, X\rangle_{\mathbb{R}^{3\times 3}}$. If $\mathbb{1}_3$ denotes the identity matrix in $\mathbb{R}^{3\times 3}$, we have $\operatorname{tr}(X) = \langle X, \mathbb{1}_3\rangle$. For an arbitrary matrix $X \in \mathbb{R}^{3\times 3}$ we define $\operatorname{sym}(X) = \frac{1}{2}(X + X^\intercal)$ and $\operatorname{skew}(X) = \frac{1}{2}(X - X^\intercal)$ as the symmetric and skew-symmetric parts, respectively and the trace-free deviatoric part is defined as $\operatorname{dev} X = X - \frac{1}{n}\operatorname{tr}(X)\mathbb{1}_n$ for all $X \in \mathbb{R}^{n\times n}$. We let $\operatorname{Sym}(n)$ and $\operatorname{Sym}^+(n)$ denote the symmetric and positive definite symmetric tensors, respectively. The Lie algebra of skew-symmetric matrices is denoted by $\mathfrak{so}(3) := \{X \in \mathbb{R}^{3\times 3} \mid X^\intercal = X\}$ and the Lie algebra of traceless tensors is defined by $\mathfrak{sl}(3) := \{X \in \mathbb{R}^{3\times 3} \mid \operatorname{tr}(X) = 0\}$. We consider the orthogonal decomposition $X = \operatorname{dev}\operatorname{sym} X + \operatorname{skew} X + \frac{1}{3}\operatorname{tr}(X)\cdot \mathbb{1}_3 = \operatorname{sym} X + \operatorname{skew} X$. The *canonical identifications* of $\mathfrak{so}(3)$ and $\mathbb{R}^3$ are given by $\operatorname{axl}\colon \mathfrak{so}(3) \to \mathbb{R}^3$ and its inverse $\operatorname{Anti}\colon \mathbb{R}^3 \to \mathfrak{so}(3)$. We note the following properties:

$$\operatorname{axl}\underbrace{\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}}_{=A} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad |A|^2_{\mathbb{R}^{3\times 3}} = 2|\operatorname{axl} A|^2_{\mathbb{R}^3}, \quad Av = \operatorname{axl}(A) \times v,$$

and

$$\operatorname{Anti}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \in \mathfrak{so}(3).$$

A matrix having the three column vectors R_1, R_2, R_3 will be written sometimes as $R = (R_1|R_2|R_3) \in \mathbb{R}^{3\times3}$. The matrix Curl and matrix Div are defined row-wise as

$$\operatorname{Curl} R = \begin{pmatrix} \operatorname{curl}(R^{\mathsf{T}}\cdot e_1) \\ \operatorname{curl}(R^{\mathsf{T}}\cdot e_2) \\ \operatorname{curl}(R^{\mathsf{T}}\cdot e_3) \end{pmatrix} \in \mathbb{R}^{3\times3}, \quad \operatorname{Div} R = \begin{pmatrix} \operatorname{div}(R^{\mathsf{T}}\cdot e_1) \\ \operatorname{div}(R^{\mathsf{T}}\cdot e_2) \\ \operatorname{div}(R^{\mathsf{T}}\cdot e_3) \end{pmatrix}.$$

For $\varphi \in C^1(\mathcal{U}, \mathbb{R}^3)$ and for every vector $(x, y, z) \in \mathbb{R}^3$, we write

$$\mathrm{D}\varphi = \begin{pmatrix} \varphi_{1,x} & \varphi_{1,y} & \varphi_{1,z} \\ \varphi_{2,x} & \varphi_{2,y} & \varphi_{2,z} \\ \varphi_{3,x} & \varphi_{3,y} & \varphi_{3,z} \end{pmatrix} = (\partial_x\varphi|\partial_y\varphi|\partial_z\varphi).$$

The mapping $m\colon \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ will always denote the deformation of the midsurface ω and we write

$$\mathrm{D}m = \begin{pmatrix} m_{1,x} & m_{1,y} \\ m_{2,x} & m_{2,y} \\ m_{3,x} & m_{3,y} \end{pmatrix} = (\partial_x m|\partial_y m), \quad \mathrm{D}^{\perp}m = \begin{pmatrix} -m_{1,y} & m_{1,x} \\ -m_{2,y} & m_{2,x} \\ -m_{3,y} & m_{3,x} \end{pmatrix} = (-\partial_y m|\partial_x m).$$

Moreover, we will use the notation

$$\operatorname{Div}(A_1|A_2) = \partial_x A_1 + \partial_y A_2, \quad \operatorname{Div}^{\perp}(A_1|A_2) = \partial_x A_2 - \partial_y A_1,$$

where A_1, A_2 may be number-, vector-, or matrix-valued functions on ω of the same type. Note that it is also customary to write Curl instead of $\operatorname{Div}^{\perp}$, but the latter underscores the symmetry of $(\mathrm{D}, \operatorname{Div})$ with $(\mathrm{D}^{\perp}, \operatorname{Div}^{\perp})$, hence we reserve Curl for three-dimensional domains.

We assume that $h > 0$ with $h \ll 1$. The three-dimensional thin flat domain $\mathcal{U}_h \subset \mathbb{R}^3$ is introduced as

$$\mathcal{U}_h := \omega \times \Big[-\frac{h}{2}, \frac{h}{2}\Big], \quad \omega \subset \mathbb{R}^2.$$

We also need to define the projection operator on the first two columns,

$$\pi_{12}\colon \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times2}, \quad \pi_{12}(X) = \pi_{12}(X_1|X_2|X_3) = (X_1|X_2) = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \\ X_{31} & X_{32} \end{pmatrix},$$

and the operator

$$\mathbb{P}_{\mu,\mu_c,\kappa}\colon \mathbb{R}^{3\times3} \to \big\{(3) \cap \operatorname{Sym}(3)\big\} \oplus \mathfrak{so}(3) \oplus \mathbb{R}\cdot \mathbb{1}_3,$$

$$\mathbb{P}_{\mu,\mu_c,\kappa}(X) = \mathbb{P}(X) = \sqrt{\mu}\,\operatorname{dev}\operatorname{sym} X + \sqrt{\mu_c}\,\operatorname{skew} X + \frac{\sqrt{\kappa}}{3}\operatorname{tr}(X)\cdot \mathbb{1}_3,$$

$$\mathbb{P}^*\mathbb{P}(X) = \mathbb{P}^2(X) = \mu\,\operatorname{dev}\operatorname{sym} X + \mu_c\,\operatorname{skew} X + \frac{\sqrt{\kappa}}{3}\operatorname{tr}(X)\mathbb{1}_3, \quad \mathbb{P}^* = \mathbb{P}.$$

3. Three-dimensional geometrically nonlinear isotropic Cosserat model

The underlying three-dimensional isotropic Cosserat model can be described in terms of the standard deformation mapping $\varphi\colon \mathcal{U} \subset \mathbb{R}^3 \to \mathbb{R}^3$ and an additional orthogonal microrotation tensor $R\colon \mathcal{U} \subset \mathbb{R}^3 \to \mathrm{SO}(3)$.

The goal is to find a minimizer of the following isotropic energy:

$$
\begin{aligned}
&E^{3\mathrm{D}}(\varphi, R)\\
&= \int_{\mathcal{U}} \mu|\operatorname{dev}\operatorname{sym}(R^{\mathsf{T}}\mathrm{D}\varphi - \mathbb{1}_3)|^2 + \mu_c|\operatorname{skew}(R^{\mathsf{T}}\mathrm{D}\varphi - \mathbb{1}_3)|^2 + \frac{\kappa_{3\mathrm{D}}}{2}\operatorname{tr}(R^{\mathsf{T}}\mathrm{D}\varphi - \mathbb{1}_3)^2\\
&\quad + \mu\frac{L_c^2}{2}\Big(a_1|\operatorname{dev}\operatorname{sym} R^{\mathsf{T}}\operatorname{Curl} R|^2 + a_2|\operatorname{skew} R^{\mathsf{T}}\operatorname{Curl} R|^2 + \frac{a_3}{3}\operatorname{tr}(R^{\mathsf{T}}\operatorname{Curl} R)^2\Big)\,\mathrm{dx}\\
&= \int_{\mathcal{U}} \mu|\operatorname{sym}(R^{\mathsf{T}}\mathrm{D}\varphi - \mathbb{1}_3)|^2 + \mu_c|\operatorname{skew}(R^{\mathsf{T}}\mathrm{D}\varphi - \mathbb{1}_3)|^2 + \frac{\lambda}{2}\operatorname{tr}(R^{\mathsf{T}}\mathrm{D}\varphi - \mathbb{1}_3)^2\\
&\quad + \mu\frac{L_c^2}{2}\Big(a_1|\operatorname{dev}\operatorname{sym} R^{\mathsf{T}}\operatorname{Curl} R|^2 + a_2|\operatorname{skew} R^{\mathsf{T}}\operatorname{Curl} R|^2 + \frac{a_3}{3}\operatorname{tr}(R^{\mathsf{T}}\operatorname{Curl} R)^2\Big)\,\mathrm{dx}\\
&= \int_{\mathcal{U}} W_{\mathrm{mp}}(R^{\mathsf{T}}\mathrm{D}\varphi) + W^{3\mathrm{D}}_{\mathrm{disloc}}(R^{\mathsf{T}}\operatorname{Curl} R)\,\mathrm{dx} \to \min \quad \text{w.r.t. } (\varphi, R).
\end{aligned}
\tag{3.1}
$$

The problem will be supplemented by Dirichlet boundary conditions for the deformation φ but the microrotations R can be left free. Here, $\mu > 0$ is the standard elastic shear modulus, $\kappa_{3\mathrm{D}} = \frac{3\lambda+2\mu}{3} > 0$ is the three-dimensional elastic bulk modulus (with λ the second elastic Lamé parameter), $\mu_c \geq 0$ is the so-called Cosserat couple modulus, a_1, a_2, a_3 are nondimensional nonnegative weights, and $L_c > 0$ is a characteristic length. The energy (3.1) is the most general isotropic quadratic representation for the Cosserat model in terms of the nonsymmetric Biot-type stretch tensor $\overline{U} = R^{\mathsf{T}}\mathrm{D}\varphi$ (first Cosserat deformation tensor [17]) and the curvature measure $R^{\mathsf{T}}\operatorname{Curl} R$ (physically linear, small strain, but geometrically nonlinear). We call

$$\boldsymbol{\alpha} := R^{\mathsf{T}}\operatorname{Curl} R,$$

the *second-order* dislocation density tensor [10]. Due to the orthogonality of dev sym, skew, and $\operatorname{tr}(.)\mathbb{1}_3$, the curvature energy provides complete control of

$$|\boldsymbol{\alpha}|^2 = |R^{\mathsf{T}}\operatorname{Curl} R|^2 \quad \text{provided } a_1, a_2, a_3 > 0.$$

For example, we can express the uni-constant isotropic curvature term

$$
\begin{aligned}
&|\mathrm{D}R|^2_{\mathbb{R}^{3\times3\times3}}\\
&\quad = |R^{\mathsf{T}}\mathrm{D}R|^2_{\mathbb{R}^{3\times3\times3}} = |R^{\mathsf{T}}\partial_x R|^2_{\mathbb{R}^{3\times3}} + |R^{\mathsf{T}}\partial_y R|^2_{\mathbb{R}^{3\times3}} + |R^{\mathsf{T}}\partial_z R|^2_{\mathbb{R}^{3\times3}}\\
&\quad = 1\cdot|\operatorname{dev}\operatorname{sym} R^{\mathsf{T}}\operatorname{Curl} R|^2_{\mathbb{R}^{3\times3}} + 1\cdot|\operatorname{skew} R^{\mathsf{T}}\operatorname{Curl} R|^2_{\mathbb{R}^{3\times3}} + \frac{1}{12}\cdot\operatorname{tr}(R^{\mathsf{T}}\operatorname{Curl} R)^2\\
&\quad = |\mathbb{P}_{1,1,\frac{1}{12}}(\alpha)|^2,
\end{aligned}
$$

where we have used (4.3) and $|\boldsymbol{\Gamma}|^2 = |\mathrm{axl}(R^\top \partial_x R)|^2 + |\mathrm{axl}(R^\top \partial_y R)|^2 + |\mathrm{axl}(R^\top \partial_z R)|^2$, together with $2|\mathrm{axl}(A)|^2_{\mathbb{R}^3} = |A|^2_{\mathbb{R}^{3\times 3}}$. Using the result in [60]

$$|\mathrm{Curl}\, R|^2_{\mathbb{R}^{3\times 3}} \geq c^+ |\mathrm{D}R|^2_{\mathbb{R}^{3\times 3\times 3}},$$

shows that (3.1) controls $\mathrm{D}R$ in $L^2(\mathcal{U}, \mathbb{R}^{3\times 3\times 3})$.

In this setting, the minimization problem is strictly convex in the strain and curvature measures $(\overline{U}, \boldsymbol{\alpha})$ but highly nonconvex with respect to (φ, R). Existence of minimizers for (3.1) with $\mu_c > 0$ was shown first in [50]; see also [10, 19, 37, 42, 50, 52, 54]. The partial regularity of minimizers/stationary solutions is investigated in [24, 41] under additional assumptions. Note also that in [24], the first author gives an example of a solution that exhibits a point singularity.

The Cosserat couple modulus μ_c controls the deviation of the microrotation R from the continuum rotation $\mathrm{polar}(\mathrm{D}\varphi)$ in the polar decomposition of $\mathrm{D}\varphi = \mathrm{polar}(\mathrm{D}\varphi) \cdot \sqrt{\mathrm{D}\varphi^\top \mathrm{D}\varphi}$; cf. [59].

For $\mu_c \to \infty$ the constraint $R = \mathrm{polar}(\mathrm{D}\varphi)$ is generated and the model would turn into a Toupin couple stress model.

3.1. Connections to the Oseen–Frank energy in nematic liquid crystals

In nematic liquid crystals one considers the unit-director field $n\colon \mathcal{U} \subset \mathbb{R}^3 \to \mathbb{S}^2$, minimizing the three-parameter frame-indifferent "curvature energy" [72]

$$\int_{\mathcal{U}} \frac{1}{2} K_1 |\mathrm{div}\, n|^2 + \frac{1}{2} K_2 |\langle n, \mathrm{curl}\, n\rangle|^2 + \frac{1}{2} K_3 |n \times \mathrm{curl}\, n|^2 \,\mathrm{dx}. \tag{3.2}$$

The uni-constant approximation $K_1 = K_2 = K_3$ leads to the Dirichlet-type integral[2]

$$\int_{\mathcal{U}} \frac{1}{2} K_1 |\mathrm{D}n|^2 \,\mathrm{dx}. \tag{3.3}$$

The corresponding Euler–Lagrange equations for the uni-constant case are (see e.g. [1])

$$\Delta n = -|\mathrm{D}n|^2 \cdot n; \tag{3.4}$$

see equation (A.21) for a self-contained derivation. Since (3.3) and (3.4) are just the energy and Euler–Lagrange equations for harmonic maps to spheres, all regularity theorems for harmonic maps apply. In the three-dimensional case, minimizers are smooth up to a discrete set of singularities. Stationary solutions have a co-dimension 1 singular set. In the two-dimensional case, all weak solutions of (3.4) are smooth; see Section 1.1 for the literature on this.

For K_1, K_2, K_3 positive and different, any minimizer to (3.2) is smooth except for a closed set of Hausdorff dimension strictly less than 1; cf. [30]. Ball and Bedford [6] consider the sublinear regime $|\mathrm{D}n|^q$, $1 < q < 2$.

[2]For this, we note the identity (see [5, eq. (2.5)] and [1, eq. (2.6)])

$$\mathrm{tr}(\mathrm{D}v)^2 + \langle v, \mathrm{curl}\, v\rangle^2 + |v \times \mathrm{curl}\, v|^2 = |\mathrm{D}v|^2 + (|v|^2 - 1)|\mathrm{curl}\, v|^2,$$

valid for all sufficiently smooth vector fields $v\colon \mathcal{U} \subset \mathbb{R}^3 \to \mathbb{R}^3$.

4. Dimensional descent towards a membrane model

4.1. Membrane Γ-limit

We are interested in a situation where the reference configuration is flat with uniform shell thickness $h > 0$, i.e. the reference configuration is taken to be of the form (see Figure 3)

$$\mathcal{U}_h = \omega \times \left[-\frac{h}{2}, \frac{h}{2}\right], \quad \omega \subset \mathbb{R}^2.$$

The goal is to derive a limit two-dimensional problem, posed over the referential midsurface $\omega \subset \mathbb{R}^2$, as $h \to 0$. This has been achieved in [58] based on Γ-convergence arguments and using the nonlinear membrane scaling. We say that the dimensionally reduced model is a membrane, since no dedicated bending terms appear in the problem.

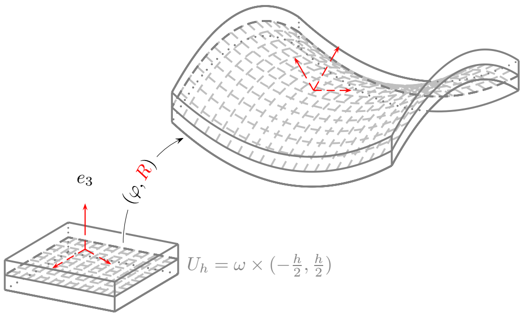


Figure 3. Process of dimensional reduction. Flat reference configuration with height h and deformed configuration.

However, since the Cosserat model already includes curvature terms (those depending on space derivatives $\mathrm{D}R$), these curvature terms "survive" in the Γ-limit procedure and scale with h, while canonical bending terms scale with h^3. This sets the Cosserat membrane model apart from more canonical membrane models [55].

For the Γ-limit procedure it is useful to re-express the curvature energy from (3.1),

$$\begin{aligned} &\mu \frac{L_c^2}{2}\Big(a_1|\operatorname{dev}\operatorname{sym}(R^\top \operatorname{Curl} R)|^2 + a_2|\operatorname{skew}(R^\top \operatorname{Curl} R)|^2 + \frac{a_3}{3}\operatorname{tr}(R^\top \operatorname{Curl} R)^2\Big) \\ &\quad = \mu \frac{L_c^2}{2}\Big(a_1|\operatorname{dev}\operatorname{sym}\boldsymbol{\alpha}|^2 + a_2|\operatorname{skew}\boldsymbol{\alpha}|^2 + \frac{a_3}{3}\operatorname{tr}(\boldsymbol{\alpha})^2\Big) = \mu\frac{L_c^2}{2}|\mathbb{P}_{a_1,a_2,a_3}(\alpha)|^2 \\ &\quad =: W_{\text{disloc}}^{\text{3D}}(\boldsymbol{\alpha}), \end{aligned} \tag{4.1}$$

in terms of the so-called second-order *wryness tensor* [18, 60] (second Cosserat deformation tensor [17])

$$\boldsymbol{\Gamma} := (\operatorname{axl}(\underbrace{R^\top \partial_x R}_{\in \mathfrak{so}(3)})|\operatorname{axl}(R^\top \partial_y R)|\operatorname{axl}(R^\top \partial_z R)) = (\Gamma_1|\Gamma_2|\Gamma_3) \in \mathbb{R}^{3\times 3}. \tag{4.2}$$

Since $R^{\mathsf{T}}\partial_{x_i} R \in \mathfrak{so}(3)$, $i = 1, 2, 3$ is skew-symmetric, we have the following relations [25, 26, 61]:

$$\boldsymbol{\Gamma} = -\boldsymbol{\alpha}^{\mathsf{T}} + \frac{1}{2}\operatorname{tr}(\boldsymbol{\alpha})\mathbb{1}_3, \quad \boldsymbol{\alpha} = -\boldsymbol{\Gamma}^{\mathsf{T}} + \operatorname{tr}(\boldsymbol{\Gamma})\mathbb{1}_3. \tag{4.3}$$

By using these formulas we note

$$\operatorname{dev}\operatorname{sym}\boldsymbol{\alpha} = -\operatorname{dev}\operatorname{sym}\boldsymbol{\Gamma}, \quad \operatorname{skew}\boldsymbol{\alpha} = \operatorname{skew}\boldsymbol{\Gamma}, \quad \operatorname{tr}(\boldsymbol{\alpha}) = 2\operatorname{tr}(\boldsymbol{\Gamma}).$$

Now using (4.1), we obtain

$$\begin{aligned}
W^{3D}_{\text{disloc}}(\boldsymbol{\alpha}) &= \mu\frac{L_c^2}{2}\Big(a_1|\operatorname{dev}\operatorname{sym}\boldsymbol{\alpha}|^2 + a_2|\operatorname{skew}\boldsymbol{\alpha}|^2 + \frac{a_3}{3}\operatorname{tr}(\boldsymbol{\alpha})^2\Big)\\
&= \mu\frac{L_c^2}{2}\Big(a_1|\operatorname{dev}\operatorname{sym}\boldsymbol{\Gamma}|^2 + a_2|\operatorname{skew}\boldsymbol{\Gamma}|^2 + 4a_3\operatorname{tr}(\boldsymbol{\Gamma})^2\Big)\\
&= \mu\frac{L_c^2}{2}(\tilde{a}_1|\operatorname{dev}\operatorname{sym}\boldsymbol{\Gamma}|^2 + \tilde{a}_2|\operatorname{skew}\boldsymbol{\Gamma}|^2 + \tilde{a}_3\operatorname{tr}(\boldsymbol{\Gamma})^2)\\
&=: W^{3D}_{\text{curv}}(\boldsymbol{\Gamma}),
\end{aligned}$$

where $\tilde{a}_1 = a_1$, $\tilde{a}_2 = a_2$, and $\tilde{a}_3 = 4a_3$. Altogether we get

$$\begin{aligned}
W^{3D}_{\text{disloc}}(\boldsymbol{\alpha}) = W^{3D}_{\text{curv}}(\boldsymbol{\Gamma}) &= \mu\frac{L_c^2}{2}(\tilde{a}_1|\operatorname{dev}\operatorname{sym}\boldsymbol{\Gamma}|^2 + \tilde{a}_2|\operatorname{skew}\boldsymbol{\Gamma}|^2 + \tilde{a}_3\operatorname{tr}(\boldsymbol{\Gamma})^2)\\
&= \mu\frac{L_c^2}{2}(\tilde{b}_1|\operatorname{sym}\boldsymbol{\Gamma}| + \tilde{b}_2|\operatorname{skew}\boldsymbol{\Gamma}|^2 + \tilde{b}_3\operatorname{tr}(\boldsymbol{\Gamma})^2),
\end{aligned}$$

with $\tilde{a}_1 = \tilde{b}_1 > 0$, $\tilde{a}_2 = \tilde{b}_2 > 0$, and $\tilde{b}_3 = \frac{\tilde{a}_1}{3} + \tilde{a}_3 > 0$. Thus, the variational problem (3.1) can be equivalently expressed as

$$E^{3D}(\varphi, R) = \int_{\mathcal{U}_h} W_{\text{mp}}(R^{\mathsf{T}}\mathrm{D}\varphi) + W^{3D}_{\text{curv}}(\boldsymbol{\Gamma})\,\mathrm{dx} \to \min \quad \text{w.r.t. } (\varphi, R).$$

Applying the nonlinear scaling [20], allows one to rewrite the problem on a domain $\mathcal{U}_1 = \omega \times [-\frac{1}{2}, \frac{1}{2}]$ with unit thickness in terms of properly scaled variables $\varphi^{\natural}$, $R^{\natural}$ in the (thickness) z-direction

$$E^{3D}_h(\varphi^{\natural}, R^{\natural}) = \int_{\mathcal{U}_1} W_{\text{mp}}(R^{\natural,T}\mathrm{D}\varphi^{\natural}) + W^{3D}_{\text{curv}}(\boldsymbol{\Gamma}^{\natural})\,\mathrm{dx}.$$

The descaled Γ-limit of E^{3D}_h as $h \to 0$ is then given by [52]

$$E^{2D}(m, R) = \int_{\omega} h\big(W^{\text{hom}}_{\text{mp}}(R^{\mathsf{T}}(\mathrm{D}m|R_3)) + W^{\text{hom}}_{\text{curv}}(\widehat{\boldsymbol{\Gamma}})\big)\,\mathrm{dx} \to \min \quad \text{w.r.t. } (m, R),$$

where $m\colon \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ describes the deformation of the midsurface, $R\colon \omega \subset \mathbb{R}^2 \to \mathrm{SO}(3)$, and

$$
\begin{aligned}
W_{\mathrm{mp}}^{\mathrm{hom}}(R^{\mathsf{T}}(\mathrm{D}m|R_3)) &:= \inf_{d\in\mathbb{R}^3} W_{\mathrm{mp}}(R^{\mathsf{T}}(\mathrm{D}m|d)) \\
&= \mu|\mathrm{sym}((R_1|R_2)^{\mathsf{T}}\mathrm{D}m - \mathbb{1}_2)|^2 + \mu_c|\mathrm{skew}((R_1|R_2)^{\mathsf{T}}\mathrm{D}m - \mathbb{1}_2)|^2 \\
&\quad + \frac{2\mu\mu_c}{\mu+\mu_c}(\langle R_3, \partial_x m\rangle^2 + \langle R_3, \partial_y m\rangle^2) \\
&\quad + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}((R_1|R_2)^{\mathsf{T}}\mathrm{D}m - \mathbb{1}_2)^2, \\
W_{\mathrm{curv}}^{\mathrm{hom}}(\widehat{\mathbf{\Gamma}}) &:= \inf_{A\in\mathfrak{so}(3)} W_{\mathrm{curv}}^{\mathrm{3D}}(\mathrm{axl}(R^{\mathsf{T}}\partial_x R), \mathrm{axl}(R^{\mathsf{T}}\partial_y R), \mathrm{axl}(A)) \\
&= \mu\frac{L_c^2}{2}\left(\tilde{b}_1\left|\mathrm{sym}\begin{pmatrix}\widehat{\Gamma}_{11} & \widehat{\Gamma}_{12} \\ \widehat{\Gamma}_{21} & \widehat{\Gamma}_{22}\end{pmatrix}\right|^2 + \tilde{b}_2\left|\mathrm{skew}\begin{pmatrix}\widehat{\Gamma}_{11} & \widehat{\Gamma}_{12} \\ \widehat{\Gamma}_{21} & \widehat{\Gamma}_{22}\end{pmatrix}\right|^2\right. \\
&\quad \left. + \frac{\tilde{b}_1\tilde{b}_3}{\tilde{b}_1+\tilde{b}_3}\,\mathrm{tr}\begin{pmatrix}\widehat{\Gamma}_{11} & \widehat{\Gamma}_{12} \\ \widehat{\Gamma}_{21} & \widehat{\Gamma}_{22}\end{pmatrix}^2 + 2\frac{\tilde{b}_1\tilde{b}_2}{\tilde{b}_1+\tilde{b}_2}\left|\begin{pmatrix}\widehat{\Gamma}_{31} \\ \widehat{\Gamma}_{32}\end{pmatrix}\right|^2\right),
\end{aligned}
$$

where the matrix $\widehat{\mathbf{\Gamma}} = (\mathrm{axl}(R^{\mathsf{T}}\partial_x R)|\mathrm{axl}(R^{\mathsf{T}}\partial_y R)) = \pi_{12}(\mathbf{\Gamma})$ is in the form (see [43])

$$
\widehat{\mathbf{\Gamma}} = (\mathrm{axl}(R^{\mathsf{T}}\partial_x R)|\mathrm{axl}(R^{\mathsf{T}}\partial_y R)) = \begin{pmatrix}\widehat{\Gamma}_{11} & \widehat{\Gamma}_{12} \\ \widehat{\Gamma}_{21} & \widehat{\Gamma}_{22} \\ \widehat{\Gamma}_{31} & \widehat{\Gamma}_{32}\end{pmatrix} \in \mathbb{R}^{3\times 2}.
$$

We set $\mathbf{\Gamma}_{\square} = \begin{pmatrix}\widehat{\Gamma}_{11} & \widehat{\Gamma}_{12} \\ \widehat{\Gamma}_{21} & \widehat{\Gamma}_{22}\end{pmatrix}$ and $\mathbf{\Gamma}_{\perp} = \begin{pmatrix}\widehat{\Gamma}_{31} \\ \widehat{\Gamma}_{32}\end{pmatrix}$. Thus we can write the Γ-limit minimization problem as[3]

$$
\begin{aligned}
E_{\mathbf{\Gamma}\text{-lim}}^{\mathrm{2D}}(m,R) = \int_\omega h\Big\{&\mu|\mathrm{sym}((R_1|R_2)^{\mathsf{T}}\mathrm{D}m - \mathbb{1}_2)|^2 + \mu_c|\mathrm{skew}((R_1|R_2)^{\mathsf{T}}\mathrm{D}m - \mathbb{1}_2)|^2 \\
&+ \frac{2\mu\mu_c}{\mu+\mu_c}(\langle R_3, \partial_x m\rangle^2 + \langle R_3, \partial_y m\rangle^2) \\
&+ \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}((R_1|R_2)^{\mathsf{T}}\mathrm{D}m - \mathbb{1}_2)^2 \\
&+ \mu\frac{L_c^2}{2}\Big(\tilde{b}_1|\mathrm{sym}\,\widehat{\mathbf{\Gamma}}_{\square}|^2 + \tilde{b}_2|\mathrm{skew}\,\widehat{\mathbf{\Gamma}}_{\square}|^2 + \frac{\tilde{b}_1\tilde{b}_3}{\tilde{b}_1+\tilde{b}_3}\,\mathrm{tr}(\widehat{\mathbf{\Gamma}}_{\square})^2 \\
&\qquad + 2\frac{\tilde{b}_1\tilde{b}_2}{\tilde{b}_1+\tilde{b}_2}|\widehat{\mathbf{\Gamma}}_{\perp}|^2\Big)\Big\}\,\mathrm{dx}. \qquad (4.4)
\end{aligned}
$$

[3] Note the fourfold appearance of the harmonic mean $\mathcal{H}$, i.e.

$$
\frac{2\mu\mu_c}{\mu+\mu_c} = \mathcal{H}(\mu,\mu_c), \quad \frac{\mu\lambda}{2\mu+\lambda} = \frac{1}{2}\mathcal{H}\Big(\mu,\frac{\lambda}{2}\Big), \quad \frac{\tilde{b}_1\tilde{b}_3}{\tilde{b}_1+\tilde{b}_3} = \frac{1}{2}\mathcal{H}(\tilde{b}_1,\tilde{b}_3), \quad \frac{2\tilde{b}_1\tilde{b}_2}{\tilde{b}_1+\tilde{b}_2} = \mathcal{H}(\tilde{b}_1,\tilde{b}_2).
$$

If we assume that in the underlying Cosserat bulk curvature energy we have the uni-constant expression

$$
\begin{aligned}
W_{\mathrm{curv}}^{\mathrm{3D}}(R^{\mathsf{T}}\mathrm{D}R) &= \frac{\mu L_c^2}{2}|R^{\mathsf{T}}\mathrm{D}R|^2 = \frac{\mu L_c^2}{2}|\mathrm{D}R|^2 \\
&= \mu L_c^2(|\mathrm{axl}(R^{\mathsf{T}}\partial_x R)|^2 + |\mathrm{axl}(R^{\mathsf{T}}\partial_y R)|^2 + |\mathrm{axl}(R^{\mathsf{T}}\partial_z R)|^2) \\
&= \frac{\mu L_c^2}{2}(|(R^{\mathsf{T}}\partial_x R)|^2 + |(R^{\mathsf{T}}\partial_y R)|^2 + |(R^{\mathsf{T}}\partial_z R)|^2),
\end{aligned}
$$

then the homogenized curvature energy is given by [10, 20, 65]

$$
\begin{aligned}
W_{\mathrm{curv}}^{\mathrm{hom}}(R^{\mathsf{T}}\mathrm{D}R) &= \inf_{A\in\mathfrak{so}(3)} W_{\mathrm{curv}}^{\mathrm{3D}}(\mathrm{axl}(R^{\mathsf{T}}\partial_x R), \mathrm{axl}(R^{\mathsf{T}}\partial_y R), \mathrm{axl}(A)) \\
&= \mu\frac{L_c^2}{2}(|R^{\mathsf{T}}\partial_x R|^2 + |R^{\mathsf{T}}\partial_y R|^2) = \mu\frac{L_c^2}{2}|R^{\mathsf{T}}\mathrm{D}R|^2 \\
&= \mu L_c^2|\widehat{\Gamma}|^2_{\mathbb{R}^{3\times 2}} = \mu L_c^2|\pi_{12}(\Gamma)|^2_{\mathbb{R}^{3\times 2}}.
\end{aligned}
$$

4.2. Alternative engineering ad hoc dimensional descent

In [49] the three-dimensional Cosserat model has been reduced to a flat shell problem by proposing an engineering ansatz for the deformation φ and the microrotation R over the shell thickness. Again we let $m\colon \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ denote the midsurface deformation, $\overline{U} := R^{\mathsf{T}}(\mathrm{D}m|R_3)$ the nonsymmetric membrane stretch tensor, and $R\colon \mathcal{U}_h \subset \mathbb{R}^2 \to \mathrm{SO}(3)$ the microrotation tensor field with $R^{\mathsf{T}}\mathrm{D}R \cong (R^{\mathsf{T}}\partial_x R, R^{\mathsf{T}}\partial_y R)$. Since we are only interested in the membrane-like response, we will neglect terms related to bending effects right away while keeping the curvature change[4] scaling with h.

The dimensionally reduced energy then reads [49, (4.5)]

$$
\begin{aligned}
E_{\mathrm{eng}}^{\mathrm{2D}} &= \int_\omega h\Big\{\mu|\mathrm{sym}(\overline{U}-\mathbb{1}_3)|^2 + \mu_c|\mathrm{skew}(\overline{U}-\mathbb{1}_3)|^2 \\
&\qquad + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}(\overline{U}-\mathbb{1}_3)^2 + \mu\frac{L_c^2}{2}|R^{\mathsf{T}}\mathrm{D}R|^2\Big\}\,\mathrm{dx} \\
&= \int_\omega h\Big\{\mu|\mathrm{dev}\,\mathrm{sym}(\overline{U}-\mathbb{1}_3)|^2 + \mu_c|\mathrm{skew}(\overline{U}-\mathbb{1}_3)|^2 \\
&\qquad + \underbrace{\left(\frac{\mu\lambda}{2\mu+3\lambda} + \frac{\mu}{3}\mu\right)}_{=:\frac{\kappa^{\mathrm{hom}}}{2}}\mathrm{tr}(\overline{U}-\mathbb{1}_3)^2 + \mu\frac{L_c^2}{2}|R^{\mathsf{T}}\mathrm{D}R|^2\Big\}\,\mathrm{dx}
\end{aligned}
$$

[4]The missing Cosserat bending terms scaling with h^3 are of the type [49, (4.5)]

$$
\frac{h^3}{12}\Big\{\mu|\mathrm{sym}(R^{\mathsf{T}}(\mathrm{D}R_3|0))|^2 + \mu_c|\mathrm{skew}(R^{\mathsf{T}}(\mathrm{D}R_3|0))|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\big(\mathrm{sym}(R^{\mathsf{T}}(\mathrm{D}R_3|0))\big)^2\Big\},
$$

and the uni-constant case would appear for $\mu = \mu_c$, $\lambda = 0$.

$$
\begin{aligned}
&= \int_\omega h\Big\{\mu|\operatorname{dev}\operatorname{sym}(R^T(\mathrm{D}m|R_3)-\mathbb{1}_3)|^2 + \mu_c|\operatorname{skew}(R^T(\mathrm{D}m|R_3)-\mathbb{1}_3)|^2 \\
&\qquad + \frac{\kappa^{\mathrm{hom}}}{2}\operatorname{tr}(R^T(\mathrm{D}m|R_3)-\mathbb{1}_3)^2 + \mu\frac{L_c^2}{2}|\mathrm{D}R|^2\Big\}\,\mathrm{dx} \\
&= \int_\omega h\Big\{\mu|\underbrace{\operatorname{sym}((R_1|R_2)^T\mathrm{D}m-\mathbb{1}_2)}_{\text{shear-stretch energy}}|^2 + \mu_c|\underbrace{\operatorname{skew}((R_1|R_2)^T\mathrm{D}m-\mathbb{1}_2)}_{\text{drill energy}}|^2 \\
&\qquad + \frac{\mu+\mu_c}{2}\underbrace{(\langle R_3,\partial_x m\rangle^2 + \langle R_3,\partial_y m\rangle^2)}_{\text{transverse shear energy}} + \frac{\mu\lambda}{2\mu+\lambda}\underbrace{\operatorname{tr}((R_1|R_2)^T\mathrm{D}m-\mathbb{1}_2)^2}_{\text{elongational stretch energy}} \\
&\qquad + \underbrace{\mu\,\frac{L_c^2}{2}|R^T\mathrm{D}R^2|}_{\text{curvature energy}}\Big\}\,\mathrm{dx}.
\end{aligned} \tag{4.5}
$$

Letting $\mu_c \to \infty$ in the reduced membrane model implies that $R_3 = n_m$ is normal to the midsurface m. Moreover, $\operatorname{skew}(R^T(\mathrm{D}m|n_m)) = 0$ implies $R = \operatorname{polar}(\mathrm{D}m|n_m)$ (trièdre caché).

In contrast to the representation of the energy in (4.5), the rigorously derived Γ-limit membrane model [55] has the energy (see equation (4.4))

$$
\begin{aligned}
E^{2\mathrm{D}}_{\boldsymbol{\Gamma}\text{-lim}}(m,R) = \int_\omega h\Big\{&\mu|\operatorname{sym}((R_1|R_2)^T\mathrm{D}m-\mathbb{1}_2)|^2 + \mu_c|\operatorname{skew}((R_1|R_2)^T\mathrm{D}m-\mathbb{1}_2)|^2 \\
&+ \frac{2\mu\mu_c}{\mu+\mu_c}(\langle R_3,\partial_x m\rangle^2 + \langle R_3,\partial_y m\rangle^2) \\
&+ \frac{\mu\lambda}{2\mu+\lambda}\operatorname{tr}((R_1|R_2)^T\mathrm{D}m-\mathbb{1}_2)^2 + W^{\mathrm{hom}}_{\mathrm{curv}}(\widehat{\boldsymbol{\Gamma}})\Big\}\,\mathrm{dx},
\end{aligned} \tag{4.6}
$$

where $\widehat{\boldsymbol{\Gamma}} = (\operatorname{axl}(R^T\partial_x R)|\operatorname{axl}(R^T\partial_y R))$. Thus, the engineering formulation in (4.5) coincides with the membrane Γ-limit if and only if

$$
\mathcal{A}(\mu,\mu_c) = \frac{\mu+\mu_c}{2} = \frac{2\mu\mu_c}{\mu+\mu_c} = \mathcal{H}(\mu,\mu_c) \iff \mu = \mu_c, \tag{4.7}
$$

and

$$
|R^T\mathrm{D}R|^2 = W^{\mathrm{hom}}_{\mathrm{curv}}(\widehat{\boldsymbol{\Gamma}}) \iff \tilde{b}_1 = \tilde{b}_2 = 1,\ \tilde{b}_3 = 0.
$$

In $(4.5)_2$, we are also led to define the appropriate modified bulk modulus κ^{hom} via[5]

$$
\frac{\kappa^{\mathrm{hom}}}{2} := \frac{\mu\lambda}{2\mu+\lambda} + \frac{\mu}{3} = \frac{2\mu}{3}\frac{2\lambda+\mu}{2\mu+\lambda} \quad \text{(effective two-dimensional bulk modulus).}
$$

[5] In linear elasticity theory for the displacement $u\colon \mathcal{U}\subset\mathbb{R}^3\to\mathbb{R}^3$, the common bulk modulus κ appears in the form $\mu|\operatorname{dev}\operatorname{sym}\mathrm{D}u|^2 + \frac{\kappa}{2}\operatorname{tr}(\mathrm{D}u)^2$ and not as $\mu|\operatorname{dev}\operatorname{sym}\mathrm{D}u|^2 + \frac{\kappa}{3}\operatorname{tr}(\mathrm{D}u)^2$, which would be more natural from the perspective of orthogonality of $\operatorname{dev}\operatorname{sym}\mathrm{D}u$ and $\operatorname{tr}(\mathrm{D}u)\cdot\mathbb{1}_3$.

Since we will need $\kappa^{\text{hom}} > 0$ for our subsequent regularity analysis, $(4.7)_2$ implies $2\mu + \lambda > 0$ and $2\lambda + \mu > 0$. One can show that the latter implies for the engineering Poisson number $\nu := \frac{\lambda}{2(\mu+\lambda)}$ the bound $\nu > -\frac{1}{2}$ (instead of $\nu > -1$ for three-dimensional linear elasticity).[6]

5. The two-dimensional Euler–Lagrange equations

Henceforth, we skip all unnecessary material parameters in (4.5) in order to arrive at a compact representation. Again, we consider the midsurface deformation $m\colon \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ and the orthogonal microrotation tensor $R\colon \omega \subset \mathbb{R}^2 \to \mathrm{SO}(3)$. We set $h = 1$ and assume the normalization $\mu \frac{L_c^2}{2} = 1$. Moreover, we set $\kappa = \frac{3\kappa^{\text{hom}}}{2}$. Thus, the corresponding energy function describing the two-dimensional membrane shell problem is

$$\begin{aligned} E(m, R) := \int_\omega & \mu |\operatorname{dev}\operatorname{sym}(R^\intercal(\mathrm{D}m|R_3) - \mathbb{1}_3)|^2 + \mu_c |\operatorname{skew}(R^\intercal(\mathrm{D}m|R_3) - \mathbb{1}_3)|^2 \\ & + \frac{\kappa}{3}\operatorname{tr}(R^\intercal(\mathrm{D}m|R_3) - \mathbb{1}_3)^2 + |\mathrm{D}R|^2 \,\mathrm{dx}. \end{aligned} \tag{5.1}$$

We assume μ, μ_c, κ to be positive. Remember that we have defined a linear operator $\mathbb{P}\colon \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ by

$$\mathbb{P}_{\mu,\mu_c,\kappa}(X) = \mathbb{P}(X) = \sqrt{\mu}\operatorname{dev}\operatorname{sym} X + \sqrt{\mu_c}\operatorname{skew} X + \frac{\sqrt{\kappa}}{3}(\operatorname{tr} X)\mathbb{1}_3.$$

Using the mutual orthogonality of $\operatorname{dev}\operatorname{sym} X$, $\operatorname{skew} X$, and $(\operatorname{tr} X)\mathbb{1}_3$, we can write down the functional in a simplified form: it reads

$$E(m, R) = \int_\omega |\mathbb{P}(R^\intercal(\mathrm{D}m|R_3) - \mathbb{1}_3)|^2 + |\mathrm{D}R|^2 \,\mathrm{dx}.$$

Now we are going to calculate the Euler–Lagrange equations for the dimensionally reduced problem based on E. The first variation of E in the direction of $(\vartheta, 0)\colon \mathcal{U} \to \mathbb{R}^3 \times \mathbb{R}^{3\times3}$ is

$$\begin{aligned} \delta E(m, R; \vartheta, 0) &= 2\int_\omega \langle \mathbb{P}(R^\intercal(\mathrm{D}m|R_3) - \mathbb{1}_3), \mathbb{P}(R^\intercal(\mathrm{D}\vartheta|0))\rangle \,\mathrm{dx} \\ &= 2\int_\omega \langle \mathbb{P}(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0)), \mathbb{P}(R^\intercal(\mathrm{D}\vartheta|0))\rangle \,\mathrm{dx}, \end{aligned}$$

and the first variation in the direction of $(0, Q)\colon \mathcal{U} \to \mathbb{R}^3 \times \mathbb{R}^{3\times3}$ with $Q(x) \in \mathrm{T}_R\,\mathrm{SO}(3)$ for almost all $x \in \mathcal{U}$ is

$$\begin{aligned} \delta E(m, R; 0, Q) &= 2\int_\omega [\langle \mathbb{P}(R^\intercal(\mathrm{D}m|R_3) - \mathbb{1}_3), \mathbb{P}(Q^\intercal(\mathrm{D}m|0))\rangle + \langle \mathrm{D}R, \mathrm{D}Q\rangle] \,\mathrm{dx} \\ &= 2\int_\omega [\langle \mathbb{P}(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0)), \mathbb{P}(Q^\intercal(\mathrm{D}m|0))\rangle + \langle \mathrm{D}R, \mathrm{D}Q\rangle] \,\mathrm{dx}. \end{aligned}$$

[6] $2\lambda + \mu > 0$ and $\mu > 0$ imply $2\lambda + 2\mu = 2(\lambda + \mu) > 0$. Therefore, $\nu = \frac{\lambda}{2(\mu+\lambda)} > -\frac{1}{2} \Leftrightarrow \frac{\lambda}{\mu+\lambda} > -1 \Leftrightarrow \lambda > -(\mu + \lambda) \Leftrightarrow 2\lambda + \mu > 0$.

Now using $\mathbb{P}^* = \mathbb{P}$ and $\mathbb{P}(X^\intercal) = \mathbb{P}(X)^\intercal$ such that $\mathbb{P}^*\mathbb{P} = \mathbb{P}^2$, and observing that $\pi_{12}(v_1|v_2|v_3) = (v_1|v_2)$, we rewrite these as

$$\begin{aligned}
\delta E(m, R; \vartheta, 0) &= 2\int_\omega \langle R\mathbb{P}^2(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0)), (\mathrm{D}\vartheta|0)\rangle_{\mathbb{R}^{3\times 3}}\, \mathrm{dx} \\
&= 2\int_\omega \big\langle \pi_{12}\big(R\mathbb{P}^2(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big), \mathrm{D}\vartheta\big\rangle_{\mathbb{R}^{3\times 2}}\, \mathrm{dx}, \\
\delta E(m, R; 0, Q) &= 2\int_\omega [\langle \mathbb{P}((\mathrm{D}m|0)^\intercal R - (\mathbb{1}_2|0)), \mathbb{P}((\mathrm{D}m|0)^\intercal Q)\rangle_{\mathbb{R}^{3\times 3}} + \langle \mathrm{D}R, \mathrm{D}Q\rangle]\, \mathrm{dx} \\
&= 2\int_\omega [\langle (\mathrm{D}m|0)\mathbb{P}^2((\mathrm{D}m|0)^\intercal R - (\mathbb{1}_2|0)), Q\rangle + \langle \mathrm{D}R, \mathrm{D}Q\rangle]\, \mathrm{dx}.
\end{aligned}$$

The pair of Euler–Lagrange equations then consists of

$$\mathrm{Div}\big[\pi_{12}\big(2R\mathbb{P}^2(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big)\big] = 0, \tag{5.2}$$

and

$$\Delta R - (\mathrm{D}m|0)\mathbb{P}^2((\mathrm{D}m|0)^\intercal R - (\mathbb{1}_2|0)) \perp \mathrm{T}_R\,\mathrm{SO}(3). \tag{5.3}$$

Note that it is *not* true that $X^\intercal\mathbb{P}^2(X) = X^\intercal\mathbb{P}^*\mathbb{P}X$ is symmetric for all matrices X; this is because $\mathbb{P}$ is not a matrix. Therefore, $(\mathrm{D}m|0)\mathbb{P}^2(\mathrm{D}m|0)^\intercal R$ is not automatically orthogonal to $\mathrm{T}_R\,\mathrm{SO}(3)$. And this term, being formally only in L^1 due to $\mathrm{D}m$ being in L^2, makes the structure of the equation interesting, as explained in Section 1.1.

For readability, we introduce a product which shares aspects of scalar products and matrix multiplication. We define $\circ\colon \mathbb{R}^{3\times 2}\times\mathbb{R}^{3\times 2}\to\mathbb{R}^{3\times 3}$ by

$$B\circ C := \frac{1}{2}BC^\intercal = \frac{1}{2}(B|0)\begin{pmatrix} C^\intercal \\ 0\end{pmatrix}. \tag{5.4}$$

Defining

$$S(\mathrm{D}m, R) := \pi_{12}\big(2R\mathbb{P}^2(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big),$$

we rewrite the second term of (5.3) as

$$\begin{aligned}
(\mathrm{D}m|0)\mathbb{P}^2((\mathrm{D}m|0)^\intercal R - (\mathbb{1}_2|0)) &= \mathrm{D}m\circ\pi_{12}\big(2\mathbb{P}^2(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big) \\
&= \mathrm{D}m\circ R^\intercal\pi_{12}\big(2R\mathbb{P}^2(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big) \\
&= \mathrm{D}m\circ R^\intercal S(\mathrm{D}m, R) = (\mathrm{D}m\circ S(\mathrm{D}m, R))R.
\end{aligned}$$

Noting that the projection of any matrix $X\in\mathbb{R}^{3\times 3}$ to $\mathrm{T}_R\,\mathrm{SO}(3)$ is $R\,\mathrm{skew}(R^\intercal X)$, we find that the projection of $(\mathrm{D}m\circ S(\mathrm{D}m, R))R$ is $R\,\mathrm{skew}(R^\intercal(\mathrm{D}m\circ S(\mathrm{D}m, R))R) = \mathrm{skew}(\mathrm{D}m\circ S(\mathrm{D}m, R))R$. This means that the pair of Euler–Lagrange equations (5.2)–(5.3) can be rewritten as

$$\mathrm{Div}\, S(\mathrm{D}m, R) = 0, \tag{5.5}$$

$$\Delta R - \mathrm{skew}(\mathrm{D}m\circ S(\mathrm{D}m, R))R \perp T_R\,\mathrm{SO}(3). \tag{5.6}$$

The latter is a relation rather than an equation, but we can rewrite it as an equation. In geometric analysis, this is usually done using the second fundamental form of SO(3), but we present the calculation in a more elementary way. Our aim is to calculate the tangential part $(\Delta R)^\top$ of ΔR.

Differentiating $RR^\intercal \equiv \mathbb{1}_3$ gives

$$0 = \partial_i(RR^\intercal) = (\partial_i R)R^\intercal + R\partial_i R^\intercal = 2\,\mathrm{sym}(R\partial_i R^\intercal). \tag{5.7}$$

Differentiating $R^\intercal R \equiv \mathbb{1}_3$ twice and summing over i, we find

$$\begin{aligned} 0 &= (\Delta R^\intercal)R + R^\intercal \Delta R + 2\sum_i \partial_i R^\intercal \partial_i R = (\Delta R)^\intercal R + R^\intercal \Delta R + 2\sum_i \partial_i R^\intercal \partial_i R \\ &= 2\,\mathrm{sym}(R^\intercal \Delta R) + 2\sum_i \partial_i R^\intercal \partial_i R, \end{aligned}$$

implying

$$\mathrm{sym}(R^\intercal \Delta R) = -\sum_i \partial_i R^\intercal \partial_i R. \tag{5.8}$$

For any fixed matrix $R \in \mathrm{SO}(3)$, we have $\mathrm{T}_R\,\mathrm{SO}(3) = R\,\mathfrak{so}(3)$, where $\mathfrak{so}(3)$ is the space of skew-symmetric matrices in $\mathbb{R}^{3\times3}$. The projections of any $X \in \mathbb{R}^{3\times3}$ to $\mathrm{T}_R\,\mathrm{SO}(3)$ or its orthogonal complement $[\mathrm{T}_R\,\mathrm{SO}(3)]^\intercal$ therefore are

$$X^\top = R\,\mathrm{skew}(R^\intercal X), \quad X^\perp = R\,\mathrm{sym}(R^\intercal X).$$

Therefore, we can calculate the orthogonal component of ΔR as

$$(\Delta R)^\perp = R\,\mathrm{sym}(R^\intercal \Delta R) = -\sum_i R\partial_i R^\intercal \partial_i R = -\sum_i \mathrm{skew}(R\partial_i R^\intercal)\partial_i R.$$

We have used (5.8) in the second "=", and (5.7) in the third. We now abbreviate

$$\begin{aligned} \Omega_R &:= \begin{pmatrix} (\Omega_R)_1 \\ (\Omega_R)_2 \end{pmatrix} = -\begin{pmatrix} R\partial_x R^\intercal \\ R\partial_y R^\intercal \end{pmatrix} = -\begin{pmatrix} \mathrm{skew}(R\partial_x R^\intercal) \\ \mathrm{skew}(R\partial_y R^\intercal) \end{pmatrix}, \\ \Omega_R \cdot \mathrm{D}R &:= \sum_{i=1}^{2} (\Omega_R)_i \partial_i R \in \mathbb{R}^{3\times3}, \end{aligned}$$

and hence have

$$(\Delta R)^\top = \Delta R - (\Delta R)^\perp = \Delta R - \Omega_R \cdot \mathrm{D}R.$$

Combining with the result of (5.6), we have calculated the tangential part of the left-hand side of (5.3) as

$$\Delta R - \Omega_R \cdot \mathrm{D}R - \mathrm{skew}(\mathrm{D}m \circ S(\mathrm{D}m, R))R,$$

and thus have derived the Euler–Lagrange equations in their final form. We summarize:

$$\operatorname{Div} S(\mathrm{D}m, R) = 0, \tag{5.9}$$

$$\Delta R - \Omega_R \cdot \mathrm{D}R - \operatorname{skew}(\mathrm{D}m \circ S(\mathrm{D}m, R))R = 0, \tag{5.10}$$

where here

$$S(\mathrm{D}m, R) := \pi_{12}\big(2R\mathbb{P}^2(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big), \quad \Omega_R := -\begin{pmatrix}\operatorname{skew}(R\partial_x R^\intercal)\\ \operatorname{skew}(R\partial_y R^\intercal)\end{pmatrix}.$$

Remark 5.1. In engineering language, (5.9) is the balance of forces, while (5.10) is the balance of angular momentum equation. The tensor

$$T(\mathrm{D}m, R) := 2\mathbb{P}^2(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))$$

is the *nonsymmetric Biot-type stress tensor* (symmetric if $\mu_c = 0$), while

$$S(\mathrm{D}m, R) = \pi_{12}(RT(\mathrm{D}m, R))$$

is the *first Piola–Kirchhoff-type force–stress tensor*. Note the analogy with the corresponding tensors in the three-dimensional Cosserat model presented in (A.4) and (A.5).

6. Regularity

The objective of this section is to prove our main theorem.

Theorem 6.1 (Interior regularity). *Every weak solution* $(m, R) \in W^{1,2}(\omega, \mathbb{R}^3 \times \mathrm{SO}(3))$ *of* (5.9)–(5.10) *is smooth on the interior of* ω.

Remark 6.2. Due to the results in [49,53,55], we know that energy minimizers to problem (5.1) exist and these are weak solutions $(m, R) \in W^{1,2}(\omega, \mathbb{R}^3 \times \mathrm{SO}(3))$ of (5.9)–(5.10). Since the problem is highly nonlinear, uniqueness cannot be shown, nor is it expected.

6.1. Hölder regularity

We observe that the last term in (5.10) is, up to "skew" and the harmless factor R, the product of a "gradient" $\mathrm{D}m$ with a divergence-free quantity $S(\mathrm{D}m, R)$, with both factors in L^2. As we know from [14], such a product is in the Hardy space $\mathcal{H}^1$ rather than just in L^1, and we will use arguments from [64] that tell us how to handle the additional R factor. A standard source for the Hardy space $\mathcal{H}^1$ is Stein's book [71, Chapter III]. Note that [64] (see also [24]) is about harmonic maps in ≥ 3 dimensions, and it is Rivière's paper [63] about two-dimensional harmonic maps that is mostly the basis of what we are doing here. Schikorra [69] found some simplification to the arguments of [63] and [64], and the most accessible account of all these arguments to date is the textbook [27] which allows us to handle the Euler–Lagrange equation (5.10) quite flexibly. Note that our equation (5.10) is more general than the equations of the form $\Delta R - \Omega \cdot DR = 0$ studied in those papers and the book [27], since we have the extra term $-R\,\mathrm{skew}(\mathrm{D}m \circ S(\mathrm{D}m, R))$ of order 0 in R. We are lucky that we have the additional structure coming from $S(\mathrm{D}m, R)$ being divergence-free, again implying that up to a bounded factor the extra term is in $\mathcal{H}^1$. Without that additional information, we would not know how to incorporate that into the existing regularity theory.

It will be crucial to use Morrey norms, at least locally. We say that $u \in L^p(U)$ is in the Morrey space $M^{p,s}(U)$ if

$$[u]^p_{M^{p,s}(U)} := \sup\Big\{r^{-s} \int_{B_r(x_0)\cap U} |u|^p \,\mathrm{dx} \;\Big|\; x_0 \in U, r \in (0,1)\Big\} < \infty.$$

Having this, we define the Morrey norm by $\|u\|_{M^{p,s}(U)} := [u]_{M^{p,s}(U)} + \|u\|_{L^p(U)}$.

We need the following lemmas. The first one is a special case of [69, Lemma A.1], in the spirit of similar estimates from [14]. This is where Hardy-BMO duality comes in as a hidden ingredient in our proof.

Lemma 6.3. *There is a constant C such that for all choices of $x_0 \in \mathbb{R}^2$, $r > 0$, and functions $a \in W^{1,2}(B_{2r}(x_0))$, $\Gamma \in L^2(B_r(x_0), (\mathbb{R}^2)^*)$, $b \in W^{1,2}_0 \cap L^\infty(B_r(x_0))$ with* $\mathrm{Div}\,\Gamma = 0$ *in the weak sense on $B_r(x_0)$, we have*

$$\left| \int_{B_r(x_0)} \langle \mathrm{D}a, \Gamma\rangle b \,\mathrm{dx} \right| \le C\|\Gamma\|_{L^2(B_r(x_0))} \|\mathrm{D}b\|_{L^2(B_r(x_0))} \|\mathrm{D}a\|_{M^{3/2,1/2}(B_{2r}(x_0))}.$$

The following is a result due to Rivière [63] and Schikorra [69], and can be found as a special case of [27, Theorem 10.57].

Lemma 6.4. *For every $\Omega \in L^2(B^2, (\mathbb{R}^2)^* \otimes \mathfrak{so}(3))$, there exists $G \in W^{1,2}(B^2, \mathrm{SO}(3))$ such that*

$$\mathrm{Div}(G^{-1}\Omega G - G^{-1}\mathrm{D}G) = 0 \quad \text{in } B^2$$

and[7]

$$\|\mathrm{D}G\|_{L^2(B^2)} + \|G^{-1}\Omega G - G^{-1}\mathrm{D}G\|_{L^2(B^2)} \le 3\|\Omega\|_{L^2(B^2)}.$$

We also need a version of the Hodge decomposition theorem. This one is a special case of [35, Corollary 10.5.1], adapted from the differential forms version to two-dimensional vector calculus as in [27, Corollary 10.70].

Lemma 6.5. *Let $p \in (1,\infty)$. On $B_r(x_0) \subset \mathbb{R}^2$, every 1-form $V \in L^p(B_r(x_0), (\mathbb{R}^2)^*)$ can be decomposed uniquely as*

$$V = \mathrm{D}\alpha + \mathrm{D}^\perp\beta + h,$$

where $\alpha \in W^{1,p}(B_r(x_0))$, $\beta \in W^{1,p}_0(B_r(x_0))$, and $h \in C^\infty(B_r(x_0), (\mathbb{R}^2)^*)$ is harmonic. Moreover, there is a constant C depending only on p, such that

$$\|\alpha\|_{W^{1,p}(B_r(x_0))} + \|\beta\|_{W^{1,p}(B_r(x_0))} + \|h\|_{L^p(B_r(x_0))} \le C\|V\|_{L^p(B_r(x_0))}.$$

[7] $(\mathbb{R}^2)^* \otimes \mathfrak{so}(3)$ is isomorphic to $\mathfrak{so}(3) \times \mathfrak{so}(3)$.

We now start our regularity proof. Our first step is local Hölder continuity.

Proposition 6.6. *Assume that* $(m, R) \in W^{1,2}(\omega, \mathbb{R}^3 \times \mathrm{SO}(3))$ *is a weak solution of* (5.9)–(5.10). *Then there is* $\beta > 0$ *such that* m *and* R *are* $C^{0,\beta}$*-Hölder continuous locally on* ω.

Proof. We write B_ρ for any ball $B_\rho(x_0) \subset \omega$. We assume r to be small enough such that $B_{2r}(x_0) \subset \omega$. We will collect more smallness conditions on r during the proof.

We choose G according to Lemma 6.4 and find, abbreviating $\Omega^G := G^{-1}\Omega_R G - G^{-1}\mathrm{D}G$,

$$\begin{aligned}\operatorname{Div}(G^{-1}\mathrm{D}R) &= \mathrm{D}(G^{-1}) \cdot \mathrm{D}R + G^{-1}\Delta R \\ &= -G^{-1}(\mathrm{D}G)G^{-1} \cdot \mathrm{D}R + G^{-1}\Omega \cdot \mathrm{D}R + G^{-1}\operatorname{skew}(\mathrm{D}m \circ S(\mathrm{D}m, R))R \\ &= \Omega^G \cdot G^{-1}\mathrm{D}R + G^{-1}\operatorname{skew}(\mathrm{D}m \circ S(\mathrm{D}m, R))R.\end{aligned}$$

Now we Hodge-decompose $G^{-1}\mathrm{D}R$ according to Lemma 6.5. We find $f \in W^{1,2}(B_r, \mathbb{R}^{3\times3})$, $g \in W_0^{1,2}(B_r, \mathbb{R}^{3\times3})$ with $dg = 0$, and a component-wise harmonic 1-form $h \in C^\infty(B_r, L(\mathbb{R}^2, \mathbb{R}^{3\times3}))$ such that

$$G^{-1}\mathrm{D}R = \mathrm{D}f + \mathrm{D}^\perp g + h \tag{6.1}$$

almost everywhere in B_r. Using the well-known relations $\operatorname{Div}\mathrm{D} = \operatorname{Div}^\perp \mathrm{D}^\perp = \Delta$ and $\operatorname{Div}\mathrm{D}^\perp = \operatorname{Div}^\perp \mathrm{D} = 0$, we calculate

$$\Delta f = \operatorname{Div}\mathrm{D}f = \operatorname{Div}(G^{-1}\mathrm{D}R) = \Omega^G \cdot G^{-1}\mathrm{D}R + G^{-1}\operatorname{skew}(\mathrm{D}m \circ S(\mathrm{D}m, R))R, \tag{6.2}$$

and

$$\Delta g = \operatorname{Div}^\perp \mathrm{D}^\perp g = \operatorname{Div}^\perp(G^{-1}\mathrm{D}R) = \mathrm{D}^\perp G^{-1} \cdot \mathrm{D}R = \operatorname{Div}((\mathrm{D}^\perp G^{-1})(R - R_0)), \tag{6.3}$$

for any constant $R_0 \in \mathbb{R}^{3\times3}$ (not necessarily a rotation). Both terms on the right-hand side, multiplied by some $\varphi \in W_0^{1,3}(B_r, \mathbb{R}^{3\times3})$, can be estimated using Lemma 6.3. Choosing $a := R^{k\ell}$, $b := (G^{-1})^{jk}\varphi^{i\ell}$, $\Gamma^s := (\Omega^G)_s^{ij}$, we find

$$\begin{aligned}&\int_{B_r} \langle \Omega^G \cdot G^{-1}\mathrm{D}R, \varphi\rangle \,\mathrm{dx} \\ &\quad \le C\|\Omega^G\|_{L^2(B_2)}(\|\mathrm{D}G\|_{L^2(B_r)}\|\varphi\|_{L^\infty(B_r)} + \|\mathrm{D}\varphi\|_{L^2(B_r)})\|\mathrm{D}R\|_{M^{3/2,1/2}(B_r)},\end{aligned} \tag{6.4}$$

and choosing $a := m^j$, $b := (G_{ij}^{-1})R^{k\ell}\varphi^{i\ell}$, $\Gamma^s := S(\mathrm{D}m, R)_s^k$, we have

$$\begin{aligned}&\int_{B_r} \big\langle \big(G^{-1}\operatorname{skew}(\mathrm{D}m \circ S(\mathrm{D}m, R))R\big), \varphi\big\rangle \,\mathrm{dx} \\ &\le C\|S(\mathrm{D}m, R)\|_{L^2(B_r)}(\|\mathrm{D}G\|_{L^2(B_r)}\|\varphi\|_{L^\infty(B_r)}\|\mathrm{D}R\|_{L^2(B_r)}\|\varphi\|_{L^\infty(B_r)} + \|\mathrm{D}\varphi\|_{L^2(B_r)}) \\ &\quad \cdot \|\mathrm{D}m\|_{M^{3/2,1/2}(B_r)}.\end{aligned} \tag{6.5}$$

We assume $\varepsilon \in (0, \varepsilon_0)$ with some $\varepsilon_0 > 0$ to be determined. Choosing $r > 0$ small enough, we may assume $\|\Omega_R\|_{L^2(B_r)} \le \varepsilon$ and $\|\mathrm{D}m\|_{L^2(B_r)} \le \varepsilon$.

We let $T := \{\varphi \in C_0^\infty(B_r, \mathbb{R}^{3\times 3}) \mid \|D\varphi\|_{L^3(B_r)} \le 1\}$. Combining the duality of $L^{3/2}$ and L^3 and (6.2) with (6.4) and (6.5), we find, using $\varphi = 0$ on ∂B_r,

$$
\begin{aligned}
&\|\mathrm{D}f\|_{L^{3/2}(B_r)} \\
&\quad \le C \sup_{\varphi\in T} \int_{B_r} \langle \mathrm{D}f, \mathrm{D}\varphi\rangle \,\mathrm{dx} \\
&\quad = C \sup_{\varphi\in T} \int_{B_r} \langle \Omega^G \cdot G^{-1}\mathrm{D}R + G^{-1}\,\mathrm{skew}(\mathrm{D}m \circ S(\mathrm{D}m, R))R, \varphi\rangle \,\mathrm{dx} \\
&\quad \le C \sup_{\varphi\in T} \|\Omega^G\|_{L^2(B_r)}(\|\mathrm{D}G\|_{L^2}\|\varphi\|_{L^\infty(B_r)} + \|\mathrm{D}\varphi\|_{L^2(B_r)})\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{2r})} \\
&\qquad + C \sup_{\varphi\in T} \|S(Dm, R)\|_{L^2(B_r)}\big(\|\mathrm{D}G\|_{L^2(B_r)}\|\varphi\|_{L^\infty(B_r)} + \|\mathrm{D}R\|_{L^2(B_r)}\|\varphi\|_{L^\infty(B_r)} \\
&\qquad\qquad + \|\mathrm{D}\varphi\|_{L^2(B_r)}\big)\|\mathrm{D}m\|_{M^{3/2,1/2}(B_{2r})} \\
&\quad \le C \sup_{\varphi\in T} \|\Omega_R\|_{L^2(B_r)}(\|\Omega_R\|_{L^2(B_r)}\|\varphi\|_{L^\infty(B_r)} + \|\mathrm{D}\varphi\|_{L^2(B_r)})\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{2r})} \\
&\qquad + C \sup_{\varphi\in T}(\|\mathrm{D}m\|_{L^2(B_r)} + r)\big(\|\Omega_R\|_{L^2(B_r)}\|\varphi\|_{L^\infty(B_r)} + \|\mathrm{D}R\|_{L^2(B_r)}\|\varphi\|_{L^\infty(B_r)} \\
&\qquad\qquad + \|\mathrm{D}\varphi\|_{L^2(B_r)}\big)\|\mathrm{D}m\|_{M^{3/2,1/2}(B_{2r})} \\
&\quad \le C(\varepsilon + r)r^{1/3}(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{2r})} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_{2r})}).
\end{aligned}
\tag{6.6}
$$

Here, in the second "≤", we have used Lemma 6.3, and in the fourth "≤", we have used $\|\varphi\|_{L^\infty(B_r)} \le Cr^{1/3}\|\mathrm{D}\varphi\|_{L^3(B_r)} \le Cr^{1/3}$, $\|\mathrm{D}\varphi\|_{L^2(B_r)} \le Cr^{1/3}\|\mathrm{D}\varphi\|_{L^3(B_r)} \le Cr^{1/3}$, $\|\Omega_R\|_{L^2(B_r)} \le \varepsilon$, and $\|\mathrm{D}m\|_{L^2(B_r)} \le \varepsilon$.

Using (6.3), we can also estimate the $L^{3/2}$-norm of $\mathrm{D}^\perp g$. We find

$$
\begin{aligned}
\|\mathrm{D}^\perp g\|_{L^{3/2}(B_r)} &\le C \sup_{\varphi\in T} \int_{B_r} \langle \mathrm{D}^\perp g, \mathrm{D}^\perp\varphi\rangle \,\mathrm{dx} \\
&= C \sup_{\varphi\in T} \int_{B_r} \langle \Delta g, \varphi\rangle \,\mathrm{dx} \\
&= C \sup_{\varphi\in T} \int_{B_r} \langle \mathrm{Div}((\mathrm{D}^\perp G^{-1})(R - R_{B_r})), \varphi\rangle \,\mathrm{dx} \\
&= C \sup_{\varphi\in T} \int_{B_r} \langle (\mathrm{D}^\perp G^{-1})(R - R_{B_r}), \mathrm{D}\varphi\rangle \,\mathrm{dx} \\
&\le C \sup_{\varphi\in T} \|\mathrm{D}\varphi\|_{L^3(B_r)}\|\mathrm{D}G\|_{L^2(B_r)}\|R - R_{B_r}\|_{L^6(B_r)} \\
&\le C\varepsilon r^{1/3}\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{2r})}.
\end{aligned}
\tag{6.7}
$$

This time, we have used $\|\mathrm{D}G\|_{L^2(B_r)} \le 3\|\Omega_R\|_{L^2(B_r)} \le 3\varepsilon$, and the Sobolev embedding $W^{1,2/3} \hookrightarrow L^6$ for R.

For h, being harmonic, we have the standard estimate

$$
\int_{B_\rho} |h|^{3/2}\,\mathrm{dx} \le C\left(\frac{\rho}{r}\right)^2 \int_{B_r} |h|^{3/2}\,\mathrm{dx},
$$

for any $0 < \rho < r$. From (6.1), and then (6.6) and (6.7), we hence infer

$$\begin{aligned}
\|\mathrm{D}R\|_{L^{3/2}(B_\rho)} &= \|G^{-1}DR\|_{L^{3/2}(B_\rho)} \\
&\le \|h\|_{L^{3/2}(B_\rho)} + \|\mathrm{D}f\|_{L^{3/2}(B_\rho)} + \|\mathrm{D}^\perp g\|_{L^{3/2}(B_\rho)} \\
&\le C\Big(\frac{\rho}{r}\Big)^{4/3}\|h\|_{L^{3/2}(B_r)} + \|\mathrm{D}f\|_{L^{3/2}(B_\rho)} + \|\mathrm{D}^\perp g\|_{L^{3/2}(B_\rho)} \\
&\le C\Big(\frac{\rho}{r}\Big)^{4/3}\|\mathrm{D}R\|_{L^{3/2}(B_r)} + C(\|\mathrm{D}f\|_{L^{3/2}(B_r)} + \|\mathrm{D}^\perp g\|_{L^{3/2}(B_r)}) \\
&\le C\Big(\frac{\rho}{r}\Big)^{4/3}\|\mathrm{D}R\|_{L^{3/2}(B_r)} \\
&\quad + C(\varepsilon + r)r^{1/3}(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{2r})} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_{2r})}). \qquad (6.8)
\end{aligned}$$

Now we are going to derive a similar estimate for $\|\mathrm{D}m\|_{L^{3/2}}$. Hodge-decompose $S(\mathrm{D}m, R)$, i.e.

$$\pi_{12}\big(2R\mathbb{P}^2(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big) = \mathrm{D}^\perp\alpha + \chi,$$

with $\alpha \in W_0^{1,2}(B_r, \mathbb{R}^{3\times 2})$, and $\chi \in W^{1,2}(B_r, L(\mathbb{R}^2, \mathbb{R}^{3\times 2}))$ harmonic. This time, there is no term of the form $\mathrm{D}\zeta$, since Div of the left-hand side is 0. This would imply that ζ is harmonic, and so would $\mathrm{D}\zeta$ be, which hence can be absorbed into χ. We have, abbreviating $\mathbb{P}_R$ for the linear mapping $\xi \mapsto 2R\mathbb{P}^2(R^\intercal(\xi))$,

$$\begin{aligned}
\Delta\alpha &= \mathrm{Div}^\perp \mathrm{D}^\perp\alpha \\
&= \mathrm{Div}^\perp\big[\pi_{12}\big(2R\mathbb{P}^2(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big)\big] \\
&= \mathrm{Div}^\perp[\pi_{12}(\mathbb{P}_R(\mathrm{D}m|0) - 2R\mathbb{P}^2(\mathbb{1}_2|0))] \\
&= (\mathrm{D}^\perp|0)\cdot[\mathbb{P}_R(\mathrm{D}m|0)] - \mathrm{Div}^\perp[\pi_{12}(2R\mathbb{P}^2(\mathbb{1}_2|0))] \\
&= \mathrm{D}^\perp\mathbb{P}_R\cdot\mathrm{D}m - \mathrm{Div}^\perp[\pi_{12}(2R\mathbb{P}^2(\mathbb{1}_2|0))] \\
&= \mathrm{Div}[(\mathrm{D}^\perp\mathbb{P}_R)(m - m_{B_r})] - \mathrm{Div}^\perp[\pi_{12}(2(R - R_{B_r})\mathbb{P}^2(\mathbb{1}_2|0))]. \qquad (6.9)
\end{aligned}$$

Using the same ideas as before, and defining $U := \{\psi \in C_0^\infty(B_r, \mathbb{R}^{3\times 2}) \mid \|\mathrm{D}^\perp\psi\|_{L^3(B_r)} \le 1\}$, we estimate

$$\begin{aligned}
\|\mathrm{D}^\perp\alpha\|_{L^{3/2}(B_r)} &\le C\sup_{\psi\in U}\int_{B_r}\langle\mathrm{D}^\perp\alpha, \mathrm{D}^\perp\psi\rangle\,\mathrm{d}x \\
&= C\sup_{\psi\in U}\int_{B_r}\big(\langle(\mathrm{D}^\perp\mathbb{P}_R)(m - m_{B_r}), \mathrm{D}\psi\rangle - \langle\pi_{12}(2(R - R_{B_r})\mathbb{P}^2(\mathbb{1}_2|0)), \mathrm{D}^\perp\psi\rangle\big)\,\mathrm{d}x \\
&\le C\sup_{\psi\in U}(\|\mathrm{D}\psi\|_{L^3(B_r)}\|\mathrm{D}\mathbb{P}_R\|_{L^2(B_r)}\|m - m_{B_r}\|_{L^6(B_r)} \\
&\qquad + \|\mathrm{D}\psi\|_{L^3(B_r)}\|R - R_{B_r}\|_{L^{3/2}(B_r)}) \\
&\le C\sup_{\psi\in U}(\|\mathrm{D}\psi\|_{L^3(B_r)}\|\mathrm{D}R\|_{L^2(B_r)}\|\mathrm{D}m\|_{L^{3/2}(B_r)} + r\|\mathrm{D}\psi\|_{L^3(B_r)}\|\mathrm{D}R\|_{L^{3/2}(B_r)}) \\
&\le C(\varepsilon + r)r^{1/3}(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{2r})} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_{2r})}). \qquad (6.10)
\end{aligned}$$

Proceeding exactly as above, we find

$$
\begin{aligned}
\|\mathrm{D}m\|_{L^{3/2}(B_\rho)} &\le C(\|\chi\|_{L^{3/2}(B_\rho)} + \|\mathrm{D}^\perp\alpha\|_{L^{3/2}(B_\rho)} + \rho^{4/3}) \\
&\le C\Big(\frac{\rho}{r}\Big)^{4/3}\|\chi\|_{L^{3/2}(B_r)} + C(\|\mathrm{D}^\perp\alpha\|_{L^{3/2}(B_\rho)} + \rho^{4/3}) \\
&\le C\Big(\frac{\rho}{r}\Big)^{4/3}\|\mathrm{D}m\|_{L^{3/2}(B_r)} + C(\|\mathrm{D}^\perp\alpha\|_{L^{3/2}(B_r)} + \rho^{4/3}) \\
&\le C\Big(\frac{\rho}{r}\Big)^{4/3}\|\mathrm{D}m\|_{L^{3/2}(B_r)} + C\rho^{4/3} \\
&\quad + C(\varepsilon + r)r^{1/3}(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{2r})} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_{2r})}). \qquad (6.11)
\end{aligned}
$$

In order to do so, we have used

$$
\begin{aligned}
C^{-1}\|\mathrm{D}m\|_{L^{3/2}(B_s)} - Cs^{4/3} &\le \big\|\pi_{12}\big(2R\mathbb{P}^2(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big)\big\|_{L^{3/2}(B_s)} \\
&\le C(\|\mathrm{D}m\|_{L^{3/2}(B_s)} + s^{4/3}). \qquad (6.12)
\end{aligned}
$$

We divide (6.8) and (6.11) by $\rho^{1/3}$ and combine them into

$$
\begin{aligned}
&\rho^{-1/3}(\|\mathrm{D}R\|_{L^{3/2}(B_\rho)} + \|\mathrm{D}m\|_{L^{3/2}(B_\rho)}) \\
&\quad \le C\,\frac{\rho}{r^{4/3}}(\|\mathrm{D}R\|_{L^{3/2}(B_r)} + \|\mathrm{D}m\|_{L^{3/2}(B_r)}) \\
&\qquad + C(\varepsilon + r)\Big(\frac{r}{\rho}\Big)^{1/3}(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{2r})} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_{2r})}) + C\rho \\
&\quad \le C\Big(\frac{\rho}{r} + (\varepsilon + r)\Big(\frac{r}{\rho}\Big)^{1/3}\Big)(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{2r})} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_{2r})}) + C\rho.
\end{aligned}
$$

We now assume $r \le \varepsilon$, where $\varepsilon > 0$ is yet to be determined. For formal reasons, we also add ρ on both sides, which gives

$$
\begin{aligned}
&\rho^{-1/3}(\|\mathrm{D}R\|_{L^{3/2}(B_\rho)} + \|\mathrm{D}m\|_{L^{3/2}(B_\rho)}) + \rho \\
&\quad \le C_0\Big(\frac{\rho}{r} + (\varepsilon + r)\Big(\frac{r}{\rho}\Big)^{1/3}\Big)(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{2r})} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_{2r})} + 2r)
\end{aligned}
$$

for some suitable constant C_0. Now we fix $\rho := \frac{r}{12C_0}$ and $\varepsilon := (12C_0)^{-4/3}$, making $C_0(\frac{\rho}{r} + (\varepsilon + r)(\frac{r}{\rho})^{1/3}) = \frac{1}{6}$. Abbreviating $\theta := \frac{1}{12C_0}$, we thus have

$$
\begin{aligned}
&(\theta r)^{-1/3}(\|\mathrm{D}R\|_{L^{3/2}(B_{\theta r})} + \|\mathrm{D}m\|_{L^{3/2}(B_{\theta r})}) + \theta r \\
&\quad \le \frac{1}{6}(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{2r})} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_{2r})} + 2r).
\end{aligned}
$$

This holds for all $B_{\theta r}(x_0)$ and $B_{2r}(x_0) \subset \omega$ which share the same center x_0. But clearly, we can replace $B_{2r}(x_0)$ with any ball $B_s(y_0) \supset B_{2r}(x_0)$ which is still in ω. All smallness assumptions made so far for $B_r(X_0)$ will now also be assumed for s, that is $s \le \varepsilon$,

$\|\Omega_R\|_{L^2(B_s(y_0))} \le \varepsilon$, and $\|\mathrm{D}m\|_{L^2(B_s(y_0))} \le \varepsilon$. We then have

$$(\theta r)^{-1/3}(\|\mathrm{D}R\|_{L^{3/2}(B_{\theta r}(x_0))} + \|\mathrm{D}m\|_{L^{3/2}(B_{\theta r}(x_0))}) + \theta r \\ \le \frac{1}{6}(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_s(y_0))} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_s(y_0))} + s),$$

which is valid for all r, s, x_0, y_0 such that $B_{2r}(x_0) \subset B_s(y_0) \subset \omega$. Then the $B_{\theta\rho}(x_0)$ cover all of $B_{\theta s/2}(y_0)$. Hence, on the left-hand-side, we can take the infimum over all feasible r and x_0, and find

$$\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{\theta s/2}(y_0))} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_{\theta s/2}(y_0))} + \frac{\theta s}{2} \\ \le \frac{1}{2}(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_s(y_0))} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_s(y_0))} + s).$$

We may replace s by $\frac{\theta}{2}s$ and iterate this, finding

$$\|\mathrm{D}R\|_{M^{3/2,1/2}(B_{(\theta/2)^k s}(y_0))} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_{(\theta/2)^k s}(y_0))} \\ \le 2^{-k}(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_s(y_0))} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_s(y_0))} + s)$$

for all $k \in \mathbb{N}$. Now, for $r \approx (\theta/2)^k s$, we have $k \approx \frac{\log r/s}{\log(\theta/2)}$, and therefore $2^{-k} \approx (r/s)^{\frac{\log 2}{\log(2/\theta)}}$ $=: (r/s)^\beta$. Hence we have proven that, for all $r \le s$, the estimate

$$\|\mathrm{D}R\|_{M^{3/2,1/2}(B_r(y_0))} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_r(y_0))} \\ \le Cr^\beta(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_s(y_0))} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_s(y_0))} + s)$$

holds. For $x_0 \in B_{s/2}(y_0)$ and $r \le s/2$, we can apply the same with $B_s(y_0)$ replaced by $B_{2r}(y_0) \subset B_s(y_0)$, and hence find

$$\|\mathrm{D}R\|_{M^{3/2,1/2}(B_r(x_0))} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_r(x_0))} \\ \le Cr^\beta(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_s(y_0))} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_s(y_0))} + s),$$

which implies

$$\|\mathrm{D}R\|_{M^{3/2,1/2+3\beta/2}(B_{s/2}(y_0))} + \|\mathrm{D}m\|_{M^{3/2,1/2+3\beta/2}(B_{s/2}(y_0))} \\ \le C(\|\mathrm{D}R\|_{M^{3/2,1/2}(B_s(y_0))} + \|\mathrm{D}m\|_{M^{3/2,1/2}(B_s(y_0))} + s).$$

This means

$$\mathrm{D}R,\ \mathrm{D}m \in M^{3/2,1/2+3\beta/2}_{\mathrm{loc}}(\omega). \tag{6.13}$$

We now use the following well-known fact, which can be found in [27, Theorem 5.7], for example.

Lemma 6.7 (Morrey' Dirichlet growth criterion). *Assume $U \subset \mathbb{R}^n$ to be open, $u \in W^{1,p}_{\mathrm{loc}}(U)$, $Du \in M^{p,n-p+\varepsilon}_{\mathrm{loc}}(U)$ for some $\varepsilon > 0$. Then $u \in C^{0,\varepsilon/p}$.*

With $p = \frac{3}{2}$, $n = 2$, the last estimate (6.13) and Lemma 6.7 imply $R, m \in C^{0,\beta}_{\mathrm{loc}}(\omega)$, which is the Hölder regularity asserted in Proposition 6.6. ∎

Remark 6.8. It is essential that we are working in the critical dimension $n = 2$ here, even though this may not be too obvious in the preceding proof which uses methods developed for supercritical dimensions. But the arithmetic of the exponents crucially uses $n = 2$. In particular, Lemma 6.3 for $n > 2$ is only available with exponents adding up to n instead of $(\frac{3}{2}, \frac{1}{2})$. But we would not succeed in finding similarly good estimates in the corresponding Morrey spaces.

6.2. Higher regularity

In this subsection, we are going to complete the proof of Theorem 6.1.

Proof. Remember we have the equations

$$\begin{aligned} \operatorname{Div} S(\mathrm{D}m, R) &= 0, \\ \Delta R - \Omega_R \cdot \mathrm{D}R - \operatorname{skew}(\mathrm{D}m \circ S(\mathrm{D}m, R))R &= 0, \end{aligned} \tag{6.14}$$

where for $\xi \in \mathbb{R}^{3\times 2}$ we have defined

$$S(\xi, R) = \pi_{12}\big(2R\mathbb{P}^2(R^\mathsf{T}(\xi, 0) - (\mathbb{1}_2|0))\big) = \pi_{12}\big(2R\mathbb{P}^\mathsf{T}\mathbb{P}(R^\mathsf{T}(\xi|R_3) - \mathbb{1}_3)\big),$$

and

$$|\Omega_R| \le C|\mathrm{D}R|.$$

Abbreviating $L_R(\xi) := \pi_{12}(2R\mathbb{P}^2(R^\mathsf{T}(\xi, 0)))$, we rewrite the first equation (6.14) as

$$\operatorname{Div} L_R(\mathrm{D}m) = \operatorname{Div}\big(\pi_{12}\big(2R\mathbb{P}^2(R^\mathsf{T}(\mathbb{1}_2|0))\big)\big). \tag{6.15}$$

For every $R \in \mathrm{SO}(3)$, $L_R\colon \mathbb{R}^{3\times 2} \to \mathbb{R}^{3\times 2}$ is a linear mapping satisfying the Legendre condition (uniform positivity) because of

$$\begin{aligned} \langle L_R(\xi), \xi\rangle &= \big\langle \pi_{12}\big(2R\mathbb{P}^2(R^\mathsf{T}(\xi, 0))\big), \xi\big\rangle = \langle 2R\mathbb{P}^2(R^\mathsf{T}(\xi, 0)), (\xi, 0)\rangle \\ &= \langle 2\mathbb{P}(R^\mathsf{T}(\xi, 0)), \mathbb{P}(R^\mathsf{T}(\xi, 0))\rangle \ge 2\hat{\lambda}|R^\mathsf{T}(\xi, 0)|^2 = 2\hat{\lambda}|\xi|^2, \end{aligned}$$

where here $\hat{\lambda} := \min\{\mu, \mu_c, \kappa\}$ is independent of R, hence we have a uniformly elliptic operator $m \mapsto \operatorname{Div} L_R(\mathrm{D}m)$. For this operator, classical Schauder theory applies once it depends Hölder continuously on x through $R(x)$. And it does, because we already know $R \in C^{0,\beta}_{\mathrm{loc}}$ for some $\beta > 0$.

We use the following version of Schauder theory. The proof is well known, a good reference is [27, Theorem 5.19] which reads as follows.

Lemma 6.9. *Let $u \in W^{1,2}_{\mathrm{loc}}(\mathcal{U}, \mathbb{R}^m)$ be a solution to*

$$\operatorname{Div}(A(x) \cdot \mathrm{D}u) = -\operatorname{Div} \mathcal{F},$$

with A satisfying the Legendre–Hadamard condition and having its components $A^{\alpha\beta}_{ij}$ in $C^{0,\sigma}_{\mathrm{loc}}(\mathcal{U})$ for some $\sigma \in (0, 1)$. If $\mathcal{F}^\alpha_i \in C^{0,\sigma}_{\mathrm{loc}}(\mathcal{U})$, then also D$u$ *is of class $C^{0,\sigma}_{\mathrm{loc}}(\mathcal{U})$.*

From what was proven in the last section, we know that both L_R and the right-hand side of (6.15) are in $C^{0,\beta}_{\mathrm{loc}}$ locally, hence Lemma 6.9 implies that $m \in C^{1,\beta}_{\mathrm{loc}}$ for some $\beta > 0$.

This simplifies the discussion of the regularity of R, because the $\mathrm{D}m$-terms in the equation for ΔR are now locally bounded. We can therefore rewrite it as

$$\Delta R + a(x, \mathrm{D}R) = 0, \tag{6.16}$$

where the function a depends on R and $\mathrm{D}m$ additionally, but those are locally bounded. The function satisfies

$$|a(x, \mathrm{D}R)| \le C(|\mathrm{D}R|^2 + 1). \tag{6.17}$$

Since $\mathrm{D}R \in L^2$, this means that ΔR is in L^1, but L^1 is just not enough to perform regularity theory for R. However, the structure of the equation almost allows one to apply the higher regularity theory for harmonic maps, where we could deal with $C|\mathrm{D}R|^2$ instead of $C(|\mathrm{D}R|^2 + 1)$ on the right-hand side. A simple formal trick will take care of that condition. Let

$$u(x) = (u_0(x), u_1(x)) := (R(x), x_1)$$

with values in $\mathrm{SO}(3) \times \mathbb{R}$. Then, letting $\tilde{a}(x, \mathrm{D}u) := (a(x, Du_0), 0)$, we have

$$\Delta u + \tilde{a}(x, \mathrm{D}u) = 0,$$

where here

$$|\tilde{a}(x, \mathrm{D}u)| = |a(x, \mathrm{D}R)| \le C(|\mathrm{D}R|^2 + 1) = C(|\mathrm{D}u_0|^2 + |\mathrm{D}u_1|^2) = C|\mathrm{D}u|^2.$$

Now we can follow the regularity theory for harmonic maps for a while. Note that [45, Lemma 3.7 and Proposition 3.2] assume u to be a harmonic map, but the proof uses only $|\Delta u| \le C|\mathrm{D}u|^2$ instead of the full harmonic map equation. We therefore can apply [45, Lemmas 3.6 and 3.7, Proposition 3.2] to our u and find that $\mathrm{D}u \in L^\infty_{\mathrm{loc}}$. This means that the second term in (6.16) is in L^p_{loc} for all $p > 1$, and standard L^p-theory gives us $u \in W^{2,p}_{\mathrm{loc}}$ for all $p > 1$. The Sobolev embedding $W^{1,p} \hookrightarrow C^{0,1-2/p}$ for $p > 2$ then gives us $\mathrm{D}u \in C^{0,\beta}_{\mathrm{loc}}$ with $\beta > 0$. Together with the result for m, we now have

$$(m, R) \in C^{1,\beta}_{\mathrm{loc}} \quad \text{for some } \beta > 0.$$

Once we have this, we can iterate the Schauder estimates, i.e. differentiate the equations and apply Lemma 6.9 to partial derivatives of m, R instead of m and R alone. Thus we find that $(m, R) \in C^{k,\beta}_{\mathrm{loc}}$ for our $\beta > 0$ and all $k \in \mathbb{N}$, which means we have proven that m and R are smooth on the interior of the domain. ∎

6.3. Body forces

It is physically reasonable to consider the equations with an additional external body force term in the first equation of balance of forces,

$$\begin{aligned} \operatorname{Div} S(\mathrm{D}m, R) &= f, \\ \Delta R - \Omega_R \cdot \mathrm{D}R - \operatorname{skew}(\mathrm{D}m \circ S(\mathrm{D}m, R))R &= 0 \end{aligned}$$

with $f \in W^{1,2}(\omega, \mathbb{R}^3)$. By integrating in one direction and setting

$$\mathcal{F}(x_1, x_2) := \left(\int_{(x_0)_1}^{x_1} f(t, x_2)\, \mathrm{dt} \mid 0_{\mathbb{R}^3} \right) \in \mathbb{R}^{3\times 2},$$

we can always assume $f = \operatorname{Div} \mathcal{F}$. Note that $\mathcal{F}$ depends on the first component $(x_0)_1$ of the center of the ball $B_r(x_0)$ on which we are momentarily working. We have $\mathcal{F} \in W^{1,2}(\omega, \mathbb{R}^{3\times 2})$, implying $\mathcal{F} \in L^p(\omega, \mathbb{R}^{3\times 2})$ for all $p \in [1, \infty)$. Now we may rewrite the first equation as

$$\operatorname{Div}(S(\mathrm{D}m, R) - \mathcal{F}) = 0.$$

We will need to estimate $\mathrm{D}\mathcal{F}$, which we calculate via $\partial_1 \mathcal{F} = (f, 0)$ and $\partial_2 \mathcal{F} = (\int_{(x_0)_1}^{x_1} \partial_2 f(t, x_2)\, \mathrm{dt}, 0)$. The latter gives

$$\begin{aligned}
\int_{B_r} |\partial_2 \mathcal{F}|^{3/2}\, \mathrm{dx} &= \int_{B_r} \left| \int_{(x_0)_1}^{x_1} \partial_2 f(t, x_2)\, \mathrm{dt} \right|^{3/2} \mathrm{dx} \\
&\le \int_{B_r} \left(\int_{(x_0)_1 - \sqrt{r^2 - x_2^2}}^{(x_0)_1 + \sqrt{r^2 - x_2^2}} |\partial_2 f(t, x_2)|\, \mathrm{dt} \right)^{3/2} \mathrm{dx} \\
&\le C r^{1/3} \int_{B_r} \int_{(x_0)_1 - \sqrt{r^2 - x_2^2}}^{(x_0)_1 + \sqrt{r^2 - x_2^2}} |\partial_2 f(t, x_2)|^{3/2}\, \mathrm{dt}\, \mathrm{dx} \\
&\le C r^{4/3} \int_{B_r} |\partial_2 f|^{3/2}\, \mathrm{dx}.
\end{aligned}$$

Since we can always assume $r \le 1$, we have proven

$$\|\mathrm{D}\mathcal{F}\|_{L^{3/2}} \le C \|f\|_{W^{1,3/2}}. \tag{6.18}$$

The regularity theory for the more general equation including forces goes pretty much along the lines of the $\mathcal{F} = 0$ case presented in Section 6.1. We only indicate the necessary modifications. We rewrite (6.2) as

$$\Delta f = \Omega^G \cdot G^{-1} \mathrm{D}R + G^{-1} \operatorname{skew}\big(\mathrm{D}m \circ (S(\mathrm{D}m, R) - \mathcal{F})\big) R + G^{-1} \operatorname{skew}(\mathrm{D}m \circ \mathcal{F}) R.$$

In (6.6), we replace $\|S(\mathrm{D}m, R)\|_{L^2(B_r)}$ by $\|S(\mathrm{D}m, R) - \mathcal{F}\|_{L^2(B_r)}$. Choosing the radius of B_r sufficiently small, we can also assume that $\|\mathcal{F}\|_{L^2(B_r)} \le \varepsilon$, hence we can estimate $\|S(\mathrm{D}m, R) - \mathcal{F}\|_{L^2(B_r)}$ by $C(\varepsilon + r)$ just as we did for $\|S(\mathrm{D}m, R)\|_{L^2(B_r)}$ in (6.6). But we also have an additional term on the right-hand side of that estimate. Using the boundedness of G^{-1} and R, it is estimated as follows, also assuming $\|\mathcal{F}\|_{L^3(B_r)} \le \varepsilon$. We have

$$\begin{aligned}
&\sup_{\varphi \in T} \int_{B_r} \langle G^{-1} \operatorname{skew}(\mathrm{D}m \circ \mathcal{F}) R, \varphi \rangle\, \mathrm{dx} \\
&\quad \le C \sup_{\varphi \in T} r^{-1/3} \|\mathrm{D}m\|_{L^{3/2}(B_r)} \|\mathcal{F}\|_{L^3(B_r)} r^{1/3} \|\varphi\|_{L^\infty(B_r)} \\
&\quad \le C \sup_{\varphi \in T} \|\mathrm{D}m\|_{M^{3/2,1/2}} \|\mathcal{F}\|_{L^3(B_r)} r^{2/3} \|\mathrm{D}\varphi\|_{L^3(B_r)} \\
&\quad \le C \varepsilon r^{1/3} \|\mathrm{D}m\|_{M^{3/2,1/2}},
\end{aligned}$$

which can be absorbed into the right-hand side of (6.6). Hence the conclusion of (6.6) continues to hold in the $\mathcal{F} \neq 0$ case also.

The second modification we have to make is that we now Hodge-decompose $S(\mathrm{D}m, R) - \mathcal{F}$, which means

$$\pi_{12}\big(2R\mathbb{P}^2(R^\mathsf{T}(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big) - \mathcal{F} = \mathrm{D}^\perp\alpha + \chi.$$

The additional term involving $\mathcal{F}$ on the right-hand side of (6.9) would be $-\operatorname{Div}^\perp \mathcal{F}$, which can be rewritten as $-\operatorname{Div}^\perp(\mathcal{F} - \mathcal{F}_{B_r})$. In (6.10), $(\mathcal{F} - \mathcal{F}_{B_r})$ can be processed exactly like $(R - R_{B_R})$, resulting in an additional $Cr\|\mathrm{D}\psi\|_{L^3(B_r)}\|\mathrm{D}\mathcal{F}\|_{L^{3/2}(B_r)}$, which can be estimated using (6.18) and $\psi \in U$ as follows, making the additional smallness assumption $\|f\|_{W^{1,2}(B_r)} \leq \varepsilon$ for r:

$$Cr\|\mathrm{D}\psi\|_{L^3(B_r)}\|\mathrm{D}\mathcal{F}\|_{L^{3/2}(B_r)} \leq Cr\|f\|_{W^{1,3/2}(B_r)} \leq Cr^{4/3}\|f\|_{W^{1,2}(B_r)} \leq \varepsilon r^{4/3}.$$

This additional term in (6.10) now contributes to the right-hand side of (6.11), but here enlarges only the $r^{4/3}$ and $\rho^{4/3}$ terms that are there anyway. By the same argument, taking $\mathcal{F}$ into account also contributes only to more $s^{4/3}$ terms in

$$\begin{aligned} C^{-1}\|\mathrm{D}m\|_{L^{3/2}(B_s)} - Cs^{4/3} &\leq \big\|\pi_{12}\big(2R\mathbb{P}^2(R^\mathsf{T}(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big) - \mathcal{F}\big\|_{L^{3/2}(B_s)} \\ &\leq C(\|\mathrm{D}m\|_{L^{3/2}(B_s)} + s^{4/3}), \end{aligned}$$

which updates (6.12). Hence the contributions of the modified versions of both (6.10) and (6.12) do not change the conclusion of (6.11).

Now that we have adapted (6.6) and (6.11) to nonvanishing body forces, we can conclude Hölder continuity just as at the end of Section 6.1, under the weak assumption of f being in $W^{1,2}$. If we assume $f \in C^\infty$ instead, both f and $\mathcal{F}$ are bounded, and the higher regularity proof from Section 6.2 goes through with hardly any modification. Note, for example, that (6.17) continues to hold.

6.4. Remarks on a special case

Our system simplifies considerably when $\mu = \mu_c = \kappa$, which makes $\mathbb{P}$ the identity.[8] Even though this assumption is not too natural from the viewpoint of applications, we would like to comment briefly on that case.

The simplified variational functional now reads

$$\begin{aligned} E(m, R) &:= \int_\omega \mu|R^\mathsf{T}(\mathrm{D}m|R_3) - \mathbb{1}_3|^2 + |\mathrm{D}R|^2\,\mathrm{dx} \\ &= \int_\omega \mu|R^\mathsf{T}(\mathrm{D}m|R_3) - R^\mathsf{T}R|^2 + |\mathrm{D}R|^2\,\mathrm{dx} \\ &= \int_\omega \mu|(\mathrm{D}m|0) - (R_1|R_2|0)|^2 + |\mathrm{D}R|^2\,\mathrm{dx}, \end{aligned}$$

[8]This case corresponds to $\mu = \lambda = \mu_c$ in the Cosserat bulk model and Poisson number $\nu = \frac{\lambda}{2(\mu+\lambda)} = \frac{1}{4}$ (nearly satisfied for magnesium).

which has the Euler–Lagrange equations (cf. (5.2))

$$\Delta m - \operatorname{Div}(R_1 | R_2 | 0) = 0 \tag{6.19}$$

and

$$\begin{aligned}\Delta R - \Omega_R \cdot \mathrm{D}R + \mu R \operatorname{skew}(R^\top (\mathrm{D}m|0)) &= \Delta R - \Omega_R \cdot \mathrm{D}R + \mu \operatorname{skew}((\mathrm{D}m|0)R^\top) \cdot R \\ &= 0. \end{aligned} \tag{6.20}$$

The point here is that the last term in the second equation now depends on $\mathrm{D}m$ only linearly, making it an L^2-term instead of L^1 (the L^1-part is cancelled by the skew-operator). But harmonic-map-type equations with a right-hand side in L^2 have been studied by Moser in quite some generality; see the book [45] for an excellent exposition of the methods.

In particular, Moser has two theorems that help us. Here, $N \subset \mathbb{R}^n$ is a compact manifold, $\mathcal{U} \subset \mathbb{R}^d$ a domain, and II is the second fundamental form of the target manifold, which corresponds to our term quadratic in $\mathrm{D}R$, i.e. $\sum_i \mathrm{II}(u)(\partial_i u, \partial_i u) = \Omega_u \cdot \mathrm{D}u$ in our case.

Theorem 6.10 ([45, Theorem 4.1]). *Suppose $u \in W^{1,2}(\mathcal{U}, N)$ is a stationary solution of*

$$\Delta u - \sum_i \mathrm{II}(u)(\partial_i u, \partial_i u) = f,$$

in $\mathcal{U}$, for a function $f \in L^p(\mathcal{U}, \mathbb{R}^n)$, where $p > \frac{d}{2}$ and $p \geq 2$. Then there exists a relatively closed set $\Sigma \subset \mathcal{U}$ of vanishing $(d-2)$-dimensional Hausdorff measure, such that $u \in C^{0,\alpha}_{\mathrm{loc}}(\mathcal{U} \setminus \Sigma, N)$ for a number $\alpha > 0$ that depends only on m, N, and p.

Theorem 6.11 ([45, Theorem 4.2]). *Under the assumptions of the previous theorem, if $n \leq 4$ and $p = 2$, we also have $u \in W^{2,2}_{\mathrm{loc}} \cap W^{1,4}_{\mathrm{loc}}(\mathcal{U} \setminus \Sigma, N)$.*

While those theorems are highly nontrivial, it is standard to deduce regularity of the solutions to our model in the special case considered here.

Theorem 6.12 (Interior regularity for $\mu = \mu_c = \kappa$). *Any solution $(m, R) \in W^{1,2}(\omega, \mathbb{R}^3 \times \mathrm{SO}(3))$ of the simplified problem (6.19)–(6.20) is smooth on the interior of the domain ω.*

Proof. We first consider equation (6.20). Since $-R \operatorname{skew}(R^\top (\mathrm{D}m|0)) \in L^2$, we can apply Theorems 6.10 and 6.11 to find $R \in C^{0,\alpha}_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}}(\omega, \mathrm{SO}(3))$. Note that $\Sigma = \emptyset$ here, since its zero-dimensional Hausdorff measure vanishes. Similarly, by L^2-theory for (6.19), we have $m \in W^{2,2}_{\mathrm{loc}}(\omega, \mathbb{R}^3)$. By the embedding $W^{2,2} \hookrightarrow W^{1,q}$ for all $q \in [2, \infty)$, we find that Δm and ΔR are in L^q_{loc} for every $q < \infty$, hence $(m, R) \in W^{2,q}_{\mathrm{loc}}(\omega, \mathbb{R}^3 \times \mathrm{SO}(3))$ for all $q < \infty$. This, in turn, embeds into $C^{1,\alpha}_{\mathrm{loc}}$ for all $\alpha \in (0, 1)$, and hence the right-hand sides are Hölder continuous. From here, we can use Schauder estimates to show that (m, R) is C^∞_{loc} on ω. ∎

7. Conclusion and open problems

We have deduced interior Hölder regularity for a Dirichlet-type geometrically nonlinear Cosserat flat membrane shell. The model is objective and isotropic but highly nonconvex. Therefore, our regularity result is astonishing and shows again the great versatility of the Cosserat approach compared to other more classical models. At present, we are limited to treating the uni-constant curvature case $|DR|^2$, since only then can sophisticated methods for harmonic functions with values in SO(3) be employed. This calls for more effort from researchers to generalize the foregoing. Progress in this direction would also allow one to consider the full Cosserat membrane-bending flat shell [9, 49–51]. Another case warrants further attention: taking the Cosserat couple modulus $\mu_c = 0$ in the model (in-plane drill allowed, but no energy connected to it) may still allow for regular minimizers. However, even the existence of minimizers remains unclear at present since it hinges on some sort of a priori regularity for the rotation field R (the nonquadratic curvature term $|DR|^{2+\varepsilon}$, $\varepsilon > 0$, together with zero Cosserat couple modulus $\mu_c = 0$ allows for minimizers [48, 53]). Finally, it is interesting to understand regularity properties of Cosserat shell models with curved initial geometry [25, 26].

We expect some boundary regularity to hold too. On the geometric analysis side, an adaptation of Rivière's boundary methods to problems with continuous Dirichlet boundary data has been performed in [46], which one could try to use. But with a view towards applications, partially free boundary problems would probably be more interesting.

Appendix

A.1. Three-dimensional Euler–Lagrange equations in dislocation tensor format

Here, for the convenience of the reader we derive the three-dimensional Euler–Lagrange equations based on the curvature expressed in the dislocation tensor $\boldsymbol{\alpha} = R^\mathsf{T} \operatorname{Curl} R$. We can write the bulk elastic energy as

$$E^{3D}(\varphi, R) = \int_{\mathcal{U}} W_{\mathrm{mp}}(\overline{U}) + W_{\mathrm{disloc}}(\boldsymbol{\alpha})\, \mathrm{dx}, \quad \overline{U} = R^\mathsf{T} \mathrm{D}\varphi, \quad \boldsymbol{\alpha} = R^\mathsf{T} \operatorname{Curl} R. \tag{A.1}$$

Taking variations of (A.1) with respect to the deformation $\varphi \in C_0^\infty(\mathcal{U}, \mathbb{R}^3)$ leads to

$$\delta E^{3D}(\varphi, R) \cdot \delta\varphi = \int_{\mathcal{U}} \langle \mathrm{D}W_{\mathrm{mp}}(\overline{U}), R^\mathsf{T} \mathrm{D}\delta\varphi \rangle_{\mathbb{R}^{3\times3}}\, \mathrm{dx} = 0$$

$$\iff \int_{\mathcal{U}} \langle R\mathrm{D}W_{\mathrm{mp}}(\overline{U}), \mathrm{D}\delta\varphi \rangle_{\mathbb{R}^{3\times3}}\, \mathrm{dx} = \int_{\mathcal{U}} \langle \operatorname{Div}[R \cdot \mathrm{D}W_{\mathrm{mp}}(\overline{U})], \delta\varphi \rangle_{\mathbb{R}^3}\, \mathrm{dx} = 0.$$

Taking variation with respect to $R \in \mathrm{SO}(3)$ results in (abbreviate $F := \mathrm{D}\varphi$)

$$\begin{aligned} \delta E^{3D}(\varphi, R) \cdot \delta R = \int_{\mathcal{U}} & \langle \mathrm{D}W_{\mathrm{mp}}(\overline{U}), \delta R^\mathsf{T} F \rangle \\ & + \langle \mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha}), \delta R^\mathsf{T} \operatorname{Curl} R + R^\mathsf{T} \operatorname{Curl} \delta R \rangle\, \mathrm{dx} \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathcal{U}} \langle \mathrm{D}W_{\mathrm{mp}}(\overline{U}), \delta R^\top R \cdot R^\top F \rangle \\
&\qquad + \langle \mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha}), \delta R^\top R \cdot R^\top \operatorname{Curl} R + R^\top \operatorname{Curl} \delta R \rangle \,\mathrm{dx} \\
&= \int_{\mathcal{U}} \langle \mathrm{D}W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^\top, \delta R^\top R \rangle \\
&\qquad + \langle \mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha}), \delta R^\top R \cdot \boldsymbol{\alpha} + R^\top \operatorname{Curl} \delta R \rangle \,\mathrm{dx} = 0. \qquad \text{(A.2)}
\end{aligned}
$$

Since $R^\top R = \mathbb{1}_3$, it follows that $\delta R^\top R + R^\top \delta R = 0$ and $\delta R^\top R = A \in \mathfrak{so}(3)$ is arbitrary. Therefore, (A.2) can be written as

$$
0 = \int_{\mathcal{U}} \langle \mathrm{D}W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^\top, A \rangle + \langle \mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha}^\top, A \rangle + \langle \mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha}), R^\top \operatorname{Curl}(R A^\top) \rangle \,\mathrm{dx}
$$

for all $A \in C_0^\infty(\mathcal{U}, \mathfrak{so}(3))$. Using that Curl is a self-adjoint operator, this means

$$
\begin{aligned}
0 &= \int_{\mathcal{U}} \langle \mathrm{D}W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^\top + \mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha})\boldsymbol{\alpha}^\top, A \rangle + \langle \operatorname{Curl}(R\mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha})), R A^\top \rangle \,\mathrm{dx} \\
&= \int_{\mathcal{U}} \langle \mathrm{D}W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^\top + \mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha})\boldsymbol{\alpha}^\top - R^\top \operatorname{Curl}(R\mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha})), A \rangle \,\mathrm{dx}.
\end{aligned}
$$

Thus, the strong form of the Euler–Lagrange equations reads

$$
\begin{aligned}
\operatorname{Div}[R\mathrm{D}W_{\mathrm{mp}}(\overline{U})] &= 0, \qquad \text{“balance of forces”}, \\
\operatorname{skew}[R^\top \operatorname{Curl}(R\mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha}))] &= \operatorname{skew}(\mathrm{D}W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^\top + \mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha}^\top), \qquad \text{(A.3)} \\
&\qquad \text{“balance of angular momentum”}.
\end{aligned}
$$

Defining the first Piola–Kirchhoff stress tensor

$$
S_1(\mathrm{D}\varphi, R) := \mathrm{D}_F[W_{\mathrm{mp}(\overline{U})}] = R\mathrm{D}W_{\mathrm{mp}}(\overline{U}) = R \cdot T_{\mathrm{Biot}}(\overline{U}), \qquad \text{(A.4)}
$$

where the nonsymmetric Biot-type stress tensor is given by

$$
T_{\mathrm{Biot}} := \mathrm{D}W_{\mathrm{mp}}(\overline{U}), \qquad \text{(A.5)}
$$

allows one to rewrite system (A.3) as

$$
\begin{aligned}
\operatorname{Div} S_1(\mathrm{D}\varphi, R) &= 0, \\
\operatorname{skew}[R^\top \operatorname{Curl}(R\mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha}))] &= \operatorname{skew}(T_{\mathrm{Biot}}(\overline{U}) \cdot \overline{U}^\top + \mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha}^\top).
\end{aligned} \qquad \text{(A.6)}
$$

Observe that $(\text{A.6})_1$ is a uniformly elliptic linear system for φ at given R. It is clear that global minimizers $\varphi \in W^{1,2}(\mathcal{U}, \mathbb{R}^3)$ and $R \in W^{1,2}(\mathcal{U}, \mathrm{SO}(3))$ are weak solutions of the Euler–Lagrange equations.

If $\mathrm{D}W_{\mathrm{disloc}}(\boldsymbol{\alpha}) \equiv 0$ (no moment stresses) then the balance of angular momentum turns into the symmetry constraint

$$
\mathrm{D}W_{\mathrm{mp}}(\overline{U}) \cdot \overline{U}^\top \in \mathrm{Sym}(3).
$$

A complete discussion of the solutions to this constraint can be found in [56, 57].

A.2. Two-dimensional Euler–Lagrange equations: Alternative derivation

Our energy functional is

$$\begin{aligned}
E^{2\mathrm{D}}(m,R) &= \int_\omega \Big(\mu|\operatorname{dev}\operatorname{sym}(R^\intercal(\mathrm{D}m|R_3)-\mathbb{1}_3)|^2 + \mu_c|\operatorname{skew}(R^\intercal(\mathrm{D}m|R_3)-\mathbb{1}_3)|^2 \\
&\qquad + \frac{\kappa}{3}\operatorname{tr}(R^\intercal(\mathrm{D}m|R_3)-\mathbb{1}_3)^2 + |\mathrm{D}R|^2\Big)\,\mathrm{dx} \\
&= \int_\omega \Big(|\mathbb{P}(R^\intercal(\mathrm{D}m|R_3)-\mathbb{1}_3)|^2 + |\mathrm{D}R|^2\Big)\,\mathrm{dx} \\
&= \int_\omega \Big(|\mathbb{P}(R^\intercal(\mathrm{D}m|R_3)-\mathbb{1}_3)|^2 + |\partial_x R|^2 + |\partial_y R|^2\Big)\,\mathrm{dx}. \qquad (\mathrm{A.7})
\end{aligned}$$

Taking free variations with respect to the midsurface deformation m in the direction of $\vartheta \in C_0^\infty(\omega,\mathbb{R}^3)$ leads to

$$\begin{aligned}
\delta E^{2\mathrm{D}}(m,R)\cdot\delta\vartheta &= \int_\omega 2\langle \mathbb{P}(R^\intercal(\mathrm{D}m|R_3)-\mathbb{1}_3), \mathbb{P}(R^\intercal(\mathrm{D}\vartheta|0))\rangle_{\mathbb{R}^{3\times3}}\,\mathrm{dx} \\
&= \int_\omega 2\langle \mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|R_3)-\mathbb{1}_3), (R^\intercal(\mathrm{D}\vartheta|0))\rangle_{\mathbb{R}^{3\times3}}\,\mathrm{dx} \\
&= \int_\omega \langle 2R\mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|R_3)-\mathbb{1}_3), (\mathrm{D}\vartheta|0)\rangle_{\mathbb{R}^{3\times3}}\,\mathrm{dx} \\
&= \int_\omega \big\langle \pi_{12}\big(2R\mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|R_3)-\mathbb{1}_3)\big), \mathrm{D}\vartheta\big\rangle_{\mathbb{R}^{3\times2}}\,\mathrm{dx} \\
&= \int_\omega \big\langle \operatorname{Div}\pi_{12}\big(2R\mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|R_3)-\mathbb{1}_3)\big), \vartheta\big\rangle_{\mathbb{R}^{3}}\,\mathrm{dx} = 0.
\end{aligned}$$

Thus the strong form of balance of forces can be expressed as

$$\operatorname{Div} S(\mathrm{D}m,R) = 0,$$

where

$$S(\mathrm{D}m,R) = \pi_{12}\big(2R\mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|R_3)) - \mathbb{1}_3\big) = \pi_{12}\big(2R\mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big)$$

is the *first Piola–Kirchhoff-type force–stress tensor* and, abbreviating $\overline{U} := R^\intercal(\mathrm{D}m|R_3)$,

$$\begin{aligned}
T(\mathrm{D}m,R) &= 2\mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0)) = 2\mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|R_3)-\mathbb{1}_3) \\
&= 2\mu\operatorname{dev}\operatorname{sym}(\overline{U}-\mathbb{1}_3) + 2\mu_c\operatorname{skew}(\overline{U}-\mathbb{1}_3) + \frac{2\kappa}{3}\operatorname{tr}(\overline{U}-\mathbb{1}_3)\cdot\mathbb{1}_3 \qquad (\mathrm{A.8})
\end{aligned}$$

is the *nonsymmetric Biot-type stress tensor* (symmetric if $\mu_c = 0$). We note the relation

$$S(\mathrm{D}m,R) = \pi_{12}(R\cdot T(\mathrm{D}m,R)),$$

resembling relation (A.4).

For balance of angular momentum we proceed similarly, but need some preparation. It is clear that

$$\begin{aligned}&(R+\delta R)^{\mathsf T}(\mathrm{D}m|(R+\delta R)e_3)-\mathbb{1}_3\\&\quad=R^{\mathsf T}(\mathrm{D}m|R_3)-\mathbb{1}_3+\underbrace{R^{\mathsf T}(0|0|\delta R_3)+(\delta R^{\mathsf T}(\mathrm{D}m|R_3))}_{\text{linear increment}}+\delta R^{\mathsf T}(0|0|\delta R_3).\end{aligned}$$

Therefore, taking variations of the energy with respect to R leads to

$$\begin{aligned}\delta E^{2\mathrm{D}}(m,R)\cdot\delta R=\int_\omega\Big(&2\langle\mathbb{P}(R^{\mathsf T}(\mathrm{D}m|R_3)-\mathbb{1}_3),\mathbb{P}(R^{\mathsf T}(0|0|\delta R_3)+\delta R^{\mathsf T}(\mathrm{D}m|R_3))\rangle\\&+2\langle\partial_x R,\partial_x\delta R\rangle+2\langle\partial_y R,\partial_y\delta R\rangle\Big)\,\mathrm{dx}=0.\end{aligned}$$

Since $R^{\mathsf T}R=\mathbb{1}_3$, we have $\delta R^{\mathsf T}R+R^{\mathsf T}\delta R=0$, hence $\delta R=RA$ for $A\in\mathfrak{so}(3)$ arbitrary. Therefore the latter turns into

$$\begin{aligned}&\int_\omega\Big(2\langle\mathbb{P}^{\mathsf T}\mathbb{P}(R^{\mathsf T}(\mathrm{D}m|R_3)-\mathbb{1}_3),R^{\mathsf T}(0|0|(RA)e_3)+(RA)^{\mathsf T}(\mathrm{D}m|R_3)\rangle\\&\quad-2\langle\partial_x^2R,RA\rangle-2\langle\partial_y^2R,RA\rangle\Big)\,\mathrm{dx}\\&\quad=\int_\omega 2\Big(\langle\mathbb{P}^{\mathsf T}\mathbb{P}(R^{\mathsf T}(\mathrm{D}m|R_3)-\mathbb{1}_3),A(0|0|e_3)-AR^{\mathsf T}(\mathrm{D}m|R_3)\rangle-\langle 2\Delta R,RA\rangle\Big)\,\mathrm{dx}\\&\quad=\int_\omega\Big(2\langle\mathbb{P}^{\mathsf T}\mathbb{P}(R^{\mathsf T}(\mathrm{D}m|R_3)-\mathbb{1}_3),-A(R^{\mathsf T}(\mathrm{D}m|R_3)-(0|0|e_3))\rangle\\&\qquad-\langle 2R^{\mathsf T}\Delta R,A\rangle\Big)\,\mathrm{dx}\\&\quad=-\int_\omega\Big(2\langle\mathbb{P}^{\mathsf T}\mathbb{P}(R^{\mathsf T}(\mathrm{D}m|R_3)-\mathbb{1}_3),AR^{\mathsf T}(\mathrm{D}m|0)\rangle+\langle 2R^{\mathsf T}\Delta R,A\rangle\Big)\,\mathrm{dx}\\&\quad=-\int_\omega\Big(\langle 2\mathbb{P}^{\mathsf T}\mathbb{P}(R^{\mathsf T}(\mathrm{D}m|R_3)-\mathbb{1}_3)(\mathrm{D}m|0)^{\mathsf T}R,A\rangle+\langle 2R^{\mathsf T}\Delta R,A\rangle\Big)\,\mathrm{dx}=0\end{aligned}\tag{A.9}$$

for all $A\in C_0^\infty(\omega,\mathfrak{so}(3))$. This implies the stationary condition in strong form

$$\begin{aligned}\operatorname{skew}(2R^{\mathsf T}\Delta R)&=-\operatorname{skew}\big(2\mathbb{P}^{\mathsf T}\mathbb{P}(R^{\mathsf T}(\mathrm{D}m|R_3)-\mathbb{1}_3)\cdot(\mathrm{D}m|0)^{\mathsf T}R\big)\\&=-\operatorname{skew}\big(2\mathbb{P}^{\mathsf T}\mathbb{P}(R^{\mathsf T}(\mathrm{D}m|0)-(\mathbb{1}_2|0))\cdot(\mathrm{D}m|0)^{\mathsf T}R\big)\\&=-\operatorname{skew}(T(\mathrm{D}m|R)\cdot(\mathrm{D}m|0)^{\mathsf T}R),\end{aligned}\tag{A.10}$$

where T is defined in (A.8)$_1$.

For $\mathbb{P}^{\mathsf T}\mathbb{P}=\mu\cdot\mathbb{1}$ the last equation simplifies to

$$\begin{aligned}\operatorname{skew}(2R^{\mathsf T}\Delta R)&=-\mu\operatorname{skew}(2R^{\mathsf T}(\mathrm{D}m|0)(\mathrm{D}m|0)^{\mathsf T}R-(\mathbb{1}_2|0)(\mathrm{D}m|0)^{\mathsf T}R)\\&=2\mu\operatorname{skew}((\mathbb{1}_2|0)(\mathrm{D}m|0)^{\mathsf T}R)\\&=2\mu\operatorname{skew}((\mathrm{D}m|0)^{\mathsf T}R)=-2\mu\operatorname{skew}(R^{\mathsf T}(\mathrm{D}m|0)).\end{aligned}$$

We can also rewrite (A.9)$_5$ as

$$\begin{aligned}0 &= \int_\omega \big(2\langle \mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|R_3) - \mathbb{1}_3)(\mathrm{D}m|0)^\intercal, AR^\intercal\rangle + 2\langle \Delta R, RA\rangle\big)\,\mathrm{dx}\\ &= \int_\omega \big(2\big\langle\big(\mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|R_3) - \mathbb{1}_3)(\mathrm{D}m|0)^\intercal\big)^\intercal, RA^\intercal\big\rangle + 2\langle \Delta R, RA\rangle\big)\,\mathrm{dx}\\ &= \int_\omega \big(2\big\langle(\mathrm{D}m|0)\big(\mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|R_3) - \mathbb{1}_3)\big)^\intercal, RA\big\rangle + 2\langle \Delta R, RA\rangle\big)\,\mathrm{dx}\end{aligned}$$

for all $A \in C_0^\infty(\omega, \mathfrak{so}(3))$, which is equivalent to

$$\Delta R - (\mathrm{D}m|0)\big(\mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|R_3) - \mathbb{1}_3)\big)^\intercal \perp \mathrm{T}_R\,\mathrm{SO}(3). \tag{A.11}$$

Since

$$\begin{aligned}\big(\mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|R_3) - \mathbb{1}_3)\big)^\intercal &= \big(\mathbb{P}^\intercal\mathbb{P}(R^\intercal(\mathrm{D}m|0) - (\mathbb{1}_2|0))\big)^\intercal\\ &= \mathbb{P}^\intercal\mathbb{P}\big((R^\intercal(\mathrm{D}m|0)(\mathbb{1}_2|0))^\intercal\big),\end{aligned}$$

we may express (A.11) also as

$$\Delta R - (\mathrm{D}m|0)\mathbb{P}^\intercal\mathbb{P}((\mathrm{D}m|0)^\intercal R - (\mathbb{1}_2|0)) \perp \mathrm{T}_R\,\mathrm{SO}(3). \tag{A.12}$$

This is the form for balance of angular momentum given in equation (5.3).

A.3. Lifting to the Δ-operator

This last equation (A.12) is not, however, the final form of the balance of angular momentum equation that we will consider. Indeed, since $R^\intercal R = \mathbb{1}_3$, we can differentiate once to obtain

$$(\partial_x R)^\intercal R + R^\intercal \partial_x R = 0, \quad (\partial_y R)^\intercal R + R^\intercal \partial_y R = 0.$$

Taking second partial derivatives, we get

$$\begin{aligned}(\partial_x^2 R)^\intercal R + (\partial_x R)^\intercal \partial_x R + (\partial_x R)^\intercal \partial_x R + R^\intercal \partial_x^2 R = 0,\\ (\partial_y^2 R)^\intercal R + (\partial_y R)^\intercal \partial_y R + (\partial_y R)^\intercal \partial_y R + R^\intercal \partial_y^2 R = 0.\end{aligned}$$

Summing shows

$$\begin{aligned}&(\Delta R)^\intercal R + R^\intercal \Delta R + 2((\partial_x R)^\intercal \partial_x R + (\partial_y R)^\intercal \partial_y R) = 0\\ \iff\; &2\,\mathrm{sym}(R^\intercal \Delta R) + 2[(\partial_x R)^\intercal \partial_x R + (\partial_y R)^\intercal \partial_R] = 0,\\ &\mathrm{sym}(R^\intercal \Delta R) = -[(\partial_x R)^\intercal \partial_x R + (\partial_y R)^\intercal \partial_y R].\end{aligned} \tag{A.13}$$

From (A.10) we have

$$\mathrm{skew}(2R^\intercal \Delta R) = -\,\mathrm{skew}(T(\mathrm{D}m, R)(\mathrm{D}m|0)^\intercal R). \tag{A.14}$$

Adding (A.13)$_2$ and (A.14) yields, due to the orthogonality of sym and skew,

$$\begin{aligned}2R^\intercal \Delta R &= \mathrm{sym}(2R^\intercal \Delta R) + \mathrm{skew}(2R^\intercal \Delta R)\\ &= -2[(\partial_x R)^\intercal \partial_x R + (\partial_y R)^\intercal \partial_y R] - \mathrm{skew}(T(\mathrm{D}m, R)(\mathrm{D}m|0)^\intercal R).\end{aligned}$$

Hence, using the isotropy of the skew-operator, we obtain

$$\begin{aligned}
2\Delta R &= -2R[(\partial_x R)^{\mathsf T}\partial_x R + (\partial_y R)^{\mathsf T}\partial_y R] - R\,\mathrm{skew}(T(\mathrm{D}m, R)(\mathrm{D}m|0)^{\mathsf T} R) \\
&= -2R[(\partial_x R)^{\mathsf T}\partial_x R + (\partial_y R)^{\mathsf T}\partial_y R] - R\,\mathrm{skew}(R^{\mathsf T} R T(\mathrm{D}m, R)(\mathrm{D}m|0)^{\mathsf T} R) \\
&= -2R[(\partial_x R)^{\mathsf T}\partial_x R + (\partial_y R)^{\mathsf T}\partial_y R] - R R^{\mathsf T}\,\mathrm{skew}(R T(\mathrm{D}m, R)(\mathrm{D}m|0)^{\mathsf T})\cdot R \\
&= -2R[(\partial_x R)^{\mathsf T}\partial_x R + (\partial_y R)^{\mathsf T}\partial_y R] + \mathrm{skew}\big((\mathrm{D}m|0)(R T(\mathrm{D}m, R))^{\mathsf T}\big)\cdot R \\
&= -2R[(\partial_x R)^{\mathsf T}\partial_x R + (\partial_y R)^{\mathsf T}\partial_y R] + \mathrm{skew}\big((\mathrm{D}m|0)\big[\big(\pi_{12}(R T(\mathrm{D}m, R))\big)\big]^{\mathsf T}\big)\cdot R \\
&= -2R[(\partial_x R)^{\mathsf T}\partial_x R + (\partial_y R)^{\mathsf T}\partial_y R] + \mathrm{skew}(\mathrm{D}m\cdot S(\mathrm{D}m, R)^{\mathsf T})\cdot R, \qquad \text{(A.15)}
\end{aligned}$$

giving

$$\begin{aligned}
\Delta R &= -R[(\partial_x R)^{\mathsf T}\partial_x R + (\partial_y R)^{\mathsf T}\partial_y R] + \frac{1}{2}\,\mathrm{skew}(\mathrm{D}m\cdot S(\mathrm{D}m, R)^{\mathsf T})\cdot R \\
&= -R[(\partial_x R)^{\mathsf T}\partial_x R + (\partial_y R)^{\mathsf T}\partial_y R] + \mathrm{skew}(\mathrm{D}m\circ S(\mathrm{D}m, R))\cdot R,
\end{aligned}$$

where we used the definition of $\circ$ given in equation (5.4).

We set

$$\begin{aligned}
-R[(\partial_x R)^{\mathsf T}\partial_x R + (\partial_y R)^{\mathsf T}\partial_y R] &= -R\partial_x R^{\mathsf T}\cdot\partial_x R - R\partial_y R^{\mathsf T}\partial_y R =: \Omega_R\cdot \mathrm{D}R, \\
(\Omega_R)_1 := -R\partial_x R^{\mathsf T} &\in \mathfrak{so}(3), \quad (\Omega_R)_2 := -R\partial_y R^{\mathsf T} \in \mathfrak{so}(3).
\end{aligned}$$

With this definition, (A.15) can be written as

$$\Delta R = \underbrace{\Omega_R\cdot \mathrm{D}R}_{\in L^1(\omega)} - \underbrace{R\,\mathrm{skew}(T(\mathrm{D}m, R)(\mathrm{D}m|0)^{\mathsf T} R)}_{\in L^1(\omega)}. \qquad \text{(A.16)}$$

Considering the special case $\mathbb{P}^{\mathsf T}\mathbb{P} = \mu\cdot\mathbb{1}$, equation (A.16) turns into

$$\Delta R = \Omega_R\cdot \mathrm{D}R + \mu R\,\mathrm{skew}((\mathrm{D}m|0)^{\mathsf T} R) = \Omega_R\cdot \mathrm{D}R - \underbrace{\mu R\,\mathrm{skew}(R^{\mathsf T}(\mathrm{D}m|0))}_{\in L^2(\omega)}.$$

We finally observe that

$$R^{\mathsf T}(\Omega_R)_i R = -(\partial_i R)^{\mathsf T} R = R^{\mathsf T}\partial_i R, \quad i = 1, 2,$$

which implies for $\Gamma_i = \mathrm{axl}(R^{\mathsf T}\partial_i R)$,

$$R^{\mathsf T}(\Omega_R)_i R = \mathrm{Anti}(\Gamma_i), \quad \mathrm{axl}(R^{\mathsf T}(\Omega_R)_i R) = \Gamma_i,$$

where Γ is the wryness tensor from equation (4.2).

A.4. A glimpse of a Reissner–Mindlin type flat membrane shell model

It is interesting to compare our Cosserat flat membrane shell model (allowing for existence of minimizers and their full regularity) with one that would appear closer to classical approaches. For this sake we consider a Reissner–Mindlin flat membrane shell model next.

In the case of the one-director geometrically nonlinear, physically linear Reissner–Mindlin flat membrane shell model without independent drilling rotations, the problem can be described as a two-field minimization for the midsurface $m\colon \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ and the unit-director field $d\colon \omega \subset \mathbb{R}^2 \to \mathbb{S}^2$ of the elastic energy[9]

$$E^{2\mathrm{D}}_{\mathrm{Reissner}}(m,d) = \int_\omega h\Big\{ \underbrace{\frac{\mu}{4}\underbrace{|\mathrm{D}m^\intercal \mathrm{D}m - \mathbb{1}_2|^2}_{\text{in-plane stretch}} + \frac{1}{8}\frac{2\mu\lambda}{2\mu+\lambda}\underbrace{\mathrm{tr}(\mathrm{D}m^\intercal \mathrm{D}m - \mathbb{1}_2)^2}_{\text{elongational stretch}}}_{\text{nonelliptic}}$$
$$+ \mu(\underbrace{\langle d, \partial_x m\rangle^2 + \langle d, \partial_y m\rangle^2}_{\text{transverse shear}}) + \underbrace{\mu\frac{L_c^2}{2}|\nabla d|^2}_{\text{curvature}}\Big\}\,\mathrm{dx}.$$

Here, the membrane energy part is not rank-one elliptic due to the presence of the membrane strain $\mathrm{D}m^\intercal \mathrm{D}m - \mathbb{1}_2$. The uni-constant curvature energy could be generalized to the Oseen–Frank form; cf. Section 3.1. We note that $\langle d, \partial_x m\rangle^2 + \langle d, \partial_y m\rangle^2 = |\mathrm{D}m^\intercal d|^2_{\mathbb{R}^2}$, and, for simplicity, consider the energy

$$\int_\omega \Big(\frac{1}{2}|\mathrm{D}m^\intercal \mathrm{D}m - \mathbb{1}_2|^2_{\mathbb{R}^{2\times 2}} + \frac{1}{2}|\mathrm{D}m^\intercal d|^2_{\mathbb{R}^2} + \frac{1}{2}|\nabla d|^2\Big)\,\mathrm{dx}$$
$$= \int_\omega \Big(\frac{1}{2}|(\mathrm{D}m|n_m)^\intercal(\mathrm{D}m|n_m) - \mathbb{1}_3|^2 + \frac{1}{2}|\mathrm{D}m^\intercal d|^2 + \frac{1}{2}|\nabla d|^2\Big)\,\mathrm{dx}.$$

The Euler–Lagrange equations are then given by

$$\begin{aligned}
&\delta E^{2\mathrm{D}}_{\mathrm{Reissner}}(m,d)\cdot\delta m\\
&\quad= \int_\omega \big(2\langle(\mathrm{D}m|n_m)^\intercal(\mathrm{D}m|n_m) - \mathbb{1}_3, (\mathrm{D}m|n_m)^\intercal(\mathrm{D}\delta m|0)\rangle_{\mathbb{R}^{3\times3}}\\
&\qquad\quad + \langle \mathrm{D}m^\intercal d, (\mathrm{D}\delta m)^\intercal d\rangle_{\mathbb{R}^2}\big)\,\mathrm{dx}\\
&\quad= \int_\omega \big(\langle(\mathrm{D}m|n_m)\cdot 2((\mathrm{D}m|n_m)^\intercal(\mathrm{D}m|n_m) - \mathbb{1}_3), (\mathrm{D}\delta m|0)\rangle_{\mathbb{R}^{3\times3}}\\
&\qquad\quad + \langle \mathrm{D}m^\intercal d \otimes \mathrm{D}\delta m^\intercal d, \mathbb{1}_2\rangle_{\mathbb{R}^{2\times2}}\big)\,\mathrm{dx}\\
&\quad= \int_\omega \big(\langle(\mathrm{D}m|n_m)\cdot 2((\mathrm{D}m|n_m)^\intercal(\mathrm{D}m|n_m) - \mathbb{1}_3), (\mathrm{D}\delta m|0)\rangle_{\mathbb{R}^{3\times3}}\\
&\qquad\quad + \langle (\mathrm{D}m^\intercal d \otimes d)\mathrm{D}\delta m, \mathbb{1}_2\rangle_{\mathbb{R}^{2\times2}}\big)\,\mathrm{dx}\\
&\quad= \int_\omega \big\langle \pi_{12}\big((\mathrm{D}m|n_m)\cdot 2((\mathrm{D}m|n_m)^\intercal(\mathrm{D}m|n_m) - \mathbb{1}_3)\big) + d\otimes \mathrm{D}m^\intercal d, \mathrm{D}\delta m\big\rangle_{\mathbb{R}^{3\times2}}\,\mathrm{dx}
\end{aligned}$$

[9]The missing Reissner–Mindlin bending contribution scaling with h^3 would be of the form [49, (7.25)]

$$\frac{h^3}{12}\Big\{\mu|\mathrm{sym}((\mathrm{D}m|d)^\intercal(\mathrm{D}d|0))|^2 + \frac{\mu\lambda}{2\mu+\lambda}\,\mathrm{tr}\big(\mathrm{sym}((\mathrm{D}m|d)^\intercal(\mathrm{D}d|0))\big)^2\Big\}.$$

Here, no choice of constitutive parameters reduces the bending energy to the uni-constant case.

$$
\begin{aligned}
&= -\int_\omega \big\langle \mathrm{Div}\big[\pi_{12}\big((\mathrm{D}m|n_m)\cdot 2((\mathrm{D}m|n_m)^\intercal(\mathrm{D}m|n_m) - \mathbb{1}_3)\big) \\
&\qquad\qquad + d \otimes \mathrm{D}m^\intercal d\big], \delta m\big\rangle_{\mathbb{R}^3}\,\mathrm{dx} \\
&= 0
\end{aligned}
$$

for all $\delta m \in C_0^\infty(\omega, \mathbb{R}^3)$. Here, we can define the first Piola–Kirchhoff-type stress tensor

$$
S(\mathrm{D}m, d) = \pi_{12}\big((\mathrm{D}m|n_m)\cdot 2((\mathrm{D}m|n_m)^\intercal(\mathrm{D}m|n_m) - \mathbb{1}_3)\big) + d \otimes \mathrm{D}m^\intercal d.
$$

For variations with respect to $d \in \mathbb{S}^2$ we note that

$$
|d + \delta d|^2 = 1 \iff |d|^2 + 2\langle d, \delta d\rangle + |\delta d|^2 = 1.
$$

Hence, the variation δd is orthogonal to d, i.e. $\langle d, \delta d\rangle = 0$. Without loss of generality, we express δd as $\delta d = d \times \delta v$ for some $\delta v \in C_0^\infty(\omega, \mathbb{R}^3)$. Therefore, taking variations with respect to d gives

$$
\begin{aligned}
\delta E^{\mathrm{2D}}_{\mathrm{Reissner}}(m, d)\cdot \delta d &= \int_\omega \langle \nabla d, \nabla \delta d\rangle + \langle \mathrm{D}m^\intercal d, \mathrm{D}m^\intercal \delta d\rangle\,\mathrm{dx} = 0 \\
\iff & -\int_\omega \langle \Delta d, d \times \delta v\rangle + \langle \mathrm{D}m\mathrm{D}m^\intercal d, d \times \delta v\rangle\,\mathrm{dx} = 0 \\
\iff & \int_\omega -\langle \Delta d, \mathrm{Anti}(d)\delta v\rangle + \langle \mathrm{D}m\mathrm{D}m^\intercal d, \mathrm{Anti}(d)\delta v\rangle\,\mathrm{dx} = 0 \\
\iff & \int_\omega \langle \mathrm{Anti}(d)\Delta d, \delta v\rangle_{\mathbb{R}^3} - \langle \mathrm{Anti}(d)\mathrm{D}m\mathrm{D}m^\intercal d, \delta v\rangle\,\mathrm{dx} = 0
\end{aligned}
$$

for all $\delta v \in C_0^\infty(\omega, \mathbb{R}^3)$. The latter leads to the strong form

$$
\mathrm{Anti}(d)(\Delta d - \mathrm{D}m\mathrm{D}m^\intercal d) = 0 \iff d \times (\Delta d - \mathrm{D}m\mathrm{D}m^\intercal d) = 0. \tag{A.17}
$$

However, since $d \in \mathbb{S}^2$, we know $|d|^2 = 1$. Therefore, in addition, taking partial derivatives, we obtain

$$
\langle \partial_x d, d\rangle = 0, \quad \langle \partial_y d, d\rangle = 0.
$$

Taking second partial derivatives yields

$$
\langle \partial_x^2 d, d\rangle + \langle \partial_x d, \partial_y d\rangle = 0, \quad \langle \partial_y^2 d, d\rangle + \langle \partial_y d, \partial_y d\rangle = 0.
$$

Summing shows

$$
\langle \Delta d, d\rangle + |\partial_x d|^2 + |\partial_y d|^2 = \langle \Delta d, d\rangle + |\mathrm{D}d|^2 = 0. \tag{A.18}
$$

Adding (A.17) and (A.18) shows

$$
d * \Delta d := \underbrace{d \times \Delta d + \langle d, \Delta d\rangle}_{\text{geometric product, Clifford product}} = \underbrace{d \times (\mathrm{D}m\mathrm{D}m^\intercal d)}_{\in \mathbb{R}^3} - \underbrace{|\mathrm{D}d|^2}_{\in \mathbb{R}}.
$$

Formally, this implies

$$\Delta d = \frac{d}{|d|^2} * [d \times (\mathrm{D}m\mathrm{D}m^\intercal d) - |\mathrm{D}d|^2].$$

In terms of equations, we have altogether from (A.17)$_1$ and (A.18), respectively

$$\mathrm{Anti}(d)\Delta d = \mathrm{D}m\mathrm{D}m^\intercal d, \quad \langle d, \Delta d\rangle = -|\mathrm{D}d|^2,$$

equivalently

$$\underbrace{\begin{pmatrix} \mathrm{Anti}(d) \\ d_1 \quad d_2 \quad d_3 \end{pmatrix}}_{=:\hat{A}\in\mathbb{R}^{4\times 3}} \begin{pmatrix} \Delta d \\ | \end{pmatrix} = \begin{pmatrix} \mathrm{D}m\mathrm{D}m^\intercal d \\ -|\mathrm{D}d|^2 \end{pmatrix}_{\mathbb{R}^4}. \tag{A.19}$$

We multiply (A.19) by $\hat{A}^\intercal$ to get

$$\hat{A}^\intercal \hat{A}\Delta d = \hat{A}^\intercal \begin{pmatrix} \mathrm{D}m\mathrm{D}m^\intercal d \\ -|\mathrm{D}d|^2 \end{pmatrix} \in \mathbb{R}^3, \quad |d| = 1.$$

Since in fact

$$\hat{A}^\intercal \hat{A} = |d|^2 \cdot \mathbb{1}_3 = \mathbb{1}_3,$$

we obtain the system of Euler–Lagrange equations

$$\mathrm{Div}\, S(\mathrm{D}m, d) = 0, \quad \text{“balance of forces”} \tag{A.20}$$

with the first Piola–Kirchhoff-type stress tensor

$$S(\mathrm{D}m, d) = \pi_{12}\big((\mathrm{D}m|n_m) \cdot 2((\mathrm{D}m|n_m)^\intercal(\mathrm{D}m|n_m) - \mathbb{1}_3)\big) + d \otimes \mathrm{D}m^\intercal d,$$

and

$$\begin{aligned} \Delta d &= (-\mathrm{Anti}(d)|d)_{\mathbb{R}^{3\times 4}} \begin{pmatrix} \mathrm{D}m\mathrm{D}m^\intercal d \\ -|\mathrm{D}d|^2 \end{pmatrix} \\ &= -\mathrm{Anti}(d)(\mathrm{D}m\mathrm{D}m^\intercal d) - |\mathrm{D}d|^2 \cdot d \qquad \text{“balance of director equilibrium”} \\ &= -\underbrace{d \times (\mathrm{D}m\mathrm{D}m^\intercal d)}_{\in L^2(\omega,\mathbb{R}^3)} - |\mathrm{D}d|^2 \cdot d. \end{aligned} \tag{A.21}$$

We observe that (A.20) constitutes a nonlinear, nonconvex problem for the midsurface m once the unit director d is determined. Therefore, existence for (A.20), (A.21) is not yet known and likely not true. We note that the right-hand side in (A.21) contains an $L^2(\omega)$-term, since $\mathrm{D}m \in L^4(\omega)$ instead of our $L^1(\omega)$-term in equation (A.16). For $\mathrm{D}m \equiv 0$ we recover from (A.21) the director equilibrium for the uni-constant liquid crystal problem equation (3.4).

A.5. Numerical experiments

We present a sequence of numerical experiments for problem (A.7). In these experiments we compare the dimensionally reduced energy (4.5) (where the transverse shear energy is multiplied by the arithmetic mean of μ and μ_c) to the energy (4.6) of the rigorously derived Γ-limit membrane model (where the transverse shear energy is multiplied by the harmonic mean of μ and μ_c). We set the Lamé parameters to $\mu = 2.7191 \cdot 10^4$, $\lambda = 4.4364 \cdot 10^4$, and vary μ_c and L_c.

For the domain we choose the unit disk, which we discretized by $6 \cdot 4^6 = 24{,}576$ triangular elements. We used Lagrange finite elements of second order for the midsurface deformation m and geodesic finite elements of second order for the microrotation field R [47, 66, 68].

To trigger the deformation process, we radially compressed the membrane to a new radius $r < 1$ by Dirichlet boundary conditions for the deformation on the entire domain boundary. The microrotation field was not subject to Dirichlet boundary conditions at all. We minimized the discrete energy using a trust-region method [68] starting from the cap function $m_0(x, y) = (x, y, 0.1 - 0.1\sqrt{x^2 + y^2})$ for all (x, y) in the interior of the unit disk and $m_0(x, y) = (rx, ry, 0)$ on the boundary. The initial microrotation was $R = \mathbb{1}$. We conducted several simulations resulting in different wrinkle patterns depending on the Cosserat couple modulus μ_c and the characteristic length L_c as shown in Tables 1, 2, and 3. The numerical algorithms were implemented in C++ using the DUNE libraries (www.dune-project.org) [67].

From the figures one can see that wrinkling only happens if the characteristic length L_c is small enough. Indeed, if $L_c = 10^{-3}$ then the deformation is largely bending dominated, with small wrinkles only appearing next to the boundary, if μ_c is large enough. With smaller values of L_c one can see wrinkling in larger parts of the domain, even if the radial compression factor r is much smaller. Note that the choice $\mu_c = 0$ does not lead to a well-posed problem when used in the energy (4.6), because there it makes the transverse shear energy term disappear.

Acknowledgments. The authors are indebted to Oliver Sander and Lisa Julia Nebel (Chair of Numerical Mathematics, Technical University of Dresden) for preparing the calculations leading to Tables 1, 2, and 3. The first author thanks Armin Schikorra (University of Pittsburgh) for an interesting discussion. The second author is grateful to Maryam Mohammadi Saem and Peter Lewintan (Faculty of Mathematics, University of Duisburg-Essen) for help in preparing the manuscript and the pictures.

Funding. Andreas Gastel acknowledges support in the framework of the DFG-Priority Programm 2256 "Variational Methods for Predicting Complex Phenomena in Engineering Structures and Materials" with the project title "Very singular solutions of a nonlinear Cosserat elasticity model for solids" with Project no. 441380936. Patrizio Neff acknowledges support in the framework of the DFG-Priority Programm 2256 "Variational

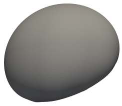
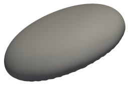
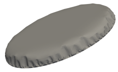
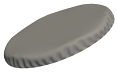
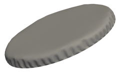
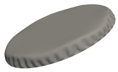
	Arithmetic mean (4.5)	Harmonic mean (4.6)
$\mu_c = 0$	 $r = 0.9$	Not well posed
$\mu_c = 10^{-5}\mu$	 $r = 0.9$	 $r = 0.9$
$\mu_c = 10^{-2}\mu$	 $r = 0.9$	 $r = 0.9$
$\mu_c = \mu$	 $r = 0.9$	

Table 1. Deformation of a radially compressed shell with $L_c = 10^{-3}$. Simulations by Lisa Julia Nebel and Oliver Sander (TU Dresden).

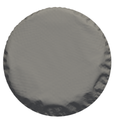
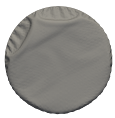
	Arithmetic mean (4.5)	Harmonic mean (4.6)
$\mu_c = 0$	 $r = 0.98$	Not well posed
$\mu_c = 10^{-2}\mu$	 $r = 0.91$	 $r = 0.94$
$\mu_c = \mu$	 $r = 0.9$	

Table 2. Deformation of a radially compressed shell with $L_c = 10^{-5}$. Simulations by Lisa Julia Nebel and Oliver Sander (TU Dresden).

	Arithmetic mean (4.5)	Harmonic mean (4.6)
$\mu_c = 0$	 $r = 0.99$	Not well posed
$\mu_c = 10^{-2}\mu$	 $r = 0.99$	 $r = 0.99$
$\mu_c = \mu$	 $r = 0.99$	

Table 3. Deformation of a radially compressed shell with $L_c = 10^{-8}$. Simulations by Lisa Julia Nebel and Oliver Sander (TU Dresden).

Methods for Predicting Complex Phenomena in Engineering Structures and Materials" with the project title "A variational scale-dependent transition scheme: From Cauchy elasticity to the relaxed micromorphic continuum" with Project no. 440935806 and also the DFG research grant Neff 902/8-1 Project no. 415894848 with the title "Modelling and mathematical analysis of geometrically nonlinear Cosserat shells with higher order and residual effects".

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Received 8 March 2023; revised 21 September 2023; accepted 23 October 2023.

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