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From pure braid groups to hyperbolic groups

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Abstract. In this note we show that any homomorphism from a pure surface braid group to a torsion-free hyperbolic group either has a cyclic image or factors through a forgetful map. This extends and gives a new proof of an earlier (2019) result of the author which works only when the target is a free group or a surface group. We also prove a similar rigidity result for the pure braid group of the disk.

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1. Introduction

Let $S_{g,p}$ be a closed surface of genus g with p punctures. Let $\mathrm{PConf}_n(S_{g,p})$ be the space of ordered n-tuples of distinct points in $S_{g,p}$. The group $PB_n(S_{g,p}) := \pi_1(\mathrm{PConf}_n(S_{g,p}))$ is called the surface braid group. For simplicity, we omit p whenever $p = 0$. We call a group Λ *hyperbolic* if Λ is finite generated and its Cayley graph with respect to a finite generating set is δ-hyperbolic for some $\delta > 0$. Examples of hyperbolic groups include free groups and $\pi_1(M)$ when M is a closed hyperbolic manifold. Surface braid groups have many homomorphisms into hyperbolic groups. Composing the map $p_i: PB_n(S_{g,p}) \to \pi_1(S_{g,p})$ with $\Phi: \pi_1(S_{g,p}) \to \Lambda$, where p_i is the induced map on the fundamental groups of the natural projection $\mathrm{PConf}_n(S_{g,p}) \to S_{g,p}$ to the ith coordinate and Λ, a hyperbolic group, is such an example. We can even obtain surjective homomorphisms from $PB_n(S_{g,p})$ to hyperbolic groups with respect to a finite generating set as g varies because $\pi_1(S_{g,p})$ has surjective homomorphisms to any finite generated group as g varies. In this paper, we will classify all homomorphisms from $PB_n(S_{g,p})$ to torsion-free hyperbolic groups.

The result of this paper has a precursor. In [4], we proved that any surjective homomorphism $PB_n(S_g) \to \Lambda$, where Λ is a nonabelian surface group or a nonabelian free group, factors through the natural projection p_i for some i. The following theorem generalizes [4] in two ways. First, the target is extended to all torsion-free, non-elementary hyperbolic groups; and second, the domain is extended to finite index normal subgroups of $PB_n(S_g)$ whose quotient is an abelian group.

Theorem 1.1 (Classification of homomorphisms for braid groups of closed surfaces). *Let $n > 0$, $g > 1$, and let $\Gamma \lhd PB_n(S_{g,p})$ be a finite index normal subgroup. Let Λ be a torsion-free, non-elementary hyperbolic group. If $\mathrm{PB}_n(S_g)/\Gamma$ is abelian, then any homomorphism $\Gamma \to \Lambda$ either factors through p_i or its image is a cyclic group.*

Understanding homomorphisms with cyclic image $\Gamma \to \mathbb{Z}$ (as we assume that the target group is torsion-free) is the same as computing $H^1(\Gamma;\mathbb{Q})$. The following theorem gives us a computation of the first Betti number of Γ.

Theorem 1.2 (First Betti number of braid groups of closed surfaces). *Let $n > 0$ and $g > 1$. Let $\Gamma \lhd PB_n(S_{g,p})$ be a finite index normal subgroup and let $\Gamma_i := p_i(\Gamma)$. If $PB_n(S_g)/\Gamma$ is abelian, then*

$$H^1(\Gamma;\mathbb{Q}) = \bigoplus p_i^*\big(H^1(\Gamma_i;\mathbb{Q})\big).$$

For the pure braid group of a punctured surface, we obtain a version of Theorem 1.1 for the whole group $PB_n(S_{g,p})$ instead of its finite index subgroups. We do not know if statements in Theorems 1.1 and 1.2 are valid or not for $PB_n(S_{g,p})$ when $p > 0$.

Theorem 1.3 (Classification of homomorphisms for braid groups of punctured surfaces). *Let $n > 0$ and $g > 1$, and let Λ be a torsion-free, non-elementary hyperbolic group. Any homomorphism $PB_n(S_{g,p}) \to \Lambda$ either factors through p_i or its image is a cyclic group.*

We now discuss the same problem for the pure braid group of the disk $PB_n := \pi_1(\mathrm{PConf}_n(\mathbb{C}))$, where $\mathrm{PConf}_n(\mathbb{C})$ is the space of ordered n-tuples of distinct points in $\mathbb{C}$. Note that the analogous statement to Theorem 1.3 is not true for PB_n.

It is well known that the center of PB_n is a cyclic group Z_n which is generated by the Dehn twist about the boundary curve. The quotient group PB_3/Z_3 is the free group of rank two F_2, which is hyperbolic. Thus, the quotient homomorphism

$$Q\colon PB_3 \to PB_3/Z_3 \cong F_2$$

is a surjective homomorphism to a hyperbolic group that does not factor through forgetful maps.

Moreover, there is a natural surjective homomorphism between braid groups that arises from a classical miracle: "resolving the quartic." Indeed, let

$$R\colon \mathrm{PConf}_4(\mathbb{C}) \to \mathrm{PConf}_3(\mathbb{C})$$

be the map given by

$$R(a,b,c,d) = (ab + cd, ac + bd, ad + bc).$$

The induced homomorphism on fundamental groups $R_*: PB_4 \to PB_3$ is a surjective homomorphism. Thus, we obtain another natural homomorphism into a torsion-free hyperbolic group

$$RQ := Q \circ R_*: PB_4 \to PB_3 \to PB_3/Z_3 \cong F_2.$$

In this paper, we will prove that RQ and Q are the only exceptional homomorphisms. We call a map $f: \mathrm{PConf}_n(\mathbb{C}) \to \mathrm{PConf}_m(\mathbb{C})$ a *forgetful map* if it is defined as $f(x_1,\dots,x_n) = (x_{\sigma(1)},\dots,x_{\sigma(m)})$ where σ is a permutation of $\{1,\dots,n\}$. For example, the map $f: \mathrm{PConf}_n(\mathbb{C}) \to \mathrm{PConf}_3(\mathbb{C})$ defined by $f(x_1,\dots,x_n) = (x_2, x_1, x_3)$ is a forgetful map.

Theorem 1.4 (Homomorphism classification for braid groups of the disk). *Let $n \geq 3$, Λ be a torsion-free, non-elementary hyperbolic group and $\phi: PB_n \to \Lambda$ be a homomorphism. Then ϕ satisfies one of the following three cases:*

(1) *The image $\phi(PB_n)$ is a cyclic group.*

(2) *There exists $\phi': F_2 \to \Lambda$ and a forgetful map $f_3: PB_n \to PB_3$ such that $\phi = \phi' \circ Q \circ f_3$.*

(3) *There exists $\phi': F_2 \to \Lambda$ and a forgetful map $f_4: PB_n \to PB_4$ such that $\phi = \phi' \circ RQ \circ f_4$.*

Theorem 1.4 is a generalization of the result [5, Theorem 3.5], in which the authors proved the same result when Λ is a free group. We remark that the statement in Theorem 1.4 is not true when Λ is a relatively hyperbolic group. Indeed, let $B_n(S^2)$ be the braid group of the two sphere S^2, which is the fundamental group of the space of unordered n-tuples of distinct points in S^2. Deligne–Mostow [6] constructed a homomorphism $E: B_n(S^2) \to SU(k,1)$ such that the image is a lattice. We also know that $B_n(S^2)$ contains PB_{n-2} as a subgroup and the restriction E to PB_{n-2} does not factor through a forgetful map. Our proof does not apply when Λ is a relatively hyperbolic group because we strongly use the fact that for hyperbolic groups the centralizer of any nontrivial element is cyclic. This property does not hold for relatively hyperbolic groups.

Finally, we ask the following natural question.

Question 1.5. *Are theorems 1.1, 1.2 and 1.4 true for all finite index subgroups of $PB_n(S_{g,p})$ and PB_n?*

Prior results. Like Theorem 1.1, there are other results in the history that address actions of the mapping class groups or braid groups on non-positive curved spaces. For example, [10, Theorem 2] and [1, Theorem D] proved that under certain conditions, an action of the mapping class group or the braid group by isometries on non-positively curved spaces always has a global fixed point. Our result deals with free actions and the action only exists when it factors through a forgetful map.

Theorems 1.1, 1.3 and 2.5 can also be compared with [8, Theorem 1.1] and [12], where the authors proved that an action of an irreducible uniform lattice in $G_1 \times \cdots \times G_n$ on certain non-positively curved spaces without global fixed points always extends to the Lie group and factors through one of G_i's.

Comparing the methods in [4], [5] and the current paper. The results in [4] and [5] only work for the free group F_m when $m > 1$ since they use the property that the $H^1(F_m; \mathbb{Z})$ is an isotropic subspace for the cup product. Since hyperbolic groups can be perfect, it seems impossible that this idea can be used to prove the results in this paper. Moreover, in [4], we use the method of F. E. A. Johnson [9] and Salter [14], which strongly uses a special property of free groups and surface groups: finitely-generated normal subgroups of either free groups or surface groups are either trivial or have finite index. This property is certainly not true for general hyperbolic groups. For example, let M be a 3-dimensional hyperbolic manifold that is a surface bundle over the circle. Then $\pi_1(M)$ contains a surface subgroup as a nontrivial finitely-generated, infinite index normal subgroup. The novelty of this paper is the observation that the rigidity results such as Theorem 1.1 and Theorem 1.4 are not consequences of the classification of isotropic subspaces of the first homology, but rather of the rich commuting and lantern relations of the subgroups of the pure braid groups.

2. Obstructing homomorphisms to hyperbolic groups

In this section, we discuss tools to obstruct homomorphisms to hyperbolic groups. We discuss the rigidity of homomorphisms to hyperbolic groups from two classes of groups: the $\mathbb{Z}$-central extensions and the direct product of groups.

2.1. The Euler class of a $\mathbb{Z}$-central extensions. For a $\mathbb{Z}$-central extension

$$1 \to \mathbb{Z} \to G \xrightarrow{p} \overline{G} \to 1, \tag{1}$$

we can associate an Euler class $\mathrm{Eu}(p) \in H^2(\overline{G}; \mathbb{Z})$ (see, e.g., [2, Chapter 4]). We know that the exact sequence (1) splits if and only if $\mathrm{Eu}(p) = 0 \in H^2(\overline{G}; \mathbb{Z})$. On the other hand, $\mathrm{Eu}(p) \neq 0$ if and only if a nontrivial element of the $\mathbb{Z}$-subgroup of G is a commutator in G.

We need the following standard fact about torsion-free, non-elementary hyperbolic groups.

Fact 2.1. *If Λ is a torsion-free hyperbolic group, then the centralizer of $1 \neq h \in \Lambda$ is a cyclic group.*

The following theorem describes the rigidity of homomorphisms from G to hyperbolic groups.

Lemma 2.2. *Let Λ be $\mathbb{Z}$ or a torsion-free, non-elementary hyperbolic group and G be the group as in the exact sequence (1). If $\mathrm{Eu}(p) \in H^2(\overline{G};\mathbb{Q})$ is nontrivial, then any homomorphism $\phi\colon G \to \Lambda$ factors through p; i.e., we have the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\ \phi\ } & \Lambda \\ {\scriptstyle p}\downarrow & \nearrow & \\ \overline{G} & & \end{array} .$$

Proof. Let α be a generator of the $\mathbb{Z}$-subgroup of G as in (1). Since α is central in G, we know $\phi(G)$ should lie in the centralizer of $\phi(\alpha)$, which is a cyclic group. When $\mathrm{Eu}(p) \neq 0 \in H^2(\overline{G};\mathbb{Q})$, we know that $H^1(G;\mathbb{Q}) = H^1(\overline{G};\mathbb{Q})$, which implies that any homomorphism $G \to \mathbb{Z}$ factors through p. Another way to see this is that the centralizer of some power of α is a product of commutators in G, which means that any homomorphism $G \to \mathbb{Z}$ factors through p. ■

When Q is a finite index subgroup of G, we obtain a similar result for Q.

Corollary 2.3. *Let G satisfy the exact sequence (1) and $Q < G$ be a finite index subgroup. Let Λ be either $\mathbb{Z}$ or a torsion-free, non-elementary hyperbolic group. If $\mathrm{Eu}(p) \neq 0 \in H^2(\overline{G};\mathbb{Q})$, then any homomorphism $Q \to \Lambda$ factors through p.*

Proof. Let Q be a finite index subgroup of G. Then we obtain a $\mathbb{Z}$-central extension

$$1 \to \mathbb{Z} \to Q \xrightarrow{p'=p|_Q} \overline{Q} \to 1.$$

If $\mathrm{Eu}(p) \neq 0 \in H^2(\overline{G};\mathbb{Q})$, then $\mathrm{Eu}(p')$ is nontrivial as well. This follows from the fact that the map $H^2(\overline{G};\mathbb{Q}) \to H^2(\overline{Q};\mathbb{Q})$ is injective when $\overline{Q}$ is a finite index subgroup in $\overline{G}$. Then some nontrivial element in the $\mathbb{Z}$-subgroup of G is also a commutator in Q. Then the corollary follows from Lemma 2.2. ■

The above method has been used in the following result of Putman [13, Theorem A] and Bridson [1, Theorem A], which we recall here. Let $\mathrm{Mod}(S_g)$ be the mapping class group of S_g; i.e., the group of connected components of the homeomorphism group of S_g. See [7] for an introduction on mapping class groups.

Corollary 2.4 (Putman, Bridson). *Let $g > 2$ and let $\Gamma < \mathrm{Mod}(S_g)$ be a finite index subgroup. Let Λ be either $\mathbb{Z}$ or a torsion-free, non-elementary hyperbolic group. Then any homomorphism $\phi\colon \Gamma \to \Lambda$ satisfies $\phi(T) = 1$ for $T \in \mathrm{Mod}(S_g)$, a power of Dehn twist that is in Γ.*

Sketch of the proof. The centralizer of a Dehn twist is a $\mathbb{Z}$-central extension of a short exact sequence where the Euler class is rationally nontrivial. See [13] for more details. ∎

Notice that Putman proved the result when the target is $\mathbb{Z}$ and Bridson proved it for actions on CAT(0)-space. The above method does not work for $h \in \mathrm{Mod}(S_g)$ when h^m is not a multi-twist (a product of powers of Dehn twist about disjoint curves) for some m. This is because if no power of h is a multi-twist, a power of h is a pseudo-Anosov element on a subsurface of S_g, which is never a product of commutators in its centralizer.

2.2. Product of groups. We now discuss homomorphisms to Λ from a direct product of groups, which is an extension of [3, Lemma 5.1].

Theorem 2.5. *Let $G_1, \dots, G_n$ be groups and let $\Gamma < G_1 \times \cdots \times G_n$ be a finite index subgroup. Let $\pi_i\colon \Gamma \to G_i$ be the ith projection map and let Γ_i be the image of π_i.*

(1) *The following decomposition holds:*

$$H^1(\Gamma;\mathbb{Q}) = \bigoplus_i \pi_i^*(H^1(\Gamma_i;\mathbb{Q})).$$

(2) *Let Λ be a torsion-free, non-elementary hyperbolic group. Then any homomorphism $\phi\colon \Gamma \to \Lambda$ either factors through π_i or its image is a cyclic group.*

Proof. We first assume that π_i is surjective for all i, otherwise we replace the group G_i by Γ_i. By induction, all we need is to prove the case when $n = 2$. Let $K_1 := \Gamma \cap (G_1 \times 1)$ and $K_2 := \Gamma \cap (1 \times G_2)$. We have the following exact sequences

$$1 \to K_1 \to \Gamma \to G_2 \to 1,$$
$$1 \to K_1 \to G_1 \to G_1/K_1 = G_2/K_2 = G_1 \times G_2/\Gamma \to 1.$$

Now,

$$H^1(K_1;\mathbb{Q})^{G_2} = H^1(K_1;\mathbb{Q})^{\Gamma} = H^1(K_1;\mathbb{Q})^{\pi_1(\Gamma)} = H^1(K_1;\mathbb{Q})^{G_1} = H^1(G_1;\mathbb{Q}),$$

which implies $H^1(\Gamma;\mathbb{Q}) = \bigoplus_i \pi_i^*(H^1(\Gamma_i;\mathbb{Q}))$ by Leray spectral sequence.

For the second statement, let $\phi\colon \Gamma \to \Lambda$ be a homomorphism. Since K_i is a finite index subgroup of G_i, it implies that Γ contains $K := K_1 \times K_2$ as a finite index subgroup. To prove that ϕ factors through either π_1 or π_2, we only need to show that ϕ is trivial on either K_1 or K_2. If not, suppose that there exists $x_i \in K_i$ such that $\phi(x_i) \neq 1$. Since x_1 and x_2 commute with each other, their centralizers are the same cyclic group C. Since K_2 commutes with x_1 and K_1 commutes with x_2, we know that $\phi(K_1 \times K_2)$ lies in C, which is a cyclic group.

Now assume that $\phi(K)$ is a cyclic group generated by $a \in \Lambda$, but $\phi(\Gamma)$ is not cyclic. There is an element $\gamma \in \Gamma$ such that $\phi(\gamma) = b$ does not commute with a. Thus, no power of a is in the group generated by b because otherwise they are both in the centralizer of that power, which should be a cyclic group. However, since K is a finite index subgroup of Γ and $\phi(K)$ is cyclic, we know that $\phi(\Gamma)$ is a finite extension of a cyclic group, which implies that some power of a should lie in the group generated by b. This is a contradiction. ∎

3. Surface braid groups

In this section, we discuss first Betti numbers of covers of surface braid groups and their homomorphisms to torsion-free, non-elementary hyperbolic groups.

3.1. The case of the braid group of a closed surface.

Proof of Theorem 1.2 and Theorem 1.1. Let

$$e\colon \mathrm{PConf}_n(S_g) \to (S_g)^n$$

be the natural embedding, where the image is the complement of the diagonal $\triangle \subset (S_g)^n$ consisting of points $(x_1, \dots, x_n)$ such that $x_i = x_j$ for some $1 \leq i < j \leq n$. By Lefschetz hyperplane theorem, the induced map e_* on fundamental group is a surjection. Furthermore, the kernel K of e_* is normally generated by $\{T_{ij}\}$, which geometrically is a loop around the diagonal

$$\triangle_{ij} := \{(x_1, \dots, x_n) \mid x_i = x_j\} \subset S_g^n.$$

We will prove a stronger theorem that for any finite index subgroup $\Gamma < PB_n(S_g)$ satisfying $K < \Gamma$, any homomorphism $\Gamma \to \Lambda$ factors through e_* for Λ a torsion-free, non-elementary hyperbolic group. Then Theorems 1.2 and 1.1 follow from Theorem 2.5, [4, Lemma 2.1] and the fact that

$$e_*\colon H^1(\mathrm{PConf}_n(S_g); \mathbb{Q}) \to H^1((S_g)^n; \mathbb{Q})$$

is an isomorphism.

Let $\rho\colon \Gamma \to \Lambda$ be a homomorphism. The goal is to prove that $\rho(gT_{ij}g^{-1}) = 0$ for any $g \in PB_n(S_g)$ (notice that g may not be in Γ, but $gT_{ij}g^{-1} \in K \subset \Gamma$).

Let us consider the centralizer of T_{ij}. The group $PB_n(S_g)$ can also be thought of as a point-pushing subgroup of the mapping class group $\mathrm{Mod}(S_{g,n})$, the connected component of the group of homeomorphisms of S_g fixing n marked points $m_1, \ldots, m_n$ (see, e.g., [7, Chapter 9] for more background on mapping class groups). Under this interpretation, the element T_{ij} can also be thought as the Dehn twist around a simple closed curve surrounding points m_i and m_j. The centralizer of T_{ij} inside $PB_n(S_g)$ satisfies the following short exact sequence:

$$1 \to \mathbb{Z} \xrightarrow{T_{ij}} C(T_{ij}) \to PB_{n-1}(S_g) \to 1. \tag{2}$$

Let US_g be the unit circle bundle of the genus g surface. The above short exact sequence is actually a pull-back of the following exact sequence:

$$1 \to \mathbb{Z} \to \pi_1(US_g) \to \pi_1(S_g) \to 1 \tag{3}$$

by a forgetful map that forgets all points except the ith and jth points. To check whether the Euler class of (2) is trivial or not, we only need to compute the pull-back of the Euler class from (3), which is a multiple of the fundamental class of S_g. By [4, Lemma 3.1], we see that the pull-back of the fundamental class by any forgetful map p_i is not rationally trivial. This implies that the Euler class of (2) is nontrivial, which shows that T_{ij} vanish under any homomorphism $C(T_{ij}) \to \Lambda$ by Proposition 2.2. The same method applies to all conjugates of T_{ij} as well. Since conjugates of all the T_{ij} generate K, we know that ρ is trivial on K. ■

3.2. The case of the braid group of a closed surface. We now prove Theorem 1.3. Even though the idea is similar to that of Theorem 1.2, but the proof is more technical.

Proof of Theorem 1.3. For the same reason as in the proof of Theorems 1.2 and 1.1, we have the following exact sequence:

$$1 \to K \to PB_n(S_{g,p}) \to \pi_1(S_{g,p}) \times \cdots \times \pi_1(S_{g,p}) \to 1.$$

We now consider $PB_n(S_{g,p})$ as a subgroup of $\mathrm{Mod}(S_{g,p+n})$. Let $\{m_1, \ldots, m_n\}$ be the marked points and $\{q_1, \ldots, q_p\}$ be the punctures. Then K is normally generated by T_{ij}, which are Dehn twists about curves surrounding the points m_i and m_j for $1 \le i, j \le n$.

The centralizer of T_{ij} satisfies the following exact sequence:

$$1 \to \mathbb{Z} \xrightarrow{T_{ij}} C(T_{ij}) \to PB_{n-1}(S_{g,p}) \to 1. \tag{4}$$

The only difference between punctured case and closed case is that the Euler class of (4) is rationally trivial but the Euler class of (2) is rationally nontrivial. So we need a different strategy.

We now work with the case $n = 2$ and the case $n = 2$ implies the rest by induction. We will show in this case that $\rho(T_{ij})$ is trivial. Assume that $\rho(T_{12}) \neq 1$, where T_{12} is the Dehn twist about a simple closed curve c surrounding m_1 and m_2. We claim that there exists a simple closed curve c' surrounding m_1 and q_k or m_2 and q_k for some k such that $\rho(T_{c'})$ is nontrivial. Otherwise, ρ is trivial over all of such simple closed curves; in which case we would know that ρ factors through $PB_2(S_g)$, since Dehn twists about all of such simple closed curves generate the kernel of the natural homomorphism $PB_2(S_{g,p}) \to PB_2(S_g)$. Then we conclude the theorem by Theorem 1.1. Without loss of generality, we assume there exists c' surrounding m_1 and q_k such that $\rho(T_{c'}) \neq 1$. We can choose c to intersect c' only at 2 points (because all curves surrounding m_1, m_2 are conjugate by the action of $PB_2(S_{g,p})$), which satisfies $\rho(T_c) \neq 1$.

For convenience, we introduce the following notations. Let $x_1 = m_1$, $x_2 = m_2$, $x_3 = q_k$ and x_4 be the set of the rest of the punctures other than q_k, which are schematically represented by Figure 1 below on a plane $P \cong \mathbb{R}^2 \subset S_{g,p}$. In Figure 1, we call a simple closed curve $c_{i_1 \ldots i_j}$ if it surrounds a convex disk with punctures $x_{i_1}, \ldots, x_{i_j}$. We position the plane P such that $c_{12} = c$ and $c_{13} = c'$. For example, Figure 1 has c_{34}, c_4, c_{13}.

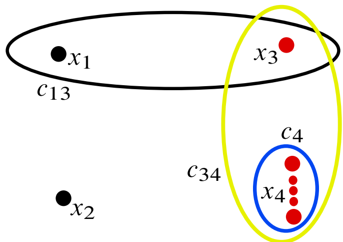

Figure 1
Notation for curves.

We continue to introduce more notations.

- Let A_{12} be the Dehn twist about c_{12}.
- Let A_{i3} be the Dehn twist about c_{i3} for $i \in \{1, 2\}$.
- Let A_{i4} be the product of Dehn twist about c_{i4} and a negative power of Dehn twist about c_4 for $i \in \{1, 2\}$.

- Let A_{123} be the Dehn twist about c_{123} and let A_{124} be the product of Dehn twist about c_{124} and a negative power of Dehn twist about c_4.
- Let A_{1234} be the product of Dehn twist about c_{1234} and a negative power of Dehn twist about c_{34}.

In defining the above list of elements, we sometimes multiply by a negative power of Dehn twist about c_4 or c_{34} so that all of A_* lie in $PB_2(S_{g,p})$. On [7, p. 97] and [7, p. 119], point-pushing map and disk-pushing map are defined as subgroups of $\mathrm{Mod}(S_{g,p+2})$. The point-pushing map of a loop is the mapping class in the isotopy class of pushing a marked point around a loop in $S_{g,p+1}$ (the other marked points are fixed); the disk-pushing map of a loop is the mapping class in the isotopy class of pushing the disk with boundary c_{12} around a loop in $S_{g,p}$. The following relations originate from the point- or disk-pushing map or the lantern relation (see [7, Chapter 5] for more details about relations in mapping class groups).

(1) $A_{12}A_{13}A_{23} = A_{123}$.

(2) $A_{12}A_{14}A_{24} = A_{124}$.

(3) $A_{123}A_{124}A_{12}^{-2} = A_{1234}A_{12}^{-1}$, which is the disk-pushing of c_{12} around c_{34}. After forgetting m_1 and m_2, the curve c_{34} is the boundary of a genus g subsurface in $S_{g,p}$. Therefore, $A_{1234}A_{12}^{-1}$ is a commutator in the centralizer of A_{12}.

(4) $A_{12}A_{14}A_{13} = A_{1234}A_{234}^{-1}$, which is the point-pushing of x_1 around c_{234}. Similarly, $A_{1234}A_{234}^{-1}$ is a commutator in the centralizer of A_{234}, A_{23} and A_{24} because the point-push of x_1 around other curves in the punctured surface do not intersect c_{234}.

(5) $A_{12}A_{23}A_{24} = A_{1234}A_{134}^{-1}$, which is the point-pushing of x_2 around c_{134} and, similarly, is a commutator in the centralizer of A_{134}, A_{13} and A_{14}.

According to the assumption, we know that both $\rho(A_{12})$ and $\rho(A_{13})$ are nontrivial. Therefore, by (3) of the above and the fact that the centralizer of $\rho(A_{12})$ is cyclic, we know that

$$\rho(A_{1234}A_{12}^{-1}) = \rho(A_{123}A_{124}A_{12}^{-2}) = 1.$$

By (5) that $A_{1234}A_{134}^{-1}$ is a commutator which commutes with A_{13} and that $\rho(A_{13}) \neq 1$, we obtain

$$\rho(A_{12}A_{23}A_{24}) = \rho(A_{1234}A_{134}^{-1}) = 1.$$

This implies that either $\rho(A_{23}) \neq 1$ or $\rho(A_{24}) \neq 1$, which further implies that the image under ρ of the commutator in (4) is trivial, as it lies in the centralizers of $\rho(A_{23})$ and $\rho(A_{24})$.

Therefore, we know that $\rho(A_{12}A_{14}A_{13}) = 1$ in (4). Multiplying (1), (2) and $\rho(A_{123}A_{124}) = \rho(A_{12}^2)$ gives us

$$\rho(A_{13}A_{14}A_{23}A_{24}) = 1.$$

It contradicts the result of the multiplication of (4) and (5) under ρ

$$\rho(A_{13}A_{14}A_{23}A_{24}A_{12}^2) = 1,$$

which implies that $\rho(A_{12}^2) = 1$. ∎

4. The pure braid groups of the disk

In the section, we will prove Theorem 1.4. We first introduce a generating set for PB_n. Recall that PB_n is the pure mapping class group of the disk with n-marked points; i.e., $\pi_0(\mathrm{Diff}(\mathbb{D}_n))$, where $\mathrm{Diff}(\mathbb{D}_n)$ is the group of diffeomorphisms of $\mathbb{D}$ fixing n marked points pointwise. Consider the disk with n marked points $\mathbb{D}_n$ in Figure 2.

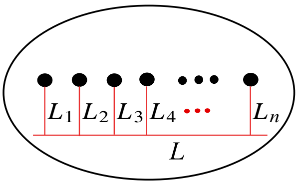

FIGURE 2
$\mathbb{D}_n$.

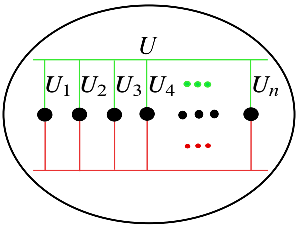

FIGURE 3
$\mathbb{D}_n$.

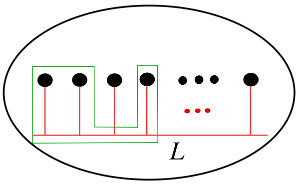

FIGURE 4
a_{124}.

Let L be a line segment below all the marked points $x_1, \dots, x_n$. Let $L_1, \dots, L_n$ be line segments connecting $x_1, \dots, x_n$ to L as in Figure 2. Similarly, let U be a line segment above all marked points and let $U_1, \dots, U_n$ be line segments connecting $x_1, \dots, x_n$ to U as shown in Figure 3.

For $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, let $a_{i_1 i_2 \dots i_k}$ (resp. $a'_{i_1 i_2 \dots i_k}$) be the boundary curve of the tubular neighborhood of $\bigcup_{m=1}^{k} L_{i_m} \cup L$ (resp. $\bigcup_{m=1}^{k} U_{i_m} \cup U$). Let $T_{i_1 i_2 \dots i_k}$ (resp. $T'_{i_1 i_2 \dots i_k}$) be the Dehn twist about $a_{i_1 i_2 \dots i_k}$ (resp. $a'_{i_1 i_2 \dots i_k}$). Figure 4 gives an example of a curve representing a_{124}. The following proposition about generating sets of PB_n is classical and can be found in [11, Theorem 2.3].

Proposition 4.1. *Both* $\{T_{ij} \mid 1 \le i < j \le n\}$ *and* $\{T'_{ij} \mid 1 \le i < j \le n\}$ *are generating sets for* PB_n.

Before proving Theorem 1.4, we analyze the map $R_*\colon PB_4 \to PB_3$.

Fact 4.2. *The map R_* satisfies the following relations:*

$$R_*(T_{12}) = T_{12}, \qquad R_*(T_{23}) = T_{23}, \qquad R_*(T_{34}) = T_{12},$$
$$R_*(T_{13}) = T_{13}, \qquad R_*(T_{24}) = T_{12}^{-1}T_{23}^{-1}T_{12}T_{13}T_{23}, \qquad R_*(T_{14}) = T_{23}.$$

Proof. The "resolving the quartic" map is also a map $R'\colon \mathrm{Conf}_4(\mathbb{C}) \to \mathrm{Conf}_3(\mathbb{C})$, where $\mathrm{Conf}_n(\mathbb{C})$ is the space of unordered n-tuples of points in $\mathbb{C}$. It induces a homomorphism on the fundamental groups $R'_*\colon B_4 \to B_3$, where $B_n := \pi_1(\mathrm{Conf}_n(\mathbb{C}))$ is the braid group.

The above relations can be computed using the map on the braid group $R_*\colon B_4 \to B_3$. Let σ_i be the standard generating set for the braid group B_n. The map R'_* satisfies that

$$R'_*(\sigma_1) = \sigma_1, \qquad R'_*(\sigma_2) = \sigma_2 \qquad \text{and} \qquad R'_*(\sigma_3) = \sigma_1.$$

The restriction of R'_* on PB_n can be computed from it. ■

The homomorphism RQ satisfies the following.

Fact 4.3. *Let $\{a, b\}$ be the natural generating set of F_2. Then the homomorphism RQ satisfies*

$$RQ(T_{12}) = a, \qquad RQ(T_{23}) = b, \qquad RQ(T_{34}) = a,$$
$$RQ(T_{13}) = b^{-1}a^{-1}, \qquad RQ(T_{24}) = a^{-1}b^{-1}, \qquad RQ(T_{14}) = b.$$

Proof. The quotient map $Q\colon PB_3 \to F_2$ satisfies that $Q(T_{12}) = a$ and $Q(T_{23}) = b$ and $Q(T_{13}) = b^{-1}a^{-1}$. Thus, by Fact 4.2, we conclude the proof. ■

We now start the proof of Theorem 1.4.

Proof of Theorem 1.4. Let Λ be a torsion-free, non-elementary hyperbolic group and let $\rho\colon PB_n \to \Lambda$ be a homomorphism such that the image is not a cyclic group. Let Z_n be the generator of the center of PB_n, which is the Dehn twist about the boundary curve. We prove this theorem by induction on n. Firstly, when $n = 3$, if $\rho(Z_3) \neq 1$, then $\rho(PB_3)$ lies in the centralizer of $\rho(Z_3)$, which is a cyclic group. Thus, we know that ρ factors through Q. We assume now that the theorem is true for $n - 1$.

Since $\{T_{ij} \mid 1 \leq i < j \leq n\}$ is a generating set of PB_n, and the fact that ρ is not trivial, there is an element $T_{i,j}$ such that $\rho(T_{ij}) \neq 1$. Observe that there exists an element $g \in B_n$ such that $gT_{ij}g^{-1} = T_{12}$ and therefore, we can assume that $\rho(T_{12}) \neq 1$. Since the image of ρ is not cyclic, we know there exist i, j such that $\rho(T_{ij})$ does not commute with a. It implies that $\{i, j\} \cap \{1, 2\} \neq \emptyset$ because otherwise T_{ij} and T_{12} commute. Observe that there exists an element $g \in B_n$ such that $g\{T_{ij}, T_{12}\}g^{-1} = \{T_{12}, T_{23}\}$. Then up to a conjugation by g, we assume that $a := \rho(T_{12})$ and $b := \rho(T_{23})$ do not commute (a conjugation by g is equivalent to a renaming of punctures).

We split the rest of the proof into two cases depending on whether $\rho(T_{34})$ is trivial or not.

The case when $\rho(T_{34}) = 1$. By the lantern relation, we have

$$T_{123}T_{34}T_{124} = T_{12}T_{1234}. \tag{5}$$

Since $\rho(T_{1234})$ commutes with both $a = \rho(T_{12})$ and $b = \rho(T_{23})$, we know that $\rho(T_{1234}) = 1$. Similarly, we know that $\rho(T_{123}) = 1$. Thus, we have $\rho(T_{124}) = a$. Since $\rho(T_{14})$ commutes with both $b = \rho(T_{23})$ and $a = \rho(T_{124})$, we know that $\rho(T_{14}) = 1$. Then by the lantern relation, we have

$$T_{12}T_{24}T_{14} = T_{124}. \tag{6}$$

Thus, we have $\rho(T_{24}) = 1$. Since $\rho(T_{4j})$ commutes with both $a = \rho(T_{12})$ and $b = \rho(T_{23})$ for any $j > 4$, we know that $\rho(T_{4j}) = 1$ for $j > 4$. Observe that $\{T_{4j} \mid 1 \leq j \neq 4 \leq n\}$ is a generating set of the kernel of the forgetful map that forgets the fourth point $F_4\colon PB_n \to PB_{n-1}$. Then we know that ρ factors through F_4, which by induction gives the result.

The case when $\rho(T_{34}) \neq 1$. We first prove the claim that $\rho(T_{34}) = a$. If not, by equation (5) and the fact that $\rho(T_{1234}) = \rho(T_{123}) = 1$, we know that $\rho(T_{124}) \neq 1$ and it commutes with a. Since $\rho(T_{14})$ commutes with two independent elements $\rho(T_{124})$ and $\rho(T_{23})$, we know that $\rho(T_{14}) = 1$. Thus, by equation (6), we know that $\rho(T_{24}) \neq 1$ and commutes with a. The element $\rho(T_{234})$ is trivial since it commutes with two independent elements $\rho(T_{23})$ and $\rho(T_{34})$. This contradicts the lantern relation $T_{23}T_{34}T_{24} = T_{234}$ because $\rho(T_{34}T_{24})$ commutes with a but $\rho(T_{23})$ does not.

We now prove that

$$\rho(T_{14}) = b, \qquad \rho(T_{13}) = b^{-1}a^{-1} \qquad \text{and} \qquad \rho(T_{24}) = a^{-1}b^{-1}.$$

Since $\rho(T_{123})$ commutes with $\rho(T_{12})$ and $\rho(T_{23})$, we know that $\rho(T_{123}) = 1$. Similarly, we know that $\rho(T_{234}) = 1$. By the lantern relation $T_{12}T_{23}T_{13} = T_{123}$, we know that $\rho(T_{13}) = b^{-1}a^{-1}$. Then by the lantern relation $T_{23}T_{34}T_{24} = T_{234}$, we know that $\rho(T_{24}) = a^{-1}b^{-1}$. Since $\rho(T_{124})$ commutes with $a = \rho(T_{12})$ and $a^{-1}b^{-1} = \rho(T_{24})$, we know that $\rho(T_{124}) = 1$. Now by the lantern relation $T_{12}T_{24}T_{14} = T_{124}$ we obtain that $T_{14} = b$. When $n = 4$, by Fact 4.3, we know that ρ factors through RQ.

When $n > 4$, we claim that $\rho(T'_{5j}) = 1$ for $1 \leq j \neq 5 \leq n$, which would imply that ρ factors through the forgetful map $F_5\colon PB_n \to PB_{n-1}$, since $\{T'_{5j} \mid 1 \leq j \neq 5 \leq n\}$ is a generating set of the kernel of F_5. The equation $\rho(T'_{51}) = 1$ follows from the commutativity of $\rho(T'_{5j})$ with $\rho(T_{23})$ and $\rho(T_{34})$. The triviality of $\rho(T'_{5j})$ for other j follows from the same reason, and the result follows. ■

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