A simple construction of finitely generated infinite torsion groups

Denis Osin

Abstract. The goal of this note is to provide yet another proof of the following theorem of Golod: there exists an infinite finitely generated group G such that every element of G has finite order. Our proof is based on the Nielsen–Schreier index formula and is simple enough to be included in a standard group theory course.

1. Introduction

A group G is said to be *torsion* (or *periodic*) if every element of G has finite order. Obviously, every finite group has this property. Infinite torsion groups can be constructed as direct products of finite groups; however, such groups are not finitely generated. The following famous problem was posed by William Burnside in 1902 and served as a catalyst for research in group theory throughout the 20th century.

Problem 1. Is every finitely generated torsion group finite?

It is easy to show that the answer is affirmative for abelian groups. By induction, this generalizes to all solvable groups. In 1911, Schur proved that every finitely generated torsion subgroup of $GL(n, \mathbb{C})$ is finite [24]. The same result holds for linear groups over arbitrary fields. Indeed, the Burnside problem for such groups can be reduced to the case of solvable groups by utilizing the well-known Tits alternative: *every finitely generated linear group contains either a solvable subgroup of finite index or a non-cyclic free subgroup* [25].

Despite these positive results, the answer to Burnside's question turns out to be negative in general as demonstrated by Golod [5] in 1964.

Theorem 2 (Golod). *There exists a finitely generated infinite torsion group.*

The proof of Theorem 2 given in [5] relied on a sufficient condition for certain graded algebras to be infinite dimensional, known as the Golod–Shafarevich inequality [6] (for a simplified version of Golod's argument, see [19]). Numerous other constructions of infinite finitely generated torsion groups have been discovered since then. Notable examples include groups generated by finite automata [2], groups of interval exchange transformations [7], inductive limits of hyperbolic groups [8, 18], and certain groups of finite exponent. The latter class of examples deserves a more detailed discussion.

Recall that a group G has *exponent* $n \in \mathbb{N}$ if every element $g \in G$ satisfies the identity

$$g^n = 1$$

and n is the smallest natural number with this property. It is well known and easy to prove that every group of exponent 2 is abelian. Therefore, finitely generated groups of exponent 2 are finite. Burnside [4] proved that finitely generated groups of exponent 3 are finite. Sanov [22] and Hall [9] obtained the same result for exponents 4 and 6, respectively. However, it remains an open problem whether a finitely generated group of exponent 5 can be infinite.

As we move on to larger exponents, we encounter infinite finitely generated groups. More precisely, let $B(m, n)$ denote the free group of rank m in the variety of all groups of exponent n. In a monumental series of papers [14–16], Novikov and Adian showed that $B(m, n)$ is infinite for all $m \geq 2$ and all odd $n \geq 4381$. An improved version of the original proof for odd $n \geq 665$ can be found in the book [1]. In [17], Olshanskii suggested a much simpler geometric proof of the Novikov–Adian theorem for odd $n > 10^{10}$. The question of whether $B(m, n)$ is infinite for all $m \geq 2$ and all sufficiently large even n remained open until the mid-1990s, when Ivanov [10] and Lysenok [13] independently gave the affirmative answer.

Despite the progress made, there is still no elementary construction of a finitely generated, infinite group of finite exponent. The easiest proof, given by Olshanskii in [17], is approximately 30 pages long and rather technical. On the other hand, the constructions of Aleshin [2] and Grigorchuk [7], as well as Olshanskii's version of Golod's proof [19], are simple enough to be discussed in a standard group theory course (in these examples, the groups are not of finite exponent).

The goal of this paper is to provide yet another elementary proof of Golod's theorem that only relies on basic properties of finitely generated abelian groups and the Nielsen–Schreier formula. In one form or another, this proof appeared in [12,20,21,23]. However, the exposition in these papers was "spoiled" by technicalities necessary to ensure certain additional properties. Here, we present the argument in its simplest form.

2. Deficiency of finitely presented groups

To make our paper as self-contained as possible, we review the standard results about abelian groups involved in the proof. Our primary reference is [11], although these results can be found in most group theory textbooks.

Recall that every finitely generated abelian group A can be decomposed as a direct sum of cyclic groups [11, Theorem 8.1.2]. We denote the number of infinite summands in this decomposition by $\mathrm{rk}(A)$. If A is free abelian, $\mathrm{rk}(A)$ is the usual rank of A. Clearly, $\mathrm{rk}(A)$ serves as the lower bound on the number of generators of A.

The standard proof of the decomposition theorem for finitely generated abelian groups goes through establishing the following fact (see [11, Theorem 8.1.1]).

For any finitely generated free abelian group B and any subgroup $C \leq B$, there exist bases $\{b_1, \ldots, b_n\}$ and $\{c_1, \ldots, c_m\}$ ($m \leq n$) of B and C, respectively, and positive integers $d_1, \ldots, d_m$ such that $c_i = b_i^{d_i}$ for all $1 \leq i \leq m$.

In this notation, the quotient $A = B/C$ decomposes as $A = \mathbb{Z}^{n-m} \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_m\mathbb{Z}$. In particular, we have

$$\mathrm{rk}(A) = n - m = \mathrm{rk}(B) - \mathrm{rk}(C).$$

A group presentation $G = \langle X \mid \mathcal{R} \rangle$ is said to be *finite* if the set of generators X and the set of relations $\mathcal{R}$ are finite. Given two elements x, y of a group G, we write x^y for $y^{-1}xy$. Given a subset $S \subseteq G$, we denote by $\langle\!\langle S \rangle\!\rangle^G$ the *normal closure* of S in G. That is, $\langle\!\langle S \rangle\!\rangle^G$ is the subgroup of G generated by the set $\{s^g \mid s \in S, g \in G\}$.

Lemma 3. *Let G be a group given by a finite presentation $\langle X \mid \mathcal{R} \rangle$ and let $d = |X| - |\mathcal{R}|$. Suppose that $d > 0$. Then, there exists a surjective homomorphism $G \to \mathbb{Z}^d$.*

Proof. Let F denote the free group with the basis X. The group $G = F/\langle\!\langle \mathcal{R} \rangle\!\rangle^F$ surjects onto the abelian group

$$A = F\big/\langle\!\langle \mathcal{R} \rangle\!\rangle^F [F, F] \cong (F/[F, F])\big/\big(\langle\!\langle \mathcal{R} \rangle\!\rangle^F [F, F]/[F, F]\big).$$

The group $B = F/[F, F]$ is free abelian of rank $|X|$ (see [11, page 99]). It is easy to see that the group $C = \langle\!\langle \mathcal{R} \rangle\!\rangle^F [F, F]/[F, F]$ is generated by the natural image of $\mathcal{R}$ in C. This implies that $\mathrm{rk}(C) \leq |\mathcal{R}|$. Therefore, $\mathrm{rk}(A) = \mathrm{rk}(B) - \mathrm{rk}(C) \geq d$ and the existence of the required surjection follows from the definition of $\mathrm{rk}(A)$. ∎

Corollary 4. *For any finitely presented group G, the difference $|X| - |\mathcal{R}|$ is uniformly bounded from above over all finite presentations $\langle X \mid \mathcal{R} \rangle$ of G.*

Proof. If G can be generated by n elements, then $|X| - |\mathcal{R}| \leq n$ for any finite presentation $\langle X \mid \mathcal{R} \rangle$ of G by Lemma 3. ∎

Corollary 4 allows us to formulate the following.

Definition 5. For a finitely presented group G, the maximum of the difference between the number of generators and the number of relations over all finite presentations of G is called the *deficiency* of G and denoted by $\mathrm{def}(G)$.

Recall that a subset T of a group G is a *left* (respectively, *right*) *transversal* of a subgroup $H \leq G$ if $G = \bigsqcup_{t \in T} tH$ (respectively, $G = \bigsqcup_{t \in T} Ht$); clearly, we have $|T| = |G : H|$ in both cases.

Lemma 6. *Let G be a finitely presented group. Every finite-index subgroup $H \leq G$ is finitely presented, and we have $\mathrm{def}(H) - 1 \geq (\mathrm{def}(G) - 1)|G : H|$.*

Proof. Let $G = F/R$, where F is free of rank r, $R = \langle\!\langle R_1, \ldots, R_s \rangle\!\rangle^F$, and $r - s = \mathrm{def}(G)$. Let H be a finite-index subgroup of G and K the full preimage of H in F. By the Nielsen–Schreier formula, K is free of rank $(r - 1)j + 1$, where $j = |F : K| = |G : H|$.

Let T be a left transversal of K in F. For every element $f \in F$, we have $f = tk$ for some $t \in T$ and $k \in K$. Hence, $R_i^f = (R_i^t)^k \in \langle\!\langle R_i^t \rangle\!\rangle^K$ for all $i = 1, \ldots, s$. This easily implies that

$$R = \langle \{R_i^f \mid i = 1, \ldots, s, f \in F\} \rangle = \langle\!\langle \{R_i^t \mid i = 1, \ldots, s, t \in T\} \rangle\!\rangle^K.$$

Thus, the group $H = K/R$ has a presentation with $(r - 1)j + 1$ generators and $s|T| = sj$ relations. Therefore, we have

$$\mathrm{def}(H) - 1 \geq (r - 1)j - sj = (r - s - 1)j = (\mathrm{def}(G) - 1)|G : H|. \qquad \blacksquare$$

3. Proof of Golod's theorem

Let $\mathcal{D}$ denote the class of all finitely presented groups that contain a finite-index normal subgroup of deficiency at least 2. For a group G, we denote by $\widehat{G}$ the quotient of G by the intersection of all finite-index subgroups of G.

The idea of the proof of the following proposition is borrowed from [3].

Proposition 7. *Let $G \in \mathcal{D}$. For every $g \in G$, there exists $m \in \mathbb{Z}$ such that, for every $\ell \in \mathbb{N}$, the quotient group $Q = G/\langle\!\langle g^{\ell m} \rangle\!\rangle^G$ belongs to $\mathcal{D}$ and the image of g in $\widehat{Q}$ has finite order.*

Proof. If the image of g in $\widehat{G}$ already has finite order, we can take $m = 0$. Henceforth, we assume that the order of the image of g in $\widehat{G}$ is infinite. That is, for any $i \in \mathbb{N}$, there exists a finite-index subgroups $K \triangleleft G$ such that $|\langle g \rangle K/K| > i$.

Let M be a finite-index normal subgroup of G such that $\mathrm{def}(M) \geq 2$. By our assumption, we can find a finite-index subgroup $K \lhd G$ such that $|\langle g \rangle K/K| > |G/M|$. Let $N = K \cap M \lhd G$. Clearly, we have

$$(1) \qquad |\langle g \rangle N/N| \geq |\langle g \rangle K/K| > |G/M|.$$

Set $m = |\langle g \rangle N/N|$ and $f = g^{\ell m}$, where ℓ is an arbitrary natural number. Obviously, $f \in N$. Let T be a right transversal of $\langle g \rangle N$ in G. Every $s \in G$ can be written as $s = g^k nt$ for some $k \in \mathbb{Z}$, $n \in N$, and $t \in T$. In this notation, we have

$$f^s = f^{g^k nt} = f^{nt} = (f^t)^{n^t} \in \langle\!\langle f^t \rangle\!\rangle^N$$

since $n^t \in N$. This easily implies that $\langle\!\langle f \rangle\!\rangle^G = \langle\!\langle \{ f^t \mid t \in T \} \rangle\!\rangle^N$. Using Lemma 6 and (1), we obtain

$$\begin{aligned}
\mathrm{def}\left(N/\langle\!\langle f \rangle\!\rangle^G\right) &\geq \mathrm{def}(N) - |T| \geq 1 + (\mathrm{def}(M) - 1)|M/N| - |T| \\
&\geq 1 + |M/N|\left(\mathrm{def}(M) - 1 - \frac{|G : \langle g \rangle N|}{|M/N|} \right) \\
&\geq 1 + |M/N|\left(\mathrm{def}(M) - 1 - \frac{|G/N|}{|\langle g \rangle N/N| \cdot |M/N|} \right) \\
&\geq 1 + |M/N|\left(\mathrm{def}(M) - 1 - \frac{|G/M|}{|\langle g \rangle N/N|} \right) > 1.
\end{aligned}$$

Therefore, $\mathrm{def}(N/\langle\!\langle f \rangle\!\rangle^G) \geq 2$ and $G/\langle\!\langle f \rangle\!\rangle^G \in \mathcal{D}$. ∎

Proof of Theorem 2. Let F denote a finitely generated free group of rank at least 2. We enumerate all elements of $F = \{ g_0 = 1, g_1, g_2, \ldots \}$ and construct a sequence of quotients $G_0, G_1, \ldots$ of F and normal subgroups $M_i \lhd G_i$ fitting into the commutative diagram with surjective arrows

$$(2) \qquad \begin{array}{ccccccc}
G_0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
G_0/M_0 & \longleftarrow & G_1/M_1 & \longleftarrow & G_2/M_2 & \longleftarrow & \cdots,
\end{array}$$

by the following inductive procedure.

Let $M_0 = G_0 = F$. Suppose that for some $k \geq 0$, we have already constructed a group G_k and a subgroup $M_k \lhd G_k$ such that the following conditions hold (note that G_0 and M_0 obviously satisfy these conditions):

(a) $G_k \in \mathcal{D}$;

(b) the natural image of g_k in $\hat{G}_k$ has finite order;

(c) $\infty > |G_k/M_k| > k$.

By (a) and Lemma 3, G_k contains subgroups of arbitrarily large finite index. In particular, we can find a subgroup $L_k \lhd G_k$ such that $L_k \leq M_k$ and $\infty > |G_k/L_k| > k + 1$. Let g denote the image of g_{k+1} in G_k and let $\ell = |G_k/L_k|$. By Proposition 7, there exists $m \in \mathbb{Z}$ such that

$$G_{k+1} = G_k/\langle\!\langle g^{\ell m} \rangle\!\rangle^{G_k} \in \mathcal{D}$$

and the image of g in $\widehat{G}_{k+1}$ has finite order. Let M_{k+1} denote the image of L_k in G_{k+1}. Note that

$$G_{k+1}/M_{k+1} \cong G_k/L_k$$

since $g^{\ell m} \in L_k$. Therefore,

$$|G_{k+1}/M_{k+1}| = |G_k/L_k| > k + 1$$

and G_{k+1}/M_{k+1} naturally surjects onto G_k/M_k since $L_k \leq M_k$.

Thus, we obtain the commutative diagram (2), where all horizontal arrows are surjective and all vertical arrows are natural homomorphisms. Let G be the direct limit of the first row of (2). That is, $G = G_0/\bigcup_{k \in \mathbb{N}} N_k$, where N_k is the kernel of the homomorphism $G_0 \to G_k$ obtained by composing the first k maps in the first row of (2). By (b), the image of g_k has finite order in $\widehat{G}_k$ for each k. Since every $\widehat{G}_k$ naturally surjects onto $\widehat{G}$, the group $\widehat{G}$ is torsion. On the other hand, $\widehat{G}$ surjects onto G_k/M_k for all k. Combining this with (c), we obtain that $\widehat{G}$ is infinite. ∎

References

[1] S. I. Adian, *The Burnside problem and identities in groups*. Ergeb. Math. Grenzgeb 95, Springer, Berlin-New York, 1979. Zbl 0417.20001 MR 0537580

[2] S. V. Aleshin, Finite automata and the Burnside problem for periodic groups. *Math. Notes* **11** (1972), 199–203. Zbl 0253.20049

[3] B. Baumslag and S. J. Pride, Groups with one more generator than relators. *Math. Z.* **167** (1979), no. 3, 279–281. Zbl 0388.20025 MR 0539110

[4] W. Burnside, On an unsettled question in the theory of discontinuous groups. *Quart. J. Math.* **33** (1902), 230–238. Zbl 33.0149.01

[5] E. S. GOLOD, On nil-algebras and finitely approximable *p*-groups. *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 273–276. Zbl 0136.02602   MR 0161878

[6] E. S. GOLOD and I. R. ŠAFAREVIČ, On the class field tower. *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 261–272. Zbl 0136.02602   MR 0161852

[7] R. I. GRIGORCHUK, On Burnside's problem on periodic groups. *Funktsional. Anal. i Prilozhen.* **14** (1980), no. 1, 53–54. Zbl 0595.20029   MR 0565099

[8] M. GROMOV, Hyperbolic groups. In *Essays in group theory*, pp. 75–263, Math. Sci. Res. Inst. Publ. 8, Springer, New York, 1987. Zbl 0634.20015   MR 0919829

[9] M. HALL, JR., Solution of the Burnside problem for exponent six. *Illinois J. Math.* **2** (1958), 764–786. Zbl 0083.24801   MR 0102554

[10] S. V. IVANOV, The free Burnside groups of sufficiently large exponents. *Internat. J. Algebra Comput.* **4** (1994), no. 1-2, ii+308. Zbl 0822.20044   MR 1283947

[11] M. I. KARGAPOLOV and J. I. MERZLJAKOV, *Fundamentals of the theory of groups*. Grad. Texts in Math. 62, Springer, New York-Berlin, 1979. Zbl 0549.20001   MR 0551207

[12] W. LÜCK and D. OSIN, Approximating the first L^2-Betti number of residually finite groups. *J. Topol. Anal.* **3** (2011), no. 2, 153–160. Zbl 1243.20039   MR 2819192

[13] I. G. LYSËNOK, Infinite Burnside groups of even period. *Izv. Ross. Akad. Nauk Ser. Mat.* **60** (1996), no. 3, 3–224. Zbl 0926.20023   MR 1405529

[14] P. S. NOVIKOV and S. I. ADJAN, Infinite periodic groups. I. *Izv. Akad. Nauk SSSR Ser. Mat.* **32** (1968), 212–244. Zbl 0194.03301   MR 0240178

[15] P. S. NOVIKOV and S. I. ADJAN, Infinite periodic groups. II. *Izv. Akad. Nauk SSSR Ser. Mat.* **32** (1968), 251–524. Zbl 0194.03301   MR 0240179

[16] P. S. NOVIKOV and S. I. ADJAN, Infinite periodic groups. III. *Izv. Akad. Nauk SSSR Ser. Mat.* **32** (1968), 709–731. Zbl 0194.03301   MR 0240180

[17] A. Y. OL'SHANSKII, The Novikov–Adyan theorem. *Math. USSR Sb.* **46** (1983), no. 2, 203–235. Zbl 0539.20020

[18] A. Y. OL'SHANSKII, On residualing homomorphisms and *G*-subgroups of hyperbolic groups. *Internat. J. Algebra Comput.* **3** (1993), no. 4, 365–409. Zbl 0830.20053   MR 1250244

[19] A. Y. OL'SHANSKII, A simplification of Golod's example. In *Groups—Korea '94 (Pusan)*, pp. 263–265, de Gruyter, Berlin, 1995. Zbl 0864.20024   MR 1476969

[20] A. Y. OLSHANSKII and D. V. OSIN, Large groups and their periodic quotients. *Proc. Amer. Math. Soc.* **136** (2008), no. 3, 753–759. Zbl 1151.20032   MR 2361846

[21] D. OSIN, Rank gradient and torsion groups. *Bull. Lond. Math. Soc.* **43** (2011), no. 1, 10–16. Zbl 1245.20044   MR 2765544

[22] I. N. SANOV, Solution of Burnside's problem for exponent 4. *Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser.* **10** (1940), 166–170. Zbl 66.0085.02   MR 0003397

[23] J.-C. Schlage-Puchta, A *p*-group with positive rank gradient. *J. Group Theory* **15** (2012), no. 2, 261–270. Zbl 1259.20046 MR 2900227

[24] I. Schur, über Gruppen periodischer substitutionen. *Sitzungsber. Preuss. Akad. Wiss.* (1911), 619–627. Zbl 42.0155.01

[25] J. Tits, Free subgroups in linear groups. *J. Algebra* **20** (1972), 250–270. Zbl 0236.20032 MR 0286898

(Reçu le 17 novembre 2022)

Denis Osin, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA; *e-mail:* denis.v.osin@vanderbilt.edu