Double Veronese cones with 28 nodes

Hamid Abban, Ivan Cheltsov, Jihun Park and Constantin Shramov

Abstract. We study nodal del Pezzo 3-folds of degree 1 (also known as double Veronese cones) with 28 singularities, which is the maximal possible number of singularities for such varieties. We show that they are in one-to-one correspondence with smooth plane quartics and use this correspondence to study their automorphism groups. As an application, we find all G-birationally rigid varieties of this kind, and construct an infinite number of non-conjugate embeddings of the group $\mathfrak{S}_4$ into the space Cremona group.

1. Introduction

After some precise surgeries dictated by the Minimal Model Program, objects in projective geometry are reduced to simpler shapes forming three main classes: varieties of general type, Calabi–Yau varieties, Mori fiber spaces. The latter consists of varieties fibered into Fano varieties. For this reason, Fano varieties inherit the title of being *building blocks* of algebraic varieties. They are defined by a positivity condition on their anticanonical line bundles. The simplest example is the projective space itself. Although Fano varieties are the closest relatives to projective spaces, they are not always rational (birational to the projective space). There is a nice interplay between admitting singularities and becoming rational for such Fano varieties; this is the starting point of the study in this article.

It is a common experience that varieties with the maximal possible number of singularities in a given class have fascinating geometric properties. Classical examples of such varieties include the Burkhardt quartic (see [10,17]), the Segre cubic, and many others. The Segre cubic, which is the unique three-dimensional cubic hypersurface with 10 isolated singular points, can be thought of as a *del Pezzo* 3-*fold*. Recall that a

del Pezzo 3-fold is a Fano 3-fold with at most terminal Gorenstein singularities whose canonical class is divisible by 2 in the Picard group; in particular, the singularities of such 3-folds are always isolated. We refer the reader to the book [39] for an excellent introduction to the general theory of Fano and del Pezzo varieties.

Del Pezzo 3-folds are classified according to their *degrees*, which are defined as the cubes of the half-anticanonical classes. Such varieties with a maximal possible number of singularities were described in [50, Theorem 7.1]. The smaller degrees such del Pezzo 3-folds have, the more intriguing their geometry is. Del Pezzo 3-folds of degree 4 can have at most six singularities, and there exists a unique such variety with exactly six singular points. It has been studied in [2]. There are many works devoted to the geometry of the Segre cubic (see [3, 5, 33], [21, §2], [13, §5]). Del Pezzo 3-folds of degree 2 with 16 singular points, which is the maximal possible number of singularities in this case, are quartic double solids branched along Kummer quartic surfaces. Thus their properties are closely related to those of the corresponding abelian surfaces. In particular, such del Pezzo 3-folds are in one-to-one correspondence with smooth curves of genus 2 (see for instance [43, Propositions 1.2 (i) and 1.3]).

The current paper studies the geometry of del Pezzo 3-folds of degree 1 with 28 singular points, which is the maximal possible number of singularities for such varieties (see [50, Theorem 1.7] and [50, Remark 1.8]). Each del Pezzo 3-fold of degree 1 is a hypersurface of degree six in the weighted projective space $\mathbb{P}(1, 1, 1, 2, 3)$ (see [39, Theorem 3.2.5 (i)]). This implies that it is a double cover of the cone over the Veronese surface in $\mathbb{P}^5$. For this reason, del Pezzo 3-folds of degree 1 are often called *double Veronese cones*.

Let V be a double Veronese cone with 28 singular points. Then $-K_V \sim 2H$, where H is an ample Cartier divisor on V such that $H^3 = 1$. The existence of the following diagram was proven in [50]:

(1.1)
$$
\begin{array}{ccc}
\widehat{\mathbb{P}^3} & \xrightarrow{\phi} & V \\
\pi \downarrow & & \downarrow \kappa \\
\mathbb{P}^3 & \dashrightarrow & \mathbb{P}^2
\end{array}
$$

In this diagram, ϕ is a small resolution of all singular points of the 3-fold V, the morphism π is the blow up of $\mathbb{P}^3$ at seven distinct points $P_1, \ldots, P_7$ that satisfy certain explicit generality conditions, the rational map κ is given by the linear system $|H|$, and the rational map from $\mathbb{P}^3$ to $\mathbb{P}^2$ is given by the linear system (net) of quadrics passing through $P_1, \ldots, P_7$. Furthermore, Prokhorov implicitly verified in [50] that each singular point of V is an isolated ordinary double point (a node).

Note that V is not $\mathbb{Q}$-factorial since $\mathrm{Cl}(V) \cong \mathbb{Z}^8$ and $\mathrm{Pic}(V) \cong \mathbb{Z}$. Nevertheless, the class group $\mathrm{Cl}(V)$ can be naturally equipped with the intersection form:

$$(1.2) \qquad\qquad (D_1, D_2) = D_1 \cdot D_2 \cdot H$$

for any two Weil divisors D_1 and D_2 on the 3-fold V. Furthermore, Prokhorov proved in [50, Theorem 1.7] (cf. [50, Remark 1.8]) that the orthogonal complement to the canonical divisor K_V in $\mathrm{Cl}(V)$ is the lattice $\mathbf{E}_7$ (cf. [24, p. 107]). The same lattice appears in the Picard group of the double cover of $\mathbb{P}^2$ branched along a smooth plane quartic curve. This suggests a possible relation between double Veronese cones with 28 nodes and smooth plane quartic curves. As far as we know, such a relation was first pointed out in [24, §IX.6] (see also [56, §1]). A goal in the present paper is to make this relation precise and detailed.

The net of quadrics in (1.1) has exactly eight distinct base points, including the points $P_1, \ldots, P_7$; such a collection of eight points is called a *regular Cayley octad* (see Definition 3.3 for details). Moreover, the Hessian curve of this net is a smooth plane quartic curve, which we denote by C. One can show that the isomorphism class of the curve C is independent of the choice of the commutative diagram (1.1). In fact, the curve C is projectively dual to the discriminant curve of the rational elliptic fibration κ (see §6 for details).

The first main result of the present paper is as follows.

Theorem 1.3. *Assigning the curve C to the 3-fold V gives a one-to-one correspondence between the isomorphism classes of smooth plane quartic curves and the isomorphism classes of 28-nodal double Veronese cones.*

Two alternative proofs of Theorem 1.3 will be presented in this paper. The first proof starts with a plane quartic curve C and produces an explicit equation that defines the corresponding 3-fold V in the weighted projective space $\mathbb{P}(1, 1, 1, 2, 3)$ (see equation (5.10) and §5 for details). Another proof uses machinery from the theory of nets of quadrics to recover a commutative diagram (1.1) and check that the resulting 3-fold does not depend on the choices we have to make on the way.

Observe that for a given 28-nodal double Veronese cone the commutative diagram (1.1) is not unique. However, we can refine Theorem 1.3 to characterize all such diagrams in terms of the corresponding smooth plane quartic curve. Essentially, the diagram (1.1) depends only on the choice of seven distinct points $P_1, \ldots, P_7$ in $\mathbb{P}^3$ up to projective transformations. We will show that there are exactly 288 diagrams (1.1) up to projective transformations of $\mathbb{P}^3$ for a given general 28-nodal double Veronese cone V, and will also give a way to compute the number of such diagrams for an arbitrary V (see Corollaries 7.19 and 7.20). Here, the number $288 = 36 \cdot 8$ equals the number of even theta characteristics of C times the number of choices for a point in

the corresponding regular Cayley octad. In other words, it is the number of Aronhold systems on C. We refer the reader to §7 for details.

Note that there are other classical constructions that associate certain 3-folds to smooth plane quartics. We recall some of them in §10. It would be interesting to find the connections between these varieties and 28-nodal double Veronese cones.

The second result of the present paper is the following.

Theorem 1.4. *Let V be a 28-nodal double Veronese cone, and let C be the smooth plane quartic curve assigned to V as above. Then one has*

$$\mathrm{Aut}(V) \cong \boldsymbol{\mu}_2 \times \mathrm{Aut}(C),$$

where the generator of the cyclic group $\boldsymbol{\mu}_2$ of order 2 acts on V by the Galois involution of the double cover of the Veronese cone.

We point out that Theorem 1.4 can be obtained from [24, §IX.6] and Torelli theorem (see for instance [59]). However, we provide a more elementary proof. Another advantage of our proof is its explicit nature, enabling an easier access to details of the group actions on the 3-folds, which will be useful later on when we study their birational properties.

Note that the commutative diagram (1.1) is not necessarily $\mathrm{Aut}(V)$-equivariant. However, the rational map κ in (1.1) always is. This gives a natural action of the group $\mathrm{Aut}(V)$ on the plane $\mathbb{P}^2$. This action agrees with the direct product structure in Theorem 1.4 (see the proof of Theorem 1.4 in §6).

The automorphism groups of smooth plane quartic curves are completely classified (see [37, 46], [20, Theorem 6.5.2]). This gives us the full list of the automorphism groups of 28-nodal double Veronese cones.

Example 1.5. Let C be the plane quartic curve given by

$$(1.6) \qquad x^4 + y^4 + z^4 + \lambda(y^2 z^2 + x^2 z^2 + x^2 y^2) = 0,$$

where λ is a constant different from -2, 2, and -1 (in these three cases, the curves are singular). The corresponding 28-nodal double Veronese cone V is the hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 2, 3)$ that is given by

$$w^2 = v^3 - g_4(s, t, u)v + g_6(s, t, u),$$

where s, t, u, v and w are weighted homogeneous coordinates on $\mathbb{P}(1, 1, 1, 2, 3)$ whose weights are equal to 1, 1, 1, 2 and 3, respectively. From §5 and Appendix A we obtain

$$g_4(s, t, u) = \frac{(\lambda^2 + 12)}{3}(s^4 + t^4 + u^4) + \frac{2(\lambda^2 + 6\lambda)}{3}(t^2 u^2 + s^2 u^2 + s^2 t^2)$$

and

$$g_6(s,t,u) = \frac{2(-\lambda^3 + 12\lambda^2 + 12\lambda)}{9}(t^4u^2 + t^2u^4 + s^4u^2 + s^2u^4 + s^4t^2 + s^2t^4)$$
$$+ \frac{2(-\lambda^3 + 36\lambda)}{27}(s^6 + t^6 + u^6) + \frac{4(8\lambda^3 - 9\lambda^2 + 36)}{9}s^2t^2u^2.$$

Let $\mathfrak{G}$ be the subgroup in $\mathrm{Aut}(V)$ that is generated by

$$[s : t : u : v : w] \mapsto [-s : -t : u : v : w],$$
$$[s : t : u : v : w] \mapsto [u : s : t : v : w],$$
$$[s : t : u : v : w] \mapsto [t : s : u : v : -w].$$

Then $\mathfrak{G} \cong \mathfrak{S}_4$. Let $\tau \in \mathrm{Aut}(V)$ be the Galois involution of the double cover of the Veronese cone. If $\lambda \neq 0$ and $\lambda^2 + 3\lambda + 18 \neq 0$, then τ and $\mathfrak{G}$ generate the group

$$\mathrm{Aut}(V) \cong \boldsymbol{\mu}_2 \times \mathfrak{S}_4$$

(see [37, 46], [20, Theorem 6.5.2]). If $\lambda = 0$, then C is the Fermat quartic, so that

$$\mathrm{Aut}(V) \cong \boldsymbol{\mu}_2 \times (\boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3),$$

where $\boldsymbol{\mu}_n$ denotes the cyclic group of order n. Finally, if $\lambda^2 + 3\lambda + 18 = 0$, then C is isomorphic to the Klein quartic, so that

$$\mathrm{Aut}(V) \cong \boldsymbol{\mu}_2 \times \mathrm{PSL}_2(\mathbb{F}_7).$$

It follows from (1.1) that 28-nodal double Veronese cones are rational. Thus, their automorphism groups are embedded into the space Cremona group, that is the group of birational selfmaps of $\mathbb{P}^3$.

For a given subgroup $G \subset \mathrm{Aut}(V)$, to study its conjugacy class in the space Cremona group, one should understand G-equivariant birational geometry of the 3-fold V. From this point of view, the most interesting groups G are such that $\mathrm{rk}\, \mathrm{Cl}(V)^G = 1$, and V is G-birationally (super-)rigid (see [14, Definition 3.1.1]). In this paper, we prove the following result.

Theorem 1.7. *Let V be a 28-nodal double Veronese cone and G be a subgroup of $\mathrm{Aut}(V)$. Then the following four conditions are equivalent:*

(1) $\mathrm{rk}\, \mathrm{Cl}(V)^G = 1$ *and V is G-birationally rigid;*

(2) $\mathrm{rk}\, \mathrm{Cl}(V)^G = 1$ *and V is G-birationally super-rigid;*

(3) $\mathrm{rk}\, \mathrm{Cl}(V)^G = 1$ *and G does not fix any point in $\mathbb{P}^2$;*

(4) *V is a double Veronese cone from Example 1.5 and G contains the subgroup $\mathfrak{G}$.*

It is known that the plane Cremona group, that is the group of birational selfmaps of $\mathbb{P}^2$, contains an infinite number of non-conjugate subgroups isomorphic to the symmetric group $\mathfrak{S}_4$ (see [22, §8], and cf. [16, Theorem 1.5]). Theorem 1.7 implies the following result.

Corollary 1.8. *The space Cremona group contains an infinite number of non-conjugate subgroups isomorphic to* $\mathfrak{S}_4$.

It is known from [12] and [1, Theorem 2.2] that the space Cremona group contains at least four non-conjugate subgroups isomorphic to $\mathrm{PSL}_2(\mathbb{F}_7)$. Applying Theorem 1.7 to the case of the Klein quartic curve, we obtain the following.

Corollary 1.9. *The space Cremona group contains at least five non-conjugate subgroups isomorphic to* $\mathrm{PSL}_2(\mathbb{F}_7)$.

Let V be the 3-fold from Example 1.5. Then $\mathrm{Aut}(V)$ contains a subgroup $\mathfrak{G}' \cong \mathfrak{S}_4$ that is not conjugate to the subgroup $\mathfrak{G}$. The subgroup $\mathfrak{G}'$ is generated by

$$[s:t:u:v:w] \mapsto [-s:-t:u:v:w],$$
$$[s:t:u:v:w] \mapsto [u:s:t:v:w],$$
$$[s:t:u:v:w] \mapsto [t:s:u:v:w].$$

In §8, we verify that $\mathrm{rk}\,\mathrm{Cl}(V)^{\mathfrak{G}'} \neq 1$, and then construct a $\mathfrak{G}'$-equivariant birational map of V onto a del Pezzo 3-fold W of degree 2 with 16 singular points such that $\mathrm{rk}\,\mathrm{Cl}(W)^{\mathfrak{G}'} = 1$. It would be interesting to determine whether W is $\mathfrak{G}'$-birationally rigid or not (cf. [4]).

We have established a relation between 28-nodal double Veronese cones and smooth plane quartics. Despite this establishment, the connection between the $\mathbf{E}_7$ lattices appearing in their class groups and the Picard groups of the double covers of $\mathbb{P}^2$ branched in the corresponding smooth quartic curves does not seem to be straightforward. We discuss this phenomenon in §10.

The plan of the paper is as follows. In §2 we collect several auxiliary results that will be used later in our proofs. In §3 we recall some assertions concerning nets of quadrics in $\mathbb{P}^3$. The base points of the nets of quadrics described therein will be utilized in order to construct 28-nodal double Veronese cones in §4. This construction is originally due to [50]. In §5 we construct a 28-nodal double Veronese cone starting from a smooth plane quartic curve. We also provide an explicit equation (5.10) for such a 3-fold in terms of covariants of the quartic. In §6 we put the previous results together to establish a one-to-one correspondence between 28-nodal double Veronese cones and smooth plane quartics, which proves Theorem 1.3. Also, in §6 we study the automorphism groups of 28-nodal double Veronese cones and prove Theorem 1.4.

In §7 we give an alternative proof of Theorem 1.3. Also, in §7 we prove Corollary 7.19 which relates commutative diagrams (1.1) with even theta characteristics on the smooth plane quartic. In §8 we investigate the $\mathfrak{S}_4$-equivariant birational geometry of double Veronese cones introduced in Example 1.5, and, in particular, describe the action of the group $\mathfrak{S}_4$ on their Weil divisor class groups. In §9 we study G-birationally rigid nodal double Veronese cones and prove Theorem 1.7. In §10 we discuss some open questions concerning 28-nodal double Veronese cones. Finally, in Appendix A we collect some formulae allowing to construct explicit equations of 28-nodal double Veronese cones from the equations of plane quartics.

2. Preliminaries

In this section we collect several auxiliary results that will be used later in our proofs.

The following fact is well known to experts. It was pointed out to us by Alexander Kuznetsov. This will be utilized in §4.

Lemma 2.1 (cf. [14, Remark 4.2.7]). *Let Y be a smooth variety, C be a smooth curve on Y, and P be a point on C. Let $\pi\colon \widehat{Y} \to Y$ be the blow up of Y at P, and let $\widehat{C}$ be the proper transform of C on $\widehat{Y}$. Then*

$$\mathcal{N}_{\widehat{C}/\widehat{Y}} \cong \mathcal{N}_{C/Y}(-P).$$

Proof. The differential of the morphism π provides the morphisms

$$d\pi\colon T_{\widehat{C}/\widehat{Y}} \to \pi^* T_{C/Y} \quad \text{and} \quad d\pi\colon \mathcal{N}_{\widehat{C}/\widehat{Y}} \to \pi^* \mathcal{N}_{C/Y}.$$

They can be included into the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T_{\widehat{C}} & \longrightarrow & T_{\widehat{C}/\widehat{Y}} & \longrightarrow & \mathcal{N}_{\widehat{C}/\widehat{Y}} & \longrightarrow & 0 \\
& & \cong \downarrow & & \downarrow d\pi & & \downarrow d\pi & & \\
0 & \longrightarrow & \pi^* T_C & \longrightarrow & \pi^* T_{C/Y} & \longrightarrow & \pi^* \mathcal{N}_{C/Y} & \longrightarrow & 0.
\end{array}
$$

Let E be the exceptional divisor of π. Note that the fiber of $\mathcal{N}_{\widehat{C}/\widehat{Y}}$ over the point $\widehat{P} = \widehat{C} \cap E$ is identified with the tangent space to E at $\widehat{P}$. Since the differential $d\pi$ vanishes on the latter space, we conclude that the morphism $d\pi\colon \mathcal{N}_{\widehat{C}/\widehat{Y}} \to \pi^* \mathcal{N}_{C/Y}$ factors through the morphism

$$\theta\colon \mathcal{N}_{\widehat{C}/\widehat{Y}} \to \pi^* \mathcal{N}_{C/Y}(-k\widehat{P})$$

for some positive integer k. Comparing the degrees of these vector bundles, we conclude that $k = 1$. Now θ is a morphism of vector bundles of the same rank and degree, and it is an isomorphism over a general point of $\widehat{C}$. This implies that θ is an isomorphism. ∎

Corollary 2.2. *Let L be a line in $\mathbb{P}^3$, and let P_1, P_2 be two points on L. Let $\pi\colon\widehat{\mathbb{P}}^3 \to \mathbb{P}^3$ be the blow up of $\mathbb{P}^3$ at P_1 and P_2, and denote by $\widehat{L}$ the proper transform of L on $\widehat{\mathbb{P}}^3$. Then*

$$\mathcal{N}_{\widehat{L}/\widehat{\mathbb{P}}^3} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Corollary 2.3. *Let C be a twisted cubic curve in $\mathbb{P}^3$, and let $P_1, \ldots, P_6$ be six distinct points on C. Let $\pi\colon\widehat{\mathbb{P}}^3 \to \mathbb{P}^3$ be the blow up at the points $P_1, \ldots, P_6$, and denote by $\widehat{C}$ the proper transform of C on $\widehat{\mathbb{P}}^3$. Then*

$$\mathcal{N}_{\widehat{C}/\widehat{\mathbb{P}}^3} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Proof. Note that

$$\mathcal{N}_{C/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5),$$

(see [32, Proposition 6]). It remains to apply Lemma 2.1.   ∎

In §9 we will need the following auxiliary result that is a special equivariant version of a result of Kawakita (see [42, Theorem 1.1]).

Lemma 2.4. *Let Y be a 3-fold, and let P be its smooth point. Let G be a subgroup of $\mathrm{Aut}(Y)$ that fixes the point P. Let $\mathcal{M}_Y$ be a G-invariant mobile linear system on Y. Suppose that P is a center of non-canonical singularities of the log pair $(Y, \lambda\mathcal{M}_Y)$ for some rational number λ. If the Zariski tangent space $T_{P,Y}$ is an irreducible representation of the group G, then*

$$\mathrm{mult}_P(\mathcal{M}_Y) > \frac{2}{\lambda}.$$

Proof. Suppose that $\mathrm{mult}_P(\mathcal{M}_Y) \leqslant 2/\lambda$. Let us show that $T_{P,Y}$ is a reducible representation of the group G. Let $\sigma\colon\widehat{Y} \to Y$ be the blow up at the point P, let E be the σ-exceptional surface, and let $\mathcal{M}_{\widehat{Y}}$ be the proper transform of the linear system $\mathcal{M}_Y$ via σ. Then

$$K_{\widehat{Y}} + \lambda\mathcal{M}_{\widehat{Y}} + \big(\lambda\,\mathrm{mult}_P(\mathcal{M}_Y) - 2\big)E \sim_{\mathbb{Q}} \sigma^*(K_Y + \lambda\mathcal{M}_Y),$$

so that E contains a non-canonical center of the log pair $(\widehat{Y}, \lambda\mathcal{M}_{\widehat{Y}})$. Then E contains a non-log canonical center of the log pair $(\widehat{Y}, E + \lambda\mathcal{M}_{\widehat{Y}})$. It then follows from the inversion of adjunction (for instance, see [44, Theorem 5.50]) that $(E, \lambda\mathcal{M}_{\widehat{Y}}|_E)$ is not log canonical.

Let μ be the positive rational number such that $(E, \mu\mathcal{M}_{\widehat{Y}}|_E)$ is strictly log canonical. Then $\mu < \lambda$, so that $\mu\,\mathrm{mult}_P(\mathcal{M}_Y) < 2$. On the other hand, we have

(2.5) $$\mu\mathcal{M}_{\widehat{Y}}|_E \sim_{\mathbb{Q}} \mu\,\mathrm{mult}_P(\mathcal{M}_Y)L,$$

where L is a line in $E \cong \mathbb{P}^2$.

Let Z be a log canonical center of the pair $(E, \mu\mathcal{M}_{\widehat{Y}}|_E)$. If Z is a curve, then the multiplicity of the restriction $\mu\mathcal{M}_{\widehat{Y}}|_E$ at this curve is at least 1 by [45, Exercise 6.18], so that (2.5) implies that Z is a line. The same arguments imply that other curves in E cannot be log canonical centers of the pair $(E, \mu\mathcal{M}_{\widehat{Y}}|_E)$ simultaneously with Z in this case, so that Z is G-invariant. This implies that $T_{P,Y}$ is a reducible representation of the group G.

To complete the proof, we may assume that the locus of log canonical singularities of the log pair $(E, \mu\mathcal{M}_{\widehat{Y}}|_E)$ consists of finitely many points. On the other hand, it follows from (2.5) that the divisor $-(K_E + \mu\mathcal{M}_{\widehat{Y}}|_E)$ is ample. Thus, using Kollár–Shokurov connectedness principle (for example, see [44, Corollary 5.49]), we see that this locus is connected. Then it consists of one point. This again means that $T_{P,Y}$ is a reducible representation of the group G. ∎

The following two lemmas will also be used in §9.

Lemma 2.6. *Let G be a finite group acting on $\mathbb{P}^2$ without fixed points. If a G-orbit on $\mathbb{P}^2$ has at least four points, then it contains four points such that no three of them are collinear.*

Proof. Let Σ be a G-orbit with at least four points. The G-action has no invariant line, otherwise it would have a fixed point outside the invariant line. Thus Σ contains at least three non-collinear points, say, P_1, P_2, and P_3. For each $i = 1, 2, 3$ denote by L_i the line determined by the two points $\{P_1, P_2, P_3\} \setminus \{P_i\}$. If Σ contains a point outside $L_1 \cup L_2 \cup L_3$, then we are done. Suppose that this is not a case. Let R_1 be a point of Σ different from P_1, P_2, and P_3. We may assume that $R_1 \in L_1$. Since the line L_1 is not G-invariant, there is a line L_1' different from L_1 and containing three points of Σ. We are then able to choose four distinct points of $\Sigma \setminus (L_1 \cap L_1')$ that lie on L_1 or L_1'. Such four points satisfy the desired property. ∎

Lemma 2.7. *Let G be a finite group acting on $\mathbb{P}^2$ without fixed points. Suppose that there is a G-invariant smooth quartic curve C in $\mathbb{P}^2$. Then*

- *the curve C is given by equation (1.6);*
- *the group G contains a subgroup isomorphic to $\mathfrak{S}_4$;*
- *the projective plane $\mathbb{P}^2$ can be identified with a projectivization of an irreducible three-dimensional representation of $\mathfrak{S}_4$.*

Proof. The list of all groups preserving plane quartics together with the equations of the corresponding quartics can be found in [20, Theorem 6.5.2], [37], or [46]. Thus, it remains to check which of them have fixed points on $\mathbb{P}^2$. ∎

3. From Aronhold heptads to smooth plane quartics

In this section we present some auxiliary assertions concerning nets of quadrics in $\mathbb{P}^3$. The theory of nets of quadrics is a classical subject. We refer the reader to [6, 24–29, 53, 55, 57, 58] and [20, §6.3.2], and references therein for many aspects of this theory.

For seven distinct points $P_1, \ldots, P_7$ in $\mathbb{P}^3$ denote by $\mathcal{L}(P_1, \ldots, P_7)$ the linear system of the quadrics passing through the points $P_1, \ldots, P_7$ in $\mathbb{P}^3$.

Before we proceed, let us employ a simple assertion that tells us the dimension of $\mathcal{L}(P_1, \ldots, P_7)$.

Lemma 3.1. *Let $P_1, \ldots, P_r$ be points in a zero-dimensional complete intersection of three quadric surfaces in $\mathbb{P}^3$.*

- *If $r \leqslant 7$, then the r points $P_1, \ldots, P_r$ impose independent conditions on quadric surfaces.*

- *If $r = 8$, then they do not impose independent conditions on quadric surfaces.*

Proof. The first assertion immediately follows from [30, Conjecture CB11]. The conjecture is partially proven in [30, §2.2], which allows us to apply it to our case. The second assertion is obvious since the eight points form a complete intersection of three quadric surfaces. ∎

The following assertion is a key observation that connects 28-nodal double Veronese cones and nets of quadrics in $\mathbb{P}^3$.

Lemma 3.2 (cf. [24, Lemma IX.5]). *For seven distinct points $P_1, \ldots, P_7$ in $\mathbb{P}^3$, the two conditions:*

(A) *every element of $\mathcal{L}(P_1, \ldots, P_7)$ is irreducible;*

(B) *the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ consists of eight distinct points,*

are satisfied if and only if the following three conditions hold:

(A′) *no four points of $P_1, \ldots, P_7$ are coplanar (and in particular no three are collinear);*

(B′) *all points $P_1, \ldots, P_7$ are not contained in a single twisted cubic;*

(C′) *for each i, the twisted cubic passing through the points of $\{P_1, \ldots, P_7\} \setminus \{P_i\}$ and the line passing through the point P_i and one point in $\{P_1, \ldots, P_7\} \setminus \{P_i\}$ meet neither twice nor tangentially.*

Proof. First, suppose that conditions (A) and (B) hold.

If a plane Π_1 contains four points in $\{P_1, \ldots, P_7\}$, then we have another plane Π_2 that contains the remaining three points. The divisor $\Pi_1 + \Pi_2$ is a reducible member in $\mathcal{L}(P_1, \ldots, P_7)$, which is a contradiction to condition (A). Therefore, condition (A') holds.

If the seven points $P_1, \ldots, P_7$ are contained in a twisted cubic, then every quadric surface passing through $P_1, \ldots, P_7$ must contain this twisted cubic. Thus, the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ is one-dimensional, which contradicts condition (B). Therefore, condition (B') must hold.

Suppose that for some i, j, say $i = 1$, $j = 2$, the twisted cubic curve C passing through the six points $P_2, \ldots, P_7$ and the line L passing through P_1 and P_2 meet at P_2 and at some point P. If they meet tangentially at P_2, then we put $P = P_2$. Since the linear system of quadric surfaces containing C is two-dimensional, there are two distinct quadric surfaces Q_1 and Q_2 that contain the curve C and the point P_1. Note that Q_1 and Q_2 belong to the linear system $\mathcal{L}(P_1, \ldots, P_7)$. The intersection of Q_1 and Q_2 is a curve of degree 4 that contains the curve C. Since L passes through three points P_1, P_2, and P of Q_i, $i = 1, 2$, it must be contained in Q_i, $i = 1, 2$. Therefore, the intersection of Q_1 and Q_2 consists of C and L. This implies that for an element Q in $\mathcal{L}(P_1, \ldots, P_7)$, the intersection of the quadrics Q_1, Q_2 and Q either is a curve or consists of 7 points (i.e., $P_1 + 2P_2 + P_3 + \cdots + P_7$), which gives a contradiction to condition (B). Therefore, condition (C') holds.

Now assume that conditions (A'), (B'), and (C') hold.

Suppose that there is a reducible member in $\mathcal{L}(P_1, \ldots, P_7)$. Then it consists of two planes containing the seven points $P_1, \ldots, P_7$. Then one of the planes must contain four of the seven points, which is a contradiction to condition (A'). Therefore, condition (A) must be satisfied.

Suppose that condition (B) does not hold, i.e., the support of the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ either consists of only seven points or contains a curve. Note that $\dim \mathcal{L}(P_1, \ldots, P_7) \geqslant 2$.

Let C be the twisted cubic curve passing through the six points $P_2, \ldots, P_7$. Then there are two distinct quadric surfaces Q_1 and Q_2 that contain the curve C and the point P_1. They are elements of $\mathcal{L}(P_1, \ldots, P_7)$. The intersection of Q_1 and Q_2 consists of the twisted cubic curve C and a line L. Since C does not contain the point P_1 by condition (B'), the line L must contain P_1. Note that L intersects C twice or tangentially.

If the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ is zero-dimensional, then we choose an element Q_3 of $\mathcal{L}(P_1, \ldots, P_7)$ that contains neither C nor L. The intersection of the quadrics Q_1, Q_2 and Q_3 then consists of the seven points $P_1, \ldots, P_7$ set-theoretically. Otherwise it would consists of eight points including $P_1, \ldots, P_7$, and every element in $\mathcal{L}(P_1, \ldots, P_7)$ would pass through the eight intersection points by Lemma 3.1. Since the three quadric surfaces meet at eight points counted with multiplicities, the

line L and C must meet at one of the six points $P_2, \ldots, P_7$; otherwise Q_3 would contain either C or L. This is a contradiction to condition (C′) since L meets C twice or tangentially.

If the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ contains a curve, then for an arbitrary member Q in $\mathcal{L}(P_1, \ldots, P_7)$ the intersection of the quadrics Q_1, Q_2 and Q contains either C or L. Since the dimension of the linear system of quadrics containing both C and P_1 is 1, the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ cannot contain C. Choose two general quadric surfaces that contain both the point P_7 and the twisted cubic curve passing through $P_1, \ldots, P_6$. They are elements in $\mathcal{L}(P_1, \ldots, P_7)$. Therefore, their intersection consists of the twisted cubic curve passing through the points $P_1, \ldots, P_6$, and the line L. Therefore, L must meet C at P_7. This contradicts condition (C′) since L and C meet twice or tangentially. Therefore, condition (B) holds. ∎

Definition 3.3. A set of seven distinct points $P_1, \ldots, P_7$ in $\mathbb{P}^3$ is called an *Aronhold heptad* if every element of $\mathcal{L}(P_1, \ldots, P_7)$ is irreducible and the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ consists of eight distinct points. The set of such eight distinct points is called a *regular Cayley octad*. To be precise, the seven points $P_1, \ldots, P_7$ are called an Aronhold heptad of the regular Cayley octad.

For an Aronhold heptad $P_1, \ldots, P_7$ in $\mathbb{P}^3$, Lemma 3.1 immediately implies that there exists a unique regular Cayley octad that contains $P_1, \ldots, P_7$. It also shows that the linear system $\mathcal{L}(P_1, \ldots, P_7)$ is a net. Conversely, seven points of a regular Cayley octad always form an Aronhold heptad.

Lemma 3.4. *Let $P_1, \ldots, P_8$ be a regular Cayley octad in $\mathbb{P}^3$. For all $1 \leqslant i < j \leqslant 8$, denote by L_{ij} the line passing through the points P_i and P_j, and denote by T_{ij} the twisted cubic passing through the six points of $\{P_1, \ldots, P_8\} \setminus \{P_i, P_j\}$. Let $\mathcal{L}_{ij}$ be the linear subsystem of $\mathcal{L}(P_1, \ldots, P_7)$ that consists of the quadrics passing through T_{ij}. Then $\mathcal{L}_{ij}$ is a pencil, and the base locus of $\mathcal{L}_{ij}$ is the union $T_{ij} \cup L_{ij}$.*

Proof. The first assertion follows from Lemma 3.1. To prove the second assertion, note that the base locus of $\mathcal{L}_{ij}$ is a union of T_{ij} and some line. On the other hand, the base locus must contain all the points $P_1, \ldots, P_8$. Since the twisted cubic T_{ij} passes neither through P_i nor through P_j, the line in the base locus of $\mathcal{L}_{ij}$ must pass through both of these points, i.e., this line must be L_{ij}. ∎

Suppose that we are given seven points $P_1, \ldots, P_7$ in $\mathbb{P}^3$ such that

$$\dim \mathcal{L}(P_1, \ldots, P_7) = 2.$$

Then there are three linearly independent quadric homogeneous polynomials F_0, F_1, F_2 over $\mathbb{P}^3$ that generate the linear system $\mathcal{L}(P_1, \ldots, P_7)$, i.e., an element in $\mathcal{L}(P_1, \ldots, P_7)$

is defined by the quadric homogeneous equation

$$(3.5) \qquad\qquad xF_0 + yF_1 + zF_2 = 0$$

for some $[x : y : z] \in \mathbb{P}^2$. We can express (3.5) as a 4×4 symmetric matrix

$$M(P_1, \ldots, P_7)$$

with entries of linear forms in x, y, z. Then $\det(M(P_1, \ldots, P_7)) = 0$ defines a plane quartic curve $H(P_1, \ldots, P_7)$ in $\mathbb{P}^2$. This plane quartic curve is called the *Hessian quartic* of the net $\mathcal{L}(P_1, \ldots, P_7)$.

Remark 3.6. The hypersurface $\mathfrak{D}$ in $\mathbb{P}(\mathrm{Sym}(4, \mathbb{C})) \cong \mathbb{P}^9$ defined by the determinant polynomial is singular exactly at non-zero 4×4 symmetric matrices of coranks at least 2. The three linearly independent quartic homogeneous polynomials F_0, F_1, F_2 determine a plane $\Pi(F_0, F_1, F_2)$ in $\mathbb{P}(\mathrm{Sym}(4, \mathbb{C}))$. Therefore, the plane quartic curve $H(P_1, \ldots, P_7)$ is singular if and only if the plane $\Pi(F_0, F_1, F_2)$ either passes through a point corresponding to a non-zero 4×4 symmetric matrix of corank at least 2, or tangentially intersects the hypersurface $\mathfrak{D}$ at a point corresponding to a 4×4 symmetric matrix of corank 1.

Lemma 3.7. *Seven distinct points $P_1, \ldots, P_7$ in $\mathbb{P}^3$ form an Aronhold heptad if and only if the linear system $\mathcal{L}(P_1, \ldots, P_7)$ is a net and the plane quartic curve $H(P_1, \ldots, P_7)$ is smooth. Moreover, every smooth plane quartic curve can be obtained in this way.*

Proof. Assume that the plane quartic curve $H(P_1, \ldots, P_7)$ is singular. Since the net $\mathcal{L}(P_1, \ldots, P_7)$ does not contain any reducible member, it follows from Remark 3.6 that the plane $\Pi(F_0, F_1, F_2)$ tangentially intersects the hypersurface $\mathfrak{D}$ at a point corresponding to a 4×4 symmetric matrix of corank 1 in $\mathbb{P}(\mathrm{Sym}(4, \mathbb{C}))$. We may assume that F_0 corresponds to such a 4×4 symmetric matrix of corank 1. Denote by A_0, A_1, A_2 the 4×4 symmetric matrices corresponding to F_0, F_1, F_2, respectively. We may also assume that the matrix A_0 is a diagonal matrix with the last diagonal entry 0. Since the lines $xA_0 + yA_1$ and $xA_0 + zA_2$ in $\mathbb{P}(\mathrm{Sym}(4, \mathbb{C}))$ tangentially intersect the surface $\mathfrak{D}$ at A_0, both the entry of A_1 at the 4th row and the 4th column and the entry of A_2 at the 4th row and the 4th column are zero. This means that the point $[0 : 0 : 0 : 1]$ is a base point of $\mathcal{L}(P_1, \ldots, P_7)$. However, F_0 is singular at $[0 : 0 : 0 : 1]$, and hence the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ cannot consist of eight distinct points. This means that the points $P_1, \ldots, P_7$ do not form an Aronhold heptad.

Now assume that the curve $H(P_1, \ldots, P_7)$ is smooth.

Suppose that the net $\mathcal{L}(P_1, \ldots, P_7)$ contains a reducible member. Then the plane

$$\Pi(F_0, F_1, F_2)$$

passes through a point corresponding to a non-zero 4×4 symmetric matrix of corank at least 2, and hence $H(P_1, \ldots, P_7)$ is singular by Remark 3.6, which gives a contradiction.

Suppose that the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ is not zero-dimensional. It then must be one-dimensional. Furthermore, each irreducible curve B of the base locus must be of degree at most 3 since $\mathcal{L}(P_1, \ldots, P_7)$ is a net.

If B is a line, then there is a point P_i outside B. Let Π be the plane determined by B and P_i. Each quadric from $\mathcal{L}(P_1, \ldots, P_7)$ cuts out in Π the line B and a line passing though P_i. Such lines form at most a pencil. Therefore, there is a member in $\mathcal{L}(P_1, \ldots, P_7)$ containing Π, which leads to a contradiction.

If B is a conic, two members in $\mathcal{L}(P_1, \ldots, P_7)$ intersect along B and another conic B'. Then either B or B' must contain four points of $P_1, \ldots, P_7$. This again implies that there is a reducible member in $\mathcal{L}(P_1, \ldots, P_7)$ and leads to a contradiction.

If B is a cubic curve, two members in $\mathcal{L}(P_1, \ldots, P_7)$ intersect along B and a line. Since a line cannot contain more than two points of $P_1, \ldots, P_7$, the curve B must contain at least 5 points of $P_1, \ldots, P_7$. If B is singular, then B is planar, and hence $\mathcal{L}(P_1, \ldots, P_7)$ contains a reducible member. Therefore, $\mathcal{L}(P_1, \ldots, P_7)$ must be the net determined by the twisted cubic B. In such a case, we can directly show that $H(P_1, \ldots, P_7)$ is singular.

Consequently, the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ is zero-dimensional. Suppose that the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ does not consist of eight distinct points. Then there is a point P_i, say P_1, with the following property: three general elements in $\mathcal{L}(P_1, \ldots, P_7)$ meet at P_1 so that their local intersection index at P_1 is at least 2. This implies that there is an element in $\mathcal{L}(P_1, \ldots, P_7)$ singular at P_1, and hence $H(P_1, \ldots, P_7)$ cannot be smooth. Indeed, we may assume that the singular quadric is defined by a 4×4 diagonal matrix A_0 with the 4th diagonal entry equal to zero. The singular point P_1 is located at $[0 : 0 : 0 : 1]$. Then the net $\mathcal{L}(P_1, \ldots, P_7)$ can be generated by the net of 4×4 symmetric matrices

$$x A_0 + y A_1 + z A_2,$$

where A_1 and A_2 are 4×4 symmetric matrices each of which has 0 for the entry at the 4th row and the 4th column. Then

$$\det(x A_0 + y A_1 + z A_2) = 0$$

defines a quartic curve singular at $[1 : 0 : 0]$. The obtained contradiction shows that the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ consists of eight distinct points, so that $P_1, \ldots, P_7$ form an Aronhold heptad.

A given smooth quartic curve can be defined by the determinant of a 4×4 symmetric linear matrix. This fact together with the first assertion of the lemma implies the second assertion. ∎

Remark 3.8. The representation of a smooth plane quartic as the determinant of a symmetric linear matrix that we used in the proof of Lemma 3.7 goes back to 1855 (see [38]) and 1902 (see [18]). For a contemporary proof, see [7, Proposition 4.2].

The following well-known fact concerning intersections of quadrics in $\mathbb{P}^3$ will be necessary for the present paper. It should be remarked here that they have been extensively researched in [53] and [55] in much wider settings.

Lemma 3.9. *Let Q_1 and Q_2 be two distinct quadrics in $\mathbb{P}^3$ and E be the intersection of Q_1 and Q_2. Then the following are equivalent.*

- *The intersection E is smooth and of codimension 2 in $\mathbb{P}^3$.*
- *The line $\mathscr{L}$ determined by Q_1 and Q_2 in $\mathbb{P}(\mathrm{Sym}(4, \mathbb{C})) \cong \mathbb{P}^9$ intersects the determinant hypersurface $\mathfrak{D}$ (see Remark 3.6) at four distinct points.*
- *After a suitable linear coordinate change, E can be defined by the equations*

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 0$$

in $\mathbb{P}^3$, where λ_i's are four distinct constants.

If these equivalent conditions hold, then the intersection E is isomorphic to the double cover of the line $\mathscr{L}$ in $\mathbb{P}(\mathrm{Sym}(4, \mathbb{C}))$ branched exactly at the four intersection points $\mathscr{L} \cap \mathfrak{D}$.

Proof. The equivalence immediately follows from [53, Proposition 2.1]. As a matter of fact, the last statement is also instantly implied by the result in [53, Theorem 4.8] that deals with much more general situation. However, in $\mathbb{P}^3$, the conditions for [53, Theorem 4.8] turn into a tangible state, so that an elementary and short proof could be presented as follows.

Suppose that E is defined in $\mathbb{P}^3$ by the equations

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 0$$

for some four distinct constants $\lambda_0, \ldots, \lambda_3$. To complete the proof, it is enough to show that E is isomorphic to a double cover of $\mathbb{P}^1$ branched at four distinct points whose cross-ratio is

(3.10)
$$\frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_0)}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_3)}$$

modulo permutation. Note that $\lambda_0 \neq \lambda_i$ for $1 \leqslant i \leqslant 3$ because E is smooth. Rewrite the equations of E as

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = (\lambda_0 - \lambda_1)x_1^2 + (\lambda_0 - \lambda_2)x_2^2 + (\lambda_0 - \lambda_3)x_3^2 = 0.$$

We first consider the projection of E to the plane Π defined by $x_0 = 0$ centered at the point $[1 : 0 : 0 : 0]$. The image is the conic R defined by

$$(\lambda_0 - \lambda_1)x_1^2 + (\lambda_0 - \lambda_2)x_2^2 + (\lambda_0 - \lambda_3)x_3^2 = 0$$

on Π. Furthermore, E is a double cover of R. Let $[0 : \alpha : \beta : \gamma]$ be a point on this conic. The equation
$$\mu^2 + \nu^2(\alpha^2 + \beta^2 + \gamma^2) = 0$$

in $[\mu : \nu] \in \mathbb{P}^1$ determines the points $[\mu : \nu\alpha : \nu\beta : \nu\gamma]$ of E over the point $[0 : \alpha : \beta : \gamma]$. Therefore, the double cover is branched at $[0 : \alpha : \beta : \gamma]$ if and only if

$$\alpha^2 + \beta^2 + \gamma^2 = 0.$$

Consequently, the four branch points on R are the intersection points of the conics

$$x_1^2 + x_2^2 + x_3^2 = 0,$$
$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 0$$

on Π. These four points are

$$\left[0 : \sqrt{\lambda_3 - \lambda_2} : \pm\sqrt{\lambda_1 - \lambda_3} : \pm\sqrt{\lambda_2 - \lambda_1} \right].$$

We now consider the projection of R centered at the point

$$\left[0 : \sqrt{\lambda_0 - \lambda_2} : \sqrt{\lambda_1 - \lambda_0} : 0 \right]$$

to the line L defined by $x_1 = 0$ on Π. This projection defines an isomorphism of R onto L. Therefore, E is the double cover of L branched at the points

$$\left[0 : 0 : \sqrt{\lambda_3 - \lambda_2}\sqrt{\lambda_1 - \lambda_0} \pm \sqrt{\lambda_0 - \lambda_2}\sqrt{\lambda_1 - \lambda_3} : \pm\sqrt{\lambda_0 - \lambda_2}\sqrt{\lambda_2 - \lambda_1} \right].$$

It is easy to check that the cross-ratio of the above four points

$$\frac{\pm\sqrt{\lambda_3 - \lambda_2}\sqrt{\lambda_1 - \lambda_0} \pm \sqrt{\lambda_0 - \lambda_2}\sqrt{\lambda_1 - \lambda_3}}{\sqrt{\lambda_0 - \lambda_2}\sqrt{\lambda_2 - \lambda_1}}$$

is equal to the desired number (3.10) modulo permutation. This completes the proof. ∎

For an Aronhold heptad $P_1, \ldots, P_7$, the projective plane $\check{\mathbb{P}}^2$ projectively dual to the net of quadrics $\mathcal{L}(P_1, \ldots, P_7) \cong \mathbb{P}^2$ can be identified with the base of the elliptic fibration $\varkappa$ that is obtained from $\mathbb{P}^3$ by blowing up the points of the corresponding regular Cayley octad. It is well known that the projectively dual curve of the Hessian quartic curve can be interpreted in terms of singular fibers of $\varkappa$. Namely, we have the following

Lemma 3.11. *Let $P_1, \ldots, P_7$ be an Aronhold heptad, and let $\check{H} \subset \check{\mathbb{P}}^2$ be the curve parametrizing singular fibers of $\varkappa$. Then $\check{H}$ is projectively dual to $H(P_1, \ldots, P_7)$.*

Proof. Choose a fiber $\widetilde{E}$ of the elliptic fibration $\varkappa$. It is the proper transform of an intersection curve E of two quadrics in $\mathcal{L}(P_1, \ldots, P_7)$. Furthermore, the fiber $\widetilde{E}$ is smooth if and only if the intersection curve E is smooth because it cannot be singular at the points of the regular Cayley octad. On the other hand, Lemma 3.9 implies that the intersection curve E is smooth if and only if the corresponding pencil of quadrics contains exactly four singular quadrics. This verifies that the projectively dual curve of $\check{H}$ coincides with the Hessian curve of the net $\mathcal{L}(P_1, \ldots, P_7)$. ∎

4. From Aronhold heptads to double Veronese cones

In this section we review the birational construction of 28-nodal double Veronese cones due to [50].

The following result is mainly a part of [50, Theorem 7.1]. We provide its proof for the reader's convenience and for clarification.

Proposition 4.1. Let $P_1, \ldots, P_7$ be seven points in $\mathbb{P}^3$ that form an Aronhold heptad. Let $\pi \colon \widehat{\mathbb{P}}^3 \to \mathbb{P}^3$ be the blow up of $\mathbb{P}^3$ at the points $P_1, \ldots, P_7$, let V be the closure of the image of $\widehat{\mathbb{P}}^3$ under the rational map given by the linear system $|-2K_{\widehat{\mathbb{P}}^3}|$, and let $\phi \colon \widehat{\mathbb{P}}^3 \dashrightarrow V$ be the corresponding rational map. Then

- the map ϕ is a birational morphism;
- the exceptional locus of ϕ is a disjoint union of the proper transforms of the lines passing through pairs of the points P_i and the twisted cubics passing through six-tuples of the points P_i;
- the variety V is a 28-nodal double Veronese cone.

Proof. Note that by Lemma 3.2 there are 21 lines passing through pairs of the points P_i, and 7 twisted cubic curves passing through six-tuples of the points P_i. Furthermore, due to Lemma 3.2 their proper transforms on $\widehat{\mathbb{P}}^3$ are disjoint. These proper transforms

intersect the anticanonical class of $\widehat{\mathbb{P}}^3$ trivially. Indeed, if $\widehat{C}$ is one of these curves on $\widehat{\mathbb{P}}^3$, then

$$-K_{\widehat{\mathbb{P}}^3} \cdot \widehat{C} = (\pi^*(-K_{\mathbb{P}^3}) - 2(F_1 + \cdots + F_7)) \cdot \widehat{C}$$
$$= 4\deg(\pi(\widehat{C})) - 2(F_1 + \cdots + F_7) \cdot \widehat{C} = 0,$$

where F_i is the exceptional surface of π over the point P_i.

We claim that any other irreducible curve intersects the anticanonical class neither trivially nor negatively. To see this, let L be an irreducible curve on $\mathbb{P}^3$ whose proper transform on $\widehat{\mathbb{P}}^3$ non-positively intersects the anticanonical class of $\widehat{\mathbb{P}}^3$. It means

(4.2)
$$\sum_{i=1}^{7} \operatorname{mult}_{P_i} L \geqslant 2\deg(L).$$

Choose a point P on L other than the base point of $\mathcal{L}(P_1, \ldots, P_7)$. Then the quadric surfaces in $\mathcal{L}(P_1, \ldots, P_7)$ passing through the extra point P form a pencil. It follows from (4.2) that every quadric surface in this pencil contains the curve L. If $\deg L = 4$, then we see from (4.2) that L must be singular at some of the points P_i. This means that the intersection of three quadrics forms $\mathcal{L}(P_1, \ldots, P_7)$ is not reduced at that point, i.e., the base locus of $\mathcal{L}(P_1, \ldots, P_7)$ consists of less than 8 points, which is a contradiction. If $\deg L = 2$, then L is a conic, and by (4.2) it contains four of the points P_i. This is impossible by Lemma 3.2. Therefore, one has either $\deg L = 3$ or $\deg L = 1$. If L is a singular cubic curve, then it is planar. However, (4.2) implies that it contains at least four of the points P_i. This is impossible by Lemma 3.2. Therefore, L is either a twisted cubic or a line. In both cases we see from (4.2) that L is one of the above 28 curves.

Let $\widetilde{\phi}$ be the morphism defined by the linear system $|-nK_{\widehat{\mathbb{P}}^3}|$ for large enough n. Now Corollaries 2.2 and 2.3 imply that $\widetilde{\phi}$ is a flopping contraction, which implies that $\widetilde{\phi}$ satisfies the properties of the first and the second assertions. They also verify that the 28 proper transforms are contracted to 28 nodes on V. Since $\widehat{\mathbb{P}}^3$ is a smooth weak Fano 3-fold and $\widetilde{\phi}$ contracts all the curves that trivially intersect the anticanonical class of $\widehat{\mathbb{P}}^3$, the 3-fold V is a Fano 3-fold with exactly 28 singular points.

Let Q be a quadric surface passing through the Aronhold heptad, then

$$-K_{\widehat{\mathbb{P}}^3} \sim 2(\pi^*(Q) - (F_1 + \cdots + F_7)).$$

This shows that $-K_V$ is divisible by 2. Since

$$(-K_V)^3 = (-K_{\widehat{\mathbb{P}}^3})^3 = 8,$$

the variety V is a del Pezzo 3-fold of degree 1. Now it follows from [39, Theorem 3.2.5] that V is a hypersurface of degree 6 in the weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, 1, 2, 3)$

that does not pass through singular points of $\mathbb{P}$. Moreover, using the adjunction formula in $\mathbb{P}$, we see that $-K_V = \mathcal{O}_\mathbb{P}(2)|_V$. So, restricting the linear system $|\mathcal{O}_\mathbb{P}(4)|$ to V, we conclude that the divisor $-2K_V$ is very ample, which implies that the morphism $\widetilde{\phi}$ coincides with the map ϕ defined by the linear system $|-2K_{\widehat{\mathbb{P}}^3}|$. This completes the proof. ∎

The following observation was explained to us by Yuri Prokhorov.

Example 4.3. Let C be a twisted cubic in $\mathbb{P}^3$. Choose six distinct points $P_1, \ldots, P_6$ on C. Let P be a point on C different from $P_1, \ldots, P_6$, and let L be the line passing through P_6 and P. Choose a point $P_7 \in L \setminus \{P_6, P\}$. Then such seven points cannot be an Aronhold heptad since they violate the conditions in Lemma 3.2. Furthermore, if we define the 3-folds $\widehat{\mathbb{P}}^3$ and V and the map ϕ as in Proposition 4.1, then ϕ contracts (in particular) the proper transforms of C and L on $\widehat{\mathbb{P}}^3$. These curves are not disjoint, which implies that the 3-fold V is not nodal. The same holds if L is chosen to be a tangent line at the point P_6 to C.

The following result is implicitly contained (but not clearly stated) in [50]. It is implied by the results of [50] together with a simple additional observation.

Theorem 4.4. *Let V be a double Veronese cone with* 28 *singular points. Then V can be constructed from some Aronhold heptad as in Proposition* 4.1.

Proof. It first follows from [50, Theorem 1.7] and [50, Remark 1.8] that the rank of the divisor class group of V is 8. Then [50, Theorem 7.1 (i)] implies that V can be obtained in the same way as in Proposition 4.1 with seven points of $\mathbb{P}^3$ such that no four of them are coplanar. It remains to verify that the seven points form a Aronhold heptad. However, [50, Theorem 7.1 (ii)] shows that having 28 singular points is equivalent to (A′), (B′), and (C′) in Lemma 3.2. Therefore, the seven points in the construction must form an Aronhold heptad by Lemma 3.2. ∎

Proposition 4.1 and Theorem 4.4 immediately imply the following.

Corollary 4.5. *If a double Veronese cone has* 28 *singular points, then they are all nodes.*

We conclude this section with another result that is a part of [50, Theorem 7.1]. Its proof is similar to that of Proposition 4.1.

Proposition 4.6. Let $P_1, \ldots, P_6$ be six points in $\mathbb{P}^3$ such that no four of them are coplanar. Let $\pi : \widehat{\mathbb{P}}^3 \to \mathbb{P}^3$ be the blow up of $\mathbb{P}^3$ at the points $P_1, \ldots, P_6$, and let $\phi : \widehat{\mathbb{P}}^3 \dashrightarrow W$ be the map given by the linear system $|-K_{\widehat{\mathbb{P}}^3}|$. Then

- the map ϕ is a birational morphism;

- the exceptional locus of ϕ is a disjoint union of the proper transforms of the lines passing through pairs of the points P_i and the twisted cubic passing through all the points P_i;

- the variety W is a 16-nodal double cover of $\mathbb{P}^3$ branched over a quartic surface.

5. From smooth plane quartics to double Veronese cones

In this section we construct a 28-nodal double Veronese cone starting from a smooth plane quartic curve. We also provide an explicit equation (5.10) for such a 3-fold in terms of covariants of the plane quartic curve.

Let $\check{\mathbb{P}}^2$ be the dual of the projective plane $\mathbb{P}^2$. We use a homogeneous coordinate system $[x : y : z]$ for $\mathbb{P}^2$ and $[s : t : u]$ for $\check{\mathbb{P}}^2$.

We start with a smooth quartic curve C in the projective plane $\mathbb{P}^2$ given by an equation

$$(5.1) \qquad H(x, y, z) = \sum_{i+j+k=4} a_{ijk} x^i y^j z^k = 0.$$

We regard $[s : t : u]$ as a general point in $\check{\mathbb{P}}^2$. Then the corresponding line on $\mathbb{P}^2$ is a general line $L_{s,t,u}$ given by

$$sx + ty + uz = 0.$$

The line $L_{s,t,u}$ hits the quartic C at four distinct points x_1, x_2, x_3, x_4 lying on $L_{s,t,u} \setminus \{z = 0\}$. We may regard these four points as points on the affine line, so that we could define their cross-ratio as follows:

$$\lambda(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_3)(x_4 - x_2)}{(x_1 - x_2)(x_4 - x_3)}.$$

This has six different values according to the order of the four points. However, the following j-function is invariant with respect to the reordering x_1, x_2, x_3, x_4:

$$(5.2) \quad j(x_1, x_2, x_3, x_4) = 256 \frac{(1 - \lambda(x_1, x_2, x_3, x_4)(1 - \lambda(x_1, x_2, x_3, x_4)))^3}{\lambda(x_1, x_2, x_3, x_4)^2(1 - \lambda(x_1, x_2, x_3, x_4))^2}$$

$$= 2^8 \frac{((x_1 - x_2)^2(x_4 - x_3)^2 - (x_1 - x_3)(x_4 - x_2)(x_4 - x_1)(x_3 - x_2))^3}{(x_1 - x_2)^2(x_1 - x_3)^2(x_1 - x_4)^2(x_2 - x_3)^2(x_2 - x_4)^2(x_3 - x_4)^2}.$$

By plugging $z = -(sx + ty)/u$ into (5.1), we obtain

$$(5.3) \qquad u^4 H\left(x, y, -\frac{sx + ty}{u}\right) = \sum_{i+j+k=4} a_{ijk}(-u)^{4-k}(sx + ty)^k x^i y^j$$

$$= \sum_{r=0}^{4} b_{4-r} x^r y^{4-r},$$

where

$$b_r = \sum_{j=0}^{r} \sum_{i+k=4-j} (-1)^k a_{ijk} \binom{k}{k+j-r} s^{k+j-r} t^{r-j} u^{4-k}.$$

Then we have the following identities for elementary symmetric functions of x_1, x_2, x_3, x_4:

$$x_1 + x_2 + x_3 + x_4 = -\frac{b_1}{b_0};$$

$$x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4 = \frac{b_2}{b_0};$$

$$x_2 x_3 x_4 + x_1 x_2 x_4 + x_1 x_2 x_4 + x_1 x_2 x_3 = -\frac{b_3}{b_0};$$

$$x_1 x_2 x_3 x_4 = \frac{b_4}{b_0}.$$

Since the denominator and the numerator of the j-function in (5.2) are symmetric polynomials in x_1, x_2, x_3, x_4, the j-function in (5.2) may be regarded as a rational function in b_0, b_1, b_2, b_3, b_4. Indeed, one has

$$j(b_0, b_1, b_2, b_3, b_4) = 1728 \frac{4h_2(b_0, b_1, b_2, b_3, b_4)^3}{4h_2(b_0, b_1, b_2, b_3, b_4)^3 - 27h_3(b_0, b_1, b_2, b_3, b_4)^2},$$

where

$$h_2(b_0, b_1, b_2, b_3, b_4) = \frac{1}{3}(-3b_1 b_3 + 12 b_0 b_4 + b_2^2);$$

$$h_3(b_0, b_1, b_2, b_3, b_4) = \frac{1}{27}(72 b_0 b_2 b_4 - 27 b_0 b_3^2 - 27 b_1^2 b_4 + 9 b_1 b_2 b_3 - 2 b_2^3)$$

(see Appendix A for a more detailed computation).

Regarding $h_2(b_0, b_1, b_2, b_3, b_4)$ and $h_3(b_0, b_1, b_2, b_3, b_4)$ as polynomials in s, t, u, we see from Appendix A that

$$h_2(b_0, b_1, b_2, b_3, b_4) = u^4 g_4(s, t, u),$$
$$h_3(b_0, b_1, b_2, b_3, b_4) = u^6 g_6(s, t, u),$$

where $g_4(s, t, u)$ and $g_6(s, t, u)$ are homogeneous polynomials of degrees 4 and 6, respectively, in s, t, u. Consequently, the rational function $j(b_0, b_1, b_2, b_3, b_4)$ may be regarded as a rational function j_C in s, t, u, so that it is a rational function on $\check{\mathbb{P}}^2$. More precisely, one has

(5.4) $$j_C(s, t, u) = 1728 \frac{4g_4(s, t, u)^3}{4g_4(s, t, u)^3 - 27 g_6(s, t, u)^2}.$$

Remark 5.5. The equation $g_4(s,t,u) = 0$ of degree 4 describes the points of $\check{\mathbb{P}}^2$ corresponding to lines that intersect C by equianharmonic quadruples of points (see for instance [20, §2.3.4] for terminology). In other words, these lines with the quadruples of points define elliptic curves of j-invariant 0. Elliptic curves of j-invariant 0 are isomorphic to the Fermat plane cubic curve. Similarly, the equation $g_6(s,t,u) = 0$ of degree 6 describes the points of $\check{\mathbb{P}}^2$ corresponding to lines that intersect C by harmonic quadruples of points. In this case, lines with the quadruples of points define elliptic curves of j-invariant 1728. Note that neither the cross-ratio nor the j-invariant is defined for the lines corresponding to the 24 intersection points (counted with multiplicities) of $g_4(s,t,u) = 0$ and $g_6(s,t,u) = 0$.

Lemma 5.6. *The j-function $j_C(s,t,u)$ is not constant. In particular, $g_4(s,t,u)$ cannot be a zero polynomial.*

Proof. Among the lines on $\mathbb{P}^2$, there are both tangent lines to C and non-tangent lines to C. This implies that the j-function, which is defined by the cross-ratios of the intersection points, is not constant. ∎

We can use the polynomials $g_4(s,t,u)$ and $g_6(s,t,u)$ to write down the equation of the projectively dual curve of the plane quartic C.

Proposition 5.7. The equation

$$(5.8) \qquad 4g_4(s,t,u)^3 - 27g_6(s,t,u)^2 = 0$$

of degree 12 defines the projectively dual curve $\check{C}$ of the quartic curve C in $\check{\mathbb{P}}^2$.

Proof. The curve $\check{C}$ has degree 12. By construction the rational function $j_C(s,t,u)$ is well defined at least outside the projectively dual curve $\check{C}$. On the other hand, we know from Lemma 5.6 that $g_4(s,t,u)$ is not a zero polynomial. Thus it follows from (5.4) that $j_C(s,t,u)$ has poles along the curves defined by some of the factors of $4g_4(s,t,u)^3 - 27g_6(s,t,u)^2$. These poles must be contained in the projectively dual curve $\check{C}$. Since $\check{C}$ is an irreducible curve of degree 12, it must be defined by the homogeneous equation (5.8) of degree 12. ∎

Recall that a bitangent line to the quartic curve C, which is not a tangent line at a hyper-inflection point of C, yields an ordinary double point on the projectively dual curve $\check{C}$. Meanwhile, the tangent line to C at an ordinary inflection point of C produces an ordinary cusp in $\check{C}$, and the tangent line to C at a hyper-inflection point of C generates a triple point of $\check{C}$ that is analytically isomorphic to a singularity defined by equation $s^3 = t^4$. These are all the possible types of singularities of $\check{C}$.

Let C_4 be the quartic curve defined by $g_4(s,t,u) = 0$, and let C_6 be the sextic curve defined by $g_6(s,t,u) = 0$ in $\check{\mathbb{P}}^2$.

Lemma 5.9. *Let P be an intersection point of the curves C_4 and C_6. Then one of the following possibilities occur.*

- *The curves C_4 and C_6 are smooth at P, and meet transversally at P; in this case, P is an ordinary cusp of the curve $\check{C}$.*

- *The curve C_4 is smooth at P, the curve C_6 has a double point at P, and the local intersection index of C_4 and C_6 at P equals 2; in this case, P is a triple point of $\check{C}$ analytically isomorphic to $s^3 = t^4$.*

Proof. The point P is a singular point of $\check{C}$. If C_4 is singular at P, then the defining equation (5.8) of $\check{C}$ shows that $\check{C}$ has a singularity of a type different from the three types described right after the proof of Proposition 5.7. Therefore, C_4 is smooth at P. Furthermore, (5.8) implies that P cannot be an ordinary double point of $\check{C}$. Thus $\check{C}$ has either an ordinary cusp or a triple point analytically isomorphic to $s^3 = t^4$ at P. Note that P is a triple point of $\check{C}$ if and only if C_6 is singular at P. Furthermore, in such a case, P must be a double point of C_6.

The defining equation (5.8) tells that P is an ordinary cusp of $\check{C}$ if C_4 and C_6 are transversal at P. On the other hand, if C_6 is smooth and tangent to C_4 at P, then $\check{C}$ has a double point worse than a simple cusp at P, in the sense that its log canonical threshold is smaller than $5/6$.

We now suppose that P is a double point of C_6. Then, (5.8) shows that $\check{C}$ has a triple point at P. Therefore, to complete the proof, it is enough to show that the singularity of $\check{C}$ at P cannot be a triple point analytically isomorphic to $s^3 = t^4$ if the tangent line to C_4 at P is one of the Zariski tangent lines of C_6 at P. This can be easily checked by blowing up. Indeed, if the tangent line to C_4 at P is one of the Zariski tangent lines of C_6 at P, then the proper transform of $\check{C}$ via the blow up of $\check{\mathbb{P}}^2$ at P is still singular at some point over P, which is not the case for the singularity $s^3 = t^4$. ■

We now consider the hypersurface V of degree 6 in $\mathbb{P}(1, 1, 1, 2, 3)$ given by an equation

$$(5.10) \qquad -w^2 + v^3 - g_4(s, t, u)v + g_6(s, t, u) = 0,$$

where $\mathrm{wt}(w) = 3, \mathrm{wt}(v) = 2$. We denote by $F(s, t, u, v, w)$ the left hand side of (5.10), and set

$$G(s, t, u) = 4g_4(s, t, u)^3 - 27g_6(s, t, u)^2.$$

Then $G(s, t, u) = 0$ defines the projectively dual curve $\check{C}$ of the smooth quartic curve C in $\check{\mathbb{P}}^2$ by Proposition 5.7.

Denote by V_v^{sing} the set of singular points of V in the section by $v = 0$ and by V_o^{sing} the set of singular points of V outside the section by $v = 0$. On the other hand, denote

by $\check{C}_v^{\text{sing}}$ the set of singular points of $\check{C}$ at which C_6 is singular, and denote by $\check{C}_o^{\text{sing}}$ the set of singular points of $\check{C}$ outside C_6.

Lemma 5.11. *Define the maps as follows:*

$$\check{C}_o^{\text{sing}} \to V_o^{\text{sing}},$$

$$[a_0 : a_1 : a_2] \mapsto \left[a_0 : a_1 : a_2 : \frac{3g_6(a_0, a_1, a_2)}{2g_4(a_0, a_1, a_2)} : 0\right];$$

$$\check{C}_v^{\text{sing}} \to V_v^{\text{sing}},$$

$$[a_0 : a_1 : a_2] \mapsto [a_0 : a_1 : a_2 : 0 : 0].$$

Then both are one-to-one correspondences.

Proof. Let $[a_0 : a_1 : a_2]$ be a point in the set $\check{C}_o^{\text{sing}}$. Observe that $g_6(a_0, a_1, a_2) \neq 0$, and hence $g_4(a_0, a_1, a_2) \neq 0$. Since $[a_0 : a_1 : a_2] \in \check{C}_o^{\text{sing}}$, we have

$$\frac{\partial F}{\partial v}\left(a_0, a_1, a_2, \frac{3g_6(a_0, a_1, a_2)}{2g_4(a_0, a_1, a_2)}, 0\right) = \frac{27g_6(a_0, a_1, a_2)^2 - 4g_4(a_0, a_1, a_2)^3}{4g_4(a_0, a_1, a_2)^2} = 0.$$

Also, since

$$12g_4(a_0, a_1, a_2)^2 \frac{\partial g_4}{\partial s}(a_0, a_1, a_2) - 54g_6(a_0, a_1, a_2)\frac{\partial g_6}{\partial s}(a_0, a_1, a_2) = 0,$$

we have

$$\frac{\partial F}{\partial s}\left(a_0, a_1, a_2, \frac{3g_6(a_0, a_1, a_2)}{2g_4(a_0, a_1, a_2)}, 0\right)$$

$$= -\frac{3g_6(a_0, a_1, a_2)}{2g_4(a_0, a_1, a_2)} \frac{\partial g_4}{\partial s}(a_0, a_1, a_2) + \frac{\partial g_6}{\partial s}(a_0, a_1, a_2)$$

$$= -\frac{27g_6(a_0, a_1, a_2)^2}{4g_4(a_0, a_1, a_2)^3} \frac{\partial g_6}{\partial s}(a_0, a_1, a_2) + \frac{\partial g_6}{\partial s}(a_0, a_1, a_2) = 0.$$

Similarly, we see that $\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial u}$ vanish at the point

$$\left[a_0 : a_1 : a_2 : \frac{3g_6(a_0, a_1, a_2)}{2g_4(a_0, a_1, a_2)} : 0\right],$$

so that this point belongs to V_o^{sing}.

Conversely, suppose that $[a_0 : a_1 : a_2 : v_0 : w_0]$ belongs to V_o^{sing}. Then $w_0 = 0$ and $v_0 \neq 0$. Furthermore, taking the partial derivative of (5.10) with respect to v, we obtain

$$g_4(a_0, a_1, a_2) = 3v_0^2.$$

Using (5.10) again, we get

$$g_6(a_0, a_1, a_2) = -v_0^3 + v_0 g_4(a_0, a_1, a_2) = 2v_0^3.$$

Then

$$\frac{\partial G}{\partial s}(a_0, a_1, a_2) = 12g_4(a_0, a_1, a_2)^2 \frac{\partial g_4}{\partial s}(a_0, a_1, a_2)$$

$$- 54g_6(a_0, a_1, a_2) \frac{\partial g_6}{\partial s}(a_0, a_1, a_2)$$

$$= 108v_0^4 \frac{\partial g_4}{\partial s}(a_0, a_1, a_2) - 108v_0^3 \frac{\partial g_6}{\partial s}(a_0, a_1, a_2)$$

$$= -108v_0^3 \frac{\partial F}{\partial s}(a_0, a_1, a_2, v_0, w_0) = 0.$$

Similarly, we obtain

$$\frac{\partial G}{\partial t}(a_0, a_1, a_2) = \frac{\partial G}{\partial u}(a_0, a_1, a_2) = 0.$$

This implies that $[a_0 : a_1 : a_2]$ belongs to $\check{C}_o^{\mathrm{sing}}$.

Suppose that $[a_0 : a_1 : a_2]$ is in $\check{C}_v^{\mathrm{sing}}$. Then

$$\frac{\partial g_6}{\partial s}(a_0, a_1, a_2) = \frac{\partial g_6}{\partial t}(a_0, a_1, a_2) = \frac{\partial g_6}{\partial u}(a_0, a_1, a_2) = 0.$$

Moreover, we have $g_6(a_0, a_1, a_2) = 0$, and thus also $g_4(a_0, a_1, a_2) = 0$. These imply

$$\frac{\partial F}{\partial w}(a_0, a_1, a_2, 0, 0) = \frac{\partial F}{\partial v}(a_0, a_1, a_2, 0, 0) = 0,$$

$$\frac{\partial F}{\partial s}(a_0, a_1, a_2, 0, 0) = \frac{\partial F}{\partial t}(a_0, a_1, a_2, 0, 0) = \frac{\partial F}{\partial u}(a_0, a_1, a_2, 0, 0) = 0.$$

Therefore, $[a_0 : a_1 : a_2 : 0 : 0]$ lies in V_v^{sing}.

Conversely, suppose that $[a_0 : a_1 : a_2 : 0 : 0]$ belongs to V_v^{sing}. Then

$$0 = \frac{\partial F}{\partial v}(a_0, a_1, a_2, 0, 0) = -g_4(a_0, a_1, a_2).$$

In particular, this gives $g_6(a_0, a_1, a_2) = 0$. We have

$$0 = \frac{\partial F}{\partial s}(a_0, a_1, a_2, 0, 0) = \frac{\partial g_6}{\partial s}(a_0, a_1, a_2),$$

$$0 = \frac{\partial F}{\partial t}(a_0, a_1, a_2, 0, 0) = \frac{\partial g_6}{\partial t}(a_0, a_1, a_2),$$

$$0 = \frac{\partial F}{\partial u}(a_0, a_1, a_2, 0, 0) = \frac{\partial g_6}{\partial u}(a_0, a_1, a_2).$$

Therefore, we conclude that the point $[a_0 : a_1 : a_2]$ belongs to $\check{C}_v^{\mathrm{sing}}$. ∎

Theorem 5.12. *The del Pezzo* 3-*fold* V *has exactly* 28 *singular points.*

Proof. Let δ_o be the number of the ordinary bitangent lines, i.e., lines tangent to C at two distinct points, of the smooth quartic curve C, and let δ_s be the number of the hyper-inflection points of the smooth quartic curve C. The tangent line to C at a hyper-inflection point is a bitangent line to C that is tangent to C at a single point with multiplicity 4. Let ι be the number of ordinary inflection points of C. Then we can derive

$$\delta_o + \delta_s = 28, \quad \iota + 2\delta_s = 24$$

from the classical Plücker formulae (for instance, see [34, §2.4]). In particular, the projectively dual curve $\check{C}$ has exactly

$$\delta_o + \delta_s + \iota = 52 - 2\delta_s$$

singular points.

The ordinary bitangent lines define δ_o ordinary double points on $\check{C}$. Such singular points cannot lie on C_6. If so, then they would also lie on C_4 by Proposition 5.7, and would not be ordinary double points of $\check{C}$ by Lemma 5.9.

Meanwhile, by Lemma 5.9 the curves C_4 and C_6 meet

- either transversally,
- or in such a way that C_6 has a double point at P, and the local intersection index of C_4 and C_6 at P is 2.

Furthermore, intersection points of the former type yield ordinary cusps, and ones of the latter type produce triple points analytically isomorphic to $s^3 = t^4$. The number of intersection points of the former type is ι and the number of intersection points of the latter type is δ_s. We may then conclude that

$$\#|\check{C}_o^{\text{sing}}| = \delta_o, \quad \#|\check{C}_v^{\text{sing}}| = \delta_s.$$

Since $\delta_o + \delta_s = 28$, Lemma 5.11 immediately implies the statement. ∎

The following assertion can be deduced from Theorem 5.12 and Corollary 4.5. However, we present a computational proof here to show how the singularities of $\check{C}$ and the singularities of V interact with each other.

Proposition 5.13. *The singular points of the del Pezzo* 3-*fold* V *are all ordinary double points.*

Proof. Let P be a point in V_v^{sing}. After a suitable coordinate changes, we may assume that

$$P = [0 : 0 : 1 : 0 : 0].$$

By Lemmas 5.9 and 5.11, the curve C_4 is smooth at $[0:0:1]$, the curve C_6 has a double point at $[0:0:1]$, and the local intersection index of C_4 and C_6 at $[0:0:1]$ is equal to 2. We may also assume that the tangent line to C_4 at $[0:0:1]$ is defined by $s = 0$ and the Zariski tangent cone of C_6 at $[0:0:1]$ is defined by $t(t + \alpha s) = 0$, where α is a (possibly zero) constant. Put

$$f(s,t,v,w) = F(s,t,1,v,w), \qquad g(s,t) = G(s,t,1),$$
$$\widetilde{g}_4(s,t,v,w) = g_4(s,t,1), \qquad \widetilde{g}_6(s,t,v,w) = g_6(s,t,1).$$

We regard s, t, v, w as local coordinates around the point P that corresponds to the origin in $\mathbb{C}^4$, which we denote by p. We then consider the Hessian of f at the point p:

$$\operatorname{Hess}(f)(p) = 2\left(\frac{\partial \widetilde{g}_4}{\partial s}(p)\right)^2 \frac{\partial^2 \widetilde{g}_6}{\partial t^2}(p) \neq 0.$$

This implies that the point P is an ordinary double point of V.

We now consider a point R in V_o^{sing}. By appropriate coordinate changes, we may assume that

$$R = [0:0:1:v_0:0],$$

where v_0 is a non-zero constant. In the affine chart defined by $u \neq 0$ in $\mathbb{P}(1,1,1,2,3)$, the point R corresponds to $q = (0,0,v_0,0)$. In the affine chart defined by $u \neq 0$ in $\check{\mathbb{P}}^2$, we denote by o the origin. Since q is a singular point of the hypersurface defined by $f = 0$, we conclude that o is a singular point of the curve defined by $g = 0$ by Lemma 5.11. Furthermore, since q is from V_o^{sing}, the origin o is an ordinary double point of $g = 0$. From this one can obtain the following relations:

$$v_0 = \frac{3\widetilde{g}_6(q)}{2\widetilde{g}_4(q)}, \qquad\qquad 27(\widetilde{g}_6(q))^3 = 4(\widetilde{g}_4(q))^3,$$
$$3\widetilde{g}_6(q)\frac{\partial \widetilde{g}_4}{\partial s}(q) = 2\widetilde{g}_4(q)\frac{\partial \widetilde{g}_6}{\partial s}(q), \quad 3\widetilde{g}_6(q)\frac{\partial \widetilde{g}_4}{\partial t}(q) = 2\widetilde{g}_4(q)\frac{\partial \widetilde{g}_6}{\partial t}(q),$$
$$9\widetilde{g}_6(q)\frac{\partial \widetilde{g}_6}{\partial s}(q) = 2\widetilde{g}_4(q)^2\frac{\partial \widetilde{g}_4}{\partial s}(q), \quad 9\widetilde{g}_6(q)\frac{\partial \widetilde{g}_6}{\partial t}(q) = 2\widetilde{g}_4(q)^2\frac{\partial \widetilde{g}_4}{\partial t}(q).$$

Using them, we are able to derive

$$48(\widetilde{g}_4(q))^4 \operatorname{Hess}(f)(q) = -\widetilde{g}_6(q) \operatorname{Hess}(g)(o).$$

(Since the computation for this is messy and tedious, it is omitted.) The point $[0:0:1]$ is an ordinary double point of $\check{C}$, so that $\operatorname{Hess}(g)(o) \neq 0$. This completes the proof. ∎

Below, we list several particular examples of 28-nodal double Veronese cones constructed from smooth plane quartics with interesting automorphism groups (cf. [22, Table 6]). In each case the 3-fold is given by equation (5.10).

Example 5.14. We consider the Klein quartic for C. It is given by the equation

$$x^3 y + y^3 z + z^3 x = 0.$$

Its automorphism group is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_7)$. We have

$$g_4(s, t, u) = s^3 t + t^3 u + u^3 s$$

and

$$g_6(s, t, u) = \frac{1}{8}(3s^5 u - 15s^2 t^2 u^2 + 3st^5 + 3tu^5).$$

Example 5.15. Let C be the Fermat plane quartic curve defined by

$$x^4 + y^4 + z^4 = 0.$$

Its automorphism group is isomorphic to $\mu_4^2 \rtimes \mathfrak{S}_3$. We have

$$g_4(s, t, u) = 4(s^4 + t^4 + u^4),$$
$$g_6(s, t, u) = 16s^2 t^2 u^2.$$

For explicit equations of smooth plane quartics with an action of the symmetric group $\mathfrak{S}_4$ and the corresponding 28-nodal double Veronese cones we refer the reader to Example 1.5.

Example 5.16. Suppose that the smooth quartic curve C is given by

$$x^4 + y^4 + z^4 + x^3 y + 2x^3 z = 0.$$

Then C has no non-trivial automorphisms. We have

$$g_4(s, t, u) = 4(s^4 - st^3 - 2su^3 + t^4 + u^4),$$
$$g_6(s, t, u) = -16s^3 t^2 u - 8s^3 tu^2 + 16s^2 t^2 u^2 - 4t^6$$
$$+ 4t^5 u - t^4 u^2 - 4t^2 u^4 + 4tu^5 - u^6.$$

6. One-to-one correspondence

In the present section we put the previous results together to establish a one-to-one correspondence between 28-nodal double Veronese cones and smooth plane quartics,

which is asserted in Theorem 1.3. We also study automorphism groups of 28-nodal double Veronese cones and prove Theorem 1.4.

In this section, for the projective plane we keep the same two notations as in §5, i.e., $\mathbb{P}^2$ with a homogeneous coordinate system $[x : y : z]$ and its projectively dual plane $\check{\mathbb{P}}^2$ with $[s : t : u]$.

Remark 6.1. For an arbitrary del Pezzo 3-fold of degree 1, let $\kappa\colon V \dashrightarrow \check{\mathbb{P}}^2$ be the rational map given by the half-anticanonical linear system on V. The half-anticanonical linear system has a unique base point. The 3-fold V is smooth at the base point, so that a general fiber of κ is an elliptic curve by Bertini theorem and adjunction formula.

In the remaining part of this section, we use κ to denote the rational half-anticanonical elliptic fibration on a double Veronese cone V.

Now we construct a smooth plane quartic from a 28-nodal double Veronese cone.

Lemma 6.2. *Let V be a 28-nodal double Veronese cone. Denote by $\check{C}$ the discriminant curve of the rational elliptic fibration κ, and let C be the projectively dual curve of $\check{C}$. Then C is a smooth plane quartic. Furthermore, for any construction of V from an Aronhold heptad as in Proposition 4.1, the curve C is isomorphic to the Hessian curve of the corresponding net of quadrics.*

Proof. It follows from Theorem 4.4 that V can be constructed out of some Aronhold heptad in $\mathbb{P}^3$ as in Proposition 4.1. Choose one construction like this, and let $\mathscr{L}$ be the net of quadrics defined by the Aronhold heptad. The rational elliptic fibration κ is given by the proper transforms of the members of the net $\mathscr{L}$. Using Lemma 3.11, we see that the projectively dual curve of $\check{C}$ is the Hessian curve of the net $\mathscr{L}$, which is a smooth quartic curve by Lemma 3.7. ∎

Recall that to any (rational) fibration whose general fiber is an elliptic curve we can associate the j-function of its fibers, which is a rational function on the base of the fibration.

Remark 6.3. Let V be a del Pezzo 3-fold of degree 1. The 3-fold V can be defined in the weighted projective space $\mathbb{P}(1, 1, 1, 2, 3)$ with weighted homogeneous coordinates s, t, u, v, and w by an equation of the form

$$w^2 = v^3 - h_4(s, t, u)v + h_6(s, t, u),$$

where h_4 and h_6 are homogeneous polynomials of degrees 4 and 6, respectively. Since the singularities of V are isolated, the polynomial $4h_4(s, t, u)^3 - 27h_6(s, t, u)^2$ is not a zero polynomial. The rational elliptic fibration κ is given by the projection to the

coordinates s, t, and u. The corresponding j-function can be written as

$$j_V(s,t,u) = 1728 \frac{4h_4(s,t,u)^3}{4h_4(s,t,u)^3 - 27h_6(s,t,u)^2}$$

(see for instance [20, §3.1.1]). In particular, the discriminant curve of κ is given in $\check{\mathbb{P}}^2$ by the equation

$$4h_4(s,t,u)^3 - 27h_6(s,t,u)^2 = 0.$$

Recall from §5 that any (smooth) quartic curve C in the projective plane $\mathbb{P}^2$ defines a j-function j_C on the dual projective plane $\check{\mathbb{P}}^2$ that computes the j-invariant of the quartuple of intersection points of C with the corresponding line. It is given by the formula (5.4).

Lemma 6.4. *Let V be a 28-nodal double Veronese cone, and let C be the smooth plane quartic curve corresponding to V via Lemma 6.2. Then $j_V(s,t,u) = j_C(s,t,u)$ for all s,t,u.*

Proof. By Theorem 4.4, the 3-fold V is constructed out of some net $\mathcal{L}$ of quadrics in $\mathbb{P}^3$ as in Proposition 4.1. Let P be a general point of $\check{\mathbb{P}}^2$. The point P determines a one-dimensional linear subsystem $\mathcal{L}_P$ of the net $\mathcal{L}$. The fiber E_P of the half-anticanonical elliptic fibration κ over P is the proper transform of the base locus of the pencil $\mathcal{L}_P$. The base locus is given by the intersection of two general quadrics Q_1 and Q_2 in $\mathcal{L}_P$. The pencil $\mathcal{L}_P$ corresponds to the line L in $\mathbb{P}^2$ dual to the point P. The singular quadrics in $\mathcal{L}_P$ correspond to the four intersection points of the line L and the Hessian curve C of the net $\mathcal{L}$. Lemma 3.9 implies that the curve E_P is an elliptic curve isomorphic to the double cover of L branched at the four intersection points of L and C. This means that $j_V(P) = j_C(P)$.   ∎

We will need the following assertion on 28-nodal double Veronese cones and their half-anticanonical elliptic fibrations.

Lemma 6.5. *Let V be a 28-nodal double Veronese cone. Then V is uniquely defined by the j-function j_V of the fibers of the rational elliptic fibration κ.*

Proof. Suppose that there exists another 28-nodal double Veronese cone V' such that $j_V = j_{V'}$. The 3-folds V and V' can be defined in the weighted projective space $\mathbb{P}(1,1,1,2,3)$ with weighted homogeneous coordinates s, t, u, v, and w by equations

(6.6) $$w^2 = v^3 - h_4(s,t,u)v + h_6(s,t,u)$$

and

(6.7) $$w^2 = v^3 - k_4(s,t,u)v + k_6(s,t,u),$$

respectively, where h_d and k_d are homogeneous polynomials of degree d. By Remark 6.3, the j-functions can be written as

$$j_V(s,t,u) = 1728\frac{4h_4(s,t,u)^3}{4h_4(s,t,u)^3 - 27h_6(s,t,u)^2},$$

$$j_{V'}(s,t,u) = 1728\frac{4k_4(s,t,u)^3}{4k_4(s,t,u)^3 - 27k_6(s,t,u)^2}.$$

Recall that neither h_4 nor k_4 is a zero polynomial by Lemmas 6.4 and 5.6. Lemma 6.2 implies that both $4h_4^3 - 27h_6^2$ and $4k_4^3 - 27k_6^2$ are irreducible, and hence h_4 and h_6 (respectively, k_4 and k_6) have no common factor. Since

$$j_V(s,t,u) = j_{V'}(s,t,u)$$

for all s, t, u, we obtain

$$h_4(s,t,u)^3\big(4k_4(s,t,u)^3 - 27k_6(s,t,u)^2\big)$$
$$= k_4(s,t,u)^3\big(4h_4(s,t,u)^3 - 27h_6(s,t,u)^2\big).$$

From this, we can easily conclude that

$$k_4(s,t,u) = ah_4(s,t,u), \quad k_6(s,t,u) = bh_6(s,t,u),$$

for some non-zero constants a and b with $a^3 = b^2$. Thus equation (6.7) takes the form

$$w^2 = v^3 - ah_4(s,t,u)v + bh_6(s,t,u).$$

Put $c = \frac{b}{a}$. Then $c^2 = a$ and $c^3 = b$. The automorphism of $\mathbb{P}(1,1,1,2,3)$ defined by

$$[s:t:u:v:w] \mapsto \left[s:t:u:\frac{v}{c}:\frac{w}{c\sqrt{c}}\right]$$

brings it to the form (6.6), which means that V and V' are isomorphic. ∎

Now we are able to prove Theorem 1.3.

First proof of Theorem 1.3. By Lemma 6.2, a 28-nodal double Veronese cone uniquely determines a smooth plane quartic curve that is the projectively dual curve of the discriminant curve $\check{C}$ of the half-anticanonical rational elliptic fibration κ.

Now let C be a smooth plane quartic curve. We then construct a 28-nodal double Veronese cone V from C as in §5 (see equation (5.10)). Using Remark 6.3 and Proposition 5.7, we see that the projectively dual curve of the discriminant curve of the rational elliptic fibration κ is isomorphic to C.

It remains to show that if V' is a 28-nodal double Veronese cone such that the projectively dual curve of the discriminant curve of its half-anticanonical rational elliptic fibration is isomorphic to C, then V' is isomorphic to V. By Lemma 6.4 the j-functions of V and V' depend only on C, so that

$$j_V(s, t, u) = j_C(s, t, u) = j_{V'}(s, t, u)$$

for all s, t, u. According to Lemma 6.5 this means that V' is isomorphic to V.   ∎

We now move our attention to automorphism groups of 28-nodal double Veronese cones. Recall that if V is such a 3-fold, we denote by τ the Galois involution of the double cover $V \to \mathbb{P}(1, 1, 1, 2)$ given by the linear system $|2H| = |-K_V|$. The automorphism τ is contained in the center of the group $\mathrm{Aut}(V)$. It is easy to see that it preserves the fibers of the rational elliptic fibration κ.

Proof of Theorem 1.4. Since the map $\kappa \colon V \dashrightarrow \check{\mathbb{P}}^2$ is $\mathrm{Aut}(V)$-equivariant, we have a group homomorphism

$$\check{\Xi} \colon \mathrm{Aut}(V) \to \mathrm{Aut}(\check{\mathbb{P}}^2).$$

Its kernel $\mathcal{K} \subset \mathrm{Aut}(V)$ acts on every fiber of κ preserving the unique base point O of $|H|$. Let E be a general fiber of κ. Then E is a general elliptic curve by Lemma 5.6, so that the stabilizer of the point O in $\mathrm{Aut}(E)$ is isomorphic to $\boldsymbol{\mu}_2$. Therefore, $\mathcal{K}$ is a subgroup of $\boldsymbol{\mu}_2$. On the other hand, $\mathcal{K}$ obviously contains the Galois involution τ of the double cover given by the anticanonical linear system of V. Thus we conclude that $\mathcal{K} \cong \boldsymbol{\mu}_2$.

Consider the action of the group $\check{\Xi}(\mathrm{Aut}(V))$ on the projective plane $\check{\mathbb{P}}^2$ which is the target of the map κ. By construction the action of $\check{\Xi}(\mathrm{Aut}(V))$ preserves the discriminant curve $\check{C}$ of the rational elliptic fibration κ, and hence its action on the projectively dual plane of the projective plane $\check{\mathbb{P}}^2$, which is $\mathbb{P}^2$, preserves the curve C projectively dual to $\check{C}$. Therefore, we obtain a group homomorphism

$$\Xi \colon \mathrm{Aut}(V) \to \mathrm{Aut}(C)$$

whose kernel $\mathcal{K}$ is isomorphic to $\boldsymbol{\mu}_2$. Note that at this moment we claim neither that Ξ is surjective nor that $\mathcal{K}$ splits as a direct factor in $\mathrm{Aut}(V)$. However, we know that $\mathcal{K}$ is contained in the center of the group $\mathrm{Aut}(V)$.

Now let G be the automorphism group of the smooth plane quartic C. Then G is a finite group, and there is a natural action of G on the plane $\check{\mathbb{P}}^2$ which is projectively dual to the plane $\mathbb{P}^2$ where the curve C sits. By Remark 5.5 the latter action preserves the curves given by equations $g_4(s, t, u) = 0$ and $g_6(s, t, u) = 0$. This means that there exist group characters $\chi_4, \chi_6 \colon G \to \mathbb{C}^*$ such that for any $\gamma \in G$, one has

$$g_4\big(\gamma(s, t, u)\big) = \chi_4(\gamma) \cdot g_4(s, t, u), \quad g_6\big(\gamma(s, t, u)\big) = \chi_6(\gamma) \cdot g_6(s, t, u).$$

On the other hand, the elements of G preserve the equivalence class of the cross-ratio of a quadruple of intersection points of a line in $\mathbb{P}^2$ and the curve C; in other words, it preserves the j-function $j_C(s, t, u)$. By (5.4) this gives

$$\chi_4(\gamma)^3 = \chi_6(\gamma)^2$$

for every $\gamma \in G$. Set

$$\chi_2(\gamma) = \frac{\chi_6(\gamma)}{\chi_4(\gamma)}, \quad \gamma \in G.$$

Then χ_2 is a character of G such that $\chi_2(\gamma)^2 = \chi_4(\gamma)$ and $\chi_2(\gamma)^3 = \chi_6(\gamma)$. Since the image $\chi_2(G) \subset \mathbb{C}^*$ is a cyclic group, there is a well-defined character

$$\chi \colon G \to \mathbb{C}^*$$

such that $\chi(\gamma)^2 = \chi_2(\gamma)$ for all $\gamma \in G$. We have

$$\chi(\gamma)^4 = \chi_4(\gamma), \quad \chi(\gamma)^6 = \chi_6(\gamma).$$

Given an element $\gamma \in G$, we write

$$\gamma\big([s : t : u]\big) = [s' : t' : u']$$

and define an automorphism of $\mathbb{P}(1, 1, 1, 2, 3)$ by

(6.8) $$[s : t : u : v : w] \mapsto [s' : t' : u' : \chi(\gamma)^2 v : \chi(\gamma)^3 w].$$

Since the 3-fold V is given by (5.10) in $\mathbb{P}(1, 1, 1, 2, 3)$, the group G preserves the hypersurface V. This provides a group homomorphism

$$\Xi' \colon \mathrm{Aut}(C) \to \mathrm{Aut}(V).$$

We see from (6.8) that $\Xi \circ \Xi' = \mathrm{id}_{\mathrm{Aut}(C)}$. This implies that

$$\mathrm{Aut}(V) \cong \mathcal{K} \rtimes \mathrm{Aut}(C).$$

Since $\mathcal{K} \cong \boldsymbol{\mu}_2$ is contained in the center of $\mathrm{Aut}(V)$, we conclude that $\mathrm{Aut}(V) \cong \mathcal{K} \times \mathrm{Aut}(C)$. ∎

Corollary 6.9. *Let V be a 28-nodal double Veronese cone, let C be the smooth plane quartic curve corresponding to V by Theorem* 1.3, *and let S be the del Pezzo surface of degree* 2 *constructed as the double cover of $\mathbb{P}^2$ branched along C. Then*

$$\mathrm{Aut}(V) \cong \boldsymbol{\mu}_2 \times \mathrm{Aut}(C) \cong \mathrm{Aut}(S).$$

7. Theta characteristics

As we have seen in §4, Aronhold heptads, which are parts of regular Cayley octads, play a role in constructing 28-nodal double Veronese cones. On the other hand, neither an Aronhold heptad nor a regular Cayley octad is uniquely associated to a 28-nodal double Veronese cone. In what follows, correspondences between regular Cayley octads, smooth plane quartic curves, and 28-nodal double Veronese cones will be investigated in more detail, and the necessary additional data that is required to have natural one-to-one correspondences will be clarified.

So far we used Proposition 4.1 to construct 28-nodal double Veronese cones starting from Aronhold heptads. However, it turns out that the construction essentially depends only on the choice of the regular Cayley octad. This can be easily understood through Geiser-type involutions.

Lemma 7.1. *Let $P_1, \ldots, P_8$ be a regular Cayley octad in $\mathbb{P}^3$. For all $i = 1, \ldots, 8$, the 28-nodal double Veronese cones defined by the Aronhold heptads $\{P_1, \ldots, P_8\} \setminus \{P_i\}$ are isomorphic to each other.*

Proof. For each $i = 1, \ldots, 8$, let $\widehat{\mathbb{P}}_i^3$ be the blow up of $\mathbb{P}^3$ at the points $P_1, \ldots, P_8$ except the point P_i. Let $\widehat{\mathbb{P}}_{ij}^3$ be the weak Fano 3-fold obtained by blowing up $\mathbb{P}^3$ at the six points of $\{P_1, \ldots, P_8\} \setminus \{P_i, P_j\}$ and let W_{ij} be its anticanonical model.

For $1 \leqslant k \neq l \leqslant 8$, denote by L_{kl} the line passing through the points P_k and P_l, and denote by T_{kl} the twisted cubic passing through the six points of $\{P_1, \ldots, P_8\} \setminus \{P_k, P_l\}$. Let $\widehat{L}_{kl}$ and $\widehat{T}_{kl}$ be the proper transforms of these curves on $\widehat{\mathbb{P}}_{ij}^3$.

Proposition 4.6 implies that W_{ij} is the double cover of $\mathbb{P}^3$ branched over a quartic surface and that the map $\phi_{ij} \colon \widehat{\mathbb{P}}_{ij}^3 \to W_{ij}$ given by the anticanonical linear system is a birational morphism that contracts exactly the 15 curves $\widehat{L}_{kl}$ for $k < l$ with $\{k, l\} \cap \{i, j\} = \varnothing$, and the curve $\widehat{T}_{ij}$. Denote by $\bar{L}_{kl}$ and $\bar{T}_{kl}$ the images on W_{ij} of the curves $\widehat{L}_{kl}$ and $\widehat{T}_{kl}$ not contracted by ϕ_{ij}, respectively.

Note that the points $\bar{P}_i$ and $\bar{P}_j$ on W_{ij} corresponding to P_i and P_j are conjugate to each other with respect to the Galois involution ε of the double cover.

The double cover $\zeta \colon W_{ij} \to \mathbb{P}^3$ is given by the half-anticanonical linear system, so that the pull-backs of planes under ζ should be the proper transforms of the quadrics in the original $\mathbb{P}^3$ passing through the six points of $\{P_1, \ldots, P_8\} \setminus \{P_i, P_j\}$. The images of the curves $\zeta(\bar{T}_{ki})$ and $\zeta(\bar{T}_{kj})$, where $k \notin \{i, j\}$, are lines in $\mathbb{P}^3$ (note however that the images $\zeta(\bar{T}_{kl})$ for $\{k, l\} \cap \{i, j\} = \varnothing$ are conics). Therefore, by Lemma 3.4 for $k \notin \{i, j\}$, one has

$$\varepsilon(\bar{T}_{ki}) = \bar{L}_{ki}, \quad \varepsilon(\bar{L}_{ki}) = \bar{T}_{ki},$$
$$\varepsilon(\bar{T}_{kj}) = \bar{L}_{kj}, \quad \varepsilon(\bar{L}_{kj}) = \bar{T}_{kj}.$$

The involution $\varepsilon\colon W_{ij} \to W_{ij}$ gives rise to a birational map

$$\widehat{\varepsilon}\colon \widehat{\mathbb{P}}_i^3 \dashrightarrow \widehat{\mathbb{P}}_j^3.$$

Since $\varepsilon(\bar{P}_i) = \bar{P}_j$, we conclude that $\widehat{\varepsilon}$ is actually an isomorphism.

Denote by $\widehat{L}_{kl}^i$ and $\widehat{T}_{kl}^i$ (respectively, $\widehat{L}_{kl}^j$ and $\widehat{T}_{kl}^j$) the proper transforms of the curves L_{kl} and T_{kl} on $\widehat{\mathbb{P}}_i^3$ (respectively, $\widehat{\mathbb{P}}_j^3$). Then for $k \notin \{i, j\}$, one has

$$\widehat{\varepsilon}(\widehat{T}_{ki}^i) = \widehat{L}_{ki}^j, \quad \widehat{\varepsilon}(\widehat{L}_{ki}^i) = \widehat{T}_{ki}^j,$$
$$\widehat{\varepsilon}(\widehat{T}_{kj}^i) = \widehat{L}_{kj}^j, \quad \widehat{\varepsilon}(\widehat{L}_{kj}^i) = \widehat{T}_{kj}^j.$$

Also, we observe that for $\{k, l\} \cap \{i, j\} = \varnothing$,

$$\widehat{\varepsilon}(\widehat{L}_{kl}^i) = \widehat{L}_{kl}^j, \quad \widehat{\varepsilon}(\widehat{T}_{ij}^i) = \widehat{T}_{ij}^j.$$

Therefore, the pluri-anticanonical maps ϕ_i and ϕ_j of $\widehat{\mathbb{P}}_i^3$ and $\widehat{\mathbb{P}}_i^3$ both factor through W_{ij}, and the pluri-anticanonical models of $\widehat{\mathbb{P}}_i^3$ and $\widehat{\mathbb{P}}_i^3$, that are obtained by the construction described in Proposition 4.1, are both isomorphic to one and the same double Veronese cone V:
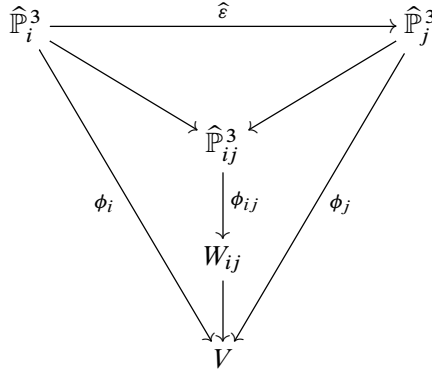
This completes the proof of the statement. ∎

Remark 7.2. A two-dimensional analog of Lemma 7.1 is the following simple observation. Let $R_1, \ldots, R_9$ be points in $\mathbb{P}^2$ such that they are the intersection of two general cubic curves. Then the del Pezzo surfaces of degree 1 obtained by blowing up the eight points of $\{R_1, \ldots, R_9\} \setminus \{R_i\}$ are isomorphic to each other for all $1 \leqslant i \leqslant 9$. For instance, to construct an isomorphism between the blow ups of $\{R_1, \ldots, R_9\} \setminus \{R_1\}$ and $\{R_1, \ldots, R_9\} \setminus \{R_2\}$, one can blow up $\mathbb{P}^2$ along the seven points $R_3, \ldots, R_9$, obtain a del Pezzo surface of degree 2, and apply the Galois involution of the morphism defined by its anticanonical linear system (which is a double cover over $\mathbb{P}^2$), similarly to the proof of Lemma 7.1.

Let $\mathcal{N}$ be the set that consists of isomorphism classes of regular Cayley octads modulo projective transformations. Let $\mathcal{T}$ be the set that consists of the pairs (C, θ), where C is a smooth plane quartic considered up to isomorphism, and θ is an even theta characteristic on C. A given regular Cayley octad defines a net $\mathcal{L}$ of quadrics in $\mathbb{P}^3$. The net $\mathcal{L}$ yields its Hessian quartic curve $H(\mathcal{L})$ in $\mathbb{P}^2$, which is smooth by Lemma 3.7. Meanwhile, the singular points of quadrics in the net $\mathcal{L}$ sweep out a smooth curve of degree 6 in $\mathbb{P}^3$ (see [6, Lemme 6.8] and [6, Proposition 6.10]), which is called the *Steinerian curve* of the net and is denoted by $S(\mathcal{L})$. There is an even theta characteristic $\theta(\mathcal{L})$ such that the linear system $|K_{H(\mathcal{L})} + \theta(\mathcal{L})|$ defines an isomorphism of $H(\mathcal{L})$ with $S(\mathcal{L})$ (see [6, Proposition 6.10] and [6, Lemme 6.12]). With such Hessian quartic curves and their even theta characteristics, we define the map

$$\Theta \colon \mathcal{N} \to \mathcal{T}$$

by assigning $\Theta(\mathcal{L}) = (H(\mathcal{L}), \theta(\mathcal{L}))$.

Theorem 7.3. *The map Θ is bijective.*

Proof. See [6, Proposition 6.23]. ∎

Suppose that the points $P_1, \ldots, P_7$ form an Aronhold heptad, and denote by P_8 the eighth base point of the net of quadrics $\mathcal{L}$ defined by the Aronhold heptad. Choose a pair of points P_i, P_j in the regular Cayley octad $P_1, \ldots, P_8$. This pair yields a pencil in the net $\mathcal{L}$ that consists of the quadrics containing the line passing through P_i and P_j. The pencil contains exactly one or two singular quadrics (see [6, Lemme 6.6 (i)]). Furthermore, the pencil defines a line in the projective plane where the Hessian quartic curve $H(\mathcal{L})$ sits, and this line is bitangent to the Hessian quartic curve (see [20, Theorem 6.3.5]). Such a bitangent line defines an odd theta characteristic of $H(\mathcal{L})$. It will be denoted by $\theta_{ij}(\mathcal{L})$. It is obvious that $\theta_{ij}(\mathcal{L}) = \theta_{kl}(\mathcal{L})$ if and only if $\{i, j\} = \{k, l\}$. In particular, every odd theta characteristic on $H(\mathcal{L})$ can be represented by $\theta_{ij}(\mathcal{L})$ for some i and j.

For each choice of four distinct indices i, j, k, l in $\{1, \ldots, 8\}$, set

(7.4) $$\theta_{i,jkl} = \theta_{ij}(\mathcal{L}) + \theta_{ik}(\mathcal{L}) + \theta_{il}(\mathcal{L}) - K_{H(\mathcal{L})}.$$

Before we proceed, notice that

(7.5) $$\theta_{ij}(\mathcal{L}) + \theta_{ik}(\mathcal{L}) + \theta_{jk}(\mathcal{L}) = K_{H(\mathcal{L})} + \theta(\mathcal{L})$$

for each choice of three distinct indices i, j, k in $\{1, \ldots, 8\}$ since the left hand side is the section of the Steinerian curve $S(\mathcal{L})$ by the plane containing the points P_i, P_j, and P_k in $\mathbb{P}^3$.

Lemma 7.6. *Let r be a fixed index in $\{1, \ldots, 8\}$.*

- *The theta characteristic given by $\theta_{r,ijk}$ is even for three distinct indices i, j, k in $\{1, \ldots, 8\} \setminus \{r\}$.*

- *Every even theta characteristic on $H(\mathcal{L})$ except $\theta(\mathcal{L})$ can be represented as $\theta_{r,ijk}$ for some choice of three distinct indices i, j, k in $\{1, \ldots, 8\} \setminus \{r\}$.*

- *One has*

(7.7)
$$\theta(\mathcal{L}) = -3K_{H(\mathcal{L})} + \sum_{i \neq r} \theta_{ri}(\mathcal{L}).$$

Proof. For the first statement, see [20, Theorem 6.3.6] or [24, Proposition IX.4].

Since there are exactly 35 even theta characteristics excluding $\theta(\mathcal{L})$, in order to prove the second statement, it is enough to verify that $\theta_{r,i_1 j_1 k_1} \neq \theta_{r,i_2 j_2 k_2}$ if $\{i_1, j_1, k_1\} \neq \{i_2, j_2, k_2\}$. Suppose that $\theta_{r,i_1 j_1 k_1} = \theta_{r,i_2 j_2 k_2}$. For convenience, assume that $r = 8$. Then

$$\theta_{8i_1}(\mathcal{L}) + \theta_{8j_1}(\mathcal{L}) + \theta_{8k_1}(\mathcal{L}) + \theta_{8i_2}(\mathcal{L}) + \theta_{8j_2}(\mathcal{L}) + \theta_{8k_2}(\mathcal{L}) - 3K_{H(\mathcal{L})}$$
$$= \theta_{r,i_1 j_1 k_1} + \theta_{r,i_2 j_2 k_2} - K_{H(\mathcal{L})} = 0.$$

If $\{i_1, j_1, k_1\} \cap \{i_2, j_2, k_2\} = \varnothing$, then the above equality and (7.5) imply

$$\theta_{i_1 i_2}(\mathcal{L}) + \theta_{j_1 j_2}(\mathcal{L}) + \theta_{k_1 k_2}(\mathcal{L}) = 3\theta(\mathcal{L}) = K_{H(\mathcal{L})} + \theta(\mathcal{L}).$$

This means that the six points P_{i_1}, P_{i_2}, P_{j_1}, P_{j_2}, P_{k_1}, P_{k_2} lie on a single plane, which is impossible by Lemma 3.2. If $\{i_1, j_1, k_1\} \cap \{i_2, j_2, k_2\} \neq \varnothing$, then we may assume that $i_1 = i_2$. Then

$$\theta_{8j_1}(\mathcal{L}) + \theta_{8k_1}(\mathcal{L}) = \theta_{8j_2}(\mathcal{L}) + \theta_{8k_2}(\mathcal{L}).$$

This together with (7.5) implies

$$\theta_{j_1 k_1}(\mathcal{L}) = \theta_{j_2 k_2}(\mathcal{L}),$$

and hence $\{i_1, j_1, k_1\} = \{i_2, j_2, k_2\}$.

Now let us prove the third statement. Suppose that the right hand side of (7.7) is an odd theta characteristic $\theta_{mn}(\mathcal{L})$. Then $m, n \neq r$; otherwise, $\theta_{r,ijk}$ would yield the same even theta characteristic for two different choices of three distinct indices i, j, k, which is not the case by the second statement. For convenience, we may assume that $r = 8$, $m = 2$, and $n = 1$. Then our assumption reads

$$\theta_{21}(\mathcal{L}) = -3K_{H(\mathcal{L})} + \sum_{i=1}^{7} \theta_{8i}(\mathcal{L}).$$

By (7.5) this yields

$$(7.8) \quad -K_{H(\mathcal{L})} + \sum_{i=3}^{7} \theta_{8i}(\mathcal{L}) = \theta_{21}(\mathcal{L}) + \theta_{81}(\mathcal{L}) + \theta_{82}(\mathcal{L}) = K_{H(\mathcal{L})} + \theta(\mathcal{L}).$$

Note that

$$K_{H(\mathcal{L})} + \theta(\mathcal{L}) = \theta_{43}(\mathcal{L}) + \theta_{83}(\mathcal{L}) + \theta_{84}(\mathcal{L})$$

by (7.5). Using this, we deduce from (7.8) that

$$-K_{H(\mathcal{L})} + \sum_{i=5}^{7} \theta_{8i}(\mathcal{L}) = \theta_{43}(\mathcal{L}).$$

This contradicts the first statement. Therefore, the right hand side of (7.7) is an even theta characteristic.

We now suppose that the right hand side of (7.7) is not $\theta(\mathcal{L})$. Then, due to the second statement, we may assume that $r = 8$ and

$$-3K_{H(\mathcal{L})} + \sum_{i=1}^{7} \theta_{8i}(\mathcal{L}) = \theta_{81}(\mathcal{L}) + \theta_{82}(\mathcal{L}) + \theta_{83}(\mathcal{L}) - K_{H(\mathcal{L})}.$$

However, this implies an absurd identity $\theta_{8,456} = \theta_{87}(\mathcal{L})$. Consequently, the right hand side of (7.7) must be $\theta(\mathcal{L})$. ∎

Let $\mathcal{L}_{\theta_{i,jkl}}$ be the net of quadrics corresponding to the plane quartic $H(\mathcal{L})$ and the even theta characteristic $\theta_{i,jkl}$ via the bijection from Theorem 7.3, and let $S_{\theta_{i,jkl}}$ be its Steinerian curve. It turns out that these curves (and also regular Cayley octads and nets of quadrics) are related by standard Cremona transformations of $\mathbb{P}^3$.

Proposition 7.9. Up to projective transformation, the curve $S_{\theta_{i,jkl}}$ is obtained from $S(\mathcal{L})$ by the standard Cremona transformation centered at the points P_i, P_j, P_k, P_l, and also by the standard Cremona transformation centered at the complementary set of points P_m, P_n, P_s, P_t.

Proof. See [24, Proposition IX.4]. ∎

Remark 7.10. By Lemma 7.6, the 35 even theta characteristics on $H(\mathcal{L})$ other than $\theta(\mathcal{L})$ are all of the form (7.4) for a fixed index i (one can take for instance $i = 8$). This means that the even theta characteristics obtained as in Proposition 7.9 by the 35 Cremona transformations associated to choices of four points out of the seven points $P_1, \ldots, P_7$ (alternatively, by the 35 Cremona transformations associated to the point P_8 and choices of three points out of $P_1, \ldots, P_7$) are pairwise different.

Using Proposition 7.9, we deduce the following.

Corollary 7.11. *Up to projective transformation, the net of quadrics $\mathcal{L}_{\theta_{i,jkl}}$ consists of the proper transforms of the quadrics from the net $\mathcal{L}$ with respect to the standard Cremona transformation ς centered at the points P_i, P_j, P_k, P_l (or with respect to the standard Cremona transformation centered at the complementary set of points P_m, P_n, P_s, P_t). Furthermore, the corresponding regular Cayley octad $P_1', \ldots, P_8'$ can be constructed as follows: the points P_i', P_j', P_k', P_l' are the images of the divisors contracted by ς, and $P_r' = \varsigma(P_r)$ for $r = m, n, s, t$.*

Proof. After a suitable projective transformation, we may assume that

$$P_i = [1:0:0:0], \quad P_j = [0:1:0:0], \quad P_k = [0:0:1:0], \quad P_l = [0:0:0:1].$$

Then the standard Cremona transformation ς is the selfmap of $\mathbb{P}^3$ defined by

$$\varsigma\left([x_0 : x_1 : x_2 : x_3]\right) = \left[\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}\right].$$

Every quadric surface in $\mathcal{L}$ is defined by a quadric homogeneous polynomial of the form

$$\sum_{\alpha \neq \beta} a_{\alpha\beta} x_\alpha x_\beta = 0,$$

where $a_{\alpha\beta} = a_{\beta\alpha}$. Its proper transform by ς is defined by the quadric homogeneous polynomial

$$\sum_{\{\alpha,\beta,\gamma,\delta\}=\{0,1,2,3\}} a_{\alpha\beta} x_\gamma x_\delta = 0.$$

The four coordinate points and the four points $\varsigma(P_m), \varsigma(P_n), \varsigma(P_s), \varsigma(P_t)$ are the eight intersection points of three quadrics. Lemma 3.1 then implies that the net $\mathcal{L}'$ of quadric surfaces passing though these eight points is exactly the net of the proper transforms of the quadric surfaces in $\mathcal{L}$. Observe that the symmetric matrix corresponding to the former quadric surface and the one corresponding to the latter have the same determinant. It then follows from Lemma 3.7 that the four coordinate points and the four points $\varsigma(P_m), \varsigma(P_n), \varsigma(P_s), \varsigma(P_t)$ form a regular Cayley octad. It also immediately follows from our observation that the Steinerian curve $S(\mathcal{L}')$ is the proper transform of the Steinerian curve $S(\mathcal{L})$ with respect to ς. Hence by Proposition 7.9, we have

$$S(\mathcal{L}') = S_{\theta_{i,jkl}}$$

up to projective transformation. By Theorem 7.3 this implies that

$$\mathcal{L}' = \mathcal{L}_{\theta_{i,jkl}},$$

and the assertion follows. ∎

For a given smooth plane quartic curve C equipped with an even theta characteristic θ, its associated net $\mathcal{L}$ of quadrics determines a regular Cayley octad $P_1, \ldots, P_8$ in $\mathbb{P}^3$, which in turn allows us to construct a unique 28-nodal double Veronese cone $V_{C,\theta}$ (see Lemma 7.1). Let θ' be another even theta characteristic on C, and let $V_{C,\theta'}$ be the corresponding 28-nodal double Veronese cone.

Lemma 7.12. *The 3-folds $V_{C,\theta}$ and $V_{C,\theta'}$ are isomorphic.*

Proof. By Remark 7.10, there are indices $1 \leqslant i < j < k \leqslant 7$ such that

$$\theta' = \theta_{8i}(\mathcal{L}) + \theta_{8j}(\mathcal{L}) + \theta_{8k}(\mathcal{L}) - K_C.$$

For convenience, let us say $i = 5, j = 6, k = 7$. Corollary 7.11 tells us that the net of quadrics $\mathcal{L}'$ associated to (C, θ') via Theorem 7.3 can be obtained by applying the standard Cremona transformation ς centered at the points P_1, P_2, P_3, P_4 to the original net $\mathcal{L}$, up to projective equivalence. Furthermore, the net $\mathcal{L}'$ is defined by the regular Cayley octad $P'_1, \ldots, P'_8$, where $P'_1, \ldots, P'_4$ are the images of the divisors contracted by ς, and set $P'_i = \varsigma(P_i)$ for $i = 5, \ldots, 8$. The 3-folds $V_{C,\theta}$ and $V_{C,\theta'}$ are constructed from the Aronhold heptads $P_1, \ldots, P_7$ and $P'_1, \ldots, P'_7$, respectively.

Let $\widehat{\mathbb{P}}^3$ be the blow up of $\mathbb{P}^3$ at $P_1, \ldots, P_7$. Denote by F_i the exceptional divisor on $\widehat{\mathbb{P}}^3$ over the point P_i. Let L_{ij} be the line in $\mathbb{P}^3$ passing through the points P_i and P_j with $i < j$, and let $\widehat{L}_{ij}$ be its proper transform on $\widehat{\mathbb{P}}^3$. Flopping the six curves $\widehat{L}_{ij}$ with $1 \leqslant i < j \leqslant 4$, we obtain a birational map ξ from $\widehat{\mathbb{P}}^3$ to another weak Fano 3-fold $\widehat{\mathbb{P}}^{3+}$. Thus, we obtain the following diagram:
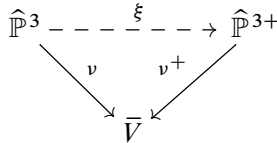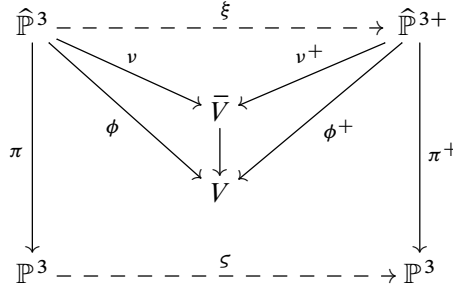

Here ν and ν^+ are contractions of the flopping curves of ξ and ξ^{-1}, respectively.

Keeping in mind the decomposition of the standard Cremona transformation into blow ups and blow downs, we see that $\widehat{\mathbb{P}}^{3+}$ can be represented as the blow up $\pi^+ \colon \widehat{\mathbb{P}}^{3+} \to \mathbb{P}^3$ at the points $P'_1, \ldots, P'_7$. More precisely, let Π_i, $1 \leqslant i \leqslant 4$, be the plane passing through the three points among $P_1, \ldots, P_4$ except P_i. The proper transforms of the four planes Π_i in $\widehat{\mathbb{P}}^{3+}$ are four disjoint surfaces Π_i^+ isomorphic to $\mathbb{P}^2$. Let F_i^+, $5 \leqslant i \leqslant 7$, be the proper transforms in $\widehat{\mathbb{P}}^{3+}$ of the surfaces F_i. Then the surfaces $\Pi_1^+, \Pi_2^+, \Pi_3^+, \Pi_4^+, F_5^+, F_6^+$, and F_7^+ are contracted to the seven points $P'_1, \ldots, P'_7$ on $\mathbb{P}^3$, respectively.

By Proposition 4.1, the 3-folds $V_{C,\theta}$ and $V_{C,\theta'}$ are the images of the pluri-anti-canonical morphisms ϕ and ϕ^+ of $\widehat{\mathbb{P}}^3$ and $\widehat{\mathbb{P}}^{3+}$, respectively. Since ξ is a flop, both ϕ

and ϕ^+ factor through $\overline{V}$, so that both $V_{C,\theta}$ and $V_{C,\theta'}$ are obtained by contracting one and the same curves on $\overline{V}$. Thus, $V_{C,\theta}$ and $V_{C,\theta'}$ are isomorphic to one and the same double Veronese cone V:

$$
\begin{array}{ccc}
\widehat{\mathbb{P}}^3 & \xrightarrow{\quad\xi\quad} & \widehat{\mathbb{P}}^{3+} \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi^+} \\
\mathbb{P}^3 & \xrightarrow{\quad\varsigma\quad} & \mathbb{P}^3
\end{array}
$$

This completes the proof of the lemma. ∎

Lemmas 7.1 and 7.12 give the following result.

Corollary 7.13. *Let C be a smooth plane quartic curve. Then for any choice of an even theta characteristic θ on C, and for any choice of an Aronhold heptad in the regular Cayley octad corresponding to θ, the 28-nodal double Veronese cones constructed from these Aronhold heptads as in Proposition 4.1 are isomorphic to each other.*

Corollary 7.13 allows us to put together the following proof.

Second proof of Theorem 1.3. Given a 28-nodal double Veronese cone, we associate to it a smooth plane quartic curve as in Lemma 6.2.

Let C be a smooth plane quartic curve. Choose an even theta characteristic on C. By Theorem 7.3, this provides us a regular Cayley octad. Choose an Aronhold heptad from this regular Cayley octad, and construct a 28-nodal double Veronese cone as in Proposition 4.1. By Corollary 7.13 the result does not depend on the choice of the even theta characteristic and the Aronhold heptad, and by Theorem 4.4 every 28-nodal double Veronese cone can be obtained in this way. Using Lemma 6.2 once again, we see that the above two constructions are mutually inverse. ∎

Recall from [24, §IX.2] that a (unordered) set of seven distinct odd theta characteristics $\theta_1, \ldots, \theta_7$ on a smooth plane quartic curve C is called an *Aronhold system* if they satisfy the condition that $\theta_i + \theta_j + \theta_k - K_C$ is an even theta characteristic for each choice of three distinct indices i, j, k. This is equivalent to the requirement that the six points of any three of $\theta_1, \ldots, \theta_7$ are not contained in a conic because

$$|2K_C - \theta_i - \theta_j - \theta_k| = |\theta_i + \theta_j + \theta_k - K_C|.$$

A given smooth quartic has exactly 288 Aronhold systems (see [24, Proposition IX.3]).

It follows from Lemma 7.6 that if $P_1, \ldots, P_8$ is a regular Cayley octad, and $\mathcal{L}$ is the corresponding net of quadrics in $\mathbb{P}^3$, then for every fixed index $r \in \{1, \ldots, 8\}$ the seven odd theta characteristics $\theta_{ri}(\mathcal{L})$, $i \neq r$, form an Aronhold system. It turns out that all Aronhold systems arise in this way.

Lemma 7.14 (cf. [20, Proposition 6.3.11]). *Let C be a smooth plane quartic, and let Ξ be an Aronhold system on C. Then there exist a unique even theta characteristic θ on C and a unique point P_8 in the regular Cayley octad $P_1, \ldots, P_8$ corresponding to (C, θ) via Theorem 7.3, such that*

$$(7.15) \qquad \Xi = \{\theta_{8i}(\mathcal{L}) \mid 1 \leqslant i \leqslant 7\},$$

where $\mathcal{L}$ is the net of quadrics defined by the regular Cayley octad $P_1, \ldots, P_8$.

Proof. For a given even theta characteristic θ on C, we obtain a regular Cayley octad $P_1, \ldots, P_8$ in $\mathbb{P}^3$ from Theorem 7.3. This gives us eight distinct Aronhold systems $\Xi_1, \ldots, \Xi_8$ on C as in (7.15). By Lemma 7.6, one has

$$\theta = -3K_C + \sum_{\vartheta \in \Xi_r} \vartheta$$

for every $r \in \{1, \ldots, 8\}$. Hence none of the Aronhold systems Ξ_r can coincide with an Aronhold system constructed in the same way starting from any other even theta characteristic $\theta' \neq \theta$ on C. Since there are exactly 288 Aronhold systems and exactly 36 even theta characteristics on C, the above construction exhausts all the Aronhold systems on C, and the assertion follows. ∎

Lemmas 7.6 and 7.14 imply the following.

Corollary 7.16. *Let C be a smooth plane quartic, and let Ξ be an Aronhold system on C. Then*

$$(7.17) \qquad \theta = -3K_C + \sum_{\vartheta \in \Xi} \vartheta$$

is an even theta characteristic on C. Moreover, for every even theta characteristic θ on C there exist exactly eight Aronhold systems such that θ is obtained from them as in (7.17), and each of these eight Aronhold systems is in turn obtained from θ as in Lemma 7.14.

To proceed we need to define a certain equivalence relation on the set of Aronhold systems. Let $\mathcal{T}_a$ be the set that consists of the pairs (C, Ξ), where C is a smooth quartic curve considered up to isomorphism, and Ξ is an Aronhold system on C. Two members (C_1, Ξ_1) and (C_2, Ξ_2) in $\mathcal{T}_a$ are considered to be equivalent if

- $C_1 = C_2$;

- one has

$$\sum_{\vartheta \in \Xi_1} \vartheta = \sum_{\vartheta \in \Xi_2} \vartheta;$$

- there is an automorphism $\sigma \colon C_1 \to C_2$ such that $\sigma^*(\Xi_2) = \Xi_1$.

In other words, the equivalence comes from the action of stabilizers of the even theta characteristic attached to (C, Ξ) by (7.17) in the automorphism group of C. Let $\mathcal{T}_A$ be the set of the equivalence classes of members in $\mathcal{T}_a$. Let $\mathcal{A}$ be the set of Aronhold heptads in $\mathbb{P}^3$ up to projective transformations. Define a map

$$\triangle \colon \mathcal{A} \to \mathcal{T}_A$$

by assigning to a given Aronhold heptad the Hessian curve of the corresponding net of quadrics together with the Aronhold system constructed as in (7.15).

Theorem 7.18. *The map $\triangle$ is bijective.*

Proof. It is enough to show that for a given smooth plane quartic curve C the map $\triangle$ induces one-to-one correspondence between members of $\mathcal{A}$ with the Hessian curve C and members of $\mathcal{T}_A$ with the first component C.

It immediately follows from Lemma 7.14 that the map $\triangle$ is surjective.

Suppose that $\triangle(X) = \triangle(Y)$ for Aronhold heptads X and Y. First of all, the Hessian curves of the nets of quadrics determined by X and Y, respectively, are one and the same smooth quartic curve C. Let Ξ_X and Ξ_Y be the Aronhold systems determined by X and Y, respectively. Then

$$-3K_C + \sum_{\vartheta \in \Xi_X} \vartheta = -3K_C + \sum_{\vartheta \in \Xi_Y} \vartheta$$

is one and the same even theta characteristic θ on C. Furthermore, since $\triangle(X) = \triangle(Y)$, there is an automorphism σ of C such that $\Xi_X = \sigma^*(\Xi_Y)$. Then the automorphism σ preserves the linear system $|K_C + \theta|$ that is the linear system of hyperplane sections of the Steinerian curve of the net of quadrics $\mathcal{L}$ in $\mathbb{P}^3$ corresponding to (C, θ) via Theorem 7.3. This means that the automorphism σ considered as an automorphism of the Steinerian curve is induced from a projective transformation of $\mathbb{P}^3$ preserving the regular Cayley octad defined by $\mathcal{L}$. Thus, we conclude that $X = Y$ in $\mathcal{A}$. ∎

Corollary 7.19. *Let V be a 28-nodal double Veronese cone, and let C be the smooth plane quartic curve corresponding to V via Theorem 1.3. Then there exists a natural one-to-one correspondence between the set of the diagrams (1.1) considered up to projective transformations of $\mathbb{P}^3$ and the set of members in $\mathcal{T}_A$ with the first component in the pair isomorphic to C.*

Proof. This immediately follows from Theorem 7.18, because by Corollary 7.13 there is an obvious one-to-one correspondence between the set of the diagrams (1.1) considered up to projective transformations of $\mathbb{P}^3$ and the set of members in $\mathcal{A}$ with the Hessian curve C. ∎

Corollary 7.20. *For a general smooth plane quartic curve C, there are exactly* 288 *diagrams* (1.1) *associated to C, up to projective transformations of $\mathbb{P}^3$.*

If the smooth plane quartic curve C is not general, the number of diagrams (1.1) associated to C up to projective transformations of $\mathbb{P}^3$ may be smaller than 288. Note however that this number is always at least 36.

Example 7.21. Let C be the Klein quartic curve given in $\mathbb{P}^2$ by the equation

$$x^3 y + y^3 z + z^3 x = 0.$$

Then $\mathrm{Aut}(C) \cong \mathrm{PSL}_2(\mathbb{F}_7)$. There exists a unique $\mathrm{Aut}(C)$-invariant even theta characteristic θ_0 on C (see [9, Example 2] or [19, Example 2.8]). The remaining 35 even theta characteristics split into three $\mathrm{Aut}(C)$-orbits of lengths 7, 7, and 21, respectively (see [23, (8.3)]). Furthermore, according to [41, Theorem 10.1] the set of 288 Aronhold systems on C splits into four $\mathrm{Aut}(C)$-orbits of lengths 8, 56, 56, and 168, respectively. This means that the stabilizer of every even theta characteristic θ in $\mathrm{Aut}(C)$ acts transitively on the set of eight Aronhold systems corresponding to θ via (7.17).

Now let V be the 28-nodal double Veronese cone corresponding to C via Theorem 1.3. Corollary 7.19 tells us that V admits exactly 36 diagrams (1.1) up to projective transformations of $\mathbb{P}^3$. One of them corresponds to the $\mathrm{Aut}(C)$-invariant theta characteristic θ_0. The group $\mathrm{Aut}(C)$ acts on the corresponding three-dimensional projective space preserving the regular Cayley octad. Note however that since the regular Cayley octad in $\mathbb{P}^3$ is a single $\mathrm{Aut}(C)$-orbit (see for instance [12, Lemma 3.2]), this diagram is not equivariant with respect to the whole group $\mathrm{Aut}(C)$. It is equivariant with respect to the subgroup $\mu_7 \rtimes \mu_3 \subset \mathrm{Aut}(C)$.

There is another natural way to recover the smooth plane quartic C corresponding to the 28-nodal double Veronese cone V.

Let $P_1, \ldots, P_7$ be seven points in $\mathbb{P}^3$ that form an Aronhold heptad. Let P_8 be the eighth base point of the net $\mathcal{L}(P_1, \ldots, P_7)$. Let $\rho \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the linear projection centered at P_8.

Lemma 7.22. *No three of the seven points $\rho(P_1), \ldots, \rho(P_7)$ are collinear (so that in particular no two of them coincide), and no six points among them are contained in a conic.*

Proof. If three points, say $\rho(P_1)$, $\rho(P_2)$, and $\rho(P_3)$, lie on a line, then the four points P_1, P_2, P_3, and P_8 lie on a plane. This implies that there is an element in $\mathcal{L}(P_1, \ldots, P_7)$ that contains the plane. Otherwise the restriction of the net $\mathcal{L}(P_1, \ldots, P_7)$ to the plane would define a net of conics passing through the four points P_1, P_2, P_3, and P_8.

We now suppose that six points, say $\rho(P_1), \ldots, \rho(P_6)$, lie on a conic. Then there is a quadric surface in $\mathbb{P}^3$ passing through the six points $P_1, \ldots, P_6$ together with P_8. It follows from Lemma 3.1 that the quadric surface should pass through the point P_7. If the base point P_8 of the net $\mathcal{L}(P_1, \ldots, P_7)$ is a singular point of a member of the net, then the base locus of the net $\mathcal{L}(P_1, \ldots, P_7)$ could not consists of eight distinct points. ∎

Remark 7.23. The heptad consisting of $\rho(P_1), \ldots, \rho(P_7)$ is the *Gale transform* of the heptad $P_1, \ldots, P_7$ (see [31], and in particular the examples following [31, Corollary 3.2]). Thus, the points $\rho(P_1), \ldots, \rho(P_7)$ uniquely define the Aronhold heptad $P_1, \ldots, P_7$ and the regular Cayley octad $P_1, \ldots, P_8$ up to projective transformations of $\mathbb{P}^3$.

Now let $\alpha\colon S \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at the seven points $\rho(P_1), \ldots, \rho(P_7)$. By Lemma 7.22 the surface S is a smooth del Pezzo surface of degree 2, so that its anti-canonical linear system yields the double cover $\varphi\colon S \to \mathbb{P}^2$ branched along some smooth plane quartic curve C. The quartic curve C coincides with the Hessian curve of the net of quadrics $\mathcal{L}(P_1, \ldots, P_7)$ (see [20, Proposition 6.3.11] or [24, Proposition IX.2]). The above maps are diagrammed as follows:

$$
(7.24) \qquad
\begin{array}{ccccc}
V & \xleftarrow{\ \phi\ } & \widehat{\mathbb{P}}^3 & & S \\
{\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\pi} & {}^{\alpha}\swarrow & \downarrow{\scriptstyle\varphi} \\
\mathbb{P}^2 & & \mathbb{P}^3 \dashrightarrow[\rho] \mathbb{P}^2 & & \mathbb{P}^2
\end{array}
$$

Here ϕ is a small resolution of all singular points of the 3-fold V, the morphism π is the blow up of $\mathbb{P}^3$ at the seven distinct points $P_1, \ldots, P_7$, and the rational map κ is given by the half-anticanonical linear system of V.

Remark 7.25. The image of the exceptional curve of α over the point $\rho(P_i)$ by the double cover morphism φ is the bitangent line of C corresponding to the odd theta characteristic $\theta_{8i}(\mathcal{L})$ (see [24, Proposition IX.2]). These seven odd theta characteristics form an Aronhold system. Moreover, this Aronhold system is the same as the one given by (7.15).

Remark 7.26. Following the diagram (7.24), one may easily figure out the last statement in Lemma 3.9. Indeed, a fiber of the rational map κ is given by the base locus

of a pencil contained in the net $\mathcal{L}(P_1, \ldots, P_7)$. In case of a smooth fiber, the fiber is the proper transform of a smooth intersection E of two quadrics in the net. The intersection E is a curve of degree 4 passing through eight points $P_1, \ldots, P_8$ which are the base points of the net. Thus the projection ρ sends E to a cubic curve passing through the seven points $\rho(P_1), \ldots, \rho(P_7)$. The proper transform of this cubic curve on the del Pezzo surface S is a smooth member of the anticanonical linear system of S. Since the double cover φ is given by $|-K_S|$, the latter proper transform is the pull-back of a line on $\mathbb{P}^2$ by φ. It is the double cover of the line branched at the distinct four points at which the line and the quartic curve C intersect. The point of the dual projective plane corresponding to this line is exactly the point over which the original fiber lies.

8. $\mathfrak{S}_4$-symmetric 28-nodal double Veronese cones

In this section we study the $\mathfrak{S}_4$-equivariant birational geometry of double Veronese cones introduced in Example 1.5.

Let C and V be the plane quartic curve and the corresponding 28-nodal double Veronese cone in Example 1.5. We consider the automorphisms of $\mathbb{P}(1, 1, 1, 2, 3)$:

$$\tau \colon [s : t : u : v : w] \mapsto [s : t : u : v : -w],$$
$$\sigma_1 \colon [s : t : u : v : w] \mapsto [-s : -t : u : v : w],$$
$$\sigma_2 \colon [s : t : u : v : w] \mapsto [u : s : t : v : w],$$
$$\sigma_3 \colon [s : t : u : v : w] \mapsto [t : s : u : v : -w].$$

Since they keep V invariant, they may be regarded as automorphisms of V. Then the involution τ is the Galois involution of the double cover of the Veronese cone. Moreover, by Theorem 1.4, we have

$$\mathrm{Aut}(V) \cong \boldsymbol{\mu}_2 \times \mathrm{Aut}(C),$$

where the subgroup $\boldsymbol{\mu}_2$ is generated by τ.

Let $\mathfrak{G}$ be the subgroup in $\mathrm{Aut}(V)$ that is generated by σ_1, σ_2, and σ_3, and let $\mathfrak{G}'$ be the subgroup in $\mathrm{Aut}(V)$ generated by σ_1, σ_2, and $\tau \circ \sigma_3$. Then $\mathfrak{G} \cong \mathfrak{G}' \cong \mathfrak{S}_4$, and both of them are projected isomorphically to the same subgroup in $\mathrm{Aut}(C)$. Meanwhile, the subgroups $\mathfrak{G}$ and $\mathfrak{G}'$ are not conjugate in $\mathrm{Aut}(V)$.

To facilitate the computations, introduce an auxiliary variable

$$(8.1) \qquad \bar{v} = v - \mu(s^2 + t^2 + u^2),$$

where $\mu = 2\lambda/3$. The defining equation of the 3-fold V may then be rewritten as

(8.2) $\quad w^2 = \overline{v}\big(\overline{v}^2 + 3\mu\overline{v}(s^2 + t^2 + u^2) + 3\mu^2(s^2 + t^2 + u^2)^2 - g_4(s, t, u)\big)$
$$+ \mu^3(s^2 + t^2 + u^2)^3 - \mu(s^2 + t^2 + u^2)g_4(s, t, u) + g_6(s, t, u).$$

One has

$$\mu^3(s^2 + t^2 + u^2)^3 - \mu(s^2 + t^2 + u^2)g_4(s, t, u) + g_6(s, t, u)$$
$$= 4(\lambda - 2)^2(\lambda + 1)s^2 t^2 u^2.$$

Set $\gamma = 2(\lambda - 2)\sqrt{\lambda + 1}$. We see from (8.2) that the 3-fold V contains the surfaces $\Pi_\pm$ given by equations

$$\overline{v} = w \mp \gamma stu = 0,$$

that is, by equations

(8.3) $\quad \begin{cases} v = \mu(s^2 + t^2 + u^2), \\ w = \pm\gamma stu. \end{cases}$

Note also that both Π_+ and Π_- are $\mathfrak{G}'$-invariant, and both of them are not $\mathbb{Q}$-Cartier divisors. In particular, we see that rk $\mathrm{Cl}(V)^{\mathfrak{G}'} \neq 1$.

Let us introduce a new coordinate

$$r = \frac{w - \gamma stu}{\overline{v}}.$$

On the 3-fold V, one has

$$r = \frac{\overline{v}^2 + 3\mu\overline{v}(s^2 + t^2 + u^2) + 3\mu^2(s^2 + t^2 + u^2)^2 - g_4(s, t, u)}{w + \gamma stu}.$$

This gives us a birational map of V to the complete intersection in $\mathbb{P}(1, 1, 1, 1, 2, 3)$ given by

(8.4) $\quad \begin{cases} r\overline{v} = w - \gamma stu, \\ r(w + \gamma stu) = \overline{v}^2 + 3\mu\overline{v}(s^2 + t^2 + u^2) \\ \qquad\qquad\qquad + 3\mu^2(s^2 + t^2 + u^2)^2 - g_4(s, t, u). \end{cases}$

Now excluding the variable w using the first equation in (8.4) and expressing $\overline{v}$ in terms of $s, t, u,$ and v by (8.1), we obtain a quartic hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$ with homogeneous coordinates $s, t, u, r,$ and v that is given by

$$r(r\overline{v} + 2\gamma stu) = \overline{v}^2 + 3\mu\overline{v}(s^2 + t^2 + u^2) + 3\mu^2(s^2 + t^2 + u^2)^2 - g_4(s, t, u),$$

or equivalently by

(8.5) $\quad v^2 + v\big(\mu(s^2 + t^2 + u^2) - r^2\big) + \mu^2(s^2 + t^2 + u^2)^2$
$$- g_4(s, t, u) + \mu r^2(s^2 + t^2 + u^2) - 2\gamma rstu = 0.$$

We denote this 3-fold by W.

We have constructed a birational map $V \dashrightarrow W$ that fits into the following $\mathfrak{G}'$-equivariant diagram

(8.6)

$$\hat{V} \longrightarrow V$$

Here the horizontal arrow is the blow up of the surface Π_+, and the vertical arrow is the blow down of the proper transform of the surface Π_- to the point

$$[s : t : u : r : v] = [0 : 0 : 0 : 1 : 0],$$

which is a smooth point of the 3-fold W.

The 3-fold W is the double cover of $\mathbb{P}^3$ branched over a quartic surface

$$\big(\mu(s^2 + t^2 + u^2) - r^2\big)^2$$
$$- 4\big(\mu^2(s^2 + t^2 + u^2)^2 - g_4(s,t,u) + \mu(s^2 + t^2 + u^2)r^2 - 2\gamma rstu\big) = 0,$$

where we consider s, t, u and r as homogeneous coordinates on $\mathbb{P}^3$. This quartic surface has exactly 16 nodes, which means that it is a Kummer surface, so that W would also have exactly 16 nodes.

Let S be the surface in W that is cut out by $r = 0$. Then S is $\mathfrak{G}'$-invariant. It is the double cover of $\mathbb{P}^2$ branched over the quartic curve

$$4g_4(s,t,u) - 3\mu^2(s^2 + t^2 + u^2)^2 = 0,$$

where we consider s, t and u as homogeneous coordinates on $\mathbb{P}^2$. Keeping in mind that $\mu = 2\lambda/3$, we see that this curve is given by

(8.7)
$$s^4 + t^4 + u^4 + \lambda(t^2u^2 + s^2u^2 + s^2t^2) = 0.$$

Therefore, it is isomorphic to our original curve C. In particular, the surface S is a smooth del Pezzo surface.

Lemma 8.8. *Let $\mathbb{U} = \mathrm{Pic}(S) \otimes \mathbb{C}$. One has an isomorphism of $\mathfrak{G}'$-representations*

(8.9)
$$\mathbb{U} \cong \mathbb{I} \oplus \mathbb{I}' \oplus \mathbb{W}_3 \oplus \mathbb{W}_3',$$

where $\mathbb{I}$ is the trivial representation, $\mathbb{I}'$ is the sign representation, and $\mathbb{W}_3$ and $\mathbb{W}_3'$ are non-isomorphic irreducible three-dimensional representations.

Proof. Recall from (8.5) that the surface S is given in $\mathbb{P}(1, 1, 1, 2)$ with weighted homogeneous coordinates s, t, u, and v by equation

$$v^2 + \frac{2\lambda}{3}v(s^2 + t^2 + u^2) + \frac{4\lambda^2}{9}(s^2 + t^2 + u^2)^2 - g_4(s, t, u) = 0,$$

where

$$g_4(s, t, u) = \frac{\lambda^2 + 12}{3}(s^4 + t^4 + u^4) + \frac{2(\lambda^2 + 6\lambda)}{3}(t^2u^2 + s^2u^2 + t^2s^2).$$

As before, the parameter λ varies in $\mathbb{C} \setminus \{\pm 2, -1\}$.

Let v be a new variable defined as

$$v = \frac{v}{2} + \frac{\lambda}{6}(s^2 + t^2 + u^2).$$

One can verify that, after this change of coordinates, S is given by the following equation in $\mathbb{P}(1, 1, 1, 2)$ with weighted homogeneous coordinates s, t, u, and v, cf. (8.7):

$$v^2 = s^4 + t^4 + u^4 + \lambda(t^2u^2 + s^2u^2 + t^2s^2).$$

Note that $\mathfrak{G}'$ is the subgroup in $\mathrm{Aut}(S)$ generated by the transformations

$$[s : t : u : v] \mapsto [-s : -t : u : v],$$
$$[s : t : u : v] \mapsto [u : s : t : v],$$
$$[s : t : u : v] \mapsto [t : s : u : v].$$

Let $\mathfrak{G}''$ be the subgroup in $\mathrm{Aut}(S)$ generated by the transformations

$$[s : t : u : v] \mapsto [-s : -t : u : v],$$
$$[s : t : u : v] \mapsto [u : s : t : v],$$
$$[s : t : u : v] \mapsto [t : s : u : -v].$$

Then $\mathfrak{G}'' \cong \mathfrak{G}'$, but these subgroups are not conjugate in $\mathrm{Aut}(S)$. They intersect by the even elements, and every odd element of $\mathfrak{G}'$ differs from the corresponding element of $\mathfrak{G}''$ by the Galois involution of the anticanonical double cover $S \to \mathbb{P}^2$. This involution acts as a multiplication by -1 on the orthogonal complement to K_S with respect to the intersection form in $\mathrm{Pic}(S) \otimes \mathbb{C}$. Thus, to prove (8.9), it is enough to show that

$$\mathbb{U} \cong \mathbb{I}^{\oplus 2} \oplus \mathbb{W}_3 \oplus \mathbb{W}_3'$$

as a representation of the group $\mathfrak{G}''$.

For ease of notation, take $\alpha, \beta \in \mathbb{C}$ so that

$$\alpha^2 = \frac{-\lambda + 2\sqrt{-\lambda - 1}}{\lambda + 2} \quad \text{and} \quad \beta = \frac{\lambda}{2}(1 + \alpha^2).$$

Consider the following twelve (-1)-curves on the surface S:

$$L_1 : s + \alpha t = \mathrm{v} - \beta t^2 - u^2 = 0; \quad L_2 : s - \alpha t = \mathrm{v} - \beta t^2 - u^2 = 0;$$
$$L_3 : t + \alpha s = \mathrm{v} + \beta s^2 + u^2 = 0; \quad L_4 : t - \alpha s = \mathrm{v} + \beta s^2 + u^2 = 0;$$
$$L_5 : t + \alpha u = \mathrm{v} - \beta u^2 - s^2 = 0; \quad L_6 : t - \alpha u = \mathrm{v} - \beta u^2 - s^2 = 0;$$
$$L_7 : s + \alpha u = \mathrm{v} + \beta u^2 + t^2 = 0; \quad L_8 : s - \alpha u = \mathrm{v} + \beta u^2 + t^2 = 0.$$
$$L_9 : u + \alpha t = \mathrm{v} + \beta t^2 + s^2 = 0; \quad L_{10} : u - \alpha t = \mathrm{v} + \beta t^2 + s^2 = 0;$$
$$L_{11} : u + \alpha s = \mathrm{v} - \beta s^2 - t^2 = 0; \quad L_{12} : u - \alpha s = \mathrm{v} - \beta s^2 - t^2 = 0.$$

They form a single $\mathfrak{G}''$-orbit. Moreover, these 12 curves split into a disjoint union of the following pairs of intersecting lines:

$$L_1 \cup L_2, \quad L_3 \cup L_4, \quad L_5 \cup L_6, \quad L_7 \cup L_8, \quad L_9 \cup L_{10}, \quad L_{11} \cup L_{12}.$$

Hence, there exists a $\mathfrak{G}''$-equivariant conic bundle $\zeta \colon S \to \mathbb{P}^1$ such that these pairs are exactly its singular fibers. This implies that $\mathrm{Pic}(S)^{\mathfrak{G}''}$ is generated by K_S and a fiber of this conic bundle. In particular, the surface S is not $\mathfrak{G}''$-minimal. With [22, Table 7], we conclude that $\mathfrak{G}''$ is the *odd lift* of $\mathfrak{S}_4$ to $\mathrm{Aut}(S)$ (see [22, §6.6] for the terminology).

Therefore, the subgroup $\mathfrak{G}'$ is the *even lift* of $\mathfrak{S}_4$ to $\mathrm{Aut}(S)$. Hence, by [22, Table 7], the surface S is $\mathfrak{G}'$-minimal, so that $\mathbb{U}$ contains a unique trivial $\mathfrak{G}'$-representation (generated by K_S). This in turn implies that $\mathbb{U}$ does not contain summands isomorphic to the sign representation of the group $\mathfrak{G}''$.

As $\mathfrak{G}''$-representation, $\mathbb{U}$ splits as

$$(8.10) \qquad\qquad\qquad \mathbb{U} \cong \mathbb{I}^{\oplus 2} \oplus \overline{\mathbb{U}},$$

where $\overline{\mathbb{U}}$ is some six-dimensional representation of $\mathfrak{G}''$ that does not contain one-dimensional subrepresentations. We conclude that $\overline{\mathbb{U}}$ splits into a sum of two- and three-dimensional irreducible $\mathfrak{G}''$-representations. If there is a two-dimensional summand in $\overline{\mathbb{U}}$, then it must be a sum of three two-dimensional summands. In this case the action of $\mathfrak{G}''$ on $\mathrm{Pic}(S)$ would be not faithful, which contradicts [20, Corollary 8.2.40]. Therefore, we see that $\overline{\mathbb{U}}$ is a sum of two three-dimensional irreducible representations of $\mathfrak{G}''$. Note that in the splitting (8.10) the summand $\mathbb{I}^{\oplus 2}$ is generated by K_S and a fiber of the conic bundle ζ, while L_1, L_3, L_5, L_7, L_9, and L_{11} form a basis in $\overline{\mathbb{U}}$.

Let σ be the element of the group $\mathfrak{G}''$ that acts by

$$[s : t : u : \mathrm{v}] \mapsto [t : s : u : -\mathrm{v}].$$

Then

$$\sigma(L_1) = L_3, \quad \sigma(L_3) = L_1, \qquad \sigma(L_5) = L_7,$$
$$\sigma(L_7) = L_5, \quad \sigma(L_9) = L_{11}, \quad \sigma(L_{11}) = L_9.$$

Thus, the trace of σ in $\overline{\mathbb{U}}$ equals 0. The element σ corresponds to a transposition in $\mathfrak{G}'' \cong \mathfrak{S}_4$. This means that $\overline{\mathbb{U}} \cong \mathbb{W}_3 \oplus \mathbb{W}_3'$ as $\mathfrak{G}''$-representation. Therefore, the isomorphism (8.9) of $\mathfrak{G}'$-representations holds. ∎

The commutative diagram (8.6) gives a surjective $\mathfrak{G}'$-module homomorphism of $\mathrm{Cl}(V)$ onto $\mathrm{Cl}(W)$ with one-dimensional kernel generated by the surface Π_-. On the other hand, restricting divisors in $\mathrm{Cl}(W)$ to S, we obtain a $\mathfrak{G}'$-module homomorphism

$$\mathrm{Cl}(W) \to \mathrm{Pic}(S),$$

which is injective (cf. [50, Corollary 3.9.3]). Note that $\mathrm{Cl}(W)$ is of rank 7, while $\mathrm{Pic}(S)$ is of rank 8.

As $\mathfrak{G}'$-representations, we have

$$\mathrm{Cl}(V) \otimes \mathbb{C} \cong \mathbb{I}^{\oplus 2} \oplus \mathbb{V},$$

where $\mathbb{I}$ is a trivial $\mathfrak{G}'$-representation, and $\mathbb{V}$ is some six-dimensional $\mathfrak{G}'$-representation. Note that the summand $\mathbb{I}^{\oplus 2}$ is generated by K_V and Π_-. We see that $\mathbb{V}$ is a subrepresentation of $\mathrm{Pic}(S) \otimes \mathbb{C}$, so that $\mathbb{V}$ is a sum of two irreducible three-dimensional $\mathfrak{G}'$-representations $\mathbb{W}_3$ and $\mathbb{W}_3'$ by Lemma 8.8. As a by-product, we obtain

Corollary 8.11. *There is an isomorphism of $\mathfrak{G}'$-representations*

$$\mathrm{Cl}(V) \otimes \mathbb{C} \cong \mathbb{I}^{\oplus 2} \oplus \mathbb{W}_3 \oplus \mathbb{W}_3'.$$

In particular, $\mathrm{rk}\,\mathrm{Cl}(V)^{\mathfrak{G}'} = 2$ *and* $\mathrm{rk}\,\mathrm{Cl}(W)^{\mathfrak{G}'} = 1$.

Recall that the subgroups $\mathfrak{G}$ and $\mathfrak{G}'$ are both isomorphic to $\mathfrak{S}_4$. They intersect by the even elements, and every odd element of $\mathfrak{G}$ differs from the corresponding element of $\mathfrak{G}'$ by the Galois involution τ. This involution acts as a multiplication by -1 on the orthogonal complement, with respect to the intersection form (1.2), to K_V in $\mathrm{Cl}(V) \otimes \mathbb{C}$. Thus, Corollary 8.11 implies that $\mathrm{Cl}(V) \otimes \mathbb{C}$ contains a unique trivial subrepresentation of the group $\mathfrak{G}$.

Corollary 8.12. *One has* $\mathrm{rk}\,\mathrm{Cl}(V)^{\mathfrak{G}} = 1$.

9. Birational rigidity

In this section we study G-birationally rigidity of nodal double Veronese cones.

Let V be a nodal del Pezzo 3-fold of degree 1 (not necessarily with 28 nodes). Then V can be given in $\mathbb{P}(1,1,1,2,3)$ with weighted homogeneous coordinates s, t, u, v, and w (of weights 1, 1, 1, 2, and 3, respectively) by equation

$$w^2 = v^3 + vh_4(s,t,u) + h_6(s,t,u)$$

for some homogeneous polynomials $h_4(s,t,u)$ and $h_6(s,t,u)$ of degree 4 and 6, respectively. Then

$$-K_V \sim 2H,$$

where H is an ample Cartier divisor on V such that $H^3 = 1$. The linear system $|H|$ has one base point, which we denote by O. Recall that the 3-fold V is smooth at the point O.

The linear system $|2H|$ is free from base points and gives a double cover

$$V \to \mathbb{P}(1,1,1,2).$$

As before, denote by τ the Galois involution of this double cover.

Let $\kappa \colon V \dashrightarrow \mathbb{P}^2$ be the projection given by

$$[s:t:u:v:w] \mapsto [s:t:u].$$

Then κ is a rational map given by the linear system $|H|$. In particular, κ is $\mathrm{Aut}(V)$-equivariant. Furthermore, there exists an $\mathrm{Aut}(V)$-equivariant commutative diagram

$$
\begin{array}{ccc}
 & \widetilde{V} & \\
{}^{\psi}\swarrow & & \searrow{}^{\widetilde{\kappa}} \\
V & \dashrightarrow{}_{\kappa} & \mathbb{P}^2
\end{array}
$$

where ψ is the blow up at the point O, and $\widetilde{\kappa}$ is a morphism whose general fiber is an elliptic curve. Note that every fiber of the morphism $\widetilde{\kappa}$ is irreducible. Moreover, we have an exact sequence of groups

$$1 \longrightarrow \Gamma \longrightarrow \mathrm{Aut}(V) \longrightarrow \mathrm{Aut}(\mathbb{P}^2),$$

where Γ is a finite subgroup in $\mathrm{Aut}(V)$ that contains τ. If $h_4(x,y,z)$ is not a zero polynomial, then Γ is generated by τ. If $h_4(x,y,z)$ is a zero polynomial, then Γ is a cyclic group of order 6 that is generated by τ and a map given by
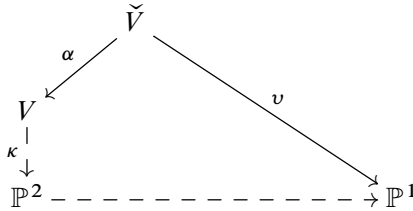
$$[s:t:u:v:w] \mapsto [s:t:u:\epsilon_3 v:w],$$

where ϵ_3 is a primitive cube root of unity (cf. [22, §6.7]).

Let G be a finite subgroup of $\mathrm{Aut}(V)$. Denote its image in $\mathrm{Aut}(\mathbb{P}^2)$ by $\bar{G}$. Suppose that

$$\mathrm{rk}\,\mathrm{Cl}(V)^G = 1,$$

so that V is a G-Mori fiber space (see [14, Definition 1.1.5]). If $\mathbb{P}^2$ contains a $\bar{G}$-fixed point, then there exists a G-equivariant commutative diagram as follows:

Here $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is the projection from the $\bar{G}$-fixed point, α is the maximal extraction whose center is the fiber of κ over the $\bar{G}$-fixed point, and υ is a fibration into del Pezzo surfaces of degree 1. Thus, in this case, the 3-fold V cannot be G-birationally rigid.

If V is smooth and $\mathbb{P}^2$ contains no $\bar{G}$-fixed points, then V is G-birationally super-rigid. This follows from [35] and [36] (cf. [11, Remark 1.19]). We conjecture that the following two conditions are equivalent for a nodal double Veronese cone V with an arbitrary number of singular points, admitting an action of a finite group G such that $\mathrm{rk}\,\mathrm{Cl}\left(V\right)^G = 1$:

- $\mathbb{P}^2$ does not contain $\bar{G}$-fixed points,
- the 3-fold V is G-birationally rigid.

At present, the proof of this conjecture is out of reach; the reason for this is that we cannot effectively control linear systems with non-canonical singularities at the curves passing through singular points unless certain restrictions on such curves can be obtained from the action of a group. Instead, we give a very simple proof of a weaker result, which implies Theorem 1.7. To state the result, note that a surface Π in the 3-fold V is said to be a *plane* if $\Pi \cong \mathbb{P}^2$ and $H^2 \cdot \Pi = 1$. For instance, by [50, Theorem 7.2] every 28-nodal double Veronese cone contains exactly 126 planes.

Theorem 9.1. *Let V be a nodal double Veronese cone with an action of a finite group G such that $\mathrm{rk}\,\mathrm{Cl}(V)^G = 1$. Suppose that the following three conditions are satisfied:*

(1) *$\mathbb{P}^2$ does not contain $\bar{G}$-fixed points;*

(2) *the singular locus $\mathrm{Sing}(V)$ does not contain G-orbits of length 3;*

(3) *for every G-irreducible curve D in V such that $H \cdot D$ is either 2 or 3, and $\kappa(D)$ is a curve in $\mathbb{P}^2$ of degree $H \cdot D$, there is a plane in V that contains D.*

Then V is G-birationally super-rigid.

Corollary 9.2. *Let V and G be as in Theorem 9.1. Suppose that $\mathbb{P}^2$ does not contain $\bar{G}$-fixed points. If $\mathbb{P}^2$ does not contain $\bar{G}$-invariant curves of degree 2 or 3, then V is G-birationally super-rigid.*

Remark 9.3. We cannot drop condition (2) in Theorem 9.1. Suppose that $\mathbb{P}^2$ does not contain $\bar{G}$-fixed points, but $\mathrm{Sing}(V)$ contains a G-orbit of length 3. Let $\beta\colon \check{V} \to V$ be the blow up at this G-orbit. Then it follows from [40] and [51, Theorem 1.5] that there exists a crepant G-birational morphism $\nu\colon \check{V} \to X$ such that X is a double cover of $\mathbb{P}^3$ branched in an irreducible sextic surface. Let ι be the involution in $\mathrm{Aut}(X)$ of this double cover, and put

$$\rho = \beta \circ \nu^{-1} \circ \iota \circ \nu \circ \beta^{-1}.$$

Then $\rho \in \mathrm{Bir}^G(V)$. Moreover, if ν is small, then ρ is not biregular.

Before proving Theorem 9.1, let us use it to prove Theorem 1.7.

Proof of Theorem 1.7. Suppose that the 28-nodal double Veronese cone V is G-birationally rigid. Then the group $\bar{G}$ does not have fixed points on $\mathbb{P}^2$, and thus also has no fixed points on its projectively dual plane. Let C be the smooth plane quartic curve that is constructed as the projectively dual curve of the discriminant curve $\check{C}$ of the half-anticanonical rational elliptic fibration $\kappa\colon V \dashrightarrow \mathbb{P}^2$ (see Theorem 1.3 or Lemma 6.2). Then the group $\bar{G}$ acts faithfully on C. We see from Lemma 2.7 that C can be given by equation (1.6), the group $\bar{G}$ must contain a subgroup isomorphic to $\mathfrak{S}_4$, and $\mathbb{P}^2$ (as well as its projective dual) can be identified with a projectivization of an irreducible three-dimensional representation of $\mathfrak{S}_4$. Using once again Theorem 1.3 and formula (5.10), we conclude that V is a double Veronese cone from Example 1.5. Furthermore, by Theorem 1.4 the group G contains a subgroup isomorphic to $\mathfrak{S}_4$. Now it follows from Corollary 8.11 that G contains a subgroup conjugate to $\mathscr{G}$.

Now we suppose that V is a double Veronese cone from Example 1.5, and $G = \mathscr{G}$. Then $\mathbb{P}^2$ does not contain $\bar{G}$-fixed points, and $\mathrm{rk}\,\mathrm{Cl}(V)^G = 1$ by Corollary 8.12. To complete the proof, we have to show that V is G-birationally super-rigid. Suppose that this is not a case. Let us seek for a contradiction.

Note that $\bar{G} \cong \mathfrak{S}_4$, and $\mathbb{P}^2$ contains a unique $\bar{G}$-orbit of length 3. This orbit consists of the points $[0:0:1]$, $[0:1:0]$ and $[1:0:0]$. Taking partial derivatives of (5.10) and using the expressions for g_4 and g_6 provided in Example 1.5, we see that $\mathrm{Sing}(V)$ does not contain points that are mapped by κ to $[0:0:1]$, $[0:1:0]$, or $[1:0:0]$. In particular, we see that the singular locus $\mathrm{Sing}(V)$ does not contain $\bar{G}$-orbits of length 3.

Using Theorem 9.1, we conclude that V contains a G-irreducible curve D on V such that $H \cdot D$ is either 2 or 3, the curve D is not contained in any plane in V, and $\kappa(D)$ is a curve in $\mathbb{P}^2$ of degree $H \cdot D$.

Observe that there exist a unique $\bar{G}$-invariant conic and a unique $\bar{G}$-invariant cubic curve in $\mathbb{P}^2$. The former curve is given by $s^2 + t^2 + u^2 = 0$, and the latter curve is given by $stu = 0$.

Suppose that $H \cdot D = 3$. Then $\kappa(D)$ is given by $stu = 0$, so that D consists of three irreducible components that are mapped isomorphically by κ to the lines $s = 0$, $t = 0$ and $u = 0$ in $\mathbb{P}^2$.

Let D_s be the irreducible component of the curve D that is mapped by κ to the line $s = 0$. Then the stabilizer G_s of D_s in $G \cong \mathfrak{S}_4$ is isomorphic to the dihedral group of order 8. However, its action on D_s is not faithful, but the action of its quotient isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ is faithful. Therefore, a general G_s-orbit in D_s has length 4. Consider the pencil $\mathcal{R}$ of G-invariant surfaces in V generated by the surfaces $v = 0$ and $s^2 + t^2 + u^2 = 0$. Since a surface from $\mathcal{R}$ has intersection number 2 with D_s, we conclude that there is a surface R in $\mathcal{R}$ that contains D_s. Since R is G-invariant, it contains the whole curve D.

The surface R is given by equation

$$v = \mu(s^2 + t^2 + u^2)$$

for some $\mu \in \mathbb{C}$. Thus, the curve D is contained in the subset in V that is cut out by

$$\begin{cases} v = \mu(s^2 + t^2 + u^2), \\ stu = 0. \end{cases}$$

If $\mu = 2\lambda/3$, then the equation $v = \mu(s^2 + t^2 + u^2)$ cuts out two planes in V, which are given by

$$\begin{cases} v = \mu(s^2 + t^2 + u^2), \\ w = \gamma stu, \end{cases}$$

where $\gamma = \pm 2(\lambda - 2)\sqrt{\lambda + 1}$ (see (8.3)). Thus, in this case, the curve D is contained in a plane, which is impossible by assumption. Hence, we conclude that $\mu \neq 2\lambda/3$.

We see that the irreducible component D_s is contained in the subset in V that is cut out by

$$\begin{cases} v = \mu(t^2 + u^2), \\ s = 0. \end{cases}$$

If this subset were irreducible and reduced, then it would coincide with D_s, which would contradict $H \cdot D_s = 1$. Hence, the above subset is either reducible or non-reduced. Algebraically, this simply means that the polynomial

$$P(t, u) = \mu^3(t^2 + u^2)^3 - \mu g_4(0, t, u)(t^2 + u^2) + g_6(0, t, u)$$

must be a complete square (this includes the possibility for the above polynomial to be zero, which corresponds to the non-reduced case). The polynomial $P(t, u)$ simplifies as

$$\left(\mu^3 - \mu\frac{\lambda^2 + 12}{3} - \frac{2\lambda(\lambda - 6)(\lambda + 6)}{27}\right)(t^2 + u^2)^3$$
$$- 4(\lambda - 2)\left(\mu - \frac{2}{3}\lambda\right)(t^2 + u^2)t^2u^2.$$

This shows that the polynomial $P(t, u)$ cannot be a zero polynomial because $\lambda \neq 2$ and $\mu \neq 2\lambda/3$. Suppose that $P(t, u)$ is a complete square. Then it must be of the form

$$P(t, u) = (a_3t^3 + a_2t^2u + a_1tu^2 + a_0u^3)^2,$$

where a_i's are constants depending on λ and μ. We can see directly from the simplified $P(t, u)$ above that $P(t, u)$ is invariant under switching variables t and u and that it contains t and u with only even exponents. We therefore obtain

$$a_0^2 = a_3^2, \quad a_1^2 + 2a_0a_2 = a_2^2 + 2a_1a_3, \quad a_0a_1 = a_2a_3 = 0, \quad a_0a_3 + a_1a_2 = 0.$$

However, these yield $a_0 = a_1 = a_2 = a_3 = 0$, which is impossible. Consequently, one has $H \cdot D \neq 3$.

We see that $H \cdot D = 2$. Then $\kappa(D)$ is given by equation $s^2 + t^2 + u^2 = 0$ in $\mathbb{P}^2$. In particular, one has $D \cong \mathbb{P}^1$. The action of $G \cong \mathfrak{S}_4$ on D is faithful, so that any G-orbit in D has length at least 6. Consider the pencil $\mathcal{R}$ as above. Since a surface from $\mathcal{R}$ has intersection number 4 with D_s, we conclude that every surface in $\mathcal{R}$ contains D. In particular, D is contained in a surface given by equation

$$v = \frac{2\lambda}{3}(s^2 + t^2 + u^2).$$

As above, we conclude that D is contained in a plane, which gives a contradiction. This completes the proof of Theorem 1.7. ∎

Now we are equipped to prove Theorem 9.1. Recall that V is a nodal double Veronese cone with an action of a finite group G such that $\operatorname{rk}\operatorname{Cl}(V)^G = 1$, and assumptions (1)–(3) of Theorem 9.1 hold. Namely, we suppose that $\mathbb{P}^2$ does not contain $\bar{G}$-fixed points; that the singular locus $\operatorname{Sing}(V)$ does not contain G-orbits of length 3; that for any G-irreducible curve D in V such that $H \cdot D$ is either 2 or 3, and $\kappa(D)$ is a curve in $\mathbb{P}^2$ of degree $H \cdot D$, there is a plane in V containing the curve D.

Remark 9.4. Since there are no $\bar{G}$-fixed points on $\mathbb{P}^2$, there are also no $\bar{G}$-invariant lines. This implies that there are no $\bar{G}$-orbits contained in a line. In other words, every $\bar{G}$-orbit contains three non-collinear points, and in particular no $\bar{G}$-orbits of length 2. Furthermore, by Lemma 2.6 every $\bar{G}$-orbit contains four points such that no three of them are collinear.

To prove Theorem 9.1, we have to show that the 3-fold V is G-birationally super-rigid. Suppose that it is not. Then there exists a G-invariant mobile linear system $\mathcal{M}$ on the 3-fold V such that

$$\mathcal{M} \sim nH,$$

and the log pair $(V, \frac{2}{n}\mathcal{M})$ is not canonical (see for instance [14, Theorem 3.3.1]).

Let Z be a G-center of non-canonical singularities of the log pair $(V, \frac{2}{n}\mathcal{M})$. Recall that O denotes the (unique) base point of the linear system $|H|$ on V, and O is a smooth point of V.

Lemma 9.5. *The G-center Z cannot be the point O.*

Proof. Let M be a general surface in $\mathcal{M}$, and let E be a general fiber of the map κ. Then $E \not\subset M$, so that

$$n = M \cdot E \geqslant \operatorname{mult}_O(E) \cdot \operatorname{mult}_O(M) \geqslant \operatorname{mult}_O(M).$$

On the other hand, the Zariski tangent space $T_{O,V}$ is an irreducible representation of the group G, because $\mathbb{P}^2$ does not contain $\bar{G}$-fixed points. Thus, using Lemma 2.4, we conclude that O is not a center of non-canonical singularities of the log pair $(V, \frac{2}{n}\mathcal{M})$. ∎

Lemma 9.6. *If Z is zero-dimensional, then $Z \subset \operatorname{Sing}(V)$.*

Proof. Suppose that the G-orbit Z consists of smooth points of the 3-fold V. Let M_1 and M_2 be general surfaces in the linear system $\mathcal{M}$. Then

$$(9.7) \qquad \operatorname{mult}_P(M_1 \cdot M_2) > n^2$$

for every $P \in Z$ by [15, Corollary 3.4]. Let us show that this leads to a contradiction.

By Lemma 9.5, we have $Z \neq O$, so that $\kappa(Z)$ is a well-defined $\bar{G}$-orbit. Moreover, $\kappa(Z)$ consists of at least 3 points, and contains three non-collinear points by Remark 9.4. Denote them by P_1, P_2 and P_3.

Let E_1, E_2 and E_3 be fibers of the map κ over the points P_1, P_2 and P_3, respectively. Then

$$M_1 \cdot M_2 = m(E_1 + E_2 + E_3) + \Delta,$$

where m is a non-negative integer, and Δ is an effective one-cycle whose support does not contain the curves E_1, E_2 and E_3. Then $m \leqslant n^2/3$, since

$$n^2 = H \cdot M_1 \cdot M_2 = H \cdot \left(m\left(E_1 + E_2 + E_3\right) + \Delta\right) = 3m + H \cdot \Delta \geqslant 3m.$$

Let O_1, O_2 and O_3 be points in Z that are mapped to P_1, P_2 and P_3, respectively. Let $\delta = \operatorname{mult}_{O_i}(E_i)$. Then either $\delta = 1$ or $\delta = 2$. Moreover, it follows from (9.7) that

$$\operatorname{mult}_{O_i}(\Delta) > n^2 - m\delta$$

for each point O_i.

Let $\mathcal{B}$ be a linear subsystem in $|2H|$ consisting of all surfaces that contain the curves E_1, E_2 and E_3. Then $\mathcal{B}$ does not have other base curves, since P_1, P_2 and P_3 are not collinear. Thus, for a general surface B in $\mathcal{B}$, we have

$$2n^2 - 6m = B \cdot \Delta \geqslant \sum_{i=1}^{3} \operatorname{mult}_{O_i}(\Delta) > 3(n^2 - m\delta).$$

This implies

$$-4m \geqslant m\delta - 6m > n^2,$$

which is a contradiction.   ∎

Lemma 9.8. *The center Z is one-dimensional.*

Proof. Suppose that $\dim Z = 0$, so that Z is a G-orbit. Then $Z \subset \operatorname{Sing}(V)$ by Lemma 9.6. Let r be the length of the $\bar{G}$-orbit $\kappa(Z)$. Note that $r \geqslant 3$ by Remark 9.4. Denote the points of $\kappa(Z)$ by $P_1, \ldots, P_r$. Let E_i be the fiber of the map κ over the point P_i. Then E_i is an irreducible curve because $E_i \cdot H = 1$. Since the arithmetic genus of E_i equals 1, we see that E_i has at most one singular point, and thus contains at most one singular point of the 3-fold V. Hence, since $E_i \cap Z \neq \varnothing$, each curve E_i contains exactly one singular point of V, which must be a singular point of the curve E_i. In particular, we see that the length of the G-orbit Z also equals r.

Let O_i be the singular point of the curve E_i. Then Z consists of the points $O_1, \ldots, O_r$. By assumption, we have $r \geqslant 4$.

The $\bar{G}$-orbit $\kappa(Z)$ contains four points in $\mathbb{P}^2$ such that no three of them are collinear by Remark 9.4. Without loss of generality, we may assume that these points are P_1, P_2, P_3, and P_4. Let $\mathcal{B}$ be a linear subsystem in $|2H|$ consisting of all surfaces that contain the curves E_1, E_2, E_3 and E_4. Then $\mathcal{B}$ does not contain other base curves.

Let $\beta \colon \bar{V} \to V$ be the blow up of the points O_1, O_2, O_3 and O_4, and let F_1, F_2, F_3, F_4 be the β-exceptional surfaces that are mapped to the points O_1, O_2, O_3, O_4, respectively. Denote by $\bar{\mathcal{B}}$ the proper transforms on $\bar{V}$ of the linear system $\mathcal{B}$. Then

$$-K_{\bar{V}} \sim \bar{\mathcal{B}} \sim \beta^*(2H) - (F_1 + F_2 + F_3 + F_4),$$

which implies that the divisor $-K_{\overline{V}}$ is nef, because it intersects the proper transform of each curve E_i trivially.

Denote by $\overline{\mathcal{M}}$ the proper transform of the linear system $\mathcal{M}$ on the 3-fold $\overline{V}$. Then

$$\overline{\mathcal{M}} \sim_{\mathbb{Q}} \beta^*(nH) - m(F_1 + F_2 + F_3 + F_4)$$

for some non-negative rational number m. Let $\overline{M}_1$ and $\overline{M}_2$ be general surfaces in $\overline{\mathcal{M}}$. Then

$$0 \leqslant \overline{B} \cdot \overline{M}_1 \cdot \overline{M}_2 = 2n^2 - 8m^2,$$

which gives $m \leqslant n/2$. This is impossible by [15, Theorem 3.10]. ∎

Thus, Lemma 9.8 concludes that Z is a G-irreducible curve. Denote by $Z_1, \ldots, Z_r$ the irreducible components of Z. Thus, if $r = 1$, then $Z = Z_1$ is an irreducible curve. In this case, $\kappa(Z)$ is neither a point nor a line, because $\mathbb{P}^2$ does not contain $\overline{G}$-fixed points. In particular, we see that $H \cdot Z \neq 1$.

Since Z is a G-center of non-canonical singularities of the log pair $(V, \frac{2}{n}\mathcal{M})$, we have

$$\mathrm{mult}_{Z_i}(\mathcal{M}) > \frac{n}{2}.$$

Let M_1 and M_2 be general surfaces in $\mathcal{M}$. Then

$$n^2 = H \cdot M_1 \cdot M_2 \geqslant \sum_{i=1}^{r} \mathrm{mult}_{Z_i}(M_1 \cdot M_2) > \frac{n^2}{4} r H \cdot Z_i,$$

so that we have the following four possibilities:

(A) $r = 1$, $H \cdot Z = 2$, and $\kappa(Z)$ is a smooth conic in $\mathbb{P}^2$;

(B) $r = 1$, $H \cdot Z = 3$, and $\kappa(Z)$ is a smooth cubic in $\mathbb{P}^2$;

(C) $r = 3$, $H \cdot Z = 3$, and $\kappa(Z)$ is a union of three lines;

(D) $r = 3$, $H \cdot Z = 3$, and $\kappa(Z)$ is a $\overline{G}$-orbit of length 3.

We claim that the case (D) is impossible. Indeed, if $\kappa(Z)$ is a $\overline{G}$-orbit of length 3, then the linear system $|H|$ contains a surface H_{12} passing through Z_1 and Z_2. Let M be a general element of the linear system $\mathcal{M}$. Then

$$n = H \cdot M \cdot H_{12} \geqslant \mathrm{mult}_{Z_1}(M) + \mathrm{mult}_{Z_2}(M) > n,$$

which is absurd.

Thus, we see that $H \cdot Z$ is either 2 or 3 and that $\kappa(Z)$ is a curve of degree $H \cdot Z$. Therefore, by assumption, the 3-fold V has a plane Π that contains the curve Z. Let L be a general line in this plane, so that $H \cdot L = 1$, the intersection $L \cap Z$ consists of

exactly $H \cdot Z$ points, and L is not contained in the base locus of the linear system $\mathcal{M}$. Then, for a general surface $M \in \mathcal{M}$, we have

$$n = L \cdot M \geqslant \mathrm{mult}_Z(M) \cdot |L \cap Z| \geqslant 2\,\mathrm{mult}_Z(M) \geqslant 2\,\mathrm{mult}_Z(\mathcal{M}) > n,$$

which is a contradiction. This completes the proof of Theorem 9.1.

10. Questions and problems

In this section we discuss several open questions concerning 28-nodal double Veronese cones.

Del Pezzo surfaces of degree 2. Let V be a 28-nodal double Veronese cone, let C be the plane quartic curve corresponding to V by Theorem 1.3, and let S be the del Pezzo surface of degree 2 constructed as the double cover of $\mathbb{P}^2$ branched along C. We know from Corollary 6.9 that

$$\mathrm{Aut}(V) \cong \boldsymbol{\mu}_2 \times \mathrm{Aut}(C) \cong \mathrm{Aut}(S),$$

although there is no obvious choice for a natural isomorphism between $\mathrm{Aut}(V)$ and $\mathrm{Aut}(S)$. Also, one has

$$\mathrm{Cl}(V) \cong \mathbb{Z}^8 \cong \mathrm{Pic}(S).$$

Recall from [50, Corollary 7.1.4] that the group $\mathrm{Aut}(V)$ acts faithfully on $\mathrm{Cl}(V)$, while by [20, Corollary 8.2.40] the group $\mathrm{Aut}(S)$ acts faithfully on $\mathrm{Pic}(S)$. It would be interesting to know if there exists a natural identification of the corresponding representations. More precisely, we ask the following question.

Question 10.1. Does there exist an isomorphism $\mathrm{Aut}(V) \cong \mathrm{Aut}(S)$ under which $\mathrm{Cl}(V) \otimes \mathbb{C}$ is isomorphic to $\mathrm{Pic}(S) \otimes \mathbb{C}$ as representations of this group?

Recall that both $\mathrm{Cl}(V)$ and $\mathrm{Pic}(S)$ have a naturally defined intersection form. The canonical class K_V is invariant with respect to the action of the group $\mathrm{Aut}(V)$ on $\mathrm{Cl}(V)$, and the canonical class K_S is invariant with respect to the action of $\mathrm{Aut}(S)$ on $\mathrm{Pic}(S)$. Moreover, in some cases these are the only trivial subrepresentations in $\mathrm{Cl}(V) \otimes \mathbb{C}$ and $\mathrm{Pic}(S) \otimes \mathbb{C}$; for instance, this is the case when $\mathrm{Aut}(C) \cong \mathrm{PSL}_2(\mathbb{F}_7)$. Therefore, if there exists a natural identification of automorphism groups as required in Question 10.1, then for any 28-nodal Veronese double cone V and the corresponding del Pezzo surface S there must exist an identification of the seven-dimensional representations of the group $\mathrm{Aut}(V) \cong \mathrm{Aut}(S)$ in the orthogonal complement $\mathrm{Cl}(V)^{\perp K_V} \otimes \mathbb{C}$ to K_V in $\mathrm{Cl}(V) \otimes \mathbb{C}$ and in the orthogonal complement $\mathrm{Pic}(S)^{\perp K_S} \otimes \mathbb{C}$ to K_S in $\mathrm{Pic}(S) \otimes \mathbb{C}$.

Recall that both lattices $\mathrm{Cl}(V)^{\perp K_V}$ and $\mathrm{Pic}(S)^{\perp K_S}$ are isomorphic to the lattice $\mathbf{E}_7$ (see [50, Theorem 1.7] and [50, Remark 1.8] for the former, and [22, §8.2.6] for the latter). We point out that despite this fact in general one cannot hope for an equivariant identification of the lattices $\mathrm{Cl}(V)^{\perp K_V}$ and $\mathrm{Pic}(S)^{\perp K_S}$ that takes into account the intersection forms, even if the answer to Question 10.1 is positive.

Indeed, let C be a smooth plane quartic curve with an action of the symmetric group $\mathfrak{S}_4$ as in Example 1.5. Let us use the notation of §8. It follows from Corollary 8.11 and Lemma 8.8 that each of the $\mathfrak{G}'$ representations $\mathrm{Cl}(V) \otimes \mathbb{C}$ and $\mathrm{Pic}(S) \otimes \mathbb{C}$ contains a unique two-dimensional subrepresentation. This implies that the lattice $\mathrm{Cl}(V)$ contains a unique $\mathfrak{G}'$-invariant primitive sublattice Λ_V of rank 2, and the lattice $\mathrm{Pic}(S)$ contains a unique $\mathfrak{G}'$-invariant primitive sublattice Λ_S of rank 2. However, the sublattice Λ_V is generated by the half-anticanonical divisor H and the plane Π_-, so that the intersection form on Λ_V is given by the matrix

$$\begin{pmatrix} H^2 & H \cdot \Pi_- \\ H \cdot \Pi_- & \Pi_-^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

On the other hand, it follows from the proof of Lemma 8.8 that the sublattice Λ_S is generated by $-K_S$ and a fiber F of a conic bundle, so that the intersection form on Λ_S is given by the matrix

$$\begin{pmatrix} K_S^2 & -K_S \cdot F \\ -K_S \cdot F & F^2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}.$$

Thus, we see that the lattices Λ_V and Λ_S are not isomorphic.

One situation when we can easily establish the isomorphism required in Question 10.1 is as follows. Let G be a subgroup in $\mathrm{Aut}(\mathbb{P}^3)$ such that there exists a G-invariant Aronhold heptad $P_1, \ldots, P_7$. Then the diagram (7.24) is G-equivariant, which implies that $\mathrm{Cl}(V) \otimes \mathbb{C}$ is isomorphic to $\mathrm{Pic}(S) \otimes \mathbb{C}$ as G-representations.

Elliptic fibration. The following construction was pointed out to us by Alexander Kuznetsov. Let C be a smooth plane quartic, and let S be the corresponding del Pezzo surface of degree 2 with the anticanonical morphism $\varphi \colon S \to \mathbb{P}^2$. Let $\check{\mathbb{P}}^2$ be the projectively dual plane of $\mathbb{P}^2$. Let p_1 and p_2 be the projections of the flag variety

$$\mathcal{F} = \mathrm{Fl}(1, 2, 3) \subset \mathbb{P}^2 \times \check{\mathbb{P}}^2$$

to the first and the second factor of $\mathbb{P}^2 \times \check{\mathbb{P}}^2$, respectively. Consider the fiber product:
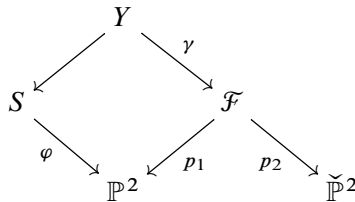
Note that there is a natural action of the group $\mathrm{Aut}(S) \cong \boldsymbol{\mu}_2 \times \mathrm{Aut}(C)$ on Y. The morphism

$$\kappa_Y = p_2 \circ \gamma \colon Y \to \check{\mathbb{P}}^2$$

is an elliptic fibration whose discriminant curve is projectively dual to C.

Now let V be the 28-nodal double Veronese cone constructed from the plane quartic C by Theorem 1.3. Then the half-anticanonical map $\kappa \colon V \dashrightarrow \check{\mathbb{P}}^2$ is a rational elliptic fibration. Let $\widetilde{V}$ be the blow up of V at the unique base point of κ, and let $\widetilde{\kappa} \colon \widetilde{V} \to \check{\mathbb{P}}^2$ be the corresponding regular elliptic fibration. Then the discriminant curve of $\widetilde{\kappa}$ is projectively dual to C. Moreover, it follows from Lemma 3.9 that over every point in $\check{\mathbb{P}}^2$ the fibers of the elliptic fibrations $\widetilde{\kappa}$ and κ_Y are isomorphic to each other (cf. Lemma 6.5).

Furthermore, it follows from Lemma 3.9 that (at least over the complement to the discriminant curve) the elliptic fibration κ_Y is the relative Pic^2 of the elliptic fibration $\widetilde{\kappa}$. Indeed, let E be a smooth fiber of $\widetilde{\kappa}$, and let P_1 and P_2 be two points on E. Recall that E is a proper transform of the base locus $E_{\mathcal{Q}}$ of some pencil $\mathcal{Q}$ of quadrics in $\mathbb{P}^3$. The two points $P_1, P_2 \in E_{\mathcal{Q}}$ define a line L passing through them. Thus they define a unique quadric Q in $\mathcal{Q}$ containing the line L, and also the family of lines on Q which contains L. Note that the fiber of the elliptic fibration κ_Y over the point $\widetilde{\kappa}(E)$ can be identified with the double cover of $\mathcal{Q} \cong \mathbb{P}^1$ branched in the locus of singular quadrics of $\mathcal{Q}$ or, in other words, with the Hilbert scheme of lines contained in the quadrics of $\mathcal{Q}$. Therefore, the pair of points (P_1, P_2) defines a point on the fiber of κ_Y over $\widetilde{\kappa}(E)$. This point depends only on the class of the pair (P_1, P_2) in $\mathrm{Sym}^2(E)$, since every other pair from the same class determines the same quadric Q and a line from the same family of lines on Q where L lies. We conclude that the fiber of κ_Y over $\widetilde{\kappa}(E)$ is identified with $\mathrm{Pic}^2(E)$. It remains to notice that this construction can be performed in the family, and thus the assertion about the fibrations follows.

One of the consequences of the above construction is as follows. Since the elliptic fibration $\widetilde{\kappa}$ is obtained by blowing up V at a single point, it comes with a distinguished section (which corresponds to the eighth base point of the net of quadrics in the construction of V from an Aronhold heptad). Therefore, there exist a natural birational map between V and Y.

Question 10.2. Is there a more explicit geometric relation between the 3-fold Y and the 28-nodal double Veronese cone V?

Intersections of quadrics and cubics. The following construction was also explained to us by Alexander Kuznetsov. Let $P_1, \ldots, P_8$ be a regular Cayley octad, and let $\mathcal{L}$ be the corresponding net of quadrics in $\mathbb{P}^3$. Then the lines contained in the quadrics of $\mathcal{L}$ are parametrized by a three-dimensional variety Z (cf. [53, §1.2]). Then Z can be

regarded as a subvariety of the Grassmannian $\mathrm{Gr}(2,4)$. Let $\mathcal{U}^\vee$ be the universal quotient bundle on $\mathrm{Gr}(2,4)$ (or, which is the same, the dual of the tautological subbundle). Furthermore, Z can be described as the degeneracy locus of the morphism of vector bundles

$$\mathcal{O}_{\mathrm{Gr}(2,4)} \to \mathrm{S}^2 \mathcal{U}^\vee$$

corresponding to $\mathcal{L}$. Therefore, the 3-fold Z is an intersection of a quadric and a cubic in $\mathbb{P}^5$. Note that Z has 28 singular points corresponding to the lines passing through pairs of points P_i and P_j. In particular, its intermediate Jacobian is trivial.

Question 10.3. Is there a natural geometric relation between the 3-fold Z and the 28-nodal double Veronese cone V constructed from the net of quadrics $\mathcal{L}$? Is the 3-fold Z rational?

Question 10.4. Does the 3-fold Z depend on the choice of the net of quadrics $\mathcal{L}$, or only on the corresponding plane quartic curve?

Other 3-folds constructed from plane quartics. There are other interesting classes of 3-folds related to plane quartic curves. For instance, a construction due to S. Mukai assigns to a general plane quartic a smooth Fano 3-fold of genus 12 (and anticanonical degree 22), see [47, 48, 52] for details.

Question 10.5. Is there a natural geometric relation between the 28-nodal double Veronese cone V and the Fano 3-fold V_{22} of genus 12 constructed from the same smooth plane quartic C?

We point out that both of the above varieties are rational, so there always exists a birational map between them. However, one cannot choose such a map to be $\mathrm{Aut}(C)$-equivariant. Indeed, if $\mathrm{Aut}(C) \cong \mathrm{PSL}_2(\mathbb{F}_7)$, then both V and V_{22} are $\mathrm{PSL}_2(\mathbb{F}_7)$-birationally super-rigid by Theorem 1.7 and [12, Theorem 1.10].

Degenerations. The following question was asked by Vyacheslav Shokurov and Yuri Prokhorov.

Question 10.6. Can one extend the one-to-one correspondence between smooth plane quartics and 28-nodal double Veronese cones given by Theorem 1.3 to the (mildly) singular case (like, double Veronese cones with $c\mathbf{A}_1$-singularities, or double Veronese cones with 26 nodes and one nice $c\mathbf{A}_2$-singularity, or plane quartics with a single node)? Furthermore, can one include in this one-to-one correspondence the plane quartic that is a union of a cuspidal cubic and its tangent line in the cusp (in other words, the quartic that gives rise to a del Pezzo surface of degree 2 with an $\mathbf{E}_7$-singularity)?

A. Computer-aided calculations

This appendix explains, from a computational point of view, how (5.4) is obtained. In particular, a way to extract explicit formulae of the covariants $g_4(s, t, u)$ and $g_6(s, t, u)$ from a given plane quartic (5.1) is described, which enables us to produce concrete examples of 28-nodal double Veronese cones from smooth quartic curves as in Example 1.5. We point out that there is a method to write down the polynomial $g_4(s, t, u)$ using a smaller amount of explicit computations. It is based on the expression for the cubic invariant in the coefficients of the plane quartic given in [54, §293] (see the end of [49, §1]).

Since the denominator and the numerator of the j-function in (5.2) are symmetric polynomials in x_1, x_2, x_3, x_4, the j-function in (5.2) may be regarded as a rational function in b_0, b_1, b_2, b_3, b_4 of (5.3). Using the simple MAGMA ([8]) function:

Computing the numerator (N) and the denominator (D) in terms of symmetric functions

```
Q<x1,x2,x3,x4> := PolynomialRing(RationalField(), 4);
Num:=(x1-x2)^2*(x4-x3)^2-(x1-x3)*(x4-x2)*(x4-x1)*(x3-x2);
Den:=(x1-x2)^2*(x1-x3)^2*(x1-x4)^2*(x2-x3)^2*(x2-x4)^2*(x3-x4)^2;
Q<b1, b2, b3, b4> := PolynomialRing(RationalField(), 4);
I, N := IsSymmetric(Num, Q);
I, D := IsSymmetric(Den, Q);
D;
N;
```

---

we are able to obtain the denominator and the numerator of the j-function (respectively, D and N), which after homogenization with respect to b_0 result in

$$(x_1 - x_2)^2(x_4 - x_3)^2 - (x_1 - x_3)(x_4 - x_2)(x_4 - x_1)(x_3 - x_2)$$
$$= \frac{1}{b_0^2}(-3b_1b_3 + 12b_0b_4 + b_2^2);$$
$$(x_1 - x_2)^2(x_1 - x_3)^2(x_1 - x_4)^2(x_2 - x_3)^2(x_2 - x_4)^2(x_3 - x_4)^2$$
$$= \frac{1}{b_0^6}\big(-27b_1^4b_4^2 + 18b_1^3b_2b_3b_4 - 4b_1^3b_3^3 - 4b_1^2b_2^3b_4 + b_1^2b_2^2b_3^2$$
$$+ 144b_0b_1^2b_2b_4^2 - 6b_0b_1^2b_3^2b_4 - 80b_0b_1b_2^2b_3b_4 + 18b_0b_1b_2b_3^3$$
$$- 192b_0^2b_1b_3b_4^2 + 16b_0b_2^4b_4 - 4b_0b_2^3b_3^2 - 128b_0^2b_2^2b_4^2 + 144b_0^2b_2b_3^2b_4$$
$$- 27b_0^2b_3^4 + 256b_0^3b_4^3\big).$$

Therefore, regarded as a rational function in b_0, b_1, b_2, b_3, b_4, the j-function in (5.2) can be expressed as follows:

$$j(b_0, b_1, b_2, b_3, b_4)$$

$$= \frac{2^8(-3b_1b_3 + 12b_0b_4 + b_2^2)^3}{\left\{\begin{array}{l} -27b_1^4b_4^2 + 18b_1^3b_2b_3b_4 - 4b_1^3b_3^3 - 4b_1^2b_2^3b_4 + b_1^2b_2^2b_3^2 + 144b_0b_1^2b_2b_4^2 \\ -6b_0b_1^2b_3^2b_4 - 80b_0b_1b_2^2b_3b_4 + 18b_0b_1b_2b_3^3 - 192b_0^2b_1b_3b_4^2 + 16b_0b_2^4b_4 \\ -4b_0b_2^3b_3^2 - 128b_0^2b_2^2b_4^2 + 144b_0^2b_2b_3^2b_4 - 27b_0^2b_3^4 + 256b_0^3b_4^3 \end{array}\right\}}$$

$$= 1728 \frac{4h_2(b_0, b_1, b_2, b_3, b_4)^3}{4h_2(b_0, b_1, b_2, b_3, b_4)^3 - 27h_3(b_0, b_1, b_2, b_3, b_4)^2},$$

where

$$h_2(b_0, b_1, b_2, b_3, b_4) = \frac{1}{3}(-3b_1b_3 + 12b_0b_4 + b_2^2);$$

$$h_3(b_0, b_1, b_2, b_3, b_4) = \frac{1}{27}(72b_0b_2b_4 - 27b_0b_3^2 - 27b_1^2b_4 + 9b_1b_2b_3 - 2b_2^3).$$

Since $b_0, \ldots, b_4$ are homogeneous polynomials of degree 4 in s, t, and u, we can obtain expressions for $h_2(b_0, b_1, b_2, b_3, b_4)$ and $h_3(b_0, b_1, b_2, b_3, b_4)$ as homogeneous polynomials of degrees 8 and 12, respectively, in s, t, u. Starting with a quartic equation

$$\sum_{i+j+k=4} a_{ijk}x^i y^j z^k = 0,$$

where a_{ijk} are independent variables, we can calculate $b_0, \ldots, b_4$, using the identities given after (5.3), as in the code below:

Calculating h_2 and h_3 in terms of s, t, u with general coefficients a_{ijk}

```
P<a400,a310,a301,a220,a202,a211,a130,a103,a121,a112,a040,a031,a022,
   a013,a004,s,t,u>:=PolynomialRing(Rationals(),18);
b0:=a004*s^4-a103*s^3*u+a202*s^2*u^2-a301*s*u^3+a400*u^4;
b1:=4*a004*s^3*t-a013*s^3*u-3*a103*s^2*t*u+a112*s^2*u^2+2*a202*s*t*u^2
   -a211*s*u^3-a301*t*u^3+a310*u^4;
b2:=6*a004*s^2*t^2-3*a013*s^2*t*u+a022*s^2*u^2
   -3*a103*s*t^2*u+2*a112*s*t*u^2-a121*s*u^3+a202*t^2*u^2
   -a211*t*u^3+a220*u^4;
b3:=4*a004*s*t^3-3*a013*s*t^2*u+2*a022*s*t*u^2-a031*s*u^3
   -a103*t^3*u+a112*t^2*u^2-a121*t*u^3+a130*u^4;
b4:=a004*t^4-a013*t^3*u+a022*t^2*u^2-a031*t*u^3+a040*u^4;
h2:=-3*b1*b3+b2^2+12*b4*b0;
h3:=8/3*b0*b2*b4-b0*b3^2-b1^2*b4+1/3*b1*b2*b3-2/27*b2^3;
```

It can be checked, for example using the command `Factorisation(h2)`, that h_2 and h_3 are divisible by u^4 and u^6, respectively. Therefore, we are able to put

$$h_2(b_0, b_1, b_2, b_3, b_4) = u^4 g_4(s, t, u) \quad \text{and} \quad h_3(b_0, b_1, b_2, b_3, b_4) = u^6 g_6(s, t, u),$$

where $g_4(s, t, u)$ and $g_6(s, t, u)$ are homogeneous polynomials of degrees 4 and 6, respectively, in s, t, u. Consequently, the rational function $j(b_0, b_1, b_2, b_3, b_4)$ can be presented as a rational function j_C in $\check{\mathbb{P}}^2$ with variables s, t, u via

$$j_C(s, t, u) = 1728 \frac{4 g_4(s, t, u)^3}{4 g_4(s, t, u)^3 - 27 g_6(s, t, u)^2}.$$

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Hamid Abban, School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK; *e-mail:* hamid.abban@nottingham.ac.uk

Ivan Cheltsov, School of Mathematics, The University of Edinburgh, Edinburgh EH9 3JZ, UK; Laboratory of Algebraic Geometry, National Research University Higher School of Economics, 6 Usacheva Street, 117312 Moscow, Russia; *e-mail:* i.cheltsov@ed.ac.uk

Jihun Park, Center for Geometry and Physics, Institute for Basic Science, 77 Cheongam-ro, Nam-gu, Pohang, 37673 Gyeongbuk; Department of Mathematics, Pohang University of Science and Technology, 77 Cheongam-ro, Nam-gu, Pohang, 37673 Gyeongbuk, Korea; *e-mail:* wlog@postech.ac.kr

Constantin Shramov, Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina Street, 119991 Moscow; Laboratory of Algebraic Geometry, National Research University Higher School of Economics, 6 Usacheva Street, 117312 Moscow, Russia; *e-mail:* shramov@mi-ras.ru; *e-mail:* costya.shramov@gmail.com