Residually finite non-linear hyperbolic groups

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Abstract. We exhibit the first examples of residually finite non-linear Gromov hyperbolic groups. Our examples are constructed as amalgamated products of torsion-free cocompact lattices in the rank 1 Lie group $\mathsf{Sp}(d, 1)$, $d \geq 2$, along maximal cyclic subgroups.

1. Introduction

Recall that a group G is called *residually finite* if for every element $g \in G \smallsetminus \{1\}$ there exists a finite group F and a group homomorphism $\phi \colon G \to F$ with $\phi(g) \neq 1$. A long standing open question of Gromov [10] asks whether every hyperbolic group is residually finite. Since every finitely generated linear group is residually finite by Malcev's theorem, a negative answer to Gromov's question should be considered among non-linear hyperbolic groups. Using the superrigidity theorems of Corlette [6] and Gromov–Schoen [11], M. Kapovich constructed in [12] the first examples of non-linear hyperbolic groups as quotients of cocompact lattices in the simple rank 1 Lie group $\mathsf{Sp}(d, 1)$, $d \geq 2$. Other examples of non-linear hyperbolic groups were constructed by Canary, Stover and the second author in [4] and by the authors of this note in [20], as amalgamated products or HNN extensions of quaternionic superrigid lattices along infinite cyclic or quasiconvex free subgroups of rank at least two.

In the light of the previous discussion, it is natural to ask which of these classes of non-linear hyperbolic groups can be shown to be residually finite. Unfortunately, this question is difficult in most cases, and deeply connected to other well-known problems in geometric group theory. For instance, if the congruence subgroup conjecture holds true for quaternionic lattices, then Kapovich's examples would be non-residually finite groups, see [15, §4]. Similarly, residual finiteness of amalgamated products along quasiconvex malnormal subgroups is related to the separability of these subgroups in the ambient lattice, see [14, §2] and [1]. In view of these difficulties, most of these non-linear examples actually provide good candidates for non-residually finite hyperbolic groups.

Nonetheless, in this note we show that one of our constructions in [20] provides hyperbolic non-linear residually finite groups. These groups are the first proven examples[1]. They are constructed as amalgamated products of cocompact lattices in $\mathsf{Sp}(d,1)$, $d \geqslant 2$, along maximal cyclic subgroups, hence they are also CAT(0) groups.

Theorem 1.1. *Let Γ_1 and Γ_2 be two cocompact lattices in $\mathsf{Sp}(d,1)$, $d \geqslant 2$. There exist finite-index subgroups Γ_1' of Γ_1 and Γ_2' of Γ_2 such that for every non-trivial primitive[2] elements $\gamma_1 \in \Gamma_1'$ and $\gamma_2 \in \Gamma_2'$ with different translation lengths in the symmetric space of $\mathsf{Sp}(d,1)$, the amalgamated product*

$$\Gamma_1' *_{\gamma_1 = \gamma_2} \Gamma_2'$$

is a residually finite non-linear Gromov hyperbolic group.

Note that there are several known examples of finitely generated residually finite non-linear groups. These examples include one-relator ascending HNN extensions of free groups exhibited by Drutu–Sapir in [7], the automorphism groups $\mathrm{Aut}(F_n)$, $n \geqslant 3$, (see [8]), Grigorchuk's groups of intermediate growth [9] and residually finite groups containing infinite torsion p and q-subgroups for two distinct primes p, q (see e.g. [16, Remark 3.4]). More recently, Chong–Wise [5] constructed an uncountable family of finitely generated residually finite groups. However, none of these examples are hyperbolic. Most of them are also not linear, since there are only countably many finitely generated linear groups, as Sami Douba pointed out to us. We would also like to thank the referee for providing an alternative explanation for this fact. Indeed, there are only countably many isomorphism classes of finitely generated groups with solvable word problem, and in particular, of finitely generated linear groups [13, 19].

Let us now provide some details on our construction. The non-linearity of the amalgamated product

$$\Delta(\gamma_1, \gamma_2) \overset{\text{def}}{=} \Gamma_1 *_{\gamma_1 = \gamma_2} \Gamma_2$$

is proved in [20] following the point of view of the constructions in [4], and relies on the superigidity theorems of Corlette [6] and Gromov–Schoen [11]. We sketch the proof in Section 2 for completeness.

In order to prove the residual finiteness of $\Delta(\gamma_1, \gamma_2)$, one needs to construct finite quotients of Γ_1 and Γ_2 in which γ_1 and γ_2 have the same arbitrarily large order. To guarantee this property, we appeal to a theorem of Platonov [17] saying that every

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[1]See M. Sapir's answer to this MathOverflow question: https://mathoverflow.net/questions/396895/examples-of-nonlinear-residually-finite-hyperbolic-groups.

[2]An element $g \in \Gamma$ is called *primitive* if whenever $g = h^m$ for some $m \in \mathbb{N}$, then $m = 1$.

finitely generated linear group admits an abundance of descending sequences of normal finite-index subgroups, where quotients between successive terms are p-groups for some primes $p \in \mathbb{N}$. We deduce the following residual finiteness result.

Theorem 1.2. *Let Γ_1 and Γ_2 be non-elementary Gromov hyperbolic groups which are linear over characteristic zero. There exist finite-index subgroups Γ_1' of Γ_1 and Γ_2' of Γ_2 with the property that for every $\gamma_1 \in \Gamma_1'$ and $\gamma_2 \in \Gamma_2'$ primitive elements, the amalgamated product*

$$\Gamma_1' *_{\gamma_1 = \gamma_2} \Gamma_2'$$

is a residually finite Gromov hyperbolic group.

2. Non-linear amalgamated products of superrigid lattices

We recall here the construction of non-linear hyperbolic groups from [20] that will answer Theorem 1.1.

Theorem 2.1 ([20, Theorem 1.3]). *Let Γ_1 and Γ_2 be two lattices in $\mathsf{Sp}(d, 1)$, $d \geqslant 2$. Assume that $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ are two infinite order elements with different translation lengths in the symmetric space of $\mathsf{Sp}(d, 1)$. Then for every field k and $r \in \mathbb{N}$, every representation $\rho \colon \Gamma_1 *_{\gamma_1 = \gamma_2} \Gamma_2 \to \mathsf{GL}_r(k)$ maps Γ_1 and Γ_2 to a finite group. In particular, the amalgamated product $\Gamma_1 *_{\gamma_1 = \gamma_2} \Gamma_2$ is not linear.*

Let us sketch the proof here for completeness. The details are in [20, §5]. The cornerstone of the proof is the superrigidity theorem of Corlette [6] and Gromov–Schoen [11], which say that a linear representation of a $\mathsf{Sp}(d, 1)$ lattice with infinite image essentially extends to $\mathsf{Sp}(d, 1)$. The other ingredient comes from the representation theory of $\mathsf{Sp}(d, 1)$: there is a constant c such that, for every proximal continuous homomorphism $\psi \colon \mathsf{Sp}(d, 1) \to \mathsf{GL}_r(\mathbb{C})$ and every $g \in \mathsf{Sp}(d, 1)$, we have

$$\log \frac{\lambda_1\big(\psi(g)\big)}{\lambda_2(\psi(g))} = c \cdot \ell_{\mathbb{H}\mathbf{H}^d}(g),$$

where $\lambda_1(\psi(g)) \geqslant \lambda_2(\psi(g)) \geqslant \cdots \geqslant \lambda_r(\psi(g))$ are the moduli of the eigenvalues of $\psi(g)$ and $\ell_{\mathbb{H}\mathbf{H}^d}(g)$ denotes the translation length of $g \in \mathsf{Sp}(d, 1)$ acting on the quaternionic hyperbolic space $\mathbb{H}\mathbf{H}^d$. In the context of Theorem 2.1, the superrigidity of Γ_1 and Γ_2 and the fact that $\gamma_1, \gamma_2 \in \mathsf{Sp}(d, 1)$ have distinct translation lengths implies that two linear representations of ρ_i of Γ_i with infinite image cannot satisfy

$$\rho_1(\gamma_1) = \rho_2(\gamma_2).$$

Thus the amalgamated product does not admit a faithful linear representation.

3. Residual finiteness of amalgamated products

There are certain ways of proving the residual finiteness of amalgamated products of residually finite groups, as soon as compatibility conditions hold for the amalgamated subgroups, see for example [2, Proposition 1]. We give here the following refinement of Baumslag's proposition for Gromov hyperbolic groups.

Lemma 3.1. *Let Γ_1 and Γ_2 be two torsion-free Gromov hyperbolic groups and let $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ be two primitive elements. Suppose that there exist decreasing sequences $(\Gamma_{1,n})_{n=1}^{\infty}$ and $(\Gamma_{2,n})_{n=1}^{\infty}$ of finite-index normal subgroups of Γ_1 and Γ_2 respectively with the following properties:*

(i) $\bigcap_{n=1}^{\infty} \Gamma_{i,n} = \{1\}$ *for $i \in \{1, 2\}$;*

(ii) $[\langle \gamma_1 \rangle : \langle \gamma_1 \rangle \cap \Gamma_{1,n}] = [\langle \gamma_2 \rangle : \langle \gamma_2 \rangle \cap \Gamma_{2,n}]$ *for every $n \in \mathbb{N}$.*

Then the amalgamated product $\Gamma_1 *_{\gamma_1 = \gamma_2} \Gamma_2$ is residually finite.

The difference between the previous lemma and Baumslag's proposition [2, Proposition 1] is that the former requires additionally, for $i = 1, 2$, that

$$\bigcap_{i=1}^{\infty} \langle \gamma_i \rangle \Gamma_{i,n} = \langle \gamma_i \rangle.$$

In our setting, this is implied by the following observation that maximal cyclic subgroups of residually finite torsion-free hyperbolic groups are separable.

Proposition 3.2. *Let Γ be a countable group. Suppose that $(\Gamma_n)_{n=1}^{\infty}$ is a decreasing sequence of finite-index normal subgroups of Γ with $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. Let $\gamma \in \Gamma$ be an element whose centraliser in Γ is the cyclic group $\langle \gamma \rangle$. Then*

$$\bigcap_{n=1}^{\infty} \langle \gamma \rangle \Gamma_n = \langle \gamma \rangle.$$

Proof. Let us set $H \overset{\text{def}}{=} \bigcap_{n=1}^{\infty} \langle \gamma \rangle \Gamma_n$. Let $h \in H$, fix $n \in \mathbb{N}$ and write $h = \eta_n \gamma^{s_n}$ for some $s_n \in \mathbb{N}$ and $\eta_n \in \Gamma_n$. Observe that $[h, \gamma] = \eta_n \gamma \eta_n^{-1} \gamma^{-1} \in \Gamma_n$ since Γ_n is a normal subgroup of Γ. We thus have

$$[h, \gamma] \in \bigcap_{n=1}^{\infty} \Gamma_n = \{1\}.$$

Hence, H centralizes the cyclic group $\langle \gamma \rangle$ and by assumption we have that $H = \langle \gamma \rangle$, concluding the proof. ∎

Proposition 3.2 applies in the case where Γ is a torsion-free hyperbolic group and $\gamma \in \Gamma$ is a primitive element, since the centraliser of $\gamma \in \Gamma$ is the cyclic group $\langle \gamma \rangle$, see [10].

In the light of Proposition 3.2, the proof of Lemma 3.1 is quite standard. We provide a proof for the reader's convenience.

Proof of Lemma 3.1. Let us set $\Delta = \Gamma_1 *_{\gamma_1 = \gamma_2} \Gamma_2$. For every $g \in \Delta$, we shall exhibit a finite-index normal subgroup N of Δ with $g \in \Delta \smallsetminus N$. First, let us assume that g does not lie in a conjugate of Γ_1 or Γ_2. Up to conjugation, we may write $g = w_1 \eta_1 \cdots w_s \eta_s$, where $w_1, \dots, w_s \in \Gamma_1 \smallsetminus \langle \gamma_1 \rangle$ and $\eta_1, \dots, \eta_s \in \Gamma_2 \smallsetminus \langle \gamma_2 \rangle$.

By assumption (i) and Proposition 3.2, there exists $m \in \mathbb{N}$ large enough such that $w_1, \dots, w_s \in \Gamma_1 \smallsetminus \langle \gamma_1 \rangle \Gamma_{1,m}$ and $\eta_1, \dots, \eta_s \in \Gamma_2 \smallsetminus \langle \gamma_2 \rangle \Gamma_{2,m}$. By assumption (ii), we have

$$[\langle \gamma_1 \rangle : \langle \gamma_1 \rangle \cap \Gamma_{1,m}] = [\langle \gamma_2 \rangle : \langle \gamma_2 \rangle \cap \Gamma_{2,m}],$$

so $\langle \overline{\gamma}_1 \rangle$ and $\langle \overline{\gamma}_2 \rangle$ are finite cyclic groups of $\Gamma_1 / \Gamma_{1,m}$ and $\Gamma_2 / \Gamma_{2,m}$ of the same order. Thus there exists a surjective homomorphism

$$\pi \colon \Delta \to \Gamma_1 / \Gamma_{1,m} *_{\overline{\gamma}_1 = \overline{\gamma}_2} \Gamma_2 / \Gamma_{2,m}$$

restricting to the quotient morphism $\Gamma_i \to \Gamma_i / \Gamma_{i,m}$ for $i \in \{1, 2\}$. By construction, $\pi(w_1), \dots, \pi(w_s) \in \Gamma_1 / \Gamma_{1,m} \smallsetminus \langle \overline{\gamma}_1 \rangle$ and $\pi(\eta_1), \dots, \pi(\eta_s) \in \Gamma_2 / \Gamma_{2,m} \smallsetminus \langle \overline{\gamma}_2 \rangle$, and we conclude that

$$\pi(g) = \pi(w_1) \pi(\eta_1) \cdots \pi(w_s) \pi(\eta_s) \neq 1.$$

Now, an amalgamated product of finite groups is virtually free (see for instance [18]) hence residually finite, so there exists a finite group F and a surjective group homomorphism

$$\varphi \colon \Gamma_1 / \Gamma_{1,m} *_{\overline{\gamma}_1 = \overline{\gamma}_2} \Gamma_2 / \Gamma_{2,m} \to F$$

with $\varphi(\pi(g)) \neq 1$. In particular, $g \in \Delta \smallsetminus \ker(\pi \circ \varphi)$.

In the case where $g = whw^{-1}$ for some $w \in \Delta$ and $h \in \Gamma_1$ (resp. $h \in \Gamma_2$) we choose $n \in \mathbb{N}$ large enough such that $h \in \Gamma_1 \smallsetminus \Gamma_{1,n}$ (resp. $h \in \Gamma_2 \smallsetminus \Gamma_{2,n}$). We obtain a surjective group homomorphism

$$\pi \colon \Delta \to \Gamma_1 / \Gamma_{1,n} *_{\overline{\gamma}_1 = \overline{\gamma}_2} \Gamma_2 / \Gamma_{2,n}$$

with $\pi(h) \neq 1$, hence $\pi(g) \neq 1$. Again, since $\Gamma_1 / \Gamma_{1,n} *_{\overline{\gamma}_1 = \overline{\gamma}_2} \Gamma_2 / \Gamma_{2,n}$ is residually finite, g survives in a finite quotient of Δ. ∎

4. Platonov's theorem

By Lemma 3.1, in order to prove residual finiteness of our amalgamated hyperbolic groups, we need to construct sufficiently many quotients of these groups in which the amalgamated cyclic subgroups have the same order. These will be given by the following theorem of Platonov [17] which shows that linear finitely generated groups are residually p-finite for some p (i.e. every non-trivial element survives in a finite quotient which is a p-group).

Theorem 4.1 (Platonov [17]). *Let k be a field of characteristic zero and Γ be a finitely generated subgroup of $\mathsf{GL}_r(k)$. Then, for all but finitely many primes $p \in \mathbb{N}$, there exists a decreasing sequence of finite-index normal subgroups $(\Gamma(p^n))_{n=1}^{\infty}$ of Γ with the following properties:*

(i) $\bigcap_{n=1}^{\infty} \Gamma(p^n) = \{1\}$;

(ii) *for $n \in \mathbb{N}$, every non-trivial element of $\Gamma(p^n)/\Gamma(p^{n+1})$ has order equal to p.*

In particular, Γ is virtually residually p-finite for all but finitely many primes p.

The sequence $(\Gamma(p^n))_{n=1}^{\infty}$ is constructed in the following way: let A be the domain generated by the matrix entries of elements of (a finite generating set of) Γ and let $I \subset A$ be a maximal ideal such that A/I is a finite field of characteristic p. The finite index normal subgroup $\Gamma(p^n)$ is then defined as the kernel of the morphism

$$\Gamma \subset \mathsf{GL}_r(A) \to \mathsf{GL}_r(A/I^n).$$

A detailed proof of Platonov's theorem is given in [16, Theorem 3.1].

5. Proof of the theorems

We now have all the tools to conclude the proof of Theorem 1.2 and Theorem 1.1.

Proof of Theorem 1.2. Note first that, by Malcev's theorem, Γ_1 and Γ_2 are residually finite, and by Selberg's lemma, up to passing to finite-index subgroups, we may assume that Γ_1 and Γ_2 are torsion-free. By Theorem 4.1 there exists a prime $p \in \mathbb{N}$ and descending sequences $\{\Gamma_1(p^n)\}_{n=1}^{\infty}$ and $\{\Gamma_2(p^n)\}_{n=1}^{\infty}$ of finite-index normal subgroups of Γ_1 and Γ_2 respectively such that:

(i) $\bigcap_{n=1}^{\infty} \Gamma_1(p^n) = \{1\}$ and $\bigcap_{n=1}^{\infty} \Gamma_2(p^n) = \{1\}$;

(ii) for $n \geqslant 1$ and $i \in \{1, 2\}$, every non-trivial element of $\Gamma_i(p^n)/\Gamma_i(p^{n+1})$ has order equal to p.

Let us set $\Gamma'_1 = \Gamma_1(p)$ and $\Gamma'_2 = \Gamma_2(p)$. Let $\gamma_1 \in \Gamma'_1$ and $\gamma_2 \in \Gamma'_2$ be two non-trivial primitive elements. We claim that the amalgamated product

$$\Delta(\gamma_1, \gamma_2) \stackrel{\text{def}}{=} \Gamma'_1 *_{\gamma_1 = \gamma_2} \Gamma'_2$$

is a residually finite hyperbolic group.

Since Γ_i is torsion-free for $i \in \{1, 2\}$, $\langle \gamma_i \rangle$ is malnormal in $\Gamma_i(p)$ and the hyperbolicity of $\Delta(\gamma_1, \gamma_2)$ follows immediately by the Bestvina–Feighn combination theorem [3].

Observe that for every $n \in \mathbb{N}$ and $i \in \{1, 2\}$ the order of $\bar{\gamma}_i$ in $\Gamma_i / \Gamma_i(p^n)$ is a power of p since $\gamma_i \in \Gamma_i(p)$ and $\Gamma_i(p)/\Gamma_i(p^n)$ is a finite p-group. For every $n \in \mathbb{N}$ and $i \in \{1, 2\}$, define $a_i(n)$ as the integer such that $\bar{\gamma}_i$ has order $p^{a_i(n)}$ in $\Gamma_i / \Gamma_i(p^n)$. Since $\Gamma_i(p^n)$ is a decreasing sequence of normal subgroups with

$$\bigcap_{n=1}^{\infty} \Gamma_i(p^n) = \{1\},$$

the sequence $a_i(n)$ is increasing and unbounded. Furthermore, by property (ii) of Platonov's theorem, $\gamma_i^{p^{a_i(n)}} \in \Gamma_i(p^n)$ has order 1 or p in $\Gamma_i(p^n)/\Gamma_i(p^{n+1})$, which implies that $a_i(n+1) - a_i(n) \in \{0, 1\}$ for $i \in \{1, 2\}$. We conclude that $a_i \colon \mathbb{N} \to \mathbb{N}$ is surjective for $i \in \{1, 2\}$.

Let $(k_n)_{n=1}^{\infty}$ and $(l_n)_{n=1}^{\infty}$ be increasing sequences such that

$$a_1(k_n) = a_2(l_n) = n.$$

Finally, set $\Gamma'_{1,n} = \Gamma_1(p^{k_n})$ and $\Gamma'_{2,n} = \Gamma_2(p^{l_n})$. By the definition of k_n and l_n, the order of $\bar{\gamma}_i$ in $\Gamma'_i / \Gamma'_{i,n}$ is p^n for $i \in \{1, 2\}$. Since (k_n) and (l_n) are unbounded, we have

$$\bigcap_{n=1}^{\infty} \Gamma_{i,n} = \{1\}.$$

The decreasing sequences of subgroups $\left(\Gamma'_{i,n}\right)_{n=1}^{\infty}$ satisfy hypotheses (i) and (ii) of Lemma 3.1 and we conclude that $\Delta(\gamma_1, \gamma_2)$ is residually finite. ∎

Now Theorem 1.1 follows straightforwardly from Theorem 1.2 and Theorem 2.1.

Proof of Theorem 1.1. Let Γ_1 and Γ_2 be two cocompact lattices of $\mathsf{Sp}(d, 1)$, $d \geqslant 2$. Since Γ_1 and Γ_2 are linear over $\mathbb{R}$, by Theorem 1.2 there exist finite-index subgroups Γ'_1 and Γ'_2 of Γ_1 and Γ_2 respectively such that for every $\gamma_1 \in \Gamma'_1$ and $\gamma_2 \in \Gamma'_2$ primitive elements, the group $\Delta(\gamma_1, \gamma_2)$ is a residually finite Gromov hyperbolic group. On the other hand, when the translation lengths of γ_1 and $\gamma_2 \in \mathsf{Sp}(d, 1)$ are different, the group $\Delta(\gamma_1, \gamma_2)$ is non-linear by Theorem 2.1. ∎

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