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# Relations between dynamical degrees, Weil's Riemann hypothesis and the standard conjectures

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**Abstract.** Let $\mathbb{K}$ be an algebraically closed field, $X$ a smooth projective variety over $\mathbb{K}$ and $f: X \to X$ a dominant regular morphism. Let $N^i(X)$ be the group of algebraic cycles, of codimension $i$, modulo numerical equivalence. Let $\chi(f)$ be the spectral radius of the pullback $f^*: H^*(X, \mathbb{Q}_l) \to H^*(X, \mathbb{Q}_l)$ on $l$-adic cohomology groups, and $\lambda(f)$ the spectral radius of the pullback $f^*: N^*(X) \to N^*(X)$. We prove in this paper, by using consequences of Deligne's proof of Weil's Riemann hypothesis, that $\chi(f) = \lambda(f)$. This answers affirmatively a question posed by Esnault and Srinivas. Consequently, the algebraic entropy $\log \chi(f)$ of an endomorphism is both a birational invariant and étale invariant. More general results are proven if either $\mathbb{K} = \overline{\mathbb{F}}_p$ or the Standard Conjecture D holds (this applies specially to Abelian varieties). Among other results in the paper, we show that if some properties of dynamical degrees, known in the case $\mathbb{K} = \mathbb{C}$, hold in positive characteristics, then simple proofs of Weil's Riemann hypothesis follow. More generally, the analogy in positive characteristic of Serre's famous result on polarized endomorphisms of compact Kähler manifolds also follows.

## 1. Introduction

The proof of Weil's conjectures by Deligne is one of the major achievements of mathematics in the 20th century. Through the visions of Weil and Grothendieck and many others, the question about counting the number of points in finite fields $\mathbb{F}_{q^n}$ (also, asymptotically as $n \to \infty$), on a smooth projective variety $X_0$ defined on $\mathbb{F}_q$, is translated to the question about the eigenvalues of the pullbacks $(\mathrm{Fr}^n)^*$ on étale cohomology groups $H^*(X, \mathbb{Q}_l)$. Here $X$ is the base change of $X_0$ to an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$, and Fr is the Frobenius map. Bombieri and Grothendieck thought of solving Weil's Riemann hypothesis via the famous standard conjectures [22], but the proof by Deligne [7, 8] was totally different and surprising. For some good historical accounts about this, see for example [27], and also see [29] for more modern updates.

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The current paper serves two purposes. First, we use either Weil's Riemann hypothesis or the Standard Conjecture D (that modulo torsions numerical and homological equivalences coincide on algebraic cycles) to extend several known results on dynamical degrees from complex dynamics to the case of base fields of positive characteristic. Conversely, the second purpose is to point out that if some stronger results on dynamical degrees (which are known when the base field is $\mathbb{K} = \mathbb{C}$) hold for base fields of positive characteristic, then we obtain natural generalisations of Weil's Riemann hypothesis. Thus, it is demonstrated here that there is a curious relation between algebraic dynamics and Weil's cohomology theories. While results on endomorphisms already have abundantly arithmetic applications, there are new situations where a more general setting - to rational maps and even correspondences - as considered here may be more useful. For example, in an approach towards the Riemann hypothesis proposed by Christopher Deninger since about 2 decades ago, analogues of Weil's Riemann hypothesis for singular foliations – closer to rational maps and correspondences than to endomorphisms – are needed (see e.g. [9] and references therein). Also, our new viewpoint allows natural generalisations to varieties and maps (or correspondences) defined over an arbitrary field of positive characteristic, not just finite fields.

This paper was inspired by the results of Esnault and Srinivas [18] on automorphisms of surfaces. In the remaining of this introduction, we pose some questions to be studied in the paper and then state the main results. To make the presentation concise, we collect some background materials on correspondences and dynamical degrees in Section 2.

**Remark.** This paper is a slight revision of [38], in response to some comments from readers. In the time between, there have been some new papers – including the author's joint work – which support the conjectures in this paper, and shed new lights on relations between the line of the conjectures and the standard conjectures (in particular, the Standard Conjecture D), and point to a more doable way than the standard conjectures to solve the positive characteristic analogue of Serre's famous result on polarized endomorphisms of compact Kähler manifolds [34] (which has been the inspiration for the standard conjectures) – see Question 2 below – as well as a generalisation of semi-simplicity conjecture to all polarized endomorphisms. In a subsequent joint paper [25], we showed that this effective version of Standard Conjecture C – for graphs of polarized endomorphisms – is a consequence of the standard conjectures. Our new approach is unconditionally applicable for example to Kummer surfaces, for which the approach through standard conjectures is not yet known to be realisable. (See the next subsection on Question 2 and other questions, and a discussion on their current situation.) However, to preserve the history, we keep the change as minimum as possible, mostly only add more explicit pointers from references to the arguments in the arXiv version and restructure the organisation for to help better understanding.

### 1.1. Questions

Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic, $X$ a smooth projective variety over $\mathbb{K}$ and $f: X \to X$ a correspondence. A correspondence is roughly an algebraic cycle of $X \times X$ whose dimension is exactly $\dim(X)$. Examples of interest include regular morphisms, rational maps and a convex combination of such. The latter means algebraic cycles of the form $\sum_{i\in I} a_i\Gamma_i$, where $I$ is a finite set, $\Gamma_i$ is the graph of an endomorphism, and $a_i$ is a positive integer. See Section 2 for a precise definition. Weil's Riemann hypothesis can be stated in terms of the following numbers $\lambda_i$ and $\chi_i$, see Theorem 1.5 and Section 4 for more details.

We first consider the groups $N^i(X)$ of algebraic cycles of codimension $i$ modulo numerical equivalence, with $\mathbb{Z}$ coefficients. These are free Abelian groups of finite ranks (see [20, Chapter 19] or [18, Section 6.2]). We let $N^i(X)_{\mathbb{R}} = N^i(X) \otimes_{\mathbb{Z}} \mathbb{R}$ be regarded as a real vector space. We define $\lambda_i(f)$ to be the numbers

$$\lambda_i(f) := \limsup_{n\to\infty} \|(f^n)^*|_{N^i(X)_{\mathbb{R}}}\|^{1/n},$$

where we fix any norm on the finite-dimensional vector space $N^i(X)_{\mathbb{R}}$. We recall here that if $A: M \to N$ is a linear map between two real vector spaces, with given norms $\|.\|_M$ and $\|.\|_N$, then

$$\|A\| := \sup_{\|v\|_M=1} \|Av\|_N.$$

In [39] (recalled in Section 2 below), we showed that in fact the lim sup can be replaced by lim, and all numbers $\lambda_i(f)$ are finite. Why $\lambda_i(f)$ is independent of the choice of the norm on $N^i(X)_{\mathbb{R}}$ can be easily seen as follows. Since $N^i(X)_{\mathbb{R}}$ is finite-dimensional, any two norms on it are equivalent, see [16, Section IV.3.1, Lemma 1]. That is, let $\|.\|_1$ and $\|.\|_2$ be two norms on $N^i(X)_{\mathbb{R}}$, then there is a constant $C > 0$ such that

$$C^{-1}\|v\|_1 \leq \|v\|_2 \leq C\|v\|_2$$

for all $v \in N^i(X)_{\mathbb{R}}$. This implies that there is a constant $D > 0$ such that for all $n \in \mathbb{N}$, we have

$$D^{-1}\|(f^n)^*|_{N^i(X)_{\mathbb{R}}}\|_1 \leq \|(f^n)^*|_{N^i(X)_{\mathbb{R}}}\|_2 \leq D\|(f^n)^*|_{N^i(X)_{\mathbb{R}}}\|_1,$$

and hence by taking the $n$th roots and limit when $n \to \infty$ we see that the limit is independent of the choice of the norm, as asserted. Moreover, if $\mathcal{L}$ is an ample divisor on $X$, then we can compute $\lambda_p(f)$ as follows (this is because a fixed multiple $c\mathcal{L}$ will bound the classes of algebraic cycles whose norm is $\leq 1$):

$$\lambda_p(f) = \lim_{n\to\infty}\big((f^n)^*(\mathcal{L}^p).\mathcal{L}^{\dim(X)-p}\big)^{1/n}. \tag{1.1}$$

Also, we define $\chi_i(f)$ to be the numbers

$$\chi_i(f) := \limsup_{n\to\infty} \|(f^n)^*|_{H^i(X,\mathbb{Q}_l)}\|^{1/n},$$

here we fix any norm on the finite-dimensional vector space $H^i(X,\mathbb{Q}_l)$. In contrast to the $\lambda_i(f)$'s, the finiteness of $\chi_i(f)$'s is not obvious, although by definition we have $\chi_{2i}(f) \geq \lambda_i(f)$. We also do not know whether the lim sup in the definition for $\chi_i(f)$ can be replaced by lim.

We may call the number

$$\log \chi(f) := \log \max_{i=0,\ldots,2\dim(X)} \chi_i(f),$$

the algebraic entropy.

These numbers $\lambda_i(f)$ and $\chi_i(f)$ have been extensively studied when $\mathbb{K} = \mathbb{C}$ in the context of complex dynamics. They are called dynamical degrees in that setting and are important to the dynamical properties of $f$, see Section 2 for more details. The known results in the case $\mathbb{K} = \mathbb{C}$ (see Section 2) and recent results of Esnault and Srinivas [18] on automorphisms of surfaces over positive characteristic inspire us to study the following questions.

**Question 1.** Is $\chi_i(f)$ finite for all $i = 0,\ldots,2\dim(X)$?

**Question 2.** Is $\chi_{2i}(f) = \lambda_i(f)$ for all $i$?

**Question 3** (Product formula). Let $f\colon X \to X$, $g\colon Y \to Y$ be dominant rational maps and $\pi\colon X \to Y$ be a dominant rational map so that $\pi \circ f = g \circ \pi$. Is it true that we can define the relative dynamical degrees $\chi_{2i}(f|\pi)$ which satisfy the relations

$$\chi_{2p}(f) = \max_{\substack{0\leq i\leq \dim(Y),\\ 0\leq p-i\leq \dim(X)-\dim(Y)}} \chi_{2i}(g)\chi_{2(p-i)}(f|\pi),$$

for all $p = 0,\ldots,\dim(X)$?

**Question 4** (Dinh's inequality). Is $\chi_i(f)^2 \leq \max_{p+q=i} \lambda_p(f)\lambda_q(f)$?

The following weaker version of Question 4 is enough for applications to dynamics (cf. Gromov–Yomdin's theorem and Gromov–Dinh–Sibony's inequality in Section 2.2).

**Question 4′.** Is $\max_{i=0,\ldots,2\dim(X)} \chi_i(f) = \max_{i=0,\ldots,\dim(X)} \lambda_i(f)$?

**Question 5.** Are $\chi_{2i}(f)$ birational invariants? This means that if $f: X \to X$ and $g: Y \to Y$ are dominant rational maps, and $\pi: X \to Y$ is a birational map such that $f = \pi^{-1} \circ g \circ \pi$, then we should have $\chi_{2i}(f) = \chi_{2i}(g)$ for all $i$.

**Remarks.** Here we provide an overview of the current situation (December 2022) of the above questions.

First, a general remark: Note that all of these questions have affirmative answers when $\mathbb{K} = \mathbb{C}$, see Section 2. A crucial advantage in working with $\mathbb{C}$ is that we have positivity notions, induced from positive closed forms and currents, on cohomology classes. More precisely, if $X$ is a compact Kähler manifold and $\omega$ is a Kähler form on $X$, and $\alpha$ is an arbitrary smooth real form of degree $2i$ on $X$, then there is a constant $C > 0$ such that $C\omega^i - \alpha$ is a positive form. As a consequence, if $f: X \to X$ is a dominant meromorphic map, then $\|f^*|_{H^{2i}(X)}\|$ is comparable to $\|f^*(\omega^i)\|_{H^{2i}(X)}$. These positivity notions are not yet available on fields of positive characteristics.

Second, another general remark: Because of properties of dynamical degrees [39], an affirmative answer to Question 2 automatically yields affirmative answers to Questions $4'$ and 5, and part of Question 1 for $\chi_{2i}$'s. Likewise, if both Question 2 and Question 4 have affirmative answers, then Question 1 has an affirmative answer also for $\chi_{2i+1}$'s.

Third, yet another general remark: In [24], it is now shown that an effective version of Standard Conjecture C for graphs of polarized endomorphisms is enough to solve Question 2 for polarized endomorphisms, and a combination of the mentioned effective version of Standard Conjecture C (again, for graphs of polarized endomorphisms) and Standard Conjecture D are enough to solve Questions 1, 2, 4, $4'$ and 5, as well as a semi-simplicity property.

*For Question* 1. It is trivially solved in the affirmative when $f$ is a regular morphism, otherwise the question is still widely open. Some special cases when it is (partly) solved in the affirmative:

(i) when Question 2 (or both Question 2 and Question 4) has an affirmative answer;

(ii) if $X$ is an Abelian variety defined over a finite field and $l$ is an appropriate chosen prime number, see the discussion after the statement of Theorem 1.1;

(iii) $X$ is a Kummer surface, see [24].

*For Question* 2. Theorem 1.4 solves Question 2 in affirmative for a large class of correspondences of surfaces. Hu [23] solves in the affirmative Question 2 for endomorphisms of Abelian varieties. More recently, [24] solves in the affirmative Question 2 for more general correspondences of Abelian varieties and Kummer surfaces.

*For Question* 3. Note that here we do not know that whether the relative dynamical degrees $\chi_{2i}(f|\pi)$ – which are more general than the dynamical degrees – can be defined on $l$-adic cohomology groups, hence the question mark. If it would be, the idea is to mimic the definition in the case of base field $\mathbb{C}$ in [11] or in the case of base field of arbitrary characteristic but only on the groups $N^i(X)_{\mathbb{R}}$ in [39]. Since the actual definition of these numbers are complicated, we refer the readers to the mentioned papers. In this paper, we do not need a precise definition of the relative dynamical degrees, but mainly only need the above relation (product formula) between dynamical degrees of $f$, dynamical degree of $g$ and the relative dynamical degrees, as well as the fact that the relative dynamical degrees should be 1 when $\pi$ has finite degree.

*For Question* 4. Hu–Truong [24] solve Question 4 in the affirmative in two cases:

(i) $X$ is an Abelian variety and $l$ is an appropriate chosen prime number;

(ii) $X$ is a Kummer surface.

*For Question* 4′. Theorem 1.1 solves Question 4′ in two cases:

(i) $f$ is a regular morphism (a different proof exploring the poles of zeta functions was presented later in [35]);

(ii) $X$ is an Abelian variety defined over a finite field and $l$ is an appropriate chosen prime number, see the discussion after the statement of Theorem 1.1. Hu–Truong [24] solve Question 4′ for Kummer surfaces.

*For Question* 5. Hu–Truong [24] solve Question 5 in the affirmative in some cases:

(i) $\chi_2(f)$;

(ii) for general $\chi_{2i}(f)$'s if $X$ is an Abelian variety defined over a finite field and $l$ is an appropriate chosen prime number;

(iii) $X$ is a Kummer surface.

### 1.2. Main results

Here we state main results of the paper. We recall that we work on an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. We use the convention that a variety is irreducible.

We mention a relevant standard conjecture on algebraic cycles. We will denote by $\mathcal{Z}^i_{\mathrm{hom}}(X)$ the set of algebraic cycles on $X$ of codimension $i$ whose image in $H^{2i}(X, \mathbb{Q}_l)$ is 0; and by $\mathcal{Z}^i_{\mathrm{num}}(X)$ the set of algebraic cycles on $X$ of codimension $i$ which are 0 under the numerical equivalence, that is those cycles $V$ for which the intersection product $V.W = 0$ for all algebraic cycles $W$ of dimension $i$; see [20] for detail. The following weaker version of the Standard Conjecture D is sufficient for our purpose.

**The numerical-homological equivalences condition.** Given a smooth projective variety $Z$ of *even* dimension $2k'$, we say that $NH(Z)$ holds if

$$\mathcal{Z}^{k'}_{\mathrm{hom}}(Z) \otimes \mathbb{Q} = \mathcal{Z}^{k'}_{\mathrm{num}}(Z) \otimes \mathbb{Q}$$

for the middle-degree cohomology group $H^{2k'}(Z, \mathbb{Q}_l)$.

The first result answers Questions 1 and 4′.

**Theorem 1.1.** *The following statements hold.*

(1) *Assume that $NH(X \times X)$ holds. Then, Questions 1 and 4′ have affirmative answers. More precisely, if $f: X \to X$ is a dominant correspondence, then*

$$\chi_i(f) \leq \max_{p=0,\ldots,\dim(X)} \lambda_p(f),$$

*for all $i = 0, \ldots, 2\dim(X)$.*

(2) *Assume that $f: X \to X$ is a* regular morphism. *Then Questions 1 and 4′ have affirmative answers.*

Part (2) of the theorem answers affirmatively a question posed in [18, Section 6.3]. Here, we do not need the "dominant" assumption, which is only needed in the case of rational maps or correspondences in order to be able to compose $f$ with itself many times. By [4], $NH(X \times X)$ holds if $X$ is an Abelian variety defined over a finite field and $l$ is a prime number appropriately chosen, hence part (1) applies. A different proof of part (2) exploring the poles of zeta functions was presented later in [35].

In the case $X$ is defined over a finite field, a weaker version of part (1) of Theorem 1.1 holds unconditionally as well.

**Theorem 1.2.** *Let $\mathbb{K} = \overline{\mathbb{F}}_p$, the algebraic closure of a finite field $\mathbb{F}_p$. Let $X$ be a smooth projective variety over $\mathbb{K}$, and $f: bX \to X$ a correspondence. Denote by* $\mathrm{sp}((f^n)^*|_{H^i(X,\mathbb{Q}_l)})$ *the spectral radius of the pullback*

$$(f^n)^*: H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l).$$

*Then,*

$$\limsup_{n\to\infty} \mathrm{sp}\big((f^n)^*|_{H^i(X,\mathbb{Q}_l)}\big)^{1/n} \leq \max_{p=0,\ldots,\dim(X)} \lambda_p(f)$$

*for all $i = 0, \ldots, 2\dim(X)$.*

As a consequence of Theorem 1.1, the algebraic entropy $\log \chi(f)$ of surjective endomorphisms is both a birational invariant and étale invariant. This means that if $f: X \to X$ and $g: Y \to Y$ are surjective endomorphisms, and $\pi: X \to Y$ is a dominant rational map of finite degree such that $\pi \circ f = g \circ \pi$, then

$$\log \chi(f) = \log \chi(g).$$

If $\pi$ is a regular morphism, then one does not need the assumption that $f$ and $g$ are surjective (this assumption is only needed to define compositions of the concerned maps). More generally, we have the following result, which is an analogue of a classical result of Bowen on topological entropy of continuous dynamical systems on compact metric spaces ([3, Theorem 17]). By the proof of the consequence, provided the Standard Conjecture D holds, the same conclusion holds for rational maps and a slightly weaker conclusion holds for all correspondences.

**Corollary 1.3.** *Let $X, Y$ be smooth projective varieties over $\mathbb{K}$ of the same dimension, $f: X \to X$ and $g: Y \to Y$ dominant* regular morphisms. *Assume that there is a dominant rational map (necessarily has generic finite fibres) $\pi: X \to Y$, so that $\pi \circ f = g \circ \pi$. Then*

$$\max_{i=0,\ldots,2\dim(X)} \chi_i(f) = \max_{i=0,\ldots,2\dim(X)} \chi_i(g).$$

*Proof.* This follows from the corresponding properties for the geometric dynamical degrees $\lambda_i(f)$ and $\lambda_i(g)$ ([39, Theorems 1.1 and 1.3]) and Theorem 1.1 (2) above. ■

As another consequence of Theorem 1.1, we answer Question 2 for a large class of correspondences on surfaces.

**Theorem 1.4.** *Let $X$ be a smooth projective surface over $\mathbb{K}$, and $f: X \to X$ a dominant correspondence with $\lambda_1(f) \geq \max\{\lambda_0(f), \lambda_2(f)\}$.*

(1) *Assume that $NH(X \times X)$ holds. Then $\chi_2(f) = \lambda_1(f)$. Moreover,*

$$\max\{\chi_1(f), \chi_3(f)\} \leq \lambda_1(f).$$

(2) *Assume that $f$ is a* regular morphism. *Then $\chi_2(f) = \lambda_1(f)$. Moreover,*

$$\max\{\chi_1(f), \chi_3(f)\} \leq \lambda_1(f).$$

If $f$ is an automorphism (or more generally, a birational map), then

$$\lambda_0(f) = \lambda_2(f) = 1 \quad \text{and} \quad \lambda_1(f) \geq 1.$$

Hence, Theorem 1.4 (2) applies. Note that this case, i.e., $f$ is an automorphism of a surface, was solved by Esnault and Srinivas in [18]. Their proof makes use of the classification of surfaces and is not purely algebraic (because at some part of the proof, they need to use the lifting to characteristic 0, and use the known results in that case).

They also mentioned an algebraic proof of their result, suggested by P. Deligne, under the assumption that the standard conjectures hold. Another case is Abelian varieties over finite fields, where it is known that for infinitely many prime numbers $l$ the conjecture $NH(X \times X)$ holds [4]. Thus, we can apply Theorem 1.4 (1). (A very recent joint paper by Fei Hu and the author [24] proves more general results for Abelian varieties.)

**Remark.** By the results in [26], all the above results are valid for any Weil's cohomology theory (recall that these theories satisfy the Poincaré duality and Weak Lefschetz axiom, as required for a "reasonable" cohomology theory in [26], this fact is also mentioned in the cited paper and in, for example, [27]).

Some other results related to Questions 2 and 3 will be proven in the last section of this paper. The last main result concerns the relation between the above questions and Weil's Riemann hypothesis.

**Theorem 1.5.** *If Question 2 or Question 3 or Question 4 has an affirmative answer, then Weil's Riemann hypothesis. More generally, the positive characteristic analogue of Serre's result* [34] *also follows.*

This theorem suggests an alternative approach towards solving Weil's Riemann hypothesis and its generalisations, such as the positive characteristic analogue of Serre's result, which may not need to go through exploring the poles of zeta functions like in Deligne's proof. See, e.g., [25] for a further development, which needs weaker assumptions than the standard conjectures approach by Bombieri and Grothendieck, where some new cases – results more general than the positive characteristic analogue of Serre's result on Abelian varieties and Kummer's surfaces – are solved.

### 1.3. Plan of the paper

Some background materials are collected in Section 2. In Section 3 we prove Theorems 1.1, 1.2 and 1.4. In Section 4 we prove Theorem 1.5. In the last section we discuss an approach toward solving Questions 2 and 3, the main result in that section is Theorem 5.2.

Two main ideas are used throughout the paper. The first one is that by working on $X \times X$, some questions about pulling back of a correspondence $f: X \to X$ on $l$-adic cohomology groups may be reduced to questions about algebraic cycles only. The second one is that given a dominant correspondence $f: X \to X$ and a dominant regular morphism with *finite* fibres $\pi: X \to Y$, we can consider the pushforward

$$g_n = \pi_*(f^n): Y \to Y$$

to study the dynamics of $f$.

## 2. Preliminaries

In this section, we recall some background on correspondences and dynamical degrees, as well as Weil's Riemann hypothesis.

### 2.1. A brief summary on correspondences

Let $\mathbb{K}$ be a field and $X, Y$ irreducible (not necessarily smooth or projective) varieties.

A correspondence $f: X \to Y$ is given by an algebraic cycle

$$\Gamma_f = \sum_{i=1}^{m} \Gamma_i$$

on $X \times Y$, where $m$ is a positive integer and $\Gamma_i \subset X \times Y$ are irreducible subvarieties of dimension exactly $\dim(X)$. We do not assume that $\Gamma_i$ are distinct, and hence may write the above sum as $\sum_j a_j \Gamma_j$, where $\Gamma_j$ are distinct and $a_j$ are positive integers. We will call $\Gamma_f$ the graph of $f$, by abusing the usual notation when $f$ is a rational map.

If $f$ is a correspondence and $a \in \mathbb{N}$, we denote by $af$ the correspondence whose graph is $a\Gamma_f$. In other words, if $\Gamma_f = \sum_i \Gamma_i$, then $\Gamma_{af} = \sum_i a\Gamma_i$. If $\Gamma_f = a\Gamma$, where $\Gamma$ is irreducible and $a \in \mathbb{N}$, we say that the correspondence $f$ is irreducible. A rational map $f$ is an irreducible correspondence, since its graph is irreducible.

A correspondence is dominant if for each $i$ in the sum, the two natural projections from $\Gamma_i$ to $X, Y$ are dominant. Dominant correspondences can be composed and the resulting correspondence is also dominant. This can be done as follows. Let $\Gamma: X \to Y$ and $\Gamma': Y \to Z$ be dominant correspondences, which we can assume to be irreducible (in the general case we can use linearity to define). Let

$$\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3: X \times Y \times Z \to Y \times Z, X \times Z, X \times Y$$

be the canonical projections. There are non-empty Zariski open sets $\Gamma_0 \subset \Gamma$ and $\Gamma'_0 \subset \Gamma'$ such that the canonical projection maps $\Gamma_0 \to X, Y$ and $\Gamma'_0 \to Y, Z$ are flat. Then we define

$$\Gamma' \circ \Gamma: X \to Z$$

as the Zariski closure in $X \times Z$ of the algebraic cycle $\hat{\pi}_{2*}(\hat{\pi}_3^{-1}(\Gamma_0) \cap \hat{\pi}_1^{-1}(\Gamma'_0))$. In case $\Gamma$ and $\Gamma'$ are the graphs of two dominant rational maps $f$ and $f'$, then $\Gamma' \circ \Gamma$ as defined above is the same as the graph of the dominant rational map $f' \circ f$.

Given two dominant correspondences $f: X \to X$ and $g: Y \to Y$, we say that they are semi-conjugate if there is a dominant rational *map* $\pi: X \to Y$ such that $\pi \circ f = g \circ \pi$. We will simply write $\pi: (X, f) \to (Y, g)$ to mean that $\pi$ is a dominant rational map semi-conjugating $(X, f)$ and $(Y, g)$.

Let $\pi: X \to Y$ be a dominant *regular morphism* with *finite* fibres, $f: X \to X$ and $g: Y \to Y$ dominant correspondences. We define $\pi^*(g)$ to be the correspondence on $X$ whose graph is $(\pi \times \pi)^*(\Gamma_g)$, and define $\pi_*(f)$ to be the correspondence on $Y$ whose graph is $(\pi \times \pi)_*(\Gamma_f)$.

**Remarks.** If $f: X \to X$ is a correspondence, and X is smooth projective, then we can define pullback and pushforward of algebraic cycles and cohomology classes in the following way. Let $\mathrm{pr}_1, \mathrm{pr}_2: X \times X \to X$ be the projections (recall that they are both proper and smooth, given the assumption on $X$). Then for an algebraic cycle $\alpha$:

$$f^*(\alpha) := (\mathrm{pr}_1)_*[\mathrm{pr}_2^*(\alpha).\Gamma_f],$$
$$f_*(\alpha) := (\mathrm{pr}_2)_*[\mathrm{pr}_1^*(\alpha).\Gamma_f].$$

Note that (in contrast to a more common use of correspondences in Algebraic Geometry), the definition of compositions of dominant correspondences in this paper is modelled after that of the compositions of rational maps. Therefore, in general we have $(f^2)^* \neq (f^*)^2$, and so on. This phenomenon of non-compatibility between pullback and iteration was first studied on projective spaces in [19], under the name of algebraic instability. One simple example is that of the standard Cremona map $f: \mathbb{P}^2 \to \mathbb{P}^2$ given by the formula

$$f[x_0, x_1, x_2] = [x_1x_2 : x_2x_0 : x_0x_1].$$

Then it can be easily computed that $f \circ f = \mathrm{id}$, and hence $(f \circ f)^* =$ the multiplicity by 1 on cohomology classes. On the other hand, $f^*$ is the multiplicity by 2 on $H^2(\mathbb{P}^2)$, and hence $f^* \circ f^*$ is the multiplicity by 4 on $H^2(\mathbb{P}^2)$.

## 2.2. Relative dynamical degrees on complex projective varieties and compact Kähler manifolds

One of the main advantages when working in dynamics over the complex field $\mathbb{C}$ is the existence of positive closed forms and currents, and consequently a positivity notion for cohomological classes.

We recall that a meromorphic map between two complex manifolds $X$ and $Y$, written $f: X \dashrightarrow Y$, is a holomorphic map $f: X \backslash I(f) \to Y$, where $I(f)$ is a proper analytic subvariety of $X$. If the image of $f$ is dense in $Y$ we say that the map is dominant. If $X = Y$ we say that the map is a selfmap. One important tool in Complex Dynamics is dynamical degrees for dominant meromorphic selfmaps. They are bimeromorphic invariants of a meromorphic selfmap $f: X \to X$ of a compact Kähler manifold $X$. The $p$th dynamical degree $\lambda_p(f)$ is the exponential growth rate of the spectral radii of the pullbacks $(f^n)^*$ on the Dolbeault cohomology group $H^{p,p}(X)$.

For a surjective holomorphic map $f$, the dynamical degree $\lambda_p(f)$ is simply the spectral radius $\mathrm{sp}(f^*|_{H^{p,p}(X)})$ of $f^*\colon H^{p,p}(X) \to H^{p,p}(X)$. Recall that for a linear map $L$ on a complex vector space, the spectral radius $\mathrm{sp}(L)$ is the maximum of the absolute values of eigenvalues of $L$. Fundamental results of Gromov [21] and Yomdin [42] expressed the topological entropy of a surjective holomorphic map in terms of its dynamical degrees

$$h_{\mathrm{top}}(f) = \log \max_{0 \le p \le \dim(X)} \lambda_p(f).$$

Since then, dynamical degrees have played a more and more important role in dynamics of meromorphic maps. In many results and conjectures in Complex Dynamics in higher dimensions, dynamical degrees play a central role.

Let $X$ be a compact Kähler manifold of dimension $k$ with a Kähler form $\omega_X$, and let $f\colon X \to X$ be a dominant meromorphic map. For $0 \le p \le k$, the $p$th dynamical degree $\lambda_p(f)$ of $f$ is defined as follows (we use the same notation as before, because indeed if $X$ is a complex projective manifold, then this dynamical degree is the same as defined before using algebraic cycles via equation (1.1), since the cohomological class of a very ample divisor represents a Kähler form)

$$\lambda_p(f) = \lim_{n\to\infty} \left( \int_X (f^n)^*(\omega_X^p) \wedge \omega_X^{k-p} \right)^{1/n}. \tag{2.1}$$

The existence of the limit in the above expression is non-trivial and has been proven by Russakovskii and Shiffman [33] when $X = \mathbb{P}^k$, and by Dinh and Sibony [13, 14] when $X$ is compact Kähler. Both of these results use regularisation of positive closed currents. The limit in (2.1) is important in showing that dynamical degrees are birational invariants. The dynamical degrees satisfy the log-concavity

$$\lambda_i(f)\lambda_{i+2}(f) \le \lambda_{i+1}(f)^2$$

for all $i = 0, \ldots, \dim(X)$. This is a consequence of the mixed Hodge-Riemann theorem. For a dominant meromorphic map, $\lambda_0(f) = 1$, while $\lambda_k(f)$ is its topological degree (i.e. the number of preimages under $f$ of a generic point). The first dynamical degree $\lambda_1(f)$ was used earlier to study Green currents in complex dynamics (first introduced by N. Sibony), see e.g. [1] for surfaces and [19] for higher dimensions. In recent work [6], it is shown that these dynamical degrees can be calculated as the spectral radius of some linear operator on an infinitely-dimensional Banach space constructed from divisors on all birational models of the concerned variety $X$.

For meromorphic maps of compact Kähler manifolds with invariant fibrations, a more general notion called relative dynamical degrees has been defined by Dinh and Nguyen in [11]. (Here, by a fibration we simply mean a dominant rational map, without any additional requirements.) The "product formulas" (see the next subsection) provide a very useful tool to check whether a meromorphic map is primitive (i.e. has no invariant fibrations over a base which is of smaller dimension and not a point, see [43]). In another direction, when $\mathbb{K} = \mathbb{C}$, Dinh and Sibony [15] defined dynamical degrees and topological entropy for meromorphic correspondences over irreducible varieties (the definition of correspondences in the analytic setting is similar to that in the algebraic setting, the only difference is one uses analytic varieties instead). For any dominant correspondence $f$ (in the analytic category) of a compact Kähler manifold, the following Gromov–Dinh–Sibony's inequality holds:

$$h_{\text{top}}(f) \leq \log \max_{0 \leq p \leq \dim(X)} \lambda_p(f).$$

Computations of dynamical degrees of so-called Hurwitz correspondences of the moduli spaces $\mathcal{M}_{0,N}$ were given in [32], wherein a proof that dynamical degrees of correspondences (over $\mathbb{K} = \mathbb{C}$, and for irreducible varieties) are birational invariants was also given.

**2.2.1. Product formula.** Let $f: X \to X$ and $g: Y \to Y$ be dominant rational maps, where $X$ and $Y$ are smooth complex projective varieties. Assume also that there is a dominant rational map $\pi: X \to Y$ so that $\pi \circ f = g \circ \pi$. Dinh and Nguyen [11] defined relative dynamical degrees $\lambda_i(f|\pi)$ for $i = 0, \ldots, \dim(X) - \dim(Y)$, which are birational invariants. In case $Y =$ a point, these relative dynamical degrees are the same as the dynamical degrees mentioned above. Moreover, they also defined relative dynamical degrees in the Kähler setting. Roughly speaking, the idea is to consider not the growth of the pullback map on the whole cohomology group $H^{2i}(X)$, but the growth relative to the fibres of the map $\pi$. The actual definition is quite involved so we refer the readers to the cited papers, and discuss in the following only some special cases which are enough for the applications in this paper. They proved the following result in the algebraic setting.

**Product formula.** For all $p = 0, \ldots, \dim(X)$, we have

$$\lambda_p(f) = \max_{\substack{0 \leq i \leq \dim(Y), \\ 0 \leq p-i \leq \dim(X)-\dim(Y)}} \lambda_i(g)\lambda_{p-i}(f|\pi).$$

The product formula in the Kähler setting for meromorphic maps was proven in [12]. For meromorphic maps, the definition of the relative dynamical degrees is rather more involved than for rational maps, so we refer the readers to the cited papers. We describe how to compute them in the following three special cases, which are enough for applications in this paper. (Here, the results are valid in both algebraic and analytic categories.)

*Case 1.* $X = Y \times Z$, $f = g \times h$ is a product map, and $\pi \colon X = Y \times Z \to Y$ is the projection onto $Y$. In this case,

$$\lambda_j(f|\pi) = \lambda_j(h)$$

for all $j$. Proving the product formula in this case is, via the Kunneth's formula, reduced to simple properties of the eigenvalues of a tensor product of linear maps.

*Case 2.* Assume that $y^0 \in Y$ is a "good" periodic point of order $m$ of $g$ (meaning that it is a periodic point of $g$, and it lies outside a proper Zariski closed set explicitly constructed in terms of $g$ and $m$, see [11] for detail). Then

$$\lambda_j(f|\pi) = \lambda_j\left(f^m|_{\pi^{-1}(y^0)}\right)^{1/m}.$$

This explains the use of the notation and also the intuitive meaning that relative dynamical degrees are the dynamical degrees of the restriction of $f$ on the fibres of $\pi$.

*Case 3.* $\dim(X) = \dim(Y)$ (equi-dimensional). In this case $\lambda_j(f) = \lambda_j(g)$ for all $j$, and the only relative dynamical degree is $\lambda_0(f|\pi) = 1$. This follows from the fact that

$$\lambda_0(f) = \lambda_0(g)\lambda_0(f|\pi)$$

(by the product formula) and

$$\lambda_0(f) = \lambda_0(g) = 1$$

(property of dynamical degrees of rational maps).

**2.2.2. Dinh's inequality.** In the analytic setting, we define

$$\chi_i(f) := \limsup_{n\to\infty} \|(f^n)^*|_{H^i(X,\mathbb{C})}\|^{1/n}.$$

Here, we can choose any norm on the finite-dimensional vector space $H^i(X, \mathbb{C})$. From the results mentioned above (i.e., (2.1)),

$$\chi_{2i}(f) = \lambda_i(f),$$

and in this case lim sup can be replaced by lim. However, when $i$ is an odd number, we do not know whether lim sup in the definition of $\chi_i(f)$ can be replaced by lim. Dinh [10] showed the following inequality, by using weakly positive closed smooth forms

$$\chi_i(f)^2 \le \max_{p+q=i} \lambda_p(f)\lambda_q(f).$$

### 2.3. Relative dynamical degrees in positive characteristics

One main difficulty in extending the results in the previous section to the positive characteristic case is that there is not yet a suitable notion of positivity on $l$-adic cohomology groups.

Recently, research on birational maps of surfaces over an algebraically closed field of arbitrary characteristic has been increased significantly. As some examples, we refer the reader to [2, 17, 18, 30, 41]. In these results, (relative) dynamical degrees also play an important role.

In the case of positive characteristic, positivity notions are not yet available on $l$-adic cohomology groups. This lets open the question of how to define cohomological dynamical degrees in the case of positive characteristic. In contrast, in [39, Theorem 1.1 (1)] we established the formula (1.1), for every correspondence $f$. While not used in this paper, we note that the definition can also be adapted to the case where $X$ is singular or not irreducible, by using de Jong's alterations and pullbacks of correspondences by equi-dimensional dominant rational maps, see [39]. Hence it is justified to call these the geometric dynamical degrees. These geometric dynamical degrees are again birational invariants ([39, Theorem 1.1 (2)]). The "product formula" is also proven in [39, Theorem 1.1 (4)] in the setting of correspondences. For some possible applications of these to topological entropy, in particular the Gromov–Yomdin's theorem, see [40]. (After sending out an earlier version of [39], we were informed by Charles Favre that Nguyen-Bac Dang had been developing an alternative approach for (relative) geometric dynamical degrees of rational maps on normal projective varieties, see [5].)

For a *regular morphism* $f$, one can check that $\chi_i(f) =$ the spectral radius of the linear map $f^*\colon H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)$. Here we use any fixed embedding of $\mathbb{Q}_l$ into $\mathbb{C}$. This is because if $f$ is a regular morphism then for the pullback on cohomology we have $(f^n)^* = (f^*)^n$ for all $n$, and hence by basic results in Linear Algebra, we have

$$\lim_{n\to\infty} \|(f^n)^*|_{H^i(X)}\|^{1/n} = \lim_{n\to\infty} \|(f^*)^n|_{H^i(X)}\|^{1/n}$$

is exactly the spectral radius of $f^*|_{H^i(X)}$.

**Remark.** As a consequence of the Riemann hypothesis for positive characteristic, which was the last and crucial part of Weil's conjectures and proven by Deligne, this $\chi_i(f)$ is independent of the embedding of $\mathbb{Q}_l$. However, we will not assume this in what follows.

Computing the $\chi_i(f)$ for the $l$-adic cohomology, even on surfaces, is quite a challenging task in practice. In contrast, as mentioned above, the geometric dynamical degrees $\lambda_i(f)$ have some good functorial properties which make computations easier. For example (see Section 4 for more details), computing the geometric dynamical degrees of the Frobenius map on any smooth projective variety $X$ can be done by applying the product formula to a dominant regular morphism $\pi: X \to \mathbb{P}^k$ with finite fibres, utilising the fact that the dynamical degrees of a regular morphism of $\mathbb{P}^k$ are very easy to describe.

For a general correspondence, taking the clues from the case $\mathbb{K} = \mathbb{C}$, we may proceed as follows. Let $\chi_i(f)$ be defined as before. We may call the number

$$\log \chi(f) := \log \max_{i=0,\dots,2\dim(X)} \chi_i(f),$$

the algebraic entropy.

Note that we always have $\chi_{2i}(f) \geq \lambda_i(f)$, but the finiteness of the above numbers $\chi_i(f)$ is not obvious. The main reason is that we have the cycle map assigning a cohomology class to algebraic cycles, which is compatible with intersection products and pullback by maps or correspondences, see [39] for detail. We expect that known results for relative dynamical degrees on $\mathbb{K} = \mathbb{C}$ should be carried out to an arbitrary field.

### 2.4. Weil's Riemann hypothesis

Weil's Riemann hypothesis is the most difficult part of the well-known Weil's conjectures, see [22]. For the convenience of the readers, we first recall some backgrounds about Weil's Riemann hypothesis. Let $X_0$ be a smooth projective variety defined over a finite field $\mathbb{F}_q$. Let $X$ be the base change of $X_0$ to an algebraic closure $\overline{\mathbb{F}}_q$. Let $\mathrm{Fr}_X: X \to X$ be the Frobenius morphism. A simple expression of it is as follows. On a projective space $\mathbb{P}^N$,

$$\mathrm{Fr}[x_0 : \cdots : x_N] = [x_0^q : \cdots : x_N^q].$$

If $X \subset \mathbb{P}^N$, then $\mathrm{Fr}_X$ is simply the restriction of $Fr$ to $X$. Weil's Riemann hypothesis is then the following statement. It was solved by Pierre Deligne [7, 8] in the 1970s.

**Weil's Riemman hypothesis.** *If $\alpha$ is an eigenvalue of $\mathrm{Fr}^*: H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)$, then $|\alpha| = q^{i/2}$.*

Serre [34] proved the following result for polarized endomorphisms of compact Kähler manifolds.

**Theorem 2.1.** *Let $X$ be a compact Kähler manifold, and $f: X \to X$ an endomorphism. Assume that $f$ is polarized, that is there is a Kähler class $\omega \in H^2(X, \mathbb{C})$ and a positive integer $q$ so that $f^*(\omega) = q\omega$. Then if $\alpha$ is an eigenvalue of $f^*: H^i(X, \mathbb{C}) \to H^i(X, \mathbb{C})$, then $|\alpha| = q^{i/2}$. Here $H^*(X, \mathbb{C})$ is the usual singular cohomology with coefficients in $\mathbb{C}$.*

If the positive characteristic analogue of Theorem 2.1 holds, then Weil's Riemann hypothesis follows since Frobenius map is a polarized endomorphism. This positive characteristic analogue inspired Grothendieck and Bombieri to propose the standard conjectures [22], however it is still open even for surfaces.

## 3. Proofs of Theorems 1.1, 1.2 and 1.4

**Convention.** Strictly speaking, for the arguments below to be extremely rigorous, we need to use the Tate twists of the $l$-adic cohomology groups in various places. For example, a subvariety of codimension $c$ of $X$ has cohomology class in $H^{2c}(X,\mathbb{Q}_l(c))$. Similarly, we also need to use a twist in the Poincaré duality. However, since

$$H^{2c}(X,\mathbb{Q}_l(c)) = H^{2c}(X,\mathbb{Q}_l) \otimes \mathbb{Q}_l(c)$$

and $\mathbb{Q}_l(c)$ is a 1-dimensional $\mathbb{Q}_l$ vector space, the computations and estimates on $H^{2c}(X,\mathbb{Q}_l(c))$ and $H^{2c}(X,\mathbb{Q}_l)$ are almost identical. For simplicity, the symbols for the twists are suppressed. (See also [28, Remark 25.5].)

Let $Z$ be a smooth projective variety of *even* dimension $2k'$. Assume that $NH(Z)$ holds. We then construct a useful decomposition on $H^{2k'}(Z,\mathbb{Q}_l)$. By Poincaré duality, the intersection product

$$(\,,\,)\colon H^{2k'}(Z,\mathbb{Q}_l) \times H^{2k'}(Z,\mathbb{Q}_l) \to \mathbb{Q}_l,$$

is symmetric and non-degenerate. Under the assumption that $NH(Z)$ holds, we will prove that there is a decomposition:

$$H^{2k'}(Z,\mathbb{Q}_l) = H^{2k'}_{\mathrm{alg}}(Z,\mathbb{Q}_l) \oplus H^{2k'}_{\mathrm{tr}}(Z,\mathbb{Q}_l).$$

Here $H^{2k'}_{\mathrm{alg}}(Z,\mathbb{Q}_l)$ (the algebraic part) is the $\mathbb{Q}_l$-vector subspace generated by the images of algebraic cycles (under the cycle map) in $H^{2k'}(Z,\mathbb{Q}_l)$; and $H^{2k'}_{\mathrm{tr}}(Z,\mathbb{Q}_l)$ (the transcendental part) is the orthogonal complement of $H^{2k'}_{\mathrm{alg}}(Z,\mathbb{Q}_l)$ under the intersection product. In other words,

$$H^{2k'}_{\mathrm{tr}}(Z,\mathbb{Q}_l) := \{\alpha \in H^{2k'}(Z,\mathbb{Q}_l) :\ \alpha.\beta = 0,\ \forall \beta \in H^{2k'}_{\mathrm{alg}}(Z,\mathbb{Q}_l)\}.$$

This decomposition is based on the following lemma, whose proof is standard and hence is skipped.

**Lemma 3.1.** *Assume that condition NH(Z) holds. Then the intersection product*

$$H^{2k'}_{\mathrm{alg}}(Z,\mathbb{Q}_l) \times H^{2k'}_{\mathrm{alg}}(Z,\mathbb{Q}_l) \to \mathbb{Q}_l$$

*is non-degenerate and symmetric.*

We let $\alpha_1, \dots, \alpha_m$ be an orthogonal basis for $H^{2k'}_{\mathrm{alg}}(X, \mathbb{Q}_l)$, with respect to the cup product (which always exists, since the characteristic of $\mathbb{Q}_l$ is 0, and the cup product is symmetric). The non-degeneracy of cup product (Lemma 3.1) implies that $\alpha_i.\alpha_i \neq 0$ for all $i$. If $x \in H^{2k'}(Z, \mathbb{Q}_l)$, we define

$$x' = \sum_{i=1}^{m} \frac{x.\alpha_i}{\alpha_i.\alpha_i}\alpha_i, \quad x'' = x - x'.$$

Then it is easy to check that $x' \in H^{2k'}_{\mathrm{alg}}(Z, \mathbb{Q}_l)$, $x'' \in H^{2k'}_{\mathrm{tr}}(Z, \mathbb{Q}_l)$ and $x = x' + x''$. Moreover, this decomposition of $x$ is unique. Hence, we have the desired decomposition. We denote by $\tau\colon H^{2k'}(Z, \mathbb{Q}_l) \to H^{2k'}_{\mathrm{alg}}(Z, \mathbb{Q}_l)$ the projection to the algebraic part.

We also present another preliminary result before the proofs of the main results. Assume that $f\colon X \to X$ is a correspondence. Let $\alpha_1, \dots, \alpha_m$ be a basis for $H^i(X, \mathbb{Q}_l)$ and $\beta_1, \dots, \beta_m$ be a basis for $H^{2\dim(X)-i}(X, \mathbb{Q}_l)$. Fix arbitrary norms on $H^i(X, \mathbb{Q}_l)$ and $H^{2\dim(X)-i}(X, \mathbb{Q}_l)$, and an embedding of $\mathbb{Q}_l$ into $\mathbb{C}$. We let $|.|$ be the induced absolute value on $\mathbb{Q}_l$. Then there are positive constants $C_1, C_2 > 0$, independent of $f$, such that

$$C_1 \sum_{p,q=1,\dots,m} |f^*(\alpha_p).\beta_q| \leq \|f^*|_{H^i(X,\mathbb{Q}_l)}\| \leq C_2 \sum_{p,q=1,\dots,m} |f^*(\alpha_p).\beta_q|. \tag{3.1}$$

Indeed, since all norms on a finite-dimensional vector space over a field of characteristic 0 (more precisely, the concerned field is $\mathbb{Q}_l$) are equivalent, see [16, Section IV.3.1, Lemma 1], we can work with special norms which are defined next. Since the intersection product

$$H^i(X, \mathbb{Q}_l) \times H^{2\dim(X)-i}(X, \mathbb{Q}_l) \to \mathbb{Q}_l$$

is non-degenerate, the following is a norm on $H^i(X, \mathbb{Q}_l)$: if $\alpha \in H^i(X, \mathbb{Q}_l)$, then

$$\|\alpha\| := \sum_{q=1}^{m} |\alpha.\beta_q|.$$

By definition of the operator norm, we then have

$$\|f^*|_{H^i(X,\mathbb{Q}_l)}\| := \sup_{\|\alpha\|=1} \|f^*(\alpha)\| = \sup_{\|\alpha\|=1} \sum_{q=1}^{m} |f^*(\alpha).\beta_q|.$$

The left-hand side inequality of (3.1) is obvious if we choose

$$\frac{1}{C_1} = \sum_{p=1}^{m} \|\alpha_p\|.$$

The right-hand side inequality of (3.1) follows provided that if

$$\alpha = \sum_{p=1,\dots,m} x_p\alpha_p \quad \text{and} \quad \|\alpha\| \leq 1,$$

then $\max_{p=1,\dots,m} |x_p| \leq C$ for some positive constant $C$. The latter claim is a simple consequence of the fact that on a finite-dimensional vector space, any two norms are equivalent. We then apply this fact to two norms. The first is the norm $\|.\|$ chosen above. The second is the one

$$\|\alpha\|' := \sum_{p=1}^{m} |x_p|.$$

Now we are ready for the proofs of the results.

*Proof of Theorem* 1.1. (1) Fix an integer $i$ between $0, \dots, 2\dim(X)$. It is sufficient to prove that given $\lambda > \max_{i=0,\dots,\dim(X)} \lambda_i(f)$, then

$$\lim_{n\to\infty} \|(f^n)^*|_{H^i(X,\mathbb{Q}_l)}\|/\lambda^n = 0.$$

Let $\alpha_1, \dots, \alpha_m$ be a basis for $H^i(X, \mathbb{Q}_l)$ and let $\beta_1, \dots, \beta_m$ be a basis for $H^{2\dim(X)-i}(X, \mathbb{Q}_l)$. Then by (3.1), we have

$$\|(f^n)^*|_{H^i(X,\mathbb{Q}_l)}\| \leq C \sum_{p,q=1,\dots,m} |(f^n)^*(\alpha_p).\beta_q|,$$

where $C$ is independent of $n$.

Hence, it is enough to show that for any $p, q = 1, \dots, m$,

$$\lim_{n\to\infty} |(f^n)^*(\alpha_p).\beta_q|/\lambda^n = 0. \tag{3.2}$$

Let $\mathrm{pr}_1, \mathrm{pr}_2 \colon X \times X \to X \times X$ be the two projections. Then, under the assumption that $NH(X \times X)$ holds, we have

$$|(f^n)^*(\alpha_p).\beta_q| = |\Gamma_{f^n}.\,\mathrm{pr}_2^*(\alpha_p).\,\mathrm{pr}_1^*(\beta_q)| = |\Gamma_{f^n}.\tau(\mathrm{pr}_2^*(\alpha_p).\,\mathrm{pr}_1^*(\beta_q))|.$$

Since $\tau(\mathrm{pr}_2^*(\alpha_p).\,\mathrm{pr}_1^*(\beta_q))$ is represented by an algebraic cycle of dimension $k$, it then follows from the results in [39] that we have the desired result in (3.2). More precisely, the results we used here are the following. First ([39, Lemma 2.2]), for any effective algebraic cycle $V$ of codimension $k$ on $X \times X$,

$$|\Gamma_{f^n}.V| \leq C \deg(\Gamma_{f^n}) \deg(V),$$

where deg(.) is the degree of an algebraic cycle in a fixed embedding of $X \times X$ into a projective space, and $C > 0$ is a positive constant independent of $n$, $f$ and $V$.

Second, let $\mathcal{L}$ be an ample divisor on $X$. Then $\mathrm{pr}_1^*(\mathcal{L}) + \mathrm{pr}_2^*(\mathcal{L})$ is an ample divisor on $X$. The number $\deg(\Gamma_{f^n})$ can be computed against $\mathrm{pr}_1^*(\mathcal{L}) + \mathrm{pr}_2^*(\mathcal{L})$, and hence is

$$\begin{aligned}\deg(\Gamma_{f^n}) &= \Gamma_{f^n}.\big(\mathrm{pr}_1^*(\mathcal{L}) + \mathrm{pr}_2^*(\mathcal{L})\big)^{\dim(X)}\\ &= \sum_{i=0}^{\dim(X)} \Gamma_{f^n}.\,\mathrm{pr}_1^*(\mathcal{L}^i).\,\mathrm{pr}_2^*(\mathcal{L}^{\dim(X)-i})\\ &= \sum_{i=0}^{\dim(X)} \Gamma_{f^n}.(\mathrm{pr}_2)_*\,\mathrm{pr}_1^*(\mathcal{L}^i).\mathcal{L}^{\dim(X)-i}\\ &= \sum_{i=0}^{\dim(X)} (f^n)^*(\mathcal{L}^i).\mathcal{L}^{\dim(X)-i}.\end{aligned}$$

By (1.1), for each $i$, we have

$$\lim_{n\to\infty} \big((f^n)^*(\mathcal{L}^i).\mathcal{L}^{\dim(X)-i}\big)^{1/n} = \lambda_i(f).$$

It easily follows that

$$\lim_{n\to\infty} \deg(\Gamma_{f^n})^{1/n} = \max_{i=0,\dots,\dim(X)} \lambda_i(f). \tag{3.3}$$

(2) The proof is similar, but here we need to use Weil's Riemann hypothesis. Let $\lambda > \max_{i=0,\dots,\dim(X)} \lambda_i(f)$. Since $f$ is a regular morphism, we obtain

$$\chi_i(f) = \mathrm{sp}(f^*|_{H^i(X,\mathbb{Q}_l)}),$$

where $\mathrm{sp}(f^*|_{H^i(X,\mathbb{Q}_l)})$ is the largest absolute value of the eigenvalues of

$$f^*\colon H^i(X,\mathbb{Q}_l) \to H^i(X,\mathbb{Q}_l).$$

(Here again the absolute value is induced from the given embedding of $\mathbb{Q}_l$ into $\mathbb{C}$.)

It suffices to consider the case where $X$ has positive characteristic $p$. Then, we may assume that $X$ and $f$ are defined on some finite field $\mathbb{F}_q$. We recall briefly this well-known "spreading out" and "specialization" argument. By [31, Theorem 3.2.1], there is (by collecting the coefficients in the defining equations for $X$ and $f$), a subring $R$ of $\mathbb{K}$, finitely generated over $\mathbb{F}_q$, so that $X$ is the generic fibre of a smooth projective scheme $\mathcal{X}$ over $\mathrm{Spec}(R)$ and $f$ is the generic fibre of a morphism $F\colon \mathcal{X} \to \mathcal{X}$ over $\mathrm{Spec}(R)$. Let $X_0$ be a special fibre defined over a finite field $\mathbb{F}_q$, and $f_0 = F|_{X_0}$. Define $\widetilde{X}_0$ and $\widetilde{f}_0$ to be the base change of $X_0$ and $f_0$ to the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. The proper-smooth base change theorem ([28, Chapter 20]) then implies that

$$\mathrm{Tr}\big[(f^n)^*\colon H^i(X,\mathbb{Q}_l) \to H^i(X,\mathbb{Q}_l)\big] = \mathrm{Tr}\big[(\widetilde{f}_0^n)^*\colon H^i(\widetilde{X}_0,\mathbb{Q}_l) \to H^i(\widetilde{X}_0,\mathbb{Q}_l)\big]$$

for all $n$. While $\widetilde{X}_0$ may have more algebraic cycles than $X$, the specialisation of algebraic cycles ([20, Chapter 20]) and (1.1) imply that geometric dynamical degrees are lower-semicontinuous, and hence $\lambda_i(f) \geq \lambda_i(\widetilde{f}_0)$ for all $i$. Indeed, let $L$ be a very ample divisor on $X$ whose specialisation to $\widetilde{X}_0$ is denoted by $L_0$. Fix a number $0 \leq i \leq \dim(X)$. Then $(f^n)^*(L^i)$ is an intersection of nef divisors. If $(f^n)^*(L^i)$ specializes to $Z_{i,n}$, then $Z_{i,n} - (\widetilde{f}_0^n)^*(L_0^i)$ is a psef class. Since specialisation preserves intersection product of algebraic cycles, we then have

$$\begin{aligned}
\lambda_i(f) &= \lim_{n\to\infty} \big((f^n)^*(L^i).L^{\dim(X)-i}\big)^{1/n} \\
&= \lim_{n\to\infty} \big(Z_{i,n}.L_0^{\dim(X)-i}\big)^{1/n} \\
&\geq \lim_{n\to\infty} \big((\widetilde{f}_0^n)^*(L_0^i).L_0^{\dim(X)-i}\big)^{1/n} = \lambda_i(\widetilde{f}_0).
\end{aligned}$$

Therefore, if we can prove the conclusion for $\widetilde{f}_0\colon \widetilde{X}_0 \to \widetilde{X}_0$, then the conclusion for $f\colon X \to X$ follows. Thus from now on we assume that $X$ is defined over a finite field.

Let $\widetilde{\mathrm{Fr}}\colon X \times X \to X \times X$ be the map $(x, y) \mapsto (x, \mathrm{Fr}(y))$, where $\mathrm{Fr}\colon X \to X$ is the Frobenius map. As a consequence of Deligne's proof of Weil's Riemann hypothesis, there is a polynomial $p_i(\widetilde{\mathrm{Fr}})$, so that we have the generalized Lefschetz Trace Formula:

$$\Gamma_{f^n}.\big[p_i(\widetilde{\mathrm{Fr}})^*\Delta\big] = (-1)^i \operatorname{Tr}\big[(f^n)^*\colon H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)\big]. \tag{3.4}$$

For more detail see [26, Theorem 2 (1)].

By a cohomological endomorphism, we mean a linear operator from $H^i(X, \mathbb{Q}_l)$ to itself, where $i = 0, \dots, \dim(X)$. If $f$ is a correspondence on $X$, then its pullback on cohomology is a cohomological endomorphism. Note that $p_i(\mathrm{Fr}^*) = p_i(\mathrm{Fr})^*$. Then, the Lefschetz Trace Formula (see e.g. [29, Theorem 2.1]) can be applied to the *cohomological* correspondence (in the usual sense, i.e. a linear map between cohomological groups, recall that if $f$ is a correspondence then $f^*$ is its pullback on cohomological groups, and hence is a linear map between cohomological groups) $(f^n)^* \circ p_i(\mathrm{Fr})^* = (p_i(\mathrm{Fr}) \circ f^n)^*$:

$$\big[p_i(\widetilde{\mathrm{Fr}})_*\Gamma_{f^n}\big].\Delta = (\Gamma_{p_i(\mathrm{Fr})\circ f^n}).\Delta = (-1)^i \operatorname{Tr}\big[(f^n)^* : H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)\big].$$

Then (3.4) is obtained by observing that by the projection formula:

$$\big[p_i(\widetilde{\mathrm{Fr}})_*\Gamma_{f^n}\big].\Delta = \Gamma_{f^n}.\big[p_i(\widetilde{\mathrm{Fr}})^*\Delta\big].$$

Since the class of $p_i(\widetilde{\mathrm{Fr}})^*\Delta$ is an algebraic cycle (with rational coefficients), the proof is completed by observing that, similarly to part (1),

$$\limsup_{n\to\infty} \big|\Gamma_{f^n}.\big[p_i(\widetilde{\mathrm{Fr}})^*\Delta\big]\big|/\lambda^n = 0,$$

and that (using $(f^n)^* = (f^*)^n$ for a regular morphism $f$)

$$\limsup_{n\to\infty} \big|\mathrm{Tr}[(f^n)^*\colon H^i(X,\mathbb{Q}_l) \to H^i(X,\mathbb{Q}_l)]\big|^{1/n} = \mathrm{sp}(f^*|_{H^i(X,\mathbb{Q}_l)}).$$

The last (elementary) equality can be deduced from the following simple claim, which we leave to the readers to verify.

*Claim.* Let $\mu_1, \ldots, \mu_m$ be complex numbers with $|\mu_1| = \cdots = |\mu_m| = 1$. For any $\varepsilon > 0$, there exist infinitely many values of positive integers $k$ such that $|\mu_i^k - 1| < \varepsilon$, and in particular $\Re(\mu_j^k) > 1 - \varepsilon$ for all $j$. ■

**Remarks 3.2.** In the original proof of part (2) above, we used the usual Lefschetz Trace Formula. Then similarly we can bound the alternating sum

$$\sum_{i=0}^{2\dim(X)} (-1)^i \,\mathrm{Tr}\big[(f^n)^*\colon H^i(X,\mathbb{Q}_l) \to H^i(X,\mathbb{Q}_l)\big],$$

in terms of the geometric dynamical degrees $\lambda_i(f)$. However, there may be some cancellations in the alternating sum of the traces which do not quite give us the inequality we need. We thank Peter O'Sullivan for pointing this out and for suggesting the correction which we used here.

There is a subtlety when applying the argument of reduction to finite fields in the proof of Theorem 1.1 (2) to iterations of correspondences (for example, the finite fields may increase when we increase the number of iterates). If $X$ is already defined over a finite field, then such a reduction is not needed. There is still another difficulty arising from the fact that in general we do not have $(f^n)^* = (f^*)^n$, and hence the eigenvalues of $(f^n)^*$ may not be related to those of $f^*$. However, this can be dealt with by a modification of the proof, and this gives us a proof of Theorem 1.2. Below we provide a detailed argument.

*Proof of Theorem 1.2.* Since $\mathbb{K} = \overline{\mathbb{F}}_p$ is the closure of a finite field, $X$ is actually defined over a finite field $\mathbb{F}_q$, where $q$ is a power of $p$. Then we have (see the proof of Theorem 1.1 (2)) that the projections $p_i\colon H^*(X,\mathbb{Q}_l) \to H^i(X,\mathbb{Q}_l)$ are all algebraic, that is $p_i = p_i(\mathrm{Fr}^*)$ for some polynomials in the pullback $\mathrm{Fr}^*$ of the Frobenius $\mathrm{Fr}\colon X \to X$. We let $\widetilde{\mathrm{Fr}}\colon X \times X \to X \times X$ be the map $(x, y) \mapsto (x, F(y))$. Then, by the proof of the Lefschetz trace formula (see e.g. [29, Theorem 2.1]), which works for all cohomological correspondences, for any *generalized* correspondence $\phi\colon X \to X$ (allowing components of the graph to have negative coefficients or not project dominantly to $X$), we have

$$\Gamma_\phi . p_i(\widetilde{\mathrm{Fr}})^*(\Delta) = (-1)^i \,\mathrm{Tr}\big[\phi\colon H^i(X,\mathbb{Q}_l) \to H^i(X,\mathbb{Q}_l)\big].$$

We will use this to prove the following claim.

*Claim.* Suppose that $f\colon X \to X$ is a dominant correspondence. Then, for all $i = 0, \dots, 2\dim(X)$, we have

$$\mathrm{sp}(f^*|_{H^i(X,\mathbb{Q}_l)}) \le C\,\deg(\Gamma_f),$$

where $C > 0$ is independent of $f$.

*Proof of Claim.* For any $n$, we consider the *cohomological* correspondence

$$\phi_n := (f^*)^n\colon H^*(X,\mathbb{Q}_l) \to H^*(X,\mathbb{Q}_l).$$

Since $\phi_1 = f^*$ is algebraic, it follows that all $\phi_n$ are algebraic. That is, we can write $\phi_n = (f_n^+)^* - (f_n^-)^*$, where $f_n^{\pm}$ are effective algebraic cycles on $X \times X$, and so $\phi_n$ are generalized correspondences. (However, note that some components of $f_n^{\pm}$ may not be dominant over $X$ under the projections $\pi_1, \pi_2$, hence $f_n^{\pm}$ may not be correspondences in the sense we use in Section 2.) Moreover, an iterated use of [39, Lemma 2.2] shows that we can arrange to have the estimates

$$\deg(f_n^{\pm}) \le (2C)^n \deg(\Gamma_f)^n,$$

for all $n$. Here $C > 0$ is the constant in [39, Lemma 2.2]. It follows again from this lemma that

$$\begin{aligned}\big|\mathrm{Tr}\big[(f^*)^n\colon H^i(X,\mathbb{Q}_l) \to H^i(X,\mathbb{Q}_l)\big]\big| &= \big|\mathrm{Tr}\big[\phi_n : H^i(X,\mathbb{Q}_l) \to H^i(X,\mathbb{Q}_l)\big]\big| \\ &= \big|(f_n^+ - f_n^-).p_i(\widetilde{\mathrm{Fr}})^*(\Delta)\big| \\ &\le C\big(\deg(f_n^+) + \deg(f_n^-)\big) \\ &\le C(2C)^n \deg(\Gamma_f)^n.\end{aligned}$$

Therefore,

$$\begin{aligned}2C\,\deg(\Gamma_f) &\ge \limsup_{n\to\infty}\big|\mathrm{Tr}\big[(f^*)^n\colon H^i(X,\mathbb{Q}_l) \to H^i(X,\mathbb{Q}_l)\big]\big|^{1/n} \\ &= \mathrm{sp}(f^*|_{H^i(X,\mathbb{Q}_l)}).\end{aligned}$$

Here the constant $2C > 0$ is independent of $f$. Thus the proof of claim is finished. ■

Now we continue the proof of the theorem. If $\chi_i(f) \le 1$, there is nothing to prove. Hence, it is sufficient to consider the case $\chi_i(f) > 1$. Applying the claim to iterates $f^n$ and using (3.3), we have

$$\begin{aligned}\limsup_{n\to\infty}\mathrm{sp}\big((f^n)^*|_{H^i(X,\mathbb{Q}_l)}\big)^{1/n} &\le \limsup_{n\to\infty}\big[C\,\deg(\Gamma_{f^n})\big]^{1/n} \\ &= \max_{p=0,\dots,\dim(X)} \lambda_p(f).\end{aligned}$$ ■

*Proof of Theorem* 1.4. (1) Since we assume that $NH(X \times X)$ holds, part 1) of Theorem 1.1 applies. Hence, we have

$$\max\{\chi_1(f), \chi_2(f), \chi_3(f)\} \leq \max\{\lambda_0(f), \lambda_1(f), \lambda_2(f)\}.$$

The right-hand side in the above inequality is $\lambda_1(f)$ by the other assumption in the theorem. From the obvious inequality $\chi_2(f) \geq \lambda_1(f)$, we obtain the conclusion of the theorem.

(2) Since we assume that $f$ is a regular morphism, Theorem 1.1 (2) applies. Then we argue similarly to part (1). ■

## 4. Dynamical degrees and Weil's Riemann hypothesis

In this section we deduce Weil's Riemann hypothesis from properties on dynamical degrees. More precisely, we provide a proof of Theorem 1.5.

**Convention.** As in Section 3, for simplicity we suppress all the Tate twists in the $l$-adic cohomology groups.

Here are some preliminary reductions of Weil's Riemann hypothesis ([28, Chapter 28]). The first reduction is that it is enough to solve the conjecture for any finite extension of $\mathbb{F}_q$. The second reduction is that, in the statement of the conjecture, it is enough to show that $|\alpha| \leq q^{i/2}$. In [28], the second reduction was proven by showing that if $\alpha$ is an eigenvalue of $\mathrm{Fr}^*$ on $H^i(X, \mathbb{Q}_l)$, then $q^{\dim(X)}/\alpha$ is an eigenvalue of $\mathrm{Fr}^*$ on $H^{2\dim(X)-i}(X, \mathbb{Q}_l)$.

In terms of the cohomological dynamical degrees, the second reduction is equivalent to the statement that $\chi_i(\mathrm{Fr}) \leq q^{i/2}$ for all $i = 0, \ldots, 2\dim(X)$. By another elementary reduction (using product of spaces and maps), we obtain the following.

**Reduction.** Weil's Riemann hypothesis is equivalent to the statement that

$$\chi_{2i}(\mathrm{Fr}) \leq q^i$$

for all $i = 0, \ldots, \dim(X)$.

The proof of Theorem 1.5 follows from the following claims. We provide the proof for the deduction of Weil's Riemann hypothesis only, but the proof for the deduction of the positive characteristic analogue of Serre's result can be obtained similarly.

**Claim 1.** Assume that we have the expected equality $\lambda_i(f) = \chi_{2i}(f)$ holds, for all smooth projective varieties $X$ and regular morphisms $f$ on $X$, and for all $i = 0, \ldots, \dim(X)$. Then Weil's Riemann hypothesis holds.

*Proof.* Applying the expected equality to iterates of the Frobenius map Fr, we find that

$$\chi_{2i}(\mathrm{Fr}) = \lambda_i(\mathrm{Fr}).$$

The $\lambda_i(\mathrm{Fr})$ is easy to compute, and in this case is $q^i$. (For a fancy proof of this fact, we can consider a dominant regular morphism $\pi\colon X \to \mathbb{P}^k$, and apply the product formula in [37, 39] for the special Case 3 recalled in Section 2.2.1. See the proof of Claim 2 below for more details.) By definition of $\chi_{2i}(\mathrm{Fr})$, any eigenvalue of Fr on $H^{2i}(X, \mathbb{Q}_l)$ has the absolute value $\leq q^i$. By the preliminary reductions mentioned above, this is enough to prove Weil's Riemann hypothesis. ∎

By Claim 1, the expected equality $\lambda_i(f) = \chi_{2i}(f)$ (which as noted before, holds in the case $\mathbb{K} = \mathbb{C}$) is a generalisation of Weil's Riemann hypothesis.

**Claim 2.** Assume that the product formula holds for the cohomological dynamical degrees $\chi_{2i}$, and where $f$ and $g$ are both regular morphisms semi-conjugated by a dominant regular morphism with finite fibres $\pi\colon X \to \mathbb{P}^k$. Then Weil's Riemann hypothesis holds.

*Proof.* Let $\dim(X) = k$. There is always a dominant regular morphism

$$\pi\colon X \to Y = \mathbb{P}^k$$

with finite fibres and which is defined on $\mathbb{F}_q$, for example by using Noether's normalisation theorem [36]. The Frobenius maps have the important property that the equality $\pi \circ \mathrm{Fr}_X = \mathrm{Fr}_Y \circ \pi$ is always satisfied. Since $\dim(X) = k = \dim \mathbb{P}^k$, by the assumptions in Claim 2, we have by using the special Case 3 in Section 2.2.1:

$$\chi_{2i}(\mathrm{Fr}_X) = \chi_{2i}(\mathrm{Fr}_Y)\chi_0(\mathrm{Fr}_X|\pi) = q^i. \tag{4.1}$$

This is the conclusion of Weil's Riemann hypothesis. Here we have used that Weil's Riemann hypothesis is true for $Y = \mathbb{P}^k$ (because the cohomology group of $\mathbb{P}^k$ is very simple and is generated by algebraic cycles) and $\chi_0(\mathrm{Fr}_X|\pi) = 1$, see Section 2.2.1 for details. ∎

By Claim 2, the product formula for cohomological dynamical degrees is also a generalisation of Weil's Riemann hypothesis.

**Claim 3.** Assume that we have Dinh's inequality $\chi_i(f)^2 \leq \max_{j+l=i} \lambda_j(f)\lambda_l(f)$ for all $i$ and regular morphisms in positive characteristic. Then Weil's Riemann hypothesis holds.

*Proof.* The proof is similar to those of the above claims, by observing that applying Dinh's inequality to the Frobenius map gives the desired inequality

$$\chi_i(\mathrm{Fr})^2 \leq \max_{j+l=i} \lambda_j(\mathrm{Fr})\lambda_l(\mathrm{Fr}) = \max_{j+l=i} q^j q^l = q^i. \qquad \blacksquare$$

By Claim 3, Dinh's inequality for cohomological dynamical degrees is yet another generalisation of Weil's Riemann hypothesis.

## 5. An approach to Questions 2 and 3

To Questions 2 and 3 in general, we propose to study the following two statements.

**Statement (A).** Let $X$ be a smooth projective variety over $\mathbb{K}$. Let $f_1, f_2 \colon X \to X$ be two correspondences. Assume that $f_1 \geq f_2$, that is there is an effective algebraic cycle $\Gamma$ on $X \times X$ so that $\Gamma_{f_1} = \Gamma_{f_2} + \Gamma$. Then there is a positive constant $C > 0$, independent of the correspondences $f_1$ and $f_2$, such that

$$C \| f_1^*|_{H^{2i}(X,\mathbb{Q}_l)} \| \geq \| f_2^*|_{H^{2i}(X,\mathbb{Q}_l)} \|$$

for all $i = 0, \ldots, \dim(X)$.

**Statement (B).** Let $X$ and $Y = \mathbb{P}^k$ be smooth projective varieties of the same dimension $k$. Let $\pi \colon X \to Y$ be a surjective regular morphism whose all fibres are *finite*. Let $g \colon Y \to Y$ be a correspondence, and let $f \colon X \to X$ be the correspondence whose graph is $\Gamma_f := (\pi \times \pi)^*(\Gamma_g)$ (i.e. the algebraic cycle pulled back from $\Gamma_g$ by the map $\pi \times \pi$). Then there exists a constant $C > 0$, independent of $g$, so that

$$\| f^*|_{H^{2i}(X,\mathbb{Q}_l)} \| \leq C \| g^*|_{H^{2i}(Y,\mathbb{Q}_l)} \|$$

for all $i = 0, \ldots, k$. Here we fix arbitrary norms on the finite-dimensional vector spaces $H^{2i}(X, \mathbb{Q}_l)$ and $H^{2i}(Y, \mathbb{Q}_l)$.

Some remarks are in order.

**Remarks 5.1.** (1) Statement (A) is true if on $H^{2i}(X, \mathbb{Q}_l)$ we have a positivity notion of cohomology classes as in the case $\mathbb{K} = \mathbb{C}$, see the end of Section 1.1 for a detailed discussion.

(2) For any projective variety $X$ of dimension $k$, there are always (by using generic projections from linear subspaces of projective spaces containing $X$) surjective regular morphisms with *finite* fibres $\pi \colon X \to \mathbb{P}^k$.

(3) The correspondence $f$ defined in Statement (B) was called the pullback of $g$ by $\pi$ and was denoted as $\pi^*(g)$ in [39, Section 3.2]. In this special case, the cohomological class of $\Gamma_f$ is exactly the same as the pullback under $\pi \times \pi$ of the cohomological class of $\Gamma_g$. For a general dominant regular morphism (more generally, a dominant rational map) with generically finite fibres $\pi: X \to Y$, we can still define the pullback $\pi^*(g)$ of any correspondence $g: Y \to Y$. However, in the general case, no relation is expected for the cohomological classes of $g$ and $\pi^*(g)$.

It can be checked that in Statement (B) that $\pi \circ f^n = \deg(\pi)^n g^n \circ \pi$ for all $n$. It is shown in [39, Theorem 1.1] that we then have $\lambda_i(f) = \deg(\pi)\lambda_i(g)$ for all $i$.

The next result concerns Questions 2 and 3.

**Theorem 5.2.** *The following statements hold.*

(1) *Statement (B) is always true.*

(2) *If Statement (A) is true, then Question 2 has an affirmative answer.*

(3) *If Question 2 has an affirmative answer, then in the definition of $\chi_{2i}(f)$ we can replace* lim sup *by* lim.

(4) *For rational maps, an affirmative answer for Question 2 implies an affirmative answer for Question 3.*

*Proof.* We first make some preparations. Let $\pi: X \to Y = \mathbb{P}^k$ be a surjective regular morphism with *finite* fibres. Let $f: X \to X$ be a correspondence. For any positive integer $n$, we define a correspondence $g_n: Y \to Y$ given by declaring

$$\Gamma_{g_n} = (\pi \times \pi)_*(\Gamma_{f^n}).$$

Note that even if $f$ is a regular morphism, $g_n$ will rarely be a regular morphism or even a rational map. Also, in general $f^n$ and $g_n$ are not semi-conjugate, even up to a multiplicative constant. We overcome this by defining a correspondence $f_n: X \to X$ by declaring

$$\Gamma_{f_n} = (\pi \times \pi)^*(\Gamma_{g_n}).$$

We note that the cohomology groups of $Y = \mathbb{P}^k$ are very simple, in particular generated by algebraic cycles:

$$H^{2i}(Y, \mathbb{Q}_l) = H^{2i}_{\mathrm{num}}(Y, \mathbb{Q}_l).$$

Here are some relations between $f^n$, $g_n$ and $f_n$. First we have $f^n \leq f_n$. Second, we have $\lambda_i(f_n) = \deg(\pi)\lambda_i(g_n)$ for all $n$ and $i$ (see [39, Theorem 1.1]). Last, we have

$$\|g_n^*|_{H^{2i}(Y,\mathbb{Q}_l)}\| \leq C\,\|(f^n)^*|_{H^{2i}(X,\mathbb{Q}_l)}\|,$$

where $C > 0$ is independent of $f$ and $n$. In fact, let $h$ be the class of a hyperplane in $Y = \mathbb{P}^k$. Then $H^{2i}(Y, \mathbb{Q}_l)$ is generated by $h^i$. Let $\mathrm{pr}_1, \mathrm{pr}_2$ denote either the projections $X \times X \to X$ or $Y \times Y \to Y$ (the meaning will be clear from the context). Then

$$\begin{aligned}
\|g_n^*|_{H^{2i}(Y,\mathbb{Q}_l)}\| &= |g_n^*(h^i).h^{k-i}| = |\Gamma_{g_n}.\,\mathrm{pr}_2^*(h^i).\,\mathrm{pr}_1^*(h^{k-i})| \\
&= |(\pi \times \pi)_*(\Gamma_{f^n}).\,\mathrm{pr}_2^*(h^i).\,\mathrm{pr}_1^*(h^{k-i})| \\
&= \big|\Gamma_{f^n}.(\pi \times \pi)^*\big(\mathrm{pr}_2^*(h^i).\,\mathrm{pr}_1^*(h^{k-i})\big)\big| \\
&= \big|\Gamma_{f^n}.\,\mathrm{pr}_2^*\,\pi^*(h^i).\,\mathrm{pr}_1^*\,\pi^*(h^{k-i})\big| \\
&= \big|(f^n)^*\big(\pi^*(h^i)\big).\pi^*(h^{k-i})\big| \\
&\leq C\|(f^n)^*|_{H^{2i}(X,\mathbb{Q}_l)}\|.
\end{aligned}$$

(1) We first observe that if $\alpha \in H^i(X, \mathbb{Q}_l)$ is such that $\pi_*(\alpha) = 0$ in $H^i(Y, \mathbb{Q}_l)$, then $f_n^*(\alpha) = (f_n)_*(\alpha) = 0$. We show for example that $f_n^*(\alpha) = 0$. To this end, it suffices to show that for any $\beta \in H^{2k-i}(X, \mathbb{Q}_l)$ then $f_n^*(\alpha).\beta = 0$. In fact, we have

$$\begin{aligned}
f_n^*(\alpha).\beta &= \Gamma_{f_n}.\,\mathrm{pr}_2^*(\alpha).\,\mathrm{pr}_1^*(\beta) \\
&= (\pi \times \pi)^*(\Gamma_{g_n}).\big(\mathrm{pr}_2^*(\alpha).\,\mathrm{pr}_1^*(\beta)\big) \\
&= \Gamma_{g_n}.(\pi \times \pi)_*\big(\mathrm{pr}_2^*(\alpha).\,\mathrm{pr}_1^*(\beta)\big) \\
&= \Gamma_{g_n}.\,\mathrm{pr}_2^*(\pi_*\alpha).\,\mathrm{pr}_1^*(\pi_*\beta).
\end{aligned}$$

The last number is 0 provided $\pi_*(\alpha) = 0$, as assumed. (Note that it is also 0 if $\pi_*(\beta) = 0$.) Here, we used that under the assumptions on $\pi$, the cohomology class of $\Gamma_{f_n}$ is the same as the cohomology class of $(\pi \times \pi)^*(\Gamma_{g_n})$. We also used that

$$(\pi \times \pi)_*\big(\mathrm{pr}_2^*(\alpha).\,\mathrm{pr}_1^*(\beta)\big) = \mathrm{pr}_2^*(\pi_*\alpha).\,\mathrm{pr}_1^*(\pi_*\beta).$$

(This can be seen very easily in the case $\alpha$ and $\beta$ are represented by irreducible subvarieties of $X$, since in this case $\mathrm{pr}_2^*(\alpha).\,\mathrm{pr}_1^*(\beta)$ is represented by the variety $\beta \times \alpha$ in $X \times X$, whose image by $\pi \times \pi$ is exactly $\pi(\beta) \times \pi(\alpha)$. In the general case, we can proceed similarly by using the Kunneth's formula for the $l$-adic cohomology.)

From the above observation and the decomposition

$$H^*(X, \mathbb{Q}_l) = \pi^*\big(H^*(Y, \mathbb{Q}_l)\big) \oplus \mathrm{Ker}(\pi_*),$$

it follows that

$$\|f_n^*|_{H^i(X,\mathbb{Q}_l)}\| = \|f_n^*|_{\pi^*(H^i(Y,\mathbb{Q}_l))}\| = \deg(\pi)^2\|g_n|_{H^i(Y,\mathbb{Q}_l)}\|,$$

provided that the norm on $\pi^*(H^i(Y, \mathbb{Q}_l)$ is induced from the norm on $H^i(Y, \mathbb{Q}_l)$. This completes the proof.

(2) Assume that Statement (A) is true. Then, from $f_n \geq f^n$ for all $n$, we have

$$C\|(f_n)^*|_{H^{2i}(X,\mathbb{Q}_l)}\| \geq \|(f^n)^*|_{H^{2i}(X,\mathbb{Q}_l)}\|.$$

Hence, from the inequalities obtained above, we get

$$\begin{aligned}\|(f^n)^*|_{H^{2i}(X,\mathbb{Q}_l)}\| &\geq \big|(f^n)^*\big(\pi^*(h^i)\big).\pi^*(h^{k-i})\big| = \|g_n^*|_{H^{2i}(Y,\mathbb{Q}_l)}\| \\ &= \frac{1}{d^2}\|f_n^*|_{H^{2i}(X,\mathbb{Q}_l)}\| \geq \frac{1}{C}\|(f^n)^*|_{H^{2i}(X,\mathbb{Q}_l)}\|.\end{aligned}$$

By (1.1), we obtain

$$\limsup_{n\to\infty}\big|(f^n)^*\big(\pi^*(h^i)\big).\pi^*(h^{k-i})\big|^{1/n} \leq \lambda_i(f).$$

The proof is thus completed.

(3) This easily follows from similar arguments.

(4) This follows from the results in [39, Theorem 1.1] for the dynamical degrees $\lambda_i$. This completes the proof of Theorem 5.2. ■

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