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A quantified local-to-global principle for Morse quasigeodesics

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Abstract. Kapovich, Leeb and Porti (2014) gave several new characterizations of Anosov representations $\Gamma \to G$, including one where geodesics in the word hyperbolic group Γ map to "Morse quasigeodesics" in the associated symmetric space G/K. In analogy with the negative curvature setting, they prove a local-to-global principle for Morse quasigeodesics and describe an algorithm which can verify the Anosov property of a given representation in finite time. However, some parts of their proof involve non-constructive compactness and limiting arguments, so their theorem does not explicitly quantify the size of the local neighborhoods one needs to examine to guarantee global Morse behavior. In this paper, we supplement their work with estimates in the symmetric space to obtain the first explicit criteria for their local-to-global principle. This makes their algorithm for verifying the Anosov property effective. As an application, we demonstrate how to compute explicit perturbation neighborhoods of Anosov representations with two examples.

Contents

1. Introduction

Anosov representations were introduced by Labourie and defined in general by Guichard and Wienhard [11,25]. An Anosov representation is a homomorphism from a word hyperbolic group Γ to a semisimple Lie group G satisfying a strong dynamical condition. These representations have come to be widely studied as an interesting source of infinite covolume discrete subgroups of higher-rank semisimple Lie groups (see the surveys [15, 22]).

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This paper is concerned with certifying the Anosov property of a given representation. For some well-studied examples of Anosov representations, such as Hitchin representations and maximal representations of surface groups, the Anosov property can be certified via coarse topological invariants [6]. However, in the most general setting, deciding whether a given representation is Anosov is difficult. Building on the work of Kapovich, Leeb and Porti in [19], we give here the first explicit, finite criteria that certify the Anosov property for a general representation.

One important property of Anosov representations is stability: Any sufficiently small perturbation of an Anosov representation remains Anosov. It can happen that a connected component of the representation space consists entirely of Anosov representations, such as the Hitchin component, or the components consisting of maximal representations of surface groups (see also [12, 31]). In these cases, the Anosov condition is closed: Every deformation of such a representation remains Anosov. However, the Anosov condition is not closed in general. For instance, given an Anosov representation of a free group, or the representations of surface groups studied by Barbot in [2], it is unclear how large to expect Anosov neighborhoods to be. As an application of our main result, we demonstrate how to construct explicit perturbation neighborhoods of a given Anosov representation with two examples (see Theorems 1.2 and 1.3).

Anosov representations have come to be viewed as the appropriate generalization to higher-rank semisimple Lie groups of convex cocompact actions on rank 1 symmetric spaces. Indeed, when G has real rank 1, a representation of a finitely generated group is Anosov if and only if it has finite kernel and the image is convex cocompact, that is, acts cocompactly on a nonempty convex subset of the associated negatively curved symmetric space. A finitely generated group of isometries of a negatively curved symmetric space is convex cocompact if and only if it is undistorted, that is, any orbit map is a quasi-isometric embedding. By the Morse lemma in hyperbolic geometry, geodesics in Γ then map within uniformly bounded neighborhoods of geodesics in the symmetric space. Moreover, the Morse lemma implies a local-to-global principle for quasigeodesics, allowing one to establish finite criteria for a finitely generated group to be undistorted. One can then exhaust the group by balls in the Cayley graph and if any such ball passes a finite check then the subgroup is undistorted. This is a semi-decidable algorithm to verify undistortion: If the subgroup is undistorted, this algorithm will eventually terminate and certify so; otherwise, it will run on forever.

The naive generalization of convex cocompactness to higher rank turns out to be too restrictive. For example, the work of Kleiner and Leeb and independently Quint implies that a Zariski dense, discrete subgroup of a higher-rank simple Lie group which acts cocompactly on a convex subset of the associated symmetric space is a uniform lattice [23, 29]. On the other hand, the naive generalization of undistortion to higher rank turns out to be too loose: In his thesis, Guichard described an example of an undistorted subgroup in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ which is unstable, in the sense that representations arbitrarily close to the inclusion fail to have discrete image [10] (see also [9]). Moreover, Kapovich, Leeb and Porti describe an example of a discrete undistorted subgroup of

$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ which is finitely generated but not finitely presentable [19], using work Baumslag and Roseblade [3]. The Anosov property strikes a balance between these two naive generalizations to give a large class of representations that still exhibit good behavior. We will be concerned with a newer characterization that directly strengthens the undistortion condition.

In [19], Kapovich, Leeb and Porti gave several new characterizations of Anosov representations generalizing some of the many characterizations of convex cocompact subgroups. We will use their characterization, called Morse actions, that strengthens the undistortion condition by requiring geodesics in Γ to map to *Morse quasigeodesics*, described below. They prove a suitable generalization of the local-to-global principle for Morse quasigeodesics in higher-rank symmetric spaces (see Theorem 1.1). They then show the Anosov property is semi-decidable by describing an algorithm which can certify the Anosov property of a given representation of a word hyperbolic group in finite time. However, some parts of their proof involve non-constructive compactness and limiting arguments, so their theorem does not explicitly quantify the size of the local neighborhoods one needs to examine to guarantee global Morse behavior. In order to implement their algorithm, one needs a quantified version of the local-to-global principle as we give here.

Roughly speaking, a quasigeodesic is *Morse* if every finite consecutive subsequence is uniformly close to a *diamond*, which plays the role of a geodesic segment in rank 1. These diamonds are intersections of Weyl cones (see Sections 3.8 and 5.1) and may also be characterized as unions of Finsler geodesic segments (see [16, 17]). An infinite Morse quasiray stays within a uniformly bounded neighborhood of a Weyl cone, which plays the role of a geodesic ray in rank 1, and a bi-infinite Morse quasigeodesic stays within a uniformly bounded neighborhood of a parallel set, which plays the role of a geodesic line in rank 1 (see Section 3.12). The precise definition of Morse quasigeodesic is given in Section 5.

The main result of this paper is a quantified version of the following theorem due to Kapovich, Leeb and Porti. We let $\mathbb{X}$ denote a symmetric space of noncompact type.

Theorem 1.1 ([19, Theorem 7.18]). *For any $\Theta < \Theta'$, D, c_1, c_2, c_3, c_4, there exists a scale L so that every L-local $(\Theta, \tau_{\mathrm{mod}}, D)$-Morse (c_1, c_2, c_3, c_4)-quasigeodesic in $\mathbb{X}$ is a $(\Theta', \tau_{\mathrm{mod}}, D')$-Morse (c_1', c_2', c_3', c_4')-quasigeodesic.*

We reprove Theorem 1.1 and obtain the first explicit estimate of L. This appears in Theorem 5.8, which depends on Theorems 5.1 and 5.5. The theorem statements involve several auxiliary parameters and inequalities, so they are too cumbersome to give here. In order to apply our quantified version of the local-to-global principle and obtain an explicit scale L, one must produce auxiliary parameters satisfying these inequalities; this process is tedious but easy, as we discuss in Section 6. Versions of Theorems 5.1 and 5.5 without explicit conditions are also proved in [19].

As a demonstration of our techniques, we compute explicit perturbation neighborhoods of two Anosov representations into $\mathrm{SL}(3,\mathbb{R})$. To quantify the distance between linear representations, we use the Frobenius norm on the generators: For a matrix A, let $|A|_{\mathrm{Fr}}^2 = \mathrm{trace}(A^T A)$. In both cases, we control the orbit map at a basepoint; the Frobenius norm is closely related to distances to that basepoint (see Section 6.3). The first example is a neighborhood of Anosov representations of a free group.

Theorem 1.2. *Let Γ_1 be the subgroup of $\mathrm{SL}(3,\mathbb{R})$ generated by*

$$g = \begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \quad h = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix},$$

with $\tanh t = 0.75$. If Γ_1' is generated by g', h' where $\max\{|g - g'|_{\mathrm{Fr}}, |h - h'|_{\mathrm{Fr}}\} \leq 10^{-15{,}309}$, then Γ_1' is Anosov.

The second example is a neighborhood of Anosov representations of a closed surface group. Let Γ_2 be the subgroup of $\mathrm{SL}(3,\mathbb{R})$ generated by

$$S = \left\{ \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \;\middle|\; \theta \in \left\{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}\right\} \right\}$$

for $\log\lambda = \cosh^{-1}(\cot\frac{\pi}{8})$. This group is isomorphic to the fundamental group of a closed surface of genus 2 (see Section 6.3). In the statement of Theorem 1.3, we control the perturbed representation on a larger generating set $S' = \{\gamma \in \Gamma_2 \mid \sqrt{6}|\log\gamma|_{\mathrm{Fr}} \leq 9.5\}$. The finite set S' contains the standard generating set S and consists of the elements of Γ_2 which move a basepoint p in the symmetric space associated with $\mathrm{SL}(3,\mathbb{R})$ by a distance of at most 9.5. This basepoint is the point stabilized by $\mathrm{SO}(3)$. Using this larger generating set allows us to perturb the initial representation farther.

Theorem 1.3. *If $\rho\colon \Gamma_2 \to \mathrm{SL}(3,\mathbb{R})$ is a representation satisfying the condition $|\rho(s) - s|_{\mathrm{Fr}} \leq 10^{-3{,}698{,}433}$ for all $s \in S'$, then ρ is Anosov.*

We briefly sketch the proof of Theorems 1.2 and 1.3. Let Γ denote either Γ_1 or Γ_2. In either case the group Γ acts cocompactly on a closed convex subset of a copy of the hyperbolic plane embedded totally geodesically in the symmetric space associated with $\mathrm{SL}(3,\mathbb{R})$. We find explicit quasi-isometry constants and by the classical Morse lemma, there exists $R > 0$ such that the orbit of any geodesic in Γ is within R of a geodesic. We slightly relax the Morse quasi-isometric parameters of Γ and apply the local-to-global principle (Theorem 5.8). This provides a lower bound on k such that any $2k$-local Morse quasigeodesic is a global Morse quasigeodesic. We control the perturbation of words of length k in terms of the perturbation of the generators, completing the proof.

We emphasize that our approach is completely general, in the following sense. Let $\rho\colon \Gamma \to G$ be any Anosov representation such that the orbit map at $p \in \mathbb{X}$ has known Morse quasi-isometry parameters with respect to a finite symmetric generating set S for Γ. We may then easily produce explicit parameters k, ϵ such that if any other representation $\rho'\colon \Gamma \to G$ satisfies $d(\rho(\gamma)p, \rho'(\gamma)p) \leq \epsilon$ for all $\gamma \in \Gamma$ of word length at most k, then ρ' is Anosov. Moreover, for linear groups we explicitly bound $d(\rho(\gamma)p, \rho'(\gamma)p)$ in terms of the word length of γ, the Frobenius norms $|\rho(s)|_{\mathrm{Fr}}$ and $|\rho(s) - \rho'(s)|_{\mathrm{Fr}}$, so we obtain a condition on ρ' just in terms of the generators.

The bulk of the paper is devoted to a proof of Theorem 1.1. We supply a number of estimates in Section 4 related to the geometry of the symmetric space $\mathbb{X}$. An important tool is the ζ-angle $\angle_p^\zeta$, a $\mathrm{Stab}_G(p)$-invariant metric on $\mathrm{Flag}(\tau_{\mathrm{mod}})$ introduced by Kapovich, Leeb and Porti in [19] (see Section 3.13 for the definition). In Lemma 4.8, we obtain explicit control on $\angle_p^\zeta(x, y)$ in terms of the Riemannian angle $\angle_p(x, y)$. The proof uses an explicit bound for the Hessian of a Morse function on $\mathrm{Flag}(\tau_{\mathrm{mod}})$ (see Proposition 3.8 and Corollary 3.15). A crucial step in the proof of the local-to-global principle is controlling the distance from the midpoint of a long regular segment to a nearby diamond. The existence of such a bound is demonstrated in the proof of Proposition 7.16 of [19] via a limiting argument. To achieve explicit control, we consider the lengths of certain curves in $\mathbb{X}$ which are images of curves in G under the orbit map (see Lemma 4.9). In Lemma 4.10, the curve in G is required to lie in a maximal compact subgroup. In Lemma 4.11, the curve is required to lie in a unipotent horocyclic subgroup. We combine these in Corollary 4.13 to obtain explicit, arbitrary control for the distance of midpoints to nearby Weyl cones (and hence diamonds). Kapovich, Leeb and Porti show that distance from a point $x \in \mathbb{X}$ to the parallel set $P(\tau_-, \tau_+)$ controls the ζ-angle $\angle_x^\zeta(\tau_-, \tau_+)$ and vice versa via a compactness argument [19, Section 2.4.5]. We give an explicit bound for $\angle_x^\zeta(\tau_-, \tau_+)$ in terms of $d(x, P(\tau_-, \tau_+))$ in Corollary 4.16. This follows from Lemma 4.14, whose proof relies on controlling the Lie derivative $\mathcal{L}_X \operatorname{grad} f_\tau$ where X is a Killing vector field and f_τ is a Busemann function. Similarly, we obtain an explicit bound for $d(x, P(\tau_-, \tau_+))$ terms of $\angle_x^\zeta(\tau_-, \tau_+)$ in Lemma 4.17 by controlling iterated derivatives of Busemann functions. In particular, we obtain an explicit uniform bound for the third derivative of the restriction of a Busemann function to a geodesic.

As in [19], the proof of Theorem 1.1 is essentially broken into two parts: Theorems 5.1 and 5.5. Theorem 5.1 guarantees that a sequence (x_n) with sufficiently spaced points forming ζ-angles sufficiently close to π is a Morse quasigeodesic. It is a quantified version of [19, Theorem 7.2] and shares the same outline. One first shows that the property of "moving away" from a simplex propagates along the sequence (see Section 5.1). This implies that we can extract a simplex τ_- that the sequence (x_n) moves away from (resp. towards) as n increases (resp. decreases), and a simplex τ_+ that the sequence (x_n) moves away from (resp. towards) as n decreases (resp. increases). One then verifies that the simplices τ_-, τ_+ are opposite and that the projections to the parallel set $P(\tau_-, \tau_+)$ define suitable diamonds, making (x_n) a Morse quasigeodesic.

Theorem 5.5 is a quantified version of [19, Proposition 7.16]. It states that sufficiently spaced points on Morse quasigeodesics have straight and spaced midpoint sequences. A crucial ingredient is Corollary 4.13, which allows us to force the midpoints to be arbitrarily close to the parallel sets in terms of the Morse and spacing parameters. This guarantees that they appear in nested Weyl cones, and makes the ζ-angles arbitrarily straight.

Armed with Theorems 5.1 and 5.5, the proof of Theorem 5.8 is similar to the proof of Theorem 7.18 in [19]. We start with an L-local Morse quasigeodesic where L is large enough to satisfy several explicit inequalities. We then replace our Morse quasigeodesic with a coarsification and take the midpoint sequence. Our assumptions together with Theorem 5.5 shows that this coarse midpoint sequence is sufficiently straight and spaced (see Section 5.1). An application of Theorem 5.1 shows that the midpoint sequence is a Morse quasigeodesic, and since it is a coarse approximation of the original sequence, the original sequence is also a Morse quasigeodesic, completing the proof.

The usual proof of the local-to-global principle in hyperbolic geometry depends on the classical Morse lemma. A higher-rank version of the Morse lemma was proved by Kapovich, Leeb and Porti in [21]. In particular, they prove that the orbit map $\Gamma \to \mathbb{X}$ of a finitely generated group is a coarsely uniformly regular quasi-isometric embedding if and only if Γ is word hyperbolic and the orbit map is a Morse quasi-isometric embedding. It would be interesting to quantify their higher-rank Morse lemma by producing an explicit Morse parameter for (coarsely) uniformly regular quasi-isometric embeddings, but we do not do this here. In the special case of the symmetric space associated with $\mathrm{SL}(d, \mathbb{R})$, another proof of the higher-rank Morse lemma appears in [4]. There, Bochi, Potrie and Sambarino give yet another characterization of Anosov representations in terms of cone-types and dominated splittings.

The organization of the paper is as follows. In Section 2, we fix some notation we use throughout the paper. In Section 3, we review some background of symmetric spaces. Much of this section is classical and may be skipped by experts on symmetric spaces, but we point the reader to our definition of regularity in Definition 3.12 and the definition of ζ-angle in Definition 3.22. The notion of regularity here is slightly different, but equivalent to, that in [19] (see Proposition 3.17). The bulk of the work is in Section 4 where we give several estimates related to the geometry of symmetric spaces. In Section 5, we supplement the proof of the local-to-global principle in [19] with our estimates from Section 4, reproving Theorem 1.1 with explicit bounds. Together with some standard geometric group theory, elementary hyperbolic geometry, and linear algebra in Section 6, this allows us to prove Theorems 1.2 and 1.3.

2. Notation

We establish our notational conventions in this paper. When possible, we have tried to keep notation consistent with [8, 19, 20].

(1) $\mathbb{X} = G/K$ will denote a symmetric space of noncompact type. Let G be the connected component of the isometry group of $\mathbb{X}$, and K be a maximal compact subgroup of G (see Section 3).

(2) We let p, q, r, c denote points or curves in $\mathbb{X}$. We let g, h, u, a denote elements or curves in G. An element or curve in K may be denoted by k.

(3) The Lie algebra of G is denoted $\mathfrak{g}$. The Lie algebra of K is denoted $\mathfrak{k}$. When a point p is given, K is the stabilizer of p in G. Usually U, V, W, X, Y, Z will denote elements of $\mathfrak{g}$.

(4) The orbit map $\mathrm{orb}_p\colon G \to \mathbb{X}$, given by $\mathrm{orb}_p(g) = gp$, has differential $\mathrm{ev}_p\colon \mathfrak{g} \to T_p\mathbb{X}$ at the identity (see Section 3).

(5) The Cartan decomposition induced by $p \in \mathbb{X}$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. It corresponds to a Cartan involution $\vartheta_p\colon \mathfrak{g} \to \mathfrak{g}$ (see Section 3.1).

(6) The Killing form on $\mathfrak{g}$ is denoted B. Each point $p \in \mathbb{X}$ induces an inner product B_p on $\mathfrak{g}$ defined by $B_p(X, Y) = -B(\vartheta_p X, Y)$ (see Section 3.1).

(7) We assume that the Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathbb{X}$ is the one induced by the Killing form (see equation (3.2)).

(8) The sectional curvature κ of $\mathbb{X}$ has image $[-\kappa_0^2, 0]$ (see Section 3.3). Note that κ_0 is the maximal norm of a restricted root vector (see Proposition 3.3).

(9) A maximal abelian subspace of $\mathfrak{p}$ will be denoted $\mathfrak{a}$. The associated restricted roots are denoted by $\Lambda \subset \mathfrak{a}^*$. A choice of simple roots is denoted by Δ (see Section 3.2).

(10) Each maximal abelian subspace $\mathfrak{a}$ has an action by the Weyl group and decomposition into Euclidean Weyl chambers denoted V (see Section 3.5).

(11) There is a *vector-valued distance function* $\vec{d}\colon \mathbb{X} \times \mathbb{X} \to V_{\mathrm{mod}}$ with image the model Euclidean Weyl chamber (see equation (3.4)). In [19, 20], this map is denoted Δ, and they let Δ denote the model Euclidean Weyl chamber we call V_{mod}. In this paper, Δ denotes a choice of simple roots.

(12) A spherical Weyl chamber σ corresponds to a set of simple roots Δ. For a face τ of σ, we have

$$\Delta_\tau = \{\alpha \in \Delta \mid \alpha(\tau) = 0\}, \quad \Delta_\tau^+ = \{\alpha \in \Delta \mid \alpha(\operatorname{int}\tau) > 0\},$$

see equation (3.6). We have

$$\tau = \sigma \cap \bigcap_{\alpha \in \Delta_\tau} \ker\alpha, \;\; \operatorname{int}_\tau \sigma = \{X \in \sigma \mid \forall \alpha \in \Delta_\tau^+, \alpha(X) > 0\}, \;\; \partial_\tau \sigma = \sigma \cap \bigcup_{\alpha \in \Delta_\tau^+} \ker\alpha.$$

(13) The visual boundary of $\mathbb{X}$ is denoted $\partial\mathbb{X}$ (see Section 3.8). We let τ, σ denote a spherical simplex/chamber in $\mathfrak{a}$ or an ideal simplex/chamber in $\partial\mathbb{X}$.

(14) There is a *type projection* $\theta\colon \partial\mathbb{X} \to \sigma_{\mathrm{mod}}$ with image the model ideal Weyl chamber (see Section 3.8).

(15) A face of σ_{mod} is called a *model simplex* and denoted τ_{mod}. There is a decomposition $\sigma_{\mathrm{mod}} = \mathrm{int}_{\tau_{\mathrm{mod}}} \sigma_{\mathrm{mod}} \sqcup \partial_{\tau_{\mathrm{mod}}} \sigma_{\mathrm{mod}}$ (see Section 3.9).

(16) The definitions of (α_0, τ)-regular and (α_0, τ)-spanning vectors and geodesics are given in Section 3.12. These notions are extended to ideal points in Section 3.9.

(17) We let $\mathrm{Flag}(\tau_{\mathrm{mod}})$ denote the set of ideal simplices in $\partial \mathbb{X}$ of type τ_{mod} (see Section 3.8). $\mathrm{Flag}(\tau_{\mathrm{mod}})$ is naturally a partial flag manifold of G.

(18) The definitions of Weyl cones $V(x, \mathrm{st}(\tau), \alpha_0)$, $V(x, \mathrm{ost}(\tau))$ and Weyl sectors $V(x, \tau)$ are given in Section 3.9.

(19) The subgroups A_τ and N_τ and the generalized Iwasawa decomposition $G = N_\tau A_\tau K$ are described in Section 3.11.

(20) A parallel set is denoted $P(\tau_-, \tau_+)$ for opposite simplices $\tau_-, \tau_+ \in \mathrm{Flag}(\tau_{\mathrm{mod}})$. A horocycle is denoted $H(p, \tau)$ (see Section 3.12). A diamond is denoted $\diamondsuit(p, q)$ and a truncated diamond is denoted $\diamondsuit_{\alpha_0}(p, q)$ (see Section 5.1).

(21) For $p \in \mathbb{X}$ and $x, y \in \overline{X} \setminus \{p\}$, $\angle_p(x, y)$ denotes the Riemannian angle at p between x and y. For $\eta, \eta' \in \partial \mathbb{X}$, we let $\angle_{\mathrm{Tits}}(\eta, \eta')$ denote their Tits angle. If px and py are τ_{mod}-regular and $\tau, \tau' \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ then we have $\angle_p^{\zeta}(\tau, \tau'), \angle_p^{\zeta}(\tau, y), \angle_p(\zeta(\tau), \zeta(py))$ denote the ζ-angles (see Section 3.13).

(22) The auxiliary model ideal point $\zeta_{\mathrm{mod}} \in \mathrm{int}(\tau_{\mathrm{mod}})$ is discussed in Section 3.13. When τ_{mod} is a minimal ι-invariant face of τ_{mod}, the regularity parameter $\zeta_0 = \min\{\alpha(\zeta_{\mathrm{mod}}) \mid \alpha \in \Delta^+_{\tau_{\mathrm{mod}}}\}$ is computed in Section 3.10.

(23) A (c_1, c_2, c_3, c_4)-*quasigeodesic* is a sequence (x_n) (possibly finite, infinite or bi-infinite) in $\mathbb{X}$ such that

$$\frac{1}{c_1}|N| - c_2 \leq d(x_n, x_{n+N}) \leq |N| c_3 + c_4.$$

A quasigeodesic is $(\alpha_0, \tau_{\mathrm{mod}}, D)$-*Morse* if for all x_n, x_m there exists a diamond $\diamondsuit = \diamondsuit_{\alpha_0}(p, q)$ such that $d(p, x_n), d(q, x_m) \leq D$ and for all $n \leq i \leq m$, $d(x_i, \diamondsuit) \leq D$ (see Section 5).

3. Background on symmetric spaces

We begin with some background on the structure of symmetric spaces of noncompact type. Experts on symmetric spaces can skip this section, but should note that we assume that the metric is induced by the Killing form (see equation (3.2)), quantify the regularity of geodesics in Definition 3.12, and define the ζ-angle in Definition 3.22. A constant ζ_0, relevant for estimates involving ζ-angles, is computed for minimal ι-invariant faces in Section 3.10. A constant κ_0, related to the lower curvature bound of $\mathbb{X}$, is discussed and computed in Section 3.3. For detailed references on symmetric spaces, see [8, 13, 14].

A *symmetric space* is a connected Riemannian manifold $\mathbb{X}$ such that for each point $p \in \mathbb{X}$, there exists a *geodesic symmetry* $S_p\colon \mathbb{X} \to \mathbb{X}$, an isometry fixing p whose differential at p is $(\mathrm{d}S_p)_p = -\mathrm{id}_{T_p\mathbb{X}}$. A symmetric space is necessarily complete with transitive isometry group. Simply connected Riemannian manifolds admit a de Rham decomposition into metric factors. If $\mathbb{X}$ is a simply connected nonpositively curved symmetric space with no Euclidean de Rham factors, $\mathbb{X}$ is called a *symmetric space of noncompact type*. Throughout the paper, $\mathbb{X}$ refers to any fixed symmetric space of noncompact type.

The isometry group of $\mathbb{X}$ is a semisimple Lie group, and we let G be the identity component of the isometry group. For each point $p \in \mathbb{X}$, the stabilizer $K = G_p = \{g \in G \mid gp = p\}$ is a maximal compact subgroup of G. Hence, $\mathbb{X}$ is diffeomorphic to G/K by the orbit-stabilizer theorem for Lie groups and homogeneous spaces. We let $\mathfrak{g}$ denote the Lie algebra of left-invariant vector fields on G.

A *Killing vector field* on a Riemannian manifold is vector field whose induced flow is by isometries. There is a natural linear isomorphism from $\mathfrak{g}$ to the space of Killing vector fields on $\mathbb{X}$ by defining for $X \in \mathfrak{g}$ the vector field X^* given by

$$X^*_p := \frac{\mathrm{d}}{\mathrm{d}t} e^{tX} p|_{t=0}. \tag{3.1}$$

The Lie bracket of two Killing vector fields is again a Killing vector field, but the map $X \mapsto X^*$ is a Lie algebra anti-homomorphism: $[X, Y]^* = -[X^*, Y^*]$.

3.1. Cartan decomposition

Each point $p \in \mathbb{X}$ induces a *Cartan decomposition* in the following way. The geodesic symmetry $S_p\colon \mathbb{X} \to \mathbb{X}$ induces an involution of G by

$$g \mapsto S_p \circ g \circ S_p.$$

The differential is a Lie algebra involution $\vartheta_p\colon \mathfrak{g} \to \mathfrak{g}$, so we may write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k} = \{X \in \mathfrak{g} \mid \vartheta_p X = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \vartheta_p X = -X\}$. Since ϑ_p preserves brackets, we have

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}\,.$$

We denote the orbit map $g \mapsto gp$ by $\mathrm{orb}_p\colon G \to \mathbb{X}$. The differential $(\mathrm{d}\,\mathrm{orb}_p)_1\colon \mathfrak{g} \to T_p\mathbb{X}$ has kernel precisely $\mathfrak{k}$. Moreover, $\mathfrak{k}$ is the Lie algebra of $K = G_p$. The restriction $(\mathrm{d}\,\mathrm{orb}_p)_1\colon \mathfrak{p} \to T_p\mathbb{X}$ is a vector space isomorphism. For any $X \in \mathfrak{g}$, $(\mathrm{d}\,\mathrm{orb}_p)_1 X = X^*_p =: \mathrm{ev}_p X$, see equation (3.1), so we use the less cumbersome notation $\mathrm{ev}_p = (\mathrm{d}\,\mathrm{orb}_p)_1\colon \mathfrak{g} \to T_p\mathbb{X}$ throughout the paper (read as "evaluation at p").

Let B denote the Killing form on $\mathfrak{g}$ and let $\langle \cdot, \cdot \rangle$ denote the Riemannian metric on $\mathbb{X}$. We will assume that for all $X, Y \in \mathfrak{p}$,

$$B(X, Y) = \langle \mathrm{ev}_p X, \mathrm{ev}_p Y \rangle_p, \tag{3.2}$$

that is, that the Riemannian metric on $\mathbb{X}$ is induced by the Killing form. Any other G-invariant Riemannian metrics on $\mathbb{X}$ only differs from this one by scaling by a global constant on each de Rham factor of $\mathbb{X}$.

Under the identification of $\mathfrak{p}$ with $T_p\,\mathbb{X}$, the Riemannian exponential map $\mathfrak{p} \to \mathbb{X}$ is given by $X \mapsto e^X p$. In particular, the constant speed geodesics at p are given by $c(t) = e^{tX} p$ for $X \in \mathfrak{p}$.

The point $p \in \mathbb{X}$ induces an inner product B_p on $\mathfrak{g}$ defined by

$$B_p(X, Y) := -B(\vartheta_p X, Y). \tag{3.3}$$

On $\mathfrak{p}$, B_p is just the restriction of the Killing form B, and we have required that the identification of $(\mathfrak{p}, B)$ with $(T_p\,\mathbb{X}, \langle , \rangle)$ is an isometry. On $\mathfrak{k}$, B_p is the negative of the restriction of B to $\mathfrak{k}$. Since $\mathfrak{k}$ and $\mathfrak{p}$ are B-orthogonal, it follows that B_p is an inner product on $\mathfrak{g}$. For each $X \in \mathfrak{p}$, $\operatorname{ad} X$ is symmetric with respect to B_p on $\mathfrak{g}$, and likewise for each $Y \in \mathfrak{k}$, $\operatorname{ad} Y$ is skew-symmetric.

3.2. Restricted root space decomposition

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Via the adjoint action, $\mathfrak{a}$ is a commuting vector space of diagonalizable linear transformations on $\mathfrak{g}$. Therefore, $\mathfrak{g}$ admits a common diagonalization called the *restricted root space decomposition*. For each $\alpha \in \mathfrak{a}^*$, define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall\, A \in \mathfrak{a}, \operatorname{ad} A(X) = \alpha(A)X\}.$$

We obtain a collection of *roots*

$$\Lambda = \{\alpha \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$$

corresponding to the nonzero root spaces. The restricted root space decomposition is then

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha\,.$$

For each root $\alpha \in \Lambda$, define the coroot $H_\alpha \in \mathfrak{a}$ by $\alpha(A) = B(H_\alpha, A)$ for all $A \in \mathfrak{a}$. This induces an inner product, also denoted B, on $\mathfrak{a}^*$ by defining $B(\alpha, \beta) := B(H_\alpha, H_\beta)$. The set Λ forms a root system in $(\mathfrak{a}^*, B)$ (see [8, Proposition 2.9.3]). Note that unlike the root systems of complex semisimple Lie algebras, the restricted root systems may be *non-reduced*, that is, it may not hold that the only multiples of α appearing in Λ are $\pm\alpha$. For example, the restricted root system of complex hyperbolic space is non-reduced. The restricted root space decomposition is B_p-orthogonal. A subset Λ^+ of the roots is *positive* if for every $\alpha \in \Lambda$, exactly one of $\alpha, -\alpha$ is contained in Λ^+ and for any $\alpha, \beta \in \Lambda^+$ such that $\alpha + \beta$ is a root, we have $\alpha + \beta \in \Lambda^+$.

The Cartan involution restricts to an isomorphism $\vartheta_p \colon \mathfrak{g}_\alpha \to \mathfrak{g}_{-\alpha}$ for each $\alpha \in \Lambda \cup \{0\}$. Thus, we have

$$\mathfrak{p}_\alpha := \mathfrak{p} \cap \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = (\operatorname{id} - \vartheta_p)\mathfrak{g}_\alpha = (\operatorname{id} - \vartheta_p)\mathfrak{g}_{-\alpha}$$

and

$$\mathfrak{k}_\alpha := \mathfrak{k} \cap \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = (\mathrm{id} + \vartheta_p)\mathfrak{g}_\alpha = (\mathrm{id} + \vartheta_p)\mathfrak{g}_{-\alpha}.$$

Note that $\mathfrak{p}_\alpha = \mathfrak{p}_{-\alpha}$ and likewise $\mathfrak{k}_\alpha = \mathfrak{k}_{-\alpha}$, so for Λ^+ a set of positive roots, we have the decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k}_0 \oplus \bigoplus_{\alpha \in \Lambda^+} \mathfrak{p}_\alpha \oplus \bigoplus_{\alpha \in \Lambda^+} \mathfrak{k}_\alpha$$

which is both B_p-orthogonal and B-orthogonal. Some authors use the notation $\mathfrak{m} = \mathfrak{k}_0$.

3.3. The lower curvature bound κ_0

Several estimates in Section 5 will rely on precise curvature estimates which we perform in the present section. These can be expressed in terms of a constant κ_0 which is closely related to the lower curvature bound of $\mathbb{X}$.

The curvature tensor R of $\mathbb{X}$ may be defined using the Levi-Civita connection ∇ by

$$R(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]},$$

for vector fields u, v on $\mathbb{X}$. In a symmetric space the curvature tensor is related to the structure of $\mathfrak{g}$ by the following formula.

Theorem 3.1 ([28, p. 242]). *Let $X, Y, Z \in \mathfrak{p}$ and write X^*, Y^*, Z^* for the corresponding Killing vector fields on $\mathbb{X}$. Then*

$$(R(X^*, Y^*)Z^*)_p = -\mathrm{ev}_p[[X, Y], Z].$$

Our convention is that the sectional curvature of a plane spanned by orthonormal unit vectors $u, v \in T_p\mathbb{X}$ is

$$\kappa(\mathrm{Span}\{u, v\}) = \langle R(u, v)v, u \rangle.$$

The following constant appears frequently throughout the paper.

Definition 3.2. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let B denote the Killing form of $\mathfrak{g}$. Consider a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and let Λ be the restricted roots. Define

$$\kappa_0 := \max\{\alpha(X) \mid \alpha \in \Lambda, X \in \mathfrak{a}, |X| = 1\}.$$

The presence of the constant κ_0 is explained by the following proposition. Moreover, it can be computed using the work of Adeboye, Wang and Wei [1] (see Theorem 3.4). We let C_1 denote the constant appearing in that theorem and we let $h^\vee$ denote the dual Coxeter number of the complexification of $\mathfrak{g}$ (see Table 1).

Proposition 3.3. *With κ_0 defined as above, we have the following:*

(1) *The image of the sectional curvature of $\mathbb{X}$ is $[-\kappa_0^2, 0]$.*

(2) $\kappa_0 = \max\{|H_\alpha| \mid \alpha \in \Lambda\}$.

(3) *In any symmetric space,* $\kappa_0 \leq \frac{1}{\sqrt{2}}$.

(4) *We have* $\kappa_0 = \frac{C_1}{\sqrt{2h^\vee}}$.

Proof. We first prove item (1). Let $X \in \mathfrak{a}$, $Y \in \mathfrak{p}$ and assume X, Y are orthogonal unit vectors. For any $Y \in \mathfrak{p}$, we may write $Y = Y_0 + \sum_{\alpha \in \Lambda^+} Y_\alpha$ where $Y_0 \in \mathfrak{a}$ and each $Y_\alpha \in \mathfrak{p}_\alpha$, and recall that this decomposition is B-orthogonal, so we have the lower curvature bound

$$\begin{aligned}
\kappa(\mathrm{Span}\{X_p^*, Y_p^*\}) &= B(-[[X,Y],Y],X) \\
&= B([X,Y],[X,Y]) \\
&= -B([X,[X,Y]],Y) \\
&= -\sum_{\alpha \in \Lambda^+} B(\alpha(X)^2 Y_\alpha, Y) \\
&= -\sum_{\alpha,\beta \in \Lambda^+} \alpha(X)^2 B(Y_\alpha, Y_\beta) \\
&= -\sum_{\alpha \in \Lambda^+} \alpha(X)^2 B(Y_\alpha, Y_\alpha) \geq -\kappa_0^2
\end{aligned}$$

since κ_0 is defined to be the maximum of $\{\alpha(X) \mid \alpha \in \Lambda, X \in \mathfrak{a}, |X| = 1\}$. By setting $Y \in \mathfrak{p}_\alpha$ and $X = H_\alpha$, we see that this bound is attained.

Item (2) follows easily from Definition 3.2, $\kappa_0 := \max\{\alpha(X) \mid \alpha \in \Lambda, X \in \mathfrak{a}, |X| = 1\}$. Since Λ is finite and the unit sphere in $\mathfrak{a}$ is compact, there exist $\alpha \in \Lambda$ and a unit vector X realizing the maximum. Such an α is maximized in the direction of the root vector H_α, so we have $\kappa_0 = \alpha\left(\frac{H_\alpha}{|H_\alpha|}\right) = |H_\alpha|$. Note that the inner product used to define H_α and its norm is the restriction of the Killing form B of $\mathfrak{g}$ to $\mathfrak{a}$.

To see item (3), we have

$$\kappa_0 = \alpha\left(\frac{H_\alpha}{|H_\alpha|}\right) = |H_\alpha|$$

for some α. By [8, Proposition 2.14.5], we have for $A, A' \in \mathfrak{a}$ that $B(A, A') = \sum_{\beta \in \Lambda} (\dim \mathfrak{g}_\beta) \beta(A) \beta(A')$, so

$$1 = B\left(\frac{H_\alpha}{|H_\alpha|}, \frac{H_\alpha}{|H_\alpha|}\right) = \sum_{\beta \in \Lambda} (\dim \mathfrak{g}_\beta) \beta\left(\frac{H_\alpha}{|H_\alpha|}\right)^2 \geq 2\alpha\left(\frac{H_\alpha}{|H_\alpha|}\right)^2 = 2\kappa_0^2.$$

We prove item (4) using the work of using the work of Adeboye, Wang and Wei [1]. By [1, equation (4.1)], $C_1 = \max\{\alpha(X) \mid \alpha \in \Lambda^+, X \in \mathfrak{a}, |X|_{B'} = 1\}$. Here, B' is a renormalizing of the Killing form B defined by

$$B = 2h^\vee B',$$

where $h^\vee$ is the *dual Coxeter number* of the complexification $\mathfrak{g}^{\mathbb{C}}$. In this normalization, the long roots of $\mathfrak{g}^{\mathbb{C}}$ have norm $\sqrt{2}$. We record the dual Coxeter numbers of complex

	$\mathfrak{g}^{\mathbb{C}}$	$\mathbf{h}^{\vee}$
A_n	$\mathfrak{sl}(n+1,\mathbb{C})$	$n+1$
B_n	$\mathfrak{so}(2n+1,\mathbb{C})$	$2n-1$
C_n	$\mathfrak{sp}(2n,\mathbb{C})$	$n+1$
D_n	$\mathfrak{so}(2n,\mathbb{C})$	$2n-2$
E_6		12
E_7		18
E_8		30
F_4		9
G_2		4

Table 1. Simple complex Lie algebras and their dual Coxeter numbers $h^{\vee}$.

simple Lie algebras in Table 1. When $\mathfrak{g}$ already admits the structure of a complex simple Lie algebra, the dual Coxeter number of $\mathfrak{g}^{\mathbb{C}}$ is twice that of $\mathfrak{g}$.

For any $\alpha \in \Lambda$ and $A \in \mathfrak{a}$, we have

$$B'(H_\alpha^{B'}, A) = \alpha(A) = B(H_\alpha^{B}, A) = 2h^{\vee} B'(H_\alpha^{B}, A),$$

so $H_\alpha^{B'} = 2h^{\vee} H_\alpha^{B}$. Moreover, for any $A \in \mathfrak{a}$, $|A|_B = \sqrt{2h^{\vee}}|A|_{B'}$.

Since the same root α realizes κ_0 and C_1, we have

$$\kappa_0 = |H_\alpha^B|_B = \sqrt{2h^{\vee}}|H_\alpha^B|_{B'} = \sqrt{2h^{\vee}}\Big|\frac{1}{2h^{\vee}} H_\alpha^{B'}\Big|_{B'} = \frac{1}{\sqrt{2h^{\vee}}}\big|H_\alpha^{B'}\big|_{B'} = \frac{C_1}{\sqrt{2h^{\vee}}}. \quad \blacksquare$$

Theorem 3.4 ([1, Theorem 4.5]). *Let G/K be a simply connected irreducible symmetric space of noncompact type. Equip G with the renormalized Killing form B'. Let C_1 be the constants defined above. Then either $C_1 = \sqrt{2}$ or $C_1 = 1$. The latter occurs exactly when G/K is one of the following:*

(1) *A rank* 1 *symmetric space other than $\mathbb{H}^2$ or $\mathbb{C}\mathbb{H}^n$, for $n \geq 2$;*

(2) $\mathrm{SU}^*(2n)/\mathrm{Sp}(n)$*, $n \geq 2$;*

(3) $\mathrm{Sp}(m,n)/(\mathrm{Sp}(m)\mathrm{Sp}(n)), m \geq n \geq 2$*; or*

(4) $E_{6(-26)}/F_4$.

In Helgason's classification, the irreducible symmetric spaces of noncompact type with $C_1 = 1$ are of type: AII = $\mathrm{SU}^*(2n)/\mathrm{Sp}(n)$, $n \geq 2$, BII = $\mathrm{SO}(2n,1)/\mathrm{SO}(2n), n \geq 2$, CII = $\mathrm{Sp}(m,n)/(\mathrm{Sp}(m)\mathrm{Sp}(n)), m \geq n \geq 2$, DII = $\mathrm{SO}(2n+1,1)/\mathrm{SO}(2n+1), n \geq 1$, EIV = $E_{6(-26)}/F_4$, and FII = $F_{4(-20)}/\mathrm{Spin}(9)$.

Example 3.5. In $\mathfrak{sl}(d,\mathbb{R})$, each root α has $|H_\alpha| = \frac{1}{\sqrt{d}}$, so we have $\kappa_0 = \frac{1}{\sqrt{d}}$ and the associated symmetric space has lower curvature bound $-\frac{1}{d}$.

3.4. Copies of hyperbolic planes

In Section 6, we will need to know the curvature of copies of the hyperbolic plane in $\mathbb{X}$. These correspond to copies of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{g}$. Let $\alpha \in \Lambda$ and $X_\alpha \in \mathfrak{g}_\alpha$ such that $B_p(X_\alpha, X_\alpha) = \frac{2}{|H_\alpha|^2}$. Set $\tau_\alpha := \frac{2}{|H_\alpha|^2} H_\alpha$ so that $\alpha(\tau_\alpha) = 2$. Set $Y_\alpha := -\vartheta_p X_\alpha \in \mathfrak{g}_{-\alpha}$. Then

$$[\tau_\alpha, X_\alpha] = 2X_\alpha, \quad [\tau_\alpha, Y_\alpha] = -2Y_\alpha, \quad \text{and} \quad [X_\alpha, Y_\alpha] = \tau_\alpha,$$

where the last equality follows from considering $B([X_\alpha, Y_\alpha], A)$ for $A \in \mathfrak{a} = \mathbb{R}\, H_\alpha \oplus \ker\alpha$. Then $\vartheta_p(X_\alpha + Y_\alpha) = \vartheta_p X_\alpha - \vartheta_p^2 X_\alpha = -(Y_\alpha + X_\alpha)$, so $X_\alpha + Y_\alpha \in \mathfrak{p}$ and $|X_\alpha + Y_\alpha|^2 = |X_\alpha|^2_{B_p} + |Y_\alpha|^2_{B_p} = \frac{4}{|H_\alpha|^2}$. So $\frac{|H_\alpha|}{2}(X_\alpha + Y_\alpha)$ and $\frac{H_\alpha}{|H_\alpha|}$ are orthonormal unit vectors in $\mathfrak{p}$, and

$$\begin{aligned}\kappa\left(\operatorname{Span}\left\{\frac{|H_\alpha|}{2}(X_\alpha + Y_\alpha), \frac{H_\alpha}{|H_\alpha|}\right\}\right) &= -\alpha\left(\frac{H_\alpha}{|H_\alpha|}\right)^2 \left|\frac{|H_\alpha|}{2}(X_\alpha + Y_\alpha)\right|^2 \\ &= -\frac{|H_\alpha|^4}{|H_\alpha|^2}\frac{|H_\alpha|^2}{4}\frac{4}{|H_\alpha|^2} = -|H_\alpha|^2\end{aligned}$$

by the formula above.

Example 3.6. In the symmetric space associated with $\mathfrak{sl}(d,\mathbb{R})$, the root spaces $\mathfrak{g}_\alpha$ are one-dimensional, so the subalgebra $\mathfrak{sl}(2,\mathbb{R})_\alpha$ spanned by $X_\alpha, Y_\alpha, \tau_\alpha$ is uniquely determined by α and we denote it by $\mathfrak{sl}(2,\mathbb{R})_\alpha$. The image of $\mathbb{R}\, H_\alpha \oplus \mathfrak{p}_\alpha$ under the Riemannian exponential map at p is a totally geodesic submanifold $\mathbb{H}^2_\alpha$ isometric to the hyperbolic plane of curvature $-\frac{1}{d}$.

3.5. Weyl chambers and the Weyl group

In this section, we describe Weyl faces as subsets of maximal abelian subspaces $\mathfrak{a} \subset \mathfrak{p}$. In Section 3.8, we will define Weyl faces as subsets of the visual boundary $\partial\mathbb{X}$, and explain how the definitions relate.

Let Λ be the roots of a restricted root space decomposition of a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. For each $\alpha \in \Lambda \subset \mathfrak{a}^*$, the kernel of α is called a *wall*, and a component C of the complement of the union of the walls is called an *open Euclidean Weyl chamber*; C is open in $\mathfrak{a}$. A vector $X \in \mathfrak{a}$ is called *regular* if it lies in an open Euclidean Weyl chamber and *singular* otherwise. The closure V of an open Euclidean Weyl chamber is a *closed Euclidean Weyl chamber*; V is closed in $\mathfrak{p}$ (see Figure 1).

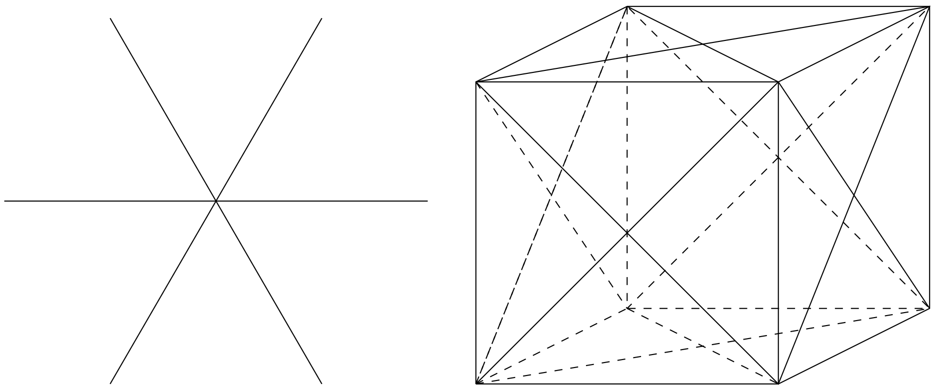
For a closed Weyl chamber V, there is an associated set of *positive roots*

$$\Lambda_+ := \{\alpha \in \Lambda \mid \forall v \in V, \alpha(v) \geq 0\}$$

and *simple roots* Δ, that is, those which cannot be written as a sum of two elements of Λ_+ (see [8, Proposition 2.9.6]).

We may define

$$N_K(\mathfrak{a}) := \{k \in K \mid \operatorname{Ad}(k)(\mathfrak{a}) = \mathfrak{a}\}, \quad Z_K(\mathfrak{a}) := \{k \in K \mid \forall A \in \mathfrak{a}, \operatorname{Ad}(k)(A) = A\}.$$



(a) The walls of a maximal flat in $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$. **(b)** The walls of a maximal flat in $\mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(4)$.

Figure 1. The walls of a maximal flat in $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ for $n = 3, 4$.

Since the adjoint action preserves the Killing form, $N_K(\mathfrak{a})$ acts by isometries on $\mathfrak{a}$ with kernel $Z_K(\mathfrak{a})$. We call the image of this action the *Weyl group*. For each reflection r_α in a wall, it is possible to find a $k \in K$ whose action on $\mathfrak{a}$ agrees with r_α [8, Proposition 2.9.7]. It is well known that the Weyl group acts simply transitively on the set of Weyl chambers, which implies it is generated by the reflections in the walls of a chosen Weyl chamber. It is convenient for us to show this fact in Proposition 3.8, since the same techniques provide Corollary 3.15.

The Riemannian exponential map identifies maximal abelian subspaces in $\mathfrak{p}$ isometrically with maximal flats through p. So we can also refer to open/closed Euclidean Weyl chambers in $\mathbb{X}$ as the images of those in some $\mathfrak{a}$ under this identification. For every $X \in \mathfrak{p}$, there exists a maximal abelian subspace $\mathfrak{a}$ containing X, and in $\mathfrak{a}$, there exists some closed Euclidean Weyl chamber V containing X.

3.6. A Morse function on flag manifolds

In this subsection, we show that the vector-valued distance function $\vec{d}$ on $\mathbb{X}$ (denoted d_Δ in [19, 20], see Definition 3.4) is well defined, and give part of a proof of Theorem 3.10, an important part of the structure theory of symmetric spaces. Along the way we prove the $\vec{d}$-triangle inequality [18–20, 27], and provide an estimate on the Hessian of a certain Morse function defined on flag manifolds embedded in $\mathfrak{p}$ (see Proposition 3.8 and Corollary 3.15).

We will use the following proposition. For $A \in \mathfrak{p}$, let $\mathfrak{e}_A$ be the intersection of all maximal abelian subspaces containing A.

Proposition 3.7 ([8, Proposition 2.20.18]). *Let p in $\mathbb{X}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let $k \in K$ and $A \in \mathfrak{p}$. If* $\mathrm{Ad}(k)(A) = A$ *then for all $E \in \mathfrak{e}_A$ we have* $\mathrm{Ad}(k)(E) = E$.

Note that there is a typo in Eberlein: The word "maximal" is omitted in the definition of $\mathfrak{e}_A$. The proof of Proposition 3.7 relies on passing to the compact real form of $\mathfrak{g}^{\mathbb{C}}$.

In this section, a *flag manifold* is the orbit of a vector $Z \in \mathfrak{p}$ under the adjoint action of $K = \mathrm{Stab}_G(p)$. The following proposition is essentially a standard part of the theory of symmetric spaces; however, we will need to extract a specific estimate, recorded in Corollary 3.15, in order to prove Lemma 4.8.

Proposition 3.8 (Cf. [14, Lemma 6.3, p. 211] and [7, Proposition 24]). *Let $X, Z \in \mathfrak{p}$ be unit vectors. Define*

$$f : K \to \mathbb{R}, \quad f(k) := B(X, \mathrm{Ad}(k)Z).$$

(1) *If k is a critical point for f, then $\mathrm{Ad}(k)Z$ commutes with X.*
(2) *If k is a local maximum for f, then $\mathrm{Ad}(k)Z$ lies in a common closed Weyl chamber with X.*
(3) *If X is regular then the function $B(X, \cdot) : \mathrm{Ad}(K)Z \to \mathbb{R}$ is Morse and has a unique local maximum.*
(4) *If X is regular then the distance function $d(X, \cdot) : \mathrm{Ad}(K)Z \to \mathbb{R}$ has a unique local minimum.*

Note that f is the composition of the orbit map $K \to \mathrm{Ad}(K)Z$ with the map $B(X, \cdot) : \mathrm{Ad}(K)Z \to \mathbb{R}$.

Proof. (1). Let $Y \in \mathfrak{k}$, viewed as a left-invariant vector field on K. If k is a critical point for f, then

$$\begin{aligned} 0 = \mathrm{d}f_k(Y) &= \frac{\mathrm{d}}{\mathrm{d}t} f(ke^{tY})|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} B(X, \mathrm{Ad}(ke^{tY})Z)|_{t=0} \\ &= B(X, \mathrm{Ad}(k)(\mathrm{ad}(Y)(Z))) \\ &= B(X, [Y', Z']) = B([Z', X], Y'), \end{aligned}$$

where we write $Y' = \mathrm{Ad}(k)Y$ and $Z' = \mathrm{Ad}(k)Z$. Since Y' is an arbitrary element of $\mathfrak{k}$, $[X, Z'] \in \mathfrak{k}$, and B is negative definite on $\mathfrak{k}$, we can conclude that $[X, Z'] = 0$, which is the claim (see Figure 2).

(2). At a critical point k for f, the Hessian of f at k is a symmetric bilinear form on $T_k K$ determined by

$$\mathrm{Hess}(f)(v, v)_k = (f \circ c)''(0)$$

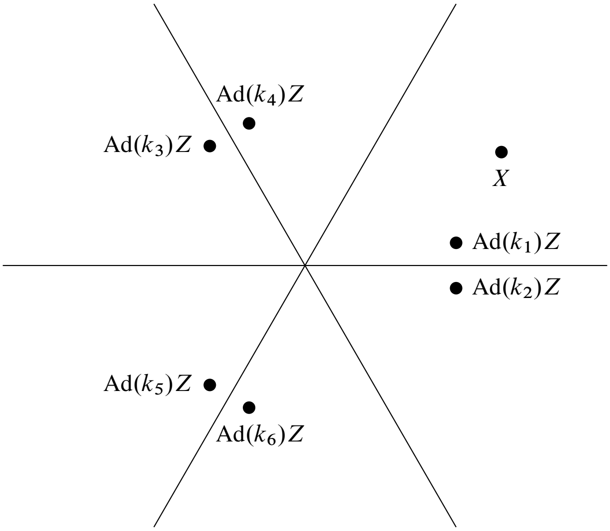


Figure 2. The intersection $\mathrm{Ad}(K)Z \cap \mathfrak{a}$.

for any curve c with $c(0) = k$ and $c'(0) = v$. Let $Y \in \mathfrak{k}$, the left-invariant vector fields on K, and choose $c(t) = ke^{tY}$. To compute the Hessian of f we only need to compute

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(ke^{tY})|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} B(X, \mathrm{Ad}(ke^{tY})(\mathrm{ad}(Y)(Z)))|_{t=0} \\ &= B(X, \mathrm{Ad}(k)([Y,[Y,Z]])) \\ &= B(X, [Y',[Y',Z']]) \\ &= B([X,Y'],[Y',Z']) \\ &= B([Z',[X,Y']],Y') \\ &= B(\mathrm{ad}(Z')\,\mathrm{ad}(X)(Y'),Y') = B(TY',Y'),\end{aligned}$$

where we write $T = \mathrm{ad}(Z') \circ \mathrm{ad}(X)$ as a linear transformation on $\mathfrak{k}$. At a critical point X and Z' commute by part (1), and we can choose a maximal abelian subspace $\mathfrak{a}$ containing both of them, and then consider the corresponding restricted root space decomposition. For $Y_\alpha \in \mathfrak{k}_\alpha$,

$$TY_\alpha = \alpha(Z')\alpha(X)Y_\alpha,$$

so the transformation T has the eigenvalue $\alpha(Z')\alpha(X)$ on its eigenspace $\mathfrak{k}_\alpha$ and acts as 0 on $\mathfrak{k}_0$. Since we assumed k is a local maximum for f, we have

$$0 \geq \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(ke^{tY})|_{t=0} = B(TY',Y')$$

for all $Y \in \mathfrak{k}$, so for each $\alpha \in \Lambda$, $\alpha(Z')\alpha(X) \geq 0$, and therefore X and Z' lie in a common closed Weyl chamber.

(3). We may assume that Z is a critical point of f by precomposing f with a left translation of K. The differential $(\mathrm{d}\,\mathrm{orb}_Z)_1 : \mathfrak{k} \to T_z\, \mathrm{Ad}(K)Z$ is given by $-\mathrm{ad}\, Z$ and has kernel $\mathfrak{k}_Z = Z_{\mathfrak{k}}(Z) = \{W \in \mathfrak{k} \mid [W,Z] = 0\}$ with orthogonal complement $\mathfrak{k}^Z = \bigoplus_{\alpha \in \Lambda : \alpha(Z) > 0} \mathfrak{k}_\alpha$. Then k is a critical point for f if and only if $Z(k) = \mathrm{Ad}(k)Z$ is a critical point for $B(X,\cdot)$. The Hessians satisfy

$$\mathrm{Hess}(B(X,\cdot))((\mathrm{d}\,\mathrm{orb}_Z)_k U, (\mathrm{d}\,\mathrm{orb}_Z)_k V)_{\mathrm{Ad}(k)Z} = \mathrm{Hess}(f)(U,V)_k,$$

so by the calculation above the critical points are nondegenerate, occur at $\mathrm{Ad}(k)Z$ when $[\mathrm{Ad}(k)Z, X] = 0$ and have index the number of positive signs in the collection $\alpha(X)\alpha(\mathrm{Ad}(k)Z)$, (weighted by $\dim \mathfrak{k}_\alpha$) as α ranges over the roots with $\alpha(Z) > 0$. These can only be nonnegative when $\mathrm{Ad}(k)Z$ lies in the closed Weyl chamber containing X.

For uniqueness, observe that any two maximizers Z', Z'' lie in the closed Weyl chamber containing X, and suppose $\mathrm{Ad}(k)(Z') = Z''$. The adjoint action takes walls to walls, so $\mathrm{Ad}(k)$ preserves the facet spanned by Z', Z'' and hence fixes its soul (i.e., its center of mass) [8, p. 65]. By Proposition 3.7, $\mathrm{Ad}(k)$ fixes each point of the face, and in particular $Z' = Z''$.

(4). Since $(\mathfrak{p}, B)$ is a Euclidean space,

$$d_{\mathfrak{p}}(X,Y)^2 = B(X-Y, X-Y) = B(X,X) + B(Y,Y) - 2B(X,Y)$$

so if X, Y are unit vectors in $\mathfrak{p}$

$$d_{\mathfrak{p}}(X,Y)^2 = 2(1 - B(X,Y))$$

and the distance function $d_{\mathfrak{p}}(X,\cdot)$ is minimized when $B(X,\cdot)$ is maximized. Then by part (3), the distance function is uniquely minimized at the unique $\mathrm{Ad}(k)Z$ in the closed Weyl chamber containing X. ■

The next two results are part of the standard theory of symmetric spaces. Since we have already proven Proposition 3.8, it is convenient to give the proofs.

Corollary 3.9 ([8, Section 2.12]). *Every K-orbit in the unit sphere $S(\mathfrak{p})$ intersects each closed spherical Weyl chamber exactly once.*

Proof. Let X be a regular vector in a chosen Weyl chamber. The K-orbit of a unit vector Z is compact and therefore the function $d_{\mathfrak{p}}(X,\cdot)$ has a global minimum on $\mathrm{Ad}(K)Z$. But that function has a unique local minimum which must lie in the chosen closed Weyl chamber. ■

For a point $p \in \mathbb{X}$, maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and closed Euclidean Weyl chamber $V \subset \mathfrak{a}$, we call $(p, \mathfrak{a}, V)$ a point-chamber triple.

Theorem 3.10 ([8, Section 2.12]). *For any two point-chamber triples $(p,\mathfrak{a},V), (p',\mathfrak{a}',V')$ there exists an isometry $g \in G$ taking $(p,\mathfrak{a},V)$ to $(p',\mathfrak{a}',V')$. If g stabilizes $(p,\mathfrak{a},V)$, then it acts trivially on it.*

Proof. The group G acts transitively on X, so we may assume that $p' = p$ and then show that an element of $K = \mathrm{Stab}_G(p)$ takes $(\mathfrak{a}, V)$ to $(\mathfrak{a}', V')$. Choose any regular unit vectors $X \in V$, $Z \in V'$. Then Proposition 3.8 implies there is an element $k \in K$ such that $\mathrm{Ad}(k)Z$ is in the same open Weyl chamber as X. Regular vectors lie in unique Weyl chambers in unique maximal abelian subspaces, so $\mathrm{Ad}(k)\mathfrak{a}' = \mathfrak{a}$ and $\mathrm{Ad}(k)V' = V$.

If g fixes p and stabilizes $(\mathfrak{a}, V)$, then it acts trivially on V by Corollary 3.9. ∎

The above isometry is not necessarily unique. For example, consider hyperbolic space $\mathbb{H}^n, n \geq 3$. There a Euclidean Weyl chamber is just a geodesic ray, which has infinite pointwise stabilizer. However, the action on V is unique.

As a corollary, we may define the *vector-valued distance function*

$$\vec{d}\colon \mathbb{X} \times \mathbb{X} \to (\mathbb{X} \times \mathbb{X})/G =: V_{\mathrm{mod}} \tag{3.4}$$

to have range a model closed Euclidean Weyl chamber. One could think of V_{mod} as some preferred Euclidean Weyl chamber, but it is better to think of it as an abstract Euclidean cone with no reference to a preferred basepoint, flat or Weyl chamber in $\mathbb{X}$. There is an "opposition involution" $\iota\colon V_{\mathrm{mod}} \to V_{\mathrm{mod}}$ induced by any geodesic symmetry S_p. On a model pointed flat $\mathfrak{a}_{\mathrm{mod}}$, the composition of $-\mathrm{id}$ with the longest element of the Weyl group restricts to ι on the model positive chamber V_{mod}. Note that $\iota\vec{d}(p,q) = \vec{d}(q,p)$.

The triangle inequality implies that for any p, p', q, q' in a metric space,

$$|d(p,q) - d(p',q')| \leq d(p,p') + d(q,q').$$

The next result is the "vector-valued triangle inequality" for symmetric spaces.

Corollary 3.11 (The $\vec{d}$-triangle inequality [18, 20, 27]). *For points p, p', q, q' in $\mathbb{X}$,*

$$|\vec{d}(p,q) - \vec{d}(p',q')| \leq d(p,p') + d(q,q').$$

Proof. In a moment we will use the proposition to prove that for any p, q, q' in $\mathbb{X}$,

$$|\vec{d}(p,q) - \vec{d}(p,q')| \leq d(q,q'), \tag{3.5}$$

from which the general inequality follows easily:

$$\begin{aligned} |\vec{d}(p,q) - \vec{d}(p',q')| &= |\vec{d}(p,q) - \vec{d}(p,q') + \vec{d}(p,q') - \vec{d}(p',q')| \\ &\leq |\vec{d}(p,q) - \vec{d}(p,q')| + |\iota\vec{d}(q',p) - \iota\vec{d}(q',p')| \\ &\leq d(q,q') + d(p,p'). \end{aligned}$$

To prove (3.5), let $X, Z \in \mathfrak{p}$ such that $e^X p = q$ and $e^Z p = q'$. Choose a closed Weyl chamber V containing X and the unique Z' in the K-orbit of Z in that Weyl chamber. The map

$\vec{d}(p, e^{(\cdot)}p): V \to V_{\mathrm{mod}}$ is an isometry. Note that $k \mapsto B(X, \mathrm{Ad}(k)Z)$ is maximized when $k \mapsto B(X, \mathrm{Ad}(k)Z)/|X||Z|$ is maximized, so by Proposition 3.8

$$\begin{aligned}|\vec{d}(p,q) - \vec{d}(p,q')|^2 = |X - Z'|^2 &= |X|^2 + |Z'|^2 - 2\langle X, Z'\rangle \\ &\leq |X|^2 + |Z|^2 - 2\langle X, Z\rangle \\ &= d_{\mathfrak{p}}(X,Z)^2 \leq d(q,q')^2\end{aligned}$$

since the Riemannian exponential map is distance non-decreasing by the nonpositive curvature of $\mathbb{X}$. ∎

3.7. Regularity in maximal abelian subspaces

A *spherical Weyl chamber* is the intersection of a Euclidean Weyl chamber with the unit sphere S in $\mathfrak{a}$. A spherical Weyl chamber σ is a spherical simplex, and each of its faces τ is called a *Weyl face*. Each Euclidean (resp. spherical) Weyl face is the intersection of walls of $\mathfrak{a}$ (resp. as well as S). The interior of a face $\mathrm{int}(\tau)$ is obtained by removing its proper faces; the interiors of faces are called *open simplices*. The unit sphere S is a disjoint union of the open simplices. If τ is the smallest simplex containing a unit vector X in its interior, we say that τ is *spanned* by X and X is τ*-spanning*.

We will quantify the regularity of tangent vectors using a parameter $\alpha_0 > 0$. We will show in Proposition 3.17 that our definition of regularity is equivalent to the definition in [19]. A similar definition appears in [21, Definition 2.6].

Definition 3.12 (Regularity). Let $p \in \mathbb{X}$ and $\mathbb{X}$ be a closed spherical Weyl chamber and let τ be a face of σ. Consider the corresponding maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and set of simple roots Δ. We define

$$\Delta_\tau = \{\alpha \in \Delta \mid \alpha(\tau) = 0\}, \quad \Delta_\tau^+ = \{\alpha \in \Delta \mid \alpha(\mathrm{int}\,\tau) > 0\}. \tag{3.6}$$

A vector $X \in \mathfrak{a}$ is called (α_0, τ)*-regular* if for each $\alpha \in \Delta_\tau^+, \alpha(X) \geq \alpha_0|X|$. A geodesic c at p is called (α_0, τ)*-regular* if $c'(0) = \mathrm{ev}_p X$ for an (α_0, τ)-regular vector $X \in \mathfrak{a}$.

It is immediate from the definition that X is (α_0, σ)-regular for some $\alpha_0 > 0$ and σ if and only if X is regular. We define

$$\Lambda_\tau := \{\alpha \in \Lambda \mid \alpha(\tau) = 0\}, \quad \Lambda_\tau^+ := \{\alpha \in \Lambda \mid \alpha(\mathrm{int}\,\tau) > 0\}. \tag{3.7}$$

Observe that X is (α_0, τ)-regular if and only if for each root $\alpha \in \Lambda_\tau^+$ we have $\alpha(X) \geq \alpha_0|X|$.

Remark 3.13. The signed distance from a vector $A \in \mathfrak{a}$ to the wall $\ker \alpha$ is $\alpha(A)/|\alpha| \geq \alpha(A)/\kappa_0$.

Definition 3.14. A unit vector X is (α_0, τ)*-spanning* if it is τ-spanning and (α_0, τ)-regular.

We may now record a mild extension of Proposition 3.8 which will appear in Lemma 4.8.

Corollary 3.15. *Suppose $X \in \mathfrak{p}$ is an (α_0, τ)-regular unit vector and $Z \in \mathfrak{p}$ is a (ζ_0, τ)-spanning unit vector. Then Z is the unique maximum of $B(X, \cdot)\colon \mathrm{Ad}(K)Z \to \mathbb{R}$, and for all $U, V \in T_Z \mathrm{Ad}(K)Z$,*

$$|\mathrm{Hess}(B(X,\cdot))(U,V)_Z| \geq \alpha_0 \zeta_0 |B_p(U,V)|.$$

Proof. The proof of Proposition 3.8 goes through in this setting, requiring only the following observation: If X is τ-regular and lies in a spherical Weyl chamber σ, then τ is a face of σ. If $U, V \in T_Z \mathrm{Ad}(K)Z$ correspond to $U', V' \in \mathfrak{k}^\tau$ under the identification $T_Z \mathrm{Ad}(K)Z = \mathfrak{k}^\tau$, we showed that $\mathrm{Hess}(B(X,\cdot))(U,V)_Z = B(\mathrm{ad}(Z)\,\mathrm{ad}(X)U', V')$. ■

3.8. The visual boundary $\partial \mathbb{X}$

A pair of unit-speed geodesic rays c_1, c_2 are called *asymptotic* if there exists a constant $D > 0$ such that

$$d(c_1(t), c_2(t)) \leq D$$

for all $t \geq 0$. The asymptote relation is an equivalence relation on unit-speed geodesic rays and the set of asymptote classes is called the *visual boundary* of $\mathbb{X}$ and denoted by $\partial \mathbb{X}$. There is a natural topology on $\partial \mathbb{X}$ called the *cone topology*, where for each point $p \in \mathbb{X}$ the map $S(T_p \mathbb{X}) \to \partial \mathbb{X}$ (which takes a unit tangent vector to the geodesic ray with that derivative) is a homeomorphism. In fact, the cone topology extends to $\overline{\mathbb{X}} := \mathbb{X} \cup \partial \mathbb{X}$, yielding a space homeomorphic to a unit ball of the same dimension as $\mathbb{X}$.

Lemma 3.16. *If c_1 and c_2 are asymptotic geodesic rays, then for all $t \geq 0$,*

$$d(c_1(t), c_2(t)) \leq d(c_1(0), c_2(0)).$$

Proof. The left-hand side, being convex [8] and bounded above, is therefore (weakly) decreasing. ■

We have a natural action of G on $\partial \mathbb{X}$: $g[c] = [g \circ c]$. For $\eta \in \partial \mathbb{X}$, we denote the stabilizer

$$G_\eta := \{g \in G \mid g\eta = \eta\}$$

and call G_η the *parabolic subgroup* fixing η. (Note that in [9, 11], G itself is a parabolic subgroup, but in this paper a parabolic subgroup is automatically a proper subgroup.) When η is regular, G_η is a *minimal parabolic* subgroup of G (sometimes called a Borel subgroup).

Let η, η' be ideal points in $\partial \mathbb{X}$, represented by the geodesics $c(t) = e^{tX} p$ and $c'(t) = e^{tY} q$. Then since G is transitive on point-chamber triples, we can find $g \in G$ such that $gq = p$ and $\mathrm{Ad}(g)Y$ lies in a (closed) Euclidean Weyl chamber in common with X. In particular, every G orbit in $\partial\mathbb{X}$ intersects every spherical Weyl chamber exactly once.

Each unit sphere $S(\mathfrak{p})$ has the structure of a simplicial complex compatible with the action of G. By Theorem 3.10, this simplicial structure passes to $\partial\mathbb{X}$, which is in fact a thick spherical building whose apartments are the ideal boundaries of maximal flats. In [19, 20], the spherical building structure on $\partial\mathbb{X}$ is used to describe the regularity of geodesic rays. We have used the restricted roots to define regularity and will show that the notions are equivalent in Proposition 3.17. When we need to distinguish between simplices in $S(\mathfrak{p})$ and simplices in $\partial\mathbb{X}$ we call the former *spherical* and the latter *ideal*. Compared to a spherical simplex, an ideal simplex lacks the data of a basepoint $p \in \mathbb{X}$.

Define the *type map* to be

$$\theta\colon \partial\mathbb{X} \to \partial\mathbb{X}/G =: \sigma_{\mathrm{mod}}$$

with range the model ideal Weyl chamber. The opposition involution $\iota\colon V_{\mathrm{mod}} \to V_{\mathrm{mod}}$ induces an opposition involution $\iota\colon \sigma_{\mathrm{mod}} \to \sigma_{\mathrm{mod}}$; see the discussion after equation (3.4) in the previous subsection. The faces of σ_{mod} are called *model simplices*. For a model simplex $\tau_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$, we define the *flag manifold* $\mathrm{Flag}(\tau_{\mathrm{mod}})$ to be the set of simplices τ in $\partial\mathbb{X}$ such that $\theta(\tau) = \tau_{\mathrm{mod}}$. If ideal points η, η' span the same simplex τ, then they correspond to the same parabolic subgroup, so we define $G_\tau := G_\eta$. A model simplex corresponds to the conjugacy class of a parabolic subgroup of G.

3.9. Regularity for ideal points

Theorem 3.10 implies that "model roots" are well defined: If $g \in G$ takes the point-chamber triple $(p, \mathfrak{a}, V)$ to $(p', \mathfrak{a}', V')$ and takes the simplex $\tau \subset \partial V$ to $\tau' \subset \partial V'$, it also takes Δ_τ to $\Delta'_{\tau'}$ and Δ_τ^+ to $\Delta'^{+}_{\tau'}$, where Δ is the simple roots in $\mathfrak{a}^*$ corresponding to V and Δ' is the simple roots in $\mathfrak{a}'$ corresponding to V'.

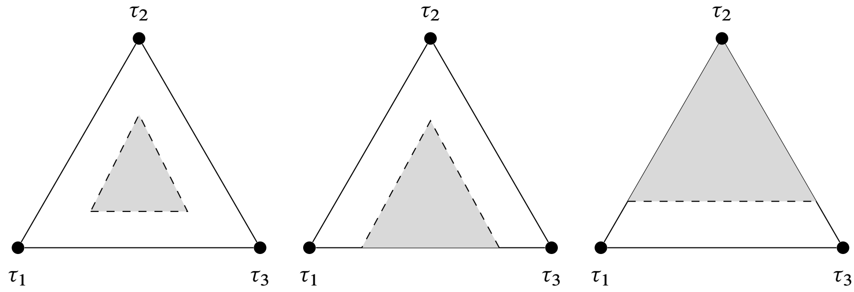
An ideal point $\eta \in \partial\mathbb{X}$ is called (α_0, τ)*-regular* if every geodesic in its asymptote class is (α_0, τ)-regular. As soon as one representative of an ideal point is (α_0, τ)-regular, every representative is. A vector, geodesic or ideal point is $(\alpha_0, \tau_{\mathrm{mod}})$*-regular* if it is (α_0, τ)-regular for some simplex τ of type τ_{mod} (see Figure 3).

Following [20], the *open star* of a simplex τ, denoted $\mathrm{ost}(\tau)$, is the union of open simplices ν whose closures intersect τ. Equivalently, it is the collection of τ-regular points in ∂X. For a model simplex, $\mathrm{int}_{\tau_{\mathrm{mod}}}(\sigma_{\mathrm{mod}})$ is the collection of τ_{mod}-regular ideal points in σ_{mod}. Equivalently, it is $\sigma_{\mathrm{mod}} \setminus \bigcup_{\alpha\in\Delta_\tau^+} \ker\alpha$.[1] We have

$$\tau = \sigma \cap \bigcap_{\alpha\in\Delta_\tau} \ker\alpha,\ \ \mathrm{int}_\tau\,\sigma = \{\eta \in \sigma \mid \forall\alpha \in \Delta_\tau^+, \alpha(\eta) > 0\},\ \ \partial_\tau\sigma = \sigma \cap \bigcup_{\alpha\in\Delta_\tau^+} \ker\alpha.$$

There is a decomposition $\sigma_{\mathrm{mod}} = \mathrm{int}_{\tau_{\mathrm{mod}}}\,\sigma_{\mathrm{mod}} \sqcup \partial_{\tau_{\mathrm{mod}}}\sigma_{\mathrm{mod}}$.

[1] In [19], the notation $\mathrm{ost}(\tau_{\mathrm{mod}})$ was used for what is called $\mathrm{int}_{\tau_{\mathrm{mod}}}(\sigma_{\mathrm{mod}})$ here and in [20].



(a) $(\alpha_0, \sigma_{\mathrm{mod}})$-regular. **(b)** (α_0, τ_{13})-regular. **(c)** (α_0, τ_2)-regular.

Figure 3. $(\alpha_0, \tau_{\mathrm{mod}})$-regularity for various choices of τ_{mod}.

The set of (α_0, τ)-regular points is called the "α_0-star of τ." The closed cone on the α_0-star of τ at p is denoted by

$$V(p, \mathrm{st}(\tau), \alpha_0) := \{c_{px}(t) \mid t \in [0, \infty), x \text{ is } (\alpha_0, \tau)\text{-regular}\},$$

the cone on the open star of τ by

$$V(p, \mathrm{ost}(\tau)) := \{c_{px}(t) \mid t \in [0, \infty), x \text{ is } \tau\text{-regular}\}$$

and the Euclidean Weyl sector by

$$V(p, \tau) := \{c_{px}(t) \mid t \in [0, \infty), x \text{ is } \tau\text{-spanning}\}.$$

It follows from Lemma 3.16 that the Hausdorff distance between $V(p, \mathrm{st}(\tau), \alpha_0)$ and $V(q, \mathrm{st}(\tau), \alpha_0)$ is bounded above by $d(p, q)$, and the same holds for the open cones $V(p, \mathrm{ost}(\tau))$ and $V(q, \mathrm{ost}(\tau))$ and for the Weyl sectors $V(p, \tau)$, $V(q, \tau)$.

We now describe the notion of regularity used in [19, 20] and show it is equivalent to our definition. We always work with respect to a fixed type τ_{mod}. A subset $\Theta \subset \sigma_{\mathrm{mod}}$ is called τ_{mod}-*Weyl convex* if its symmetrization $W_{\tau_{\mathrm{mod}}}\Theta \subset a_{\mathrm{mod}}$ is a convex subset of the model apartment a_{mod}. Here we think of the Weyl group W as acting on the visual boundary a_{mod} of a model flat $\mathfrak{a}_{\mathrm{mod}}$ with distinguished Weyl chamber σ_{mod} and $W_{\tau_{\mathrm{mod}}}$ is the subgroup of W stabilizing the simplex τ_{mod}. One then quantifies τ_{mod}-regular ideal points by fixing an auxiliary compact τ_{mod}-Weyl convex subset Θ of $\mathrm{int}_{\tau_{\mathrm{mod}}}(\sigma_{\mathrm{mod}}) \subset \sigma_{\mathrm{mod}}$.

An ideal point η is Θ-*regular* if $\theta(\eta) \in \Theta$. It is easy to see that the notions of Θ-regularity and $(\alpha_0, \tau_{\mathrm{mod}})$-regularity are equivalent.

Proposition 3.17. *Let $\Delta_{\tau_{\mathrm{mod}}} \subset \Delta$ be the model simple roots corresponding to a simplex $\tau_{\mathrm{mod}} \subset \sigma_{\mathrm{mod}}$. Then*

(1) *If Θ is a compact subset of $\mathrm{int}_{\tau_{\mathrm{mod}}}(\sigma_{\mathrm{mod}})$ then every Θ-regular ideal point is $(\alpha_0, \tau_{\mathrm{mod}})$-regular for $\alpha_0 = \min_{\alpha \in \Delta^+_{\tau_{\mathrm{mod}}}} \alpha(\Theta)$.*

(2) *Every* $(\alpha_0, \tau_{\mathrm{mod}})$*-regular ideal point is* Θ*-regular for* $\Theta = \{\xi \in \sigma_{\mathrm{mod}} \mid \forall \alpha \in \Delta^+_{\tau_{\mathrm{mod}}}, \alpha(\zeta) \geq \alpha_0\}$.

Proof. We first prove (1). Since Θ is a compact subset of $\sigma_{\mathrm{mod}} \setminus \bigcup_{\alpha \in \Delta^+_{\tau_{\mathrm{mod}}}} \ker \alpha$, the quantity $\min\{\alpha(\zeta) \mid \alpha \in \Delta^+_{\tau_{\mathrm{mod}}}, \zeta \in \Theta\}$ exists and is positive.

We now prove (2). The subset $\Theta = \{\zeta \in \sigma_{\mathrm{mod}} \mid \forall \alpha \in \Delta^+_{\tau_{\mathrm{mod}}}, \alpha(\zeta) \geq \alpha_0\}$ has symmetrization $W_{\tau_{\mathrm{mod}}}\Theta = \{\xi \in a_{\mathrm{mod}} \mid \forall \alpha \in \Delta^+_{\tau_{\mathrm{mod}}}, \alpha(\xi) \geq \alpha_0\}$ which is an intersection of finitely many half-spaces together with the unit sphere, so it is compact and convex. Furthermore, $\Theta = \sigma_{\mathrm{mod}} \cap W_{\tau_{\mathrm{mod}}}\Theta$ is a compact subset of $\mathrm{int}_{\tau_{\mathrm{mod}}}(\sigma_{\mathrm{mod}}) \cap \sigma_{\mathrm{mod}}$. ∎

3.10. Choosing ζ_{mod} and computing ζ_0

Throughout the paper it will be essential to choose an auxiliary ι-invariant model ideal point $\zeta_{\mathrm{mod}} \in \mathrm{int}(\tau_{\mathrm{mod}})$. The regularity parameter ζ_0 of ζ_{mod} will appear in many estimates below. In this subsection, we explain how to compute ζ_0 when τ_{mod} is a minimal ι-invariant face of σ_{mod}. In this case, there is a unique choice of ζ_{mod}. In Subsection 3.10.1, we compute $\widetilde{\zeta_0}$, which agrees with the regularity parameter ζ_0 up to renormalizing the longest simple restricted root to have $\sqrt{2}$. These numbers are presented in Table 3. In the present subsection, we explain how to compute the renormalizing constant.

Proposition 3.18. *Let B' be the renormalized Killing form $B = 2h^\vee B'$. If the restricted root system is reduced, then the longest norm of a simple root with respect to B' is C_1. In the two rank 1 non-reduced restricted root systems, the longest norm of a simple root with respect to B' is $\frac{C_1}{2}$. In the four remaining cases, which are AIIIa, CIIa, DIIb and EIIIa, the longest norm of a simple root with respect to B' is $\frac{C_1}{\sqrt{2}}$.*

Proof. If the restricted root system is reduced, then every restricted root is in the Weyl group orbit of a simple root.

The non-reduced cases can be analyzed by consulting [13, Table VI, Ch. X]. According to this table, there are six cases of non-reduced restricted root systems.

The cases of AIV and FII have real rank 1, so there is a unique simple restricted root λ and this root is non-reduced. Then 2λ is a restricted root of maximal length, so $C_1 = |2\lambda|_{B'}$.

The remaining non-reduced cases are AIIIa, CIIa, DIIb and EIIIa. In each of these cases, there is a unique non-reduced simple restricted root λ. Moreover, the root λ is the unique short simple restricted root, and there exists at least one strictly longer simple restricted root. Since the long restricted roots are reduced, $C_1 = |2\lambda|_{B'}$, and in each case the long simple roots have norm $\sqrt{2}|\lambda|_{B'} = \sqrt{2}C_1$. ∎

To give a succinct description of ζ_0 in each case, we introduce the constant C_3 (see Table 2). We are avoiding the notation C_2 since this constant appears in [1], but we do not need it here. We set C_3 to be 1 in all cases except the non-reduced restricted root systems. For non-reduced restricted root systems, we set C_3 to be 2 for the rank 1 cases and $\sqrt{2}$ otherwise.

$\mathfrak{g}$	Restricted Root System	C_3
AIV, FII	Non-reduced and rank 1	2
AIIIa, CIIa, DIIb, EIIIa	Non-reduced and rank ≥ 2	$\sqrt{2}$
All others	Reduced	1

Table 2. C_3 in terms of the restricted root system of $\mathfrak{g}$.

Lemma 3.19. *Consider a further renormalization of the Killing form*

$$B = \frac{4C_3^2 h^\vee}{C_1^2} B''.$$

With respect to B'', the longest norm of a simple restricted root is $\sqrt{2}$.

Proof. For any simple restricted root α, we have

$$\frac{C_1}{2C_3\sqrt{h^\vee}} |H_\alpha^{B''}|_{B''} = |H_\alpha^B|_B = \frac{1}{\sqrt{2h^\vee}} |H_\alpha^{B'}|_{B'},$$

so

$$|H_\alpha^{B''}|_{B''} = \frac{\sqrt{2}C_3}{C_1} |H_\alpha^{B'}|_{B'}$$

which has maximum $\sqrt{2}$ by Proposition 3.18 and the definition of C_3. ∎

Proposition 3.20. *Let Δ be the set of simple roots in a restricted root system and let $\alpha_k \in \Delta$. Let ζ_{mod} be the unique ι-invariant unit vector in the face τ_{mod} corresponding to $\{\alpha_k, \iota(\alpha_k)\}$. Then ζ_{mod} is $(\zeta_0, \tau_{\mathrm{mod}})$-spanning where*

$$\zeta_0 = \frac{C_1}{2C_3\sqrt{h^\vee}} \widetilde{\zeta_0},$$

and $\widetilde{\zeta_0}$ is recorded in Table 3. Moreover, the regularity parameter ζ_0 is optimal.

3.10.1. $\widetilde{\zeta_0}$ in standardized root systems. Below we will give a brief description of each irreducible reduced root system. Each root system is considered to be a subset of the Euclidean space $\mathbb{R}^n$ with the standard basis $e_1, \ldots, e_n$, standard inner product and dual basis $e^1, \ldots, e^n$. We choose a scaling of the roots so that the longest simple root has norm $\sqrt{2}$. We list the simple roots and describe the opposition involution ι. In Table 3, we depict the Dynkin diagram with labeled nodes. For each minimal ι-invariant subset of simple roots Θ, we record $\widetilde{\zeta_0} := \min\{\alpha(\zeta_{\mathrm{mod}}) \mid \alpha \in \Theta\}$ where ζ_{mod} is the unique ι-invariant unit vector in the face τ_{mod} corresponding to Θ. Note that minimal ι-invariant subsets are singletons or pairs, so the corresponding faces are vertices or edges.

In order to give a precise description of ζ_{mod}, we need a precise description of the vertices of σ_{mod}. For this purpose we consider the *fundamental weights ω_i* of the root system, which are uniquely defined by

$$2\frac{\langle \omega_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}$$

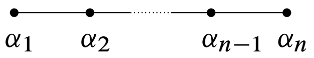
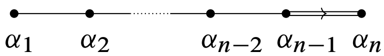
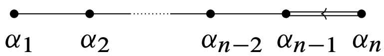
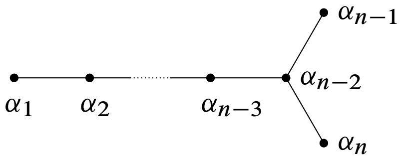
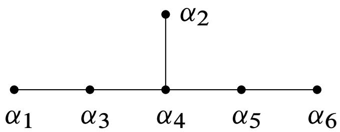
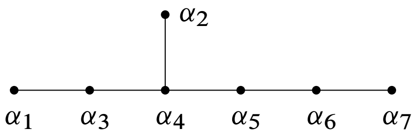
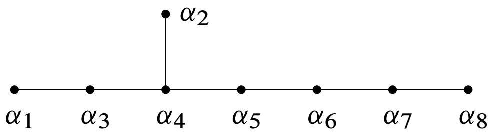
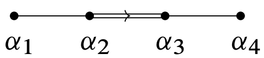
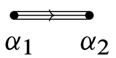
Name	**Diagram**	$\mathbf{1/\widetilde{\zeta_0}(\{\alpha_k, \iota(\alpha_k)\})}$
A_n		$\sqrt{2k}$, $2k < n+1$; $\sqrt{n+1}/2$, $2k = n+1$.
B_n		$\sqrt{k}$
C_n		$\sqrt{2k}$, $k < n$; $\sqrt{n/2}$, $k = n$.
D_n		$\sqrt{k}$, $k \leq n-2$; $2\sqrt{n}$, $k \geq n-1, n$ even; $\sqrt{n-1}$, $k = n-1, n$ odd.
E_6		$\sqrt{34}/2$, $k = 1$; $5\sqrt{2}/4$, $k = 2$; $\sqrt{354}/4$, $k = 3$; $\sqrt{42}/2$, $k = 4$.
E_7		$\sqrt{2}$, $k = 1$; $\sqrt{7/2}$, $k = 2$; $\sqrt{6}$, $k = 3$; $2\sqrt{3}$, $k = 4$; $\sqrt{15/2}$, $k = 5$; 2, $k = 6$; $\sqrt{3/2}$, $k = 7$.
E_8		2, $k = 1$; $2\sqrt{2}$, $k = 2$; $\sqrt{14}$, $k = 3$; $\sqrt{30}$, $k = 4$; $2\sqrt{5}$, $k = 5$; $2\sqrt{3}$, $k = 6$; $\sqrt{6}$, $k = 7$; $\sqrt{2}$, $k = 8$.
F_4		$\sqrt{2}$, $k = 1$; $\sqrt{6}$, $k = 2$; $\sqrt{12}$, $k = 3$; 2, $k = 4$.
G_2		$\sqrt{2}$, $k = 1$; $\sqrt{6}$, $k = 2$.

Table 3. $\widetilde{\zeta_0}$ for minimal ι-invariant subsets of irreducible root systems, normalized so that the longest simple roots have norm $\sqrt{2}$.

where the simple roots are $\Delta = \{\alpha_i\}$. Then the dual vectors $H_{\omega_i} \in \mathfrak{a}$ defined by $\langle H_{\omega_i}, A\rangle = \omega_i(A)$ are proportional to the vertices of σ_{mod}. If α_k is an ι-invariant simple root, then we set $\zeta_{\mathrm{mod}} = H_{\omega_k}/|H_{\omega_k}|$ and

$$\widetilde{\zeta_0} = \alpha_k(H_{\omega_k}/|H_{\omega_k}|) = |\alpha_k|^2/(2|\omega_k|).$$

If α_k is not ι-invariant, then we set $\zeta_{\mathrm{mod}} = (H_{\omega_k} + \iota(H_{\omega_k}))/|H_{\omega_k} + \iota(H_{\omega_k})|$ and

$$\widetilde{\zeta_0} = \alpha_k(H_{\omega_k}/|H_{\omega_k} + \iota(H_{\omega_k})|) = |\alpha_k|^2/(2|H_{\omega_k} + \iota(H_{\omega_k})|).$$

The fundamental weights of irreducible root systems can be found, for example, in [24].

- A_n. $E = \{v \in \mathbb{R}^{n+1} \mid \langle v, e_1 + \cdots + e_{n+1}\rangle = 0\}$. The simple roots are $\Delta = \{\alpha_i\} = \{e^i - e^{i+1}\}_{i=1}^n$. The fundamental weights are $\omega_i = e^1 + \cdots + e^i$, restricted to E. The opposition involution ι takes α_i to α_{n+1-i}.
- B_n. $E = \mathbb{R}^n$. The simple roots are $\Delta = \{e^i - e^{i+1}\}_{i=1}^{n-1} \cup \{e^n\}$. The fundamental weights are $\omega_i = e^1 + \cdots + e^i$ for $i < n$ and $\omega_n = \frac{1}{2}(e^1 + \cdots + e^n)$. The opposition involution is trivial.
- C_n. $E = \mathbb{R}^n$. The simple roots are $\Delta = \{\frac{1}{\sqrt{2}}(e^i - e^{i+1})\}_{i=1}^{n-1} \cup \{\sqrt{2}e^n\}$. For this scaling, the long root has norm $\sqrt{2}$. The fundamental weights are $\omega_i = \frac{1}{\sqrt{2}}(e^1 + \cdots + e^i)$ for all $i \leq n$. The opposition involution is trivial.
- D_n. $E = \mathbb{R}^n$. The simple roots are $\Delta = \{e^i - e^{i+1}\}_{i=1}^{n-1} \cup \{e^{n-1} + e^n\}$. The fundamental weights are $\omega_i = e^1 + \cdots + e^i$ for $i \leq n-2$, $\omega_{n-1} = \frac{1}{2}(e^1 + \cdots + e^{n-1} - e^n)$, $\omega_n = \frac{1}{2}(e^1 + \cdots + e^n)$. When n is even, the opposition involution is trivial. When n is odd, $\iota(\alpha_n) = \alpha_{n-1}$, and ι fixes the other simple roots.
- E_6. $E = \{v \in \mathbb{R}^8 \mid \langle v, e_6 - e_7\rangle = \langle v, e_7 + e_8\rangle = 0\}$. The simple roots are

  $$\Delta = \Big\{\frac{1}{2}(e^8 - e^7 - e^6 - e^5 - e^4 - e^3 - e^2 + e^1), \\ e^2 + e^1, e^2 - e^1, e^3 - e^2, e^4 - e^3, e^5 - e^4\Big\}.$$

  The fundamental weights are

  $$\{(0,0,0,0,0,-1/6,-7/6,2/3), (1/2,1/2,1/2,1/2,1/2,1/4,-5/4,1/2), \\ (-1/2,1/2,1/2,1/2,1/2,1/6,-11/6,5/6), (0,0,1,1,1,1/2,-5/2,1), \\ (0,0,0,1,1,7/12,-23/12,2/3), (0,0,0,0,1,2/3,-4/3,1/3)\}.$$

  The opposition involution takes α_1 to α_6, α_3 to α_5, and fixes α_2 and α_4. We have $\widetilde{\zeta_0}(\{\alpha_1,\alpha_6\}) = 1/|\omega_1 + \omega_6| = 2/\sqrt{34}$, $\widetilde{\zeta_0}(\{\alpha_2\}) = 1/|\omega_2| = 4/5\sqrt{2}$, $\widetilde{\zeta_0}(\{\alpha_3,\alpha_5\}) = 1/|\omega_3 + \omega_5| = 4/\sqrt{354}$ and $\widetilde{\zeta_0}(\{\alpha_4\}) = 1/|\omega_4| = 2/\sqrt{42}$.
- E_7. $E = \{v \in \mathbb{R}^8 \mid \langle v, e_7 + e_8\rangle = 0\}$. The simple roots are

  $$\Delta = \Big\{\frac{1}{2}(e^8 - e^7 - e^6 - e^5 - e^4 - e^3 - e^2 + e^1), \\ e^2 + e^1, e^2 - e^1, e^3 - e^2, e^4 - e^3, e^5 - e^4, e^6 - e^5\Big\}.$$

The opposition involution is trivial. The fundamental weights are

$$\Big\{e^8 - e^7, \frac{1}{2}(e^1 + e^2 + e^3 + e^4 + e^5 + e^6 - 2e^7 + 2e^8),$$
$$\frac{1}{2}(-e^1 + e^2 + e^3 + e^4 + e^5 + e^6 - 3e^7 + 3e^8),$$
$$e^3 + e^4 + e^5 + e^6 - 2e^7 + 2e^8, e^4 + e^5 + e^6 - \frac{3}{2}e^7 + \frac{3}{2}e^8,$$
$$e^5 + e^6 - e^7 + e^8, e^6 - \frac{1}{2}e^7 + \frac{1}{2}e^8\Big\}.$$

Their norms are $\sqrt{2}$, $\sqrt{7/2}$, $\sqrt{6}, 2\sqrt{3}$, $\sqrt{15/2}, 2$, $\sqrt{3/2}$.

- E_8. $E = \mathbb{R}^8$. The simple roots are

$$\Delta = \Big\{\frac{1}{2}(e^8 - e^7 - e^6 - e^5 - e^4 - e^3 - e^2 + e^1),$$
$$e^2 + e^1, e^2 - e^1, e^3 - e^2, e^4 - e^3, e^5 - e^4, e^6 - e^5, e^7 - e^6\Big\}.$$

The opposition involution is trivial. The fundamental weights are

$$\Big\{2e^8, \frac{1}{2}(e^1 + e^2 + e^3 + e^4 + e^5 + e^6 + e^7 + 5e^8),$$
$$\frac{1}{2}(-e^1 + e^2 + e^3 + e^4 + e^5 + e^6 + e^7 + 7e^8),$$
$$e^3 + e^4 + e^5 + e^6 + e^7 + 5e^8, e^4 + e^5 + e^6 + e^7 + 4e^8,$$
$$e^5 + e^6 + e^7 + 3e^8, e^6 + e^7 + 2e^8, e^7 + e^8\Big\}.$$

Their norms are $2, 2\sqrt{2}$, $\sqrt{14}$, $\sqrt{30}, 2\sqrt{5}, 2\sqrt{3}$, $\sqrt{6}$, $\sqrt{2}$.

- F_4. $E = \mathbb{R}^4$. The simple roots are $\Delta = \{e^1 - e^2, e^2 - e^3, e^3, \frac{1}{2}(-e^1 - e^2 - e^3 - e^4)\}$. The fundamental weights are: $\omega_1 = e^1 - e^4$ of norm $\sqrt{2}$, $\omega_2 = e^1 + e^2 - 2e^4$ of norm $\sqrt{6}$, $\omega_3 = \frac{1}{2}(e^1 + e^2 + e^3 - 3e^4)$ of norm $\sqrt{3}$, and $\omega_4 = -e^4$ of norm 1. The opposition involution is trivial.
- G_2. $E = \{v \in \mathbb{R}^3 \mid \langle v, e_1 + e_2 + e_3\rangle = 0\}$. The simple roots are $\Delta = \frac{1}{\sqrt{3}}\{e^1 - e^2, -2e^1 + e^2 + e^3\}$. For this scaling, the short root has norm $\sqrt{\frac{2}{3}}$ and the long root has norm $\sqrt{2}$. The fundamental weights are $\omega_1 = \frac{1}{\sqrt{3}}(-e^2 + e^3)$ and $\omega_2 = \frac{1}{\sqrt{3}}(-e^1 - e^2 + 2e^3)$, with norms $\sqrt{\frac{2}{3}}$ and $\sqrt{2}$, respectively. The opposition involution is trivial.

Example 3.21. In the symmetric space associated with $\mathfrak{sl}(d, \mathbb{R})$, the root system is of type A_{d-1} and the opposition involution takes the simple root α_i to α_{d-i}. The subset $\{\alpha_1, \alpha_{d-1}\}$ is a minimal ι-invariant subset. In this case ζ_{mod} is given by

$$\zeta_{\mathrm{mod}} = (H_{\omega_1} + H_{\omega_{d-1}})/|H_{\omega_1} + H_{\omega_{d-1}}|$$

which may be represented as a diagonal matrix with its first and last entries opposite and all other entries 0. One can compute directly or apply Proposition 3.20 to see that $\zeta_0 = \frac{1}{2\sqrt{d}}$. In this case τ_{mod}-Anosov subgroups of $\mathrm{SL}(d,\mathbb{R})$ are sometimes called *projective* Anosov subgroups.

3.11. Generalized Iwasawa decomposition

Let p be a point in $\mathbb{X}$, $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ and let $X \in \mathfrak{p}$ be τ-spanning. Choose a Cartan subspace $X \in \mathfrak{a} \subset \mathfrak{p}$, with restricted roots Λ and a choice of simple roots Δ associated with $\sigma \supset \tau$. Recalling the notation in (3.7) following Definition 3.12, we define

(1) $\mathfrak{a}_\tau = Z(X) \cap \mathfrak{p} = \{Y \in \mathfrak{p} \mid [X,Y] = 0\}$ and $A_\tau = \exp(\mathfrak{a}_\tau)$. Note that $\mathfrak{a}_\tau$ and A_τ depend on p.

(2) The *(nilpotent) horocyclic subalgebra* $\mathfrak{n}_\tau = \bigoplus_{\alpha \in \Lambda_\tau^+} \mathfrak{g}_\alpha$ and the *(unipotent) horocyclic subgroup* $N_\tau = \exp(\mathfrak{n}_\tau)$.

(3) The *generalized Iwasawa decomposition* of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_\tau \oplus \mathfrak{n}_\tau$.

(4) The *generalized Iwasawa decomposition* of G is $G = KA_\tau N_\tau = N_\tau A_\tau K$. The indicated decomposition is unique.

Note that our notation differs from [20], where N_τ denotes the full horocyclic subgroup at τ and A_τ is the group of translations of the flat factor of the parallel set defined by p and τ (see Section 3.12). In our notation, N_τ is the unipotent radical of the parabolic subgroup G_τ (see [8, Section 2.17]).

3.12. Antipodal simplices, parallel sets and horocycles

A pair of points ξ, η in $\partial\mathbb{X}$ are said to be *antipodal* if there exists a geodesic c with $c(-\infty) = \xi$ and $c(+\infty) = \eta$. Equivalently, ξ, η are antipodal if there exists a geodesic symmetry S_p taking ξ to η.

A pair of simplices $\tau_\pm$ are *antipodal* if there exists some $p \in \mathbb{X}$ such that $S_p\tau_- = \tau_+$, or equivalently if there exists a geodesic c with $c(-\infty) \in \mathrm{int}(\tau_-)$ and $c(+\infty) \in \mathrm{int}(\tau_+)$. If a model simplex τ_{mod} is ι-invariant, then every simplex τ of type τ_{mod} has the same type as any of its antipodes.

For antipodal simplices $\tau_\pm$, the *parallel set* $P(\tau_-,\tau_+)$ is the union of (images of) geodesics c with $c(-\infty) \in \tau_-$ and $c(+\infty) \in \tau_+$. Given one such geodesic c, we may alternatively define $P(\tau_-,\tau_+) = P(c)$ to be the union of geodesics parallel to c, or equivalently to be the union of maximal flats containing c. Antipodal τ_{mod}-regular points ξ, η lie in the boundary of a unique parallel set $P = P(\tau(\xi), \tau(\eta))$, where $\tau(\xi)$ (resp. $\tau(\eta)$) is the unique simplex of type τ_{mod} in some/every Weyl chamber containing ξ (resp. η). We say that $P(\tau_-,\tau_+)$ *joins* τ_- and τ_+. The parallel set joining a pair of antipodal Weyl chambers is a maximal flat.

The *horocycle* centered at $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ through $p \in \mathbb{X}$ is denoted $H(p,\tau)$ and is defined to be the orbit $N_\tau \cdot p$. For any $p \in \mathbb{X}$ and $\hat\tau$ antipodal to τ, the horocycle

$H(p, \tau)$ intersects the parallel set $P(\hat{\tau}, \tau)$ in exactly one point. A horocycle is the union of basepoints of strongly asymptotic Weyl sectors/geodesic rays [19, 20].

3.13. The ζ-angle and Tits angle

We follow [19] in defining the ζ-*angle* between two simplices at a point $p \in \mathbb{X}$. For fixed $p \in X$ and ζ, the ζ-angle provides a metric on $\mathrm{Flag}(\tau_{\mathrm{mod}})$ by viewing it as embedded in the tangent space at p and restricting the angle metric $\angle_p$ to the vectors of type ζ. The ζ-angle also makes sense for τ_{mod}-regular directions by projecting to $\mathrm{Flag}(\tau_{\mathrm{mod}})$. To make this definition, we first fix the auxiliary data of a $(\zeta_0, \tau_{\mathrm{mod}})$-spanning ι-invariant model ideal point $\zeta = \zeta_{\mathrm{mod}} \in \mathrm{int}(\tau_{\mathrm{mod}})$. We recall from Definition 3.14 that $(\zeta_0, \tau_{\mathrm{mod}})$-spanning means that ζ is in the interior of τ_{mod} and all simple roots $\alpha \in \Delta^+_{\tau_{\mathrm{mod}}}$ positive on the interior of τ_{mod} satisfy $\alpha(\zeta) \geq \zeta_0$.

Definition 3.22 (ζ-angle, cf. [19, Definitions 2.3 and 2.4]). For ζ as above, define:

(1) For a simplex $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$, let $\zeta(\tau)$ denote the unique point in $\mathrm{int}(\tau)$ of type ζ.

(2) For a τ_{mod}-regular ideal point $\xi \in \partial\mathbb{X}$, let $\zeta(\xi) = \zeta(\tau(\xi))$ where $\tau(\xi)$ is the simplex spanned by ξ.

(3) Let $p \in \mathbb{X}$, let τ, τ' be Weyl chambers in $\partial\mathbb{X}$ and let $x, y \in \overline{\mathbb{X}}$ with px and py τ_{mod}-regular. The ζ-*angle* is given by

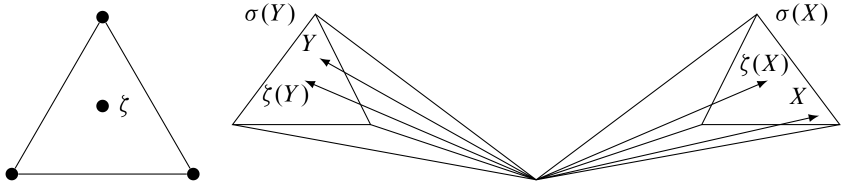
$$\angle_p^\zeta(\tau, \tau') := \angle_p(\zeta(\tau), \zeta(\tau')),$$
$$\angle_p^\zeta(\tau, y) := \angle_p(\zeta(\tau), \zeta(py)),$$
$$\angle_p^\zeta(x, y) := \angle_p(\zeta(px), \zeta(py)).$$

Note that there is a typo in the definition of ζ-angle in [19, Definition 7.5] (see Figure 4).

For $\xi, \eta \in \partial\mathbb{X}$, the *Tits angle* is

$$\angle_{\mathrm{Tits}}(\xi, \eta) := \sup_{p \in \mathbb{X}} \angle_p(\xi, \eta).$$

Ideal points ξ, η are antipodal if and only if their Tits angle is π. For $p \in \mathbb{X}$, $\xi, \eta \in \partial\mathbb{X}$, the equality $\angle_p(\xi, \eta) = \angle_{\mathrm{Tits}}(\xi, \eta)$ holds if and only if there is a maximal flat F containing p



(a) $\zeta \in \sigma_{\mathrm{mod}}$. **(b)** The ζ-angle between X and Y.

Figure 4. ζ-angles.

with $\xi, \eta \in \partial F$ and moreover for any $\xi, \eta \in \partial \mathbb{X}$, there exists some maximal flat F with $\xi, \eta \in \partial F$ [8].

For simplices τ, τ' in $\mathrm{Flag}(\tau_{\mathrm{mod}})$, we may define

$$\angle^{\zeta}_{\mathrm{Tits}}(\tau, \tau') := \angle_{\mathrm{Tits}}(\zeta(\tau), \zeta(\tau')).$$

There are only finitely many possible Tits angles between ideal points of fixed type. Therefore, there exists a bound $\varepsilon(\zeta_{\mathrm{mod}})$ such that if $\angle^{\zeta}_{\mathrm{Tits}}(\tau, \tau') > \pi - \varepsilon(\zeta_{\mathrm{mod}})$ then τ and τ' are antipodal, as observed in [19, Remark 2.42]. By Remark 3.13, we have

$$\sin\left(\frac{1}{2}\varepsilon(\zeta_{\mathrm{mod}})\right) = \min_{\alpha \in \Lambda^{+}_{\tau_{\mathrm{mod}}}} \frac{\alpha(\zeta_{\mathrm{mod}})}{|\alpha|} \geq \frac{\zeta_0}{\kappa_0}.$$

By the definition of Tits angle, the same holds if the ζ-angle at any point is strictly within $\varepsilon(\zeta_{\mathrm{mod}})$ of π: The inequality

$$\angle^{\zeta}_{\mathrm{Tits}}(\tau, \tau') \geq \angle^{\zeta}_{p}(\tau, \tau') > \pi - \varepsilon(\zeta_{\mathrm{mod}})$$

implies that τ and τ' are antipodal. Since $\zeta_0 \leq \kappa_0 < 2\kappa_0$ (recall κ_0 from Definition 3.2), we have

$$\sin \frac{1}{2} \frac{\zeta_0^2}{\kappa_0^2} \leq \frac{1}{2} \frac{\zeta_0^2}{\kappa_0^2} < \frac{\zeta_0}{\kappa_0} \leq \sin \frac{1}{2}\varepsilon(\zeta_{\mathrm{mod}}),$$

and we obtain the estimate $\frac{\zeta_0^2}{\kappa_0^2} < \varepsilon(\zeta_{\mathrm{mod}})$. We record this observation in the following lemma.

Lemma 3.23 (Cf. [19, Remark 2.42]). *If the inequality $\angle^{\zeta}_{p}(\tau_-, \tau_+) \geq \pi - \frac{\zeta_0^2}{\kappa_0^2}$ holds for some $p \in \mathbb{X}$, then τ_- is antipodal to τ_+. In other words, $\frac{\zeta_0^2}{\kappa_0^2} < \varepsilon(\zeta_{\mathrm{mod}})$.*

4. Estimates

This section contains the main contributions of the paper. We prove several explicit estimates in the symmetric space that we will use in Section 5 to give a quantified version of the local-to-global principle for Morse quasigeodesics. Qualitative versions of these estimates appear in [19, 20], but there the proofs rely on topological arguments that do not produce explicit bounds. For example, in Subsection 4.4, Lemma 4.8, we consider the natural projection from $(\alpha_0, \tau_{\mathrm{mod}})$-regular vectors in $\mathfrak{p}$ to $\mathrm{Flag}(\tau_{\mathrm{mod}})$. This map is the restriction of a smooth map to a compact submanifold with boundary, so an abstract proof of the existence of a Lipschitz constant is not hard. However, that approach is not suitable for our purposes, so we apply Corollary 3.15 to obtain an explicit local Lipschitz constant. Note that such an estimate cannot be uniform for all $\alpha_0 > 0$ and therefore must depend on α_0.

A crucial notion, introduced in [19], is the ζ-angle, denoted $\angle^{\zeta}$ (see Section 3.13). Recall that $\zeta = \zeta_{\mathrm{mod}}$ is a fixed type in the interior of τ_{mod}. Moreover, we assume that ζ is

$(\zeta_0, \tau_{\mathrm{mod}})$-regular and that ζ and τ_{mod} are ι-invariant (see Definition 3.14 and Section 3.8). For fixed $p \in X$ and ζ, the ζ-angle provides a metric on $\mathrm{Flag}(\tau_{\mathrm{mod}})$ by viewing it as embedded in the tangent space at p and restricting the angle metric $\angle_p$ to the vectors of type ζ. The ζ-angle also makes sense for τ_{mod}-regular directions by projecting to $\mathrm{Flag}(\tau_{\mathrm{mod}})$.

The organization of the section is as follows. In Subsection 4.1, we relate the Riemannian metric on $\mathbb{X}$ to algebraic data on $\mathfrak{g}$, for example, the Killing form B and the canonical inner product B_p. In Subsection 4.2, we use the vector-valued triangle inequality to control the regularity of bounded perturbations of long regular geodesic segments. In Subsection 4.4, we prove Lemma 4.8, which allows us to bound $\angle_p^{\zeta}(x, y)$ in terms of α_0, ζ_0 and $\angle_p(x, y)$. In Subsection 4.5, we prepare a technique for the subsequent subsections, where we bound the lengths of certain non-geodesic curves in $\mathbb{X}$ which are images of curves in G under the orbit map. In Subsection 4.6, the curve lies in the subgroup stabilizing a point, and we bound the distance the midpoint of a segment can move when we move one endpoint a bounded amount, assuming the segment is long enough. Subsection 4.7 is roughly similar; there we bound the distance between points far along on strongly asymptotic geodesic rays (so the curve in G lies in a unipotent horocyclic subgroup). These combine to yield a crucial estimate in Corollary 4.13, which implies that if a pair of points are in the D-neighborhood of a diamond, then their midpoint is close to the diamond; moreover, the distance from the midpoint to the diamond becomes arbitrarily small as the points move farther apart. In the remaining subsections, we show that distance to a corresponding parallel set controls the corresponding ζ-angles (Corollary 4.16) and vice versa (Lemma 4.17). Along the way we provide some control for the Lie derivatives of gradients of Busemann functions with respect to Killing vector fields (see the proofs of Lemmas 4.14 and 4.17).

4.1. Useful properties of the inner product B_p on $\mathfrak{g}$

We remind the reader that our convention is that the Riemannian metric on $\mathbb{X}$ is the one induced by the Killing form (see equation (3.2)). Recall that each point $p \in \mathbb{X}$ induces an inner product B_p on $\mathfrak{g}$ and the evaluation map $\mathrm{ev}_p \colon \mathfrak{g} \to T_p\mathbb{X}$ (see Section 3.1). We first relate the inner product B_p, the Killing form B on $\mathfrak{g}$ and the Riemannian metric $\langle \cdot, \cdot \rangle$ at p.

Lemma 4.1. *For any $X, Y \in \mathfrak{g}$ and $p \in \mathbb{X}$,*

$$2\langle \mathrm{ev}_p X, \mathrm{ev}_p Y \rangle = B(X, Y) + B_p(X, Y).$$

In particular, any U in $\mathfrak{n}_\tau$ or $\mathfrak{g}_\alpha$ is ad*-nilpotent, so $B(U, U) = 0$ and $|U|_{B_p} = \sqrt{2}|\mathrm{ev}_p U|$ (see Section 3.11).*

Recall that ϑ_p is a Lie algebra automorphism, so $\vartheta_p[X, Y] = [\vartheta_p X, \vartheta_p Y]$ and $B(\vartheta_p X, \vartheta_p Y) = B(X, Y)$.

Proof. The kernel of ev_p is the $+1$-eigenspace for ϑ_p, so for any $X \in \mathfrak{g}$, $2\mathrm{ev}_p X = \mathrm{ev}_p(X - \vartheta_p X)$ and

$$\begin{aligned} 4\langle \mathrm{ev}_p X, \mathrm{ev}_p Y\rangle_p &= \langle \mathrm{ev}_p(X - \vartheta_p X), \mathrm{ev}_p(Y - \vartheta_p Y)\rangle_p \\ &= B(X - \vartheta_p X, Y - \vartheta_p Y) \\ &= B(X,Y) + B(\vartheta_p X, \vartheta_p Y) - B(\vartheta_p X, Y) - B(X, \vartheta_p Y) \\ &= 2B(X,Y) + 2B_p(X,Y). \end{aligned}$$ ∎

Next we show that the transpose on $\operatorname{End}\mathfrak{g}$ with respect to B_p restricts to $-\vartheta_p$ on the image of the adjoint representation.

Lemma 4.2. *For $X, Y, Z \in \mathfrak{g}$, $B_p(\operatorname{ad} X(Y), Z) = B_p(Y, \operatorname{ad}(-\vartheta_p X)(Z))$.*

Proof. We have

$$\begin{aligned} B_p(\operatorname{ad} X(Y), Z) &= -B(\vartheta_p \operatorname{ad} X(Y), Z) \\ &= -B(\operatorname{ad}(\vartheta_p X)(\vartheta_p Y), Z) \\ &= -B(\vartheta_p Y, \operatorname{ad}(-\vartheta_p X)(Z)) \\ &= B_p(Y, \operatorname{ad}(-\vartheta_p X)(Z)), \end{aligned}$$

where we have used that $\operatorname{ad} \vartheta_p X$ is skew-symmetric relative to B. ∎

Third, we bound $B(\operatorname{ad} X(Y), Z)$ by the product of the B_p-norms of X, Y and Z and bound the operator norm of $\operatorname{ad} X$ by $|X|_{B_p}$ along the way.

Lemma 4.3. *Let $X, Y, Z \in \mathfrak{g}$ and let $p \in \mathbb{X}$ induce the inner product B_p on $\mathfrak{g}$. Consider the operator norm $|\cdot|_{op}$ and Frobenius norm $|\cdot|_{\mathrm{Fr}}$ on $\operatorname{End}\mathfrak{g}$ induced by B_p. Then*

(1) $|\operatorname{ad} Y|_{op} \le |\operatorname{ad} Y|_{\mathrm{Fr}} = |Y|_{B_p}$,

(2) $B(X, \operatorname{ad} Y(Z)) \le |X|_{B_p}|Y|_{B_p}|Z|_{B_p}$, *and*

(3) *for $Y \in \mathfrak{p}$, $|[Y, X]|_{B_p} \le \kappa_0 |Y|_{B_p}|X|_{B_p}$.*

Proof. Recall that the operator norm of a linear transformation is the largest singular value, while the Frobenius norm is the square root of the sum of the singular values squared. Therefore,

$$|\operatorname{ad} X|_{op}^2 \le |\operatorname{ad} X|_{\mathrm{Fr}}^2 = \operatorname{trace}_{\mathfrak{g}}(\operatorname{ad}(-\vartheta_p X) \circ \operatorname{ad} X) = B_p(X, X)$$

by Lemma 4.2, proving the first claim. Using this, we have

$$\begin{aligned} B(X, \operatorname{ad} Y(Z)) &= -B_p(\vartheta_p X, \operatorname{ad} Y(Z)) \\ &\le |\vartheta_p X|_{B_p} |\operatorname{ad} Y(Z)|_{B_p} \\ &\le |X|_{B_p} |\operatorname{ad} Y|_{op} |Z|_{B_p} \\ &\le |X|_{B_p} |Y|_{B_p} |Z|_{B_p}. \end{aligned}$$

If $Y \in \mathfrak{p}$, we may choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ containing Y and decompose $X = \sum_{\alpha \in \Lambda \cup \{0\}} X_\alpha$ according to the associated restricted root space decomposition, which is B_p-orthogonal. Therefore,

$$|[Y, X]|_{B_p}^2 = \left| \sum_{\alpha \in \Lambda} \alpha(Y) X_\alpha \right|_{B_p}^2 = \sum_{\alpha \in \Lambda} \alpha(Y)^2 |X_\alpha|_{B_p}^2 \leq \kappa_0^2 |Y|_{B_p}^2 |X|_{B_p}^2,$$

where κ_0 is the maximum of $\{\alpha(A) \mid \alpha \in \Lambda, A \in \mathfrak{a}, |A| = 1\}$ (see Definition 3.2). ∎

Fourth, we need to compare the norms induced by $p, q \in \mathbb{X}$ in terms of $d(p, q)$.

Lemma 4.4. *Let $p, q \in \mathbb{X}$, $g \in G$ and $X \in \mathfrak{g}$. Then*

(1) $\vartheta_{gp} \circ \mathrm{Ad}(g) = \mathrm{Ad}(g) \circ \vartheta_p$,
(2) $|X|_{B_p} = |\mathrm{Ad}(g) X|_{B_{gp}}$, *and*
(3) $|X|_{B_p} \leq e^{\kappa_0 d(p,q)} |X|_{B_q}$.

Proof. The point stabilizer G_{gp} is $g G_p g^{-1}$ and it follows that $\mathrm{Ad}(g)$ takes ϑ_p to ϑ_{gp}. This, together with the Ad invariance of the Killing form implies (2). For the last point, choose a maximal flat F containing p and q, let $\mathfrak{a} \subset \mathfrak{p}$ be the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ corresponding to $p \in F$, and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus \mathfrak{g}_\alpha$ be the corresponding restricted root space decomposition. There is a unique $A \in \mathfrak{a}$ such that $e^A p = q$, and then

$$|X|_{B_p} = |e^{\mathrm{ad}\, A} X|_{B_q} = \left| \sum_{\alpha \in \Lambda \cup \{0\}} e^{\alpha(A)} X_\alpha \right|_{B_q} \leq e^{\kappa_0 d(p,q)} |X|_{B_q},$$

using the restricted root space decomposition of X and the fact that the restricted root space decomposition is B_q-orthogonal. ∎

4.2. Perturbations of long, regular segments

We will need to control the regularity of bounded perturbations of long regular geodesic segments. The following lemma is an explicit version of [21, Lemma 3.6]. This assertion also appears in the proof of Lemma 7.10 in [19].

Lemma 4.5. *Suppose xy is an $(\alpha_0, \tau_{\mathrm{mod}})$-regular geodesic segment with $d(x, y) \geq l$ and let x', y' be points in $\mathbb{X}$ satisfying $d(x, x') \leq \delta_x$ and $d(y, y') \leq \delta_y$. If*

$$\alpha_0 - \frac{(\delta_x + \delta_y)(\alpha_0 + \kappa_0)}{l - \delta_x - \delta_y} \geq \alpha_0',$$

then $x'y'$ is $(\alpha_0', \tau_{\mathrm{mod}})$-regular.

We will often apply this lemma in the case $\delta_x = \delta_y = D$.

Proof. We apply Corollary 3.11, the triangle inequality for $\vec{d}$-distances:

$$\left|\vec{d}(x,y)-\vec{d}(x',y')\right| \le d(x,x')+d(y,y') \le \delta_x+\delta_y.$$

Similarly, $|d(x,y)-d(x',y')| \le d(x,x')+d(y,y') \le \delta_x+\delta_y$, so $d(x',y') \ge l-\delta_x+\delta_y$ and

$$\frac{d(x,y)}{d(x',y')} \ge 1-\frac{\delta_x+\delta_y}{d(x',y')} \ge 1-\frac{\delta_x+\delta_y}{l-\delta_x-\delta_y}.$$

For any $\alpha \in \Delta^{+}_{\tau_{\mathrm{mod}}}$,

$$\begin{aligned}\frac{\alpha(\vec{d}(x',y'))}{d(x',y')} &\ge \frac{\alpha_0 d(x,y)-\delta_x\kappa_0-\delta_y\kappa_0}{d(x',y')}\\ &\ge \alpha_0\left(1-\frac{\delta_x+\delta_y}{l-\delta_x-\delta_y}\right)-\frac{(\delta_x+\delta_y)\kappa_0}{l-\delta_x-\delta_y}\\ &= \alpha_0-\frac{(\delta_x+\delta_y)(\alpha_0+\kappa_0)}{l-\delta_x-\delta_y} \ge \alpha_0'.\end{aligned}$$

■

It is also straightforward to control the regularity of segments in terms of τ_{mod}-Weyl convex subsets $\Theta \subset \sigma_{\mathrm{mod}}$.

Lemma 4.6. *Suppose $\Theta, \Theta' \subset \sigma_{\mathrm{mod}}$ satisfy $N_A(\Theta) \subset \Theta'$ where $N_A(\Theta)$ denotes the A-neighborhood of A with respect to the angular metric. Let xy be a Θ-regular geodesic segment with $d(x,y) \ge l$ and suppose x', y' satisfy $d(x,x') \le \delta_x$ and $d(y,y') \le \delta_y$. If*

$$\sin(A) \le \frac{\delta_x+\delta_y}{l},$$

then $x'y'$ is Θ'-regular.

Proof. As before, we have

$$\left|\vec{d}(x,y)-\vec{d}(x',y')\right| \le \delta_x+\delta_y$$

and by assumption $d(x,y) \ge l$, so

$$\sin\angle(\vec{d}(x,y),\vec{d}(x',y')) \le \frac{\delta_x+\delta_y}{l}.$$

■

4.3. Angle comparison to Euclidean space

When p, q, r are points in $\mathbb{X}$ such that $d(p,q)$ is much larger than $d(q,r)$, we provide an upper bound for the Riemannian angle $\angle_p(q,r)$ by comparing to Euclidean space. The following estimate is surely not new, but we could not find a direct reference so we give a proof.

Lemma 4.7. *Let p, q, r be non-collinear points in $\mathbb{X}$. Then*

$$\sin \angle_p(q,r) \leq \frac{d(q,r)}{d(p,q)}.$$

The convenience of this estimate is that the third possible distance $d(p,r)$ does not appear.

Proof. Let $X, Y \in \mathfrak{p}$ such that $e^X p = q$ and $e^Y p = r$. Then $|X| = d(p,q)$ and $d(X,Y) \leq d(q,r)$, and we may assume that $d(p,q) > d(q,r)$. In Euclidean space, the comparison holds: Among vectors Y' with $d(X,Y') \leq d(X,Y)$, the largest angle occurs for a vector Y' forming a right triangle with X as hypotenuse. Then

$$\sin \angle(X,Y) \leq \sin \angle(X,Y') = \frac{d(X,Y')}{|X|} \leq \frac{d(q,r)}{d(p,q)}. \qquad \blacksquare$$

4.4. Projecting regular vectors to flag manifolds

Recall that we have a fixed type $\zeta = \zeta_{\mathrm{mod}}$ which is $(\zeta_0, \tau_{\mathrm{mod}})$-spanning (see Definition 3.14). For a τ_{mod}-regular $X \in \mathfrak{p}$, define $\zeta(X)$ to be the unique vector in a common closed Weyl chamber as X of type ζ. Note that $\zeta(X)$ is the unique maximizer for $B(X, \cdot)\colon \mathrm{Ad}(K)Z \to \mathbb{R}$ where $Z \in \mathfrak{p}$ is any vector of type ζ by Corollary 3.15. In the next lemma we show that nearby τ_{mod}-regular points project to nearby points on $\mathrm{Ad}(K)Z$ in the metric induced by viewing $\mathrm{Ad}(K)Z$ as a Riemannian submanifold of $\mathfrak{p}$. Note that one expects a local Lipschitz constant proportional to $\frac{1}{\alpha_0}$ by considering vectors near the walls $\ker \alpha$ for $\alpha \in \Delta_\tau^+$.

Lemma 4.8. *Let X, X' be (α_0, τ)-regular unit vectors in $\mathfrak{p}$ with $d_{\mathfrak{p}}(X, X') \leq \alpha_0$. Write $Z = \zeta(X)$ and $Z' = \zeta(X')$. Then the Riemannian distance on $\mathrm{Ad}(K)Z$ from Z to Z' is bounded by the distance in $\mathfrak{p}$ from X to X':*

$$d_{\mathrm{Ad}(K)Z}(Z, Z') \leq \frac{1}{\alpha_0 \zeta_0} d_{\mathfrak{p}}(X, X').$$

Proof. Let $t \mapsto X_t$ be a unit-speed line segment from X to X' in $\mathfrak{p}$. Let $\{X^i\}_{i=1}^{\dim \mathfrak{p}}$ be linear coordinates on $\mathfrak{p}$, and we may assume that the derivative of $t \mapsto X_t$ is $\frac{\partial}{\partial X^1}$. Since $d_{\mathfrak{p}}(X, X') \leq \alpha_0$ each X_t is $\left(\frac{\alpha_0}{2}, \tau_{\mathrm{mod}}\right)$-regular. Write $Z_t = \zeta(X_t)$ and note that $t \mapsto Z_t$ is a smooth curve on $\mathrm{Ad}(K)Z$. To prove the claim we will show that $\left|\frac{\mathrm{d}}{\mathrm{d}Z_t} t\right| \leq \frac{1}{\alpha_0 \zeta_0}$, where we restrict the inner product on $\mathfrak{p}$ to a Riemannian metric on $\mathrm{Ad}(K)Z$.

Restricting the domain of B, we write $B\colon \mathfrak{p} \times \mathrm{Ad}(K)Z \to \mathbb{R}$. Near $(X_0, Z_0) = (X_{t_0}, Z_{t_0})$, we have coordinates $\{Z^j\}_{j=1}^{\dim \mathrm{Ad}(K)Z}$ on $\mathrm{Ad}(K)Z$. We may assume that Z_t is an immersion at Z_0 because the set $\left\{t \mid \left|\frac{\mathrm{d}}{\mathrm{d}Z_t} t\right| = 0\right\}$ does not contribute to the arclength of Z_t and furthermore up to a change of coordinates we may assume that $\frac{\mathrm{d}}{\mathrm{d}Z_t} t = \frac{\partial}{\partial Z^1}$. On this coordinate patch U, we obtain the function $B_j\colon \mathfrak{p} \times U \to \mathbb{R}$ defined by $B_j(X'', Z'') := \mathrm{d}B_{(X'', Z'')}\left(\frac{\partial}{\partial Z^j}\right)$. Along the curve $t \mapsto (X_t, Z_t)$, the function B_j is identically 0 (where defined) since Z_t maximizes $B(X_t,\cdot)$ on $\mathrm{Ad}(K)Z$. Differentiating $B_j(X_t,Z_t)=0$ in t, we obtain

$$0=\mathrm{d}B_{j\,(X_t,Z_t)}\left(\frac{\partial}{\partial X^1},\frac{\partial}{\partial Z^1}\right)=\frac{\partial}{\partial B_j}X^1+\frac{\partial}{\partial B_j}Z^1.$$

Observe that

$$\begin{aligned}\frac{\partial}{\partial B_j}Z^1{}_{(X_t,Z_t)}&=\left(\frac{\partial}{\partial Z^1}\frac{\partial}{\partial Z^j}B\right)_{(X_t,Z_t)}\\&=\mathrm{Hess}(B)\left(\frac{\partial}{\partial Z^1},\frac{\partial}{\partial Z^j}\right)_{(X_t,Z_t)}\\&=\mathrm{Hess}(B(X_t,\cdot))\left(\frac{\partial}{\partial Z^1},\frac{\partial}{\partial Z^j}\right)_{Z_t},\end{aligned}$$

so by Corollary 3.15 we have

$$\left|\frac{\partial}{\partial B_j}Z^1\right|\ge\alpha_0\zeta_0\left|\left\langle\frac{\partial}{\partial Z^1},\frac{\partial}{\partial Z^j}\right\rangle\right|.$$

In particular, along (X_t,Z_t) and setting $j=1$, we have

$$\alpha_0\zeta_0\left|\frac{\partial}{\partial Z^1}\right|^2\le\left|\frac{\partial}{\partial B_1}X^1{}_{(X_t,Z_t)}\right|=\left|B_1\left(\frac{\partial}{\partial X^1},Z_t\right)\right|=\left|B\left(\frac{\partial}{\partial X^1},\frac{\partial}{\partial Z^1}\right)\right|\le\left|\frac{\partial}{\partial Z^1}\right|$$

since $\frac{\partial}{\partial X^1}$ is a unit vector. We obtain for all t

$$\left|\frac{\partial}{\partial Z^1}\right|\le\frac{1}{\alpha_0\zeta_0}$$

and the claim is proven. ■

4.5. Projecting curves in G to $\mathbb{X}$

In this subsection, we prepare to estimate the length of curves in $\mathbb{X}$ which are images of curves in G under the orbit map. We begin by comparing the speeds of two such curves related by right translation. We apply this result in the next section to Lemma 4.10 for a curve in K, and in the following section to Lemma 4.11 for a curve in the subgroup N_τ.

For an element $g\in G$, we let $l_G\colon G\to G$, $l_g(h)=gh$ denote left translation and $r_g\colon G\to G$, $r_g(h)=hg$ denote right translation. We denote by $\mathrm{conj}_g\colon G\to G$ the conjugation map $\mathrm{conj}_g(h)=ghg^{-1}$.

Lemma 4.9. *Let $g\colon\mathbb{R}\to G$ be a curve in G, let $h\in G$ and let $p\in\mathbb{X}$. Write $q_h(s)=g(s)hp$. If $\dot g(s)=(\mathrm{d}l_{g(s)})_1X_s$, then*

$$|\dot{q_h}(s)|=|\mathrm{ev}_p\,\mathrm{Ad}(h^{-1})X_s|.$$

Proof. The curve $q_h(s) = g(s)hp$ has the same speed as $c_h(s) = h^{-1}g(s)hp$ since h^{-1} is an isometry. Writing

$$c_h(s) = p \circ \mathrm{conj}_{h^{-1}} \circ g(s)$$

and differentiating with respect to s we have

$$\dot{c_h}(s) = (\mathrm{d}\,\mathrm{orb}_p)_{h^{-1}gh} \circ (\mathrm{d}\,\mathrm{conj}_{h^{-1}})_{g(s)} \circ \dot{g}(s).$$

For any $a, b \in G$ and $X \in T_1G$, we have

$$\begin{aligned}(\mathrm{d}\,\mathrm{conj}_a)_b(\mathrm{d}l_b)_1X &= (\mathrm{d}l_a)_{ba^{-1}}(\mathrm{d}r_a^{-1})_b(\mathrm{d}l_b)_1X \\ &= (\mathrm{d}l_a)_{ba^{-1}}(\mathrm{d}l_b)_{a^{-1}}(\mathrm{d}l_a^{-1})_1(\mathrm{d}l_a)_{a^{-1}}(\mathrm{d}r_a^{-1})_1X \\ &= \mathrm{d}l_{aba^{-1}}\,\mathrm{Ad}(a)X.\end{aligned}$$

We also have $(\mathrm{d}\,\mathrm{orb}_p)_a(\mathrm{d}l_a)_1 = \mathrm{d}a_p(\mathrm{d}\,\mathrm{orb}_p)_1$, so if $\dot{g}(s) = \mathrm{d}l_{g(s)}X_s$, then

$$\dot{c_t}(s) = (\mathrm{d}\,\mathrm{orb}_p)_{h^{-1}gh} \circ (\mathrm{d}l_{h^{-1}gh})_1\,\mathrm{Ad}(h^{-1})X_s = (\mathrm{d}h^{-1}gh)_p(\mathrm{d}\,\mathrm{orb}_p)_1\,\mathrm{Ad}(h^{-1})X_s.$$

This implies

$$|\dot{q_h}(s)| = |\dot{c_h}(s)| = |(\mathrm{d}\,\mathrm{orb}_p)_1\,\mathrm{Ad}(h^{-1})X_s| = |\mathrm{ev}_p\,\mathrm{Ad}(h^{-1})X_s|$$

and completes the proof. ■

4.6. Weyl cones forming small angles

In this subsection, we show that if $q \in V(p, \mathrm{st}(\tau), \alpha_0)$ and $r \in V(p, \mathrm{st}(\tau'), \alpha_0)$ with $d(p,q)$ much larger than $d(q,r)$, the midpoint of pq is close to $V(p, \mathrm{st}(\tau'), \alpha_0)$. Recall that the Weyl cone $V(p, \mathrm{st}(\tau), \alpha_0)$ is defined to be the closed cone at p of the α_0-star of τ (see Section 3.9).

Lemma 4.10. *Let $p, q, r \in \mathbb{X}$. Suppose that pq is an (α_0, τ)-regular geodesic ray with $d(p,q) \geq 2l$ and $d(q,r) \leq D$. Let $m = \mathrm{mid}(p,q)$, $K = \mathrm{Stab}_G(p)$ and suppose moreover that*

$$\alpha_0 - \frac{D(\kappa_0 + \alpha_0)}{2l - D} \geq \alpha_0' > 0$$

and

$$\frac{1}{2}\big(e^{2\kappa_0 D} - 1\big)[\sinh(\alpha_0'(2l - D))]^{-2} \leq 3e^{2\kappa_0 D}.$$

Then there exists $k \in K$ such that $km \in V(p, \mathrm{st}(\tau(pr)), \alpha_0)$ and $d(m, km)$ is at most $2De^{\kappa_0 D - \alpha_0 l}$.

The first inequality guarantees that pr is τ_{mod}-regular so that $\tau(pr)$ is well defined. The second requirement looks strange and involves an arbitrary choice, but is extremely mild and serves our purposes well. (When we apply this lemma, we will have a bounded D and a large l.) Compared to other variations of Lemma 4.10 we could present here, the given version has a less cumbersome upper bound in the conclusion of the lemma.

Proof. We may assume that $d(p,q) = 2l$ and $d(q,r) = D$. Let $c\colon [0,D] \to \mathbb{X}$ be the unit-speed geodesic from q to r. We have l large enough that Lemma 4.5 implies that each ray $pc(t)$ is $(\alpha_0', \tau_{\mathrm{mod}})$-regular and defines a simplex $\tau_t := \tau(pc(t))$. We may decompose

$$\dot{c}(t) = N_{c(t)} + T_{c(t)},$$

so that $T_{c(t)}$ is tangent to $V_t := V(p, \mathrm{ost}(\tau_t))$ and $N_{c(t)}$ is normal to V_t. Recall that for $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ we let $\mathfrak{k}_\tau$ denote the infinitesimal stabilizer in $\mathfrak{k}$ and let $\mathfrak{k}^\tau$ denote $(\mathfrak{k}_\tau)^\perp$ with respect to the restriction of the Killing form to $\mathfrak{k}$. For each t there is a unique $X_t \in \mathfrak{k}^{\tau_t} \subset T_1 K$ such that $\mathrm{ev}_{c(t)} X_t = N_{c(t)}$, and we extend each X_t to a *right*-invariant vector field on K. We may view this time-dependent vector field as vector field supported on a compact neighborhood of $[0,D] \times K$, so it defines a flow and in particular a curve $k\colon [0,D] \to K$ with $k(0) = 1$ and $\dot{k}(t) = (X_t)_{k(t)} = (\mathrm{d}r_{k(t)})_1 X_t$.

Viewing $\mathfrak{k}$ as $T_1 K$, it is convenient to set $X_t = \mathrm{Ad}(k(t)) Y_t$ and work with the time-dependent tangent vector $Y_t \in \mathfrak{k}^\tau$. We have $\dot{k}(t) = (\mathrm{d}l_{k(t)})_1 Y_t$, so we may extend Y_t to the unique *left*-invariant vector field agreeing with X_t along $k(t)$.

We may now write $c(t) = k(t)v(t)$ where $v(t) \in V(p, \mathrm{st}(\tau), \alpha_0')$ (see Figure 5). Since $T_{c(t)} = (\mathrm{d}k(t))_{v(t)} \dot{v}(t)$ we have $|\dot{v}| \le |\dot{c}|$, so

$$d(k(t)v(0), k(t)v(t)) = d(v(0), v(t)) \le t \le D.$$

Setting $q(t) = k(t)q$, we have $|\dot{q}(t)| = |\mathrm{ev}_q Y_t|$ by Lemma 4.9, and by Lemma 4.4 (3) we have

$$2|\mathrm{ev}_q Y_t|^2 - |Y_t|_B^2 = |Y_t|_{B_q}^2 \le e^{2\kappa_0 t} |Y_t|_{B_{v(t)}}^2 = e^{2\kappa_0 t}(2|\mathrm{ev}_{v(t)} Y_t|^2 - |Y_t|_B^2), \tag{4.1}$$

where $|Y_t|_B^2 = B(Y_t, Y_t)$ is nonpositive.

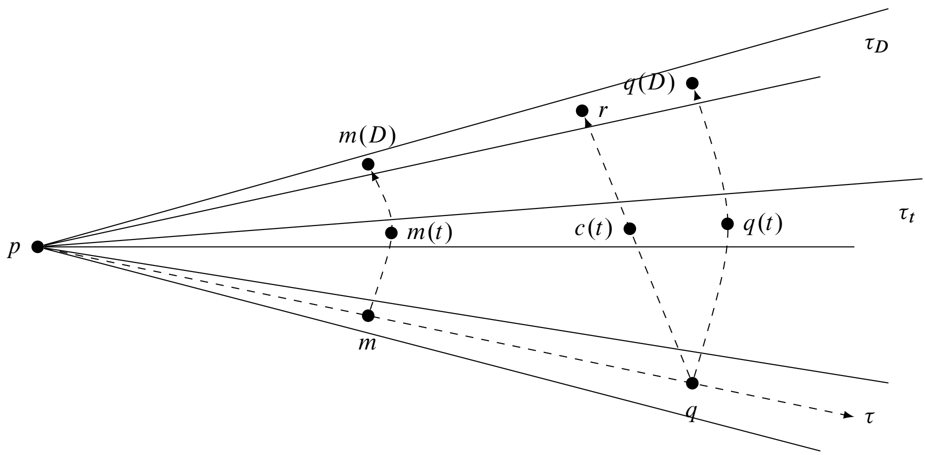


Figure 5. Weyl cones forming a small angle.

For large l, the evaluation of Y_t at v bounds the Killing form norm of Y_t: We choose a maximal flat containing p and $v = e^A p$ and, suppressing t, write $Y_t = \sum_{\alpha \in \Lambda_\tau^+} Y_\alpha + Y_{-\alpha}$ with $Y_\alpha \in \mathfrak{g}_\alpha$ and compute

$$\begin{aligned}
|\mathrm{ev}_v\, Y_t|^2 &= |\mathrm{ev}_p\, \mathrm{Ad}(e^{-A}) Y_t|^2 && \text{by Lemma 4.4 (2)} \\
&= \sum_{\alpha \in \Lambda_\tau^+} \left| \left(e^{\alpha(A)} - e^{-\alpha(A)}\right) \mathrm{ev}_p Y_\alpha \right|^2 && \text{since } Y_\alpha + Y_{-\alpha} \in \ker \mathrm{ev}_p \\
&= \frac{1}{2} \sum_{\alpha \in \Lambda_\tau^+} \left(e^{\alpha(A)} - e^{-\alpha(A)}\right)^2 |Y_\alpha|_{B_p}^2 && \text{since the restricted root space decomposition is } B_p\text{-orthogonal} \\
&\geq \frac{1}{2} \sum_{\alpha \in \Lambda_\tau^+} \left[2 \sinh(\alpha_0'(2l - t))\right]^2 |Y_\alpha|_{B_p}^2 && \text{since } \alpha(A) \geq \alpha_0'(2l - t) \text{ by regularity} \\
&= \frac{1}{2} \left[2 \sinh(\alpha_0'(2l - t))\right]^2 \sum_{\alpha \in \Lambda_\tau^+} |Y_\alpha|_{B_p}^2 \\
&= \frac{1}{4} \left[2 \sinh(\alpha_0'(2l - t))\right]^2 (-|Y|_B^2).
\end{aligned}$$

This bound $-[\sinh(\alpha_0'(2l - t))]^2 |Y_t|_B^2 \leq |\mathrm{ev}_{v(t)}\, Y_t|^2$ together with (4.1) implies

$$\begin{aligned}
2|\mathrm{ev}_q\, Y_t|^2 &\leq e^{2\kappa_0 t} 2|\mathrm{ev}_{v(t)}\, Y_t|^2 - (e^{2\kappa_0 D t} - 1)|Y_t|_B^2 \\
&\leq 2|\mathrm{ev}_{v(t)}\, Y_t|^2 \left[e^{2\kappa_0 t} + \frac{1}{2}(e^{2\kappa_0 t} - 1)[\sinh(\alpha_0'(2l - t))]^{-2}\right].
\end{aligned}$$

We now write $m(t) = k(t)m$ where $m = \mathrm{mid}(p, q) = e^{lW} p$ for $W \in \mathfrak{p}$. For $t \geq 0$, using $\alpha(W) \geq \alpha_0 > 0$ for all $\alpha \in \Lambda_\tau^+$ and Lemma 4.9, we have

$$\begin{aligned}
|\dot{m}(t)|^2 &= |\mathrm{ev}_p\, \mathrm{Ad}(e^{-lW}) Y_t|^2 \\
&= \frac{1}{2} \sum_{\alpha \in \Lambda_\tau^+} \left(e^{l\alpha(W)} - e^{-l\alpha(W)}\right)^2 |Y_\alpha|_{B_p}^2 \\
&\leq \frac{1}{2} \sum_{\alpha \in \Lambda_\tau^+} \left[\left(e^{2l\alpha(W)} - e^{-2l\alpha(W)}\right) e^{-l\alpha(W)}\right]^2 |Y_\alpha|_{B_p}^2 \\
&\leq \frac{1}{2} \sum_{\alpha \in \Lambda_\tau^+} \left(e^{2l\alpha(W)} - e^{-2l\alpha(W)}\right)^2 e^{-2\alpha_0 l} |Y_\alpha|_{B_p}^2 \\
&= e^{-2\alpha_0 l} |\dot{q}(t)|^2 \\
&\leq e^{-2\alpha_0 l} \left[e^{2\kappa_0 t} + \frac{1}{2}(e^{2\kappa_0 t} - 1)[\sinh(\alpha_0'(2l - t))]^{-2}\right].
\end{aligned}$$

The length of m is then

$$\int_0^D |\dot{m}(t)|\mathrm{d}t \leq \int_0^D e^{-\alpha_0 l}\sqrt{e^{2\kappa_0 t} + \frac{1}{2}(e^{2\kappa_0 t} - 1)[\sinh(\alpha_0'(2l - t))]^{-2}}\mathrm{d}t$$
$$\leq \int_0^D e^{-\alpha_0 l}\sqrt{e^{2\kappa_0 D} + 3e^{2\kappa_0 D}}\mathrm{d}t \leq 2De^{\kappa_0 D - \alpha_0 l},$$

and $k(D)$ is the desired isometry. ∎

It is possible to give a slightly stronger upper bound in Lemma 4.10, but the improvement would be inconsequential when we apply this lemma in Section 5 while making the already cumbersome statements even harder to read.

4.7. Strongly asymptotic geodesics and Weyl cones

The next estimate says that a point far along an (α_0, τ)-regular geodesic ray gets arbitrarily close to any given parallel set $P(\hat{\tau}, \tau)$. The following lemma is a quantified version of [19, Lemma 2.39].

Lemma 4.11. *Let $q \in \mathbb{X}$ and let $\eta \in \partial X$ be (α_0, τ)-regular. Let $P = P(\hat{\tau}, \tau)$ be a parallel set with $d(q, P) \leq D$, and let $p \in P$ be the unique point on the horocycle $H(q, \tau)$. Then for all $l \geq 0$ the geodesic rays $p\eta$ and $q\eta$ satisfy*

$$d(p\eta(l), q\eta(l)) \leq De^{\kappa_0 D - \alpha_0 l}.$$

It is possible to prove (a slightly weaker variation of) Lemma 4.11 as a limiting case of Lemma 4.10, or to construct a curve in N_τ in a similar way as we constructed a curve in K in Lemma 4.10. However, we give a direct proof here using the generalized Iwasawa decomposition (see Section 3.11).

Proof. We may assume that $d(q, P) = D$. By abuse of notation, let $q\colon [0, D] \to \mathbb{X}$ be the unit-speed geodesic segment from q to its nearest point $\overline{q} \in P$. Let $G = N_\tau A_\tau K$ be the generalized Iwasawa decomposition associated with p and τ (see Section 3.11). Since $N_\tau \times A_\tau \to \mathbb{X}$, $(u, a) \mapsto uap$ is a diffeomorphism, we may write $q(s) = u(s)a(s)p$ for unique curves $u\colon [0, D] \to N_\tau$ and $a\colon [0, D] \to A_\tau$. Note that $u(D) = 1 = a(0)$, since horocycles at τ meet parallel sets $P(\hat{\tau}, \tau)$ in exactly one point.

Writing $c_t(s) = C(s, t) = u(s)a(t)p$ we have $q(s) = C(s, s) = c_s(s)$, so

$$\dot{q}(s_0) = \frac{\partial}{\partial C}s|_{s_0,s_0} + \frac{\partial}{\partial C}t|_{s_0,s_0} = \dot{c}_{s_0}(s_0) + \frac{\partial}{\partial C}t|_{s_0,s_0}$$

and these vectors are orthogonal, so each has norm bounded by 1. The curve $t \mapsto a(t)p$ has speed bounded by 1 since

$$\frac{\partial}{\partial C}t|_{s_0,t_0} = \mathrm{d}u(s_0)\frac{\mathrm{d}}{\mathrm{d}t}a(t)p|_{t=t_0},$$

so $d(p, a(t)p) \leq t \leq D$. We write $\dot{u}(s) = \mathrm{d}l_{u(s)} U_s$ and use Lemmas 4.1, 4.4 and 4.9 to obtain

$$|\dot{c}_0(s)| = |\mathrm{ev}_p U_s| = \frac{1}{\sqrt{2}} |U_s|_{B_p} \leq \frac{1}{\sqrt{2}} e^{\kappa_0 d(p,a(s)p)} |U_s|_{B_{a(s)p}}$$
$$= e^{\kappa_0 d(p,a(s)p)} |\dot{c}_s(s)| \leq e^{\kappa_0 s}.$$

We next need to push this horocyclic curve towards τ and check that the length shrinks by at least $e^{-\alpha_0 l}$. Let $X \in \mathfrak{p}$ be the unit vector so that $q\eta(t) = u(0)e^{tX} p$. By abuse of notation, define the curve $r_t(s) = u(s)e^{tX} p$ from $q\eta(t)$ to $p\eta(t)$ and note that $r_l(0) = u(0)e^{lX} p = q\eta(l)$ (see Figure 6). We have shown that the speed of $r_0 = c_0$ is at most $e^{\kappa_0 s}$, and we may conclude after we show that

$$|\dot{r_t}(s)| \leq e^{-\alpha_0 t} |\dot{r_0}(s)|$$

in the next paragraph.

Define curves $U_\alpha(s) \in \mathfrak{g}_\alpha$ by $\dot{u}(s) = (\mathrm{d}l_{u(s)})_1 \sum_{\alpha \in \Lambda_\tau^+} U_\alpha(s)$ and using Lemma 4.9 write

$$\begin{aligned}
|\dot{r_t}(s)|_{T_{r_t(s)} \mathbb{X}} &= \left| \mathrm{ev}_p \operatorname{Ad}(e^{-tX}) \sum_{\alpha \in \Lambda_\tau^+} U_\alpha(s) \right|_{T_p \mathbb{X}} \\
&= \left| \mathrm{ev}_p \sum_{\alpha \in \Lambda_\tau^+} e^{-t\alpha(X)} U_\alpha(s) \right|_{T_p \mathbb{X}} \\
&= \frac{1}{\sqrt{2}} \left| \sum_{\alpha \in \Lambda_\tau^+} e^{-t\alpha(X)} U_\alpha(s) \right|_{B_p} \\
&\leq \frac{1}{\sqrt{2}} e^{-\alpha_0 t} \left| \sum_{\alpha \in \Lambda_\tau^+} U_\alpha(s) \right|_{B_p} \\
&= e^{-\alpha_0 t} |\dot{r_0}(s)|_{T_{c(s)} \mathbb{X}}.
\end{aligned}$$

Integrating this inequality bounds the length of r_l by $De^{\kappa_0 D - \alpha_0 l}$ and completes the proof. ∎

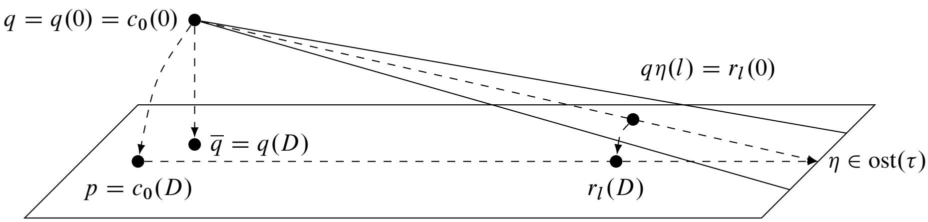


Figure 6. Strongly asymptotic geodesics get close at an exponential rate.

It is possible to give a slightly stronger upper bound in Lemma 4.11, but the improvement would be inconsequential when we apply this lemma in Section 5 while making the already cumbersome statements even harder to read.

The following lemma is a quantified version of [19, Lemma 2.40].

Lemma 4.12. *Let $p, q, x \in \mathbb{X}$ with pq an (α_0, τ)-regular geodesic segment and $d(p,q) \geq l$ and $d(p,x) \leq D$. If*

$$\alpha_0 - \frac{D(\alpha_0 + \kappa_0)}{l - D} \geq \alpha_0' \quad \text{and} \quad \frac{1}{\alpha_0' \zeta_0} \frac{D}{l} \leq \frac{\zeta_0^2}{\kappa_0^2},$$

then

$$d(q, V(x, \mathrm{st}(\tau), \alpha_0')) \leq D e^{\kappa_0 D - \alpha_0 l}.$$

Proof. Let $\eta \in \mathrm{ost}(\tau)$ such that $pq(+\infty) = \eta$. Let y be the unique point in the intersection $P(S_x\tau, \tau) \cap H(p, \tau)$. The point q' on the image of $y\eta$ such that $\vec{d}(y, q') = \vec{d}(p, q)$ satisfies $d(q, q') \leq D e^{\kappa_0 D - \alpha_0 l}$ by Lemma 4.11. We will prove the lemma by showing that xq' is (α_0', τ)-regular.

Choose chambers σ, σ' so that $yq' \in V(y, \sigma)$ and $xq' \in V(x, \sigma')$. Then there is a unique (restricted) isometry $g\colon V(y, \sigma) \to V(x, \sigma')$ by Theorem 3.10 and

$$d(gq', q') = |\vec{d}(x, gq') - \vec{d}(x, q')| = |\vec{d}(y, q') - \vec{d}(x, q')| \leq d(x, y) \leq D.$$

Now both q' and gq' lie in the same Euclidean Weyl cone $V(x, \sigma')$ with $d(q', gq') \leq D$ and the geodesic segment from x to gq' is length at least l and $(\alpha_0, \tau_{\mathrm{mod}})$-regular, so Lemma 4.5 implies that xq' is $(\alpha_0', \tau_{\mathrm{mod}})$-regular.

We conclude by showing that xq' is τ-regular. By Lemmas 4.7 and 4.8, we have that $\angle_{q'}^{\zeta}(x, y) \leq \frac{1}{\alpha_0' \zeta_0} \frac{D}{l} \leq \frac{\zeta_0^2}{\kappa_0^2}$, so $\angle_{q'}^{\zeta}(x, \tau) \geq \pi - \varepsilon(\zeta_{\mathrm{mod}})$ by Lemma 3.23. Since $S_x\tau = S_{q'}\tau$ is the unique antipode of τ in the boundary of $P(S_x\tau, \tau)$, it follows that xq' is τ-regular. ■

4.8. Projecting midpoints to Weyl cones

We combine the previous Lemmas 4.10, 4.11 and 4.12 to show that a long regular geodesic segment in a bounded neighborhood of a Weyl cone has its midpoint arbitrarily close to the Weyl cone.

Corollary 4.13. *Let $p, q, x \in \mathbb{X}$ with pq an $(\alpha_0, \tau_{\mathrm{mod}})$-regular geodesic segment with midpoint m, let $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ and let $V = V(x, \mathrm{st}(\tau))$. Assume that $d(p,x) \leq D$, $d(q, V) \leq D$ and $d(p,q) \geq 2l$. Suppose that*

(1)
$$\alpha_0 - \frac{2D(\alpha_0 + \kappa_0)}{l - 2D} \geq \alpha_0' > 0,$$

(2)
$$\frac{1}{2}(e^{4\kappa_0 D} - 1)[\sinh(\alpha_0'(2l - 2D))]^{-2} \leq 3e^{4\kappa_0 D}, \text{ and}$$

(3)
$$\frac{2}{\alpha_0'\zeta_0}\frac{D}{l} \leq \frac{\zeta_0^2}{\kappa_0^2}$$

then
$$d(m, V(x, \mathrm{st}(\tau), \alpha_0')) \leq 5De^{2\kappa_0 D - \alpha_0 l}.$$

Proof. Since $d(q, V) \leq D$ and the Hausdorff distance from V to $V(p, \mathrm{st}(\tau))$ is at most D, we have $d(q, V(p, \mathrm{st}(\tau))) \leq 2D$. We may now apply Lemma 4.10 together with assumptions (1) and (2) to see that there exists $m' \in V(p, \mathrm{st}(\tau), \alpha_0)$ with $d(m, m') \leq 4De^{2\kappa_0 D - \alpha_0 l}$ and $d(p, m') = d(p, m) \geq l$.

By assumption (1) and (3), the bound $d(m', V(x, \mathrm{st}(\tau), \alpha_0')) \leq De^{\kappa_0 D - \alpha_0 l}$ follows from Lemma 4.12. By the triangle inequality,
$$\begin{aligned} d(m, V(x, \mathrm{st}(\tau), \alpha_0')) &\leq d(m, m') + d(m', V(x, \mathrm{st}(\tau), \alpha_0')) \\ &\leq 4De^{2\kappa_0 D - \alpha_0 l} + De^{\kappa_0 D - \alpha_0 l} \leq 5De^{2\kappa_0 D - \alpha_0 l}. \end{aligned}$$
■

4.9. Simplex displacement after a short flow

Recall that we have fixed a model type $\zeta = \zeta_{\mathrm{mod}}$ spanning τ_{mod} (see Definition 3.14 and Section 3.9).

Lemma 4.14. *For any point $p \in \mathbb{X}$, simplex $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ and transvection vector $X \in \mathfrak{p}$, it holds that*
$$\sin \frac{1}{2}\angle_p^{\zeta}(\tau, e^X\tau) \leq \frac{\kappa_0}{2}|X|_{B_p}.$$

Proof. Denote by f_τ the Busemann function associated with the ray from p to $\zeta(\tau)$ and write $\mathrm{grad} f_\tau$ for its gradient (see Figure 7). Then
$$\angle_p^{\zeta}(\tau, e^X\tau) = \angle_p(\mathrm{grad} f_\tau, \mathrm{grad} f_{e^X\tau})$$
and
$$\sin \frac{1}{2}\angle_p(\mathrm{grad} f_\tau, \mathrm{grad} f_{e^X\tau}) = \frac{1}{2}d_{T_p\mathbb{X}}(\mathrm{grad} f_\tau, \mathrm{grad} f_{e^X\tau}).$$

Let $Z \in \mathfrak{p}$ be the unit vector so that $\mathrm{ev}_p Z = (\mathrm{grad} f_\tau)_p$. Decompose $X = U + Y$ according to the generalized Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\tau + \mathfrak{n}_\tau$ so that flowing by Y fixes τ and therefore commutes with $\mathrm{grad} f_\tau$, and flowing by U fixes p (see Section 3.11). We may write $X = A + \sum_{\alpha \in \Lambda^+}(-X_\alpha + \vartheta_p X_\alpha)$ and $U = \sum_{\alpha \in \Lambda^+}(X_\alpha + \vartheta_p X_\alpha)$, so

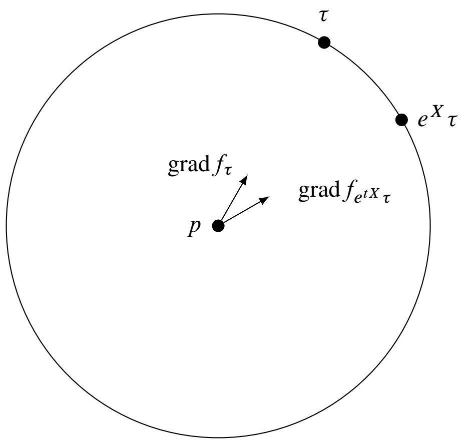


Figure 7. Simplex displacement.

$|K|_{B_p} \leq |X|_{B_p}$. At p we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\operatorname{grad} f_{e^{tX}\tau})_p|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\big((e^{tX})_* \operatorname{grad} f_\tau\big)_p|_{t=0} \\
&= (\mathcal{L}_{-X^*} \operatorname{grad} f_\tau)_p \\
&= [-X^*, \operatorname{grad} f_\tau]_p \\
&= [(-X+Y)^*, \operatorname{grad} f_\tau]_p \\
&= [-U^*, \operatorname{grad} f_\tau]_p \\
&= (\mathcal{L}_{-U^*} \operatorname{grad} f_\tau)_p \\
&= \lim_{t\to 0} \frac{(\mathrm{d}e^{tU})(\operatorname{grad} f_\tau)_{e^{-tU}p} - (\operatorname{grad} f_\tau)_p}{t} \\
&= \lim_{t\to 0} \frac{(\mathrm{d}e^{tU})(\operatorname{grad} f_\tau)_p - (\operatorname{grad} f_\tau)_p}{t} \\
&= \lim_{t\to 0} \frac{(\mathrm{d}e^{tU})\mathrm{ev}_p Z - \mathrm{ev}_p Z}{t} \\
&= \lim_{t\to 0} \frac{\mathrm{ev}_p \operatorname{Ad}(e^{tU})Z - \mathrm{ev}_p Z}{t} \\
&= \mathrm{ev}_p [U, Z]_{\mathfrak{g}}.
\end{aligned}$$

Since we assumed nothing about the relationship of X and τ, we see that for all $t' \in [0,1]$,

$$\begin{aligned}
\Big|\frac{\mathrm{d}}{\mathrm{d}t}(\operatorname{grad} f_{e^{tX}\tau})_p|_{t=t'}\Big| &= \Big|\frac{\mathrm{d}}{\mathrm{d}t}((e^{tX})_* \operatorname{grad} f_{e^{t'X}\tau})_p|_{t=0}\Big| \\
&\leq |[U, Z]|_{B_p} \leq \kappa_0 |U|_{B_p} \leq \kappa_0 |X|_{B_p},
\end{aligned}$$

where we used Lemma 4.3 in the second inequality. Finally, we obtain

$$|\operatorname{grad} f_\tau - \operatorname{grad} f_{e^X\tau}|_{T_p\mathbb{X}} \le \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{grad} f_{e^{tX}\tau}\right|_{T_p\mathbb{X}} \mathrm{d}t \le \kappa_0 |X|_{B_p},$$

which completes the proof. ∎

4.10. The distance to a parallel set bounds the ζ-angle

Corollary 4.15. *Let p, q be points $\mathbb{X}$ and $\tau, \tau' \in \mathrm{Flag}(\tau_{\mathrm{mod}})$. If $d(p,q) \le \frac{2}{\kappa_0}$, then*

$$|\angle_p^\zeta(\tau,\tau') - \angle_q^\zeta(\tau,\tau')| \le 4\sin^{-1}\left(\frac{\kappa_0}{2} d(p,q)\right).$$

Proof. Write $q = e^{-X}p$ for $X \in \mathfrak{p}$. We use that ζ-angles are G-invariant, the triangle inequality for quadruples in $(\mathrm{Flag}(\tau_{\mathrm{mod}}), \angle_p^\zeta)$ and the simplex displacement estimate given by Lemma 4.14:

$$\begin{aligned}\left|\angle_p^\zeta(\tau,\tau') - \angle_q^\zeta(\tau,\tau')\right| &= \left|\angle_p^\zeta(\tau,\tau') - \angle_p^\zeta(e^X\tau, e^X\tau')\right| \\ &\le \angle_p^\zeta(\tau, e^X\tau) + \angle_p^\zeta(\tau', e^X\tau') \le 4\sin^{-1}\left(\frac{\kappa_0}{2}|X|_{B_p}\right).\end{aligned}$$

Since $|X|_{B_p} = d(p,q)$, we are done. ∎

We will often apply Corollary 4.15 in the following form. This result is a quantified version of [19, Lemma 2.43 (i)].

Corollary 4.16. *Let τ_+, τ_- be antipodal simplices in $\mathrm{Flag}(\tau_{\mathrm{mod}})$ and let $P = P(\tau_-,\tau_+)$ be the parallel set joining them. Let p be any point in $\mathbb{X}$ such that $d(p,P) \le \frac{2}{\kappa_0}$. Then*

$$\angle_p^\zeta(\tau_-,\tau_+) \ge \pi - 4\sin^{-1}\left(\frac{\kappa_0}{2} d(p,P)\right).$$

Proof. Since $\angle_q^\zeta(\tau_-,\tau_+) = \pi$ for any $q \in P$, and in particular the projection of p to P, the assertion follows immediately from Corollary 4.15. ∎

4.11. The ζ-angle bounds the distance to the parallel set

We continue to work with a fixed $(\zeta_0, \tau_{\mathrm{mod}})$-spanning type $\zeta = \zeta_{\mathrm{mod}}$ and from now on assume that ζ is ι-invariant (see the discussion after Theorem 3.10). The next lemma complements Corollary 4.16: When the ζ-angle at $q \in \mathbb{X}$ between simplices $\tau_\pm \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ is near π, the point q is near the parallel set $P(\tau_-,\tau_+)$. In the proof we use the fact that a vector field X is Killing (if and) only if for all vector fields V, W on $\mathbb{X}$, we have

$$X\langle V, W\rangle = \langle [X,V], W\rangle + \langle V, [X,W]\rangle,$$

see [26, Proposition 9.25]. The result in the following lemma is a quantified version of [19, Lemma 2.43 (ii)].

Lemma 4.17. *Let* $\tau_\pm \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ *and let* $q \in \mathbb{X}$. *If* $\delta \le \frac{\zeta_0^2}{2\kappa_0^2}$ *and* $\angle_q^\zeta(\tau_-, \tau_+) \ge \pi - \delta$, *then* $\tau_\pm$ *are antipodal and* $d(q, P(\tau_-, \tau_+)) \le \delta/\zeta_0$.

Proof. Since $\angle_q^\zeta(\tau_-, \tau_+) \ge \pi - \frac{\zeta_0^2}{2\kappa_0^2} > \pi - \frac{\zeta_0^2}{\kappa_0^2}$, Lemma 3.23 implies that the simplices τ_-, τ_+ are antipodal.

Write $\zeta_\pm$ for the unique ideal points $\tau_\pm$ of type ζ, and choose Busemann functions $f_\pm$ at $\zeta_\pm$. For all $p \in \mathbb{X}$ we have $\cos \angle_p^\zeta(\tau_-, \tau_+) = \cos \angle_p(\zeta_-, \zeta_+) = \langle \mathrm{grad}\, f_-, \mathrm{grad}\, f_+ \rangle_p$. Let $\overline{q} \in P = P(\tau_-, \tau_+)$ be the nearest point on P to q, and let $X \in \mathfrak{p}_{\overline{q}}$ such that $c(t) = e^{tX}\overline{q}$ is the unit-speed geodesic from $\overline{q}$ to q (see Figure 8). Either $\angle_q(\zeta_-, \overline{q}) \ge \frac{\pi}{2} - \frac{\delta}{2}$ or $\angle_q(\overline{q}, \zeta_+) \ge \frac{\pi}{2} - \frac{\delta}{2}$, so without loss of generality we may assume the second inequality holds. Let $f : (-\infty, \infty) \to [-1, 1]$ be defined by $f(s) = \langle -X^*, \mathrm{grad}\, f_+ \rangle_{c(s)}$ and note that $f(s) = \cos \angle_{c(s)}(\overline{q}, \zeta_+)$ for all $s > 0$. We first show that $f'(s) \ge 0$ for all s, so f is (weakly) monotonic.

At the point $c(s)$, we have $X \in \mathfrak{p}_{c(s)}$ since X is a transvection along c. The point $c(s)$ together with a fixed choice of chamber containing τ_+ allows us to decompose X according to the restricted root space decomposition. Suppressing the dependence on s, we have $X = A + \sum_{\alpha \in \Lambda^+} -X_\alpha + \vartheta X_\alpha$. Then for $U = \sum_{\alpha \in \Lambda^+} X_\alpha + \vartheta X_\alpha$ and the unit vector $Z \in \mathfrak{p}_{c(s)}$ pointing to ζ_+, we see that

$$\begin{aligned} f'(s) &= X^* \langle -X^*, \mathrm{grad}\, f_+ \rangle_{c(s)} \\ &= \langle -X^*, [X^*, \mathrm{grad}\, f_+] \rangle_{c(s)} \\ &= \langle -X^*, [U, Z]^* \rangle_{c(s)} \\ &= B(-X, -[U, Z]_{\mathfrak{g}}) \\ &= B\Big(A + \sum_{\beta \in \Lambda^+} -X_\beta + \vartheta X_\beta, \sum_{\alpha \in \Lambda^+} \alpha(Z)(X_\alpha - \vartheta X_\alpha)\Big) \\ &= \sum_{\alpha \in \Lambda^+} \alpha(Z) B(X_\alpha - \vartheta X_\alpha, X_\alpha - \vartheta X_\alpha) \\ &\ge \zeta_0 \sum_{\alpha \in \Lambda_\tau^+} |-X_\alpha + \vartheta X_\alpha|_B^2. \end{aligned}$$

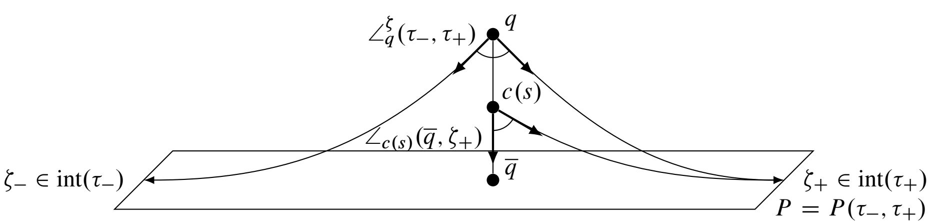


Figure 8. The ζ-angle at q bounds the distance to P.

The third line follows from the reasoning in the proof of Lemma 4.14. This calculation shows that $f'(s) \geq 0$ for all s. Moreover, since X^* is orthogonal to $P(\overline{q}, \tau)$ at $s = 0$, we have $1 = |X^*_{\overline{q}}|^2 = \sum_{\alpha \in \Lambda_\tau^+} |-X_\alpha + \vartheta X_\alpha|_B^2$, so $f'(0) \geq \zeta_0$.

We next bound the norm of

$$\begin{aligned}
f''(s) &= X^*(X^*\langle -X^*, \operatorname{grad} f_+ \rangle)_{c(s)} \\
&= \langle -X^*, [X^*, [X^*, \operatorname{grad} f_+]]\rangle_{c(s)} \\
&= \langle -X^*, [X^*, [U^*, \operatorname{grad} f_+]]\rangle_{c(s)} \\
&= \langle -X^*, [U^*, [X^*, \operatorname{grad} f_+]]\rangle_{c(s)} - \langle X^*, [[U^*, X^*], \operatorname{grad} f_+]\rangle_{c(s)} \\
&= \langle -X^*, [U^*, [U^*, \operatorname{grad} f_+]]\rangle_{c(s)} + \langle X^*, [U'^*, \operatorname{grad} f_+]\rangle_{c(s)} \\
&= B_{c(s)}(-X, [U, [U, Z]]) + B_{c(s)}(X, [U', Z]) \\
&= B_{c(s)}([U, X], [U, Z]) + B_{c(s)}([X, Z], U'),
\end{aligned}$$

where $[U, X] = U' + A' + N'$ according to the KAN decomposition for $c(s)$ and τ_+. We get the bound

$$\begin{aligned}
|f''(s)| &= |B_{c(s)}([U, X], [U, Z]) + B_{c(s)}([X, Z], U')| \\
&\leq |B_{c(s)}([U, X], [U, Z])| + |B_{c(s)}([X, Z], U')| \\
&\leq |[U, X]|_{B_{c(s)}} |[U, Z]|_{B_{c(s)}} + |[X, Z]|_{B_{c(s)}} |U'|_{B_{c(s)}} \\
&\leq 2\kappa_0^2
\end{aligned}$$

by applying Lemma 4.3.

Since $f'(0) \geq \zeta_0$ and $|f''(s)| \leq 2\kappa_0^2$, we have $f(s) \geq s\zeta_0 - \kappa_0^2 s^2$. Since f is monotonic, if $s \geq \frac{\zeta_0}{2\kappa_0^2}$ then $f(s) \geq f\left(\frac{\zeta_0}{2\kappa_0^2}\right) \geq \frac{\zeta_0^2}{4\kappa_0^2}$. On the other hand, if $s \leq \frac{\zeta_0}{2\kappa_0^2}$ we have $f(s) \geq \zeta_0 s - \kappa_0^2\left(\frac{\zeta_0}{2\kappa_0^2}\right)s \geq \frac{1}{2}\zeta_0 s$. This implies

$$\frac{1}{2}\zeta_0 d(q, P) \leq f(d(q, P)) = \cos \angle_q^\zeta(\overline{q}, \tau_+) \leq \cos\left(\frac{\pi}{2} - \frac{\delta}{2}\right) = \sin\left(\frac{\delta}{2}\right) \leq \frac{\delta}{2}$$

unless $d(q, P) > \frac{\zeta_0}{2\kappa_0^2}$, which yields $\frac{\zeta_0^2}{2\kappa_0^2} < \delta$ and contradicts our assumption. ∎

5. Quantified local-to-global principle

In this section, we augment the theorems of [19, Section 7] with quantitative estimates. We obtain a precise version of the local-to-global principle which allows us to perturb known Anosov representations by a definite amount, producing new Anosov representations in Section 6.

In rank 1, local quasigeodesics of sufficiently good quality are global quasigeodesics. This naive version of the local-to-global principle fails in the Euclidean plane, hence in higher rank, so we must use *Morse quasigeodesics* as defined in [19]. The strategy here, as in [19], is to show that local Morse quasigeodesics of sufficiently good quality have straight and spaced midpoint sequences which are then globally Morse quasigeodesics. First we give an explicit local criterion for a sequence to be a Morse quasigeodesic.

5.1. Sufficiently straight and spaced sequences are Morse quasigeodesics

We recall some definitions from [19]. A sequence of points (x_n) in $\mathbb{X}$ is $(\alpha_0, \tau_{\mathrm{mod}}, \epsilon)$-*straight* if each geodesic segment $x_n x_{n+1}$ is $(\alpha_0, \tau_{\mathrm{mod}})$-regular and if

$$\angle^{\xi}_{x_n}(x_{n-1}, x_{n+1}) \geq \pi - \epsilon$$

for all n. The sequence is s-*spaced* if $d(x_n, x_{n+1}) \geq s$ for all n. A sequence (x_n) is said to *move ϵ-away from a simplex τ* if for all n

$$\angle^{\xi}_{x_n}(\tau, x_{n+1}) \geq \pi - \epsilon.$$

In this paper, we are only interested in discrete sequences of points in $\mathbb{X}$. For us, a (c_1, c_2, c_3, c_4)-*quasigeodesic* is a sequence (x_n) (possibly finite, infinite or bi-infinite) such that

$$\frac{1}{c_1}|N| - c_2 \leq d(x_n, x_{n+N}) \leq |N|c_3 + c_4.$$

A sequence (x_n) is (c_1, c_2)-*coarsely spaced* (or *lower-quasigeodesic*) if

$$\frac{1}{c_1}|N| - c_2 \leq d(x_n, x_{n+N}).$$

Likewise, (x_n) is (c_3, c_4)-*coarsely Lipschitz* (or *upper-quasigeodesic*) if

$$d(x_n, x_{n+N}) \leq |N|c_3 + c_4.$$

For an $(\alpha_0, \tau_{\mathrm{mod}})$-regular segment pq, the $(\alpha_0, \tau_{\mathrm{mod}})$-*diamond* is the intersection

$$\diamondsuit_{\alpha_0}(p, q) := V(p, \mathrm{st}(\tau(pq)), \alpha_0) \cap V(q, \mathrm{st}(\tau(qp)), \alpha_0).$$

A quasigeodesic is $(\alpha_0, \tau_{\mathrm{mod}}, D)$-*Morse* if for all x_n, x_m there exists a diamond $\diamondsuit = \diamondsuit_{\alpha_0}(p, q)$ such that $d(p, x_n), d(q, x_m) \leq D$ and for all $n \leq i \leq m$, $d(x_i, \diamondsuit) \leq D$ (see Figure 9). In rank 1, quasigeodesics are automatically Morse by the Morse lemma. In higher-rank symmetric spaces of noncompact type, the following theorem allows us to establish the Morse property for sufficiently straight and spaced sequences.

There are a few variations of the precise definition of Morse quasigeodesic in the literature. The definition of Morse quasigeodesic here is the same as that given in [20, Definition 5.50], except that we keep track of more constants in the definition of quasi-geodesic. This is the same as [19, Definition 7.14] except that we work with sequences rather than paths. Likewise, [17, Definition 6.13] defines paths to be Morse quasigeodesics

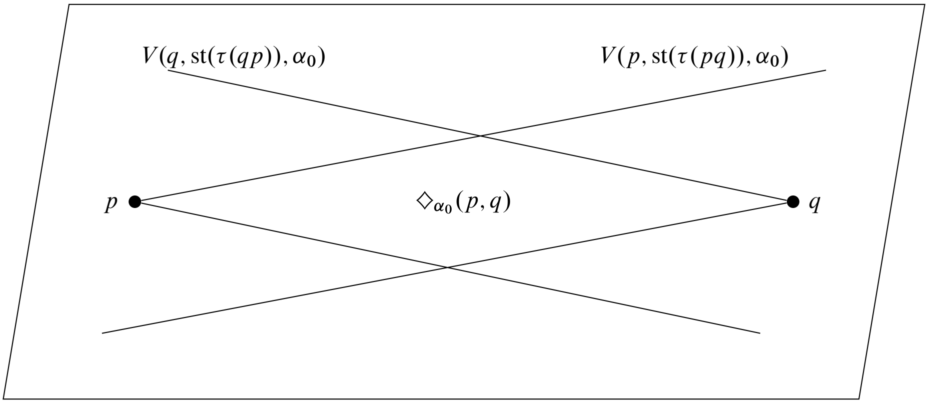


Figure 9. The $(\alpha_0, \tau_{\mathrm{mod}})$-diamond with endpoints p and q.

when they satisfy a similar and equivalent, but not identical, property as the one we have given here (the constants will be different).

Define the constant

$$c_0 := \sum_{\alpha \in \Lambda^+_{\tau_{\mathrm{mod}}}} \dim \mathfrak{g}_\alpha,$$

equal to the codimension of any parallel set of type τ_{mod}. The inequality $c_0 \geq 1$ always holds. Theorem 5.1 is a quantified version of [19, Theorem 7.2]. The constant κ_0 is defined and computed in Section 3.3. Any choice of ζ_{mod} has some regularity parameter $\zeta_0 = \min\{\alpha(\zeta_{\mathrm{mod}}) \mid \alpha \in \Delta^+_{\tau_{\mathrm{mod}}}\}$. For minimal ι-invariant faces τ_{mod} these are computed in Section 3.10.

Theorem 5.1. *Fix $\alpha_{\mathrm{new}} < \alpha_0, \delta$ and assume ϵ is small and s is large. Precisely, we assume that:*

(1) $5\epsilon \leq \frac{\zeta_0^2}{2\kappa_0^2}$, *so that we may apply the angle-to-distance estimate in Lemma 4.17;*

(2)
$$\frac{\epsilon \kappa_0}{\zeta_0} e^{2\kappa_0 \epsilon/\zeta_0 - \alpha_0 s} \leq \sin\left(\frac{\epsilon}{4}\right)$$
so that we may apply the distance-to-angle estimate in Lemma 4.16;

(3)
$$\frac{5\epsilon}{\zeta_0} \leq \delta$$
to control the distance from the sequence to the parallel set;

(4)
$$\alpha_0 - \frac{2\delta(\alpha_0 + \kappa_0)}{s - 2\delta} \geq \alpha_{\mathrm{new}}$$
so that certain projections are $(\alpha_{\mathrm{new}}, \tau_{\mathrm{mod}})$-regular by Lemma 4.5;

(5)

$$2\epsilon + \sin^{-1}\left(\frac{2\delta}{\alpha_0\zeta_0 s}\right) < \varepsilon(\zeta)$$

so that certain simplices are antipodal (see Section 3.13).

Then every $(\alpha_0, \tau_{\mathrm{mod}}, \epsilon)$-straight s-spaced sequence (x_n) in $\mathbb{X}$ is δ-close to a parallel set $P(\tau_-, \tau_+)$ such that

$$\overline{x}_{n\pm m} \in V(\overline{x}_n, \mathrm{st}(\tau_\pm), \alpha_{\mathrm{new}})$$

for all n and $m \geq 1$. It follows that the sequence is coarsely spaced:

$$d(x_n, x_{n\pm m}) \geq 2\alpha_{\mathrm{new}}\zeta_0 c_0(s - 2\delta)m - 2\delta,$$

and if (x_n) is coarsely Lipschitz it is then an $(\alpha_{\mathrm{new}}, \tau_{\mathrm{mod}}, \delta)$-Morse quasigeodesic.

Our proof closely follows [19, Section 7], who prove the same theorem without the explicit assumptions (1) through (5) and without the explicit estimates we obtained in Section 4. Note that the resulting sequence will always be $\frac{\zeta_0}{2\kappa_0^2}$-close to the parallel set, even if δ is chosen larger than that quantity.

Proof. Step 1. Propagation, cf. [19, Lemma 7.6]. For sufficiently straight and spaced sequences, the property of moving away from a simplex propagates along the sequence.

Assume that for some simplex τ in Flag(τ_{mod}) we have $\angle^{\zeta}_{x_0}(\tau, x_1) \geq \pi - 2\epsilon$. Since $2\epsilon < \frac{\zeta_0^2}{2\kappa_0^2}$ by assumption (1), Lemma 3.23 implies that the simplex τ_{01} containing $x_0x_1(+\infty)$ is antipodal to τ and together they define a parallel set $P = P(\tau, \tau_{01})$ (see Figure 10). Moreover, assumption (1) and our angle-to-distance estimate, Lemma 4.17, imply that $d(x_0, P) \leq \frac{2\epsilon}{\zeta_0}$. By Lemma 4.11, the geodesic ray from x_0 through x_1 gets arbitrarily close to P and in particular

$$d(x_1, P) \leq \frac{2\epsilon}{\zeta_0} e^{2\kappa_0\epsilon/\zeta_0 - \alpha_0 s},$$

and by assumption (2) and the distance-to-angle estimate, Corollary 4.16, we have

$$\angle^{\zeta}_{x_1}(\tau, \tau_{01}) \geq \pi - 4\sin^{-1}\left(\frac{\epsilon\kappa_0}{\zeta_0} e^{2\kappa_0\epsilon/\zeta_0 - \alpha_0 s}\right) \geq \pi - \epsilon,$$

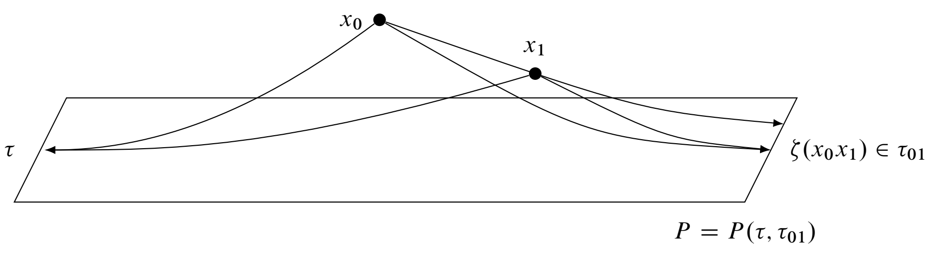


Figure 10. "Moving away from τ" propagates along the sequence.

which then implies that $\angle^{\zeta}_{x_1}(\tau, x_0) = \pi - \angle^{\zeta}_{x_1}(\tau, \tau_{01}) \le \epsilon$. Straightness and an application of the triangle inequality for $(S(T_{x_1}\mathbb{X}), \angle_{x_1})$ imply $\angle^{\zeta}_{x_1}(\tau, x_2) \ge \pi - 2\epsilon$. By induction, we have that $\angle^{\zeta}_{x_n}(\tau, x_{n+1}) \ge \pi - 2\epsilon$ for all $n \ge 1$.

Step 2. Extraction, cf. [19, Lemma 7.7]. We extract antipodal simplices that the sequence moves away/towards. It follows that the sequence stays near the corresponding parallel set.[2]

For each n, define the compact subsets $C^{\pm}_n \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$:

$$C^{\pm}_n := \{\tau_{\pm} \mid \angle^{\zeta}_{x_n}(\tau_{\pm}, x_{n\mp 1}) \ge \pi - 2\epsilon\}.$$

Each of these is nonempty since $\angle^{\zeta}_{x_n}(x_{n\mp 1}x_n, x_{n\mp 1}) = \pi$ implies $\tau(x_{n\mp 1}x_n) \in C^{\pm}_n$. By Step 1, $C^-_n \subset C^-_{n+1}$ so there exists $\tau_- \in \bigcap_n C^-_n$. Similarly, there exists some $\tau_+ \in \bigcap_n C^+_n$. Straightness and the triangle inequality imply $\angle^{\zeta}_{x_n}(\tau_-, \tau_+) \ge \pi - 5\epsilon$, and by assumption (1) we have $5\epsilon \le \frac{\zeta_0^2}{2\kappa_0^2}$. Therefore, the angle-to-distance estimate, Lemma 4.17, implies that $\tau_{\pm}$ are antipodal and define the parallel set $P = P(\tau_-, \tau_+)$ and moreover

$$d(x_n, P) \le \frac{5\epsilon}{\zeta_0} \le \delta$$

with the last inequality from assumption (3).

Step 3. Morseness, cf. [19, Lemmas 7.9 and 7.10, Corollary 7.13]. We verify that the sequence is a Morse quasigeodesic. We have already shown the angles are straight enough to guarantee that the distance to P is bounded. We show that projected rays land in nested cones; it follows that projecting further to the ζ-ray yields a monotonic sequence which makes progress bounded away from zero.

By assumption (4), and Lemma 4.5, we have that the projections $(\overline{x}_n)$ to P are $(\alpha_{\mathrm{new}}, \tau_{\mathrm{mod}})$-regular. Let ξ be the ideal point corresponding to the ray $\overline{x}_n\overline{x}_{n+1}$ (see Figure 11). Since the rays $x_n\xi$ and $\overline{x}_n\xi$ are asymptotic, their Hausdorff distance is at most $d(x_n, \overline{x}_n) \le \delta$, so x_{n+1} is at most 2δ from $x_n\xi$. Then

$$\begin{aligned}\angle^{\zeta}_{\mathrm{Tits}}(\tau_-, \xi) \ge \angle^{\zeta}_{x_n}(\tau_-, \xi) &\ge \angle^{\zeta}_{x_n}(\tau_-, x_{n+1}) - \angle^{\zeta}_{x_n}(x_{n+1}, \xi)\\ &\ge \pi - 2\epsilon - \angle^{\zeta}_{x_n}(x_{n+1}, \xi).\end{aligned}$$

By Lemmas 4.7 and 4.8, we may guarantee that

$$\sin \angle^{\zeta}_{x_n}(x_{n+1}, \xi) \le \frac{1}{\alpha_0\zeta_0}\frac{2\delta}{s},$$

so by assumption (5) this Tits angle is within $\varepsilon(\zeta)$ of π, so $\zeta(\tau_-)$ is antipodal to $\zeta(\xi)$, but the only simplex in ∂P antipodal to τ_- is τ_+, so $\tau(\xi) = \tau_+$ and

$$\angle^{\zeta}_{\overline{x}_n}(\tau_-, \overline{x}_{n+1}) = \angle^{\zeta}_{\overline{x}_n}(\tau_-, \xi) = \pi.$$

[2]The simplices are unique when the sequence is bi-infinite (see [19, Lemmas 5.15 and 7.19]), but this theorem also applies when the sequence is finite or a Morse quasiray.

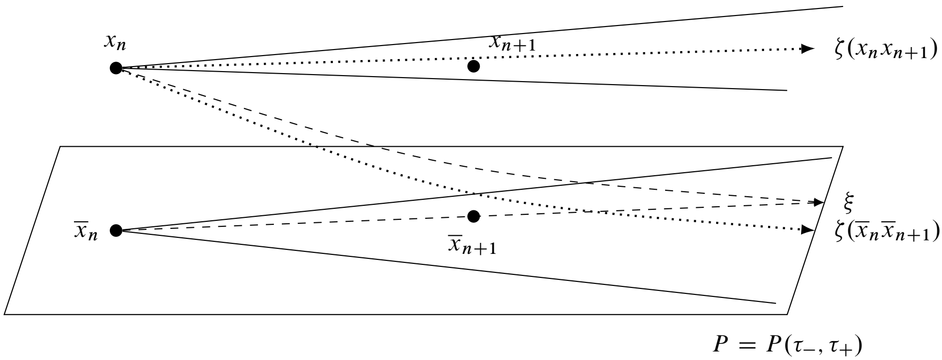


Figure 11. The projection $\overline{x}_{n+1}$ lands in the Weyl cone $V(\overline{x}_n, \mathrm{st}(\tau_+), \alpha_{\mathrm{new}})$.

We know that $\overline{x}_n\overline{x}_{n+1}$ is $(\alpha_{\mathrm{new}}, \tau_{\mathrm{mod}})$-regular and $\angle^{\xi}_{\overline{x}_n}(\tau_-, \xi) = \pi$ and these two properties are equivalent to $\overline{x}_{n+1} \in V(\overline{x}_n, \mathrm{st}(\tau_+), \alpha_{\mathrm{new}})$. Using the convexity of Weyl cones and induction, we get that for all n and all $m \geq 1$

$$\overline{x}_{n\pm m} \in V(\overline{x}_n, \mathrm{st}(\tau_\pm), \alpha_{\mathrm{new}}).$$

Finally, we want to show the sequence is coarsely spaced. The bound

$$d(x_n, x_{n+m}) \geq 2\alpha_{\mathrm{new}}\zeta_0 c_0(s - 2\delta)m - 2\delta$$

will follow from

$$d(\overline{x}_n, \overline{x}_{n+m}) \geq 2\alpha_{\mathrm{new}}\zeta_0 c_0(s - 2\delta)m.$$

Indeed, the sequence $(\overline{x}_n)$ in P is $(s - 2\delta)$-spaced and has a monotonic projection $(\overline{\overline{x}}_n)$ to the geodesic line $\overline{x}_n\zeta(\tau_+)$ for any n by the nestedness of Weyl cones (see Figure 12). By [8, Proposition 2.14.5],

$$\begin{aligned} B(\zeta, \vec{d}(\overline{x}_n, \overline{x}_{n+1})) &= \sum_{\alpha\in\Lambda} \alpha(\zeta)\alpha(\vec{d}(\overline{x}_n, \overline{x}_{n+1})) \dim \mathfrak{g}_\alpha \\ &\geq 2\alpha_{\mathrm{new}}\zeta_0 d(\overline{x}_n, \overline{x}_{n+1}) \sum_{\alpha\in\Lambda^+_\tau} \dim \mathfrak{g}_\alpha \\ &= 2\alpha_{\mathrm{new}}\zeta_0 c_0 d(\overline{x}_n, \overline{x}_{n+1}). \end{aligned}$$

It follows that the projection $\overline{\overline{x}}_{n+1}$ lies at least $2\alpha_{\mathrm{new}}\zeta_0 c_0(s - 2\delta)$ along the ray $\overline{x}_n\zeta$. ∎

In the final step of the proof we used the regularity of the projections to obtain the linear lower-quasigeodesic constant. When the angular radius of σ_{mod} with respect to ζ is strictly less than $\pi/2$, the linear lower-quasigeodesic bound can be chosen independent of the regularity. By [16, Lemma 5.8], this happens exactly when ζ is not contained in a factor of a nontrivial spherical join decomposition of σ_{mod}. In particular, this is always possible when $\mathbb{X}$ is irreducible.

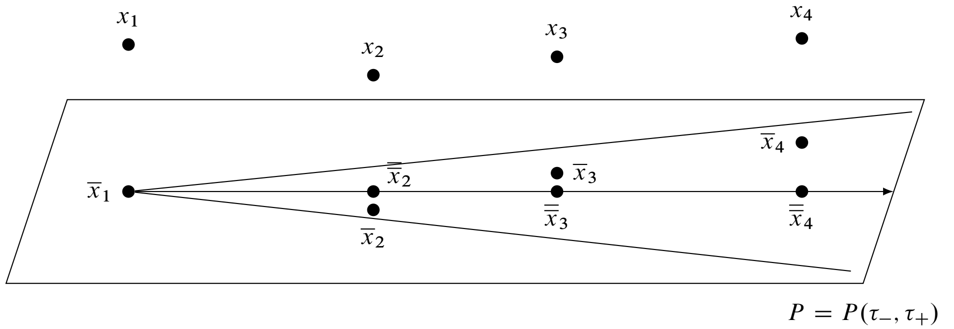


Figure 12. Sufficiently straight and spaced sequences have monotonic projections to a geodesic ray.

Remark 5.2. To provide suitable auxiliary parameters to apply Theorem 5.1, we may first choose ϵ small enough to satisfy assumptions (1) and (3) subsequently, we can choose s large enough to satisfy assumptions (2), (4) and (5). When we apply Theorem 5.1 in Section 6, we will choose $\delta = \frac{\zeta_0}{2\kappa_0^2}$ and $\epsilon = \frac{\zeta_0^2}{10\kappa_0^2}$ and then find a large enough parameter s to satisfy the conditions of Theorem 5.1.

Remark 5.3. Theorem 5.1 can be modified to deal with arbitrary τ_{mod}-Weyl convex subsets Θ, Θ' as well. Let $\alpha_0 = \min\{\alpha(\Theta) \mid \alpha \in \Delta^+_{\tau_{\mathrm{mod}}}\}$ and suppose that a Θ-regular sequence satisfies the hypotheses of Theorem 5.1. If in addition it holds that Θ' is contained in the $\sin^{-1}\left(\frac{2\delta}{s}\right)$-neighborhood of Θ, then Lemma 4.6 implies that the sequence is (Θ', δ)-Morse.

5.2. Morse quasigeodesics have straight and spaced midpoints

In this section, we show that Morse quasigeodesics of sufficiently good quality have straight and spaced midpoint sequences.

Definition 5.4 (Cf. [19, Definition 7.14]). For points p, q in $\mathbb{X}$ we let $\mathrm{mid}(p, q)$ denote the midpoint of the geodesic segment pq. A sequence $(p_n)_{n=t_0}^{n=t_{\max}}$ in $\mathbb{X}$ satisfies the $(\alpha_0, \tau_{\mathrm{mod}}, \epsilon, s, k)$-*quadruple condition* if for all $t_1, t_2, t_3, t_4 \in [t_0, t_{\max}] \cap \mathbb{Z}$ with $t_2 - t_1, t_3 - t_2, t_4 - t_3 \geq k$ the triple of midpoints

$$(\mathrm{mid}(p_1, p_2), \mathrm{mid}(p_2, p_3), \mathrm{mid}(p_3, p_4))$$

is $(\alpha_0, \tau_{\mathrm{mod}}, \epsilon)$-straight and s-spaced. (Here $p(t_i) = p_i$.)

Our next theorem says that sufficiently spaced points on Morse quasigeodesics have straight and spaced midpoint sequences. In an effort to make Theorem 5.5 readable, we have given up some control over the required spacing. For example, we use only one auxiliary parameter α_{aux} to control the regularity as well as the crude estimate $\sin^{-1}(x) \leq \frac{\pi}{2}x$ for $0 \leq x \leq 1$ (this follows from the fact that $\sin^{-1}$ is convex). The following result is a quantified version of [19, Proposition 7.16].

Theorem 5.5. *Assume that k is large enough in terms of $\alpha_{\mathrm{new}} < \alpha_0, D, \epsilon, c_1, c_2$ and s. To make this precise, we use auxiliary constants $l, \delta, \alpha_{\mathrm{aux}}$ and make the following assumptions.*

(1) *Let k be large enough in terms of the quasigeodesic parameters so that if $|N| \geq k$ then $d(x_n, x_{n+N}) \geq 2l$. Precisely, let $k \geq c_1(2l + c_2)$. Our requirements on k will manifest as requirements on l.*

(2)

$$1 \leq 6\sinh(\alpha_{\mathrm{aux}}(2l - 2D))^2, \quad \frac{1}{\alpha_{\mathrm{aux}}\zeta_0}\frac{D}{l} \leq \frac{\zeta_0^2}{\kappa_0^2} \quad \text{and} \quad 5De^{2\kappa_0 D - \alpha_{\mathrm{aux}} l} \leq \delta$$

so that midpoints are δ-close to diamonds by Lemma 4.13.

(3) *We assume that $\frac{2\alpha_{\mathrm{aux}}}{\kappa_0}\big(l - \delta - D\big) \geq s$ to ensure that the midpoints are appropriately spaced.*

(4) *We use an auxiliary parameter α_{aux} such that $\alpha_{\mathrm{new}} < \alpha_{\mathrm{aux}} < \alpha_0$,*

$$\frac{\alpha_0\delta + 3\alpha_0 D + 2\kappa_0 D}{l - \delta - 2D} \leq \alpha_0 - \alpha_{\mathrm{aux}}$$

and

$$\frac{2\kappa_0\delta(\alpha_{\mathrm{aux}} + \kappa_0)}{2\alpha_{\mathrm{aux}}(l - \delta - D) - 2\kappa_0\delta} \leq \alpha_{\mathrm{aux}} - \alpha_{\mathrm{new}},$$

so that certain perturbations of regular segments are regular by Lemma 4.5.

(5) *We assume that*

$$\frac{1}{\alpha_{\mathrm{aux}}\zeta_0}\frac{D}{l} + \frac{1}{\alpha_{\mathrm{new}}\zeta_0}\frac{\kappa_0\delta}{2\alpha_{\mathrm{aux}}(l - \delta - D) - \delta\kappa_0} + \frac{1}{2\alpha_{\mathrm{aux}}\zeta_0}\frac{\delta}{l - D} + \frac{1}{2\alpha_{\mathrm{new}}\zeta_0}\frac{\delta}{l - \delta} + 2\kappa_0\delta \leq \frac{\epsilon}{\pi}$$

to ensure that the midpoint sequence is straight.

Every $(\alpha_0, \tau_{\mathrm{mod}}, D)$-Morse (c_1, c_2)-lower-quasigeodesic satisfies the $(\alpha_{\mathrm{new}}, \tau_{\mathrm{mod}}, \epsilon, s, k')$-quadruple condition for every $k' \geq k$.

Note that in assumption (5), we have in particular assumed $2\pi\kappa_0\delta < \epsilon$, so the δ which appears in the proof is quite small. Our proof follows [19, Proposition 7.16] closely.

Proof. Let $(q_n)_{n=t_0}^{n=t_{\max}}$ be an $(\alpha_0, \tau_{\mathrm{mod}}, D)$-Morse quasigeodesic and let $t_1, t_2, t_3, t_4 \in [t_0, t_{\max}] \cap \mathbb{Z}$ such that $t_2 - t_1, t_3 - t_2, t_4 - t_3 \geq k$. We abbreviate $p_i := q_{t_i}$ and $m_i := \mathrm{mid}(p_i, p_{i+1})$. We have $d(p_i, p_{i+1}) \geq 2l$, $d(m_i, p_i) \geq l$ and $d(m_i, p_{i+1}) \geq l$.

To show that the midpoint sequence is $(\alpha_{\mathrm{new}}, \tau_{\mathrm{mod}}, \epsilon)$-straight, it suffices to show that the segment m_2m_1 is $(\alpha_{\mathrm{new}}, \tau_{\mathrm{mod}})$-regular and that $\angle^{\zeta}_{m_2}(p_2, m_1) \leq \epsilon/2$ under our assumptions on k.

The Morse property implies the existence of a diamond $\diamondsuit_{\alpha_0}(x_1, x_3)$ such that $d(x_1, p_1), d(x_3, p_3) \leq D$ and p_2 is in the D-neighborhood of $\diamondsuit_{\alpha_0}(x_1, x_3)$. The diamond spans a unique parallel set $P = P(\tau_-, \tau_+)$. We denote by $\overline{p}_i$ and $\overline{m}_i$ the projections of p_i and m_i to P.

We begin by observing that m_1 is δ-close to P as determined by the midpoint projection estimate Lemma 4.13: We have $d(p_1, x_1) \leq D$, $d(p_2, V(x_1, \mathrm{ost}(\tau(x_1x_3)))) \leq d(p_2, \diamondsuit_{\alpha_0}(x_1, x_3)) \leq D$ and p_1p_2 is $(\alpha_{\mathrm{aux}}, \tau_{\mathrm{mod}})$-regular with $d(p_1, p_2) \geq 2l$ and l large enough by assumptions (2) and (4):

$$d(m_1, P) \leq 5De^{2\kappa_0 D - \alpha_{\mathrm{aux}} l} \leq \delta.$$

Next we look at the directions of the segments $\overline{m}_2\overline{m}_1$ and $\overline{m}_2\overline{p}_2$ and show that they have the same τ-direction. Let x_2 be a point in $\diamondsuit_{\alpha_0}(x_1, x_3)$ within D of p_2. We have

$$\begin{aligned} d(\overline{p}_2, V(\overline{p}_1, \mathrm{st}(\tau_+), \alpha_0)) &\leq d(p_2, V(\overline{p}_1, \mathrm{st}(\tau_+), \alpha_0)) \\ &\leq d(p_2, x_2) + d(x_2, V(\overline{p}_1, \mathrm{st}(\tau_+), \alpha_0)) \leq 2D \end{aligned}$$

since projecting to a closed convex subset is distance-non-increasing. If c_1 is the geodesic from p_1 through p_2, the function $t \mapsto d(c_1(t), V(\overline{p}_1, \mathrm{st}(\tau_+), \alpha_0))$ is convex, which implies $\overline{m}_1$ is $2D$-close to $V(\overline{p}_1, \mathrm{st}(\tau_+), \alpha_0)$. We have $d(\overline{m}_1, \overline{p}_1) \geq l - \delta - D$, so by using the point in $V(\overline{p}_1, \mathrm{st}(\tau_+), \alpha_0)$ within $2D$ of $\overline{m}_1$ and Lemma 4.5 in the presence of assumption (4), we obtain that $\overline{m}_1 \in V(\overline{p}_1, \mathrm{st}(\tau_+), \alpha_{\mathrm{aux}})$. Similar arguments show that $\overline{m}_1 \in V(\overline{p}_2, \mathrm{st}(\tau_-), \alpha_{\mathrm{aux}})$, or equivalently (by using the geodesic symmetry at $\mathrm{mid}(\overline{p}_1, \overline{p}_2)$) that $\overline{p}_2 \in V(\overline{m}_1, \mathrm{st}(\tau_+), \alpha_{\mathrm{aux}})$. By the nestedness of Weyl cones, $\overline{p}_1 \in V(\overline{p}_2, \mathrm{st}(\tau_-), \alpha_{\mathrm{aux}})$ and $\overline{p}_2 \in V(\overline{p}_1, \mathrm{st}(\tau_+), \alpha_{\mathrm{aux}})$. Similarly, $\overline{m}_2 \in V(\overline{p}_2, \mathrm{st}(\tau_+), \alpha_{\mathrm{aux}})$ and $\overline{p}_2 \in V(\overline{m}_2, \mathrm{st}(\tau_-), \alpha_{\mathrm{aux}})$ (see Figure 13). The convexity of Weyl cones implies that also $\overline{m}_1 \in V(\overline{m}_2, \mathrm{st}(\tau_-), \alpha_{\mathrm{aux}})$. In particular, $\angle^{\zeta}_{\overline{m}_2}(\overline{p}_2, \overline{m}_1) = 0$.

It is convenient to show that the midpoint sequence is appropriately spaced at this point in the proof, so that we can use the resulting estimate to control the regularity parameters α_{aux} and α_{new} and the straightness parameter ϵ. The inclusions $\overline{m}_1 \in V(\overline{p}_2, \mathrm{st}(\tau_-), \alpha_{\mathrm{aux}})$ and $\overline{m}_2 \in V(\overline{p}_2, \mathrm{st}(\tau_+), \alpha_{\mathrm{aux}})$ imply that $d(\overline{m}_1, \overline{m}_2) \geq \frac{\alpha_{\mathrm{aux}}}{\kappa_0}(d(\overline{m}_1, \overline{p}_2) + d(\overline{p}_2, \overline{m}_1))$.

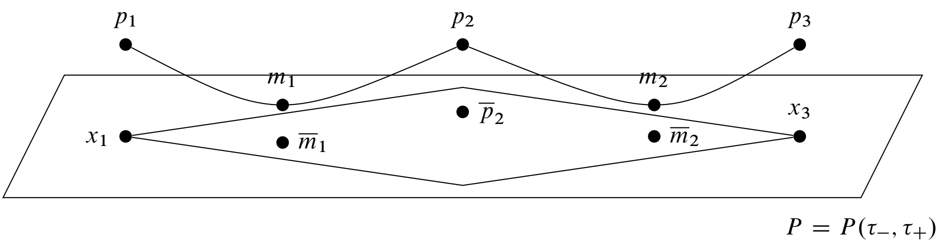


Figure 13. The projections satisfy $\overline{p}_2 \in V(\overline{m}_1, \mathrm{st}(\tau_+), \alpha_{\mathrm{aux}})$ and $\overline{m}_2 \in V(\overline{p}_2, \mathrm{st}(\tau_+), \alpha_{\mathrm{aux}})$.

Therefore by assumption (3), the midpoint sequence is appropriately spaced:

$$d(m_1, m_2) \geq d(\overline{m}_1, \overline{m}_2) \geq \frac{\alpha_{\text{aux}}}{\kappa_0}(d(\overline{m}_1, \overline{p}_2) + d(\overline{p}_2, \overline{m}_1)) \geq \frac{2\alpha_{\text{aux}}}{\kappa_0}(l - \delta - D) \geq s.$$

Using the previous estimate, Lemma 4.5 and assumption (4), we see that $m_2 m_1$ and $m_2 \overline{m}_1$ are $(\alpha_{\text{new}}, \tau_{\text{mod}})$-regular and $m_2 \overline{p}_2$ is $(\alpha_{\text{aux}}, \tau_{\text{mod}})$-regular.

We may now demonstrate the bound $\angle^{\zeta}_{m_2}(p_2, m_1) \leq \epsilon/2$. We have

$$\begin{aligned}
\angle^{\zeta}_{m_2}(p_2, m_1) &= |\angle^{\zeta}_{m_2}(p_2, m_1) - \angle^{\zeta}_{m_2}(\overline{p}_2, \overline{m}_1)| \\
&\leq |\angle^{\zeta}_{m_2}(p_2, m_1) - \angle^{\zeta}_{m_2}(\overline{p}_2, \overline{m}_1)| \\
&\quad + |\angle^{\zeta}_{m_2}(\overline{p}_2, \overline{m}_1) - \angle^{\zeta}_{m_2}\big(\tau(\overline{m}_2\overline{p}_2), \tau(\overline{m}_2\overline{m}_1)\big)| \\
&\quad + |\angle^{\zeta}_{m_2}\big(\tau(\overline{m}_2\overline{p}_2), \tau(\overline{m}_2\overline{m}_1)\big) - \angle^{\zeta}_{\overline{m}_2}(\overline{p}_2, \overline{m}_1)|.
\end{aligned}$$

By the triangle inequality for quadruples (in the metric space $(\text{Flag}(\tau_{\text{mod}}), \angle^{\zeta}_{m_2})$), we have

$$\begin{aligned}
\left|\angle^{\zeta}_{m_2}(p_2, m_1) - \angle^{\zeta}_{m_2}(\overline{p}_2, \overline{m}_1)\right| &\leq \angle^{\zeta}_{m_2}(p_2, \overline{p}_2) + \angle^{\zeta}_{m_2}(m_1, \overline{m}_1) \\
&= 2\sin^{-1}\left(\frac{1}{2}d_{\mathfrak{p}}(Z_1, Z_2)\right) + 2\sin^{-1}\left(\frac{1}{2}d_{\mathfrak{p}}(Z_3, Z_4)\right),
\end{aligned}$$

where Z_1, Z_2, Z_3, Z_4 are the unit vectors at m_2 in the directions $\zeta(m_2 p_2)$, $\zeta(m_2 \overline{p}_2)$, $\zeta(m_2 m_1)$, $\zeta(m_2 \overline{m}_1)$, respectively. Let X_1, X_2, X_3, X_4 be the unit vectors at m_2 which in the directions $p_2, \overline{p}_2, m_1, \overline{m}_1$, respectively. Then by Lemma 4.8 and the angle comparison to Euclidean space Lemma 4.7, we have

$$d(Z_1, Z_2) \leq \frac{1}{\alpha_{\text{aux}}\zeta_0} d(X_1, X_2) = \frac{2}{\alpha_{\text{aux}}\zeta_0} \sin\frac{1}{2}\angle_{m_2}(p_2, \overline{p}_2) \leq \frac{1}{\alpha_{\text{aux}}\zeta_0}\frac{D}{l}.$$

Similarly,

$$\begin{aligned}
d(Z_3, Z_4) \leq \frac{1}{\alpha_{\text{new}}\zeta_0} d(X_3, X_4) &= \frac{2}{\alpha_{\text{new}}\zeta_0} \sin\frac{1}{2}\angle_{m_2}(m_1, \overline{m}_1) \\
&\leq \frac{1}{\alpha_{\text{new}}\zeta_0}\frac{\kappa_0\delta}{2\alpha_{\text{aux}}(l - \delta - D) - \delta\kappa_0}.
\end{aligned}$$

Again by the triangle inequality on $(\text{Flag}(\tau_{\text{mod}}), \angle^{\zeta}_{m_2})$,

$$\begin{aligned}
&\left|\angle^{\zeta}_{m_2}(\overline{p}_2, \overline{m}_1) - \angle^{\zeta}_{m_2}(\tau(\overline{m}_2\overline{p}_2), \tau(\overline{m}_2\overline{m}_1))\right| \\
&\quad \leq \angle^{\zeta}_{m_2}(\overline{p}_2, \tau(\overline{m}_2\overline{p}_2)) + \angle^{\zeta}_{m_2}(\overline{m}_1, \tau(\overline{m}_2\overline{m}_1)).
\end{aligned}$$

Asymptotic geodesic rays are bounded by the distance of their tips, so if we let c_2 be the geodesic ray from m_2 to $\overline{m}_2\overline{p}_2(+\infty)$ we may use Lemma 4.8 to obtain

$$\sin\frac{1}{2}\angle^{\zeta}_{m_2}(\overline{p}_2, \tau(\overline{m}_2\overline{p}_2)) \leq \frac{1}{2\alpha_{\text{aux}}\zeta_0}\frac{d(\overline{p}_2, \operatorname{im} c_2)}{d(m_2, \overline{p}_2)} \leq \frac{1}{2\alpha_{\text{aux}}\zeta_0}\frac{\delta}{l - D}.$$

Similarly, by considering the geodesic ray c_3 from $\overline{m}_2$ through $\overline{m}_1$,

$$\sin \frac{1}{2}\angle^{\zeta}_{m_2}(\overline{m}_1, \tau(\overline{m}_2\overline{m}_1)) \le \frac{1}{2\alpha_{\mathrm{new}}\zeta_0} \frac{d(\overline{m}_1, \operatorname{im} c_3)}{d(m_2, \overline{m}_1)} \le \frac{1}{2\alpha_{\mathrm{new}}\zeta_0} \frac{\delta}{l-\delta}.$$

Write $\tau = \tau(\overline{m}_2\overline{p}_2)$ and $\tau' = \tau(\overline{m}_2\overline{m}_1)$. By the distance-to-angle estimate Corollary 4.15,

$$\begin{aligned} &\left|\angle^{\zeta}_{m_2}(\tau(\overline{m}_2\overline{p}_2), \tau(\overline{m}_2\overline{m}_1)) - \angle^{\zeta}_{\overline{m}_2}(\overline{p}_2, \overline{m}_1)\right| \\ &\quad = \left|\angle^{\zeta}_{m_2}(\tau, \tau') - \angle^{\zeta}_{\overline{m}_2}(\tau, \tau')\right| \le 4\sin^{-1}\left(\frac{\kappa_0}{2} d(\overline{m}_2, m_2)\right) \le 4\sin^{-1}\left(\frac{\kappa_0\delta}{2}\right). \end{aligned}$$

Combining these estimates with the fact that $\sin^{-1}(x) \le \frac{\pi}{2}x$ for $0 \le x \le 1$ yields

$$\begin{aligned} \angle^{\zeta}_{m_2}(p_2, m_1) \le \frac{\pi}{2}\Big[&\frac{1}{\alpha_{\mathrm{aux}}\zeta_0}\frac{D}{l} + \frac{1}{\alpha_{\mathrm{new}}\zeta_0}\frac{\kappa_0\delta}{2\alpha_{\mathrm{aux}}(l-\delta-D) - \delta\kappa_0} \\ &+ \frac{1}{2\alpha_{\mathrm{aux}}\zeta_0}\frac{\delta}{l-D} + \frac{1}{2\alpha_{\mathrm{new}}\zeta_0}\frac{\delta}{l-\delta} + 2\kappa_0\delta\Big] \le \frac{\epsilon}{2} \end{aligned}$$

by assumption (5). For similar reasons $\angle^{\zeta}_{m_2}(p_3, m_3) \le \frac{\epsilon}{2}$, so $\angle^{\zeta}_{m_2}(m_1, m_3) \ge \pi - \epsilon$ as desired. We have already shown that $m_2 m_1$ is $(\alpha_{\mathrm{new}}, \tau_{\mathrm{mod}})$-regular and s-spaced. For similar reasons the same holds for $m_2 m_3$. This concludes the proof. ∎

Remark 5.6. To provide suitable auxiliary parameters to apply Theorem 5.5, we may first choose any $\delta < \frac{\epsilon}{2\pi\kappa_0}$ and any $\alpha_{\mathrm{new}} < \alpha_{\mathrm{aux}} < \alpha_0$. Then we may choose l large enough to satisfy assumptions (2) through (5), which provides a suitable k via assumption (1). When we apply Theorem 5.5 in Section 6, we set $\delta = \frac{\epsilon}{20\pi\kappa_0}$ and $\alpha_{\mathrm{aux}} = 0.8\alpha_0 + 0.2\alpha_{\mathrm{new}}$.

Remark 5.7. Theorem 5.5 can be modified to deal with arbitrary τ_{mod}-Weyl convex subsets as well. Let $\alpha_0 = \min\{\alpha(\Theta) \mid \alpha \in \Delta^+_{\tau_{\mathrm{mod}}}\}$ and suppose that (x_n) is a (Θ, D)-Morse (c_1, c_2)-lower-quasigeodesic. Let Θ_{aux} and Θ_{new} be τ_{mod}-Weyl convex subsets with $\alpha_{\mathrm{aux}} = \min\{\alpha(\Theta_{\mathrm{aux}}) \mid \alpha \in \Delta^+_{\tau_{\mathrm{mod}}}\}$ and $\alpha_{\mathrm{new}} = \min\{\alpha(\Theta_{\mathrm{new}}) \mid \alpha \in \Delta^+_{\tau_{\mathrm{mod}}}\}$ such that Θ_{aux} is contained in the $\sin^{-1}\left(\frac{2D}{l-\delta-D}\right)$-neighborhood of Θ and Θ_{new} is contained in the $\sin^{-1}\left(\frac{\kappa_0\delta}{\alpha_{\mathrm{aux}}(l-\delta-D)}\right)$-neighborhood of Θ. If in addition (x_n) satisfies the hypotheses of Theorem 5.5, then Lemma 4.6 implies that it satisfies the $(\Theta_{\mathrm{new}}, \epsilon, s, k')$-quadruple condition.

5.3. Local-to-global principle for Morse quasigeodesics

An *L-local $(\alpha_0, \tau_{\mathrm{mod}}, D)$-Morse (c_1, c_2, c_3, c_4)-quasigeodesic* is a sequence $(x_n)_{n=t_0}^{n=t_{\max}}$ in $\mathbb{X}$ such that for $t_0 \le t_1 \le t_2 \le t_{\max}$ with $t_2 - t_1 \le L$, the subsequence $(x_n)_{n=t_1}^{n=t_2}$ is an $(\alpha_0, \tau_{\mathrm{mod}}, D)$-Morse (c_1, c_2, c_3, c_4)-quasigeodesic.

We now come to the main result of the paper. The following result is a quantified local-to-global principle for Morse quasigeodesics. Theorem 5.8 says that for any fixed quality of Morse quasigeodesic, there exists a large enough scale so that a local Morse quasigeodesic of that scale and quality is a global Morse quasigeodesic. It is a quantified version of [19, Theorem 7.18], stated as Theorem 1.1 in the introduction. We will apply Theorems 5.1 and 5.5. While these theorems have cumbersome statements, finding auxiliary parameters which satisfy the required inequalities is easy, as we discussed in Remarks 5.2 and 5.6, and as we demonstrate in the next section.

Theorem 5.8. *For any $\alpha_{\mathrm{new}} < \alpha_0, D, c_1, c_2, c_3, c_4$, there exists a scale L so that every L-local $(\alpha_0, \tau_{\mathrm{mod}}, D)$-Morse (c_1, c_2, c_3, c_4)-quasigeodesic in $\mathbb{X}$ is an $(\alpha_{\mathrm{new}}, \tau_{\mathrm{mod}}, D')$-Morse (c_1', c_2', c_3', c_4')-quasigeodesic. Precisely, $L = 3k$ is large enough if auxiliary parameters $\alpha_{\mathrm{aux}}, k, \delta, s, \epsilon$ satisfy:*

1. *ϵ is small enough and s is large enough to satisfy the conditions of Theorem 5.1 for $\alpha_{\mathrm{new}} < \alpha_{\mathrm{aux}}, \delta$,*
2. *k is large enough in terms of $\alpha_{\mathrm{aux}} < \alpha_0, D, \epsilon, c_1, c_2$ and s to satisfy the conditions of Theorem 5.5,*

and the sequence has global Morse parameters

1. *$D' = c_3 k + \frac{3}{2} c_4 + \delta$,*
2. *$(c_1')^{-1} = 2\alpha_{\mathrm{new}} \zeta_0 c_0 (s - 2\delta) k^{-1}$,*
3. *$c_2' = 2\alpha_{\mathrm{new}} \zeta_0 c_0 (s - 2\delta) + 2\delta + 2c_3 k + 3c_4$,*
4. *$c_3' = c_3 + \frac{c_4}{L}$,*
5. *$c_4' = c_4$.*

Proof. Let $(x_n)_{n=-\infty}^{n=+\infty}$ be an L-local $(\alpha_0, \tau_{\mathrm{mod}}, D)$-Morse (c_1, c_2, c_3, c_4)-quasigeodesic. Theorem 5.5 and assumption (2) imply that each subsequence $(x_n)_{n=t_0}^{n=t_0+3k}$ satisfies the $(\alpha_{\mathrm{aux}}, \tau_{\mathrm{mod}}, \epsilon, s, k)$-quadruple condition. In particular, the coarse midpoint sequence $m_n = \mathrm{mid}(x_{nk}, x_{nk+k})$ is $(\alpha_0, \tau_{\mathrm{mod}}, \epsilon)$-straight and s-spaced. By Theorem 5.1 and assumption (1), the midpoint sequence (m_n) is an $(\alpha_{\mathrm{new}}, \tau_{\mathrm{mod}}, \delta)$-Morse $((2\alpha_{\mathrm{new}} \zeta_0 c_0 (s - 2\delta))^{-1}, 2\delta)$-lower-quasigeodesic. We now use the midpoint sequence as a coarse approximation of the original sequence to show that (x_n) is a global Morse quasigeodesic.

The subsequences $x_{nk}, x_{nk+1}, \ldots, x_{nk+k-1}, x_{nk+k}$ are (c_3, c_4)-upper-quasigeodesics (because $L \geq k$), so they lie in uniform neighborhoods of each m_n: If $|t - nk| \leq \frac{k}{2}$, then

$$
\begin{aligned}
d(m_n, x_t) &\leq d(m_n, x_{nk}) + d(x_{nk}, x_t) \\
&\leq \frac{d(x_{nk}, x_{nk+k})}{2} + d(x_{nk}, x_t) \\
&\leq \frac{c_3}{2} k + \frac{c_4}{2} + c_3 \frac{k}{2} + c_4 = c_3 k + \frac{3}{2} c_4.
\end{aligned}
$$

In particular, (x_n) is $(\alpha_{\mathrm{new}}, \tau_{\mathrm{mod}}, D')$-Morse for $D' = c_3 k + \frac{3}{2} c_4 + \delta$. The midpoint sequence is coarsely spaced:

$$
d(m_n, m_{n+N}) \geq 2\alpha_{\mathrm{new}} \zeta_0 c_0 (s - 2\delta)|N| - 2\delta,
$$

so the original sequence is also coarsely spaced:

$$\begin{aligned} d(x_t, x_{t'}) &\geq d(m_n, m_{n'}) - d(m_n, x_t) - d(m_{n'}, x_{t'}) \\ &\geq 2\alpha_{\text{new}}\zeta_0 c_0(s - 2\delta)|n - n'| - 2\delta - 2c_3 k - 3c_4 \\ &\geq 2\alpha_{\text{new}}\zeta_0 c_0(s - 2\delta)k^{-1}|t - t'| - 2\alpha_{\text{new}}\zeta_0 c_0(s - 2\delta) - 2\delta - 2c_3 k - 3c_4. \end{aligned}$$

Finally, if a sequence is (c_3, c_4)-coarsely Lipschitz on intervals of length L, it then satisfies $d(x_n, x_{n+N}) \leq |N|(c_3 + \frac{c_4}{L}) + c_4$ and is $(c_3 + \frac{c_4}{L}, c_4)$-coarsely Lipschitz. ■

6. Applications of the local-to-global principle

In this section, we give two applications of the main result (Theorem 5.8). We describe two explicit neighborhoods of Anosov representations in $\mathrm{SL}(3, \mathbb{R})$, one for free groups and another for closed surface groups. Each of them is constructed by perturbing a group acting cocompactly on a convex subset of a totally geodesic hyperbolic plane in the associated symmetric space.

We will need some further estimates in order to quantify these neighborhoods. First we recall a standard proof of the Milnor–Schwarz lemma so that we may use the explicit quasi-isometry constants it produces. We then give a version of the classical Morse lemma that will be used in Section 6.3. In Section 6.1.3, we use elementary linear algebra to control the perturbations of long words in a linear group that results from perturbing the generators. We also relate the Frobenius norm on $d \times d$ matrices to the distance in the symmetric space associated with $\mathrm{SL}(d, \mathbb{R})$. In the final two sections, we apply the local-to-global principle, Theorem 5.8, to describe explicit neighborhoods of Anosov representations.

As one might expect, straightforward applications of Theorem 5.8 as we have done here will yield only very small perturbations. This is partially explained by the following geometric difficulty. The Morse condition implies that the image of each geodesic in the Cayley graph fellow-travels a unique parallel set. After perturbing the representation, one expects the image of the geodesic to fellow-travel a new parallel set. For geodesics through the identity, our techniques merely bound the distance from the perturbed geodesic to its previous parallel set, so for it to fellow-travel for a long time, the perturbation has to be extremely small. If we could identify the new parallel set it fellow-travels and bound the distance to that parallel set, we expect that the perturbation bounds would improve significantly.

6.1. Preliminary estimates

6.1.1. The Milnor–Schwarz lemma. In this subsection, we state and prove a standard result in geometric group theory called the Milnor–Schwarz lemma. It is a source of concrete quasi-isometry parameters for nice enough actions of finitely generated groups, such as those we consider in Sections 6.2 and 6.3. The proof given here is taken directly from Sisto's lecture notes [30].

Lemma 6.1 (Milnor–Schwarz lemma). *Let G be a group acting properly discontinuously, cocompactly and by isometries on a proper geodesic space X. Choose any $p \in X$. Then the group G has a finite generating set S so that the orbit map at p is a quasi-isometry from G with the word metric induced by S. In fact,*

$$wl(g) \leq d(p, gp) + 1, \quad \text{and} \quad d(p, gp) \leq \max_{s \in S}\{d(p, sp)\} wl(g).$$

Proof. Since the action is cocompact, there exists a constant R so that the G-translates of $B_R(p)$ cover X. Let $S := \{g \in G \mid d(p, gp) \leq 2R + 1\}$. Since X is proper, the closed ball of radius $R + \frac{1}{2}$ centered at p is compact, and since the action is properly discontinuous, $S = \{g \in G \mid B_{R+\frac{1}{2}}(p) \cap B_{R+\frac{1}{2}}(gp)\}$ is finite. Now let $g \in G$. Choose a minimal geodesic from p to gp, and subdivide it with points p_i so that $p = p_0, p_1, p_2, \ldots, p_{n-1}, p_n = gp$ occur monotonically and for $i = 0, 1, 2, \ldots, n-2$, we have $d(p_i, p_{i+1}) = 1$ and $d(p_{n-1}, p_n) \leq 1$. For each $1 \leq i \leq n-1$ choose $g_i \in G$ so that $d(g_i p, p_i) \leq R$ and set $g_0 = \mathrm{id}$ and $g_n = g$. Then for all $0 \leq i \leq n-1$, we have

$$d(g_i p, g_{i+1} p) \leq d(g_i p, p_i) + d(p_i, p_{i+1}) + d(p_{i+1}, g_{i+1} p) \leq 2R + 1,$$

which implies that there exists $s_{i+1} \in S$ so that $g_{i+1} = g_i s_{i+1}$. For all $1 \leq i \leq n$, it follows that $g_i = s_1 s_2 s_3 \cdots s_i$. Therefore, g can be written as a product of n elements of S, with $n - 1 \leq d(p, gp)$. It follows that S is a finite generating set for G and the word length of g with respect to S is bounded above by $d(p, gp) + 1$.

We have shown that S is a finite generating set for G. Write $g = g_1 \cdots g_n$ with $g_i \in S$. Then

$$\begin{aligned} d(p, g_1 g_2 g_3 \cdots g_n p) &\leq d(p, g_1 \cdots g_{n-1} p) + d(g_1 \cdots g_{n-1} p, g_1 \cdots g_{n-1} g_n p) \\ &= d(p, g_1 \cdots g_{n-1} p) + d(p, g_n p) \\ &\leq d(p, g_1 p) + \cdots + d(p, g_n p) \\ &\leq \max_{s \in S}\{d(p, sp)\} n, \end{aligned}$$

so the orbit map at p is $\max_{s \in S}\{d(p, sp)\}$-Lipschitz with respect to the generating set S. Note that by the definition of S, $\max_{s \in S}\{d(p, sp)\} \leq 2R + 1$. ∎

The previous lemma provides quasi-isometry constants in terms of only the constant R so that the image of an R-ball covers the quotient. In return, we give up control over the generating set. In particular, when we apply Lemma 6.1 to an action of a closed surface group on the hyperbolic plane in Section 6.3, we will give quasi-isometry parameters with a nonstandard generating set for the Cayley graph. We will need to control the Frobenius norm of the matrices in our generating set by using Lemma 6.7.

6.1.2. The classical Morse lemma. In Section 6.3, we will use the following version of the classical Morse lemma to provide Morse quasi-isometry parameters for the orbit map of a surface group acting on a copy of the hyperbolic plane. The following proof is adapted from Bridson–Haefliger [5].

Theorem 6.2 (Classical Morse lemma, cf. [5, Theorem III.H.1.7]). *Let D_0 be an upper bound for*

$$\{D \mid D - 1 \leq \delta|\log_2(2D + 2M^2 l + 6DMl + aM)|\}$$

and set $R = D_0 + lMD_0 + lM^2 + \frac{a}{2}$. Then:

If $(y_i)_{i=0}^{i=N}$ is a sequence in a δ-hyperbolic geodesic space $\mathbb{Y}$ with

$$d(y_i, y_j) \leq M|j - i| \quad \text{and} \quad |j - i| \leq l d(y_i, y_j) + a$$

then for all $0 \leq n \leq N$, the distance from y_n to a geodesic segment from y_0 to y_N is bounded above by R.

Proof. Let $c\colon [0, N] \to \mathbb{Y}$ be the piecewise geodesic curve with $c(i) = y_i$. Let D be minimal so that the closed D-neighborhood of $\operatorname{im} c$ covers the geodesic from $p = y_0$ to $q = y_N$. Choose a point x_0 on pq realizing D, and choose y, z on pq at distance $2D$ from x_0 so that y, x_0, z occurs in order (if x_0 is too close to p, use p for y, and likewise for z). Choose y' on $\operatorname{im} c$ within D of y, and choose z' similarly. Choose i, j so that y' is on $y_i y_{i+1}$ and z' is on $y_{j-1} y_j$. If $c(t) = y'$ and $c(t') = z'$, then the length of c restricted to the $[t, t']$ is at most

$$\operatorname{length}(c|_{[t,t']}) \leq \operatorname{length}(c|_{[i,j]}) \leq M|j - i| \leq M[l d(y_i, y_j) + a].$$

Also,

$$d(y_i, y_j) \leq d(y_i, y') + d(y', y) + d(y, z) + d(z, z') + d(z', y_j) \leq 2M + 6D,$$

and it follows that the curve c' formed by following a geodesic segment from y to y' then along c to z' then along a geodesic segment to z has length at most $2D + M[l(2M + 6D) + a]$. Proposition III.H.1.6 in [5] bounds D in terms of the length of c' and δ. In particular,

$$D - 1 \leq \delta|\log_2(2D + 2M^2 l + 6DMl + aM)|,$$

which implies an upper bound D_0 on D.

Now suppose that $(y_n)_{n=a'}^{n=b'}$ is a maximal (consecutive) subsequence outside the D_0-neighborhood of pq. There exist s, s' such that $0 \leq s \leq a'$ and $b' \leq s' \leq N$ within D_0 of the same point on pq, so $d(c(s), c(s')) \leq 2D_0$. As before, by choosing m, n so that $c(s)$ lies on $y_m y_{m+1}$ and $c(s')$ lies on $y_n y_{n+1}$, we have that

$$\operatorname{length}(c|_{[s,s']}) \leq \operatorname{length}(c|_{[m,n]}) \leq M|m - n| \leq M(l d(y_m, y_n) + a)$$

and

$$d(y_m, y_n) \leq d(y_m, c(s)) + d(c(s), c(s')) + d(c(s'), y_n) \leq 2M + 2D_0,$$

so we obtain

$$\text{length}\, c|_{[s,s']} \leq M[l(2D_0 + 2M) + a].$$

It follows that $R = D_0 + M[l(D_0 + M) + \frac{a}{2}]$ is an upper bound for the distance from any y_n to pq. ∎

6.1.3. Matrix estimates. In this subsection, we establish a few elementary estimates related to the symmetric space associated with $\mathrm{SL}(d, \mathbb{R})$. We will control perturbations of long words in a generating set in terms of the Frobenius norm of the generators. As noted above, we use a nonstandard generating set for the closed surface group, so we also prepare to control the Frobenius norm of the generators in that case. In Sections 6.2 and 6.3, we combine these estimates with the local-to-global principle Theorem 5.8 to guarantee that the Morse subgroups under consideration remain Morse after certain explicit perturbations.

In the rest of the paper, we identify the symmetric space associated with $\mathrm{SL}(d, \mathbb{R})$ with the space of real, symmetric, positive-definite matrices of determinant 1. We remind the reader that we take the Riemannian metric to be induced by the Killing form, so at the identity matrix, the Riemannian metric is $2d$ times the Frobenius inner product $\langle X, Y\rangle_{\mathrm{Fr}} = \mathrm{trace}(X^T Y)$.

Lemma 6.3. *Let $|\cdot|$ be any submultiplicative norm on $d \times d$ matrices. Let $w = g_1 g_2 \cdots g_{k-1} g_k$ be a product of k matrices, and let $w' = (g_1 + \epsilon_1)(g_2 + \epsilon_2) \cdots (g_{k-1} + \epsilon_{k-1})(g_k + \epsilon_k)$ be a product of perturbed matrices. Suppose that for all $1 \leq i \leq k$, $|g_i| \leq A$ and $|\epsilon_i| \leq \epsilon$. If $k \geq 3$ and $\frac{k-1}{2}\frac{\epsilon}{A} \leq 1$, then $|w' - w| \leq 2kA^{k-1}\epsilon$.*

Proof. We have

$$\begin{aligned}
|w' - w| &= \left| \prod_{i=1}^{k} (g_i + \epsilon_i) - \prod_{i=1}^{k} g_i \right| \\
&= \left| \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_j \leq k \\ j=1}}^{j=k} g_1 g_2 \cdots g_{i_1 - 1} \epsilon_{i_1} g_{i_1+1} \cdots g_{i_j - 1} \epsilon_{i_j} g_{i_j + 1} \cdots g_k \right| \\
&\leq kA^{k-1}\epsilon + \binom{k}{2} A^{k-2}\epsilon^2 + \cdots + \binom{k}{j} A^{k-j}\epsilon^j + \cdots + \epsilon^k \\
&= A^k \left[\left(1 + \frac{\epsilon}{A}\right)^k - 1 \right] \\
&\leq 2kA^{k-1}\epsilon,
\end{aligned}$$

where the last line follows from the Taylor approximation $(1 + \frac{\epsilon}{A})^k - 1 \leq k\frac{\epsilon}{A} + \frac{k(k-1)}{2}\left(\frac{\epsilon}{A}\right)^2$, valid when $\frac{\epsilon}{A} \leq 1$. ∎

We next relate the Riemannian distance in $\mathbb{X}$ to the B_p-norm on the space of matrices. Recall that when p is the identity matrix, B_p is $2d$ times the Frobenius inner product. We let B_p be defined on all of $\mathfrak{gl}(d,\mathbb{R})$ as $2d$ times the Frobenius inner product.

Lemma 6.4. *Let $g \in \mathrm{SL}(d,\mathbb{R})$ and $p \in \mathbb{X}$ be the identity matrix. Then*

$$d_{\mathbb{X}}(gp, p) \leq \sqrt{d}(d-1)|g-1|_{B_p}.$$

Proof. $K = \mathrm{SO}(d)$ acts on $(\mathfrak{gl}(d,\mathbb{R}), B_p)$ by isometries on the left and the right, so $|g-1|_{B_p} = |e^A - 1|_{B_p}$ where

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}$$

is the Cartan projection of g. That is, A is the unique diagonal matrix with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ and $\lambda_1 + \cdots + \lambda_d = 0$ such that $g = ke^Ak'$ for some $k, k' \in \mathrm{SO}(d)$. We have $|A|_{B_p} = d(gp, p)$. Since $-\lambda_d \leq (d-1)\lambda_1$ and $\lambda_1^2 \leq (e^{\lambda_1} - 1)^2$,

$$\begin{aligned} d(gp,p)^2 = |A|_{B_p}^2 &= 2d\sum_{i=1}^{d}\lambda_i^2 \\ &\leq 2d^2(d-1)^2\lambda_1^2 \\ &\leq 2d^2(d-1)^2\sum_{i=1}^{d}(e^{\lambda_i}-1)^2 \\ &= d(d-1)^2|e^A - 1|_{B_p}^2 \\ &= d(d-1)^2|g-1|_{B_p}^2. \end{aligned}$$

■

In the following corollary, we control the distance between $\rho'(\gamma)p$ and $\rho(\gamma)p$ in terms of the word length of γ and the distance between $\rho'(\gamma_i)$ and $\rho(\gamma_i)$ for a generating set $\{\gamma_i\}$.

Corollary 6.5. *Let Γ be a group with symmetric generating set $S = \{\gamma_1, \ldots, \gamma_n\}$ and let ρ and ρ' be two representations of Γ into $\mathrm{SL}(d,\mathbb{R})$. Assume that*

(1) *for $i \in \{1,\ldots,n\}$, $|\rho(\gamma_i)|_{\mathrm{Fr}} \leq A$ and $|\rho(\gamma_i) - \rho'(\gamma_i)|_{\mathrm{Fr}} \leq \epsilon$; and*

(2) *$k \geq 3$ and $\frac{k-1}{2}\frac{\epsilon}{A} \leq 1$.*

Then for any $\gamma \in \Gamma$ with $d_S(\gamma,1) \leq k$, it holds that

$$d_{\mathbb{X}}(\rho'(\gamma)p, \rho(\gamma)p) \leq \sqrt{8}d(d-1)kA^{2k-1}\epsilon.$$

Proof. Let $g = \rho(\gamma)$ and $g' = \rho'(\gamma)$ for $d_S(\gamma,1) \leq k$. Since the Frobenius norm is submultiplicative, we have $|g^{-1}|_{\mathrm{Fr}} \leq A^k$ and moreover because of the assumptions, Lemma 6.3

applies and we obtain $|g - g'|_{\mathrm{Fr}} \leq 2kA^{k-1}\epsilon$. We see that

$$\begin{aligned} |g^{-1}g' - 1|_{\mathrm{Fr}} &= |g^{-1}(g' - g)|_{\mathrm{Fr}} \\ &\leq |g^{-1}|_{\mathrm{Fr}}|g' - g|_{\mathrm{Fr}} \\ &\leq A^k|g' - g|_{\mathrm{Fr}} \leq 2kA^{2k-1}\epsilon. \end{aligned}$$

Then by applying Lemma 6.4 to $g^{-1}g'$, we obtain

$$d(g'p, gp) = d(g^{-1}g'p, p) \leq \sqrt{d}(d-1)|g^{-1}g' - 1|_{B_p} \leq \sqrt{8}d(d-1)kA^{2k-1}\epsilon. \quad \blacksquare$$

In the next lemma we give a precise, quantitative version of the following statement: If a representation ρ induces a Morse quasi-isometric embedding, then its perturbation ρ' induces a local Morse quasi-isometric embedding.

Lemma 6.6. *Let $\rho, \rho' \colon \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be representations and let S be a symmetric generating set for Γ. If $d(\rho(\gamma)p, \rho'(\gamma)p) \leq \epsilon$ for all $d_S(\gamma, 1) \leq k$ and if the orbit map of ρ at p is an $(\alpha_0, \tau_{\mathrm{mod}}, D)$-Morse (c_1, c_2, c_3, c_4)-quasi-isometric embedding then the orbit map of ρ' at p is a $2k$-local $(\alpha_0, \tau_{\mathrm{mod}}, D + \epsilon)$-Morse $(c_1, c_2 + \epsilon, c_3, c_4 + \epsilon)$-quasi-isometric embedding.*

Proof. If $d(\rho(\gamma)p, \rho'(\gamma)p) \leq \epsilon$ for all $d_S(\gamma, 1) \leq k$, then for every geodesic $(\gamma_n)_{n=-k}^{n=k}$ in Γ of length $2k$,

$$d(\rho'(\gamma_n)p, \rho'(\gamma_0)p) = d(\rho'(\gamma_0^{-1})\rho'(\gamma_n)p, p)$$

is within ϵ of $d(\rho(\gamma_n)p, \rho(\gamma_0)p)$. Additionally, if $(\rho(\gamma_n)p)$ is within D of $\diamondsuit_{\alpha_0}(q, r)$, then $(\rho'(\gamma_n)p)$ is within $D + \epsilon$ of $\diamondsuit_{\alpha_0}(\rho'(\gamma_0)\rho(\gamma_0^{-1})q, \rho'(\gamma_0)\rho(\gamma_0^{-1})r)$. In particular, if ρ induces an $(\alpha_0, \tau_{\mathrm{mod}}, D)$-Morse (c_1, c_2, c_3, c_4)-quasi-isometric embedding, then ρ' induces a $2k$-local $(\alpha_0, \tau_{\mathrm{mod}}, D + \epsilon)$-Morse $(c_1, c_2 + \epsilon, c_3, c_4 + \epsilon)$-quasi-isometric embedding. $\blacksquare$

When we apply the Milnor–Schwarz lemma we use the generating set $S = \{s \in \Gamma \mid d(p, sp) \leq 2R + 1\}$, and when we apply Corollary 6.5 we need to bound the size of the generating set. The following lemma helps us do just that.

Lemma 6.7. *Let p be the identity matrix in $\mathbb{X}_d$ and let $g \in \mathrm{SL}(d, \mathbb{R})$ such that $d(p, gp) \leq 2R + 1$. Let $|\cdot|_{\mathrm{Fr}}$ denote the Frobenius norm. Then*

$$|g|_{\mathrm{Fr}} \leq \exp\Big(\frac{2R+1}{\sqrt{2d}}\Big).$$

Proof. Combine

$$|g|_{\mathrm{Fr}}^2 = \big|gg^T\big|_{\mathrm{Fr}} = \big|\exp\log gg^T\big|_{\mathrm{Fr}} \leq \exp\big|\log gg^T\big|_{\mathrm{Fr}}$$

and

$$\sqrt{\frac{d}{2}}\left|\log gg^T\right|_{\mathrm{Fr}} = \frac{1}{2}\left|\log gg^T\right|_{B_p} = \left|\log\sqrt{gg^T}\right|_{B_p} = d(p, gp) \leq 2R + 1$$

to obtain

$$|g|_{\mathrm{Fr}} \leq \exp\frac{1}{2}\left|\log gg^T\right|_{\mathrm{Fr}} \leq \exp\left(\frac{2R+1}{\sqrt{2d}}\right). \qquad \blacksquare$$

6.2. An explicit neighborhood of Anosov free groups

In this subsection, we obtain an explicit nonempty neighborhood of Anosov free groups. Let Γ_1 be the subgroup of $\mathrm{SL}(3,\mathbb{R})$ generated by

$$g = \begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \quad h = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix}.$$

As in Section 6.1.3 we identify the associated symmetric space with the space of real, symmetric, positive-definite matrices of determinant 1. Let $p \in \mathbb{X}$ be the identity matrix. Observe that Γ_1 is a subgroup of a reducible copy of $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(3,\mathbb{R})$ preserving a copy of $\mathbb{H}^2 \subset \mathbb{X}$ containing p of curvature $-\frac{1}{3}$ (see Section 3.4). We will directly estimate the Morse quasi-isometry parameters of the orbit map at p on Γ_1.

The points p, gp, hp form an isosceles right triangle:

$$d(p, gp) = \left|\left|\begin{bmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -t \end{bmatrix}\right|\right|_{B_p} = 2\sqrt{3}t = \left|\left|\begin{bmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{bmatrix}\right|\right|_{B_p} = d(p, hp).$$

Write $T = \tanh(t)$. If $\sqrt{2}T > 1$, then Γ_1 acts cocompactly on a closed convex subset C of $\mathbb{H}^2$, with a Dirichlet domain C_p (see Figure 14). The domain C_p is an octagon with geodesic boundary and neighbors $gC_p, g^{-1}C_p, hC_p, h^{-1}C_p$ in C. Since C is convex, the minimum distance between any pair of neighbors is bounded below by the length of an arc in C_p joining non-adjacent edges. This has lower bound

$$c_1^{-1} = \sqrt{3}\min\left\{t, \frac{1}{2}\log\left(\frac{T^2 + \sqrt{2T^2 - 1}}{T^2 - \sqrt{2T^2 - 1}}\right), \frac{1}{2}\log\left(\frac{1 + 2T\sqrt{1 - T^2}}{1 - 2T\sqrt{1 - T^2}}\right)\right\}.$$

We also set $c_3 = 2\sqrt{3}t$. The orbit map is a $(c_1, 0, c_3, 0)$ quasi-isometry. Set $R = \sqrt{3}\tanh^{-1}\left(\sqrt{T^{-2} - 2 + 2T^2}\right)$. Then C is within the R-neighborhood of $\Gamma_1 \cdot p$ and the diameter of C_p is $2R$. The orbit map is R-Morse.

We are now in position to prove Theorem 1.2.

Theorem 1.2. *Let Γ_1 be the subgroup of* $\mathrm{SL}(3,\mathbb{R})$ *generated by*

$$g = \begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}, \quad h = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix},$$

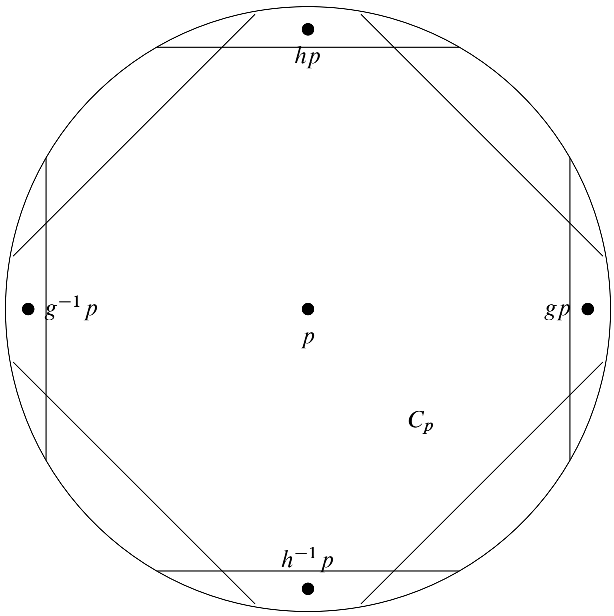


Figure 14. The Dirichlet domain C_p in the projective model for $\mathbb{H}^2$.

with $\tanh t = 0.75$*. If* Γ_1' *is generated by* g', h' *where* $\max\{|g - g'|_{\mathrm{Fr}}, |h - h'|_{\mathrm{Fr}}\} \leq 10^{-15{,}309}$*, then* Γ_1' *is Anosov.*

Before proceeding to the proof, we discuss how to choose suitable parameters in the application of Theorem 5.8. There are a number of auxiliary parameters appearing in Theorems 5.1 and 5.5. We will choose these auxiliary parameters in the same way in Section 6.3. Because of the large number of auxiliary parameters, it is not clear how to obtain optimal estimates, even when treating Theorems 5.1, 5.5 and 5.8 as black boxes. The choices we make here are simply the result of selecting auxiliary parameters in a few different ways and choosing the best result (smallest k) we achieved. We used a Mathematica notebook to verify the system of inequalities for each theorem. Recall that the constants κ_0 and ζ_0 have been computed in Examples 3.5 and 3.21.

First we choose auxiliary parameters $\delta = \frac{\zeta_0}{2\kappa_0^2}$ and $\alpha_{\mathrm{aux}} := 0.5\alpha_0 + 0.5\alpha_{\mathrm{new}}$. We apply Theorem 5.1 with $\alpha_{\mathrm{aux}} < \alpha_0$ and $\delta = \frac{\zeta_0}{2\kappa_0^2}$ by setting $\epsilon = \frac{\zeta_0^2}{10\kappa_0^2}$ and then choosing s large enough to satisfy the assumptions of the theorem. In Theorem 5.5, for any choice of auxiliary parameters $\delta_{\mathrm{aux}} < \frac{\epsilon}{2\pi\kappa_0}$ and any $\alpha_{\mathrm{aux}} < \alpha'_{\mathrm{aux}} < \alpha_0$, there is a large enough auxiliary parameter l to satisfy the assumptions. We select $\delta_{\mathrm{aux}} := 0.1\frac{\epsilon}{2\pi\kappa_0} = 0.1\frac{\zeta_0^2}{20\pi\kappa_0^3}$ and $\alpha'_{\mathrm{aux}} := 0.8\alpha_0 + 0.2\alpha_{\mathrm{aux}}$.

Proof. As discussed earlier in this section, the orbit map of Γ_1 is a $(\zeta_0, \sigma_{\mathrm{mod}}, 3.18)$-Morse $((1.28)^{-1}, 0, 3.38, 0)$-quasi-isometric embedding. We relax the parameters, asking the perturbation to induce a 33,602-local $(\zeta_0, \sigma_{\mathrm{mod}}, 3.28)$-Morse $(1, 0.1, 3.38, 0.1)$-quasi-isometric embedding. According to Theorem 5.8, such an orbit map is a global $(0.95\zeta_0; \sigma_{\mathrm{mod}}; 37{,}858)$-Morse $(91; 75{,}838; 3.38; 0)$-quasi-isometric embedding.

If $g', h' \in \mathrm{SL}(3,\mathbb{R})$ satisfy $|g - g'|_{\mathrm{Fr}}, |h - h'|_{\mathrm{Fr}} \le 10^{-15{,}309}$, then for $d_{\Gamma_1}(w, 1) \le k = 16{,}801$ we have $d(\rho(w)p, \rho'(w)p) \le 0.1$ by Corollary 6.5, so ρ' also induces a 33,602-local $(\zeta_0, \sigma_{\mathrm{mod}}, 3.28)$-Morse $(1, 0.1, 3.38, 0.1)$-quasi-isometric embedding and therefore its orbit map at p is a (global) Morse quasi-isometric embedding. In particular, g', h' generate an Anosov subgroup of $\mathrm{SL}(3,\mathbb{R})$ and our proof of Theorem 1.2 is complete. ■

6.3. An explicit neighborhood of Anosov surface groups

Let Γ_2 be the subgroup of $\mathrm{SL}(3,\mathbb{R})$ generated by

$$S = \left\{ \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \middle| \, \theta \in \left\{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}\right\} \right\}$$

for $\log\lambda = \cosh^{-1}(\cot\frac{\pi}{8})$. This group acts cocompactly on a complete, totally geodesic submanifold of $\mathbb{X}$ of constant curvature $-\frac{1}{3}$, see Section 3.4, with quotient a closed surface of genus 2. A fundamental domain for this action is given by a regular octagon in $\mathbb{H}^2$ with center p, the identity matrix in $\mathbb{X}$. This octagon decomposes into 16 triangles with vertices at the center, the vertices of the octagon and the midpoints of the edges. These triangles are isosceles with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{\pi}{8}$. By the hyperbolic law of cosines (for curvature $-\frac{1}{3}$),

$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cosh\left(\frac{1}{\sqrt{3}}c\right),$$

we see that the distance from the center p to the vertex is $R = \sqrt{3}\cosh^{-1}\left(\cot^2\frac{\pi}{8}\right)$. The Γ_2 translates of $B_R(p)$ cover $\mathbb{H}^2$, so by the Milnor–Schwarz lemma the orbit map $\mathrm{orb}_p\colon \Gamma_2 \to \mathbb{H}^2$ is a $(1, 1, 2R+1, 0)$-quasi-isometric embedding. One checks that $2R + 1 \le 9.5$. Here, we use the symmetric generating set $S' = \{\gamma \in \Gamma_2 \mid d(p, \gamma p) \le 9.5\}$. Note that the S' here agrees with the one in the introduction because $d(p, \gamma p) = \sqrt{6}|\log\gamma|_{\mathrm{Fr}}$. Every geodesic in this copy of $\mathbb{H}^2$ is $\left(\frac{1}{2\sqrt{3}}, \sigma_{\mathrm{mod}}\right)$-regular in $\mathbb{X}$. Representations of this form were studied by Barbot in [2].

We may now prove the following.

Theorem 1.3. *If* $\rho\colon \Gamma_2 \to \mathrm{SL}(3,\mathbb{R})$ *is a representation satisfying*

$$|\rho(s) - s|_{\mathrm{Fr}} \le 10^{-3{,}698{,}433}$$

for all $s \in S'$*, then* ρ *is Anosov.*

Proof. From the classical Morse lemma (Theorem 6.2), we get a Morse constant of $D = 163$. Thus, the orbit map at p is a $\big(\frac{1}{2\sqrt{3}}, \sigma_{\mathrm{mod}}, 163\big)$-Morse $(1, 1, 9.5, 0)$-quasi-isometric embedding. We relax the additive parameters by 10 and ask a perturbation to be a (2.2×10^6)-local $\big(\frac{1}{2\sqrt{3}}, \sigma_{\mathrm{mod}}, 173\big)$-Morse $(1, 11, 9.5, 10)$-quasi-isometric embedding. By Theorem 5.8, such an orbit map is a global $\big(\frac{1}{4\sqrt{3}}, \sigma_{\mathrm{mod}}, 6.8 \times 10^6\big)$-Morse $(108{,}214; 1.4 \times 10^7; 9.5; 0)$-quasi-isometric embedding.

If $\rho\colon \Gamma_2 \to \mathrm{SL}(3, \mathbb{R})$ is another representation such that $|\rho(s) - s|_{\mathrm{Fr}} \le 10^{-3{,}698{,}433}$, then for $d_{S'}(w, 1) \le k = 1.1 \times 10^6$ we have $d_{\mathbb{X}}(\rho(w)p, wp) \le 10$ by Corollary 6.5 so ρ also induces a (2.2×10^6)-local $\big(\frac{1}{2\sqrt{3}}, \sigma_{\mathrm{mod}}, 173\big)$-Morse $(1, 11, 9.5, 10)$-quasi-isometric embedding. Therefore, the orbit map is a global $\big(\frac{1}{4\sqrt{3}}, \sigma_{\mathrm{mod}}, 6.8 \times 10^6\big)$-Morse $(108{,}214; 1.4 \times 10^7; 9.5; 0)$-quasi-isometric embedding. In particular, ρ is Anosov and our proof of Theorem 1.3 is complete. ■

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