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Transitivity of normal subgroups of the mapping class groups on character varieties

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Abstract. We prove that the action of any non-trivial normal subgroup of the mapping class group of a closed surface of genus $g \geqslant 2$ is almost minimal on the character variety $X(\pi_1\Sigma_g, \mathrm{SU}_2)$: the orbit of almost every point is dense.

1. Introduction

For every $g \geqslant 2$, let $\pi_1\Sigma_g$ denote the fundamental group of a compact, connected, orientable surface of genus g, and $\mathrm{Mod}(\Sigma_g)$ its mapping class group. In [10], Goldman proved that $\mathrm{Mod}(\Sigma_g)$ acts ergodically on the character variety $X(\pi_1\Sigma_g, \mathrm{SU}_2)$, and subsequently, Previte and Xia [16] proved that for every conjugacy class of representation $\rho\colon \pi_1\Sigma_g \to \mathrm{SU}_2$ with dense image, the orbit $\mathrm{Mod}(\Sigma_g)\cdot[\rho]$ is dense in $X(\pi_1\Sigma_g, \mathrm{SU}_2)$.

Goldman then raised (see [11]) the question of whether smaller subgroups of $\mathrm{Mod}(\Sigma_g)$ still act ergodically on $X(\pi_1\Sigma_g, \mathrm{SU}_2)$, and with Xia he proved [12] that when Σ is a twice punctured torus, the Torelli group acts ergodically on the relative SU_2 character varieties. This question was addressed by Funar and Marché [7], who proved that the Johnson subgroup, generated by the Dehn twists along separating curves, acts ergodically on this character variety. Provided $g \geqslant 3$, Bouilly (see [1]) gave a simpler proof that the Torelli group acts ergodically on this character variety, and in fact on the topological components of the character variety $X(\pi_1\Sigma_g, G)$ for any compact Lie group G.

It is natural to ask how small a subgroup of $\mathrm{Mod}(\Sigma_g)$ acting ergodically on the character variety can be. We do not know, for example, if there exists a tower of subgroups $\Gamma_0 \supset \Gamma_1 \supset \cdots$ whose intersection is trivial and such that each term acts ergodically. Similar towers were investigated at the end of [7]. The strongest result we can imagine in this direction would be the existence of a single pseudo-Anosov element acting ergodically. Yet, such elements are known not to exist in the case of the relative character varieties of a one-holed torus, see [2, 6]; the question is open for the case of closed surfaces, see [10, Problem 2.8].

In this note, when a group Γ acts on a topological space X endowed with a Radon measure μ, we will say that the action is *almost minimal* if the orbit of almost every point

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is dense. We say the action is minimal if every orbit is dense, and ergodic if for every measurable Γ-invariant set U, either U or its complement has measure 0. These two latter properties are independent in general, while both imply almost minimality.

The main result of this note is the following.

Theorem 1. *Suppose $g \geqslant 2$. Let Γ be a noncentral, normal subgroup of* $\mathrm{Mod}(\Sigma_g)$. *Then the action of Γ on $X(\pi_1\Sigma_g, \mathrm{SU}_2)$ is almost minimal.*

When $g \geqslant 3$, the centre of $\mathrm{Mod}(\Sigma_g)$ is trivial, while if $g = 2$, this centre is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and generated by the hyperelliptic involution. The hypothesis "noncentral" simply rules out the cases when Γ is trivial or equal to this central $\mathbb{Z}/2\mathbb{Z}$ subgroup. Thus Theorem 1 applies, for example, to every term of the lower central series of $\mathrm{Mod}(\Sigma_g)$.

When Γ is normal, it follows from the ergodicity of the action of $\mathrm{Mod}(\Sigma_g)$ that the set of characters whose Γ-orbit is dense, has measure 0 or 1. This general observation does not give much hint about the *existence* of a dense orbit, which may be considered as the main result of this article.

The mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by Dehn twists, while the Torelli group is generated by products of the form $\tau_\gamma\tau_\delta^{-1}$, where (γ, δ) is a *bounding pair*, i.e., a pair of disjoint curves bounding a subsurface. Bouilly's approach to the ergodicity of the Torelli group uses the idea that, for almost every conjugacy class of representation $[\rho]$ and for every bounding pair (γ, δ), the product $\tau_\gamma\tau_\delta^{-1}$ acts as a totally irrational rotation along a torus embedded in the character variety $X(\pi_1\Sigma_g, \mathrm{SU}_2)$. Thus, for an appropriate sequence of powers, $\tau_\gamma^N\tau_\delta^{-N}$ approximates the effect of the Dehn twist τ_γ. This reduces the ergodicity properties of the Torelli group to those of the whole mapping class group, and these are well understood.

The key ingredient in the proof of Theorem 1 is Lemma 8 below. It consists of extending Bouilly's trick to the case when γ and δ are no longer disjoint. We manage to control the action of $\tau_\gamma^n\tau_\delta^{-n}$ for some sequences of integers n dictated by classical theorems in Diophantine approximation theory. This works for SU_2-characters, but our method does not extend to characters in higher rank compact Lie groups, as this would require a stronger (but false) result of Diophantine approximation (see Remark 4). At present, we do not know if Theorem 1 is true even for $\mathrm{SU}_2 \times \mathrm{SU}_2$ or SU_3-characters.

2. Proof of Theorem 1

We first set up some notation.

2.1. Notation and reminders

The space $\mathrm{Hom}(\pi_1\Sigma_g, \mathrm{SU}_2)$ of morphisms from $\pi_1\Sigma_g$ to SU_2 is naturally endowed with the product topology and the *character variety* $X(\pi_1\Sigma_g, \mathrm{SU}_2)$ is the quotient of this representation space by the conjugation action of SU_2. From now on, we will denote it simply by X.

The mapping class group $\mathrm{Mod}(\Sigma_g) = \pi_0(\mathrm{Diff}_+(\Sigma_g))$ is, by the Dehn–Nielsen–Baer theorem, isomorphic to an index two subgroup of $\mathrm{Out}(\pi_1\Sigma_g)$. It acts naturally on X, by setting, for $\phi \in \mathrm{Aut}(\Sigma_g)$ and $[\rho] \in X$, $\phi \cdot [\rho] = [\rho \circ \phi^{-1}]$: this descends to an action of $\mathrm{Out}(\pi_1\Sigma_g)$.

The mapping class group is generated by the Dehn twists: when $\gamma \subset \Sigma$ is a simple closed curve, we denote by τ_γ the Dehn twist along γ; see, e.g., [5, Chapter 3] for a definition, and numerous properties. Given such a curve, we may choose a representant in $\pi_1\Sigma_g$: such a representant is well defined up to conjugacy and up to passing to the inverse. Yet, we will often use the same notation, γ for the corresponding elements of $\pi_1\Sigma_g$.

For every element $A \in \mathrm{SU}_2$, we will write $\theta(A) = \frac{1}{\pi}\arccos(\frac{1}{2}\mathrm{tr}(A)) \in [0,1]$. Note that this is also invariant by conjugation and by taking the inverse. Thus, when γ is an unoriented closed curve, or an element of $\pi_1\Sigma_g$, we also define $\theta_\gamma\colon X \to [0,1]$ by $\theta_\gamma([\rho]) = \theta(\rho(\gamma))$. This function is continuous, and smooth on $\theta_\gamma^{-1}((0,1))$.

It is well known that the subspace of irreducible representations in X forms a Zariski open subset X^{irr}, which is the smooth part of X. Moreover, there is a $\mathrm{Mod}(\Sigma_g)$-invariant symplectic form on X^{irr} and the Hamiltonian flow of θ_γ on $X^{\mathrm{irr}} \cap \theta_\gamma^{-1}((0,1))$, denoted by Φ_γ^t, is 1-periodic. This flow can be extended to $\theta_\gamma^{-1}((0,1))$ and it satisfies the crucial identity

$$\tau_\gamma([\rho]) = \Phi_\gamma^{\theta_\gamma([\rho])}([\rho])$$

for all $[\rho] \in \theta_\gamma^{-1}((0,1))$. We refer to [9, 10] for all these facts.

2.2. Simultaneous Diophantine approximation

In the following definition, and subsequently in this note, for all $x \in \mathbb{R}/\mathbb{Z}$ we will denote by $|x|$ its distance to 0 in $\mathbb{R}/\mathbb{Z}$.

Definition 2. A pair (x, y) of irrational elements of $\mathbb{R}/\mathbb{Z}$ will be said *appropriately approximable* if there exists a strictly increasing sequence (q_n) of integers such that $q_n x$ converges to 0 faster than $\frac{1}{q_n}$ (i.e., $|q_n x| = o(\frac{1}{q_n})$) and $q_n y$ converges to y in $\mathbb{R}/\mathbb{Z}$.

A classical theorem of Khinchin [15] states that if (ψ_n) is a decreasing sequence of real numbers and if $\sum \psi_n$ diverges, then for almost every $x \in \mathbb{R}/\mathbb{Z}$ there are infinitely many integers q such that $|qx| \leqslant \psi_q$. In particular, for example, for almost every x, there are infinitely many integers q satisfying $|qx| \leqslant \frac{1}{q \ln q}$.

Now, a classical theorem of Hardy and Littlewood [13, Theorem 1.40] states that for every strictly increasing sequence of integers (q_n), for almost every $y \in \mathbb{R}/\mathbb{Z}$ the set $\{q_n y, n \geqslant 0\}$ is dense in $\mathbb{R}/\mathbb{Z}$. In particular, for almost every y, the number y is an accumulation point of the sequence $(q_n y)$.

These two theorems together imply the following observation.

Observation 3. *The set* $\mathrm{App} \subset (\mathbb{R}/\mathbb{Z})^2$ *of appropriately approximable pairs has full measure.*

Remark 4. Following our strategy (and, in particular, Lemma 8 below) with characters in a compact Lie group of rank d would lead to replace the approximable pairs by tuples $(x_1, \dots, x_d, y_1, \dots, y_d)$ with the property that there exists a sequence $(q_n)_{n \geqslant 1}$ such that $|q_n x_i| = o(\frac{1}{q_n})$ and $q_n y_i \to y_i \mod \mathbb{Z}$ for all $i = 1, \dots, d$. But as soon as $d \geqslant 2$, this condition holds on a set of measure 0; see [3, Theorem II] or [8, Theorem 1].

We continue with some preliminary observations concerning mapping class groups and character varieties.

2.3. Preliminary observations

In the next statements, we denote by P the set of pairs (γ, δ) of isotopy classes of nonseparating and non-isotopic simple curves.

Observation 5. *Let $\gamma \subset \Sigma_g$ be an unoriented, nonseparating simple closed curve. Then there exists $\varphi \in \Gamma$ such that $(\gamma, \varphi(\gamma)) \in P$.*

Proof. Since Γ is not central, and since $\mathrm{Mod}(\Sigma_g)$ is generated by Dehn twists along nonseparating curves, there exist a nonseparating simple closed curve δ, and $\psi \in \Gamma$, such that ψ and the Dehn twist τ_δ do not commute. There exists $\phi \in \mathrm{Mod}(\Sigma_g)$ mapping δ to γ, so $\phi \tau_\delta \phi^{-1} = \tau_\gamma$. Now $\varphi = \phi \psi \phi^{-1}$ is in Γ since Γ is normal, and φ does not commute with τ_γ; this implies the statement. ■

For every $(\gamma, \delta) \in P$, we denote by $\mathrm{Ind}(\gamma, \delta)$ the subset of X consisting of those $[\rho]$ such that $(\theta(\rho(\gamma)), \theta(\rho(\delta))) \in \mathrm{App}$. As we will see below, this condition gives some *independence* of the traces of $\rho(\gamma)^n$ and $\rho(\delta)^n$ for large n.

Observation 6. *Let $(\gamma, \delta) \in P$. Then* $\mathrm{Ind}(\gamma, \delta)$ *has full measure in X.*

Proof. Consider the map $\Theta = (\theta_\gamma, \theta_\delta) \colon X \to [0,1]^2$. We want to show that $\Theta^{-1}(\mathrm{App})$ has full measure in X. If Θ is a submersion at $[\rho]$, the implicit function theorem implies that $\Theta^{-1}(\mathrm{App})$ has full measure locally around $[\rho]$. Hence it suffices to show that Θ is a submersion in a dense Zariski open subset of X. As $t_\gamma = 2\cos(\pi \theta_\gamma)$ is an algebraic function, all the reasonings below work as if θ_γ were itself algebraic. Consider the Zariski open set $U = \Theta^{-1}(0,1)^2$: it is well known that $d\theta_\gamma$ and $d\theta_\delta$ are smooth non-vanishing forms on U. If γ and δ are disjoint, Θ can be extended to a system of action-angle coordinate, which implies that Θ is a submersion everywhere in U, see, for instance, [14]. If γ, δ do intersect, then it is known that their Poisson bracket does not vanish identically, see, for instance, [4, Corollary 5.2] where it is proved that $\{t_\gamma, t_\delta\} \neq 0$. As X is irreducible, it follows that $d\theta_\gamma, d\theta_\delta$ are linearly independent in a Zariski open subset of U, proving the lemma. ■

From the Observation 6, it follows that the set

$$\mathrm{Ind} = \bigcap_{(\gamma,\delta) \in P} \mathrm{Ind}(\gamma, \delta)$$

has full measure in X. It is obviously $\mathrm{Mod}(\Sigma_g)$-invariant, and for any $[\rho] \in \mathrm{Ind}$ and any nonseparating simple curve γ, we have $\theta_\gamma(\rho) \in (0,1)$; in fact, $\theta_\gamma(\rho)$ is irrational.

2.4. The proof

Since the action of $\mathrm{Mod}(\Sigma_g)$ on X is ergodic (by Goldman [10]), the set

$$D = \{[\rho] \mid \mathrm{Mod}(\Sigma_g) \cdot [\rho] \text{ is dense in } X\}$$

has full measure in X. In fact, this set is known explicitly from the work of Previte and Xia [16]; it is the set of those $[\rho]$ such that the image of ρ is dense in SU_2. Thus, the set $D \cap \mathrm{Ind}$ also has full measure, and Theorem 1 will follow from the following statement.

Proposition 7. *For all* $[\rho] \in D \cap \mathrm{Ind}$, *the set* $\Gamma \cdot [\rho]$ *is dense in* X.

The proof resides on the following lemma.

Lemma 8. *Let* γ *be a nonseparating simple closed curve, and let* $[\rho] \in \mathrm{Ind}$. *Then* $\tau_\gamma \cdot [\rho]$ *is in the closure of* $\Gamma \cdot [\rho]$.

Proof. Consider an element $\varphi \in \Gamma$ as in Observation 5 and set $\delta = \varphi(\gamma)$. We observe that for any $n \in \mathbb{N}$, $\tau_\gamma^n \tau_\delta^{-n} = \tau_\gamma^n \varphi \tau_\gamma^{-n} \varphi^{-1}$ belongs to Γ.

Write $\alpha = \theta_\delta(\rho)$ and $\beta = \theta_\gamma(\rho)$. As $\alpha \in (0,1)$, the twist flow $(\Phi_\gamma^s)_{s\in\mathbb{R}/\mathbb{Z}}$ is well defined on $\Phi_\delta^{-t}([\rho])$ for all t in a neighbourhood I of 0 in $\mathbb{R}/\mathbb{Z}$. We set $f(t) = \theta_\gamma(\Phi_\delta^{-t}([\rho]))$ and $F(t,s) = \Phi_\gamma^s \Phi_\delta^{-t}([\rho])$ for $(t,s) \in I \times \mathbb{R}/\mathbb{Z}$. From the identity $\tau_\gamma = \Phi_\gamma^{\theta_\gamma}$, we get for all n such that $n\alpha \in I$,

$$\tau_\gamma^n \tau_\delta^{-n}[\rho] = F(n\alpha, nf(n\alpha)).$$

As $\beta \in (0,1)$, the function θ_γ is smooth at $[\rho]$ hence f is smooth at 0. To prove the lemma, it is sufficient to show that one has $(n\alpha, nf(n\alpha)) \to (0,\beta)$ for a sequence of n's going to infinity.

Since $(\alpha,\beta) \in \mathrm{App}$, there exists a sequence (q_n) of integers as in Definition 2. We have $q_n\alpha \to 0$, so we consider the Taylor expansion of f at 0: since $|q_n\alpha| = o(\frac{1}{q_n})$, this gives

$$f(q_n\alpha) = f(0) + o\Big(\frac{1}{q_n}\Big),$$

so $q_n f(q_n\alpha) = q_n\beta + o(1)$. Now, $q_n\beta$ tends to β, by Definition 2. ∎

We are ready to conclude the proof of Theorem 1.

Proof of Proposition 7. Recall that $D \cap \mathrm{Ind}$ is $\mathrm{Mod}(\Sigma_g)$-invariant. Let $[\rho] \in D \cap \mathrm{Ind}$, and let γ_1, γ_2 be two nonseparating simple closed curves. By Lemma 8, there exists a sequence (φ_n) of elements of Γ such that $\varphi_n \cdot [\rho] \to \tau_{\gamma_2} \cdot [\rho]$. For all n, we may apply Lemma 8 to $\varphi_n \cdot [\rho]$, and now we can apply a diagonal argument to show that $\tau_{\gamma_1}\tau_{\gamma_2} \cdot [\rho]$ is in the closure of $\Gamma \cdot [\rho]$. We proceed by induction: for all $[\rho] \in D \cap \mathrm{Ind}$, and all curves $\gamma_1, \dots, \gamma_n$, the representation $\tau_{\gamma_1} \cdots \tau_{\gamma_n} \cdot [\rho]$ is in the closure of $\Gamma \cdot [\rho]$.

We notice[1] that Lemma 8 works equally well for τ_γ^{-1} instead of τ_γ, hence we deduce that the whole orbit $\mathrm{Mod}(\Sigma_g) \cdot [\rho]$ (and hence, also its closure) is contained in the closure of $\Gamma \cdot [\rho]$. As $[\rho] \in D$, this implies that $\Gamma \cdot [\rho]$ is dense in X. ■

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