Groups Geom. Dyn. 19 (2025), 195–226
DOI 10.4171/GGD/793

A constructive proof that many groups with non-torsion 2-cohomology are not matricially stable

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Abstract. A discrete group is matricially stable if every function from the group to a complex unitary group that is "almost multiplicative" in the point-operator norm topology is "close" to a genuine unitary representation. It follows from a recent result due to Dadarlat that all amenable groups with non-torsion integral 2-cohomology are not matricially stable, but the proof does not lead to explicit examples of asymptotic representations that are not perturbable to genuine representations. The purpose of this paper is to give an explicit formula, in terms of cohomological data, for asymptotic representations that are not perturbable to genuine representations for a class of groups that contains all finitely generated groups with a non-torsion 2-cohomology class that corresponds to a central extension where the middle group is residually finite. This class includes polycyclic groups with non-torsion 2-cohomology.

1. Introduction

An *asymptotic representation* of a discrete group Γ is a sequence of functions $\rho_n : \Gamma \to U(k_n)$ so that for all $g, h \in \Gamma$, we have $\|\rho_n(gh) - \rho_n(g)\rho_n(h)\| \to 0$ as n goes to infinity, where $\|\cdot\|$ is the operator norm. We say an asymptotic representation is *perturbable to a genuine representation* if there is a sequence of representations $\widetilde{\rho}_n : \Gamma \to U(k_n)$ so that for all $g \in \Gamma$, we have $\|\rho_n(g) - \widetilde{\rho}_n(g)\| \to 0$ as n goes to infinity. Recall that a countable discrete group, Γ, is *matricially stable* if every asymptotic representation of Γ is perturbable to a genuine representation of Γ [5].

In [21] Voiculescu shows that $\mathbb{Z}^2$ is not matricially stable by constructing an explicit sequence of pairs of unitaries that commute asymptotically in the operator norm, but remain far from pairs of unitaries that commute. In [14] Kazhdan independently uses the same sequence of pairs of unitaries to show that a particular surface group is not Ulam stable, where Ulam stability is defined similarly to matricial stability, but the pointwise convergence is replaced by uniform convergence. Kazhdan also connects his argument to the 2-cohomology of the group. In [5] Eilers, Shulman, and Sørensen give explicit asymptotic representations that are not perturbable to genuine representations for non-cyclic torsion-free finitely generated 2-step nilpotent groups and several other groups.

Mathematics Subject Classification 2020: 20J06 (primary); 46L05, 46M20 (secondary).
Keywords: group theory, operator algebras, matricial stability, group cohomology.

In [3] Dadarlat shows that a large class of countable discrete groups with non-vanishing rational even cohomology are not matricially stable, including amenable and, hence, polycyclic groups with non-vanishing rational even cohomology. In [4] he connects this obstruction on the level of 2-cohomology to the "winding number argument" used by Kazhdan. However, the proof in [3] uses Voiculescu's theorem, so it cannot lead to an explicit construction of an asymptotic representation that is not perturbable to a genuine representation. In this paper we will give an alternate proof that a group with a 2-homology class that pairs non-trivially with a 2-cohomology class x satisfying an additional condition is not matricially stable, and give a formula, in terms of cohomological data. The following result is what we aim to prove:

Theorem 1.1. *Suppose that Γ is a countable discrete group and $x \in H^2(\Gamma;\mathbb{Z})$ is a cohomology class represented as the central extension*

$$e \longrightarrow \mathbb{Z} \xrightarrow{\iota} \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow e.$$

If x is not in the kernel of the map $h : H^2(\Gamma;\mathbb{Z}) \to \mathrm{Hom}(H_2(\Gamma;\mathbb{Z}),\mathbb{Z})$ induced by the Kronecker pairing and there is a sequence of finite index subgroups $\Gamma_n \leq \widetilde{\Gamma}$ so that $\iota(\mathbb{Z}) \cap \bigcap \Gamma_n = \{e\}$, then Γ is not matricially stable. The sequence of functions ρ_n we will define in Proposition 3.17 is an asymptotic representation of Γ that cannot be perturbed to a genuine representation. In fact, the asymptotic representation may not be perturbed to any representation, let alone a unitary one.

In particular, if $\widetilde{\Gamma}$ is residually finite and x is not in the kernel of the map $h : H^2(\Gamma;\mathbb{Z}) \to \mathrm{Hom}(H_2(\Gamma;\mathbb{Z}),\mathbb{Z})$, then we can create an explicit formula for an asymptotic representation that cannot be perturbed to a genuine representation in terms of finite quotients of Γ and cocycle representatives of x. If Γ is finitely generated, the condition that x is not in the kernel of h is equivalent to the condition that x is non-torsion. In particular, it follows that any virtually polycyclic group with non-torsion 2-cohomology is not matricially stable and an explicit formula for the relevant asymptotic representation can be found in terms of cohomological data.

Our construction is similar to another construction of projective representations of subgroups of $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ that come from factoring cocycles through finite quotients; see the proof of [12, Corollary B].

Three virtues of our proof compared to Dadarlat's broader result are as follows: first, our proof leads to a formula for asymptotic representations that cannot be perturbed to genuine representations (Proposition 3.17). We use this formula to construct new examples of asymptotic representations that cannot be perturbed to genuine representations in Section 5. Second, our proof is relatively elementary and uses only basic group cohomology, instead of employing techniques used in the Novikov conjecture. Third, because we do not use these techniques, we do not require the existence of a γ-element.

This paper is organized as follows: in Section 2 we will review relevant background information. In Section 3 we prove the main result (Theorem 1.1), and find that a formula for an asymptotic representation that cannot be perturbed to a genuine representation for a group satisfies the assumptions of the main theorems (Proposition 3.17). We show that the main results apply to virtually polycyclic groups with non-torsion 2-cohomology (Corollary 3.23) and, hence, non-cyclic finitely generated torsion-free nilpotent groups. In Section 4 we give an alternate proof that non-cyclic torsion-free finitely generated nilpotent groups satisfy the cohomological conditions required for the main result, which is useful for computing examples. In Section 5 we illustrate our formula for the following groups: $\mathbb{Z}^2$, to show our methods can recover Voiculescu's matrices; a 3-step nilpotent group; and the polycyclic group $\mathbb{Z}^2 \rtimes \mathbb{Z}$, where the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ is induced by "Arnold's cat map."

2. Background information

2.1. Group homology and cohomology

There are many ways to characterize group homology and cohomology, but to us the most useful will be to describe them as the homology and cohomology of an explicit chain complex, described below. We will only use homology with coefficients in $\mathbb{Z}$ and cohomology with trivial action in this paper. For more about this construction, see [1, Chapter II.3].

Definition 2.1. Let Γ be a discrete group. We define $C_n(\Gamma)$ to be the free abelian group generated by elements of Γ^n. We may write an element of Γ^n as $[a_1|a_2|\cdots|a_n]$ with $a_i \in \Gamma$. We thus write a typical element of $C_n(\Gamma)$ as

$$c = \sum_{i=1}^{N} x_i [a_{i1}|a_{i2}|\cdots|a_{in}]$$

with $x_i \in \mathbb{Z}$ and $a_{ij} \in \Gamma$. We define the boundary map ∂_n from $C_n(\Gamma) \to C_{n-1}(\Gamma)$ by

$$\partial_n[a_1|\cdots|a_n] = [a_2|\cdots|a_n] + \sum_{i=1}^{n-1}(-1)^i[a_1|\cdots|a_{i-1}|a_i a_{i+1}|a_{i+2}|\cdots|a_n] + (-1)^n[a_1|\cdots|a_{n-1}].$$

Often we will just write ∂ where the domain is clear from context. The group *homology* of Γ is the homology group of the chain complex $(C_\bullet(\Gamma), \partial_\bullet)$, that is to say, $H_n(\Gamma) := \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$.

Definition 2.2. If A is any abelian group, then the *cohomology of* Γ *with coefficients in* A is the cohomology of $(C_\bullet, \partial_\bullet)$ with coefficients[1] in A.

[1]In general, A may be taken to be a left $\mathbb{Z}[\Gamma]$ module, but we will only consider the case where the action of Γ on A is trivial here, so we may consider A to only have the structure of an abelian group.

To be more explicit, we use the notation

$$C^n(\Gamma; A) := \mathrm{Hom}(C_n(\Gamma), A)$$

and note that this is isomorphic to the group of functions from Γ^n to A. Then, we define $\partial^n : C^n(\Gamma; A) \to C^{n+1}(\Gamma; A)$ to be the adjoint[2] of ∂_{n+1}, that is,

$$(\partial^n \sigma)(a_1, \dots, a_{n+1}) = \sigma(a_2, \dots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \sigma(a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_{n+1}) + (-1)^{n+1} \sigma(a_1, \dots, a_n).$$

Then, we define $H^n(\Gamma; A) := \ker(\partial^n)/\operatorname{im}(\partial^{n-1})$. As with homology, we will suppress the n in ∂^n if the dimension is obvious from context.

Suppose that f is a group homomorphism from Γ_1 to Γ_2. This induces a map $f_\#$ from $C_n(\Gamma_1)$ to $C_n(\Gamma_2)$ and a map $f_* : H_n(\Gamma_1) \to H_n(\Gamma_2)$. Similarly, the adjoint of $f_\#$ from $C^n(\Gamma_2) \to C^n(\Gamma_1)$ called $f^\#$ descends to a well-defined map from $f^* : H^2(\Gamma_2; A) \to H^2(\Gamma_1; A)$. Similarly, if $g : A_1 \to A_2$ is a homomorphism of abelian groups, there is a map $g_\#$ from $C^n(\Gamma; A_1) \to C^n(\Gamma; A_2)$. This map descends to a well-defined map $g_* : H^n(\Gamma; A_1) \to H_n(\Gamma; A_2)$. All maps defined in this paragraph are functorial.

Because $C^n(\Gamma; A)$ is isomorphic to $\mathrm{Hom}(C_n(\Gamma), A)$, there is a natural bilinear map, called the Kronecker pairing, from $C^n(\Gamma; A) \times C_n(\Gamma) \to A$ defined by

$$\left\langle \sigma, \sum_{i=1}^{N} x_i [a_{i1}| \cdots |a_{in}] \right\rangle = \sum_{i=1}^{N} x_i \sigma(a_{i1}, \dots, a_{in}).$$

This descends to a well-defined bilinear map from $H^n(\Gamma; A) \times H_n(\Gamma) \to A$. We will use the notation $\langle \cdot, \cdot \rangle$ for both maps.

2.2. 2-cohomology and central extensions

The 2-cohomology has an alternative characterization.

Definition 2.3. If Γ is a discrete group and A is an abelian group, then a *central extension* of Γ by A is a short exact sequence

$$e \longrightarrow A \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow e,$$

where the image of A in $\widetilde{\Gamma}$ is central in $\widetilde{\Gamma}$. We say two central extensions are *equivalent* if we can make a commutative diagram as follows:

$$\begin{array}{ccccccccc} e & \longrightarrow & A & \longrightarrow & \widetilde{\Gamma}_1 & \longrightarrow & \Gamma & \longrightarrow & e \\ & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mathrm{id}_\Gamma} & & \\ e & \longrightarrow & A & \longrightarrow & \widetilde{\Gamma}_2 & \longrightarrow & \Gamma & \longrightarrow & e. \end{array}$$

[2] Actually, [1] defines the coboundary map to be $(-1)^{n+1}$ times the adjoint, but this does not change the image or kernel boundary of the maps, so it leads to an equivalent definition of the cohomology groups.

Theorem 2.4 ([1, Theorem IV.3.12]). *As a set, $H^2(\Gamma; A)$ is in bijection with the equivalence classes of central extensions of Γ by A.*

Given an explicit central extension, we may find a cocycle representative of the corresponding element of $H^2(\Gamma; A)$ as follows: pick θ to be a set-theoretic section from Γ to $\widetilde{\Gamma}$. Then, viewing A as a subset of $\widetilde{\Gamma}$, define

$$\sigma(g,h) = \theta(g)\theta(h)\theta(gh)^{-1} \in A.$$

By [1, IV (3.3)], this is a cocycle representative of the cohomology class corresponding to this central extension.

2.3. Polycyclic and nilpotent groups

Definition 2.5. A group Γ is called *polycyclic* if there is a sequence of subgroups

$$\Gamma = \Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_m \geq \Gamma_{m+1} = \{e\}$$

so that $\Gamma_i \triangleright \Gamma_{i+1}$ and Γ_i / Γ_{i+1} is cyclic. A sequence of subgroups obeying this condition is called a *polycyclic sequence of subgroups*. We may pick this sequence so that each quotient is non-trivial. We may pick a_i to be a representative of a generator of Γ_i / Γ_{i+1}. We call these generators a *polycyclic sequence* for Γ, and they generate Γ.

Definition 2.6. A group is called *virtually polycyclic* if it has a finite index polycyclic subgroup.

In this case we may assume that there is a *normal* finite index polycyclic subgroup. This may be constructed by intersecting over each conjugate of the subgroup and using the fact that a subgroup of a polycyclic group is polycyclic (see [20, Proposition 9.3.7]). By [9, Theorem 3], polycyclic groups are residually finite. From this, it follows that virtually polycyclic groups are residually finite as well. The following proposition is an elementary exercise:

Proposition 2.7. *Suppose that Γ is virtually polycyclic, and*

$$e \longrightarrow \mathbb{Z} \xrightarrow{\iota} \widetilde{\Gamma} \xrightarrow{\varphi} \Gamma \longrightarrow e$$

is an extension of Γ. Then, $\widetilde{\Gamma}$ is virtually polycyclic as well.

By [20, Proposition 9.3.4], all finitely generated nilpotent groups are polycyclic. Let Γ be a torsion-free finitely generated nilpotent group.

Definition 2.8. A *Mal'cev basis* for a torsion-free finitely generated nilpotent group is an m-tuple of elements $(a_1, \ldots a_m) \in \Gamma^m$ that obeys the following conditions:

- For all $g \in \Gamma$, g can be written uniquely as $g = a_1^{x_1} \cdots a_m^{x_m}$ for some $(x_1, \ldots, x_m) \in \mathbb{Z}^m$. We call this presentation the *canonical form* of g.

- The subgroups $\Gamma_i = \langle a_i, \ldots, a_m \rangle$ form a central series for Γ.

Every finitely generated torsion-free nilpotent group has a Mal'cev basis by [11, Lemma 8.23]. It also follows that the Γ_is make a polycyclic sequence of subgroups.

2.4. Rational 2-cohomology of a torsion-free finitely generated nilpotent group

We will need the following result:

Theorem 2.9 ([2, 19]). *Let Γ be a torsion-free finitely generated nilpotent group that is neither $\mathbb{Z}$ nor the trivial group. Then, $H^2(\Gamma; \mathbb{Q}) \not\simeq \{0\}$.*

Proof. By a result of Pickel [19], we have that $H^\bullet(\Gamma; \mathbb{Q})$ can be calculated in terms of the of the cohomology of an associated rational Lie algebra. By a result of Ado explained on [2, p. 86], the 2-cohomology of the algebra is non-zero. ∎

Corollary 2.10. *If Γ is a torsion-free finitely generated nilpotent group that is neither $\mathbb{Z}$ nor trivial, there is a pair $([\sigma], c) \in H^2(\Gamma; \mathbb{Z}) \times H_2(\Gamma)$ so that $\langle [\sigma], c \rangle \neq 0$.*

Proof. The rational 2-cohomology of Γ is non-trivial, by Theorem 2.9. By the universal coefficient theorem [16, Theorem 53.1], we have a sequence

$$0 \longrightarrow \mathrm{Ext}(H_1(\Gamma), \mathbb{Q}) \longrightarrow H^2(\Gamma; \mathbb{Q}) \longrightarrow \mathrm{Hom}(H_2(\Gamma), \mathbb{Q}) \longrightarrow 0.$$

Since $\mathrm{Hom}(\cdot, \mathbb{Q})$ is exact, $\mathrm{Ext}(H_1(\Gamma), \mathbb{Q})) \simeq \{0\}$. From this, it follows that $H_2(\Gamma)$ is non-torsion. Next we need to show that $H_2(\Gamma)$ is finitely generated. First we will show that $H^n(\Gamma; \mathbb{Z})$ is finitely generated for all n and for all torsion-free finitely generated nilpotent groups, by induction on the number of elements in the Mal'cev basis. If this number is zero, this is obvious. For the inductive step, let $x_1, \ldots, x_m$ be a Mal'cev basis. Then, by the inductive hypothesis $H^n(\Gamma/\langle x_m \rangle; \mathbb{Z})$ is finitely generated for all n. By [10, Theorem 5.3], there is an exact sequence

$$\cdots H^{n+2}(\Gamma/\langle x_m \rangle; \mathbb{Z}) \longrightarrow H^{n+2}(\Gamma; \mathbb{Z}) \longrightarrow H^{n+1}(\Gamma/\langle x_m \rangle) \cdots$$

which shows that $H^n(\Gamma; \mathbb{Z})$ is finitely generated for $n \geq 2$. For $n = 0$, this is obvious. For $n = 1$, this follows from the fact that $H^1(\Gamma; \mathbb{Z}) = \mathrm{Hom}(H_1(\Gamma), \mathbb{Z}) = \mathrm{Hom}(\Gamma/[\Gamma, \Gamma], \mathbb{Z})$ [1, p. 36]. Next, the fact that the homology is finitely generated as well follows from [8, Proposition 3F.12]. Because $H_2(\Gamma)$ is non-torsion and finitely generated, we know that $\mathrm{Hom}(H_2(\Gamma), \mathbb{Z})$ is non-zero. In particular, there must be a cocycle $[\sigma]$ that is not in the kernel of the map from $H^2(\Gamma; \mathbb{Z})$ to $\mathrm{Hom}(H_2(\Gamma), \mathbb{Z})$. ∎

2.5. Log of a matrix

Throughout this article, if m is a matrix, we define $\log(m)$ to be the power series for log centered at 1, that is,

$$\log(m) := \sum_{j=1}^{\infty} (-1)^{j-1} \frac{(m - \mathrm{id})^n}{n},$$

which converges and is continuous for $\|m - \mathrm{id}\| < 1$. We will consider this to be well-defined when $\|m - \mathrm{id}\| < 1$. By looking at the Jordan form for m, we can see that the eigenvalues of $\log(m)$ are the logs of eigenvalues of m. This justifies the formula $\exp(\mathrm{Tr}(\log(m)) = \det(m)$.

2.6. Voiculescu's matrices and the winding number argument

The classical example due to Voiculescu in [21] for an asymptotic representation of $\mathbb{Z}^2$ that is not perturbable to a genuine representation comes in the form

$$\rho_n(a,b) = u_n^a v_n^b,$$

where u_n and v_n are $n \times n$ matrices such that

$$u_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad v_n = \begin{bmatrix} \exp\left(\frac{2\pi i}{n}\right) & 0 & 0 & \cdots & 0 \\ 0 & \exp\left(\frac{4\pi i}{n}\right) & 0 & \cdots & 0 \\ 0 & 0 & \exp\left(\frac{6\pi i}{n}\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The argument that we summarize here was first applied to this problem by Exel and Loring in [6], and had previously independently been used by Kazhdan in [14]. It can be computed that $u_n v_n u_n^{-1} v_n^{-1} = \exp(\frac{-2\pi i}{n}) \mathrm{id}_{\mathbb{C}^n}$. It is not difficult to show that the fact that this gets arbitrarily close to $\mathrm{id}_{\mathbb{C}^n}$ in the operator norm implies asymptotic multiplicativity of the associated representation. A sketch of the argument that this asymptotic representation cannot be perturbed follows.

The path $p(t) = \det(t \exp(\frac{-2\pi i}{n}) \mathrm{id}_{\mathbb{C}^n} + (1-t) \mathrm{id}_{\mathbb{C}^n})$ is a path in $\mathbb{C}^\times$ with winding number -1. Suppose, towards a contradiction, that u_n is close enough in the operator norm to a u'_n and v_n is close enough to v'_n so that $v'_n u'_n = u'_n v'_n$. We make a contradiction as follows: we define

$$\hat{u}_n(s) = s u_n + (1-s) u'_n,$$
$$\hat{v}_n(s) = s v_n + (1-s) v'_n.$$

Then,

$$h(t,s) = \det(t \hat{u}_n(s) \hat{v}_n(s) \hat{u}_n(s)^{-1} \hat{v}_n(s)^{-1} + (1-t) \mathrm{id}_{\mathbb{C}^n}).$$

It can be shown that $h(t,s) \neq 0$ for all $s, t \in [0,1]$. It follows that h is a homotopy from the path p to the trivial loop centered at 1. This is a contradiction, since p has non-zero winding number.

A more general statement of this type of invariant can be found in [5, Theorem 3.9] or in [4]. Essentially, the relation $u_n v_n u_n^{-1} v_n^{-1}$ can be replaced with another product of commutators. Note that the winding number of p could also be computed by calculating

$\mathrm{Tr}(\log(u_n v_n u_n^{-1} v_n^{-1}))/(2\pi i)$, where log is defined to be a power-series centered at 1. We will phrase our analogous argument in terms of computing the trace of log instead of computing the winding number directly, but it is inspired by this more classical argument. Kazhdan develops this example independently to show that a certain surface group is not uniformly stable [14]. He develops them as a representation of a central extension of the group in question, thereby connecting asymptotic representations to 2-cohomology.

3. Main results

In Section 3.1 we prove some analytic lemmas that we use later. In Section 3.2 we develop a pairing between between almost multiplicative functions from Γ to M_n and 2-chains (Definition 3.3). We show that if an almost multiplicative function is close enough to a genuine unitary representation, its pairing with a 2-cycle is zero (Theorem 3.7). In Section 3.3 we introduce a "finite type" condition on cohomology classes (Definition 3.9), and give some alternate characterizations of the definition (Proposition 3.14). In Section 3.4 we develop a formula for an asymptotic representation (Proposition 3.17), and show that if the right cohomological conditions hold, it is well-defined and cannot be perturbed to a genuine representation (Theorem 1.1). We show that polycyclic groups with non-torsion 2-cohomology meet this condition (Corollary 3.23).

3.1. Analytic Lemmas

We will use the following elementary results:

Lemma 3.1. *Let A be a C^*-algebra. Then, the following hold:*

(1) *If $a_i, b_i \in A$ for $i \in \{1, \ldots, N\}$ so that for all i, $\|a_i - b_i\| < \varepsilon$ and $\|a_i\|, \|b_i\| < M$, we have that*
$$\Big\|\prod_{i=1}^{N} a_i - \prod_{i=1}^{N} b_i\Big\| < NM^{N-1}\varepsilon.$$

(2) *If $a \in A$ and u is a unitary in A so that $\|a - u\| \le \frac{1}{2}$, we have*
$$\|a^{-1} - u^*\| \le 2\|a - u\|.$$
In particular, $\|a^{-1}\| \le 2$.

Proof. Proof of (1)*:* We compute

$$\begin{aligned}\sum_{j=1}^{N}\Big(\prod_{i=1}^{j-1} a_i\Big)(a_j - b_j)\prod_{i=j+1}^{N} b_i &= \sum_{j=1}^{N}\Big(\prod_{i=1}^{j} a_i\Big)\prod_{i=j+1}^{N} b_i - \sum_{j=1}^{N}\Big(\prod_{i=1}^{j-1} a_i\Big)\prod_{i=j}^{N} b_i\\ &= \sum_{j=1}^{N}\Big(\prod_{i=1}^{j} a_i\Big)\prod_{i=j+1}^{N} b_i - \sum_{j=0}^{N-1}\Big(\prod_{i=1}^{j} a_i\Big)\prod_{i=j+1}^{N} b_i\end{aligned}$$

$$= \prod_{i=1}^{N} a_i - \prod_{i=1}^{N} b_i.$$

Applying the triangle inequality and submultiplicativity to the first term gives us the desired inequality.

Proof of (2)*:* Let $b = u^*a$. Note that $\|b-1\| = \|a-u\| < 1$, so by [17, Theorem 1.2.2], we have that

$$b^{-1} = \sum_{k=0}^{\infty}(1-b)^k.$$

Thus,

$$\begin{aligned}\|a^{-1}-u^*\| &= \|a^{-1}u-1\| = \|b^{-1}-1\| \\ &\leq \sum_{k=1}^{\infty}\|1-b\|^k = \frac{\|1-b\|}{1-\|1-b\|} = \frac{\|a-u\|}{1-\|a-u\|} \\ &\leq 2\|a-u\|.\end{aligned}$$

We use that $\|a-u\| \leq \frac{1}{2}$ in the last step. ∎

Lemma 3.2. *Let $m_1, m_2 \in U(n)$ so that $\|m_i - \mathrm{id}_{\mathbb{C}^n}\| < \frac{1}{2}$. Then, if* log *is defined as a power series centered at 1, we have that*

$$\mathrm{Tr}(\log(m_1m_2)) = \mathrm{Tr}(\log(m_1)) + \mathrm{Tr}(\log(m_2)).$$

Proof. First note that $\|m_1m_2 - 1\| < \frac{1}{2}\cdot 2 = 1$ by Lemma 3.1, so the expression is well-defined. Then,

$$\exp(\mathrm{Tr}(\log(m_1m_2)) - \mathrm{Tr}(\log(m_1)) - \mathrm{Tr}(\log(m_2)) = \frac{\det(m_1m_2)}{\det(m_1)\det(m_2)} = 1,$$

so

$$\mathrm{Tr}(\log(m_1m_2)) - \mathrm{Tr}(\log(m_1)) - \mathrm{Tr}(\log(m_2)) \in 2\pi i\mathbb{Z}.$$

Then,

$$\mathrm{Tr}(\log((m_1t + (1-t)\,\mathrm{id}_{\mathbb{C}^n})m_2) - \mathrm{Tr}(\log(m_1t + (1-t)\,\mathrm{id}_{\mathbb{C}^n}) - \mathrm{Tr}(\log(m_2))$$

is well-defined for all $t \in [0,1]$ by and in $2\pi i\mathbb{Z}$ by the same argument above. Because this expression depends continuously on t, we must have that it is constant in t. Plugging in $t = 0$, we must have that the expression is uniquely zero. ∎

3.2. A homological version of the winding number argument

The idea of this section is to find a pairing between maps from Γ to M_n that are almost multiplicative and elements of 2-cycles in $C_2(\Gamma)$. In general, how "close" a map is to being multiplicative depends on the specific element of $C_2(\Gamma)$ we pair with.

Definition 3.3. Suppose that $c \in C_2(\Gamma)$ is expressed by the formula

$$c = \sum_{i=1}^{N} x_i [a_i | b_i],$$

so that $(a_i, b_i) = (a_j, b_j)$ implies $i = j$. The *support* of c is the set of ordered pairs $\{(a_i, b_i)\}_{i=1}^N$. The *boundary support of* c is the set of elements of Γ, $\{a_i, b_i, a_i b_i\}_{i=1}^N$. Then, we say $\rho : \Gamma \to M_n(\mathbb{C})$ is *ε-almost multiplicative on the support of* c if $\rho(a_i), \rho(b_i) \in \mathrm{GL}_n(\mathbb{C})$ and

$$\|\rho(a_i b_i)\rho(a_i)^{-1}\rho(b_i)^{-1} - \mathrm{id}_{\mathbb{C}^n}\| < \varepsilon$$

for each i. In this case, if additionally $\varepsilon \leq 1$, we define

$$\boldsymbol{(}\rho, c\boldsymbol{)} = \frac{1}{2\pi i} \sum_{j=1}^{N} x_j \operatorname{Tr}(\log(\rho(a_j b_j)\rho(b_j)^{-1}\rho(a_j)^{-1})),$$

where log is defined as a power series centered at 1.

This is clearly $\mathbb{Z}$-linear in the second entry, in the sense that

$$\boldsymbol{(}\rho, c_1 \pm c_2\boldsymbol{)} = \boldsymbol{(}\rho, c_1\boldsymbol{)} \pm \boldsymbol{(}\rho, c_2\boldsymbol{)}$$

when the right side is well-defined. Due to potential cancellation, the support of $c_1 + c_2$ may be smaller than the support of c_1 union the support of c_2. It is also "linear" in the first entry in the sense that

$$\boldsymbol{(}\rho_1 \oplus \rho_2, c\boldsymbol{)} = \boldsymbol{(}\rho_1, c\boldsymbol{)} + \boldsymbol{(}\rho_2, c\boldsymbol{)};$$

if one side of this equality is well-defined, then so is the other, because

$$\|(\rho_1 \oplus \rho_2)(gh) - (\rho_1 \oplus \rho_2)(g)(\rho_1 \oplus \rho_2)(h)\| = \max_i \|\rho_i(gh) - \rho_i(g)\rho_i(h)\|.$$

Proposition 3.4. *If* $\partial c = 0$*, then*

$$\boldsymbol{(}\rho, c\boldsymbol{)} \in \mathbb{Z}.$$

Proof. Let F be the boundary support of c and let $C_1(F)$ be the subgroup of $C_1(\Gamma)$ spanned by elements of the form $[g]$, where $g \in F$. Define a homomorphism $\varphi : C_1(F) \to \mathbb{C}^\times$ by taking $[g] \mapsto \det(\rho(g))$. This is well-defined because $C_1(F)$ is a free abelian group and $\det(\rho(g)) \in \mathbb{C}^\times$, since $\rho(F) \subset \mathrm{GL}_n(\mathbb{C})$. Then, we see that

$$\exp(2\pi i \boldsymbol{(}\rho, c\boldsymbol{)}) = \prod_{j=1}^{N} \det((\rho(a_j b_j)\rho(b_j)^{-1}\rho(a_j)^{-1})^{x_i}) = \varphi(-\partial c) = \varphi(0) = 1. \qquad \blacksquare$$

Definition 3.5. If c is a 2-cycle on Γ with boundary support F and ρ_0 and ρ_1 are maps from Γ to $\mathrm{GL}_n(\mathbb{C})$ that are 1-almost multiplicative on the support of c, we say that ρ_0

and ρ_1 are *homotopy equivalent on the boundary support of c* if the following conditions are met: there is a family of functions

$$\rho^t : \Gamma \to M_n(\mathbb{C})$$

continuous in t so that $\rho^0(g) = \rho_0(g)$ and $\rho^1(g) = \rho_1(g)$ for all $g \in F$; and for all t, ρ^t is 1-almost multiplicative on the support of c.

Proposition 3.6. *Let c be a 2-cycle on Γ and let $0 < \varepsilon < 1$. Let $\rho_0 : \Gamma \to \mathrm{GL}_n(\mathbb{C})$ be ε-multiplicative on the support of c and $\rho_1 : \Gamma \to U(n)$. Then, the following hold:*

(1) *If*

$$\|\rho_0(g) - \rho_1(g)\| < \frac{1-\varepsilon}{24}$$

for all g in the boundary support of c, then ρ_0 and ρ_1 are homotopy equivalent in the boundary support of c and ρ_1 is 1-almost multiplicative on the support of c.

(2) *If ρ_0 and ρ_1 are homotopy equivalent in the boundary support of c, then* $(\rho_0, c) = (\rho_1, c)$.

Proof. Proof of (1)*:* Define ρ^t to be

$$t\rho_1 + (1-t)\rho_0.$$

Then, for each g in the boundary support of c, we must have

$$\|\rho^t(g) - \rho_0(g)\| < \frac{1-\varepsilon}{24}.$$

Applying Lemma 3.1(2), this gives us

$$\|\rho^t(g)^{-1} - \rho_0(g)^{-1}\| < \frac{1-\varepsilon}{12}.$$

Then, for (a_i, b_i) in the support of c, we have

$$\begin{aligned}&\|\rho^t(a_i b_i)\rho^t(b_i)^{-1}\rho^t(a_i)^{-1} - 1\| \\ &\quad < \|\rho^t(a_i b_i)\rho^t(b_i)^{-1}\rho^t(a_i)^{-1} - \rho_0(a_i b_i)\rho_0(b_i)^{-1}\rho_0(a_i)^{-1}\| + \varepsilon \le 1.\end{aligned}$$

The last step is using Lemma 3.1(1) with where the N, M, and ε in the statement of the lemma are 3, 2, and $\frac{1-\varepsilon}{12}$, respectively. Applying this to $t = 1$, we get that ρ_1 is 1-almost multiplicative on the support of c.

Proof of (2)*:* The function

$$t \mapsto (\rho^t, c)$$

is a continuous function from $[0, 1]$ to $\mathbb{Z}$, so it must be constant. ∎

Theorem 3.7. *If ρ_0 is a genuine representation, ρ_1 is a function from Γ to $U(n)$, c is a 2-cocycle on Γ, and*

$$\|\rho_1(g) - \rho_0(g)\| < \frac{1}{24}$$

for all g in the boundary support of c, then ρ_1 is 1-multiplicative on the support of c and $\langle\rho_1, c\rangle = 0$.

Proof. This follows from Proposition 3.6, taking the limit as $\varepsilon \to 0$. ■

This is sufficient for our purposes, but for conceptual clarity it would be nice to show that this pairing depends only on homology class, not on the choice of cycle representative. We do not have a result that is quite this strong, but we can show that it is "eventually true" for asymptotic homomorphisms.

Theorem 3.8. *Let $c = \partial d$ be a 2-boundary in $C^2(\Gamma)$ and let $\rho_n : \Gamma \to U(N_n)$ be an asymptotic homomorphism. Then, for large enough n, we have $\langle\rho_n, c\rangle = 0$.*

Proof. By linearity, we may reduce the case that $c = \partial[g_1|g_2|g_3]$. In this case we have that

$$c = -[g_1|g_2] + [g_1|g_2g_3] - [g_1g_2|g_3] + [g_2|g_3].$$

For large enough n, we have that ρ_n is multiplicative enough that we may apply Lemma 3.2. Thus,

$$\begin{aligned} 2\pi i\langle\rho_n, c\rangle &= \mathrm{Tr}(\log(\rho_n(g_1)\rho_n(g_2)\rho_n(g_1g_2)^{-1}\rho_n(g_1g_2)\rho_n(g_3)\rho_n(g_1g_2g_3)^{-1} \\ &\qquad \cdot \rho_n(g_1g_2g_3)\rho_n(g_2g_3)^{-1}\rho_n(g_1)^{-1}\rho_n(g_2g_3)\rho_n(g_3)^{-1}\rho_n(g_2)^{-1})) \\ &= \mathrm{Tr}(\log(\rho_n(g_1)\rho_n(g_2)\rho_n(g_3)\rho_n(g_2g_3)^{-1}\rho_n(g_1)^{-1}\rho_n(g_2g_3)\rho_n(g_3)^{-1} \\ &\qquad \cdot \rho_n(g_2)^{-1})). \end{aligned}$$

Now, since the complex unitary group is path connected, we can make a path $u_t : [0,1] \to U(n)$ so that $u_0 = \rho(g_1)$ and $u_1 = \mathrm{id}_{\mathbb{C}^n}$. Then,

$$\begin{aligned} \|u_t\rho_n(g_2)\rho_n(g_3)\rho_n(g_2g_3)^{-1}u_t^{-1} - \mathrm{id}_{\mathbb{C}^n}\| &= \|\rho_n(g_2)\rho_n(g_3)\rho_n(g_2g_3)^{-1} - u_t^{-1}u_t\| \\ &= \|\rho_n(g_2)\rho_n(g_3)\rho_n(g_2g_3)^{-1} - \mathrm{id}_{\mathbb{C}^{N_n}}\|. \end{aligned}$$

For large enough n, this will be less than 1, so

$$\log(u_t\rho_n(g_1)\rho_n(g_2)\rho_n(g_2g_3)^{-1}u_t^{-1}\rho_n(g_2g_3)\rho_n(g_3)^{-1}\rho_n(g_2)^{-1})$$

will be well-defined. Moreover, since

$$\det(u_t\rho_n(g_2)\rho_n(g_3)\rho_n(g_2g_3)^{-1}u_t^{-1}\rho_n(g_2g_3)\rho_n(g_3)^{-1}\rho_n(g_2)^{-1}) = 1,$$

we must have that

$$\mathrm{Tr}(\log(u_t\rho_n(g_2)\rho_n(g_3)\rho_n(g_2g_3)^{-1}u_t^{-1}\rho_n(g_2g_3)\rho_n(g_3)^{-1}\rho_n(g_2)^{-1}) \in 2\pi i\mathbb{Z}.$$

Because this is a discrete space, the values cannot depend on t. We conclude that

$$2\pi i\langle\rho_n, c\rangle = \mathrm{Tr}(\log(\rho_n(g_2)\rho_n(g_3)\rho_n(g_2g_3)^{-1}\rho_n(g_2g_3)\rho_n(g_3)^{-1}\rho_n(g_2)^{-1}) = 0.$$ ■

3.3. Finite type cohomology

Definition 3.9. Let Γ be a countable discrete group and $[\sigma] \in H^2(\Gamma; \mathbb{Z})$ be given by the central extension

$$e \longrightarrow \mathbb{Z} \xrightarrow{\iota} \widetilde{\Gamma} \xrightarrow{\varphi} \Gamma \longrightarrow e.$$

We say that $[\sigma]$ is of *finite type* if $\widetilde{\Gamma}$ has a sequence of finite index subgroups $\{\Gamma_k\}_{k\in\mathbb{N}}$ so that

$$\iota(\mathbb{Z}) \cap \bigcap_k \Gamma_k = \{e\}.$$

Remark 3.10. Clearly, if $\widetilde{\Gamma}$ is residually finite, $[\sigma]$ is of finite type.

Remark 3.11. We may assume that the subgroups in Definition 3.9 are normal and decreasing. To achieve normality, replace Γ_k with the kernel of the action of $\widetilde{\Gamma}$ on $\widetilde{\Gamma}/\Gamma_k$. To achieve a decreasing sequence, replace Γ_k with the cumulative intersection of Γ_k.

To develop our formula, we will develop an alternate characterization of finite type cohomology classes that can be expressed in terms of the cohomology cochain complex.

Let Γ be a discrete group and let Q be a finite quotient of Γ. Call q the quotient map from Γ to Q and f^n the canonical map from $\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$. Then, q induces a cochain map $q^\#$ from $C^k(Q; \mathbb{Z}/n\mathbb{Z})$ to $C^k(\Gamma; \mathbb{Z}/n\mathbb{Z})$ and f induces a map $f^n_\#$ from $C^k(\Gamma; \mathbb{Z})$ to $C^k(\Gamma; \mathbb{Z}/n\mathbb{Z})$. These in turn induce maps $q^* : H^*(Q; \mathbb{Z}/n\mathbb{Z}) \to H^k(\Gamma; \mathbb{Z}/n\mathbb{Z})$ and $f^n_* : H^*(\Gamma; \mathbb{Z}) \to H^k(\Gamma; \mathbb{Z}/n\mathbb{Z})$.

Definition 3.12. If σ is a $\mathbb{Z}$-valued k-cocycle on Γ, we say that σ is of *n-Q type* if $f^n_\#(\sigma) = q^\#(\sigma')$ for some $\mathbb{Z}/n\mathbb{Z}$-valued k-cocycle on Q, σ'. We say that a cohomology class $[\sigma] \in H^k(\Gamma; \mathbb{Z})$ is *n-Q type* if there is a cohomology class $[\sigma'] \in H^k(Q; \mathbb{Z}/n\mathbb{Z})$ so that $f^n_*([\sigma]) = q^*([\sigma'])$. See the below diagram for a picture of f^n_* and q^*.

$$\begin{array}{ccc} & & [\sigma'] \in H^k(Q; \mathbb{Z}/n\mathbb{Z}) \\ & & \downarrow q^* \\ [\sigma] \in H^k(\Gamma; \mathbb{Z}) & \xrightarrow{f^n_*} & H^k(\Gamma; \mathbb{Z}/n\mathbb{Z}). \end{array}$$

Example 3.13. Consider $\Gamma = \mathbb{Z}^2$ and the cocycle $\sigma((x_1, x_2), (y_1, y_2)) = x_2 y_2$. That this is a cocycle is easy to check:

$$\partial\sigma((x_1, x_2), (y_1, y_2), (z_1, z_2)) = x_2 y_1 - x_2(y_1 + z_1) + (x_2 + y_2)z_1 - y_2 z_1 = 0.$$

This is not a coboundary because

$$c = [(0,1)|(1,0)] - [(1,0)|(0,1)]$$

is a 2-chain such that

$$\langle \sigma, c \rangle = 1.$$

Then, set $Q_n = (\mathbb{Z}/n\mathbb{Z})^2$ and let $q_n : \mathbb{Z}^2 \to Q_n$ be the obvious quotient map. We have that σ and, hence, $[\sigma]$ is of n-Q_n type. To show this, note that the same formula used for σ defines a 2-cochain, $\sigma' \in C^2(Q_n; \mathbb{Z}/n\mathbb{Z})$. The same computations that show that σ is a cocycle also show that σ' is a cocycle and, clearly, $q_n^{\#}(\sigma) = f_{\#}^n(\sigma')$, where f^n is the quotient map from $\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$.

Proposition 3.14. *Suppose that Γ is a discrete group and $[\sigma] \in H^2(\Gamma; \mathbb{Z})$. Let the central extension corresponding to $[\sigma]$ be as follows:*

$$e \longrightarrow \mathbb{Z} \xrightarrow{\iota} \widetilde{\Gamma} \xrightarrow{\varphi} \Gamma \longrightarrow e.$$

The following are equivalent:

(1) *$[\sigma]$ is of finite type;*

(2) *there are infinitely many $n \in \mathbb{N}$ so that there is a finite quotient $\widetilde{Q}_n$ of $\widetilde{\Gamma}$ so that $\iota(1)$ has order n in the quotient;*

(3) *there are infinitely many $n \in \mathbb{N}$ so that Γ has a finite quotient Q_n so that $[\sigma]$ is of n-Q_n type.*

Proof. (1) $\Longrightarrow$ (2): Let Γ_k be a sequence of subgroups as in Definition 3.9 and assume that they are normal as in Remark 3.11. For $\ell \in \mathbb{N}$, pick Γ_k so that $\iota(1)^{\ell!} \notin \Gamma_k$. Then, let n be the order of $\iota(1)$ in $\widetilde{\Gamma}/\Gamma_k$. Call $\widetilde{Q}_n = \widetilde{\Gamma}/\Gamma_k$. Note that $n > \ell$, so letting $\ell \to \infty$ we get the desired family of subgroups for infinitely many distinct $n \in \mathbb{N}$.

(2) $\Longrightarrow$ (1): Let Γ_n be the kernel of the map from $\widetilde{\Gamma}$ to $\widetilde{Q}_n$. Then, for any $\ell \in \mathbb{Z} \setminus \{0\}$, we can pick $n > |\ell|$ so that $\iota(1)^\ell \notin \Gamma_n$.

(2) $\Longrightarrow$ (3): By assumption, for infinitely many n, we have the diagram

$$\begin{array}{ccccccccc} e & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota} & \widetilde{\Gamma} & \xrightarrow{\varphi} & \Gamma & \longrightarrow & e \\ & & \big\downarrow{\scriptstyle f^n} & & \big\downarrow{\scriptstyle \tilde{q}_n} & & \big\downarrow{\scriptstyle q_n} & & \\ e & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \widetilde{Q}_n & \xrightarrow{\varphi_n'} & Q_n & \longrightarrow & e. \end{array} \tag{3.1}$$

Then, we may factor the vertical maps as follows:

$$\begin{array}{ccccccccc} e & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota} & \widetilde{\Gamma} & \xrightarrow{\varphi} & \Gamma & \longrightarrow & e \\ & & \big\downarrow{\scriptstyle f^n} & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\ e & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \widetilde{\Gamma}/\langle \iota(1)^n \rangle & \longrightarrow & \Gamma & \longrightarrow & e \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow & & \big\downarrow{\scriptstyle q_n} & & \\ e & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \widetilde{Q}_n & \xrightarrow{\varphi_n'} & Q_n & \longrightarrow & e. \end{array} \tag{3.2}$$

By [1, Chapter IV.3, Exercise 1], $f_*^n([\sigma])$ corresponds to the middle row of diagram (3.2). Let $[\sigma']$ be the element of $H^2(Q_n; \mathbb{Z}/n\mathbb{Z})$ corresponding to the bottom row of diagram (3.2). Again, using [1, Chapter IV.3, Exercise 1], we get that $q_n^*([\sigma'])$ corresponds to the middle row as well. It follows that $f_*^n([\sigma]) = q_n^*([\sigma'])$.

(3) $\Longrightarrow$ (2): Using [1, Chapter IV.3, Exercise 1] as we did for the other direction, we get that diagram (3.2) must exist for infinitely many n. Composing the vertical maps, we get that quotients such as diagram (3.1) must exist for infinitely many n as well. ∎

This theorem motivates a definition that extends the finite type concept to cocycle representatives of the cohomology class.

Definition 3.15. If σ is a $\mathbb{Z}$-valued 2-cocycle on Γ, we say that σ is of *finite type* if for infinitely many $n \in \mathbb{N}$, there is a finite quotient Q_n of Γ so that σ is of n-Q_n type.

The cocycle defined in Example 3.13 is of finite type. From Proposition 3.14, it follows that its cohomology class is as well.

Remark 3.16. Obviously, if σ is of n-Q type, then so is $[\sigma]$. Conversely, if $[\sigma] \in H^k(\Gamma;\mathbb{Z})$ is of n-Q type, then there exists some $\omega \in [\sigma]$ that is of n-Q type. To show this, let $q : \Gamma \to Q$ be the quotient map and let $f : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the usual map. We can take $\dot\omega \in f_*([\sigma])$ so that $\dot\omega \in \operatorname{im}(q^\#)$. Let $\dot\omega = \partial\dot\alpha + f_\#(\sigma)$, where $\dot\alpha \in C^1(\Gamma;\mathbb{Z}/n\mathbb{Z})$. Then, find $\alpha \in C^1(\Gamma;\mathbb{Z})$ so that $f_\#(\alpha) = \dot\alpha$. Then, letting $\omega = \partial\alpha + \sigma$, we see that $f_\#(\omega) = \dot\omega \in \operatorname{im}(q^\#)$. Thus, ω is of n-Q type. However, a finite-type cohomology class does *not* obviously have a finite-type representative, because the choice of representative of $[\sigma]$ might depend on n.

3.4. Constructing asymptotic representations from cocycles

We start by defining our formula for an asymptotic representation.

Proposition 3.17. *Suppose that Γ is a discrete group and $[\sigma] \in H^2(\Gamma;\mathbb{Z})$. Let Q_n be a finite quotient of Γ so that $[\sigma]$ is of n-Q_n type; this exists by Remark 3.16. Let σ_n be a representative of $[\sigma]$ so that σ_n is of n-Q_n type. Let α_n be a 1-cochain so that $\sigma_n + \partial\alpha_n = \sigma$. Let $\widehat{\chi}_n(g_1,g_2) = \exp(\frac{2\pi i}{n}(\alpha_n(g_1) + \sigma_n(g_1,g_2)))$. Then, define $V_n = \ell^2(Q_n)$. Treat $\overline{g}$ as a basis element for V_n, where $g \in \Gamma$ and $\overline{g}$ is its image in Q_n. Then, there is a well-defined function $\rho_n : \Gamma \to U(V_n)$ that obeys the formula*

$$\rho_n(g_1)\overline{g}_2 = \widehat{\chi}_n(g_1,g_2)\overline{g}_1\overline{g}_2.$$

Proof. We will show that ρ_n is well-defined. Suppose that $\overline{g}_2 = \overline{g_2'}$. If we show that

$$\sigma_n(g_1,g_2) \equiv \sigma_n(g_1,g_2') \mod n,$$

we will have shown that ρ_n is well-defined, because $\widehat{\chi}_n$ only depends on σ_n up to equivalence mod n. If $\dot\sigma_n$ is σ_n reduced mod n, then we have by assumption that $\dot\sigma_n = q^\#(\sigma_n')$, where q is the quotient map from Γ to Q and σ_n' is a 2-cocycle in $C^2(Q;\mathbb{Z}/n\mathbb{Z})$. Thus,

$$\dot\sigma_n(g_1,g_2) = \sigma_n'(\overline{g}_1,\overline{g}_2) = \sigma_n'(\overline{g}_1,\overline{g_2'}) = \dot\sigma_n(g_1,g_2').$$

Note that $\rho_n(g)$ maps the orthonormal basis $\{h : h \in Q_n\}$ to another orthonormal basis, so it is unitary. ∎

Definition 3.18. Suppose $\rho : \Gamma \to U(k)$ and χ is an S^1-valued 2-cocycle on Γ. If ρ obeys the formula $\rho(gh)\rho(h)^{-1}\rho(g)^{-1} = \chi(g,h)\,\mathrm{id}_{\mathbb{C}^n}$, it is called a *projective representation* with cocycle χ.

We will show that ρ_n is a projective representation in Lemma 3.20.

In the case that $\alpha_n = 0$, our formula reduces to what is known as the *projective left regular representation* for Q_n and χ_n; for example, see [18, p. 2]. The proof of [12, Corollary B] also uses a cohomology class that "behaves well" with respect to finite quotients, to make projective representations.

We now give an alternate justification for the formula. Suppose that there is a finite quotient of a central extension of Γ as follows:

$$\begin{array}{ccccccccc} e & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota} & \widetilde{\Gamma} & \underset{\theta}{\overset{\varphi}{\rightleftarrows}} & \Gamma & \longrightarrow & e \\ & & \big\downarrow f^n & & \big\downarrow \widetilde{q}_n & & \big\downarrow q_n & & \\ e & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\iota'} & \widetilde{Q}_n & \xrightarrow{\varphi_n'} & Q_n & \longrightarrow & e, \end{array}$$

and a set-theoretic section θ of the extension. Let π_n be the induced representation of $\widetilde{Q}_n$ from the character on $\iota'(\mathbb{Z}/n\mathbb{Z})$ that takes $\iota'(1) \mapsto \exp(2\pi i/n)$. Then, $\rho_n = \pi_n \circ \widetilde{q}_n \circ \theta$. Deriving the formula from here is technical.

The discussion at the start of [13, Chapter 3.3] explains how one should expect a projective representation to come from a splitting and representation of $\widetilde{\Gamma}$ as described above.

Remark 3.19. The existence proof technically only uses the fact that $\dot{\sigma}(g_1, g_2)$ depends only on g_1 and the reduction of g_2 in Q_n, rather than the image of both g_1 and g_2 in Q_n. We will use this fact in examples to reduce the asymptotics of the dimension of the asymptotic representation.

Lemma 3.20. *Let Γ, n, σ, Q_n, and ρ_n be as above. Define $\chi_n \in C^2(\Gamma; S^1)$ by*

$$\chi_n(g_1, g_2) = \exp\Big(\frac{2\pi i}{n}\sigma(g_1, g_2)\Big).$$

Then, ρ_n obeys the formula

$$\rho_n(g_1 g_2)\rho_n(g_2)^{-1}\rho_n(g_1)^{-1} = \chi_n(g_1, g_2)^{-1}\,\mathrm{id}_{V_n}.$$

Proof. Define $\widehat{\sigma}_n(g_1, g_2) = \alpha_n(g_1) + \sigma_n(g_1, g_2)$. We compute

$$\begin{aligned} -\widehat{\sigma}_n(g_1, g_2 g_3) &+ \widehat{\sigma}_n(g_1 g_2, g_3) - \widehat{\sigma}_n(g_2, g_3) \\ &= \alpha_n(g_1 g_2) - \alpha_n(g_1) - \alpha_n(g_2) - \sigma_n(g_1, g_2 g_3) + \sigma_n(g_1 g_2, g_3) - \sigma_n(g_2, g_3) \\ &= -\partial\alpha_n(g_1, g_2) - \sigma_n(g_1, g_2) = -\sigma(g_1, g_2). \end{aligned} \tag{3.3}$$

The second equality follows from the fact that σ_n is a cocycle. Exponentiating both sides of (3.3), we get

$$\widehat{\chi}_n(g_1, g_2 g_3)^{-1}\widehat{\chi}_n(g_1 g_2, g_3)\widehat{\chi}_n(g_2, g_3)^{-1} = \chi_n(g_1, g_2)^{-1}. \tag{3.4}$$

Next we claim that

$$\rho_n(g_1)^{-1}\overline{g}_2 = \widehat{\chi}_n(g_1, g_1^{-1}g_2)^{-1}\overline{g_1^{-1}g_2}.$$

To check this, it suffices to compute that

$$\begin{aligned}\rho_n(g_1)\widehat{\chi}_n(g_1, g_1^{-1}g_2)\overline{g_1^{-1}g_2} &= \widehat{\chi}_n(g_1, g_1^{-1}g_2)^{-1}\rho_n(g_1)\overline{g_1^{-1}g_2}\\ &= \widehat{\chi}_n(g_1, g_1^{-1}g_2)^{-1}\widehat{\chi}_n(g_1, g_1^{-1}g_2)\overline{g}_2 = \overline{g}_2.\end{aligned}$$

Using this, we can compute

$$\begin{aligned}\rho_n(g_1g_2)\rho_n(g_2)^{-1}\rho_n(g_1)^{-1}\overline{g}_3 &= \rho_n(g_1g_2)\rho_n(g_2)^{-1}\widehat{\chi}_n(g_1, g_1^{-1}g_3)^{-1}\overline{g_1^{-1}g_3}\\ &= \rho_n(g_2g_2)\widehat{\chi}_n(g_2, g_2^{-1}g_1^{-1}g_3)^{-1}\widehat{\chi}_n(g_1, g_1^{-1}g_3)^{-1}\overline{g_2^{-1}g_1^{-1}g_3}\\ &= \widehat{\chi}_n(g_1g_2, g_2^{-1}g_1^{-1}g_3)\widehat{\chi}_n(g_2, g_2^{-1}g_1^{-1}g_3)^{-1}\widehat{\chi}_n(g_1, g_1^{-1}g_3)^{-1}\overline{g}_3\\ &= \chi_n(g_1, g_2)^{-1}\overline{g}_3.\end{aligned}$$

Here the last step follows from (3.4) applied to g_1, g_2, and $g_2^{-1}g_2^{-1}g_3$. ■

Now we are ready to prove Theorem 1.1.

Proof. First note that by Proposition 3.14 and Remark 3.16, there are infinitely many n, so the formula given in Proposition 3.17 is well-defined.

Now we will show that ρ_n is asymptotically multiplicative. Noting that, since σ does not depend on n, we have that $\chi_n(g_1, g_2)$, defined as in Lemma 3.20, goes to 1 as n goes to infinity. Thus, Lemma 3.20 implies asymptotic multiplicativity.

Now we will show that for large enough n, ρ_n is not close to any genuine representation of Γ on a particular finite subset of Γ. From the fact that $[\sigma] \notin \ker(h)$, there is some 2-cycle $c \in C_2(\Gamma)$ written as

$$c = \sum_{i=1}^{N} x_i[a_i|b_i]$$

so that

$$\langle\sigma, c\rangle \neq 0.$$

Then, we compute that

$$\begin{aligned}(\rho_n, c) &= \frac{1}{2\pi i}\sum_{j=1}^{N} x_j \operatorname{Tr}(\log(\rho_n(a_jb_j)\rho_n(b_j)^{-1}\rho_n(a_j)^{-1}))\\ &= \frac{1}{2\pi i}\sum_{j=1}^{N} x_j \operatorname{Tr}(\log(\chi_j(a_jb_j)^{-1})\operatorname{id}_{V_n}) \quad \text{(by Lemma 3.20)}\\ &= -\frac{1}{2\pi i}\sum_{j=1}^{N} x_j \frac{2\pi i}{n}\sigma(a_j, b_j)\operatorname{Tr}(\operatorname{id}_{V_n}) = -\langle\sigma, c\rangle\frac{\dim(V_n)}{n} \neq 0.\end{aligned}$$

By Theorem 3.7, it follows that ρ_n cannot be within $\frac{1}{24}$ of a genuine representation on the boundary support of c and, thus, cannot be perturbed to a genuine representation. ■

Remark 3.21. If Γ is finitely generated, then pairing non-trivially with a cohomology class is equivalent to being non-torsion. To see this, note that from the universal coefficient theorem (see [16, Theorem 53.1]), we have a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_1(\Gamma), \mathbb{Z}) \longrightarrow H^2(\Gamma; \mathbb{Z}) \longrightarrow \mathrm{Hom}(H_2(\Gamma), \mathbb{Z}) \longrightarrow 0.$$

Clearly, any torsion element cannot fit the condition, since $\mathrm{Hom}(H_2(\Gamma), \mathbb{Z})$ is a torsion-free group. To see the converse, note that since Γ is finitely generated, $H_1(\Gamma) \simeq \Gamma/[\Gamma, \Gamma]$ [1, p. 36] is finitely generated as well. Thus, all elements of $\mathrm{Ext}(H_1(\Gamma), \mathbb{Z}))$ have finite order by [16, Theorem 52.3] and the table on [16, p. 331].

Remark 3.22. In many examples we will have the stronger condition that there is a particular cocycle representative σ of $[\sigma]$ that is of finite type. In this case the formula simplifies to

$$\rho_n(g_1)\overline{g}_2 = \chi_n(g_1, g_2)\overline{g}_1\overline{g}_2,$$

where $\chi_n(g_1, g_2) = \exp(\frac{2\pi i}{n}\sigma(g_1, g_2))$.

Corollary 3.23. *Suppose that Γ is a virtually polycyclic group with non-torsion 2-cohomology. Then, Γ meets the conditions of Theorem 1.1 and is thus not matricially stable.*

Proof. First we note that if $[\sigma]$ is a non-torsion cohomology class, it is not in the kernel of the map from $H^2(\Gamma; \mathbb{Z})$ to $\mathrm{Hom}(H_2(\Gamma), \mathbb{Z})$, by Remark 3.21. Let

$$e \longrightarrow \mathbb{Z} \xrightarrow{\iota} \widetilde{\Gamma} \xrightarrow{\varphi} \Gamma \longrightarrow e$$

be the central extension corresponding to $[\sigma]$. We have that $\widetilde{\Gamma}$ is also virtually polycyclic, by Proposition 2.7. Now because $\widetilde{\Gamma}$ is virtually polycyclic, it is residually finite [9, Theorem 3]. Thus, $[\sigma]$ is of finite type. ∎

4. Torsion-free finitely generated nilpotent groups

The purpose of this section is to provide an alternate proof that torsion-free finitely generated nilpotent groups fit the conditions of Theorem 1.1 (Theorem 4.2). While this follows from Corollary 3.23, the alternate proof gives rise to a simple formula for the asymptotic representation.

Proposition 4.1. *Suppose that Γ is a torsion-free finitely generated nilpotent group with a Mal'cev basis $(a_1, \ldots, a_m)$ and a central extension given by*

$$e \longrightarrow \mathbb{Z} \xrightarrow{\iota} \widetilde{\Gamma} \xrightarrow{\varphi} \Gamma \longrightarrow e.$$

Then, if $\widetilde{a}_i$ is a lift of a_i for $i \in \{1, \ldots, m\}$ and $\widetilde{a}_{m+1} = \iota(1)$, $(\widetilde{a}_1, \ldots, \widetilde{a}_{m+1})$ is a Mal'cev basis for $\widetilde{\Gamma}$.

Proof. Let θ be the set-theoretic section for φ defined by $\theta(a_1^{x_1}\cdots a_m^{x_m}) = \widetilde{a}_1^{x_1}\cdots\widetilde{a}_m^{x_m}$. First, we claim that any element $g \in \widetilde{\Gamma}$ can be written in the form $\widetilde{a}_1^{x_1}\cdots\widetilde{a}_m^{x_m}\cdot\widetilde{a}_{m+1}^{x_{m+1}}$. To see this, note that $g = (\theta\circ\varphi(g))\widetilde{a}_{m+1}^{x_{m+1}}$. Then, the claim follows from the definition of θ. Next, to show that this is unique, we suppose that $\widetilde{a}_1^{x_1}\cdots\widetilde{a}_{m+1}^{x_{m+1}} = \widetilde{a}_1^{y_1}\cdots\widetilde{a}_{m+1}^{y_{m+1}}$. Noting that

$$a_1^{x_1}\cdots a_m^{x_m} = \varphi(\widetilde{a}_1^{x_1}\cdots\widetilde{a}_{m+1}^{x_{m+1}}) = \varphi(\widetilde{a}_1^{y_1}\cdots\widetilde{a}_{m+1}^{y_{m+1}}) = a_1^{y_1}\cdots a_m^{y_m},$$

we get that $x_i = y_i$ for $i \neq m+1$. Then, equality for $i = m+1$ follows by canceling the other terms and noting that $\widetilde{a}_{m+1}$ is non-torsion. Next, we define $\widetilde{\Gamma}_i = \langle\widetilde{a}_i,\ldots,\widetilde{a}_{m+1}\rangle$ and similarly define $\Gamma_i = \langle a_i,\ldots,a_m\rangle$. By our assumptions, the Γ_is form a central series for Γ. Note that for $i \leq m$,

$$[\widetilde{\Gamma},\widetilde{\Gamma}_i] \subseteq \varphi^{-1}(\varphi([\widetilde{\Gamma},\widetilde{\Gamma}_i])) \subseteq \varphi^{-1}([\Gamma,\Gamma_i]) \subseteq \varphi^{-1}(\Gamma_{i+1}) \subseteq \langle\widetilde{a}_{m+1},\widetilde{\Gamma}_{i+1}\rangle = \widetilde{\Gamma}_{i+1}.$$

For $i = m+1$, we have $[\widetilde{\Gamma},\widetilde{\Gamma}_{m+1}] = \{e\} = \widetilde{\Gamma}_{m+2}$. This shows that $\widetilde{\Gamma}_i$ is a central series, which completes our proof. ■

Theorem 4.2. *Suppose that Γ is a torsion-free finitely generated nilpotent group that is not $\mathbb{Z}$ or trivial. Then, Γ has a cohomology class that meets the conditions of Theorem 1.1, and the asymptotic representation can be expressed as follows: Γ can be viewed as $\mathbb{Z}^m$ with multiplication defined by*

$$\mathbf{x} * \mathbf{y} = (\eta_1(\mathbf{x},\mathbf{y}),\ldots,\eta_m(\mathbf{x},\mathbf{y})),$$

where $\eta_1,\ldots,\eta_n$ are rational[3] polynomials in $\mathbf{x} = (x_1,\ldots,x_m)$ and $\mathbf{y} = (y_1,\ldots,y_m)$. In addition, we have a non-torsion cocycle $\sigma(\mathbf{x},\mathbf{y})$ that is also a rational polynomial in the entries. Then, the underlying vector space is $(\mathbb{C}^n)^{\otimes m}$. Then, for n co-prime to the denominators of coefficients in the η_is and σ, we have

$$\rho_n(\mathbf{x})e_{y_1}\otimes e_{y_2}\otimes\cdots\otimes e_{y_m} = \exp\Big(\frac{2\pi i}{n}\sigma(\mathbf{x},\mathbf{y})\Big)e_{\eta_1(\mathbf{x},\mathbf{y})}\otimes\cdots\otimes e_{\eta_m(\mathbf{x},\mathbf{y})},$$

where we have the convention that $e_{j+n} = e_j$.

Proof. By Corollary 2.10, there is a cohomology class $[\sigma] \in H^2(\Gamma;\mathbb{Z})$ which pairs non-trivially with a homology class. By [1, Theorem IV.3.12], we have the following central extension corresponding to $[\sigma]$:

$$e \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma} \xrightarrow{\ \varphi\ } \Gamma \longrightarrow e.$$

By Proposition 4.1, we have a Mal'cev basis $(\widetilde{a}_1,\ldots,\widetilde{a}_{m+1})$ for $\widetilde{\Gamma}$ and a Mal'cev basis $(a_i,\ldots,a_m)$ for Γ where for $i \neq m+1$, we have $\varphi(\widetilde{a}_i) = a_i$. Any element in $\widetilde{\Gamma}$ can be written uniquely as $\widetilde{a}_1^{x_1}\cdots\widetilde{a}_{m+1}^{x_{m+1}}$. It follows that

$$\widetilde{a}_1^{x_1}\cdots\widetilde{a}_{m+1}^{x_{m+1}}\cdot\widetilde{a}_1^{y_1}\cdots\widetilde{a}_{m+1}^{y_{m+1}} = \widetilde{a}_1^{z_1}\cdots\widetilde{a}_{m+1}^{z_{m+1}},$$

[3] The polynomials will always take integer values if given integer inputs, but in general the coefficients may not be integers.

where z_i is a function of $x_1, \ldots, x_{m+1}, y_1, \ldots y_{m+1}$. Hall has shown that these functions are rational polynomials [7, Theorem 6.5]. They can be computed by the methods given in [15]. Then, if we pick a section $\theta : a_1^{x_1} \cdots a_m^{x_m} \mapsto \widetilde{a}_1^{x_1} \cdots \widetilde{a}_m^{x_m}$, by [1, IV (3.3)], there is a cocycle σ' with $[\sigma] = [\sigma']$ so that

$$\theta(g)\theta(h) = \theta(gh)\widetilde{a}_{m+1}^{\sigma'(g,h)}.$$

Since $\theta(gh)$ is in canonical form and has no power of $\widetilde{a}_{m+1}$, it follows that $\sigma'(g,h)$ must be a rational polynomial of the powers in the canonical forms of g and h. Note that by writing elements of Γ in canonical form, we get that Γ can be viewed as $\mathbb{Z}^m$ with multiplication given by polynomial formulas of the entries. Thus, if we take n to be co-prime to the denominator of each polynomial in the multiplication for Γ and the denominator of σ', we may define a quotient Q_n of Γ by reducing each entry of Γ mod n. Then, we may also reduce the formula for σ' mod n, thus showing that σ' is of n-Q_n type. To justify the formula, note that $\ell^2(Q_n) \simeq (\mathbb{C}^n)^{\otimes n}$ and the isomorphism sends $(y_1, \ldots, y_m) \mapsto e_{y_1} \otimes \cdots \otimes e_{y_m}$. Then, applying Proposition 3.17, we get the formula mentioned here, with $\sigma_n = \sigma$, and so $\alpha_n = 0$. ∎

Remark 4.3. Additionally, we may modify this construction to get an asymptotic representation of dimension n^{m-1} instead of dimension n^m. Note that the power of $\widetilde{a}_{m+1}$ in the product $(\widetilde{a}_1^{x_1} \cdots \widetilde{a}_m^{x_m})(\widetilde{a}_1^{y_1} \cdots \widetilde{a}_m^{y_m})$ can be computed by using the relations to put all terms in order. If we leave the $\widetilde{a}_m^{y_m}$ term at the end until the last step, we notice that we may have to switch the positions of the $\widetilde{a}_m$ and $\widetilde{a}_{m+1}$, but these commute because the extension is central. From this, it follows that σ does not depend on y_m. Thus, we may replace our quotient Q_n with $Q_n' = Q_n/\langle q_n(a_m)\rangle$ and by Remark 3.19, we may use the formula for ρ_n except with Q_n' instead of Q_n. The rest of the proof for asymptotic multiplicativity and non-perturbability flows the same way. An alternative explanation is that because $\sigma(\mathbf{x}, \mathbf{y})$ does not depend on y_m, the formula for ρ_n commutes with the projection $r_n = \mathrm{id}_{\mathbb{C}^n} \otimes \cdots \otimes \mathrm{id}_{\mathbb{C}^n} \otimes p_n$, where p_n is defined by the formula $p_n e_i = \sum_j \frac{1}{\sqrt{n}} e_j$. Thus, $r_n \rho_n r_n$ defines an n^{m-1}-dimensional asymptotic representation.

5. Examples

5.1. $\mathbb{Z}^2$ Revisited

In this subsection we will apply our results to $\mathbb{Z}^2$, the simplest non-trivial example. We will compare the result of our algorithm to the classical results and show that we get Voiculescu's matrices tensored against another representation. Using Remark 4.3, we obtain Voiculescu's matrices precisely. In Example 3.13 we have shown that

$$\sigma((x_1, x_2), (y_1, y_2)) = x_2 y_1$$

is a cocycle on $\mathbb{Z}^2$ and

$$c = [(0,1)|(1,0)] - [(1,0)|(0,1)]$$

is a 2-chain such that

$$\langle \sigma, c \rangle = 1.$$

Moreover, since σ is a polynomial with integer coefficients, it follows that σ is $\mathbb{Z}/n\mathbb{Z}$-compatible with $(\mathbb{Z}/n\mathbb{Z})^2$. Then, applying Theorem 4.2, we get that ρ_n acts on $V_n = \ell^2((\mathbb{Z}/n\mathbb{Z})^2) \simeq \mathbb{C}^n \otimes \mathbb{C}^n$. Then, pick the basis $\{e_j \otimes e_k\}$ with e_j defined for all $j \in \mathbb{Z}$ by the formula $e_{j+n} = e_j$. Using the formula for Theorem 4.2, we get that

$$\rho_n(a,b)e_j \otimes e_k = \exp\Big(\frac{2\pi i}{n} bj\Big) e_{a+j} \otimes e_{b+k}.$$

Note that we may write $\rho_n(a,b) = \rho_n^1(a,b) \otimes \rho_n^2(a,b)$, where

$$\rho_n^1(a,b)e_j = \exp\Big(\frac{2\pi i}{n} bj\Big) e_{a+j}$$

and

$$\rho_n^2(a,b)e_j = e_{b+j}.$$

Note that ρ_n^1 is precisely the asymptotic representation given by Voiculescu's matrices, while ρ_n^2 is a genuine representation. This is unsurprising because Remark 4.3 allows us to reduce the dimension by "ignoring" the second tensor coordinate and the resulting formula is precisely ρ_n^1.

5.2. A 3-step nilpotent group

Consider the group Γ generated by $a_1, \ldots, a_5$ with the following relations:

$$\begin{aligned} a_2 a_1 &= a_1 a_2 a_3, \\ a_3 a_1 &= a_1 a_3 a_4^2, \\ a_3 a_2 &= a_2 a_3 a_5, \\ a_i a_j &= a_j a_i \quad \text{for all other pairs } \{i, j\}. \end{aligned}$$

We first state a simplified version of our formula, then we explain how to compute it. We first compute the general version, then explain how it simplifies in this case. Our asymptotic representation is defined for n co-prime to 6 and sends generators to the following $\mathbb{C}^n$-spanning e_i with $i \in \mathbb{Z}$ and $e_{i+n} = e_i$:

$$\begin{aligned} \rho_n'(a_1)e_j &= e_{j+1}, \\ \rho_n'(a_2)e_j &= \exp\Big(\frac{4\pi i}{n}\binom{j}{3}\Big)e_j, \\ \rho_n'(a_3)e_j &= \exp\Big(\frac{4\pi i}{n}\binom{j}{2}\Big)e_j, \\ \rho_n'(a_4)e_j &= \exp\Big(\frac{2\pi i}{n} j\Big)e_j, \\ \rho_n'(a_5) &= \mathrm{id}_{\mathbb{C}^n}, \end{aligned}$$

and sends the element written uniquely in the form $a_1^{x_1}\cdots a_5^{x_5}\mapsto \rho_n'(a_1)^{x_1}\cdots\rho_n'(a_5)^{x_5}$. Here $\binom{j}{3}$ is the polynomial $\frac{1}{6}j(j-1)(j-2)$. This group can be concretely realized as $\mathbb{Z}^5$ with multiplication given by

$$(x_1,\ldots,x_5)*(y_1,\ldots y_5)=(\eta_1(\mathbf{x},\mathbf{y}),\ldots,\eta_5(\mathbf{x},\mathbf{y})),$$

where

$$\begin{aligned}
\eta_1(\mathbf{x},\mathbf{y}) &= x_1+y_1,\\
\eta_2(\mathbf{x},\mathbf{y}) &= x_2+y_2,\\
\eta_3(\mathbf{x},\mathbf{y}) &= x_3+y_3+x_2y_1,\\
\eta_4(\mathbf{x},\mathbf{y}) &= x_4+y_4+2x_3y_1+2x_2\binom{y_1}{2},\\
\eta_5(\mathbf{x},\mathbf{y}) &= x_5+y_5+y_1\binom{x_2}{2}+x_3y_2+x_2y_1y_2,
\end{aligned}$$

by the isomorphism $a_1^{x_1}\cdots a_5^{x_5}\mapsto(x_1,\ldots,x_5)$. In general, these polynomials may be calculated by the methods given in [15]. We will explain how to verify these polynomials using a computer. There is a full description of the code in the appendix, but we will summarize the main steps here.

(1) Verify that the operation "$*$" defined above is associative.

(2) Calling a_i the element with a 1 in the ith entry, and zeroes elsewhere, verify that $a_1^{x_1}\cdots a_5^{x_5}=(x_1,\ldots,x_5)$ under the operation $*$.

(3) Use $*$ to compute $a_5^{-x_5}a_4^{-x_4}\cdots a_1^{-x_1}$.

(4) Verify that the formula computed in the previous step is both a left and a right inverse to $(x_1,\ldots,x_5)$.

(5) Verify that $a_1,\ldots,a_5$ satisfies the relations of the group.

In order to compute a non-torsion cocycle, we will develop one as a central extension. We will do this by "blowing up" the relation $[a_4,a_1]=e$. Thus, we get a group $\widetilde{\Gamma}$ generated by $\widetilde{a}_1,\ldots,\widetilde{a}_6$ with the relations

$$\begin{aligned}
\widetilde{a}_2\widetilde{a}_1 &= \widetilde{a}_1\widetilde{a}_2\widetilde{a}_3,\\
\widetilde{a}_3\widetilde{a}_1 &= \widetilde{a}_1\widetilde{a}_3\widetilde{a}_4^2,\\
\widetilde{a}_3\widetilde{a}_2 &= \widetilde{a}_2\widetilde{a}_3\widetilde{a}_5,\\
\widetilde{a}_4\widetilde{a}_1 &= \widetilde{a}_1\widetilde{a}_4\widetilde{a}_6,\\
\widetilde{a}_i\widetilde{a}_j &= \widetilde{a}_j\widetilde{a}_i \quad \text{for all other pairs } \{i,j\}.
\end{aligned}$$

Remark 5.1. The reader is warned that such an extension cannot be made for any homogeneous relation in any torsion-free finitely generated nilpotent group. Consider the group Λ generated by $b_1, \ldots, b_4$ with the relations

$$[b_1, b_2] = b_3,$$
$$[b_i, b_j] = e \quad \text{for all other pairs } \{i, j\}.$$

In this case the relation $[b_3, b_4] = 1$ follows from the other three relations, so we cannot construct a central extension by "blowing it up." We will explain why this issue does not arise in our example below. In general, an algorithm for finding when relations of this form make a nilpotent group where each "a_i" has infinite order is described in [20, Propositions 9.8.3 and 9.9.1].

Then, $\widetilde{\Gamma}$ can be identified with $\mathbb{Z}^6$ with multiplication given by

$$(x_1, \ldots, x_6) * (y_1, \ldots, y_6) = (\gamma_1(\mathbf{x}, \mathbf{y}), \ldots, \gamma_6(\mathbf{x}, \mathbf{y})),$$

where

$$\gamma_i(\mathbf{x}, \mathbf{y}) = \eta_i(x_1, \ldots x_5, y_1, \ldots y_5) \quad \text{for } i < 6,$$
$$\gamma_6(\mathbf{x}, \mathbf{y}) = x_6 + y_6 + x_4 y_1 + 2x_3 \binom{y_1}{2} + 2x_2 \binom{y_1}{3}.$$

We have verified the fact that these polynomials give rise to a group operation satisfying the relations of the group with similar code to what we used to verify these things for Γ. Since the element $(0, 0, 0, 0, 0, 1)$ has infinite order in the group determined by these polynomials, it follows that $\widetilde{a}_6$ has infinite order as well.

From this, it follows that a cocycle corresponding to the central extension is given by

$$\sigma(\mathbf{x}, \mathbf{y}) = x_4 y_1 + 2x_3 \binom{y_1}{2} + 2x_2 \binom{y_1}{3}.$$

Let c be the 2-cycle

$$c = [a_1|a_4] - [a_4|a_1] = [(1, 0, 0, 0, 0)|(0, 0, 0, 1, 0)] - [(0, 0, 0, 1, 0)|(1, 0, 0, 0, 0)].$$

Then, $\langle \sigma, c \rangle = 1$.

For any n co-prime to 6, we may define

$$Q_n = (\mathbb{Z}/n\mathbb{Z})^5$$

with multiplication given by

$$(x_1, \ldots, x_n) * (y_1, \ldots, y_n) = (\overline{\eta}_1(\mathbf{x}, \mathbf{y}), \ldots, \overline{\eta}_5(\mathbf{x}, \mathbf{y})),$$

where $\overline{\eta}_i$ is η_i with each coefficient reduced mod n.[4] Then, reducing each coefficient of σ mod n, we get $\overline{\sigma}$. Note that the fact that $\overline{\sigma}$ is a $\mathbb{Z}/n\mathbb{Z}$-valued cocycle on Q_n implies

[4] Any $\frac{1}{2}$ coefficient will be the corresponding inverse of 2 mod n, which exists since n is co-prime to 6.

that σ is of Q_n-n type. Then, we can use the formula from Remark 3.22 to define our asymptotic representation. Note that $\ell^2(Q_n) \simeq \ell^2(\mathbb{Z}/n\mathbb{Z})^{\otimes 5}$. Treat $\{e_i\}_{i=0,\dots,n-1}$ as a basis for $\ell^2(\mathbb{Z}/n\mathbb{Z})$. For ease of notation we will extend e_i to be well-defined for all $i \in \mathbb{Z}$ by the formula $e_{i+n} = e_i$. Thus, we get

$$\rho_n(x_1,\dots,x_5)e_{y_1} \otimes e_{y_2} \otimes \cdots \otimes e_{y_5} = \exp\Big(\frac{2\pi i \sigma(\mathbf{x},\mathbf{y})}{n}\Big)e_{\eta_1(\mathbf{x},\mathbf{y})} \otimes e_{\eta_2(\mathbf{x},\mathbf{y})} \otimes \cdots \otimes e_{\eta_5(\mathbf{x},\mathbf{y})}.$$

Theorem 1.1 guarantees that this formula is well-defined. As in Remark 4.3, we can "ignore" the last tensor coordinate, since $\sigma(\mathbf{x},\mathbf{y})$ does not depend on y_5 and neither does η_i for $i < 5$. This gives us an n^4-dimensional asymptotic representation. In this particular case, we may go much further. It turns out that σ depends only on $\mathbf{x}$ and y_1, so we can ignore every tensor coordinate except the first. This gives us the asymptotic representation ρ'_n we introduced in the start of the section.

5.3. A polycyclic group

Let $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}$, where the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ is given by "Arnold's Cat Map"

$$1 \mapsto T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^2).$$

The generators are a_1, a_2, a_3 with the relations

$$\begin{aligned} a_2a_1 &= a_1a_2, \\ a_3a_1 &= a_1^2a_2a_3, \\ a_3a_2 &= a_1a_2a_3. \end{aligned}$$

A simplified version of our asymptotic representation is given on $\mathbb{C}^n \otimes \mathbb{C}^n$ with basis $\{e_j \otimes e_k\}_{j,k\in\mathbb{Z}}$ and the convention that $e_{j+n} = e_j$. With this notation the generators are sent to the following operators:

$$\begin{aligned} \rho'_n(a_1)e_j \otimes e_k &= e_{j+1} \otimes e_k, \\ \rho'_n(a_2)e_j \otimes e_k &= \exp\Big(\frac{2\pi i}{n}j\Big)e_j \otimes e_{k+1}, \\ \rho'_n(a_3)e_j \otimes e_k &= \exp\Big(\frac{2\pi i}{n}\Big(jk + j^2 + \frac{1}{2}k^2 - j - \frac{1}{2}k\Big)\Big)e_{2j+k} \otimes e_{j+k}. \end{aligned}$$

As in the last chapter, we will explain how to compute the asymptotic representation given by Proposition 3.17, then explain how to derive the simpler formula ρ'_n.

We will compute a non-torsion cocycle in $H^2(\Gamma;\mathbb{Z})$. We explain our reasoning about how to find the cocycle without formal proof, then show formally that it obeys the cocycle condition. The idea is as in the previous section: to find a central extension of Γ, compute the multiplication in the middle group of the central extension. We may consider the following presentation of Γ: each element in the group can be written uniquely as $a_1^{x_1}a_2^{x_2}a_3^{x_3}$ and this element will be sent to the corresponding product of matrices.

We may make an extension by "blowing up" the relation $[a_1, a_2] = 1$. Then, we may consider $\widetilde{\Gamma}$ to be the group generated by $b_1, \dots, b_4$ and the relations

$$\begin{aligned} b_2 b_1 &= b_4, \\ b_3 b_1 &= b_1^2 b_2 b_3, \\ b_3 b_2 &= b_1 b_2 b_3, \\ b_4 b_i &= b_i b_4 \quad \text{for all } i. \end{aligned}$$

Using these relations, we may write any element of $\widetilde{\Gamma}$ uniquely in the form $b_1^{x_1} \cdots b_4^{x_4}$ with $x_i \in \mathbb{Z}$. As the reader is warned in Remark 5.1, it is not always the case that "blowing up" a relation like this leads to a sensible extension. In this case when we verify the cocycle condition, we will also have verified that this makes a sensible extension.

We will describe an element of Γ implicitly by a pair (v, k) with $v = (v^1, v^2) \in \mathbb{Z}^2$ and $k \in \mathbb{Z}$, then, by the definition of the semi-direct product, the multiplication is given implicitly by $(v_1, k_1) * (v_2, k_2) = (v_1 + T^{k_1} v_2, k_1 + k_2)$. Our goal is to implicitly describe $\widetilde{\Gamma}$ similarly. To that end, we will describe an element of $\widetilde{\Gamma}$ as a triple (v, k, d) with $v = (v^1, v^2) \in \mathbb{Z}^2$ and $k, d \in \mathbb{Z}$. This represents the element $b_1^{v^1} b_2^{v^2} b_3^k b_4^d$. To that end, we make the following observations:

$$b_2^{v^2} b_1^{v^1} = b_1^{v^1} b_2^{v^2} b_3^{v^1 v^2}, \tag{5.1}$$

$$\begin{aligned} b_3 a_1^{v^1} &= (b_1^2 b_2)^{v^1} b_3 \\ &= b_1^{2v^1} b_2^{v^1} b_3 b_4^{v^1(v^1-1)}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} b_3 a_2^{v^2} &= (b_1 b_2)^{v^2} b_3 \\ &= b_1^{v^2} b_2^{v^2} b_3 b_4^{\frac{1}{2} v^2(v^2-1)}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} b_3 b_1^{v^1} b_2^{v^2} &= b_1^{2v^1} b_2^{v^1} b_1^{v^2} b_2^{v^2} b_3 a_4^{v^1(v^1-1)+\frac{1}{2} v^2(v^2-1)} \\ &= b_1^{2v^1+v^2} b_2^{v^1+v^2} b_3 a_4^{v^1(v^1-1)+\frac{1}{2} v^2(v^2-1)} b_4^{v^1 v^2}. \end{aligned} \tag{5.4}$$

These essentially describe the ways we can get "b_4 terms." We will informally refer to the contributions "(5.1) terms," "(5.2) terms," "(5.3) terms," and "(5.4) terms." The "(5.4) terms" will refer to the terms in (5.4) that do not appear in (5.2) or (5.3). In order to capture these terms, we define $\alpha, \beta : \mathbb{Q}^2 \otimes \mathbb{Q}^2 \to \mathbb{Q}$ and $\gamma : \mathbb{Q}^2 \to \mathbb{Q}$ as follows:

$$\begin{aligned} \alpha(v_1 \otimes v_2) &= v_1^1 v_2^2, \\ \beta(v_1 \otimes v_2) &= \frac{1}{2} v_1^2 v_2^2 + v_1^1 v_2^1, \\ \gamma(v_1) &= v_1^1 + \frac{1}{2} v_1^2. \end{aligned}$$

Thus, we have shown

$$(v_1, 1, 0) * (v_2, 0, 0) = (v_1 + T v_2, 1, \alpha(v_2 \otimes v_1 + v_2 \otimes v_2) + \beta(v_2 \otimes v_2) - \gamma(v_2)).$$

Note that although β and γ take rational values in general, $\beta(v_2 \otimes v_2) - \gamma(v_2)$ is always an integer. Here the input to α comes from the (5.1) and (5.4) terms while the inputs to β and γ come from the (5.2) and (5.3) terms. To compute the product in general for positive values of k_1, the steps would look like

$$\begin{aligned}(v_1, k_1, 0) * (v_2, k_2, 0) &= (v_1, k_1 - 1, 0) * (Tv_2, k_2 + 1, (\alpha + \beta)(v_2 \otimes v_2) - \gamma(v_2)) \\ &= (v_1, k_1 - 2, 0) \\ &\quad * (T^2 v_2, k_2{+}2, \alpha(v_2 \otimes v_2{+}Tv_2 \otimes Tv_2){-}\gamma(v_2 + Tv_2)) \\ &= (v_1, k_1 - 3, 0) \\ &\quad * (T^3 v_2, k_2{+}3, \alpha(v_2 \otimes v_2{+}Tv_2 \otimes Tv_2{+}T^2 v_2 \otimes T^2 v_2) \\ &\quad - \gamma(v_2 + Tv_2 + T^2 v_2)),\end{aligned}$$

and so on. This motivates the definition

$$S_k = \begin{cases} \sum_{j=0}^{k-1} (T \otimes T)^j & k \geq 0, \\ -\sum_{j=k+1}^{-1} (T \otimes T)^j & k < 0. \end{cases}$$

We cannot use the exponential sum formula to get a closed form for S_k, because $T \otimes T - 1$ is not invertible. However, $T - 1$ is invertible, so we may write a closed form for the analogue of S_k in the linear terms. We have done enough to motivate our definition of the cocycle. It comes from keeping track of each of the "b_3 terms" when computing multiplications in $\widetilde{\Gamma}$. Note an element of Γ given by $g_i = (v_i, k_i) = ((v_i^1, v_i^2), k_i)$ so that $v_i \in \mathbb{Z}^2$ represent the element $a_1^{v_i^1} a_2^{v_i^2} a_3^{k_i}$. We define

$$\begin{aligned}\sigma_1((v_1, k_1), (v_2, k_2)) &= \alpha(T^{k_1} v_2 \otimes v_1), \\ \sigma_2((v_1, k_1), (v_2, k_2)) &= \alpha((S_{k_1}(v_2 \otimes v_2)), \\ \sigma_3((v_1, k_1), (v_2, k_2)) &= \beta(S_{k_1}(v_2 \otimes v_2)), \\ \sigma_4((v_1, k_1), (v_2, k_2)) &= \gamma((T - 1)^{-1}(T^{k_1} - 1)v_2)).\end{aligned}$$

Then, we define our cocycle

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3 - \sigma_4.$$

Note that σ_4 is subtracted, unlike the others. Here σ_1 comes from the (5.1) terms, σ_2 comes from the (5.4) terms, and $\sigma_3 - \sigma_4$ comes from the (5.2) and (5.3) terms. Before we compute $\partial\sigma$, we observe the following identities about S_k:

$$\begin{aligned}S_{k_1+k_2} &= S_{k_1} + (T \otimes T)^{k_1} S_{k_2}, \\ (T \otimes T - 1)S_k &= (T \otimes T)^k - 1.\end{aligned}$$

Now we compute $\partial\sigma$ piece by piece. We will let $g_i \in \Gamma$ be represented as the pair (v_i, k_i). Then, we compute

$$\begin{aligned}\partial\sigma_1(g_1, g_2, g_3) = \alpha(&-T^{k_1} v_2 \otimes v_1 + T^{k_1}(v_2 + T^{k_2} v_3) \otimes v_1 \\ &- T^{k_1+k_2} v_3 \otimes (T^{k_1} v_2 + v_1) + T^{k_2} v_3 \otimes v_2)\end{aligned}$$

$$\begin{aligned}&= \alpha(-T^{k_1+k_2}v_3 \otimes T^{k_1}v_2 + T^{k_2}v_3 \otimes v_2)\\ &= -\alpha(((T \otimes T)^{k_1} - 1)(T^{k_2}v_3 \otimes v_2))\\ &= -\alpha((T \otimes T - 1)S_{k_1}(T^{k_2}v_3 \otimes v_2)).\end{aligned}$$

Next

$$\begin{aligned}\partial(\sigma_2+\sigma_3)(g_1, g_2, g_3) &= (\alpha+\beta)(-S_{k_1}(v_2 \otimes v_2)+S_{k_1}((v_2+T^{k_2}v_3) \otimes (v_2+T^{k_2}v_3))\\ &\quad - S_{k_1+k_2}(v_3 \otimes v_3) + S_{k_2}(v_3 \otimes v_3))\\ &= (\alpha+\beta)(-S_{k_1}(v_2 \otimes v_2)+S_{k_1}((v_2+T^{k_2}v_3) \otimes (v_2+T^{k_2}v_3))\\ &\quad - S_{k_2}(v_3 \otimes v_3) - S_{k_1}(T^{k_2}v_3 \otimes T^{k_2}v_3) + S_{k_2}(v_3 \otimes v_3))\\ &= (\alpha + \beta)(S_{k_1}(v_2 \otimes T^{k_2}v_3 + T^{k_2}v_3 \otimes v_2)).\end{aligned}$$

Finally,

$$\begin{aligned}\partial\sigma_4(g_1, g_2, g_3) = \gamma((T - 1)^{-1}&(-(T^{k_1} - 1)v_2 + (T^{k_1} - 1)(T^{k_2}v_3 + v_2)\\ &- (T^{k_1+k_2} - 1)v_3 + (T^{k_2} - 1)v_3) = 0.\end{aligned}$$

Let

$$S_{k_1}(T^{k_2}v_3 \otimes v_2) = \sum_{i=1}^{2} u_i \otimes w_i.$$

Note that since S_{k_1} commutes with the map $u \otimes w \mapsto w \otimes u$ by construction, we have that

$$S_{k_1}(v_2 \otimes T^{k_2}v_3) = \sum_{i=1}^{2} w_i \otimes u_i.$$

Now we have

$$\begin{aligned}\partial\sigma(g_1, g_2, g_3) = \sum_{i=1}^{2}(\alpha(-(T \otimes T)(u_i \otimes w_i) &+ 2u_i \otimes w_i + w_i \otimes u_i)\\ &+ \beta(u_i \otimes w_i + w_i \otimes u_i)).\end{aligned}$$

Next we see from the definition of T and α that

$$\alpha((T \otimes T)u_i \otimes w_i) = (2u_i^1 + u_i^2)(w_i^2 + w_i^1) = 2u_i^1 w_i^1 + 2u_i^1 w_i^2 + u_i^2 w_i^1 + u_i^2 w_i^2.$$

Similarly,

$$\begin{aligned}\alpha(2u_i \otimes w_i) &= 2u_i^1 w_i^2,\\ \alpha(w_i \otimes u_i) &= u_i^2 w_i^1,\\ \beta(u_i \otimes w_i + w_i \otimes u_i) = 2\beta(u_i \otimes w_i) &= 2u_i^1 w_i^1 + u_i^2 w_i^2.\end{aligned}$$

Thus, $\partial\sigma = 0$. Secondly, we have the 2-chain

$$c = [a_2|a_1] - [a_1|a_2] = [((0,1),0)|((1,0),0)] - [((1,0),0)|((0,1),0)].$$

Then, since $S_0 = 0$ and $T^0 - 1 = 0$, we have that σ_1 is the only one of the forms to pair non-trivially with c. Thus, we see that

$$\langle \sigma, c \rangle = 1.$$

We next investigate finite quotients of Γ. If $n, m \in \mathbb{N}^+$ so that the order of T reduced mod n in $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ divides m, then a finite quotient of Γ can be of the form

$$(\mathbb{Z}/n\mathbb{Z})^2 \rtimes \mathbb{Z}/m\mathbb{Z}$$

with the action described by the reduction of T mod n. Our goal is to find finite quotients Q_n of this form so that σ is n-Q_n compatible. In order to do this, note that the pair $(S_{k\pm1}, T^{k\pm1})$ can be determined from the pair (S_k, T^k) and the entries of $(S_{k\pm1}, T^{k\pm1})$ are polynomials in the entries of (S_k, T^k). These polynomials may be reduced mod n, so, it follows that if we pick m (depending on n) so that $S_m \equiv S_0 \mod n$ and $T^m \equiv 1 \mod n$, we have

$$(S_k, T^k) \equiv (S_{k+m}, T^{k+m}) \mod n$$

for all k, by induction on $|k|$. It follows that the order of the order of T reduced mod n in $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ divides m. Thus, for odd n, we define

$$Q_n = (\mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}/m\mathbb{Z}.$$

Since n is odd, we may express the $\frac{1}{2}$s in the definition of σ as the inverse of 2 mod n. We may reduce the rest of the operations mod n easily and the operators T^k and S_k are m-periodic, so the formula for σ determines a cocycle σ' which defines a cohomology class in $H^2(Q_n; \mathbb{Z}/n\mathbb{Z})$. Thus, σ is of finite type. Our formula for the representation then acts on the space $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^m$. We can get a formula from Proposition 3.17, but this formula is messy, since our definition for σ is messy. For that reason we will simply check what the generators do, as follows:

$$\rho_n(a_1) e_j \otimes e_k \otimes e_\ell = e_{j+1} \otimes e_k \otimes e_\ell,$$
$$\rho_n(a_2) e_j \otimes e_k \otimes e_\ell = \exp\Big(\frac{2\pi i}{n} j\Big) e_j \otimes e_{k+1} \otimes e_\ell,$$
$$\rho_n(a_3) e_j \otimes e_k \otimes e_\ell = \exp\Big(\frac{2\pi i}{n}\Big(jk + j^2 + \frac{1}{2}k^2 - j - \frac{1}{2}k\Big)\Big) e_{2j+k} \otimes e_{j+k} \otimes e_{\ell+1}.$$

We can see that this must be an asymptotic representation tensored against a genuine representation in the third tensor coordinate. Thus, we can pick a smaller representation by "ignoring" the e_ℓ part. This gives us the list of formulas ρ'_n from the start of the subsection.

A. Code to check multiplication polynomials

We will show Sage code that verifies these polynomials give rise to an associative relation that obeys the given presentation of the group, by the isomorphism described above. We first enter the polynomials, as shown in Figure 1.

```
Eta1(x1,x2,x3,x4,x5,y1,y2,y3,y4,y5)=x1+y1
Eta2(x1,x2,x3,x4,x5,y1,y2,y3,y4,y5)=x2+y2
Eta3(x1,x2,x3,x4,x5,y1,y2,y3,y4,y5)=x3+y3+x2*y1
Eta4(x1,x2,x3,x4,x5,y1,y2,y3,y4,y5)=x4+y4+2*x3*y1+x2*y1*(y1-1)
Eta5(x1,x2,x3,x4,x5,y1,y2,y3,y4,y5)=x5+y5+y1*x2*(x2-1)/2+x3*y2+x2*y1*y2

def Eta(x,y):
    return
[Eta1(x[0],x[1],x[2],x[3],x[4],y[0],y[1],y[2],y[3],y[4]),Eta2(x[0],x[1]
,x[2],x[3],x[4],y[0],y[1],y[2],y[3],y[4]),Eta3(x[0],x[1],x[2],x[3],x[4]
,y[0],y[1],y[2],y[3],y[4]),Eta4(x[0],x[1],x[2],x[3],x[4],y[0],y[1],y[2]
,y[3],y[4]),Eta5(x[0],x[1],x[2],x[3],x[4],y[0],y[1],y[2],y[3],y[4])]
```

Figure 1. Entering the polynomials.

Here the function "Eta" should take in 2 lists of 5 numbers or algebraic expressions (representing two elements of the group) and apply the 5 polynomials to them, outputting another list of 5 elements (representing the product of those 2 elements). Next we check associativity, as shown in Figure 2.

```
var('z1','z2','z3','z4','z5','z6')
    (z1, z2, z3, z4, z5, z6)

Associative1=Eta([x1,x2,x3,x4,x5],Eta([y1,y2,y3,y4,y5],[z1,z2,z3,z4,z5]))
Associative2=Eta(Eta([x1,x2,x3,x4,x5],[y1,y2,y3,y4,y5]),[z1,z2,z3,z4,z5])

for i in range(5):
    print(expand(Associative1[i]-Associative2[i]))
    0
    0
    0
    0
    0
```

Figure 2. Checking associativity.

```
Eta([0,0,0,0,-x5],Eta([0,0,0,-x4,0],Eta([0,0,-x3,0,0],Eta([0,-x2,0,0,0],[-x1,0,0,0,0]))))
```

```
[-x1, -x2, x1*x2 - x3, -(x1 + 1)*x1*x2 + 2*x1*x3 - x4,
-1/2*x1*(x2 + 1)*x2 + x2*x3 - x5]
```

```
InverseCheck=Eta([-x1, -x2, x1*x2 - x3, -(x1 + 1)*x1*x2 + 2*x1*x3 - x4, -1/2*x1*(x2 + 1)*x2 + x2*x3 - x5],[x1,x2,x3,x4,x5])
for entry in InverseCheck:
    print(expand(entry))
```

```
0
0
0
0
0
```

```
InverseCheck2=Eta([x1,x2,x3,x4,x5],[-x1, -x2, x1*x2 - x3, -(x1 + 1)*x1*x2 + 2*x1*x3 - x4, -1/2*x1*(x2 + 1)*x2 + x2*x3 - x5])
for entry in InverseCheck:
    print(expand(entry))
```

```
0
0
0
0
0
```

Figure 3. Checking for the existence of inverses.

```
Eta([0,1,0,0,0],[1,0,0,0,0])
```

```
[1, 1, 1, 0, 0]
```

```
Eta([0,0,1,0,0],[1,0,0,0,0])
```

```
[1, 0, 1, 2, 0]
```

```
Eta([0,0,0,1,0],[1,0,0,0,0])
```

```
[1, 0, 0, 1, 0]
```

```
Eta([0,0,0,0,1],[1,0,0,0,0])
```

```
[1, 0, 0, 0, 1]
```

Figure 4. The first few computations involved in verifying that the relations satisfy the presentation of the group.

The first line of code in Figure 2 is needed to make Sage treat $z_1, \ldots, z_6$ as algebraic expressions. The next two lines compute $\mathbf{x} * (\mathbf{y} * \mathbf{z})$ and $(\mathbf{x} * \mathbf{y}) * \mathbf{z}$, respectively. The "for" loop checks that these expressions are equivalent in each coordinate. It is easy to see by inspecting the polynomials that $(1,0,0,0,0)^{x_1} = (x_1,0,0,0,0)$, $(0,1,0,0,0)^{x_2} = (0, x_2, 0, 0, 0)$, and so on. The code in Figure 5 assumes this fact and checks that the relation $(x_1, \ldots, x_5) = a_1^{x_1} \cdots a_5^{x_5}$ makes sense with a_i corresponding to the vector that has a 1 in the ith place and zeroes elsewhere.

```
Eta([x1,0,0,0,0],Eta([0,x2,0,0,0],Eta([0,0,x3,0,0],Eta([0,0,0,x4,0],[0,0,0,0,x5]))))
    [x1, x2, x3, x4, x5]
```

Figure 5. Checking that the relation $(x_1, \ldots, x_5) = a_1^{x_1} \cdots a_5^{x_5}$ makes sense with a_i corresponding to the vector that has a 1 in the ith place and zeroes elsewhere.

Next we must check the existence of inverses; see Figure 3 for the corresponding code.

The first line of code in Figure 3 computes what the inverse of $(x_1, \ldots, x_5)$ must be if it exists. The next two lines verify that this is in fact both a left and right inverse, respectively. Finally, we need to check that these relations satisfy the presentation of the group. The first few of the relevant computations are shown in Figure 4.

These lines verify the relations $a_2 a_1 = a_1 a_2 a_3$, $a_3 a_1 = a_1 a_3 a_4^2$, and $a_5 a_1 = a_1 a_5$, respectively. The rest of the relations may be checked with similar code.

Acknowledgments. I would like to thank my advisor Marius Dadarlat for introducing me to the problem and for all his other advice. I would like to thank Ben McReynolds and Mark Pengitore for answering questions I had. Finally, I would like to thank an anonymous referee for their hard work and many helpful comments.

Funding. I would like to thank the Purdue University mathematics department for supporting me in the summer of 2021 with the summer research grant.

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Received 23 July 2022.

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