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Amenability of quadratic automaton groups

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Abstract. We give lower bounds for the electrical resistance between vertices in the Schreier graphs of the action of the linear (degree 1) and quadratic (degree 2) mother groups on the orbit of the 0-ray. These bounds, combined with results of Juschenko et al. (2016), show that every quadratic activity automaton group is amenable. The resistance bounds use an apparently new "weighted" version of the Nash-Williams criterion which may be of independent interest.

1. Introduction

Automaton groups are a rich family of groups, with a simple definition which exhibit rich behaviour. They include many groups with interesting properties, including the Grigorchuk group of intermediate growth, the Basilica group, the Hanoi towers group, lamplighter groups, and many others.

Consider a finite state automaton. At each state s, it receives an input letter i from some alphabet $[m] = \{0, \dots, m-1\}$. It then outputs a letter from $[m]$ which is determined by some permutation $\sigma_s \in S_m$ applied to i. The automaton then jumps to a state s' which is some function of s and i, before receiving the next input letter. For an initial state s, this defines an action on the space of sequences in $[m]$, where for each n, the first n letter of the output is determined by the first n letters of the input. This can also be viewed as an action on the rays of the m-ary tree T_m, which is an automorphism of the tree. The corresponding automaton group is the group generated by the actions corresponding to the states. Many interesting groups are generated by very small automata.

The action described above yields that automaton groups are certain subgroups of the automorphism group $\mathrm{Aut}(T_m)$ of the rooted infinite m-ary tree for some m. For any vertex $v \in T_m$, and automorphism g, there is an induced action of g on the sub-tree above v. This action is called the *section* of g at v, denoted g_v. An action maps a sub-tree to another sub-tree, but since a sub-tree is isomorphic to the whole tree, a section can be viewed naturally as an automorphism of the whole tree. In this terminology, automaton groups are the finitely generated subgroups of $\mathrm{Aut}(T_m)$ with generators for which all sections are also among the generating set. We refer the reader to, for example, Nekrashevych's monograph [13] for detailed definitions and further history.

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The *activity* of $g \in \operatorname{Aut}(T_m)$ at level n, denoted $a_n(g)$, is the number of v in level n of the tree such that g_v is not the identity. Automaton groups have the property that for any g the activity sequence $a_n(g)$ grows either polynomially or exponentially. An automaton group Γ is said to have *degree* d, if every $g \in \Gamma$ has activity $a_n(g) = O(n^d)$. Activity of g was introduced by Sidki [14] as a measure of the complexity of the action of g, and the degree of an automaton group as a measure of the complexity of the group.

There exist exponential activity automaton groups that are isomorphic to the free group, as shown by Glasner and Mozes [9], and separately Vorobets and Vorobets [16]. However, one expects polynomial activity automaton groups to be smaller. In particular, in contrast to most examples of finitely generated non-amenable groups, Sidki [14] showed that polynomial activity automaton groups have no free subgroups. This prompted Sidki [15] to ask the following natural question.

Question 1.1. Are all polynomial activity automaton groups amenable?

This was answered affirmatively for degree 0 in [5] by Bartholdi, Kaimanovich and Nekrashevych and for degree 1 in [3] by the present authors. These results were also reproved by Juschenko, Nekrashevych and de la Salle [10] (see the discussion below). Our main result resolves Sidki's question for degree 2.

Theorem 1.2. *Every automaton group of degree* 2 *is amenable.*

For degrees 0 and 1, the proofs of [3, 5] proceed as follows. First, for each degree d and m, a certain specific automaton group acting on T_m, called the *mother group*, is constructed. It is then shown that every automaton group Γ of degree d is isomorphic to a subgroup of the mother group of degree d for some m' (see [3, Theorem 5.1]). Next, it is proved that the mother groups of degrees 0 and 1 are Liouville with respect to a carefully chosen random walk on them. Since the Liouville property implies amenability, and amenability is inherited by subgroups, this implies amenability of all bounded or linear activity automaton groups.

It is shown by Amir et al. [4] that for $d \geq 3$ the mother groups are not Liouville.[1] Thus the method of [5] cannot be extended to degree $d > 2$. This raises the natural question of the Liouville property for degree $d = 2$.

Conjecture 1.3 ([3]). *The mother groups of degree* 2 *are Liouville with respect to some (or even every) random walk on them. Moreover, the same holds for every automaton group of degree* 2.

The Liouville property of the mother groups is established in the papers above by showing that a certain random walk on the group has sublinear entropy. By results of

[1]Except for the case $d = 3$ and $m = 2$ which remains open.

Kaimanovich and Vershik [11], sublinear entropy growth is equivalent to having the Liouville property (with respect to this random walk). In a sense, entropy bounds can be thought of quantitative versions of the Liouville property. One should note that – while amenability is inherited by subgroups – it is not known whether the Liouville property passes onto subgroups. Consequently, the results of [3, 5] do not imply that all automata groups of degrees 0 or 1 are Liouville, nor that the mother groups are Liouville with respect to other generating sets.

The fact that degree 0 automata groups are Liouville with respect to any measure on them was proved in [2] by giving explicit entropy bounds for random walks on these groups. These entropy bounds come from resistance lower bounds in the Schreier graphs associated with the action of the automata group on a ray of the tree. To get such lower bounds, it is enough to attain lower bounds on the resistance for the Schreier graphs of the mother groups, since resistance can only increase when going into subgraphs. Thus resistance estimates on the Schreier graph of the mother groups imply entropy estimates and the Liouville property for bounded automata. We believe that a similar approach can be used to show that automata groups of degree 1 also have the Liouville property for any measure. For higher degree automata groups, the situation is different. In [4], it was shown that the Schreier graphs for degree 3 and up mother groups are transient. This was used to show (as noted above) that these groups do not have the Liouville property.

Upper and lower bounds for resistances in the Schreier graphs of the mother groups were given in [4] and [2], respectively. These bounds are tight for degree 0 mother groups and were enough to deduce transience for degree 3 and up mother groups. For degrees 1 and 2, there are significant gaps between the upper and lower bounds on resistances. In particular, these bounds were not enough to deduce recurrence of the Schreier graphs for degree 2 mother groups.

Since the Liouville property is harder to establish for $d = 2$ and false for $d > 2$, new methods are needed for further progress on Sidki's conjecture. In [10], Juschenko, Nekrashevych and de la Salle proved that (under some general conditions), if the action of a group G on a set X is significant enough, and the Schreier graph of the action is recurrent, then the group is amenable. In the context of polynomial activity automaton groups, their method yields a second proof of the amenability of degrees 0 and 1 automata groups that does not pass via the Liouville property. More precisely, they state the following (paraphrased).

Theorem 1.4 ([10, Theorem 5.2]). *If G is a quadratic activity automaton group and the Schreier graph of its action on the tree is recurrent, then G is amenable.*

Unfortunately, we do not prove recurrence of all components of the Schreier graph, but just the Schreier graph on the orbit of the 0-ray. However, using the methods of [10], this is sufficient for our needs, as we now explain. In the proof of [10, Theorem 5.1], they show that all rays of the tree that are not eventually 0 are not singular (as defined in Theorem 3.1 there). While Theorem 5.1 is stated for degree 1 groups, that part of the proof is written for general degree. Therefore, Theorem 3.1 together with the proof of Theorem 5.1 implies that if the Schreier graph of a quadratic activity automaton group G on the orbit of the 0-ray is recurrent, then G is amenable. In light of this, Theorem 1.2 is a corollary of the following result.

Theorem 1.5. *For any degree* 2 *automaton group, the Schreier graph of its action on the* 0*-ray of* T_m *is recurrent.*

The bulk of the work in this paper is actually in a more general context of groups of automorphisms of a spherically symmetric tree (i.e., a tree where every vertex at distance k from the root has the same number m_k of children). Such groups were used in [1,4,7] by Brieussel, Amir and Virág to construct groups where the random walk has varied speeds and entropy growth.

1.1. Mother groups on spherically symmetric trees

Let $\overline{m} = (m_0, m_1, \ldots)$ be some infinite *bounded* sequence with $m_i \in \{2, 3, \ldots\}$. We consider the spherically symmetric tree $T_{\overline{m}}$ defined as follows. At level 0 there is a single vertex $\emptyset$ (the root). Each vertex at level i has m_i children at level i, so that the size of level ℓ is $\prod_{i<\ell} m_i$. A vertex at level ℓ is naturally encoded by a word $x_\ell \cdots x_1 x_0$ where $x_i \in [m_i] = \{0, 1, \ldots, m_i - 1\}$. Since we will later have a group acting on $T_{\overline{m}}$ on the right, it is more useful to write the digits with x_0 on the right. The set of *ends of the tree* $T_{\overline{m}}$, denoted $\mathcal{E}$, are the infinite rays in $T_{\overline{m}}$, and are naturally encoded by infinite sequences (which we again write with x_0 on the right) $\cdots x_2 x_1 x_0$, with $x_i \in [m_i]$. The subset of ends with only finitely many non-zero digits is denoted $\mathcal{E}_0$.

We remark that the case of $\overline{m}$ constant is already new and of interest. This case is of particular significance since the corresponding groups (as defined below) are the automaton groups discussed above. The confused reader may well restrict to the case where $\overline{m}$ is the constant sequence, and the tree is the m-ary tree, without losing much.

We consider *automorphisms* of the rooted tree (which preserve the root $\emptyset$, and hence each level). (For some sequences $\overline{m}$ such as $(3, 2, 2, 2, \ldots)$ there are automorphisms which do not preserve the root, but we do not consider such automorphisms in this work.) An automorphism acts naturally on the set of ends of the tree, and is determined by this action. A bijection f of the set of ends with itself corresponds to an automorphism of the tree if for every i, the ith digit of $f(x)$ is determined by $x_i, \ldots, x_0$.

Towards defining our groups, we need notation for the locations of non-zero digits in an end of the tree. For a (finite or infinite) word x, let $\ell_{-1}(x) = -1$, and inductively let

$$\ell_t(x) = \inf\{n > \ell_{t-1}(x) : x_n \neq 0\}.$$

For some fixed degree d, the *mother group* of degree d, denoted $\mathcal{M}_{d,\overline{m}}$, is a subgroup of the automorphism group with the following set of generators. Each generator is specified by a degree $t \in \{-1, 0, \ldots, d\}$, and a sequence $(\sigma_i)_{i \le \max \overline{m}}$, where σ_i is a permutation in the symmetric group S_i for each i. For an end x, let $k = 1 + \ell_t(x)$. The generator applies σ_{m_k} to the digit x_k, and leaves all other digits unchanged. If $k = \infty$ (which happens for some ends in $\mathcal{E}_0$), then x is a fixed point of the generator.

As an example, suppose $x = \dots 201300010020$. Then $\ell_0 = 1$, $\ell_1 = 4$, $\ell_2 = 8$, etc. If (for whatever value of m_9) σ_{m_9} maps 1 to 0, then the corresponding generator with $t = 2$ will map this x to $\dots 200300010020$.

Note that the subset $\mathcal{E}_0$ of ends is preserved by all actions of generators of the mother group, and hence by actions of the group. Moreover, $\mathcal{E}_0$ is dense in the set of all ends, and so the action on $\mathcal{E}_0$ determines an automorphism of the tree. Finally, we remark that the mother groups act transitively on $\mathcal{E}_0$. (This is not hard but requires some observation and is also the basis of some mechanical puzzles such as the *Chinese rings*.) For $d = 1, 2$, these groups are referred to as the *linear* and *quadratic* mother groups, respectively.

Recall that the *Schreier graph* for the action of a group G generated by S on a set A is the graph with vertex set A and an edge (x, xg) for any $x \in A$ and generator $g \in S$. The set of infinite rays in the tree is uncountable, and the Schreier graph corresponding to the action on the full tree is not connected. Instead of considering the entire Schreier graph, our main object of study in this work is the Schreier graph $\mathcal{G}_{d,\overline{m}}$ for the action of $\mathcal{M}_{d,\overline{m}}$ on $\mathcal{E}_0$. It is not hard to see that $\mathcal{G}_{d,\overline{m}}$ is connected, and is the connected component of the 0-ray in the Schreier graph for the action on $\mathcal{E}$. We shall also consider the finite Schreier graphs for the action on level n of the tree, which will be denoted $\mathcal{G}_{d,\overline{m},n}$.

Note that for any x, y, if there is a generator g with $xg = y$, then there are many such generators. This is since the action of g on x only takes into account one of the entries in one of the permutations defining g. This means that the Schreier graphs we consider all have many parallel edges. However, the multiplicity of each edge is bounded, and so the effect of this multiplicity on electrical properties is at most a constant factor. From here on we ignore multiplicity of edges in the original Schreier graph. Note that in Section 2 we apply a projection to the Schreier graph which will create parallel edges with unbounded multiplicity, which has to be taken into account.

1.2. Results for mother groups

Theorem 1.6. *For $d \leq 2$ and any bounded sequence $\overline{m}$, the Schreier graph $\mathcal{G}_{d,\overline{m}}$ is recurrent.*

We expect other components of the Schreier graph on $\mathcal{E}$ to have a very similar geometry to the component on $\mathcal{E}_0$, and in particular to also be recurrent. This is not needed for the application to amenability of the mother groups, and the combinatorial ingredients in the analysis of the geometry of the graph are easier for $\mathcal{E}_0$, and so we restrict our attention to that component.

In the case $d = 0$, the Schreier graph has been analysed in [4]. When $d = 0$ and $m_i \equiv 2$, the graph is simply the half-line $\mathbb{N}$. (Other components of the Schreier graph on $\mathcal{E}$ in this case are isomorphic to $\mathbb{Z}$.) For general $\overline{m}$, it is easily seen to be recurrent as it contains infinitely many cutsets of bounded size. Resistances in $\mathcal{G}_{0,\overline{m}}$ are studied in [4].

The following theorems give bounds on the effective resistance between certain sets in the graphs $\mathcal{G}_{d,\overline{m}}$. We assume here the reader has basic familiarity with the theory of electrical networks, and refer the reader to, for example, [8, 12] for detailed background. Recall that a graph can be considered as an electrical network with resistors represented by edges. Electrical properties of the network are intimately related to behaviour of the random walk on the graph. A fundamental result is that the random walk on a graph is recurrent if and only if the effective resistance from any vertex (or finite set) to infinity is infinite, or equivalently, the resistance to some far away set can be made arbitrarily large.

For $d = 1, 2$, recurrence of $\mathcal{G}_{d,\overline{m}}$ is a direct consequence of the quantitative estimates in the following theorem, which require some additional notation. In Section 2 we describe an explicit projection $\widehat{\pi} : \mathcal{G}_{d,\overline{m}} \to \mathbb{N}$ with the following properties: The only vertex with $\widehat{\pi}(v) = 0$ is the 0-ray, and each $n \in \mathbb{N}$ has a finite non-empty pre-image. In the case $m_i \equiv 2$, $\widehat{\pi}$ is a bijection.

Theorem 1.7. *Fix a bounded sequence $\overline{m}$. There exists a constant C, depending only on $\sup \overline{m}$ such the following holds. For any $0 < s < t$, the effective resistance in $\mathcal{G}_{d,\overline{m}}$ satisfies*

$$\mathcal{R}(\widehat{\pi}^{-1}[0, 2^s) \leftrightarrow \widehat{\pi}^{-1}[2^t, \infty)) \geq \begin{cases} C(t-s) & \text{for } d = 1, \\ C(\log t - \log s) & \text{for } d = 2. \end{cases}$$

Remark 1.8. Note that by monotonicity, if $a < 2^s < 2^t < b$ then

$$\mathcal{R}(\widehat{\pi}^{-1}[0, a) \leftrightarrow \widehat{\pi}^{-1}[b, \infty)) \geq \mathcal{R}(\widehat{\pi}^{-1}[0, 2^s) \leftrightarrow \widehat{\pi}^{-1}[2^t, \infty)).$$

Thus Theorem 1.7 implies a similar bound for such resistances (i.e., $\log(b/a)$ and $\log\log(b/a)$ in the two cases, respectively) as long as $b \geq 4a$. For b close to a, the result might fail. Indeed, if $b = a + 1$, then the resistance can be of order $a^{-\delta}$ for some δ depending on $\overline{m}$, which can be large if $\overline{m}$ has large entries.

As mentioned in the introduction, any automaton group of degree d is conjugate to a subgroup of the mother group of the same degree d, possibly on a larger alphabet [3, Theorem 5.1]. This was first proved in the degree 0 case by Bartholdi, Kaimanovich and Nekrashevych [5]. Brieussel used a version of the degree 0 mother groups in spherically symmetric trees to establish amenability of certain automorphism groups in those cases [6, Theorem 3.1].

Since resistances in subgraphs are larger than resistances in a graph, we get the following corollary, which in turn implies amenability of the groups.

Corollary 1.9. *The Schreier graph for the natural action of any automaton group of degree at most 2 on the ends of the regular tree has a recurrent component.*

Structure of the paper. In Section 2, we give a combinatorial description of the Schreier graphs of the mother groups. In Section 3, we give a generalisation of the Nash-Williams resistance bound for collections of non-disjoint cutsets. Unlike the Nash-Williams bound, the generalised version always achieves the actual resistance if the correct cutsets and weights are used. While this generalisation is not difficult, we have not found a reference for it, and it is of some independent interest. Finally, in Section 4 we define a collection of cutsets in $\mathcal{G}_{d,\overline{m}}$, assign them weights and deduce Theorem 1.7.

2. Combinatorial description of the graphs

In this section we give a more explicit description of the Schreier graphs $\mathcal{G}_{d,\overline{m}}$ and $\mathcal{G}_{d,\overline{m},n}$. Recall that a vertex x of $\mathcal{G}_{d,\overline{m}}$ is an end in $\mathcal{E}_0$ of $T_{\overline{m}}$, and so is naturally described by a sequence $(x_i)_{i\geq 0}$ where $x_i \in [m_i]$ such that eventually $x_i = 0$. For $\mathcal{G}_{d,\overline{m},n}$, the vertices are finite sequences $x_{n-1}\cdots x_1 x_0$.

We write $x \sim y$ to denote that x, y are connected by an edge. Edges are of $d+2$ different *types*, denoted by $t \in \{-1, 0, \ldots, d\}$, corresponding to the types of the generator associated with the edge. In all cases, an edge connects vertices x and y which differ only in a single coordinate (though not all such pairs are connected). We denote that coordinate by $k = k(x, y)$, so that $x_k \neq y_k$, and $x_i = y_i$ for all $i \neq k$. For such a pair x, y, we have that

- (x, y) is an edge of type -1 if $k = 0$.
- (x, y) is an edge of type $t \geq 0$ if $k > 0$, and $x_{k-1} = y_{k-1} \neq 0$, and moreover there are precisely t indices $i < k-1$ for which $x_i \neq 0$.
- Otherwise, (x, y) is not an edge.

For example, $x = 0340020$ is connected to $y = 0140020$ by an edge of type 1 (here $k = 5$), since x_1 and x_4 are non-zero. The same x is not adjacent to $z = 0320020$, since $x_{k-1} = 0$. See Figure 1 for some small examples.

Clearly the graphs $\mathcal{G}_{d,\overline{m},n}$ are monotone in d, m_i, n: Reducing any m_i restricts to a subset of the vertices, while reducing d to d' removes all edges of type $t > d'$. Extending a vertex of $\mathcal{G}_{d,\overline{m},n}$ by 0s gives a vertex of $\mathcal{G}_{d,\overline{m},n'}$ for any $n' > n$. Extending by infinitely many 0s gives a vertex of $\mathcal{G}_{d,\overline{m}}$. This gives a canonical embedding of $\mathcal{G}_{d,\overline{m},n}$ in the graphs for larger n and in $\mathcal{G}_{d,\overline{m}}$.

Clearly for each i, the graphs $\mathcal{G}_{d,\overline{m}}$ and $\mathcal{G}_{d,\overline{m},n}$ are also invariant to permuting the letters $\{1, \ldots, m_i - 1\}$. Consider two vertices x, y to be *equivalent* if they have the same non-zero coordinates, that is, $\{i : x_i \neq 0\} = \{i : y_i \neq 0\}$. From each such equivalence class we take as representative the vertex in $\{0, 1\}^n$. The *hamming weight* of a vertex x, denoted $|x|$, is the number of non-zero coordinates. The equivalence class of x has $\prod_{i:x_i\neq 0}(m_i - 1)$ vertices. The projection from $\mathcal{G}_{d,\overline{m},n}$ to $\{0, 1\}^n$ is denoted by π. In the limit $n \to \infty$, this projection extends to a projection from $\mathcal{G}_{d,\overline{m}}$ to $\bigoplus_{i\in\mathbb{N}}\{0, 1\}$, namely the set of $\{0, 1\}$ sequences with finitely many ones. Since these are the vertices of $\mathcal{G}_{d,2,n}$ (or $\mathcal{G}_{d,2}$), we can see π as a map from $\mathcal{G}_{d,\overline{m},n}$ to $\mathcal{G}_{d,2,n}$, which preserves much of the graph structure.

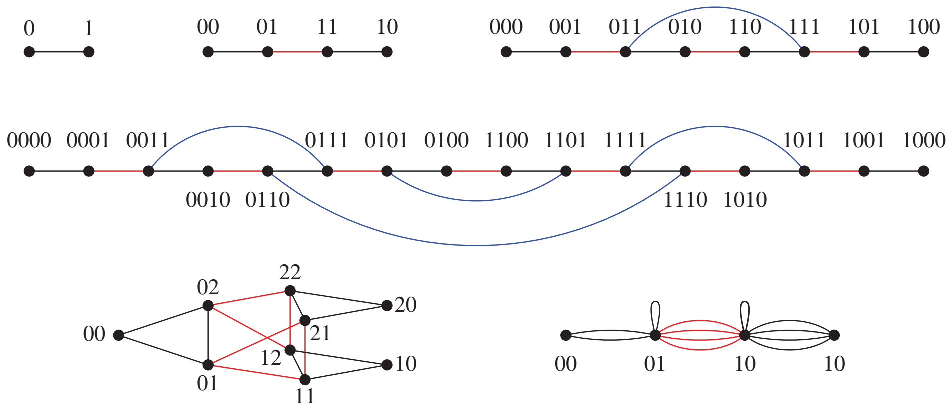

Figure 1. Top and middle rows: The graphs $\mathcal{G}_{d,2,n}$ for $d = 1$ and $n = 1, 2, 3, 4$. Bottom row: The graph $\mathcal{G}_{1,3,2}$ and its quotient $\mathcal{G}_{1,2,2}$ with multiple edges and self-loops. Edge colour denotes its type: Black for $t = -1$, red for $t = 0$ and blue for $t = 1$. The bold loop on the bottom right has multiplicity 4, not all of the same type, but of course has no effect on resistances. Vertices are laid out according to the linear order $\hat{x}$ from left to right.

The resistances we shall consider are between sets that are themselves invariant to such permutations, and thus can be studied by looking at resistances on the quotient graph. Many edges become self-loops under this projection, and do not affect the resistance. If (x, y) is an edge where $x_k = 0$ and $y_k \neq 0$, then the edge x, y is projected onto a proper edge of $\mathcal{G}_{d,2,n}$. If (x, y) is an edge where $x_k, y_k \neq 0$, then the edge x, y is projected onto a self-loop. Thus π maps $\mathcal{G}_{d,\overline{m},n}$ to $\mathcal{G}_{d,2,n}$ with self-loops. Each edge of $\mathcal{G}_{d,2,n}$ can have multiple pre-images under π. The number of pre-images of an edge (x, y) is $\prod_{i:y_i \neq 0}(m_i - 1)$ (assuming $x_k = 0$ and $y_k \neq 0$). We will therefore consider the graph $\mathcal{G}_{d,2,n}$ where edges have conductance given by this multiplicity. We remark that $\prod_{i:x_i \neq 0}(m_i - 1)$ and $\prod_{i:y_i \neq 0}(m_i - 1)$ differ by a bounded factor of $m_j - 1$ for some j, so up to constant factors either can be used for the conductance of the edge. The same holds for the projection from $\mathcal{G}_{d,\overline{m}}$ to $\mathcal{G}_{d,2}$. We will work below primarily with the graphs $\mathcal{G}_{d,2,n}$ and $\mathcal{G}_{d,2}$ with edge conductances coming from these multiple pre-images.

The graphs in the case $d = 0$ are particularly simple. Each vertex of $\mathcal{G}_{0,2,n}$ is incident to one edge of type -1 and one edge of type 0, except for the root $o = 00 \cdots 0$ and one other vertex $o' = 10 \cdots 0$ which have degree 1. It is not hard to verify that the graph $\mathcal{G}_{0,2,n}$ is a path of length $2^n - 1$ from o to o'. Since increasing d does not remove any edges, this path is contained in $\mathcal{G}_{d,2,n}$ for any d. It will be useful to keep track of the position of a vertex x along this path, which shall be denoted $\hat{x} \in [0, 2^n - 1)$.

Given a finite binary string $x = x_n x_{n-1} \cdots x_0$, we can represent its *linear position* as $\hat{x}_n \hat{x}_{n-1} \cdots \hat{x}_0$ by $\hat{x}_k = \sum_{i=k}^{n} x_i \pmod 2$. Note that this map is a bijection from $\{0, 1\}^n$ to itself, and the inverse transform is given by $x_k = \hat{x}_k + \hat{x}_{k+1} \pmod 2$. This extends to infinite sequences $x \in \bigoplus_{i \in \mathbb{N}} \{0, 1\}$, since the infinite sum contains finitely many ones. We also define the projection $\hat{\pi}$ on $\mathcal{G}_{d,\overline{m}}$ and $\mathcal{G}_{d,\overline{m},n}$ by composing this transformation with the projection π, namely $\widehat{\pi}(x) = \widehat{\pi(x)}$. For example, if $x = 0340020$, then $\pi(x) = 0110010$ and $\widehat{\pi}(x) = 0100011$.

We remark that $\bigoplus_{i \in \mathbb{N}}\{0, 1\}$ is naturally in bijection with $\mathbb{N}$, where any integer corresponds to its binary expansion. This gives a natural order on $\mathcal{G}_{d,2}$ and a partial order on $\mathcal{G}_{d,\overline{m}}$, where $x < y$ if $\widehat{\pi}(x) < \widehat{\pi}(y)$ as integers. The root is the unique minimal vertex, mapped to $\widehat{o} = 0$.

Recall that for a vertex x we denote by $\ell_i(x)$ the position of the $(i+1)$th non-zero digit in i from the right, and we let $\ell_{-1}(x) = -1$. Thus for an edge $e = (x, y)$ of type t we have $\ell_i(x) = \ell_i(y)$ for all $i \le t$. We denote the part of x (or y) strictly to the left of position $k = k(x, y)$ by $z(x, y)$. When x and y are fixed, we shorten notation and denote the above simply by k, ℓ_i and z. We shall make use of the following description of edges in terms of the linear order on vertices.

Remark 2.1. Edges of type -1 and 0 always connect adjacent points in the linear order. That is, 0 is connected to 1 by an edge of type -1, and every other $x \in \{0, 1\}^*$ $\widehat{x}$ is connected to $\widehat{x} \pm 1$ by edges of types -1 and 0. The edge types alternate along the resulting path (see Figure 1).

3. Weighted Nash-Williams

In this section we give a generalisation of the classical Nash-Williams bound on resistances in electrical networks, which applies for collections of not-necessarily disjoint cutsets. This generalisation, which is also of some independent interest, will be used to give lower bounds on resistances in the Schreier graphs of the mother groups.

As noted, we assume here a basic familiarity with the theory of electrical networks, and refer the reader to, for example, [8,12] for detailed background. We recall the notations we use below. An electrical network is a graph $G = (V, E)$ with edge weights or conductances $C_e \in \mathbb{R}_+$. The resistance of an edge is denoted $R_e = C_e^{-1}$. An unweighted graph is seen as a network with $C_e \equiv 1$. We denote the resulting effective resistance between vertices a, b by $\mathcal{R}(a \leftrightarrow b)$. This is extended to resistance between sets A, B, denoted $\mathcal{R}(A \leftrightarrow B)$.

Recall the classical Nash-Williams inequality: In any graph G with vertices a, b, if $\{S_i\}_{i \in I}$ are disjoint edge cutsets (i.e., S_i separates a from b), then $\mathcal{R}(a \leftrightarrow b) \ge \sum |S_i|^{-1}$. This extends in the natural way to the resistance between sets A, B, as well as to electrical networks, where $|S_i|$ is replaced by the total conductance of S_i. In general networks, there is no collection of disjoint cutsets for which this bound achieves the actual resistance. For example, in the triangle $\{a, b, c\}$ there can only be one cutset of degree 2 between a, b, giving a bound of $1/2$, to the actual value $\mathcal{R}(a \leftrightarrow b) = 2/3$. There is not even a bound on how far from $\mathcal{R}(A \leftrightarrow B)$ the optimal collection of cutsets might be: If a single edge e of a long cycle has r_e much larger than the other edges, the Nash-Williams bound will be far from the actual effective resistance. However, it turns out that there is a weighted version of Nash-Williams that can achieve the resistance on any graph, which we describe below.

Let $\{S_i\}_{i \in I}$ be some collection of cutsets (not necessarily disjoint). A *resistance allocation* is an assignment, where each edge splits its resistance between the cutsets containing it. More explicitly, for each edge e and $i \in I$, we have partial resistances $R_{e,i} \geq 0$ which satisfy $\sum_i R_{e,i} \leq R_e$ so that $R_{e,i} = 0$ if $e \notin S_i$. Define the *split conductance* of S_i by $C(S_i) = \sum_{S_i} R_{e,i}^{-1}$.

The following is the generalisation of Nash-Williams to non-disjoint cutsets. While it is fairly simple to prove, we are not aware of a reference for it in the literature.

Proposition 3.1. *With the above notations, for any collection of cutsets and resistance allocation we have* $\mathcal{R}(A \leftrightarrow B) \geq \sum_i C(S_i)^{-1}$. *Moreover, in any finite graph,* $\mathcal{R}(A \leftrightarrow B)$ *is the maximum of* $\sum_i C(S_i)^{-1}$ *over all resistance allocations on some collection of cutsets.*

Remark 3.2. In an infinite graph, the supremum of $\sum_i C(S_i)^{-1}$ is the free effective resistance between A, B. This follows from the above, by applying the proposition to the restriction of G to a large set Λ and taking a limit as Λ exhausts the graph.

A particular way of allocating resistances is to assign each cutset S_i a weight $K_i \geq 0$ and allocate resistances in proportion to these weights. Formally this means to set $R_{e,i} = \frac{R_e K_i}{\sum_{j:e \in S_j} K_j}$. Plugging this in yields the following bound.

Corollary 3.3. *For any collection* $\{S_i\}_{i \in I}$ *of cutsets between* A, B, *and any non-negative weights* (K_i), *we have*

$$\mathcal{R}(A \leftrightarrow B) \geq \sum_i \left(\sum_{e \in S_i} \frac{\sum_{j:e \in S_j} K_j}{R_e K_i} \right)^{-1}.$$

Remark 3.4. If the graph is infinite, the weighted cutset method still gives a lower bound on the resistances. If we consider the resistance from a vertex (or set) to infinity, $\mathcal{R}(A \leftrightarrow \infty)$ is a limit of the resistance to the complement of an arbitrary exhaustion G_n. It follows that $\mathcal{R}(A \leftrightarrow \infty)$ is again the supremum over weighted cutsets as above. In particular, a graph is recurrent if and only if there exist a collection of cutsets S_i and resistance allocations $R_{e,i}$ such that $\sum C(S_i)^{-1} = \infty$. We omit further details.

Proof of Proposition 3.1. Assume first that there are only finitely many cutsets S_i in the collection. Given a resistance allocation, we construct a new network G' from G, where each edge e is replaced by several edges in series, with resistances $\{R_{e,i}\}_{i \in I}$. (The order of these edges in the series is arbitrary.) Since $\sum_i R_{e,i} \leq R_e$, effective resistances in G' are all smaller than in G. In G' we can construct a collection of disjoint cutset: For each i take the edges of resistance $R_{e,i}$. The classical Nash-Williams applied to these disjoint cutsets gives the claimed bound. If there are infinitely many cutsets, just note that any finite partial sum gives a finite resistance allocation, and thus gives a lower bound on $\mathcal{R}(A \leftrightarrow B)$.

To see that some weighted cutsets achieve the resistance, we give an explicit construction. Consider the induced equilibrium voltage with $V = 0$ on A and $V = 1$ on B, and let f

be the equilibrium flow from a to b, so that $f(x,y) = \frac{V_y - V_x}{R_{xy}}$. Let $0 = a_0 < a_1 < \cdots < a_m = 1$ be the different values taken by V. For each $i \le m$, let $U_i = \{x \in G : V_x < a_i\}$, so that $A \subset U_i$ and $B \subset U_i^c$. Define the cutsets S_i of edges x, y with $x \in U_i$ and $y \notin U_i$.

We use the weighted resistance allocation as defined above. Assign S_i weight $K_i = a_i - a_{i-1}$, so that $\sum K_i = V_B - V_A = 1$. For an edge $e = (x,y)$ with $V_x < V_y$, we have that

$$\sum_{j:e \in S_j} K_j = V_y - V_x,$$

and therefore $R_{e,i}^{-1} = R_e^{-1} \frac{V_y - V_x}{K_i} = f(e)/K_i$. Since the flow is always in the direction of increasing voltage, the total flow across any cutset S_i is exactly $1/\mathcal{R}(A \leftrightarrow B)$. Thus $C(S_i)^{-1} = K_i \mathcal{R}(A \leftrightarrow B)$. Summing over i we get the claim. ■

4. Cutsets in $\mathcal{G}$

We now use the linear order on vertices of $\mathcal{G} = \mathcal{G}_{d,2,n}$ or $\mathcal{G}_{d,2}$ to define a collection of cutsets. We remind that we work here with the graphs resulting from projecting $\mathcal{G}_{d,\overline{m},n}$ so that edges have unequal conductances. The conductance of an edge (x,y) is either $\prod_{i:x_i=1}(m_i - 1)$ or the corresponding product for y. The two are equivalent up to a bounded multiplicative factor.

For $\hat{a} \in \mathbb{N}$, we let S_a be the set of all edges (x,y) with $\hat{x} < \hat{a} \le \hat{y}$. Note that these cutsets are not disjoint for $d > 0$. (If $d = 0$, then $\mathcal{G}$ is a path, and each of these cutsets is a single edge.) As with x, each $\hat{a}$ is associated with a sequence $a \in \{0,1\}^*$. Note that even for general sequences m we take $a \in \{0,1\}^*$.

For our analysis, it will be convenient to enlarge these cutsets slightly. Edges of type -1 and 0 will not be added to the cutsets. However, to some cutsets we will add an edge of type 1 and possibly several edges of type 2, as described below. The enlarged cutsets will be denoted $\overline{S}_a$, and are defined formally after the proof of Lemma 4.1.

For a sequence $q = (q_i)$, we denote $\beta^q := \prod_i \beta_i^{q_i}$, where $\beta_i = 1/(m_i - 1)$. An integer $a \in \mathbb{N}$ is interpreted as a sequence using its binary representation, so we can write $\beta^a := \prod_{i:a_i=1} \beta_i$. To a cutset $\overline{S}_a$ we associate weight β^a. We also use below the notation $\beta^{-a} = 1/\beta^a$. We then allocate the resistance R_e of an edge e between the cutsets in proportion to their weight, that is, for $e \in \overline{S}_a$ let

$$R_{e,a} = \frac{R_e \beta^a}{\sum_{b:e \in \overline{S}_b} \beta^b}. \tag{4.1}$$

Our immediate goal is therefore to understand which of the cutsets S_a include a given edge and which edges are included in any cutset. The enlarged cutsets $\overline{S}_a$ will be defined so that $R_{e,a}$ is easier to analyse.

Lemma 4.1. *Consider an edge $e = (x,y)$ of type t with $\hat{x} < \hat{y}$.*

(1) *If $t \in \{-1, 0\}$, then $e \in S_a$ if and only if $\{\hat{x}, \hat{y}\} = \{\hat{a} - 1, \hat{a}\}$ as an unordered pair.*

(2) *If $t = 1$ and $a \neq x$, then we have $e \in S_a$ if and only if $a_i = x_i = y_i$ for all $i > \ell_0$ except $i = k$.*

(3) *If $t = 2$ and $e \in S_a$, then $a_i = x_i = y_i$ for all $i > \ell_1$ except $i = k$.*

See Figure 2 for examples of the type 1 and type 2 cases. Note that in the case $t = 1$ we have $k = \ell_1 + 1 > \ell_0$. In that case the condition on the digits of a, x, y is satisfied when $a = x$ but $e \notin S_a$ due to the strict inequality in the definition of S_a. In the case $t = 2$, we do not provide a sufficient criterion for $e \in S_a$ but only a necessary condition. In the case of $t = 2$, one could give a necessary and sufficient condition for e to be in S_a, which would be more cumbersome and would not lead to a significant improvement in the estimates below.

Proof. The cases $t = -1$ and $t = 0$ follow immediately from Remark 2.1.

Let $e = (x, y)$ be a type 1 edge with $\hat{x} < \hat{y}$. Since $k = 1 + \ell_1$ is the unique index where $x_k \neq y_k$, we have that x and y have the following form (from left to right): They start with the same sequence of bits w, until position k. At position k, one of them is 0 and the other is 1. Which is 0 depends on the parity of the number of 1s in w. At position $\ell_1 = k - 1$ both are 1, and the rest of the bits are 0 except for a single position l_0 where also $x_{\ell_0} = y_{\ell_0} = 1$ (see Figure 2).

Therefore, their linear order representations $\hat{x}, \hat{y}$ have the following structure: Both begin (on the left) with $\hat{w}$ till position k. At position k, $\hat{x} = 0$ and $\hat{y} = 1$ (since we assumed $\hat{x} < \hat{y}$). This is followed in $\hat{x}$ by $\ell_1 - \ell_0$ ones, and ℓ_0 zeros. In $\hat{y}$, the final ones and zeros are reversed: There are $\ell_1 - \ell_0$ zeros followed by ℓ_0 ones.

$$
\begin{array}{cccccccccc}
 & & k & l_1 & & & l_0 & & & \\
x & w & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
y & w & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hat{x} & \hat{w} & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hat{y} & \hat{w} & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\multirow{2}{*}{$\hat{a}$} \left\{ \right. & \hat{w} & 0 & 1 & 1 & 1 & * & * & * & * \\
 & \hat{w} & 1 & 0 & 0 & 0 & * & * & * & * \\
a & w & * & 1 & 0 & 0 & * & * & * & *
\end{array}
\qquad
\begin{array}{ccccccccccccc}
 & & k & l_2 & & & l_1 & & & & l_0 & & \\
x & w & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
y & w & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hat{x} & \hat{w} & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\hat{y} & \hat{w} & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\multirow{2}{*}{$\hat{a}$} \left\{ \right. & \hat{w} & 0 & 1 & 1 & 1 & * & * & * & * & * & * & * \\
 & \hat{w} & 1 & 0 & 0 & 0 & * & * & * & * & * & * & * \\
a & w & * & 1 & 0 & 0 & * & * & * & * & * & * & *
\end{array}
$$

Figure 2. Examples for Lemma 4.1. Left: (x, y) is a type 1 edge. x and y differ only in a single position k, and have a one at position $k - 1 = \ell_1$. The digits $x_i = y_i$ for $i > k$ form a sequence w. If w has an even number of ones, then $\hat{x} < \hat{y}$; otherwise, it would be reversed. If $(x, y) \in S_a$, then $\hat{x} < \hat{a} \leq \hat{y}$, and so $\hat{a}$ must take one of two forms, depending on its kth digit. The $*$s indicate digits that could take any value in $\{0, 1\}$. However, in the first form of $\hat{a}$, if all $*$s are 0 then $\hat{x} = \hat{a}$, which is excluded. In either case, a_i agrees with either x_i or y_i for all $i > \ell_0$. Right: a type 2 edge. Here, $\hat{x} < \hat{a} \leq \hat{y}$ implies that $\hat{a}$ and a have the form shown. However, even more cases are excluded; for example, if $\hat{a} = \hat{w}10001111010$, then $\hat{a} > \hat{y}$ and $(x, y) \notin S_a$. The enlarged cutsets $\bar{S}_a$ contain (x, y) whenever a has one of the forms above, even if $\hat{a} \notin (\hat{x}, \hat{y}]$.

Since we assume $a \neq x$, then also $\widehat{a} \neq \widehat{x}$. Therefore, the assumption $(x, y) \in S_a$ is equivalent to $\widehat{x} \leq \widehat{a} \leq \widehat{y}$. Then $\widehat{a}$ must agree with both $\widehat{x}$ and $\widehat{y}$ in all positions left of k. If $\widehat{a}_k = 0$, then $\widehat{a} \leq \widehat{y}$ must hold, and the condition $\widehat{x} \leq \widehat{a}$ is equivalent to the next $\ell_1 - \ell_0$ digits being 1 (and the final ℓ_0 digits can be anything). Similarly, if $\widehat{a}_k = 1$, then $\widehat{x} \leq \widehat{a}$ must hold, and the condition $\widehat{a} \leq \widehat{y}$ is equivalent to the next $\ell_1 - \ell_0$ digits all being 0.

Converting this description of $\widehat{a}$ to a yields the claim for the case $t = 1$ (recall $a_i = \widehat{a}_i + \widehat{a_{i+1}} \bmod 2$).

The case $t = 2$ is similar. Let $e = (x, y)$ be a type 2 edge with $\widehat{x} < \widehat{y}$. Since $k = 1 + \ell_2$ is the unique index where $x_k \neq y_k$, we have that x and y have the following form (from left to right): They start with the same sequence w, until position k. At position k one of them is 0 and the other is 1. Subsequently, their non-zero digits are precisely in positions ℓ_2, ℓ_1, ℓ_0.

In the linear order representation, $\widehat{x}, \widehat{y}$ have the following structure: Both begin (on the left) with $\widehat{w}$ till position k. At position k, $\widehat{x} = 0$ and $\widehat{y} = 1$. This is followed in $\widehat{x}$ by a block of 1s, a block of 0s, and another block of 1s, and in $\widehat{y}$ by blocks of the same lengths, but starting and ending with 0s.

Suppose $(x, y) \in S_a$, and in particular $\widehat{x} \leq \widehat{a} \leq \widehat{y}$. Then $\widehat{a}$ must agree with both $\widehat{x}$ and $\widehat{y}$ in all positions left of k. If $\widehat{a}_k = 0$, then $\widehat{a} \leq \widehat{y}$ holds, and the assumption $\widehat{x} \leq \widehat{a}$ implies that the next $\ell_2 - \ell_1$ digits of $\widehat{a}$ are all 1s. Similarly, if $\widehat{a}_k = 1$, then the assumption $\widehat{a} \leq \widehat{y}$ implies that the next $\ell_2 - \ell_1$ digits are 0s. Converting this description of $\widehat{a}$ to a yields the claim for the case $t = 2$. ∎

In light of Lemma 4.1, we define the *enlarged cutsets* $\overline{S}_a$ which contain all edges (x, y) which satisfy the condition in the corresponding clause of the Lemma 4.1. Explicitly, an edge (x, y) of type t with $\widehat{x} < \widehat{y}$ is in $\overline{S}_a$ if

- $t \in \{-1, 0\}$, and $\widehat{x} = \widehat{a} - 1$, $\widehat{y} = \widehat{a}$, or
- $t = 1$ and $a_i = x_i = y_i$ for all $i > \ell_0$ except possibly $i = k$, or
- $t = 2$ and $a_i = x_i = y_i$ for all $i > \ell_1$ except possibly $i = k$.

From here on we work with the cutsets $\overline{S}_a$.

Lemma 4.2. *For an edge $e = (x, y)$ of type t, we have that*

$$\sum_{a: e \in \overline{S}_a} \beta^a \asymp \begin{cases} \beta^x & t = -1 \text{ or } t = 0, \\ \beta^x \prod_{i \leq \ell_0} (1 + \beta_i) & t = 1, \\ \beta^x \prod_{i \leq \ell_1} (1 + \beta_i) & t = 2, \end{cases}$$

where the implicit constants depend only on $\max \overline{m}$.

Proof. The case $t \leq 0$ is trivial since $a = x$ or $a = y$ and β^x and β^y differ by the bounded ratio β_k.

In the case $t = 1$, from the definition of $\overline{S}_a$ we have $a_i = x_i = y_i$ for $i > \ell_0$, except $i = k = \ell_1 + 1$. In particular, $a_i = 0$ for $i \in (\ell_0, \ell_1)$. We are interested in the sum of β^a over all a with $e \in \overline{S}_a$. Since for $i \le \ell_0$ we can have any combination of 0s and 1s, this satisfies

$$\sum_{a:e\in\overline{S}_a} \beta^a = \Big(\prod_{i>k} \beta_i^{x_i}\Big)(1+\beta_k)\beta_{\ell_1}\Big(\prod_{i\le\ell_0}(1+\beta_i)\Big).$$

Since β is bounded from 0 and above, and since $x_i = 0$ for $i < k$ except $i = \ell_0, \ell_1$, the first term in this product is (up to constants) β^x, and the next two terms are bounded, giving the lemma.

The case $t = 2$ is almost identical, with ℓ_1 replacing ℓ_0. ■

The next step is to compute the total conductance of a cutset. Whereas previously we were interested in which cutsets contain an edge, now this requires the dual question: Which edges are in a cutset. Each cutset contains a unique edge of type 0 or -1, but can contain more edges of higher types. For any a, the conductance of $\overline{S}_a$ is given by $C_a := \sum_{e\in\overline{S}_a} R_{e,a}^{-1}$. Note that for an integer a we have $[\log_2 a] = \max\{i : a_i \ne 0\}$.

Lemma 4.3. *Fix $a \in \mathbb{N}$. For each $\ell_0 \le \log_2 a$, there is a unique edge (x, y) of type 1 in $\overline{S}_a$ with the given ℓ_0, whereas for $\ell_0 > \log_2 a$ there are no edges of type 1 in $\overline{S}_a$ with that value of ℓ_0.*

Proof. This is seen directly from the definition of $\overline{S}_a$. Let us fix ℓ_0 and a, we try to recover the edge (x, y). First we find ℓ_1, which must be the minimal $i > \ell_0$ with $a_i = 1$. This is the only choice, since a_{ℓ_1} cannot be 0 if $(x, y) \in \overline{S}_a$, and since $a_i = 0$ for all $i \in (\ell_0, \ell_1)$. Such ℓ_1 can be found if and only if $\ell_0 < \log a$. We now can identify x and y, since $x_i = y_i = 0$ for all $i \le \ell_1$ except $i = \ell_0, \ell_1$ where $x_i = y_i = 1$. Moreover, $x_i = y_i = a_i$ for all $i > \ell_1 + 1$. Finally, for $i = \ell_1 + 1$ we have that x_i, y_i are 0 and 1 in some order. ■

Lemma 4.4. *In the degree 1 mother group, we have that*

$$C_a \asymp \beta^{-a} \prod_{i<\log_2 a} (1+\beta_i),$$

where the constants depend only on $\max \overline{m}$.

Proof. Fix some $a \in \mathbb{N}$, and consider the cutset $\overline{S}_a$. The cutset contains exactly one edge of type -1 or 0 (with $\hat{x} + 1 = \hat{y} = \hat{a}$). This edge has conductance $R_e^{-1} = \beta^{-x}$ or β^{-y} (equivalent up to constants to β^{-a}), and assigns all of it to the cutset $\overline{S}_a$. Recall the definition (4.1) of the resistance allocation $R_{e,a}$. The contribution to the cutset conductance C_a from edges $e = (x, y)$ of type 1 is given (using Lemma 4.2) by

$$\sum_{e\in\overline{S}_a} R_{e,a}^{-1} = \sum_{e\in\overline{S}_a} \frac{\sum_{b:e\in\overline{S}_b}\beta^b}{R_e\beta^a} \asymp \sum_{e\in\overline{S}_a} \frac{\beta^x\prod_{i\le\ell_0}(1+\beta_i)}{\beta^x\beta^a} = \beta^{-a}\sum_{e\in\overline{S}_a}\prod_{i\le\ell_0}(1+\beta_i).$$

By Lemma 4.3, there is a unique edge $\bar{S}_a$ with each $\ell_0 \le \log_2 a$, so we have

$$C_a \asymp \beta^{-a} + \beta^{-a} \sum_{\ell_0 \le \log_2 a} \prod_{i \le \ell_0} (1+\beta_i)$$

where the first term is from the edge of type 0 or -1 and the sum from edges of type 1. Since β_i are bounded away from 0, the sum is dominated up to a constant factor by the largest term and we get

$$C_a \asymp \beta^{-a} \prod_{i < \log_2 a} (1+\beta_i)$$

as claimed. ■

Lemma 4.5. *In $\mathcal{G}_{2,\bar{m}}$ we have that $C_a \asymp \beta^{-a}(\log_2 a)\prod_{i<\log_2 a}(1+\beta_i)$, where the constants depend only on* $\max \bar{m}$.

Proof. This is very similar to the proof of Lemma 4.4. The main change and additional contribution now is from edges of type 2, and so we need to understand edges of type 2 in $\bar{S}_a$. We first count edges of type 2 in $\bar{S}_a$. (This is just as the type 1 case in Lemma 4.3.) We claim that for any $\ell_1 < \log_2 a$, there are exactly ℓ_1 edges $e \in \bar{S}_a$ with that ℓ_1. To see this note that given $\ell_1 < \log a$, and any $\ell_0 < \ell_1$, there is a unique edge x, y of type 2 with those ℓ_0, ℓ_1 in the cutset. Thus the contribution to C_a from edges of type 2 is up to constants

$$\beta^{-a} \sum_{\ell_1 < \log_2 a} k \prod_{i \le \ell_1} (1+\beta_i) \asymp \beta^{-a}(\log_2 a) \prod_{i<\log_2 a} (1+\beta_i),$$

since the sum is again dominated by its largest term. This dominates the contribution from edges of type $-1, 0, 1$, and so gives the claimed total conductance. ■

Proof of Theorem 1.7. To bound the resistance from $\pi^{-1}([0,2^s))$ to $\pi^{-1}([2^t,\infty))$, we separate cutsets into groups according to $[\log_2 a] \in [s,t)$. For $k \in [s,t)$, there are 2^k choices for a with $[\log_2 a] = k$.

In the case $d = 1$, the contribution from the cutsets $\bar{S}_a$ with $[\log_2 a] = k$ is given by Lemma 4.4:

$$\sum_{a=2^k}^{2^{k+1}-1} C_a^{-1} \asymp \sum_{a=2^k}^{2^{k+1}-1} \beta^a \prod_{i \le k} (1+\beta_i)^{-1} = \beta_k.$$

Since β_k is bounded, the total resistance is at least $c(t-s)$.

In the case $d = 2$, Lemma 4.5 gives an extra factor of $1/k$ in the kth term, so

$$\sum_{a=2^k}^{2^{k+1}-1} C_a^{-1} \asymp \sum_{a=2^k}^{2^{k+1}-1} \beta^a k^{-1} \prod_{i \le k} (1+\beta_i)^{-1} = \beta_k k^{-1}.$$

Therefore,

$$\mathcal{R}(\hat{\pi}^{-1}[0,2^s) \leftrightarrow \hat{\pi}^{-1}[2^t,\infty)) \ge \sum_{k=s}^{t-1} c/k \asymp \log(t) - \log(s)$$

as claimed. ∎

Proof of Theorem 1.6. Fix $s = 0$ in Theorem 1.7. We find that $\mathcal{R}(\hat{\pi}^{-1}(0) \leftrightarrow \hat{\pi}^{-1}[2^t, \infty))$ is unbounded as $t \to \infty$. Recurrence of the graph follows. ∎

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