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On $\mathbb{Z}^d$-odometers associated to integer matrices

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Abstract. We extend the results by Giordano, Putnam, and Skau (2019) on characterization of conjugacy, isomorphism, and continuous orbit equivalence of $\mathbb{Z}^d$-odometers to dimensions $d > 2$. We then apply these extensions to the case of odometers defined by matrices with integer coefficients.

1. Introduction

In this note, we first supply an argument that extends the main result of the paper "$\mathbb{Z}^d$-odometers and cohomology" by Giordano, Putnam, and Skau [1, Theorem 1.5 (1)–(3)] to dimensions greater than 2. More precisely, the proofs of conjugacy, isomorphism, and continuous orbit equivalence characterizations of [1, Theorem 1.5] are based on [1, Theorem 4.4], the only result in that paper where the dimension restriction $d = 1, 2$ is important. Our Proposition 2.3 below lifts this restriction and therefore leads to the characterizations in arbitrary dimension $d \geq 1$. In turn, our proof of Proposition 2.3 uses Lemma 2.1 that made the relevant computations of the image of the first cohomology group under a natural map simple and possible in higher dimensions. In the last section, we apply these results along with the earlier results by the second author [4] in the setting of odometers defined by matrices with integer coefficients. Below, we use the notations and terminology from [1, 3, 4] and refer to these papers for more details.

If $d \in \mathbb{N}$ and G is a subgroup of $\mathbb{Z}^d$, the group $\mathbb{Z}^d$ acts on $\mathbb{Z}^d/G$ by

$$\phi_G^k(l + G) = k + l + G, \quad k, l \in \mathbb{Z}^d. \tag{1.1}$$

Let $\mathcal{G} = \{G_1, G_2, \dots\}$ be a decreasing sequence of subgroups of $\mathbb{Z}^d$ of finite index, i.e.,

$$\mathbb{Z}^d = G_1 \supseteq G_2 \supseteq \cdots, \quad [\mathbb{Z}^d : G_n] < \infty,\ n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ and $\mathbb{Z}^d \supseteq G_n \supseteq G_{n+1}$, let $q_n\colon \mathbb{Z}^d/G_{n+1} \to \mathbb{Z}^d/G_n$ denote the quotient map $q_n(k + G_{n+1}) = k + G_n$, $k \in \mathbb{Z}^d$.

Definition 1.1. Let $\mathcal{G} = \{G_1, G_2, \dots\}$ be as above, and let $X_{\mathcal{G}}$ denote the inverse limit system

$$\mathbb{Z}^d/G_1 \xleftarrow{q_1} \mathbb{Z}^d/G_2 \xleftarrow{q_2} \cdots.$$

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A $\mathbb{Z}^d$*-odometer* is a pair $(X_{\mathcal{G}}, \phi_{\mathcal{G}})$, where $\phi_{\mathcal{G}}$ is the natural action of $\mathbb{Z}^d$ on $X_{\mathcal{G}}$ induced by $\phi_{G_n}^k$, $k \in \mathbb{Z}^d$, $n \in \mathbb{N}$. The natural projection from $X_{\mathcal{G}}$ to $\mathbb{Z}^d/G_n$ is denoted by π_n.

If $G_n \neq G_{n+1}$ for infinitely many $n \in \mathbb{N}$, then $X_{\mathcal{G}}$ is a Cantor set and $\phi_{\mathcal{G}}$ is a minimal action of $\mathbb{Z}^d$ on it, which is free if and only if $\bigcap_{n=1}^{\infty} G_n = \{0\}$. The action $\phi_{\mathcal{G}}$ is also isometric in the metric $d_{\mathcal{G}}$ given by

$$d_{\mathcal{G}}(x, y) = \sup\Big\{0, \frac{1}{n} \;\Big|\; \pi_n(x) \neq \pi_n(y)\Big\}, \quad x, y \in X_{\mathcal{G}}.$$

Moreover, $X_{\mathcal{G}}$ supports a unique $\phi_{\mathcal{G}}$-invariant probability measure $\mu_{\mathcal{G}}$ such that

$$\mu_{\mathcal{G}}(\pi_n^{-1}(k + G_n)) = \frac{1}{[\mathbb{Z}^d : G_n]}, \quad n \in \mathbb{N},\ k \in \mathbb{Z}^d.$$

In [1], the authors proposed a system equivalent to $(X_{\mathcal{G}}, \phi_{\mathcal{G}})$ in the sense of conjugacy defined below, using Pontryagin duality. Namely, let H be a subgroup of $\mathbb{Q}^d$ so that H contains $\mathbb{Z}^d$. Let $Y_H = \widehat{H/\mathbb{Z}^d}$ be the Pontryagin dual of the quotient. Here, the groups H and $H/\mathbb{Z}^d$ are endowed with the discrete topology, and hence Y_H is compact. Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ denote the d-torus, and let $\rho\colon H/\mathbb{Z}^d \to \mathbb{T}^d$ be the map induced by the inclusion $\mathbb{Q}^d \hookrightarrow \mathbb{R}^d$, i.e., $\rho((r_1, \dots, r_d) + \mathbb{Z}^d) = (e^{2\pi i r_1}, \dots, e^{2\pi i r_d})$, $(r_1, \dots, r_d) \in H$. Identifying the Pontryagin dual of $\mathbb{T}^d$ with $\mathbb{Z}^d$, we have the dual map $\widehat{\rho}\colon \mathbb{Z}^d \to Y_H$. Then the action ψ_H of $\mathbb{Z}^d$ on Y_H is defined via

$$\psi_H^k(x) = x + \widehat{\rho}(k), \quad k \in \mathbb{Z}^d,\ x \in Y_H.$$

The action (Y_H, ψ_H) is free if and only if H is dense in $\mathbb{Q}^d$.

Definition 1.2. Let X, Y be topological spaces, let G be a group, and let ϕ, ψ be actions of G on X, Y, respectively, by homeomorphisms. The actions (X, ϕ) and (Y, ψ) of G are said to be *conjugate* if there exists a homeomorphism $h\colon X \to Y$ such that

$$h \circ \phi^g = \psi^g \circ h$$

for all $g \in G$. In this case, we refer to h as a *conjugacy* between the two actions.

If H is an increasing union of finite index extensions H_n of $\mathbb{Z}^d$, $n \in \mathbb{N}$, then, up to conjugacy, (Y_H, ψ_H) is the inverse limit of

$$(Y_{H_1}, \psi_{H_1}) \xleftarrow{\widehat{i_1}} (Y_{H_2}, \psi_{H_2}) \xleftarrow{\widehat{i_2}} \cdots, \tag{1.2}$$

where $i_n\colon H_n/\mathbb{Z}^d \to H_{n+1}/\mathbb{Z}^d$ is the inclusion, $n \in \mathbb{N}$. The correspondence between $\mathbb{Z}^d$-odometers and $\mathbb{Z}^d$-actions (Y_H, ψ_H), up to conjugacy, is established by realizing (Y_H, ψ_H) as an inverse limit (1.2) and by passing to dual lattices; see [1, Theorems 2.5 and 2.6]. Here, if K is a lattice in $\mathbb{R}^d$, its dual lattice K^* is

$$K^* = \{x \in \mathbb{R}^d \mid \langle k, x\rangle \in \mathbb{Z} \text{ for all } k \in K\}.$$

We now consider odometers associated to integer matrices. For a non-singular $d \times d$ matrix A with integer coefficients, $A \in \mathrm{M}_d(\mathbb{Z})$, define

$$G_A = \{A^{-k}x \mid x \in \mathbb{Z}^d,\ k \in \mathbb{Z}\}, \quad \mathbb{Z}^d \subseteq G_A \subseteq \mathbb{Q}^d.$$

One can readily check that G_A is a subgroup of $\mathbb{Q}^d$. Applying the process described above to the group $H = G_A$, we get an associated $\mathbb{Z}^d$-odometer Y_{G_A}, defined up to conjugacy. In [3], the second author classified groups G_A in the 2-dimensional case and applied the results to 2-dimensional odometers Y_{G_A} using [1, Theorem 1.5]. In [4], the second author studied the question of when G_A, G_B are isomorphic as abstract groups for non-singular $A, B \in \mathrm{M}_d(\mathbb{Z})$ for an arbitrary d. We combine the results from [4] with the results of this paper to analyze when Y_{G_A}, Y_{G_B} are equivalent with respect to conjugacy, isomorphism, continuous orbit equivalence, and orbit equivalence.

2. Conjugacy and isomorphism

In what follows, $\mathrm{M}_d(\mathbb{Z})$ denotes the ring of $d \times d$ matrices with integer coefficients, and $\mathrm{GL}_d(\mathbb{Z})$ denotes the group of non-singular matrices $A \in \mathrm{M}_d(\mathbb{Z})$ with $\det A = \pm 1$. Proposition 2.3 below extends [1, Theorem 4.4] to higher dimensions and its proof requires the following lemma.

Lemma 2.1. *Let $d \in \mathbb{N}$ and let G be a subgroup of $\mathbb{Z}^d$ of finite index. Then for each $i = 1, \dots, d$, there exists a free basis $\mathcal{B} = \{f_1, \dots, f_d\}$ of G such that $f_i = a_i e_i$, where $a_i \in \mathbb{N}$ and e_i is the i-th vector of the standard basis of $\mathbb{Z}^d$. Equivalently, for any non-singular $M \in \mathrm{M}_d(\mathbb{Z})$ and each $i = 1, \dots, d$, there exists $P(i) \in \mathrm{GL}_d(\mathbb{Z})$ such that the i-th column of $MP(i)$ is $a_i e_i$ for some $a_i \in \mathbb{N}$.*

Proof. Since G is a subgroup of $\mathbb{Z}^d$ of finite index, G is a free group of rank d. Thus, the columns of M form a basis of G. We first prove the following.

Lemma 2.2. *For any j there exists a basis $g_1, \dots, g_d$ of G such that the j-th component $[g_l]_j$ of g_l is zero for any $l = 2, \dots, d$ and $[g_1]_j = a_j$, $a_j \in \mathbb{N}$. Equivalently, the j-th row of the matrix $(g_1 \quad g_2 \quad \dots \quad g_d)$ (we write coordinates of each g_i as a column) is $(a_j \quad 0 \quad \dots \quad 0)$.*

Proof. Let $g_1, \dots, g_d$ be an arbitrary basis of G, and let

$$M = (g_1 \quad g_2 \quad \dots \quad g_d) = (g_{ik}),$$

$M \in \mathrm{M}_d(\mathbb{Z})$ is non-singular. Note that there exists at least one non-zero element in the j-th row of M since $\det M \neq 0$. Assume there are two non-zero elements in the j-th row of M. Without loss of generality (we can interchange columns of M), $g_{j1} \neq 0$, $g_{j2} \neq 0$. By multiplying g_1, g_2 by -1 if necessary, we can assume $g_{j1} > 0$, $g_{j2} > 0$. If $g_{j1} = g_{j2}$, then $g'_1, \dots, g'_d$ is a new basis of G with $g'_2 = g_2 - g_1$, $g'_l = g_l$, $l \neq 2$, and $[g'_2]_j = 0$.

If $g_{j1} \neq g_{j2}$, then without loss of generality, we can assume $g_{j1} > g_{j2}$. Let $a_1 = g_{j1}$, $a_2 = g_{j2}$, and $a_1 = a_2 k + a_3$, for some $k, a_3 \in \mathbb{Z}$, $0 \le a_3 < a_2$. Then $g'_1, \dots, g'_d$ is a new basis of G with $g'_1 = g_2$, $g'_2 = g_1 - k g_2$, $g'_l = g_l$, $l \neq 1, 2$, and $[g'_1]_j = a_2$, $[g'_2]_j = a_3$, $0 \le a_3 < a_2 < a_1$. Continuing this way, we get a sequence $0 \le \dots < a_h < \dots < a_3 < a_2 < a_1$ of non-negative integer numbers, which in finitely many steps has to reach zero (this is essentially the Euclidean algorithm). This shows that there exists a basis $g'_1, \dots, g'_d$ of G such that $[g'_2]_j = 0$. Repeating the process for any other g'_s, g'_t with $s \neq t$, $[g'_s]_j \neq 0$, $[g'_t]_j \neq 0$, we conclude that there exists a basis $g''_1, \dots, g''_d$ of G such that $[g''_1]_j \in \mathbb{N}$, $[g'_l]_j = 0$ for any $l \neq 1$. ■

We now use induction on d to prove Lemma 2.1. Let $d = 1$. Then $e_1 = 1$, $G = a\mathbb{Z}$ for some $a > 0$, $f_1 = a e_1$, and the claim follows. Let $d > 1$. We consider two cases: $i > 1$ and $i = 1$. Let $i > 1$. By Lemma 2.2 applied to $j = 1$, there exists a basis $g_1, \dots, g_d$ of G such that

$$M = \begin{pmatrix} g_1 & \dots & g_d \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ * & M' \end{pmatrix}, \quad a_1 \in \mathbb{N},\ M' \in \mathrm{M}_{d-1}(\mathbb{Z}),\ \det M' \neq 0.$$

By induction on d, there exists $S \in \mathrm{GL}_{d-1}(\mathbb{Z})$ such that the $(i-1)$-st column of $M'S$ is $b e'_{i-1}$, where $b \in \mathbb{N}$ and e'_{i-1} is the $(i-1)$-st vector of the standard basis of $\mathbb{Z}^{d-1}$. Let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} \in \mathrm{GL}_d(\mathbb{Z}).$$

Then the i-th column of MP is $b e_i$, and the claim follows for $i > 1$.

Let $i = 1$. By Lemma 2.2 applied to $j = d$, there exists a basis $g_1, \dots, g_d$ of G such that

$$M = \begin{pmatrix} g_d & \dots & g_1 \end{pmatrix} = \begin{pmatrix} M'' & * \\ 0 & a_d \end{pmatrix}, \quad a_d \in \mathbb{N},\ M'' \in \mathrm{M}_{d-1}(\mathbb{Z}),\ \det M'' \neq 0.$$

As in the case $i > 1$, we apply the induction on d to M''. Thus, there exists $S' \in \mathrm{GL}_{d-1}(\mathbb{Z})$ such that the 1-st column of $M''S'$ is $b'e'_1$, where $b' \in \mathbb{N}$ and e'_1 is the 1-st vector of the standard basis of $\mathbb{Z}^{d-1}$. Let

$$P' = \begin{pmatrix} S' & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_d(\mathbb{Z}).$$

Then the 1-st column of MP' is $b'e_1$, and the claim follows in the case $i = 1$ as well.

This completes the proof of Lemma 2.1. ■

For a system (X, ϕ), where X is a topological space and ϕ is an action of $\mathbb{Z}^d$ on X by homeomorphisms, the first cohomology group $H^1(X, \phi)$ is defined as follows. A 1-cocycle θ is a continuous function $\theta: X \times \mathbb{Z}^d \to \mathbb{Z}$ such that

$$\theta(x, m+n) = \theta(x, m) + \theta(\phi^m(x), n), \quad x \in X,\ m, n \in \mathbb{Z}^d.$$

A 1-cocycle θ is a coboundary if and only if there exists a continuous function $h: X \to \mathbb{Z}$ such that

$$\theta(x,n) = h(\phi^n(x)) - h(x), \quad x \in X,\ n \in \mathbb{Z}^d.$$

Let $Z^1(X,\phi)$ and $B^1(X,\phi)$ denote the groups of 1-cocycles and coboundaries, respectively, and let $H^1(X,\phi) = Z^1(X,\phi)/B^1(X,\phi)$ be the first cohomology group.

We now recall the definition of the map τ_μ^1, where μ is an invariant probability measure on a $\mathbb{Z}^d$-action (X,ϕ). If θ is a 1-cocycle, $\tau_\mu^1(\theta) \in \mathrm{Hom}(\mathbb{Z}^d,\mathbb{R})$ is given by

$$\tau_\mu^1(\theta)(n) = \int_X \theta(x,n)\,d\mu(x), \quad n \in \mathbb{Z}^d.$$

Since $\tau_\mu^1(\theta) = 0$ if θ is a coboundary, τ_μ^1 passes to a well-defined group homomorphism

$$\tau_\mu^1\colon H^1(X,\phi) \to \mathrm{Hom}(\mathbb{Z}^d,\mathbb{R}). \tag{2.1}$$

The space $\mathrm{Hom}(\mathbb{Z}^d,\mathbb{R})$ is identified with $\mathbb{R}^d$ via the map that takes $\alpha \in \mathrm{Hom}(\mathbb{Z}^d,\mathbb{R})$ to $(\alpha(e_1),\dots,\alpha(e_d)) \in \mathbb{R}^d$.

We also denote by $B_\mu^1(X,\phi)$ the group of 1-cocycles $\theta(x,n) \in Z^1(X,\phi)$ such that

$$\tau_\mu^1(\theta) = 0,$$

and by $H_\mu^1(X,\phi)$ the group $Z^1(X,\phi)/B_\mu^1(X,\phi)$. The group $H_\mu^1(X,\phi)$ is a quotient of $H^1(X,\phi)$. By definition of $B_\mu^1(X,\phi)$, the map τ_μ^1 factors through the quotient $H_\mu^1(X,\phi)$, and, by abuse of notation, we denote the resulting map also by τ_μ^1:

$$\tau_\mu^1\colon H_\mu^1(X,\phi) \to \mathrm{Hom}(\mathbb{Z}^d,\mathbb{R}). \tag{2.2}$$

Proposition 2.3. *Let $d \in \mathbb{N}$, and let H be a dense subgroup of $\mathbb{Q}^d$ such that $\mathbb{Z}^d \subseteq H$. Let μ be the unique invariant probability measure for $\mathbb{Z}^d$-action (Y_H,ψ_H). Then the map*

$$\tau_\mu^1\colon H_\mu^1(Y_H,\psi_H) \to H$$

given by (2.2) is an isomorphism.

Proof. We supplement the proof of [1, Theorem 4.4] with Lemma 2.1 above. This allows to simplify computations and also extends [1, Theorem 4.4] to an arbitrary dimension.

It is immediate from the definitions above that τ_μ^1 is an injective group homomorphism. We write $H = \bigcup_{n\in\mathbb{N}} H_n$, the union of an increasing sequence of finite index extensions H_n of $\mathbb{Z}^d$. Let $G_n = H_n^*$ be the dual lattice. Then $\mathcal{G} = \{G_1, G_2, \dots\}$ is a decreasing sequence of finite index subgroups of $\mathbb{Z}^d$ such that (Y_H,ψ_H) and $(X_{\mathcal{G}},\phi_{\mathcal{G}})$ are conjugate.

The group $H^1(Y_H,\psi_H)$ is the direct limit of the sequence $H^1(\mathbb{Z}^d/G_n,\phi_{G_n})$, where ϕ_{G_n} is the natural action of $\mathbb{Z}^d$ on $\mathbb{Z}^d/G_n$ given by (1.1), $n \in \mathbb{N}$. Therefore, if $[\theta] \in H_\mu^1(Y_H,\psi_H)$, then θ is a cocycle in $Z^1(\mathbb{Z}^d/G_n,\phi_{G_n})$ for some n. We can write

$$\tau_\mu^1(\theta) = [\mathbb{Z}^d : G_n]^{-1} \sum_{k\in F} (\theta(k+G_n,e_1),\dots,\theta(k+G_n,e_d)),$$

where F is a fundamental domain for G_n. Since $\theta(k + G_n, e_i) \in \mathbb{Z}$ for any $i = 1, \dots, d$, we have $\tau_\mu^1(\theta) \in H_n$ and hence $\tau_\mu^1(\theta) \in H$.

Clearly, to show that $\tau_\mu^1\colon H_\mu^1(Y_H, \psi_H) \to H$ given by (2.2) is surjective, it is enough to show that $\tau_\mu^1\colon H^1(Y_H, \psi_H) \to H$ given by (2.1) is surjective. Let $h \in H$. Then $h \in H_n$ for some $n \in \mathbb{N}$. We show that there exists $[\theta] \in H^1(Y_H, \psi_H)$ such that $\tau_\mu^1(\theta) = h$. According to [1, Lemma 4.2], for each $\theta \in Z^1(\mathbb{Z}^d/G_n, \phi_{G_n})$, the map $\alpha(\theta)$ given by

$$\alpha(\theta)(g) = \theta(G_n, g), \quad g \in G_n,$$

induces an isomorphism $\alpha\colon H^1(\mathbb{Z}^d/G_n, \phi_{G_n}) \to \mathrm{Hom}(G_n, \mathbb{Z})$. The group $\mathrm{Hom}(G_n, \mathbb{Z})$ is identified with H_n via the canonical inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$, and thus there exists $\theta \in Z^1(\mathbb{Z}^d/G_n, \phi_{G_n})$ with $\alpha(\theta) = \langle \cdot, h \rangle$.

It remains to show $\tau_\mu^1(\theta) = h$, i.e.,

$$\sum_{k \in F} \theta(k + G_n, e_i) = [\mathbb{Z}^d : G_n] \langle e_i, h \rangle \quad \text{for each } i = 1, \dots, d. \tag{2.3}$$

For each fixed $i = 1, \dots, d$, we apply Lemma 2.1 to find a basis $\mathcal{B} = \{f_1, \dots, f_d\}$ for G_n such that $f_i = a e_i$, where $a \in \mathbb{N}$. We choose a fundamental domain F for G_n in equation (2.3) to consist of elements from $\mathbb{Z}^d$ inside a parallelepiped in $\mathbb{R}^d$ determined by the basis $\mathcal{B}$. Namely,

$$F = \{t_1 f_1 + \dots + t_d f_d \mid 0 \le t_1, \dots, t_d < 1\} \cap \mathbb{Z}^d.$$

The map θ is defined by

$$\theta(k_1 + G_n, k_2 + g) = \langle g', h \rangle$$

for $k_1, k_2 \in F$, $g \in G_n$, where $g' \in G_n$ is the unique element such that $k_1 + k_2 + g = k' + g'$ and $k' \in F$. One can check that so defined θ is a 1-cocycle. To compute the left-hand side of (2.3), we choose $k_1 = k \in F$, $k_2 = e_i$, and $g = 0$. Note that $k_2 \in F$ because of our choice of F. If $k \in F$ is such that $k + e_i$ is also in F, then $g' = 0$, and thus the term $\theta(k + G_n, e_i)$ does not contribute anything to the sum in (2.3).

Now assume $k \in F$ is such that $k + e_i$ is not in F. Let F_i denote the set of all such k. In this case, $g' = f_i$ because $k' = k + e_i - f_i \in F$. Indeed, let $P_{k,i}$ consist of all elements $n \in F$ such that for each $j = 1, \dots, d$, $j \neq i$, the j-th component of n coincides with the j-th component of k. Since F is a parallelepiped with side $f_i = a e_i$, $a \in \mathbb{N}$, the i-th coordinates of elements $n \in P_{k,i}$ form a sequence of a consecutive integers $m, m+1, \dots, m+a-1$. Now, by the choice of k, its i-th component is $m + a - 1$. For $j = 1, \dots, d$, $j \neq i$, the j-th component of k' equals the j-th component of k. The i-th component of k' equals m, and thus $k' \in P_{k,i} \subseteq F$.

We conclude that $\theta(k + G_n, e_i) = \langle f_i, h \rangle = a \langle e_i, h \rangle$ for each $k \in F_i$. The number of elements k in F_i equals the number of elements in the projection P_i of F to $\mathbb{Z}^{d-1}$ obtained from $\mathbb{Z}^d$ by omitting the i-th coordinate. Indeed, by omitting the i-th coordinate of $k \in F_i \subseteq F$, we get an element in P_i. Conversely, for any $k_i \in P_i$, the i-th coordinates of all elements in F that project to k_i form a sequence $m, m+1, \ldots, m+a-1$, since F is a parallelepiped with side $f_i = ae_i, a \in \mathbb{N}$. The element k that projects to k_i and whose i-th coordinate is $m+a-1$ is in F_i. The projection P_i is itself a parallelepiped in $\mathbb{Z}^{d-1}$. Let b denote the number of elements in P_i, which is the same as the number of elements in F_i. We therefore conclude

$$\sum_{k\in F} \theta(k+G_n, e_i) = \sum_{k\in F_i} \theta(k+G_n, e_i) = ab\langle e_i, h\rangle.$$

Note that ab is the number of elements in F, which is equal to $[\mathbb{Z}^d : G_n]$. Thus, (2.3) holds. ■

A characterization of a $\mathbb{Z}^d$-action (Y_H, ψ_H) up to conjugacy now follows from Proposition 2.3 and the following elementary lemma.

Lemma 2.4. *Let $d \in \mathbb{N}$. If $\mathbb{Z}^d$-actions (Y_{H_1}, ψ_{H_1}) and (Y_{H_2}, ψ_{H_2}) are conjugate, then*

$$\tau^1_{\mu_1}(H^1_{\mu_1}(Y_{H_1}, \psi_{H_1})) = \tau^1_{\mu_2}(H^1_{\mu_2}(Y_{H_2}, \psi_{H_2})),$$

where μ_1, μ_2 are the unique probability measures for (Y_{H_1}, ψ_{H_1}), (Y_{H_2}, ψ_{H_2}), respectively.

Proof. Let h be a conjugacy from (Y_{H_1}, ψ_{H_1}) to (Y_{H_2}, ψ_{H_2}). It induces an isomorphism

$$h^*\colon H^1_{\mu_2}(Y_{H_2}, \psi_{H_2}) \to H^1_{\mu_1}(Y_{H_1}, \psi_{H_1})$$

given by $h^*([\theta]) = [h^*(\theta)]$, where $h^*(\theta(y,n)) = \theta(h(x), n)$, $y = h(x)$, $x \in Y_{H_1}$. The invariance of μ_1 and uniqueness of μ_2 imply $h_*(\mu_1) = \mu_2$, where $h_*(\mu_1)$ denotes the push-forward measure of μ_1 under h. Therefore, for $[\theta(y,n)] \in H^1_{\mu_2}(Y_{H_2}, \psi_{H_2})$, one has

$$\begin{aligned}\tau^1_{\mu_2}(\theta)(n) &= \int_{Y_{H_2}} \theta(y,n)\, d\mu_2 = \int_{Y_{H_2}} \theta(y,n)\, dh_*(\mu_1)\\ &= \int_{Y_{H_1}} \theta(h(x), n)\, d\mu_1 = \tau^1_{\mu_1}(h^*(\theta))(n).\end{aligned}$$ ■

Corollary 2.5. *Let H_1, H_2 be dense subgroups of $\mathbb{Q}^d$ such that $\mathbb{Z}^d \subseteq H_1$, $\mathbb{Z}^d \subseteq H_2$. Two $\mathbb{Z}^d$-actions (Y_{H_1}, ψ_{H_1}) and (Y_{H_2}, ψ_{H_2}) are conjugate if and only if $H_1 = H_2$.*

Proof. If $H_1 = H_2$, then the conjugacy is trivial. Assume (Y_{H_1}, ψ_{H_1}) and (Y_{H_2}, ψ_{H_2}) are conjugate. By Proposition 2.3,

$$\tau^1_{\mu_1}(H^1_{\mu_1}(Y_{H_1}, \psi_{H_1})) = H_1, \quad \tau^1_{\mu_2}(H^1_{\mu_2}(Y_{H_2}, \psi_{H_2})) = H_2,$$

where μ_1 and μ_2 are the unique invariant probability measures for the actions (Y_{H_1}, ψ_{H_1}) and (Y_{H_2}, ψ_{H_2}), respectively. By Lemma 2.4, we have $H_1 = H_2$. ■

Definition 2.6. Let (X, ϕ) be an action of a group G, and let (Y, ψ) be an action of a group H. An *isomorphism* between the actions is a pair (h, α), where $h: X \to Y$ is a homeomorphism and $\alpha: G \to H$ is a group isomorphism, such that

$$h \circ \phi^g = \psi^{\alpha(g)} \circ h.$$

If such a pair (h, α) exists, then (X, ϕ) and (Y, ψ) are said to be *isomorphic*.

Proposition 2.7. *Let $d_1, d_2 \in \mathbb{N}$, and let H_1 (resp. H_2) be a dense subgroup of $\mathbb{Q}^{d_1}$ (resp. $\mathbb{Q}^{d_2}$) that contains $\mathbb{Z}^{d_1}$ (resp. $\mathbb{Z}^{d_2}$). A $\mathbb{Z}^{d_1}$-action (Y_{H_1}, ψ_{H_1}) is isomorphic to a $\mathbb{Z}^{d_2}$-action (Y_{H_2}, ψ_{H_2}) if and only if $d_1 = d_2$, the common value being denoted by d, and there exists $A \in \mathrm{GL}_d(\mathbb{Z})$ such that $AH_1 = H_2$.*

Proof. If $d = d_1 = d_2$ and $AH_1 = H_2$ for some $A \in \mathrm{GL}_d(\mathbb{Z})$, then (Y_{AH_1}, ψ_{AH_1}) is trivially conjugate to (Y_{H_2}, ψ_{H_2}). Moreover, the actions (Y_{AH_1}, ψ_{AH_1}) and (Y_{H_1}, ψ_{H_1}) are isomorphic via the automorphism of $\mathbb{Z}^d$ defined by A (see [1, Proposition 2.8]).

Conversely, assume (Y_{H_1}, ψ_{H_1}) and (Y_{H_2}, ψ_{H_2}) are isomorphic. The existence of a group isomorphism $\alpha: H_1 \to H_2$ implies $d_1 = d_2$. Indeed, d_i is the rank of H_i, $i = 1, 2$, and a group isomorphism preserves the rank. We denote the common value of d_1, d_2 by d. By [1, Proposition 2.8], there is $A \in \mathrm{GL}_d(\mathbb{Z})$ such that (Y_{AH_1}, ψ_{AH_1}) is conjugate to (Y_{H_2}, ψ_{H_2}). We now apply Corollary 2.5 to conclude $AH_1 = H_2$. ■

3. Continuous orbit equivalence

Definition 3.1. An action (X, ϕ) of a group G and an action (Y, ψ) of another group G' are said to be *orbit equivalent* if there exists a homeomorphism $h: X \to Y$ such that for each $x \in X$ one has

$$h(\{\phi^g(x) \mid g \in G\}) = \{\psi^{g'}(h(x)) \mid g' \in G'\}. \tag{3.1}$$

In [1], the authors characterize orbit equivalence of $\mathbb{Z}^d$-action (Y_H, ψ_H), where H is a dense subgroup of $\mathbb{Q}^d$ containing $\mathbb{Z}^d$, using superindex $[\![H : \mathbb{Z}^d]\!]$. The *superindex* is defined as

$$[\![H : \mathbb{Z}^d]\!] = \{[H' : \mathbb{Z}^d] \mid \mathbb{Z}^d \subseteq H' \subseteq H,\ [H' : \mathbb{Z}^d] < \infty\}.$$

Theorem 3.2 ([1, Corollary 5.5]). *Let $d, d' \in \mathbb{N}$, let H be a dense subgroup of $\mathbb{Q}^d$ that contains $\mathbb{Z}^d$, and let H' be a dense subgroup of $\mathbb{Q}^{d'}$ that contains $\mathbb{Z}^{d'}$. The $\mathbb{Z}^d$-action (Y_H, ψ_H) and the $\mathbb{Z}^{d'}$-action $(Y_{H'}, \psi_{H'})$ are orbit equivalent if and only if*

$$[\![H : \mathbb{Z}^d]\!] = [\![H' : \mathbb{Z}^{d'}]\!].$$

Let (X, ϕ) be an action of a group G, and let (Y, ψ) be an action of a group G'. Assume that (X, ϕ) and (Y, ψ) are orbit equivalent. By definition, there exists a homeomorphism

$h: X \to Y$ satisfying (3.1). If the actions (X, ϕ), (Y, ψ) are free, then there exist unique maps $\alpha: X \times G \to G'$ and $\beta: Y \times G' \to G$ such that

$$h(\phi^g(x)) = \psi^{\alpha(x,g)}(h(x))$$

for all $x \in X$, $g \in G$, and also

$$h^{-1}(\psi^{g'}(y)) = \phi^{\beta(y,g')}(h^{-1}(y))$$

for all $y \in Y$, $g' \in G'$. The maps α, β are called *orbit cocycles*.

Definition 3.3. Let (X, ϕ) and (Y, ψ) be free actions of groups G and G', respectively, such that (X, ϕ) and (Y, ψ) are orbit equivalent. Then the actions (X, ϕ) and (Y, ψ) are called *continuously orbit equivalent* if there is a homeomorphism $h: X \to Y$ such that the associated orbit cocycles α and β are continuous in the corresponding product topologies.

Proposition 3.4. *Let $d_1, d_2 \in \mathbb{N}$, let (Y_{H_1}, ψ_{H_1}) be a free $\mathbb{Z}^{d_1}$-action, and let (Y_{H_2}, ψ_{H_2}) be a free $\mathbb{Z}^{d_2}$-action. Then (Y_{H_1}, ψ_{H_1}) and (Y_{H_2}, ψ_{H_2}) are continuously orbit equivalent if and only if $d = d_1 = d_2$ and there exists $A \in \mathrm{GL}_d(\mathbb{Q})$ with $\det A = \pm 1$ and $AH_1 = H_2$.*

Proof. Here $\mathrm{GL}_d(\mathbb{Q})$ denotes the group of non-singular $d \times d$ matrices with rational coefficients. The proof follows the lines of the proof of [1, Theorem 5.7], where one uses Proposition 2.7 in place of [1, Corollary 5.1]. ∎

Remark 3.5. Investigation of continuous orbit equivalence in the general setting was carried out in [2]. In particular, conditions on when continuous orbit equivalence implies isomorphic equivalence were given in that paper.

4. Odometers defined by matrices

In this section, we generalize the results in [3] on $\mathbb{Z}^2$-odometers defined by matrices with integer coefficients to the d-dimensional case, $d > 2$. For convenience, we first put together the results from previous sections in one theorem. To simplify notation, we denote a $\mathbb{Z}^d$-action (Y_H, ψ_H) defined above for a subgroup H of $\mathbb{Q}^d$ containing $\mathbb{Z}^d$ by Y_H.

Theorem 4.1. *Let $d \in \mathbb{N}$, and let H_1, H_2 be dense subgroups of $\mathbb{Q}^d$ such that $\mathbb{Z}^d \subseteq H_1$, $\mathbb{Z}^d \subseteq H_2$. Then*

(1) *$\mathbb{Z}^d$-actions Y_{H_1}, Y_{H_2} are conjugate if and only if $H_1 = H_2$.*

(2) *$\mathbb{Z}^d$-actions Y_{H_1}, Y_{H_2} are isomorphic if and only if there is $T \in \mathrm{GL}_d(\mathbb{Z})$ such that $TH_1 = H_2$.*

(3) *$\mathbb{Z}^d$-actions Y_{H_1}, Y_{H_2} are continuously orbit equivalent if and only if there is $T \in \mathrm{GL}_d(\mathbb{Q})$ such that $\det T = \pm 1$ and $TH_1 = H_2$.*

(4) *Assume H_1, H_2 are dense in $\mathbb{Q}^d$. Then $\mathbb{Z}^d$-actions Y_{H_1}, Y_{H_2} are orbit equivalent if and only if $[\![H_1 : \mathbb{Z}^d]\!] = [\![H_2 : \mathbb{Z}^d]\!]$.*

Proof. This is the content of Corollary 2.5, Proposition 2.7, Theorem 3.2, and Proposition 3.4 above. ■

Recall that for a non-singular $d \times d$ matrix A with integer coefficients, $A \in \mathrm{M}_d(\mathbb{Z})$,

$$G_A = \{A^{-k}x \mid x \in \mathbb{Z}^d,\, k \in \mathbb{Z}\}, \quad \mathbb{Z}^d \subseteq G_A \subseteq \mathbb{Q}^d.$$

One can check that G_A is a subgroup of $\mathbb{Q}^d$. For simplicity, we denote Y_{G_A} by Y_A. Note that G_A is naturally isomorphic to the inductive limit of the system $(\mathbb{Z}^d, f_j)_{j\in\mathbb{N}}$, where each $f_j\colon \mathbb{Z}^d \to \mathbb{Z}^d$ is given by multiplication by A, $f_j(\mathbf{x}) = A\mathbf{x}$, $\mathbf{x} \in \mathbb{Z}^d$, $j \in \mathbb{N}$.

We start with a characterization of dense groups G_A in $\mathbb{Q}^d$. Let $h_A \in \mathbb{Z}[t]$ be the characteristic polynomial of A, and let $h_A = h_1 h_2 \cdots h_s$, where $h_1, h_2, \ldots, h_s \in \mathbb{Z}[t]$ are non-constant and irreducible.

Lemma 4.2 ([4, Lemma 8.1]). *The group G_A is dense in $\mathbb{Q}^d$ if and only if $h_i(0) \neq \pm 1$ for all $i \in \{1, 2, \ldots, s\}$.*

Next, we describe orbit equivalent odometers.

Lemma 4.3 ([4, Lemma 8.2]). *Let $A, B \in \mathrm{M}_d(\mathbb{Z})$ be non-singular such that G_A (resp. G_B) is dense in $\mathbb{Q}^d$. Then $\mathbb{Z}^d$-actions Y_A, Y_B are orbit equivalent if and only if* $\det A$, $\det B$ *have the same prime divisors (in $\mathbb{Z}$).*

The next lemma is a special (simple) case when all the equivalences hold at the same time.

Lemma 4.4. *Let $A, B \in \mathrm{M}_d(\mathbb{Z})$ be non-singular such that G_A (resp. G_B) is dense in $\mathbb{Q}^d$. Assume that for any prime $p \in \mathbb{N}$ that divides* $\det A$ *we have*

$$h_A \equiv t^d \pmod p.$$

Then the following are equivalent:

(1) *$\mathbb{Z}^d$-actions Y_A, Y_B are conjugate;*

(2) *$\mathbb{Z}^d$-actions Y_A, Y_B are isomorphic;*

(3) *$\mathbb{Z}^d$-actions Y_A, Y_B are continuously orbit equivalent;*

(4) $\det A$, $\det B$ *have the same prime divisors, and for any prime $p \in \mathbb{N}$ that divides* $\det B$, *we have*

$$h_B \equiv t^d \pmod p.$$

Proof. Follows from Theorem 4.1 and [3, Lemma 3.10]. ■

Let

$$\det A = a p_1^{s_1} p_2^{s_2} \cdots p_l^{s_l}$$

be the prime-power factorization of $\det A$, where $p_1, p_2, \ldots, p_l \in \mathbb{N}$ are distinct primes, $a = \pm 1$, and $s_1, s_2, \ldots, s_l \in \mathbb{N}$. Let

$$\mathcal{P} = \mathcal{P}(A) = \{p_1, p_2, \ldots, p_l\}.$$

If $\mathcal{P} = \emptyset$, equivalently, $\det A = \pm 1$, then $G_A = \mathbb{Z}^d$ and Y_A is trivial. Moreover, for a non-singular $B \in \mathrm{M}_d(\mathbb{Z})$, we know that $TG_A = G_B$ for some $T \in \mathrm{GL}_d(\mathbb{Q})$ if and only if $\det B = \pm 1$ and hence $G_B = \mathbb{Z}^d$ [3, Lemma 3.2 (i)].

In what follows, we assume $\mathcal{P} \neq \emptyset$. Denote

$$\mathcal{P}' = \mathcal{P}'(A) = \{p \in \mathcal{P},\ h_A \not\equiv t^d \ (\mathrm{mod}\, p)\},$$

where $h_A \in \mathbb{Z}[t]$ denotes the characteristic polynomial of A. The case $\mathcal{P}' = \emptyset$ is settled in Lemma 4.4, so in what follows we assume $\mathcal{P}' \neq \emptyset$. Finally, for a prime $p \in \mathbb{N}$ let $t_p = t_p(A)$ denote the multiplicity of zero in the reduction of the characteristic polynomial of A modulo p, $0 \leq t_p \leq d$. Thus, in our notation $t_p \neq d$ if and only if $p \in \mathcal{P}'$.

Even though the results in [4] apply to isomorphisms between groups G_A, G_B for arbitrary non-singular $A, B \in \mathrm{M}_d(\mathbb{Q})$, to avoid making this paper too technical, we only consider a generic case when the characteristic polynomials of A, B are irreducible. An interested reader could use [4] together with Theorem 4.1 to treat other cases. Another additional assumption we make is the condition that there exists $p \in \mathcal{P}'$ such that the greatest common divisor (t_p, d) of t_p and d is 1. It seems that this condition provides the right setting for the generalization of the 2-dimensional case to higher dimensions. In particular, in the 2-dimensional case, if h_A is irreducible and $TG_A = G_B$ for some $T \in \mathrm{GL}_d(\mathbb{Q})$, then h_B is irreducible and T takes an eigenvector of A to an eigenvector of B [3]. It turns out these facts remain true under the assumption $(t_p, d) = 1$ and not true in general, e.g., all $t_p = 2$ and $d = 4$ [4].

Let $\overline{\mathbb{Q}}$ denote a fixed algebraic closure of $\mathbb{Q}$. Recall that $\overline{\mathbb{Q}}$ consists of algebraic numbers, roots of non-zero polynomials in one variable with rational coefficients. The eigenvalues of a matrix A with integer coefficients are algebraic numbers since they are roots of the characteristic polynomial of A. Let $A, B \in \mathrm{M}_d(\mathbb{Z})$ be non-singular, and let $\lambda_1, \ldots, \lambda_d \in \overline{\mathbb{Q}}$ (resp. $\mu_1, \ldots, \mu_d \in \overline{\mathbb{Q}}$) denote eigenvalues of A (resp. B). Assume there exists $T \in \mathrm{GL}_d(\mathbb{Q})$ that satisfies $TG_A = G_B$. Suppose further that the characteristic polynomial $h_A \in \mathbb{Z}[t]$ of A is irreducible and there exists a prime $p \in \mathcal{P}(A)$ such that $(t_p(A), d) = 1$. The following facts follow from [4, Proposition 5.7]. Namely, $h_B \in \mathbb{Z}[t]$ is irreducible. Moreover (up to rearrangement of eigenvalues),

$$\mathbb{Q}(\lambda_1) = \mathbb{Q}(\mu_1) \text{ and } \lambda_1, \mu_1 \text{ have the same prime divisors in the ring of integers.} \tag{4.1}$$

In particular, the splitting fields of h_A, h_B coincide. Furthermore, both A and B are diagonalizable (over $\overline{\mathbb{Q}}$) and there exist $M, N \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ such that

$$A = M \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_d \end{pmatrix} M^{-1}, \quad B = N \begin{pmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_d \end{pmatrix} N^{-1}, \tag{4.2}$$

and $T = NM^{-1}$. If $\mathbb{Z}^d$-actions Y_A, Y_B are conjugate (or isomorphic, or continuously orbit equivalent), then by Theorem 4.1, we have $T = I_d$, the identity matrix (or $T \in \mathrm{GL}_d(\mathbb{Z})$, or

$\det T = \pm 1$, respectively) and, hence, $M = N$ (or $NM^{-1} \in \mathrm{GL}_d(\mathbb{Z})$, or $\det NM^{-1} = \pm 1$, respectively). This proves the following lemma.

Lemma 4.5. *Let $A, B \in \mathrm{M}_d(\mathbb{Z})$ be non-singular such that G_A (resp. G_B) is dense in $\mathbb{Q}^d$. Assume the characteristic polynomial $h_A \in \mathbb{Z}[t]$ of A is irreducible and $(t_p, d) = 1$ for some prime $p \in \mathcal{P}$.*

(i) *If $\mathbb{Z}^d$-actions Y_A, Y_B are conjugate, then (4.1) holds, there exist $M, N \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ such that (4.2) holds, and $M = N$.*

(ii) *If $\mathbb{Z}^d$-actions Y_A, Y_B are isomorphic, then (4.1) holds, there exist $M, N \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ such that (4.2) holds, and $NM^{-1} \in \mathrm{GL}_d(\mathbb{Z})$.*

(iii) *If $\mathbb{Z}^d$-actions Y_A, Y_B are continuously orbit equivalent, then (4.1) holds, there exist $M, N \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ such that (4.2) holds, and $\det NM^{-1} = \pm 1$.*

As in the 2-dimensional case, the conditions in Lemma 4.5 are also sufficient in the cases of conjugacy and isomorphism.

Lemma 4.6. *Let $A, B \in \mathrm{M}_d(\mathbb{Z})$ be non-singular such that G_A (resp. G_B) is dense in $\mathbb{Q}^d$. Assume $h_A \in \mathbb{Z}[t]$ is irreducible and $(t_p, d) = 1$ for some prime $p \in \mathcal{P}$.*

(i) *If (4.1) holds and there exists $M \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ such that (4.2) holds for $N = M$, then $\mathbb{Z}^d$-actions Y_A, Y_B are conjugate.*

(ii) *If (4.1) holds, there exist $M, N \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ such that $NM^{-1} \in \mathrm{GL}_d(\mathbb{Z})$ and (4.2) holds, then $\mathbb{Z}^d$-actions Y_A, Y_B are isomorphic.*

Proof. First, (ii) follows easily from (i) as in the proof of [3, Lemma 8.10]. We repeat the argument for the sake of completeness. Assume (i) holds, let $X = NM^{-1} \in \mathrm{GL}_d(\mathbb{Z})$, and assume (4.1) and (4.2) hold. Then $XM = N$ and

$$XG_A = G_{XAX^{-1}} = G_{N\Lambda N^{-1}} = G_B, \quad \Lambda = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_d \end{pmatrix}.$$

Here, $XAX^{-1} = N\Lambda N^{-1} \in \mathrm{M}_d(\mathbb{Z})$, since $X \in \mathrm{GL}_d(\mathbb{Z})$. Also, $G_{N\Lambda N^{-1}} = G_B$ by (i) and Theorem 4.1. Thus, $\mathbb{Z}^d$-actions Y_A, Y_B are isomorphic by Theorem 4.1.

We now prove (i). Assume (4.1), (4.2) hold and $N = M$. Since h_A is irreducible, (4.1) implies that h_B is also irreducible. Indeed, h_B is irreducible if and only if $[\mathbb{Q}(\mu_1) : \mathbb{Q}] = d$. From (4.2) with $N = M$, we see that A, B share the same eigenvector $\mathbf{u} \in \overline{\mathbb{Q}}^d$ such that $A\mathbf{u} = \lambda_1 \mathbf{u}$, $B\mathbf{u} = \mu_1 \mathbf{u}$. Since h_A is irreducible, the Galois group $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts transitively on the eigenvalues of A, i.e., there exist $\sigma_2, \dots, \sigma_d \in G$ such that $\lambda_i = \sigma_i(\lambda_1)$, $\sigma_1 = \mathrm{id}$, $1 \le i \le d$. Since A, B have integer coefficients, by applying σ_i to $A\mathbf{u} = \lambda_1 \mathbf{u}$, $B\mathbf{u} = \mu_1 \mathbf{u}$, we conclude that $\sigma_i(\mathbf{u})$ is an eigenvector of A (resp. B) corresponding to λ_i (resp. $\sigma_i(\mu_1)$), $1 \le i \le d$, and $\sigma_1(\mu_1), \dots, \sigma_d(\mu_1)$ are all (distinct) eigenvalues of B. By abuse of notation, let $\mu_i = \sigma_i(\mu_i)$, $1 \le i \le d$. Moreover, it follows from (4.1) that h_A, h_B share the same splitting field K and for each i, $1 \le i \le d$, λ_i, μ_i have the same prime ideal divisors in the ring of integers $\mathcal{O}_K$ of K. Therefore, $\mathcal{P} = \mathcal{P}(A) = \mathcal{P}(B)$, $\mathcal{P}' = \mathcal{P}'(A) = \mathcal{P}'(B)$, and $t_p = t_p(A) = t_p(B)$ for any prime $p \in \mathbb{N}$. In [4], we discuss the characteristic $\{\alpha_{pij}\}_{p,i,j}$ of a group G_A with respect to a free basis $\{\mathbf{f}_1, \dots, \mathbf{f}_d\}$ of $\mathbb{Z}^d$, where $p \in \mathcal{P}'(A)$, $\alpha_{pij} \in \mathbb{Z}_p$, $1 \le i \le t_p(A)$, $t_p(A) + 1 \le j \le d$. The system

$$\mathcal{S}(A) = \{\mathbf{f}_1, \dots, \mathbf{f}_d, \alpha_{pij} \mid p,\ i,\ j\}$$

determines all generators of G_A over $\mathbb{Z}$ [4, Lemma 3.5]. Furthermore, we show that $\mathcal{S}(A)$ can be calculated from eigenvectors of A corresponding to eigenvalues divisible by a prime ideal of $\mathcal{O}_K$ that divides p, $p \in \mathcal{P}'$ [4, Remark 4.4]. Since for each i, $1 \le i \le d$, λ_i, μ_i have the same prime ideal divisors and A, B share the same eigenvectors corresponding to λ_i, μ_i, respectively, we see that G_A, G_B share the same set of generators. Therefore, $G_A = G_B$. ■

Remark 4.7. It follows from its proof that in Lemma 4.6 (i), instead of (4.2) with $N = M$, it is enough to assume that A, B share the same eigenvector $\mathbf{u} \in \overline{\mathbb{Q}}^d$ such that $A\mathbf{u} = \lambda_1 \mathbf{u}$, $B\mathbf{u} = \mu_1 \mathbf{u}$.

It is well known (it follows from the Latimer–MacDuffee–Taussky theorem) that for a fixed monic irreducible polynomial $h \in \mathbb{Z}[t]$ of degree d, there are finitely many $\mathrm{GL}_d(\mathbb{Z})$-conjugacy classes $[A]$ of $A \in \mathrm{M}_d(\mathbb{Z})$ with characteristic polynomial h. Let $n = n(h)$ denote the number of conjugacy classes. Also, for $S \in \mathrm{GL}_d(\mathbb{Z})$ we have

$$SG_A = G_{SAS^{-1}}.$$

As was discussed above, we know that if there exists t_p with $(t_p, d) = 1$, $A, B \in \mathrm{M}_d(\mathbb{Z})$ share the same irreducible characteristic polynomial and

$$TG_A = G_B$$

for some $T \in \mathrm{GL}_d(\mathbb{Q})$, then $TAT^{-1} = B$ [4]. This implies that there are exactly n isomorphism classes $[Y_A]_{\mathrm{isom}}$ of odometers Y_A, where $A \in \mathrm{M}_d(\mathbb{Z})$ has characteristic polynomial h, and there are less or equal than n continuously orbit equivalent classes $[Y_A]_{\mathrm{cont.orb}}$ of odometers Y_A, where $A \in \mathrm{M}_d(\mathbb{Z})$ has characteristic polynomial h.

Remark 4.8. Note that $M = N$ implies A, B commute. However, $AB = BA$ does not imply conjugacy, since the ordering of eigenvalues matters. For example,

$$G_A \ne G_B \quad \text{for } A = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}, \ B = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}.$$

Remark 4.9. We know from the 2-dimensional case that for odometers Y_A defined by $A \in \mathrm{M}_d(\mathbb{Z})$, continuous orbit equivalence is more subtle than conjugacy and isomorphism. Even in the 2-dimensional case, general sufficient conditions for $\mathbb{Z}^2$-actions Y_A, Y_B to be continuously orbit equivalent under the conditions of Lemma 4.5 become rather technical. However, in each particular example, it is possible to resolve the question using Theorem 4.1 and the results in [3, 4].

Example 4.10. In this example, $d = 3$ and $A, B, C \in \mathrm{M}_3(\mathbb{Z})$ have the same irreducible characteristic polynomial $h(t) = t^3 - 39t - 91$,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 91 & 39 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 0 & 0 \\ 5 & 1 & 0 \\ -24 & -4 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 49 & 0 & 0 \\ 33 & 1 & 0 \\ 4 & -4 & 1 \end{pmatrix}.$$

Since $\det A = \det B = \det C$, by Lemma 4.3, $\mathbb{Z}^d$-actions Y_A, Y_B, Y_C are orbit equivalent. All three A, B, and C are conjugate to each other in $\mathrm{GL}_3(\mathbb{Q})$. Moreover, A is a companion matrix of B and C. The three matrices above give (all) three equivalence classes of integer matrices with characteristic polynomial h up to conjugation by elements in $\mathrm{GL}_3(\mathbb{Z})$, i.e., any matrix in $\mathrm{M}_3(\mathbb{Z})$ with characteristic polynomial h is $\mathrm{GL}_3(\mathbb{Z})$-conjugate to A, B, or C, and any two matrices out of A, B, and C are not $\mathrm{GL}_3(\mathbb{Z})$-conjugate to each other (this can be verified using [5, 6]). Using the methods of [4], we can prove that any two out of Y_A, Y_B, and Y_C are not continuously orbit equivalent.

Example 4.11. In this example, $d = 4$, $A, B \in \mathrm{M}_4(\mathbb{Z})$ have the same irreducible characteristic polynomial $h(t) = t^4 + t^2 + 9$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 0 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 9 & 0 & 2 & 1 \\ 9 & 0 & 1 & 1 \\ -18 & -9 & 7 & -1 \end{pmatrix}.$$

One can show that Lemma 4.6 (i) holds for A, B, so that $G_A = G_B$ (see also [4, Example 11]). Thus, $\mathbb{Z}^d$-actions Y_A, Y_B are conjugate.

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