Mathematisches Forschungsinstitut Oberwolfach

Report No. 34/2024

DOI: 10.4171/OWR/2024/34

Topologie

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21 July – 26 July 2024

Abstract. The aim of the recurring workshop is to inform topologists of various spcialization about major advances in other parts of topology, foster collaboration and possibly advance this broad and far reaching field of mathematics.

Mathematics Subject Classification (2020): 55-XX, 57-XX.

Introduction by the Organizers

The lectures in the workshop *Topologie* covered various topics in modern topology, including manifold and geometric topology, homotopy theory, and geometric group theory. Connections to neighboring areas, most prominently to gauge theory, was also in the center of focus. The following current research topics received special attention during the workshop: manifolds and K-theory, Seiberg–Witten invariants and other incarnations of gauge theory, generalizations of hyperbolic techniques in geometric group theory, and stable homotopy theory. The aim of the various topics was to foster communication and provide chances for participants to see and experience driving questions and important methods in nearby fields within the realm of topology.

In the final program we had 5 lectures on Monday and Tuesday, 3 on Wednesday and 3 on Friday (due to the traditional hike on Wednesday and some early departures on Friday). On Thursday we had 3 hour-long talks in the morning and 4 shorter talks in the afternoon. The format seemed to work very well.

The workshop opened with a survey talk by **Wolfang Lück** on L^2-invariants and their applications to algebra, geometry and group theory, touching on a variety of topics discussed later in the week.

The workshop featured three lectures by **Daniel Ruberman** these gave an overview of the study of diffeomorphism groups (and the comparison between the homeomorphism and the diffeomorphism groups) of four-dimensional manifolds. Gauge theoretic methods in this subject started to play a significant role in the 90's, and the application of family Seiberg–Witten invariants led to several breakthroughs in the recent past. Ruberman's lecture series was nicely complemented by a lecture of **Hokuto Konno**, where specific examples of the fundamental difference between the groups of homeomorphisms vs diffeomorphisms was demonstrated. Similar results (from yet another perspective and using slightly different tools) were presented by **Danica Kosanović**. Various other versions of gauge theoretic techniques (like Heegaard Floer homology, Pin(2)-equivariant Seiberg–Witten theory) played important roles in several further lectures.

In even lower dimensional topology, **Corey Bregman** presented a breakthrough result on diffeomorphism groups of 3-manifolds, resolving a conjecture of Kontsevich. Regarding the topology of high-dimensional manifolds, **Fabian Hebestreit** presented the state-of-the-art concerning surgery and homology manifolds, and **Manuel Krannich** explained a strengthening of Weiss' theorem of topological Pontryagin classes.

There were two complementary talks on scissors congruence, by **Inna Zakharevich** and **Alexander Kupers**, respectively explaining a topological point of view on a chain complex featuring in a conjecture on Goncharov, and a way of describing the homology of scissors automorphism groups in terms of assembler K-theory. The latter employed methods of homological stability, which was also discussed by **Nathalie Wahl**, as well as by **Ishan Levy** who explained an exciting new method which completes a programme of Ellenberg–Venkatesh–Westerland on the Cohen–Lenstra heuristics in number theory.

In the area of homotopy theory, **Gabriel Angelini-Knoll** detailed how the recently developed methods of syntomic cohomology are resulting in computations of the algebraic K-theory of ring spectra which were previously inaccessible. **Robert Burklund** surveyed the consequences of the results of him and his collaborators disproving the telescope conjecture. **Gregory Arone** discussed the computation of Ext groups between polynomial functors, with applications to the stable cohomology of $\mathrm{Aut}(F_n)$. **Connor Malin** lectured on a categorical form of Koszul duality, and used it to establish a chain rule for orthogonal calculus. **Hana Jia Kong** discussed real motivic analogs of a formula of Milnor's concerning the relations in the Steenrod algebra, with the goal of machine implementation to systematically extend low dimensional computations of real motivic stable stems.

There were two talks on topics in geometric group theory. **Stefanie Zbinden** discussed Morse boundaries of 3-manifolds. Morse boundaries are a generalization of Gromov boundaries designed to encode hyperbolic aspects of a space. Zbinden gave a complete description of the topology of the Morse boundary of all 3-manifold groups. **Elia Fioravanti**'s talk concerned the growth of automorphisms of right-angled Artin groups, and more generally, of fundamental groups of special cube complexes. The growth rate asks how fast the length of a group element grows as we apply powers of an automorphism. Fioravanti proves a variety of new results about these growth rates.

Acknowledgement: The workshop organizers would like to thank the MFO staff for their constant help and support in putting this event together.

Workshop: Topologie

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Abstracts

A survey on L^2-invariants and their applications to algebra, geometry and group theory

Wolfgang Lück

In this survey talk we will discuss L^2-invariants and their applications to algebra, geometry, group theory, and topology and some open problems.

We will state several theorems concerning group theory, algebra, differential geometry, algebraic geometry, and algebraic K-theory which on the first glance do not involve L^2-invariants but whose proofs rely on L^2-methods. Here are some examples due to Cheeger, Cochrane, Gromov, Kielak, Lück, Orr, and Teichner

- Let G be a group which contains a normal infinite amenable subgroup. Suppose that there is a finite model for BG. Then its Euler characteristic $\chi(BG)$ vanishes.
- Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ be an exact sequence of infinite groups. Suppose that G is finitely presented and H is finitely generated. Then its deficiency satisfies
$$\operatorname{defi}(G) \leq 1.$$
- Let M be a closed oriented 4-manifold. Suppose that there is an exact sequence of infinite groups $1 \to H \xrightarrow{i} \pi_1(X) \xrightarrow{q} K \to 1$ such that H is finitely generated. Then its signature and Euler characteristic satisfies
$$|\operatorname{sign}(M)| \leq \chi(M).$$
- Let G be a group and $H \subseteq G$ be a normal finite subgroup. Then the canonical map for the Whitehead groups
$$\mathbb{Z} \otimes_{\mathbb{Z}G} \operatorname{Wh}(H) \to \operatorname{Wh}(G)$$
is rationally injective.
- Any S^1-action on a hyperbolic closed manifold is trivial.
- Let M be a closed Kähler manifold. Suppose that it admits some Riemannian metric with negative sectional curvature, or, more generally, that $\pi_1(M)$ is hyperbolic (in the sense of Gromov) and $\pi_2(M)$ is trivial.
  Then M is a projective algebraic variety.
- There are non-slice knots in 3-space whose Casson-Gordon invariants are all trivial.
- Let G be an infinite finitely generated group which is virtually RFRS.
  Then G is virtually fibered in the sense that it admits a finite index subgroup mapping onto $\mathbb{Z}$ with a finitely generated kernel, if and only if its first L^2-Betti number $b_1^{(2)}(G)$ vanishes.

Finally we discuss some of the main open problems or future projects concerning L^2-invariants. Examples are the Atiyah Conjecture about the integrality of L^2-Betti numbers, from which Kaplansky's Zero-Divisor Conjecture and Malcev's

Embedding Conjecture follow, the Singer Conjecture, and the L^2-torsion and its applications to 3-manifolds, group automorphisms, and homological growth,

Homology manifolds and euclidean bundles

Fabian Hebestreit
(joint work with M. Land, C. Winges, M. Weiss)

It is one of the classical results of geometric topology that concordance classes of smooth structures on a topological manifold M are in one-to-one correspondence with vector bundle reductions of the stable euclidean normal bundle of M. Relatedly, for a Poincaré complex P degree 1 normal maps from a manifold M are in one-to-one correspondence with bundles reduction of its Spivak fibration, a stable spherical fibration. Given this level of control the structure group of the normal "bundle" exerts over the regularity of a Poincaré complex or manifold, it is perhaps surprising that Ferry and Pedersen in [2] claimed that every closed homology manifold M admits a reduction of its Spivak fibration to a stable euclidean bundle; recall that a homology manifold is a compact topological space X for which the sheaf

$$\mathrm{Ouv}(X)^{\mathrm{op}} \longrightarrow \mathcal{D}(\mathbb{Z}), \quad U \longmapsto C_*(X, X \setminus U)$$

is invertible in addition to some mild point-set assumptions (namely being an ENR, or equivalently being sublocally contractible, second countable and of finite Lebesque covering dimension). The point of my talk was to explain our result from [3] that this statement is in fact not correct.

To explain our approach to the counterexamples a little, recall first that Ranicki (with a small injection of work by Weiss and Williams) recast the topological surgery sequence for an oriented Poincaré complex P of dimension $d \geq 5$ as an identification

$$\widetilde{S}^{\mathrm{top}}(P) \subset \mathrm{fib}_{\mathrm{sig}^{\mathrm{vs}}(P)}\left(\mathrm{asbl}\colon \Omega^{\infty+d}(P \otimes \mathrm{L}^{\mathrm{s}}(\mathbb{Z})) \to \Omega^{\infty+d}\mathrm{L}^{\mathrm{vs}}(P)\right)$$

of the topological block structure space $\widetilde{S}^{\mathrm{top}}(P)$ (whose set of components consists of h-cobordism classes of closed topological manifolds with a homotopy equivalence to P) as the collection of those components of the fibre of the assembly map in visible symmetric L-theory that are spanned by the $\mathrm{L}^{\mathrm{s}}(\mathbb{Z})$-fundamental classes of P; Ranicki's total surgery obstruction precisely tracks whether such a lift of $\mathrm{sig}^{\mathrm{vs}}(P)$ exists.

Since the occuring L-spectra are 4-periodic this in particular implies that the homotopy groups of $\widetilde{S}^{\mathrm{top}}(P)$ are also 4-periodic away from degree 0. Since

$$\pi_i \widetilde{S}^{\mathrm{top}}(P) = \pi_0 \widetilde{S}^{\mathrm{top}}_{\partial}(P \times \mathrm{D}^i)$$

this is (among other things) a strong existence result for manifolds with non-empty boundary, a statement known as Siebenmann periodicity.

In degree 0 one only finds an injection

$$\pi_0 \widetilde{S}^{\mathrm{top}}(P) \longrightarrow \pi_0 \widetilde{S}^{\mathrm{top}}_{\partial}(P \times D^4)$$

and the search for the closed manifolds missing in the source finally concluded with celebrated work of Bryant, Ferry, Mio and Weinberger: In [1] they extended Ranicki's result to an identification

$$\widetilde{S}^{\mathrm{H}}(P) \simeq \mathrm{fib}_{\mathrm{sig}^{\mathrm{vs}}(P)}\left(\mathrm{asbl}\colon \Omega^{\infty+d}(P \otimes \mathrm{L}^{\mathrm{s}}(\mathbb{Z})) \to \Omega^{\infty+d}\mathrm{L}^{\mathrm{vs}}(P)\right),$$

where the left hand side denotes the homology manifold block structure space, whose set of components consists of h-cobordism classes of homology manifolds with a homotopy equivalence to P. In particular, this result gives a method for producing homology manifolds with prescribed homotopy type.

Our simplest counterexample to the Ferry-Pedersen claim then arises as follows: Take a generator of the 3-torsion in $\pi_4(\mathrm{BS}\mathcal{G}) = \pi_3(\mathbb{S}) \cong \mathbb{Z}/24$ ($\mathrm{BS}\mathcal{G} \simeq \mathrm{colim}_n \mathrm{BS}\mathcal{G}(S^n)$ denotes the classifying space for oriented stable spherical fibrations) and extend it to a map $\mathrm{M}(\mathbb{Z}/3, 4) \to \mathrm{BS}\mathcal{G}$ and factor it through $\alpha\colon \mathrm{M}(\mathbb{Z}/3, 4) \to \mathrm{BS}\mathcal{G}(S^n)$ for large enough n. By surgery below the middle dimension we can further pick a stably framed closed 9-or-higher-dimensional manifold M with a 4-equivalence to $\mathrm{M}(\mathbb{Z}/3, 4)$. Let $E \to M$ denote the S^n-fibration classified by the composite $M \to \mathrm{BS}\mathcal{G}(S^n)$. Then E is a Poincaré duality complex by a simple calculation with the Serre spectral sequence and the result we prove is:

Theorem. *The visible symmtetric signature of any of the Poincaré complexes E obtained in this fashion lifts along the assembly map $\Omega^{\infty+d}(E \otimes \mathrm{L}^{\mathrm{s}}(\mathbb{Z})) \to \Omega^{\infty+d}\mathrm{L}^{\mathrm{vs}}(E)$, but its Spivak fibration does not admit a reduction to a stable euclidean bundle.*

The theorem of Bryant-Ferry-Mio-Weinberger then implies that E is realised by a homology manifold, giving the desired counterexample to the claim of Ferry and Pedersen.

The proof of the second claim is simple and motivates the choice of α: Because M is stably parallelisable, the Spivak fibration of E is classified by the composite

$$E \longrightarrow M \longrightarrow \mathrm{M}(\mathbb{Z}/3, 4) \xrightarrow{\alpha} \mathrm{BS}\mathcal{G}(S^n) \longrightarrow \mathrm{BS}\mathcal{G}.$$

On π_4 this map induces an injection $\mathbb{Z}/3 \to \mathbb{Z}/24$, which cannot factor through $\pi_4(\mathrm{BSTop}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. This implies that the Spivak fibration of E does not admit a reduction to a euclidean bundle. Our proof that the visible symmetric signature of E lifts through assembly is slightly more complicated (and in particular beyond the scope of this extended abstract) but works uniformly for all odd-order elements in $\pi_{4k}(\mathrm{BS}\mathcal{G})$, that lie in the image of the J-homomorphism $\mathrm{BSO} \to \mathrm{BS}\mathcal{G}$. For example the statement also applies for all generators of the p-torsion of $\pi_{2p-2}(\mathrm{BS}\mathcal{G})$; the case $p = 3$ is then the one explicitly considered above.

Remark.

(1) As written, the proof of the identification of $\widetilde{S}^{\mathrm{H}}(P)$ in L-theoretic terms in [1] in fact relies on the result of Ferry and Pedersen (a corresponding erratum to [1] is set to appear soon). Weinberger has asserted in private conversation, that he is certain that the description of $\widetilde{S}^{\mathrm{H}}(P)$ is correct as stated (and I at least am inclined to believe this as well, if for no

other reason than its inherit beauty), but it is not currently proven in full. Assuming the remainder of the deduction in [1] correct, the existence of counterexamples to the Ferry-Pedersen statement nevertheless follows unconditionally: If on the one hand their result is false, then it is certainly false, and if on the other hand it is correct, the construction above provides a counterexample (I'll eat my hat if it is independent enough to be able to choose between these options).

(2) Our counterexample is philosophically in line with another conjecture of Weinberger's: Every homology manifold X with $\dim(X) \geq 5$, that satisfies the disjoint discs property (a very mild version of transversality for 2-discs in X) should be homogeneous, i.e. acted on transitively by its homeomorphism group. The assumption about 2-discs in particular rules out manifolds with cone-singularities whose links are integral homology spheres (two 2-discs that cone off a non-trivial element in the fundamental group of a link do not have disjoint embeddings nearby); these examples are evidently not homogeneous.

If true one might expect the structure group of a normal "bundle" of X to be controlled by the automorphisms of small enough open subsets (a "local model") of X rather than euclidean space. In fact, associated to X is Quinn's invariant $q(X) \in 1 + 8\mathbb{Z}$, with $q(M) = 1$ for every topological manifold, and spinning this narrative further one might even more optimistically predict that there is one such local model R_k^d for each $k \in 1+8\mathbb{Z}$ in dimension d with $R_1^d = \mathbb{R}^d$ and $R_k^d \times R_{k'}^{d'} \cong R_{k\cdot k'}^{d+d'}$, and bundle reductions of the Spivak fibration of X to $\mathrm{colim}_d \mathrm{Aut}(R_{q(X)}^d)$ rather that euclidean ones.

References

[1] J. Bryant, S. Ferry, W. Mio & S. Weinberger, *Topology of homology manifolds*, Ann. of Math. **143** no. 3 (1996), 435–467.

[2] S.C. Ferry & E.K. Pedersen, *Epsilon surgery theory*, Novikov conjectures, index theorems and rigidity, London Math. Soc. Lecture Note Ser. **227**, Cambridge Univ. Press, Cambridge (1995), 167–226.

[3] F. Hebestreit, M. Land, C. Winges & M. Weiss, *Homology manifolds and euclidean bundles*, arXiv: 2406:14677 (2024)

Moduli spaces of 3-manifolds with boundary are finite

Corey Bregman

(joint work with Rachael Boyd, Jan Steinebrunner)

Let M be a smooth, compact, connected, orientable 3-manifold, and let $\mathrm{Diff}(M)$ be the diffeomorphism group of M with the C^∞-topology. Homotopy classes of maps from a CW complex X into the classifying space $B\mathrm{Diff}(M)$ are in one-to-one correspondence with equivalence classes of smooth M-bundles over X. In particular, the cohomology ring of $B\mathrm{Diff}(M)$ yields characteristic classes of such bundles, making it an object of interest in geometric and low-dimensional topology.

If the boundary ∂M is non-empty, let $\mathrm{Diff}_\partial(M)$ denote the subgroup of diffeomorphisms which are pointwise fixed on ∂M. The corresponding space $B\mathrm{Diff}_\partial(M)$ classifies M-bundles equipped with a trivialization along ∂M. We prove the following theorem, which verifies a conjecture of Kontsevich [22, Kirby 3.48] in the orientable case:

Theorem (Theorem 6.1, [4]). *Let M be a compact, connected, orientable 3-manifold. If $\partial M \neq \emptyset$, then $B\mathrm{Diff}_\partial(M)$ has the homotopy type of a finite CW complex.*

When M is irreducible (*i.e.* every embedded S^2 bounds D^3), the above theorem was proven by Hatcher–McCullough [16]. In this case, $B\mathrm{Diff}_\partial(M)$ was known to be aspherical [13, 19], hence they prove the classifying space of the corresponding mapping class group $\pi_0 \mathrm{Diff}_\partial(M)$ is homotopy equivalent to a finite complex. For surfaces, the analogous result is due to Earle-Schatz [8] and Gramain [12]. However, in higher dimensions, Kontsevich's theorem is false in general: for $n = 4$ this is due to Budney–Gabai [5], while for $n \geq 6$ it is due to Hatcher–Wagoner [17]. In each case, it is shown that there exist manifolds M with non-empty boundary such that $\pi_0 \mathrm{Diff}_\partial(M)$ is not finitely generated, hence $B\mathrm{Diff}_\partial(M)$ cannot be homotopy equivalent to a complex with finite 1-skeleton.

The theorem implies that the cohomology ring of $B\mathrm{Diff}_\partial(M)$ is finitely generated with any coefficient ring. Previously, finite (co)homological generation was proved under additional assumptions by Nariman [25]. We apply Kontsevich's theorem to deduce the following result for all 3-manifolds, which implies finite (co)homological generation in each degree:

Theorem (Theorem 6.14, [4]). *If M is any compact, oriented 3-manifold, then $B\mathrm{Diff}(M)$ is of finite type, i.e. it has the homotopy type of a CW complex with finite n-skeleton for every n.*

In this generality this theorem cannot be improved; for example, if M is a closed hyperbolic 3-manifold then it follows from work of Gabai [9] that $\mathrm{Diff}(M) \simeq \mathrm{Isom}(M)$, which is a finite group. Thus, $B\mathrm{Diff}(M) \simeq K(\mathrm{Isom}(M), 1)$ is necessarily infinite-dimensional. For surfaces, the analogous result follows from classical results of Earle–Eells [7]. In higher dimensions, Bustamante–Krannich–Kupers [6] have shown that if $\dim(M) = 2n \geq 6$ and M has finite fundamental group, then $B\mathrm{Diff}(M)$ is of finite type.

We now discuss some of the ideas that go into the proof of Kontsevich's Theorem. If M is a compact, connected, oriented 3-manifold, we will denote the *spherical closure of* M by $\widehat{M}$; this is the manifold obtained from M by capping each S^2 boundary component with disk D^3. The classical Kneser–Milnor theorem states that $\widehat{M}$ admits a minimal connected sum decomposition

$$\widehat{M} = P_1 \sharp \cdots \sharp P_n \sharp (S^1 \times S^2)^{\sharp g},$$

where the P_i are irreducible, and the factors are unique up to reordering (recall that $S^1 \times S^2$ is prime in the sense that it cannot be written as a nontrivial connected sum, but not irreducible). The embedded 2-spheres which yield the above connected sum decomposition above are all separating. More generally, we consider decompositions arising from any collection of essential 2-spheres.

Definition (Definition 3.8, [4]). A *separating system for M* is a submanifold $\Sigma \cong \amalg_k S^2 \subset M$ such that

- No component of Σ bounds D^3 in $\widehat{M}$.
- No two components of Σ cobound $S^2 \times I$.
- Every component of $\widehat{M \setminus \Sigma}$ is irreducible.

Typically, M may have infinitely many isotopy classes of separating systems. To account for this, we introduce a topological poset $\mathrm{Sep}(M)$ consisting of all possible separating systems for M, ordered by inclusion. Because there is a bound on the number of disjointly embedded spheres in M such that no two are isotopic, the nerve $N_\bullet \mathrm{Sep}(M)$ has finite height. Although $\mathrm{Sep}(M)$ need not even be connected, we nevertheless show:

Theorem (Theorem 3.20, [4]). *If $M \ncong S^1 \times S^2$, the geometric realization of $\|N_\bullet(\mathrm{Sep}(M))\|$ is contractible.*

This theorem is proved using an adaptation of the powerful discretization technique of Galatius–Randal-Williams [11]. This reduces contractibility to that of discrete simplicial complex of essential spheres, which we prove is contractible by a surgery argument. Since $\mathrm{Diff}_\partial(M)$ acts on $\mathrm{Sep}(M)$ and preserves inclusions, this theorem gives us a natural model for $B\mathrm{Diff}_\partial(M)$, namely the homotopy quotient $\|N_\bullet(\mathrm{Sep}(M))\| /\!\!/ \mathrm{Diff}_\partial(M)$. (Recall that if G is a topological group, and X is any G-space, then $X/\!\!/G$ is defined as $X \times_G EG$.) We emphasize, however, that the theorem holds regardless of whether $\partial M = \emptyset$, hence may serve as a starting point for further investigation into the homotopy type of $B\mathrm{Diff}(M)$ more generally.

Given a separating system Σ, let its $\mathrm{Diff}_\partial(M)$-stabilizer be $\mathrm{Diff}_\partial(M, \Sigma)$, the subgroup of $\mathrm{Diff}_\partial(M)$ which preserves Σ setwise. We show the action of $\mathrm{Diff}_\partial(M)$ on $\mathrm{Sep}(M)$ has only finitely many orbits. Together with the fact that $N_\bullet \mathrm{Sep}(M)$ has finite height, this allows us to reduce finiteness of $B\mathrm{Diff}_\partial(M)$ to finiteness of $B\mathrm{Diff}_\partial(M, \Sigma)$ ranging over all orbits, via a technical result known as the homotopy orbit-stabilizer theorem.

If $\partial M \neq \emptyset$, by various fiber sequences one can further reduce finiteness of $B\mathrm{Diff}_\partial(M, \Sigma)$ to two base cases:

(1) M is irreducible, and $\partial M \neq \emptyset$.
(2) $\widehat{M}$ is closed, irreducible, and $M = \widehat{M} \setminus \mathring{D}^3$.

As mentioned above, (1) is covered by work of Hatcher–McCullough, while (2) presents a previously untackled case.

Suppose now that M is a closed, oriented and irreducible 3-manifold. Let $D^3 \subset M$ be an embedded disk and let $\mathrm{Diff}_{D^3}(M)$ be the collection of diffeomorphisms which fix D^3 pointwise. Using contractibility of collars, case (2) is equivalent to the statement that $B\mathrm{Diff}_{D^3}(M)$ has the homotopy type of a finite complex. For technical reasons, we prove the following more general result:

Theorem (Theorem 4.1, [4]). *Let M^3 be oriented, irreducible with empty or incompressible toroidal boundary, and let $D^3 \subset \mathring{M}$ be an embedded disk. Then $B\mathrm{Diff}_{D^3}(M)$ has the homotopy type of a finite CW complex.*

Combining work of Jaco–Shalen [20] and Johannson [21] with the Geometrization theorem of Thurston [29] and Perelman [26, 28, 27], any M as in the theorem contains minimal collection T of disjointly embedded, 2-sided tori such that each component of M is either hyperbolic or Seifert-fibered. The submanifold T provides a further decomposition of M called the JSJ decomposition of M, and in contrast to the case of separating systems, T is unique up isotopy. In fact, as long as M is not the total space of T^2-bundle over S^1 with Anosov monodromy, using a result of Hatcher [15] we show that the inclusion $\mathrm{Diff}(M,T) \hookrightarrow \mathrm{Diff}(M)$ is a homotopy equivalence. By cutting along T, we can then reduce th theorem to the components of $M \setminus T$.

Delooping the fiber sequence $\mathrm{Diff}_{D^3}(M) \to \mathrm{Diff}(M) \to \mathrm{Emb}(D^3, M)$, we can identify $B\mathrm{Diff}_{D^3}(M)$ with the homotopy quotient $\mathrm{Emb}(D^3, M) /\!\!/ \mathrm{Diff}(M)$. Since the embedding space $\mathrm{Emb}(D^3, M)$ is in turn homotopy equivalent to the frame bundle $\mathrm{Fr}(M)$, thus $B\mathrm{Diff}_{D^3}(M) \simeq \mathrm{Fr}(M) /\!\!/ \mathrm{Diff}(M)$.

Many components of $M \setminus T$ support one of the 8 Thurston geometries. If $\mathrm{Isom}(M) \simeq \mathrm{Diff}(M)$, we can replace $\mathrm{Fr}(M) /\!\!/ \mathrm{Diff}(M)$ by $\mathrm{Fr}(M)/\mathrm{Isom}(M)$, which is homotopy equivalent to a compact manifold, hence has finite homotopy type. The statement that $\mathrm{Diff}(M) \simeq \mathrm{Isom}(M)$ is known as the (*strong*) *generalized Smale conjecture* (SGSC), after Smale's original conjecture that $\mathrm{Diff}(S^3) \simeq O(4)$, which was proved by Hatcher [14]. It turns out that for irreducible M, the SGSC holds exactly when $\pi_0 \mathrm{Diff}(M)$ is finite, and was fully resolved in all cases only recently by combining work of many authors (see [24, 9, 10, 18, 23, 1, 2, 3] and the references therein). The cases for which SGSC fails are all Haken Seifert-fibered; for these we leverage the structure of the Seifert fibering and work of Hong–Kalliongis–McCullough–Rubinstein [18] to prove the theorem.

References

[1] Richard Bamler and Bruce Kleiner. Ricci flow and contractibility of spaces of metrics. arXiv:1909.08710, 2019.

[2] Richard H. Bamler and Bruce Kleiner. Ricci flow and diffeomorphism groups of 3-manifolds. *J. Amer. Math. Soc.*, 36(2):563–589, 2023.

[3] Richard H. Bamler and Bruce Kleiner. Diffeomorphism groups of prime 3-manifolds. *J. Reine Angew. Math.*, 806:23–35, 2024.

[4] Rachael Boyd, Corey Bregman, and Jan Steinebrunner. Moduli spaces of 3-manifolds with boundary are finite. arXiv:2404.12748, 2024.

[5] Ryan Budney and David Gabai. Knotted balls in S^4. arXiv:1912.09029, 2021.

[6] Mauricio Bustamante, Manuel Krannich, and Alexander Kupers. Finiteness properties of automorphism spaces of manifolds with finite fundamental group. *Math. Ann.*, 388(4):3321–3371, 2024.

[7] Clifford J. Earle and James Eells. A fibre bundle description of Teichmüller theory. *J. Differential Geometry*, 3:19–43, 1969.

[8] Clifford J. Earle and Alfred Schatz. Teichmüller theory for surfaces with boundary. *J. Differential Geometry*, 4:169–185, 1970.

[9] David Gabai. The Smale conjecture for hyperbolic 3-manifolds: $\mathrm{Isom}(M^3) \simeq \mathrm{Diff}(M^3)$. *J. Differential Geom.*, 58(1):113–149, 2001.

[10] David Gabai, G. Robert Meyerhoff, and Nathaniel Thurston. Homotopy hyperbolic 3-manifolds are hyperbolic. *Ann. of Math. (2)*, 157(2):335–431, 2003.

[11] Søren Galatius and Oscar Randal-Williams. Homological stability for moduli spaces of high dimensional manifolds. I. *J. Amer. Math. Soc.*, 31(1):215–264, 2018.

[12] André Gramain. Le type d'homotopie du groupe des difféomorphismes d'une surface compacte. *Ann. Sci. École Norm. Sup. (4)*, 6:53–66, 1973.

[13] Allen Hatcher. Homeomorphisms of sufficiently large P^2-irreducible 3-manifolds. *Topology*, 15(4):343–347, 1976.

[14] Allen Hatcher. A proof of the Smale conjecture $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$. *Ann. of Math.* (2)3:553–607, 1983.

[15] Allen Hatcher. Spaces of incompressible surfaces. arXiv:9906074, 1999.

[16] Allen Hatcher and Darryl McCullough. Finiteness of classifying spaces of relative diffeomorphism groups of 3-manifolds. *Geom. Topol.*, 1:91–109, 1997.

[17] Allen Hatcher and John Wagoner. *Pseudo-isotopies of compact manifolds*, volume No. 6 of *Astérisque*. Société Mathématique de France, Paris, 1973.

[18] Sungbok Hong, John Kalliongis, Darryl McCullough, and J. Hyam Rubinstein. *Diffeomorphisms of elliptic 3-manifolds*, Volume 2055 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[19] Nikolai V. Ivanov. Groups of diffeomorphisms of Waldhausen manifolds. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 66:172–176, 209, 1976.

[20] William H. Jaco and Peter B. Shalen. Seifert fibered spaces in 3-manifolds. *Mem. Amer. Math. Soc.*, 21(220):viii+192, 1979.

[21] Klaus Johannson. *Homotopy equivalences of* 3*-manifolds with boundaries*, volume 761 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[22] Rob Kirby. Problems in low-dimensional topology. (Edited by Rob Kirby). Kazez, William H. (ed.), Geometric topology. 1993 Georgia international topology conference, August 2–13, 1993, Athens, GA, USA. Providence, RI: American Mathematical Society. AMS/IP Stud. Adv. Math. 2(pt.2), 35-473, 1997.

[23] Darryl McCullough and Teruhiko Soma. The Smale conjecture for Seifert fibered spaces with hyperbolic base orbifold. *J. Differential Geom.*, 93(2):327–353, 2013.

[24] William H. Meeks III and Peter Scott. Finite group actions on 3-manifolds. *Invent. math.*, 86:287-346, 1986

[25] Sam Nariman. On the finiteness of the classifying space of diffeomorphisms of reducible three manifolds. arXiv:21041.2338, 2021.

[26] Grigori Perelman. The entropy formula for the Ricci flow and its geometric applications. arXiv:0211159, 2002.

[27] Grigori Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. arXiv:0307245, 2003.

[28] Grigori Perelman. Ricci flow with surgery on three-manifolds. arXiv:0303109, 2003.

[29] William P. Thurston. Hyperbolic structures on 3-manifolds. I. Deformation of acylindrical manifolds. *Ann. of Math. (2)*, 124(2):203–246, 1986.

Isotopy classification of 1/2-disks in 4-manifolds

Peter Teichner

Starting with an introduction to the classical light bulb theorem (LBT), we generalized it to a homotopy equivalence in all dimensions and focused on the isotopy classification (i.e. connected components of an embedding space) in dimension 4.

The classical LBT says that a knotted electric cord, hanging from the ceiling and ending with a light bulb, can be unknotted by an isotopy in a way that the light bulb stays at the same place at all times. One can formulate this as saying that homotopy implies isotopy for 1/2-arcs in any 3-manifold M. Here a 1/2-arc is an embedded arc in M (the electric cord) that has one endpoint fixed on the boundary of M (the ceiling) and one endpoint fixed in the interior of M (the light bulb), i.e. mixed boundary conditions (as compared to the usual use of "neat" arcs, where the entire boundary lies in the boundary of M).

In fact, one can compute the homotopy type of all 1/2-arcs in M in terms of the based loop space of the unit tangent bundle of M, giving the first well known "space level" LBT. In joint work with Danica Kosanovic, we generalized this result to all dimensions, where we study the space of embeddings of a 1/2-disk. This is an n-disk embedded in a d-manifold M, where one half of the boundary, an $(n-1)$-disk, lies on the boundary of M and the other half lies in its interior.

As a consequence, we prove a semi-direct product structure for certain diffeomorphism groups, which leads to very interesting nilpotent subgroups of mapping class groups of certain 4-manifolds. These nilpotent groups arise as isotopy classes of 1/2-disks (n=2) in a 4-manifold M which we completely determine by a short exact sequence involving $\pi_2(X)$ and a certain quotient of the group ring $\mathbb{Z}[\pi_1 M]$ by the image of the Dax invariant. This invariant goes back to Jean-Pierre Dax, a student of Cerf, who introduced these tricks in the special case that $d = n$.

References

[1] A new approach to light bulb tricks: Disks in 4-manifolds, with Danica Kosanovic, to appear in Duke Math. Journal 2024.

[2] A space level light bulb theorem in all dimensions, with Danica Kosanovic, to appear in Commentarii Math. Helv. 2024.

Syntomic cohomology of ring spectra

Gabriel Angelini-Knoll

(joint work with Christian Ausoni, John Rognes, Jeremy Hahn, Dylan Wilson)

The algebraic K-theory of the sphere spectrum provides information about diffeomorphism groups $\mathrm{Diff}(D^n)$ of discs for large n by the parametrized s-cobordism theorem [22]. This shows that it is useful to generalize the notion of algebraic K-theory from rings to ring spectra for geometric applications. Waldhausen [23], suggested approaching the algebraic K-theory of the sphere spectrum by the algebraic K-theory of its telescopic localizations $L_n^f\mathbb{S}$.

The telescopically localized algebraic K-theory of the ring of integers in a totally real number field $\mathcal{O}_F$ has a close relationship to special values of Dedekind zeta functions by work of Thomason [21] and the resolution of the Iwasawa main conjecture by Wiles [24]. Lichtenbaum and Quillen [16, 19] conjectured that for suitable R the map from algebraic K-theory of R to its telescopically localized algebraic K-theory is an isomorphism in sufficiently large degrees. This allows one to relate quotients of orders of algebraic K-theory groups to special values of Dedekind zeta functions. This prediction, the Lichtenbaum–Quillen property, was generalized by Ausoni–Rognes [6] to the setting of ring spectra in one of their famous redshift conjectures.

In the 1980's, Ravenel [20] made several predictions that shaped the field of chromatic homotopy theory. All but one of Ravenel's conjectures were proven by Devinatz, Hopkins, and Smith [11, 15]. The remaining conjecture, known as the telescope conjecture, relates telescopic localizations L_n^f to more computable localizations L_n. Burklund–Hahn–Levy–Schlank [9] disproved this conjecture for $n \geq 2$, so it becomes an interesting question of which spectra $L_n^f X \simeq L_n X$. For example, Mahowald–Rezk [17] suggested that for spectra with finitely presented mod p cohomology, the telescope conjecture should hold.

The Ausoni–Rognes conjecture [6], generalizing the Lichtenbaum–Quillen conjecture, is one part of a larger program of studying redshift phenomena in algebraic K-theory. Perhaps the simplest form of redshift can be phrased in terms of $K(n)$-acyclicity. Here $K(n)$ is Morava K-theory, one of the minimal p-local skew-fields in homotopy theory. We say a spectrum X has height n if $K(n)_*X \neq 0$ and $K(n+k)_*X = 0$ for all $k > 0$. It is now known by work of Burklund–Schlank–Yuan [10], that if R is a commutative ring spectrum and it has height exactly n then its algebraic K-theory has height exactly $n+1$. This leaves open the question of whether algebraic K-theory increases height by exactly one for non-commutative ring spectra; i.e does algebraic K-theory of non-commutative ring spectra satisfy redshift. There are degenerate examples where this doesn't hold, for example when $K(R) = 0$, but in many fundamental examples of interest it does.

Syntomic and prismatic cohomology unifies various cohomology theories in p-adic geometry, specializing to de Rham cohomology, étale cohomology, and crystalline cohomology. Originally, it was defined in terms of sheaf cohomology of a scheme with respect to the syntomic topology. The term prismatic comes from the notion of a prism, which consists of a δ-ring R and a Cartier divisor I satisfying the prism condition. Groundbreaking work of Bhatt–Morrow–Scholze [8] demonstrated that syntomic cohomology can be produced as the associated graded of a filtration on topological cyclic homology and prismatic cohomology can be produced as the associated graded of a filtration on topological periodic cyclic homology. Topological cyclic homology, topological negative cyclic homology, and topological periodic cyclic homology sit in a long exact sequence by Nikolaus–Scholze [18] and topological cyclic homology is a very close approximation to algebraic K-theory by the Dundas–Goodwillie–McCarthy theorem [12].

Work of Hahn–Raksit–Wilson [13] beautifully extends work of Bhatt–Morrow–Scholze [8] on syntomic cohomology to the setting of ring spectra. This produces an efficient new tool for computing algebraic K-theory. For example, in work with Ch. Ausoni and J. Rognes [3], I used syntomic cohomology to compute

$$A(1)_* \mathrm{K}(\mathrm{ko})$$

where ko denotes real topological K-theory and $A(1) := ((\mathbb{S}/2)/\eta)/v_1$ denotes a finite spectrum whose mod 2 cohomology is the sub-algebra $A(1)$ of the 2-primary Steenrod algebra.

This also provides an elegant approach to studying the Lichtenbaum-Quillen property, redshift, and the telescope conjecture for algebraic K-theory of ring spectra. For example, if R is an arbitrary $\mathbb{E}_1$-MU-algebra form of truncated Brown—Peterson spectra $\mathrm{BP}\langle n\rangle$ or Morava K-theory $K(n)$, I prove the Lichtenbaum–Quillen property, telescope conjecture, and redshift conjecture for the algebraic K-theory of R in joint work with J. Hahn and D. Wilson [4, 5]. In the case of $\mathrm{BP}\langle n\rangle$, this extends previous work of Hahn–Wilson [14] to arbitrary $\mathbb{E}_1$-MU-algebra forms of $\mathrm{BP}\langle n\rangle$. As a consequence, we also extend computations of

$$V(2)_* \mathrm{K}(\mathrm{BP}\langle 2\rangle)$$

from [2] to the prime $p = 5$ and to arbitrary $\mathbb{E}_1$-MU-algebra forms of $\mathrm{BP}\langle 2\rangle$. Here $V(2) := ((\mathbb{S}/p)/v_1)/v_2$. We also extend computations of

$$V(1)_* \mathrm{K}(k(1))$$

from [7] to the prime $p = 3$ and to arbitrary $\mathbb{E}_1$-MU-algebra forms of $k(1)$. Here $V(1) := (\mathbb{S}/p)/v_1$. In joint work in progress with Ch. Ausoni, R. Bruner, J. Davies, J. Rognes, and T. Yang [1], we are working towards computing

$$A(2)_* \mathrm{K}(\mathrm{tmf}_{(2)})$$

using syntomic cohomology, where $A(2) := (A(1)/\nu)/\gamma)/v_2$ is a finite spectrum whose mod 2 cohomology is the sub-algebra $A(2)$ of the 2-primary Steenrod algebra and $\mathrm{tmf}_{(2)}$ is 2-local topological modular forms.

References

[1] G. Angelini-Knoll, Ch. Ausoni, R. Bruner, J. Davies, J. Rognes, and T. Yang, *Algebraic K-theory of 2-local topological modular forms*, In progress.

[2] G. Angelini-Knoll, Ch. Ausoni, D. Culver, E. Höning, and J. Rognes, *Algebraic K-theory of elliptic cohomology*, To appear in Geom. Topol. (2024).

[3] G. Angelini-Knoll, Ch. Ausoni, and J. Rognes, *Algebraic K-theory of real topological K-theory*, arXiv:2309.11463 (2023).

[4] G. Angelini-Knoll, J. Hahn, and D. Wilson, *Syntomic cohomology of Morava K-theory*, In progress.

[5] G. Angelini-Knoll, J. Hahn, and D. Wilson, *Syntomic cohomology of truncated Brown–Peterson spectra*, In progress.

[6] Ch. Ausoni and J. Rognes, *The chromatic red-shift in algebraic K-theory*, Enseign. Math. 54, **2** (2008), 9–11.

[7] Ch. Ausoni, and J. Rognes, *Algebraic K-theory of the first Morava K-theory*, J. Eur. Math. Soc. (JEMS), **14** (2012) 1041–1079.

[8] B. Bhatt, M. Morrow, and P. Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, Publ. Math., Inst. Hautes Étud. Sci. **129** (2019), 199–310.
[9] R. Burklund, J. Hahn, I. Levy, and T. Schlank, *K-theoretic counterexamples to Ravenel's telescope conjecture*, arXiv:2310.17459 (2023).
[10] R. Burklund, T. Schlank, and A. Yuan, *The Chromatic Nullstellensatz*, arXiv:2207.09929 (2022).
[11] E. Devinatz, M. Hopkins, and J. Smith, *Nilpotence and stable homotopy theory. I*, Ann. Math. (2), **128** (1988), 207–241.
[12] B. Dundas, T. Goodwillie, and R. McCarthy, *The local structure of algebraic K-theory*, London: Springer (2013).
[13] J. Hahn, A. Raksit, and D. Wilson, *A motivic filtration on the topological cyclic homology of commutative ring spectra*, arXiv:2206.11208 (2022).
[14] J. Hahn, and D. Wilson, *Redshift and multiplication for truncated Brown-Peterson spectra*, Ann. Math. (2), **196** (2022), 1277–1351.
[15] M. Hopkins, and J. Smith, *Nilpotence and stable homotopy theory. II*, Ann. Math. (2), **148** (1998), 1–49.
[16] S. Lichtenbaum, *Values of zeta-functions, étale cohomology, and algebraic K-theory*, Algebraic K-Theory II, Proc. Conf. Battelle Inst. 1972, Lect. Notes Math. **342** (1973), 489–501.
[17] M. Mahowald, M. and C. Rezk, *Brown-Comenetz duality and the Adams spectral sequence*, Am. J. Math. **121** (1999), 1153–1177.
[18] T. Nikolaus, and P. Scholze, *On topological cyclic homology*, Acta Math. **221** (2018), 203–409.
[19] D. Quillen, *Higher algebraic K-theory*, Proc. int. Congr. Math., Vancouver 1974, Vol. **1** (1975), 171–176.
[20] D. Ravenel, *Localization with respect to certain periodic homology theories*, Am. J. Math. **106** (1984), 351–414.
[21] R. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Sci. Éc. Norm. Supér. (4), **18** (1985), 437–552.
[22] F. Waldhausen, B. Jahren, and J. Rognes, *Spaces of PL manifolds and categories of simple maps*, Annals of Mathematics Studies, **186** (2013).
[23] F. Waldhausen, *Algebraic K-theory of spaces, localization, and the chromatic filtration of stable homotopy*, Algebraic topology, Proc. Conf., Aarhus 1982, Lect. Notes Math. **1051** (1984), 173–195.
[24] A. Wiles, *The Iwasawa conjecture for totally real fields*, Ann. Math. (2), **131** (1990), 493-540.

Growth of automorphisms of special groups

Elia Fioravanti

One of the most fundamental problems in group theory is to understand, given a finitely generated group G, the behaviour of its automorphisms. A typical question could be whether, for a certain group G, the outer automorphism group $\mathrm{Out}(G)$ is finitely generated, or how fast the complexity of an element $g \in G$ can increase when we apply to it iterates of some fixed automorphism.

We focus on the latter problem. As a measure of complexity, it is customary to consider *conjugacy length*. Fixing a finite generating set $S \subseteq G$, this quantity is defined as

$$\|g\|_S := \min\{n \in \mathbb{N} \mid \exists x \in G,\ \exists s_1, \dots, s_n \in S^{\pm} \text{ s.t. } g = x(s_1 s_2 \dots s_n)x^{-1}\}.$$

Geometrically, if G is the fundamental group of a negatively-curved closed Riemannian manifold, then $\|g\|_S$ is roughly equal to the length of the closed geodesic in the free homotopy class determined by g, up to multiplicative constants depending on S but not on g.

For any outer automorphism $\phi \in \mathrm{Out}(G)$, we define the *stretch factor* of ϕ as

$$\Lambda(\phi) := \sup_{g\in G} \limsup_{n\to+\infty} \|\phi^n(g)\|^{1/n} \geq 1.$$

Here the choice of the generating set S does not affect the value of the stretch factor, so we omit the subscript S from $\|\cdot\|$.

For a topological example, suppose that G is the fundamental group of a closed surface S, so that the outer automorphism group $\mathrm{Out}(G)$ is identified with the extended mapping class group $\mathrm{Mod}^{\pm}(S)$. The stretch factor $\Lambda(\phi)$ is then simply the highest stretch factor of a pseudo-Anosov component in the Nielsen–Thurston decomposition of ϕ (or equal to 1 if no such component exists). One can say more: if $g \in G$, then the sequence $n \mapsto \|\phi^n(g)\|$ is either bounded, or it grows linearly, or it grows roughly like λ^n, where λ is the stretch factor of one of the pseudo-Anosov components of ϕ. Thus, fixing $\phi \in \mathrm{Out}(G)$ and varying $g \in G$, one sees at most N possible speeds for the sequence $n \mapsto \|\phi^n(g)\|$ (up to bi-Lipschitz equivalence of sequences), where N only depends on the complexity of the surface S.

Growth of automorphisms is fully understood and similarly-behaved also when:

- G is a free group, due to the work of Bestvina, Handel, Feighn and Levitt on train tracks [4, 3, 11]
- G is a negatively curved group (a.k.a. a Gromov-hyperbolic group) exploiting the canonical JSJ decomposition [12] and Rips theory [2].

Unfortunately, this comes close to being our entire understanding of growth of group automorphisms. Beyond negatively curved groups (and closely related ones, such as relatively hyperbolic groups), the picture remains much murkier. Coulon constructed very exotic groups that have automorphisms growing super-polynomially but sub-exponentially [8].

What is particularly embarrassing is that almost nothing is known on growth of automorphisms of *non-positively curved groups*, although these might not sound very far from negatively curved groups.

Perhaps the simplest examples of non-positively curved groups with interesting automorphisms (excluding negatively curved groups) are provided by right-angled Artin groups. These are groups A_Γ determined by a finite simplicial graph Γ via the presentation

$$A_\Gamma = \langle \Gamma^{(0)} \mid vw = wv \Leftrightarrow \ v \text{ and } w \text{ span an edge in } \Gamma\rangle.$$

It has been shown in recent years that automorphisms of right-angled Artin groups share some superficial similarities with automorphisms of free groups; for instance, $\mathrm{Out}(A_\Gamma)$ has a finite (rational) classifying space with a particularly simple description [5], analogous to Outer Space [9]. However, the growth of their automorphisms is understood only in very specific cases [6].

Our key insight is that it is easier to study automorphism growth if one works within a much broader class of non-positively curved groups, namely the *special groups* introduced by Haglund and Wise [10]. This class includes free groups, free abelian groups, surface groups and all right-angled Artin groups, but also many more surprising examples, such as fundamental groups of hyperbolic 3–manifolds [1]. This level of generality allows us to study elements of $\mathrm{Out}(G)$ by decomposing the group G into simpler pieces; these will remain special if G is special, but they might stop being right-angled Artin groups even if G initially was.

We prove the following two theorems.

Theorem. *If G is a special group, the stretch factor of each element $\phi \in \mathrm{Out}(G)$ is an algebraic integer (a weak Perron number). Moreover, if the stretch factor equals 1, then ϕ grows at most polynomially fast.*

Every special group comes equipped with some *coarse median structures*; these are well-behaved notions of a coarse barycentre for triples of group elements. When an outer automorphism leaves one such coarse median structure invariant, one can prove stronger properties of growth rates. For context, all automorphisms of hyperbolic groups are coarse-median preserving, and automorphisms of right-angled Artin groups are coarse-median preserving if and only if they are "untwisted" in the sense of [7].

To be precise, we say that a *growth rate* of an outer automorphism $\phi \in \mathrm{Out}(G)$ is the equivalence class of a sequence $n \mapsto \|\phi^n(g)\|$, for some $g \in G$, up to the equivalence relation that identifies two sequences a_n and b_n if there exists a constant $C \geq 1$ such that $\frac{1}{C}a_n \leq b_n \leq Ca_n$ for all $n \in \mathbb{N}$.

Theorem. *Let G be special and let $\phi \in \mathrm{Out}(G)$ coarsely preserve one of the coarse median structures on G. Then:*

1. *ϕ has at most $N(G)$ growth rates, where the constant $N(G)$ only depends on the group G;*
2. *each growth rate is equivalent to $n \mapsto n^p\lambda^n$ for some integer $p \in \mathbb{N}$ and some algebraic integer $\lambda \geq 1$;*
3. *the maximal subgroups of G all of whose elements grow at most at a given speed are finitely generated, and there are only finitely many G–conjugacy classes of such subgroups of G.*

References

[1] N. Bergeron and D. T. Wise, *A boundary criterion for cubulation*, Amer. J. Math., 134(3):843–859, 2012.

[2] M. Bestvina and M. Feighn, *Stable actions of groups on real trees*, Invent. Math., 121(2):287–321, 1995.

[3] M. Bestvina, M. Feighn and M. Handel, *The Tits alternative for* $\mathrm{Out}(F_n)$*. I. Dynamics of exponentially-growing automorphisms*, Ann. of Math. (2) 151(2):517–623, 2000.

[4] M. Bestvina and M. Handel, *Train tracks and automorphisms of free groups*, Ann. of Math. (2), 135(1):1–51, 1992.

[5] C. Bregman, R. Charney and K. Vogtmann, *Outer space for RAAGs*, Duke Math. J., 172(6):1033–1108, 2023.

[6] C. Bregman and Y. Qing, *Dilatation of outer automorphisms of right-angled Artin groups*, arXiv:1810.06499, 2018.

[7] R. Charney, N. Stambaugh and K. Vogtmann, *Outer space for untwisted automorphisms of right-angled Artin groups*, Geom. Topol. 21(2):1131–1178, 2017.

[8] R. Coulon, *Examples of groups whose automorphisms have exotic growth*, Algebr. Geom. Topol. 22(4):1497–1510, 2022.

[9] M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent. Math. 84(1):91–119, 1986.

[10] F. Haglund and D. T. Wise, *Special cube complexes*, Geom. Funct. Anal. 17(5):1551–1620, 2008.

[11] G. Levitt, *Counting growth types of automorphisms of free groups*, Geom. Funct. Anal., 19(4):1119–1146, 2009.

[12] E. Rips and Z. Sela, *Cyclic splittings of finitely presented groups and the canonical JSJ decomposition*, Ann. of Math. (2), 146(1):53–109, 1997.

Diffeomorphism groups of 4-manifolds

Daniel Ruberman

These three lectures surveyed some aspects of the diffeomorphism group of 4-manifolds. This area has been quite active in recent years, and the referenced papers represent only a portion of recent developments.

Acknowledgement. Thanks to Dave Auckly and Hokuto Konno for their comments on this outline in advance of the lectures, and to the organizers for inviting me to give these talks.

Lecture I

Overview. The main concern in these lectures will be simply connected smooth compact X^4. Mostly these will be closed, but some of the results are of particular interest for manifolds with boundary. A particular focus will be the relations between $\mathrm{Diff}(X)$, $\mathrm{Homeo}(X)$, $\mathrm{HE}(X)$ (self-homotopy equivalences) and $\mathrm{Aut}(Q_X)$. We will always assume that such automorphisms are orientation preserving and so omit any indication of that in the notation; Diff^0 and Homeo^0 are the identity components of the respective topological groups.

In summary, we will see big differences between behavior in dimension 4 and higher dimensions, and big differences between Diff and Homeo in dimension 4.

Classical existence results. Whitehead-Milnor [Mil58, Whi49] showed that simply connected closed 4-manifolds X are determined up to homotopy type by the intersection form Q_X. Moreover, homotopy equivalences determined up to homotopy by effect on $H_2(X)$ (Cochran-Habegger [CH90]; also Kreck [Kre01]).

A foundational series of papers by Wall [Wal64a, Wal62, Wal64] showed that in many circumstances, an automorphism of Q_X can be realized by a diffeomorphism. Wall [Wal64b] used these results to prove that homotopy equivalent simply connected closed 4-manifolds are h-cobordant, setting the stage for Freedman's topological classification of 4-manifolds.

Wall's realization results were proved in stages. First, we have some easy diffeomorphisms: Reflection r in S^2 gives a diffeomorphism of $S^2 \times S^2$; note $r \times r$ is orientation preserving. Complex conjugation on $\mathbb{C}P^2$ or $\overline{\mathbb{C}P}^2$ is likewise orientation preserving. Also, we can permute diffeomorphic summands in a connected sum.

In general, if $f_i \colon X_i \overset{\cong}{\to} X_i$ are orientation preserving and each preserve a ball (easily arranged by an isotopy), then we can form $f_1 \# f_2$. So if X contains a 2-sphere S of square ± 1, then it has a connected summand of $\mathbb{C}P^2$ or $\overline{\mathbb{C}P}^2$ containing S. it has a reflection supported near S. A similar construction is possible for S with $S \cdot S = \pm 2$.

The most interesting diffeomorphism in [Wal64a] realizes (and, to a topologist, explained) an automorphism algebraically defined by Eichler. The idea is that surgery along a loop sweeping out a 2-sphere in X induces a self-diffeomorphism of $X \# S^2 \times S^2$ or $X \# S^2 \tilde{\times} S^2$. Such diffeomorphisms, in more elaborate form, appear in recent work of Budney-Gabai, Watanabe, and others. See the recent systematic study by Kosanović [Kos24].

Putting these together, Wall [Wal64] shows that for X indefinite, every automorphism of $Q_{X \# S^2 \times S^2}$ is realized by a diffeomorphism. The key step is algebraic: Wall shows that maps induced by above diffeomorphisms generate $\mathrm{Aut}(Q_{X \# S^2 \times S^2})$. This is the first of many results showing that 4-manifold theory is simpler after stabilization, or connected sum with $S^2 \times S^2$.

The topological case has a more complete answer. For closed X, Freedman showed how to realize all automorphisms by homeomorphisms [Fre82]. (The starting point is the existence of a smooth h-cobordism realizing the automorphism.) When X has non-empty boundary, the story is somewhat more complicated: see Boyer [Boy86] and Orson-Powell [OP22].

Classic results II: Uniqueness. For uniqueness question, the most intuitive relation is isotopy. Slightly less intuitive is pseudoisotopy, where f_0 and f_1 are pseudoisotopic if there is a diffeomorphism of $X \times I$ restricting to f_0 on $X \times \{0\}$ and f_1 on $X \times \{1\}$. These notions make sense in topological case as well.

Kreck [Kre79] showed that for simply connected closed X, pseudoisotopy class determined by action on $H_2(X)$. In particular, homotopy implies pseudoisotopy; this was reproved by Quinn [Qui86]. Following Cerf's proof [Cer70] in higher dimensions, Perron [Per86] and Quinn [Qui86] show pseudoisotopy implies TOP isotopy; see the correction by Gabai et al [GGH$^+$23]. Quinn [Qui86] claimed pseudoisotopy implies isotopy after stabilization; corrected in [GGH$^+$23]. Often one stabilization suffices: [AKMR15]; [AKMRS19].

The non-simply connected case is also of interest. There are K-theory obstructions to isotopy and pseudoisotopy defined by Hatcher-Wagoner [HW73]. Igusa [Igu21a, Igu21b] and Singh [Sin22] have proved that some of these are actually realized.

Gauge theory–first consequences. Friedman-Morgan [FM88] and Donaldson [Don90] showed not all automorphisms are realized by diffeomorphisms, using

Donaldson theory. These results were reproved more easily using Seiberg-Witten theory [FM97] using the principle that basic classes must be preserved by any diffeomorphism).

Ruberman [Rub98] showed pseudoisotopy does not imply smooth isotopy. Let us call a diffeomorphism that is topologically isotopic to the identity but not smoothly so an *exotic diffeomorphism*. These ideas have been greatly extended in recent years, and will be the subject of the next lectures.

Lecture II

Background on gauge theory and 4-manifolds. The theme for this lecture is the use of gauge theory methods to detect non-trivial topology of $\mathrm{Diff}(X^4)$ and $\mathrm{BDiff}(X^4)$.

First, some brief background on gauge theory. One can use Yang-Mills equations (Donaldson theory) or Seiberg-Witten theory but we will stick to the latter. I introduced Spin^c structures, SW equations, and gauge transformations; see Donaldson's lovely survey article [Don96] for a quick introduction or [Nic00] for many more details. The Seiberg-Witten equations (following conventions in [Nic00]) are

$$\begin{aligned} \sqrt{2}(F_A^+ + i\eta^+) - \frac{1}{2}\mathbf{c}^{-1}(q(\psi)) &= 0 \\ D_A^+\psi) &= 0. \end{aligned}$$

Here A is a connection on a Spin^c bundle $\mathcal{S}^+$, F_A is the curvature of $\det(\mathcal{S}^+)$, ψ is a spinor (a section of $\mathcal{S}^+$), and D_A^+ is the associated Dirac operator. The quadratic term $\mathbf{c}^{-1}(q(\psi))$ arises from Clifford multiplication. We are interested in solutions to these equations modulo a natural symmetry–the action of the gauge group. This quotient is called the Seiberg-Witten moduli space. The basic point we use is that the moduli space depends on auxiliary choices: a Riemannian metric g on the bundle, and a closed 2-form η. Note that these live in a contractible space. The formal dimension of the moduli space (ie the index of the linearization of the Seiberg-Witten equations modulo gauge transformations) is given by

$$d(\mathfrak{s}) = \frac{c_1^2(\mathfrak{s}) - \sigma}{4} - (1 + b_2^+).$$

If $d(\mathfrak{s}) = 0$, we count solutions to get the *Seiberg-Witten invariant* $\mathrm{SW}_{X,\mathfrak{s}} \in \mathbb{Z}$. That this gives a smooth invariant is proved by Donaldson's cobordism argument: for a path (g_t, η_t), $t \in I$ get 1-parameter moduli space

$$\mathcal{M}_{X,\mathfrak{s}}(\{(g_t, \eta_t)\}) = \bigcup_{t \in I} \mathcal{M}_{X,\mathfrak{s}}((g_t, \eta_t))$$

giving a cobordism between $\mathcal{M}_{X,\mathfrak{s}}(\{(g_0, \eta_0)\})$ and $\mathcal{M}_{X,\mathfrak{s}}(\{(g_1, \eta_1)\})$.

Assume $d(\mathfrak{s}) = -1$; this means that generically $\mathcal{M}_{X,\mathfrak{s}}((g, \eta))$ is empty. But for finitely many $t \in I$ have an isolated solution. Donaldson's cobordism argument extends to say that this is an invariant (1-parameter invariant) of the path rel endpoints.

Extension to families and application to diffeomorphisms. A key idea from [Rub98] is to apply this 1-parameter invariant to a diffeomorphism f that preserves a Spin^c structure $\mathfrak{s}$. For generic (g_0, η_0) let $(g_1, \eta_1) = f^*(g_0, \eta_0)$ and join by path; compute 1-parameter family invariant $\mathrm{SW}_{X,\mathfrak{s}}(f)$. There is an important technical point: to get $\mathrm{SW}_{X,\mathfrak{s}}(f) \in \mathbb{Z}$ we have to assume that f preserves orientation of the moduli space; this is determined by $\mathfrak{s}$ and the action of f on cohomology. Otherwise $\mathrm{SW}_{X,\mathfrak{s}}(f) \in \mathbb{Z}_2$.

The requirement that f preserve $\mathfrak{s}$ is restrictive and causes some technical issues related to compositions. The mod 2 and integral versions are both isotopy invariants; when restricted to the Torelli group (or more generally diffeomorphisms preserving $\mathfrak{s}$ and (respectively) orientation of the moduli space) they give homomorphisms to $\mathbb{Z}_2$ and $\mathbb{Z}$, respectively.

A similar procedure gives rise to invariants for $(k+1)$-dimensional families when $d(\mathfrak{s}) = -k+1$ to get homomorphisms on the homology and homotopy groups of $\mathrm{Diff}(X)$. To get well defined invariants $\mathrm{SW}^{\pi_k}_{X,\mathfrak{s}}$ for (say) $\pi_k(\mathrm{Diff}(X))$, we need to take $b^2_+(X) > k+2$. This has to do with avoiding reducible solutions (for which the spinor $\psi \equiv 0$) and will be assumed going forward without further comment.

Cohomological family invariants. As suggested by Donaldson [Don89, Don96], a more global way to organize invariants of the sort we are using is to define cohomology classes on $\mathrm{BDiff}(X)$ that give invariants for non-trivial families. This was carried out in some detail in [Kon21]. Baraglia-Konno proved a useful gluing formula for such invariants [BK20]. A related approach is to use the Bauer-Furuta extension of the Seiberg-Witten invariants to a stable homotopy invariant; see for instance [Szy10, Xu04, BK22, Bar21].

Lecture III

Exotic families detected by family gauge theory. Non-isotopic (but pseudoisotopic and topologically isotopic) diffeomorphisms were first constructed in the series: [Rub98]; [Rub99]; [Rub01]. The basic idea is that there are manifolds X_0 and X_1 that are homeomorphic but not diffeomorphic, and such that $X_0 \# N \cong X_1 \# N$ where N is a simple manifold such as $S^2 \times S^2$ or $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2$. Such N support simple diffeomorphisms, say ρ_N, from reflections in spheres as described in lecture I. The key computation is a gluing theorem showing that for appropriate Spin^c structures, $\mathrm{SW}_{X\#N, \mathfrak{s}_X \# \mathfrak{s}_N}(1_X \# \rho_N) = \mathrm{SW}_{X,\mathfrak{s}_X}$. (Gluing theorems for other family invariants were proved by Baraglia-Konno [BK20].) Hence if X_0 and X_1 are distinguished by their Seiberg-Witten invariants, then $1_{X_0} \# \rho_N$ and $1_{X_1} \# \rho_N$ are not smoothly isotopic. With a little care (an application of Wall's theorems from Lecture I) it can be arranged that they are pseudoisotopic and hence topologically isotopic.

A more recent example: Kronheimer-Mrowka[KM20]; show the non-triviality of the Dehn twist along the separating 3-sphere in $K3 \# K3$. Lin [Lin23] showed that this persists even after connected sum with $S^2 \times S^2$. Other recent works include [Bar23a]; [KMT23]. This was a surprise; the examples from [Rub98] described above become isotopic after a single stabilization [AKMR15], a fact which is crucial in the higher-parameter constructions of [AR24]. An interesting question is to find exotic diffeomorphism on irreducible 4-manifolds (not a connected sum, except perhaps with a homotopy sphere.)

There are now interesting examples of non-trivial higher homotopy groups. Lin [Lin22] and Smirnov [Smi22a] show that $\pi_1(\mathrm{Diff}(X))$ can be large whenever X has non-trivial Seiberg-Witten invariants and contains spheres of self-intersection -1 or -2. Forthcoming work of Auckly and Ruberman [AR24] shows that the groups $\pi_k(\mathrm{Diff})$ (and $H_k(\mathrm{Diff}^0)$) can be infinitely generated for $k \geq 1$. The elements that are detected are in the kernel of the natural map to $\pi_k(\mathrm{Homeo})$ and $H_k(\mathrm{Homeo})$. The non-trivial elements in $\pi_k(\mathrm{Diff}(X))$ are 1-stably trivial (ie trivial after connected sum with $1_{S^2\times S^2}$). Roughly speaking, the contraction of these elements gives rise to new elements of $\pi_{k+1}(\mathrm{Diff}(X\#S^2\times S^2))$. Gluing techniques relate $\mathrm{SW}^{\pi_k}_{X,\mathfrak{s}}$ and $\mathrm{SW}^{\pi_{k+1}}_{X\#S^2\times S^2,\mathfrak{s}}$; we all this the 'suspension theorem'. The same construction gives infinite generation of

$$\ker[H_{k+1}(\mathrm{BDiff}^0(X)) \to H_{k+1}(\mathrm{BHomeo}^0(X))].$$

In all of these results, the topological triviality is built out of the topological isotopies constructed by Quinn and Perron, and in particular rely on the recursive nature of the constructions from [AR24]. As far as I know, there are no know methods to show triviality of other sorts of elements in $\pi_k(\mathrm{Homeo}(X^4))$.

An important variation, essential to the study of manifolds and diffeomorphisms in higher dimensions, is the idea of moduli spaces of manifolds, or in effect the classifying space BDiff. Konno-Lin [KL22] show that analogues of homology stability for BDiff (with respect to adding copies of $S^2 \times S^2$) fail in dimension 4. A key idea is a refinement $\mathrm{SW}_{\mathrm{half-tot}}$ of the invariant $\mathrm{SW}_{\mathrm{tot}}$ from [Rub01], a sum of SW invariants over various Spin^c structures, to better take account of the 'charge conjugation' action. Baraglia [Bar23b] and Konno [Kon23] use related ideas to show that the mapping class group can be infinitely generated. The work of Auckly-Ruberman [AR24] shows that homology stability fails for BDiff^0 and the classifying space of the Torelli group, but this doesn't seem to hold in higher dimensions either.

Some additional applications of family Seiberg-Witten theory. This is not an all-inclusive list, and was not really covered in the lectures.

(1) Embedded surfaces and stabilizations: [Auc23].
(2) Embedded 3-manifolds: [AR24]; [KMT22]; [IKMT22]
(3) Group actions: [BK23]; [Bar19]
(4) Applications to the topology of the space of positive scalar curvature metrics: [Rub01, BW22, AR24].
(5) Symplectomorphism groups/symplectic families: [Kro97, Smi22b]

Some other methods. Topological results and Kontsevich integrals (again, only lightly touched on.)

(1) Light bulb trick [Gab20, KT24]
(2) Dax invariant, $\mathrm{Diff}(S^1 \times S^3)$, and barbell diffeomorphisms [BG19]

(3) Families of diffeomorphisms of S^4 detected by Kontsevich integrals; failure of the 4-dimensional version of the Smale conjecture [Wat18, Wat20]
(4) Formal smoothings of bundles of 4-manifolds in relation to Watanabe's work [LX23]
(5) Localization of exotic diffeomorphisms along topologically simple (eg contractible) submanifolds [KMPW24].

References

[AKMRS19] Dave Auckly, Hee Jung Kim, Paul Melvin, Daniel Ruberman, and Hannah Schwartz. *Isotopy of surfaces in 4-manifolds after a single stabilization.* Adv. Math., **341** (2019), 609–615. doi:10.1016/j.aim.2018.10.040.

[AKMR15] Dave Auckly, Hee Jung Kim, Paul Melvin, and Daniel Ruberman. *Stable isotopy in four dimensions.* J. Lond. Math. Soc. (2), **91**(2) (2015), 439–463. http://dx.doi.org/10.1112/jlms/jdu075.

[AR24] Dave Auckly and Daniel Ruberman. *Exotic phenomena in dimension four: diffeomorphism groups and embedding spaces.* Manuscript in preparation, 2024.

[AR24] D. Auckly and D. Ruberman, *Exotic families of embeddings*, in "Frontiers in Geometry and Topology", L. Ng and P. Ozsváth, eds., Proceedings of Symposia in Pure Mathematics, American Mathematical Society, 2024.

[Auc23] David Auckly. *Smoothly knotted surfaces that remain distinct after many internal stabilizations*, 2023. https://arxiv.org/abs/2307.16266.

[Bar19] David Baraglia. *Obstructions to smooth group actions on 4-manifolds from families Seiberg-Witten theory.* Adv. Math., **354** (2019), 106730, 32 pages. https://doi.org/10.1016/j.aim.2019.106730.

[Bar21] David Baraglia. *Constraints on families of smooth 4-manifolds from Bauer-Furuta invariants.* Algebr. Geom. Topol., **21**(1) (2021), 317–349. doi:10.2140/agt.2021.21.317.

[Bar23a] David Baraglia. *Non-trivial smooth families of $K3$ surfaces.* Math. Ann., **387**(3-4) (2023), 1719–1744. doi:10.1007/s00208-022-02508-3.

[Bar23b] David Baraglia. *On the mapping class groups of simply-connected smooth 4-manifolds*, 2023. https://arxiv.org/abs/2310.18819.

[BG19] Ryan Budney and David Gabai. *Knotted 3-balls in S^4*, 2019. https://arxiv.org/abs/1912.09029.

[BK20] David Baraglia and Hokuto Konno. *A gluing formula for families Seiberg-Witten invariants.* Geom. Topol., **24**(3) (2020), 1381–1456. doi:10.2140/gt.2020.24.1381.

[BK22] David Baraglia and Hokuto Konno. *On the Bauer-Furuta and Seiberg-Witten invariants of families of 4-manifolds.* J. Topol., **15**(2) (2022), 505–586. doi:10.1112/topo.12229.

[BK23] David Baraglia and Hokuto Konno. *A note on the Nielsen realization problem for $K3$ surfaces.* Proc. Amer. Math. Soc., **151**(9) (2023), 4079–4087. doi:10.1090/proc/15544.

[Boy86] Steven Boyer. *Simply-connected 4-manifolds with a given boundary.* Trans. Amer. Math. Soc., **298**(1) (1986), 331–357. http://dx.doi.org/10.2307/2000623, doi:10.2307/2000623.

[BW22] Boris Botvinnik and Tadayuki Watanabe. *Families of diffeomorphisms and concordances detected by trivalent graphs*, 2022. https://arxiv.org/abs/2201.11373. arXiv:arXiv:2201.11373.

[Cer70] Jean Cerf. *La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie.* Publ. I.H.E.S., **39** (1970), 5–173.

[CH90] T.D. Cochran and N. Habegger. *On the homotopy theory of simply connected four manifolds.* Topology, **29** (1990), 419–440.

[Don89] S. K. Donaldson. *Yang-Mills invariants of four-manifolds*, in "Geometry of low-dimensional manifolds, 1 (Durham, 1989)", Cambridge Univ. Press, Cambridge, 1990, 5–40.

[Don90] S. K. Donaldson. *Polynomial invariants of smooth 4-manifolds.* Topology, **29** (1990), 257–315.

[Don96] S. K. Donaldson. *The Seiberg-Witten equations and 4-manifold topology*, Bull. Amer. Math. Soc. (N.S.), **33** (1996), 45–70.

[FM88] Robert Friedman and John W. Morgan. *On the diffeomorphism types of certain algebraic surfaces. I.* J. Differential Geom., **27**(2) (1988), 297–369. http://projecteuclid.org/euclid.jdg/1214441784.

[FM97] R. Friedman and J. W. Morgan, *Algebraic surfaces and Seiberg-Witten invariants*, J. Algebraic Geom., **6** (1997), 445–479.

[Fre82] Michael H. Freedman. *The topology of four–dimensional manifolds.* J. Diff. Geo., **17** (1982), 357–432.

[Gab20] David Gabai. *The 4-dimensional light bulb theorem.* J. Amer. Math. Soc., **33**(3) (2020), 609–652. doi:10.1090/jams/920.

[GGH+23] David Gabai, David T. Gay, Daniel Hartman, Vyacheslav Krushkal, and Mark Powell. *Pseudo-isotopies of simply connected 4-manifolds,* 2023. https://arxiv.org/abs/2311.11196.

[HW73] Allen Hatcher and John Wagoner. *Pseudo-isotopies of compact manifolds*, volume No. 6 of Astérisque. Société Mathématique de France, Paris, 1973. With English and French prefaces.

[Igu21a] Kiyoshi Igusa. *Second obstruction to pseudoisotopy I*, 2021. https://arxiv.org/abs/2110.09659.

[Igu21b] Kiyoshi Igusa. *Second obstruction to pseudoisotopy in dimension 3,* 2021. https://arxiv.org/abs/2112.08293.

[IKMT22] N. Iida, H. Konno, A. Mukherjee, and M. Taniguchi, *Diffeomorphisms of 4-manifolds with boundary and exotic embeddings*, 2022. newblock https://arxiv.org/abs/2203.14878

[KL22] Hokuto Konno and Jianfeng Lin. *Homological instability for moduli spaces of smooth 4-manifolds,* 2022. https://arxiv.org/abs/2211.03043.

[KMT22] H. Konno, A. Mukherjee, and M. Taniguchi, *Exotic codimension-1 submanifolds in 4-manifolds and stabilizations*, 2022. https://arxiv.org/abs/2210.05029.

[KM20] P. B. Kronheimer and T. S. Mrowka. *The Dehn twist on a sum of two $K3$ surfaces.* Math. Res. Lett., **27**(6) (2020), 1767–1783. doi:10.4310/MRL.2020.v27.n6.a8.

[KMT23] Hokuto Konno, Abhishek Mallick, and Masaki Taniguchi. *Exotic Dehn twists on 4-manifolds,* 2023. https://arxiv.org/abs/2306.08607.

[Kon21] Hokuto Konno. *Characteristic classes via 4-dimensional gauge theory.* Geom. Topol., **25**(2) (2021), 711–773. doi:10.2140/gt.2021.25.711.

[Kon23] Hokuto Konno. *The homology of moduli spaces of 4-manifolds may be infinitely generated*, 2023. https://arxiv.org/abs/2310.18695.

[Kos24] Danica Kosanović. *On fundamental groups of spaces of (framed) embeddings of the circle in a 4-manifold,* 2024. https://arxiv.org/abs/2407.06923.

[KT24] D. Kosanović and P. Teichner, *A new approach to light bulb tricks: disks in 4-manifolds*, Duke Math. J., **173** (2024), 673–721.

[Kre79] M. Kreck. *Isotopy classes of diffeomorphisms of $(k-1)$-connected almost-parallelizable $2k$-manifolds.* In Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), pages 643–663. Springer, Berlin, 1979.

[Kre01] Matthias Kreck. *h-cobordisms between 1-connected 4-manifolds.* Geom. Topol., **5** (2001), 1–6. doi:10.2140/gt.2001.5.1.

[Kro97] P.B. Kronheimer. *Some non-trivial families of symplectic structures.* Preprint, available from https://people.math.harvard.edu/~kronheim/diffsymp.pdf, 1997.

[KMPW24] V. Krushkal, A. Mukherjee, M. Powell, and T. Warren, *Corks for exotic diffeomorphisms*, 2024. https://arxiv.org/abs/2407.04696.

[Lin22] Jianfeng Lin. *The family Seiberg-Witten invariant and nonsymplectic loops of diffeomorphisms,* 2022. https://arxiv.org/abs/2208.12082.

[Lin23] Jianfeng Lin. *Isotopy of the Dehn twist on $K3 \# K3$ after a single stabilization.* Geom. Topol., **27** (5) (2023), 1987–2012. doi:10.2140/gt.2023.27.1987.

[LX23] Jianfeng Lin and Yi Xie. *Configuration space integrals and formal smooth structures,* 2023. https://arxiv.org/abs/2310.14156.

[Mil58] John Milnor. *On simply connected* 4*-manifolds.* In Symposium internacional de topología algebraica International symposium on algebraic topology, pages 122–128. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.

[Nic00] Liviu Nicolaescu. Notes on Seiberg-Witten theory American Mathematical Society, Providence, RI, 2000.

[OP22] Patrick Orson and Mark Powell. *Mapping class groups for simply-connected 4-manifolds with boundary,* 2022. https://arxiv.org/abs/2207.05986.

[Per86] Bernard Perron. *Pseudo-isotopies et isotopies en dimension quatre dans la catégorie topologique.* Topology, **25** (1986), 381–397.

[Qui86] Frank Quinn. *Isotopy of* 4*-manifolds.* J. Differential Geom., **24**(3) (1986), 343–372. http://projecteuclid.org/euclid.jdg/1214440552.

[Rub98] Daniel Ruberman. *An obstruction to smooth isotopy in dimension* 4*.* Math. Res. Lett., **5**(6) (1998), 743–758. https://doi.org/10.4310/MRL.1998.v5.n6.a5. .

[Rub99] Daniel Ruberman. *A polynomial invariant of diffeomorphisms of 4-manifolds.* In Proceedings of the Kirbyfest (Berkeley, CA, 1998), pages 473–488 (electronic). Geom. Topol., Coventry, 1999. https://doi.org/10.2140/gtm.1999.2.473.

[Rub01] Daniel Ruberman. *Positive scalar curvature, diffeomorphisms and the Seiberg-Witten invariants.* Geom. Topol., **5** (2001), 895–924 (electronic). https://doi.org/10.2140/gt.2001.5.895.

[Sin22] Oliver Singh. *Pseudo-isotopies and diffeomorphisms of 4-manifolds,* 2022. https://arxiv.org/abs/2111.15658, arXiv:2111.15658.

[Smi22a] Gleb Smirnov. *From flops to diffeomorphism groups.* Geom. Topol., 26(2) (2022), 875–898. doi:10.2140/gt.2022.26.875.

[Smi22b] Gleb Smirnov. *Symplectic mapping class groups of K3 surfaces and Seiberg-Witten invariants,* Geom. Funct. Anal., **32** (2022), 280–301.

[Szy10] M. Szymik, *Characteristic cohomotopy classes for families of 4-manifolds*, Forum Math., **22** (2010), 509–523. https://doi.org/10.1515/FORUM.2010.027

[Wal62] C. T. C. Wall. *On the orthogonal group of unimodular quadratic forms.* Math. Ann., **147** (1962), 328–338.

[Wal64a] C. T. C. Wall. *Diffeomorphisms of* 4*-manifolds.* J. London Math. Soc., **39** (1964), 131–140.

[Wal64b] C. T. C. Wall. *On simply-connected* 4*-manifolds*, J. London Math. Soc., **39** (1964), 141–149.

[Wal64] C. T. C. Wall. *On the orthogonal groups of unimodular quadratic forms. II.* J. Reine Angew. Math., **213** (1963/64), 122–136. doi:10.1515/crll.1964.213.122.

[Wat18] Tadayuki Watanabe. *Some exotic nontrivial elements of the rational homotopy groups of* $\mathrm{Diff}(S^4)$*,* 2018. https://arxiv.org/abs/1812.02448.

[Wat20] Tadayuki Watanabe. *Theta-graph and diffeomorphisms of some 4-manifolds*, 2020. https://arxiv.org/abs/2005.09545.

[Whi49] J. H. C. Whitehead. *On simply connected* 4*-dimensional polyhedra.* Comm. Math. Helv., **22** (1949), 48–92.

[Xu04] M. Xu, *The Bauer-Furuta invariant and a cohomotopy refined Ruberman invariant*, ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–State University of New York at Stony Brook.

Triangulation and Homology Cobordisms

Matthew Stoffregen
(joint work with Irving Dai, Jen Hom, Linh Truong)

In [8, 7], Galewski-Stern and Matumoto developed machinery to study triangulations of higher-dimensional manifolds, and proved the following remarkable result:

Theorem ([8, 7]). *Let M be a topological manifold with either*

(1) M is closed, and dimension of M is at least 5,
(2) M is a compact manifold possibly with boundary, of dimension at least 6,
(3) M is of dimension at least 7

Then M is triangulable if and only if $\beta(\Delta(M)) = 0$, where:

(1) $\Delta(M)$ is the Kirby-Siebenmann invariant of M, a class in $H^4(M;\mathbb{Z}/2)$.
(2) $\Theta^3_{\mathbb{Z}}$ is the integral homology cobordism group of smooth three-manifolds. That is, objects are closed, oriented 3-manifolds $[Y]$ with $H_*(Y;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$, where we quotient by the relation $Y_1 \sim Y_2$ if there exists a compact oriented 4-manifold W with $\partial W = Y_1 \amalg -Y_2$ with $H_*(Y_i;\mathbb{Z}) \to H_*(W;\mathbb{Z})$ is an isomorphism for $i = 1, 2$.
(3) There is a homomorphism $\mu\colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2$, the Rokhlin invariant, *determined by taking $Y = \partial W$, for W some compact, spin 4-manifold, and setting*

$$\mu(Y) = \sigma(W)/8 \bmod 2,$$

where $\sigma(W)$ is the signature of W.
(4)

$$\beta\colon H^4(-;\mathbb{Z}/2) \to H^5(-;\ker\mu)$$

is the Bockstein operator associated to the short-exact sequence:

$$0 \to \ker\mu \to \Theta^3_{\mathbb{Z}} \xrightarrow{\mu} \mathbb{Z}/2 \to 0. \tag{1}$$

The above theorem relates questions about triangulations of higher-dimensional manifolds, where here higher-dimensional means M as in the theorem statement, to understanding the homology cobordism $\Theta^3_{\mathbb{Z}}$.

We note some important riders to the theorem. Indeed, that the theorem does not actually imply that there are non-triangulable higher-dimensional manifolds, even though it does identify exactly when a particular higher-dimensional manifold is triangulable! The problem lies in that the exact sequence (1) is not easy to determine, a priori. For instance, if there were a class $Y \in \Theta^3_{\mathbb{Z}}$ for which $\mu(Y) = 1$ and $2Y = 0 \in \Theta^3_{\mathbb{Z}}$, then the short exact sequence would split, so that its Bockstein operator would be zero (we note that the short exact sequence splits exactly if such a class Y exists). In this case, all higher-dimensional manifolds would be triangulable. Galewski-Stern and Matumoto [8, 7] provide a kind of converse to this statement:

Theorem. *All higher-dimensional topological manifolds are triangulable if and only if there exists a class $Y \in \Theta^3_{\mathbb{Z}}$ so that $\mu(Y) = 1$ and $2Y = 0 \in \Theta^3_{\mathbb{Z}}$.*

As remarked previously, the 'if' part follows from the first theorem, while the 'only if' is established by constructing a particular topological manifold M for which $\beta(\Delta(M))$ vanishes only if (1) is split.

However, independently, the group $\Theta^3_{\mathbb{Z}}$ has proven difficult to understand; the first theorem provides strong motivation for its study. In light of that theorem, we are especially interested in studying the isomorphism class of the arrow $\Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2$ as an object in the arrow category of the category of abelian groups. So far, the only techniques available to understand the structure of $\Theta^3_{\mathbb{Z}}$ have been *monopole Floer homology*, *Heegaard Floer homology*, and *instanton Floer theory*, where the former two are very closely related. Prior to the advent of Pin(2)-equivariant monopole Floer homology, introduced in [6] (see also [4]) and its close cousin *involutive Heegaard Floer homology*, monopole Floer and Heegaard Floer theory give (using either theory) a homomorphism:

$$d\colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}.$$

To be more precise, for each coefficient field characteristic, there is such a homomorphism d (by convention, usually d refers to the d-invariant defined for a field of characteristic 0); it would be natural to expect that these homomorphisms are distinct, but this is not known (and, at present, seems to be a very difficult problem). None of the d-invariants from monopole or Heegaard Floer theory are related directly to the μ invariant. That is, μ does not factor through any d. We note that $\Theta^3_{\mathbb{Z}}$ is an infinitey-generated abelian group. Such basic questions as whether or not $\Theta^3_{\mathbb{Z}}$ contains a torsion element or a divisible element are unanswered.

Instanton theory provides powerful tools to study $\Theta^3_{\mathbb{Z}}$ and can often be used to distinguish classes $Y_1, Y_2 \in \Theta^3_{\mathbb{Z}}$; the reader should note that a priori it is very difficult to tell if 3-manifolds Y_1, Y_2 actually represent the same element in $\Theta^3_{\mathbb{Z}}$. However, instanton theory does not seem to naturally produce invariants of $\Theta^3_{\mathbb{Z}}$ through which the Rokhlin invariant factors.

In [6], Manolescu introduced Pin(2)-equivariant monopole Floer homology; this is an enhancement of ordinary monopole Floer homology. The reader familiar with Borel homology should think of monopole Floer homology, defined by Kronheimer-Mrowka, as the S^1-equivariant Borel homology of some space or spectrum, while the Pin(2)-equivariant theory is a Pin(2)-equivariant Borel homology. Modules over $H^*(B\mathrm{Pin}(2))$ are much more complicated than modules over $H^*(BS^1)$, and so Pin(2)-equivariant monopole Floer homology lives in a much richer 'algebraic' category than ordinary monopole Floer homology does. This allows one to define more subtle invariants of homology cobordism using this theory. Prior to [6], it had already been observed by Kronheimer-Mrowka and Furuta that in the presence of a spin structure, the Seiberg-Witten equations admit additional symmetry, this resulted, among other things, in the 10/8-Theorem by Furuta.

Manolescu defined an invariant $\beta\colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$ using Pin(2)-equivariant monopole Floer homology and showed that

(1) $\beta(Y) = \mu(Y) \bmod 2$.
(2) $\beta(Y) = -\beta(-Y)$.

Note that β is not a homomorphism. In particular, Manolescu showed:

Theorem ([6]). *Let $Y \in \Theta^3_{\mathbb{Z}}$. If $\mu(Y) = 1$, then $2Y \neq 0 \in \Theta^3_{\mathbb{Z}}$. In particular, there are non-triangulable higher-dimensional topological manifolds.*

However, the story of which higher-dimensional topological manifolds are triangulable is not completed, because for a given topological manifold M, together with knowledge of $\Delta(M) \in H^4(M;\mathbb{Z}/2)$, the first of these three theorems does *not* provide a practical way to determine $\beta(\Delta(M))$ - indeed, the exact sequence (1) is somewhat inscrutable.

As a particular question, it is natural to ask the following:

Question. *For topological, higher-dimensional non-triangulable manifolds M, what constraints are there on $H^4(M;\mathbb{Z})$?*

We note that all higher-dimensional manifolds constructed so far have a $\mathbb{Z}/2$ summand in $H^4(M;\mathbb{Z})$. In fact, we make the following simple observation (which the reader is invited to check):

Lemma. *Let E denote the exact sequence (1), and let E_k denote the exact sequence:*

$$0 \to \mathbb{Z}/2^{k-1} \to \mathbb{Z}/2^k \to \mathbb{Z}/2 \to 0$$

If there is a morphism $E_k \to E$ of exact sequences which is an isomorphism in the last $\mathbb{Z}/2$, then $\beta\colon H^4(M;\mathbb{Z}/2) \to H^5(M;\ker\mu)$ vanishes if the 2-primary part $H^4(M;\mathbb{Z})_2$ has only $\mathbb{Z}/2^t$-summands for $t \geq k$.

To put it in more parseable terms:

Corollary. *Fix $k > 0$. If there is a class $Y \in \Theta^3_{\mathbb{Z}}$ with $\mu(Y) = 1$ and $2^kY = 0$, then:*

For all topological higher-dimensional manifolds M with $H^4(M;\mathbb{Z})_2$ (the 2-primary part) has summands $\mathbb{Z}/2^t$ with $t \geq k$, then M is triangulable.

That is to say, the existence of torsion in the homology cobordism group with nontrivial Rokhlin invariant guarantees that large collections of higher-dimensional manifolds are triangulable.

In this talk, we outline a strategy, in joint work with Dai, Hom, and Truong to show:

Conjecture. *Let $Y \in \Theta^3_{\mathbb{Z}}$ with $\mu(Y) = 1$. Then Y is not torsion in $\Theta^3_{\mathbb{Z}}$.*

This would mean that, in a very rough sense, as many topological manifolds as possible 'should' be nontriangulable. If our conjecture is resolved positively, it would be interesting to show that there are indeed non-triangulable manifolds with homotopy types that, say, have any H^4 with non-trivial 2-primary part, among other questions.

We say a few words about the strategy. Manolescu's proof resulted from studying the algebraic structure of the target category of Pin(2)-equivariant monopole

Floer homology. One can extract (easily) a 'universal' invariant from this target category [9] to define a homomorphism

$$SWF\colon \Theta^3_{\mathbb{Z}} \to \mathfrak{CLE},$$

where the latter term is the *local equivalence group* of Pin(2)-chain complexes. It follows from the construction that $\mu\colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2$ is a composite $\Theta^3_{\mathbb{Z}} \to \mathfrak{CLE} \to \mathbb{Z}/2$, where the latter map $\mathfrak{CLE} \to \mathbb{Z}/2$ is also denoted μ by abuse of notation.

The group $\mathfrak{CLE}$ is formed from a certain collection of $C_*(\mathrm{Pin}(2))$-chain complexes by identifying those related by *local equivalence*; this latter is defined in order that homology cobordant manifolds Y_1 and Y_2 with have locally equivalent monopole Floer complexes. In fact, $\mathfrak{CLE}$ is naturally a partially-ordered group.

In the present work, we study the morphism $\mathfrak{CLE} \to \mathbb{Z}/2$ - this is a, roughly speaking, 'algebraic' question, which is separate from understanding the actual gauge-theoretic construction $SWF\colon \Theta^3_{\mathbb{Z}} \to \mathfrak{CLE}$. In previous work, we had studied other local equivalence groups, analogous to $\mathfrak{CLE}$, and had found how to find totally-ordered quotients of various local equivalence groups [1, 3, 2]. The study of such totally-ordered quotients was initiated by Hom in [5].

In the present work, we find what amount to the most-natural totally-ordered quotients of $\mathfrak{CLE}$. An interesting wrinkle, in comparison to the work in [1, 3, 2], is that the analogously-constructed totally-ordered quotient sets of $\mathfrak{CLE}$ are not naturally groups. More usefully, we show that the Rokhlin invariant factors through a certain totally-ordered quotient F of $\mathfrak{CLE}$. The projection map $\mathfrak{CLE} \to F$ does not appear to be a homomorphism - if it were a homomorphism, our conjecture would follow directly. In work in progress, we study the projection map $\mathfrak{CLE} \to F$; with more work we hope to establish that even though it is likely not to be a homomorphism, that the projection map is sufficiently well-behaved to establish the conjecture.

References

[1] I. Dai, J. Hom, M. Stoffregen, L. Truong, *An infinite-rank summand of the homology cobordism group*, arXiv preprint arXiv:1810.06145

[2] I. Dai, J. Hom, M. Stoffregen, L. Truong, *Homology concordance and knot Floer homology*, arXiv preprint arXiv:2110.14803

[3] I. Dai, J. Hom, M. Stoffregen, L. Truong, *More concordance homomorphisms from knot Floer homology*, Geometry & Topology, **25**, (2021), 275-338

[4] F. Lin, *A Morse-Bott approach to monopole Floer homology and the triangulation conjecture*, 225 (2018), American Math Society.

[5] J. Hom *An infinite-rank summand of topologically slice knots*, Geom. Topol. **19**. (2015), 1063-1110.

[6] C. Manolescu, Pin(2)*-equivariant Seiberg-Witten Floer homology and the triangulation conjecture*, J. Amer. Math. Soc., **1**, (2016), 147-176.

[7] T. Matumoto, *Triangulation of manifolds*, Proc. Sympos. Pure Math. XXXII, (1978), 3-6.

[8] D. Galewski, R. Stern, *Classification of simplicial triangulations of topological manifolds*, Ann. of Math. (2), (1980), 1-34

[9] M. Stoffregen, Pin(2)*-equivariant Seiberg–Witten Floer homology of Seifert fibrations*, Compositio Mathematica, **156**, (2020), 199-250.

Homological stability for symplectic groups via disordered arcs

Nathalie Wahl
(joint work with Ismael Sierra)

We prove the following result:

Theorem. [9] *Let R by a PID. Then the homology of the symplectic group stabilises in the following range*

$$H_i(\mathrm{Sp}_{2g}(R);\mathbb{Z}) \xrightarrow{\cong} H_i(\mathrm{Sp}_{2g+2}(R);\mathbb{Z}) \quad \text{for } i \leq \frac{2g-2}{3}$$

The result holds more generally for rings satisfying a unitary and dual unitary stable rank condition, for example Dedekind domains, with an appropriate shift of the stability bound. This theorem improves the previous known slope $\frac{1}{2}$ bounds of [2, 6] to slope $\frac{2}{3}$.

A similar looking result is the following:

Theorem. [3, 1, 7] *Let $S_{g,1}$ be a surface of genus g with one boundary component. The homology of the mapping class group $\Gamma_{g,1} := \pi_0 \operatorname{Diff}(S_{g,1})$ stabilises in the following range*

$$H_i(\Gamma_{g,1};\mathbb{Z}) \xrightarrow{\cong} H_i(\Gamma_{g+1,1};\mathbb{Z}) \quad \text{for } i \leq \frac{2g-2}{3}$$

The idea of the proof of the first theorem is to mimic algebraically a recent proof of the second one, given in [4]. In the case of the mapping class groups of surfaces, the idea of the proof is to identify the mapping class group $\Gamma_{g,1}$ as the automorphism group of $D^{\#2g+1}$ in the monoidal category of *bidecorated surfaces*, with D a disc with two intervals marked in its boundary. Applying the stability machine of [8, 5], one finds out that this particular stability problem is ruled by the connectivity of the complex of *disordered arcs*. In the case of symplectic groups, the category of bidecorated surfaces is replaced by that of *formed spaces with boundary* (M, λ, ∂), where M is an R-module, λ an alternating form on M, and $\partial : M \to R$ a linear map, with an appropriate monoidal structure. The disc D is replaced by the formed space $(R, 0, \mathrm{id})$, and there is an identification

$$\mathrm{Aut}(R^{\#2g+1}) \cong \mathrm{Sp}_{2g}(R),$$

with $\mathrm{Aut}(R^{\#2g})$ an *odd* symplectic group. Applying the same stability machine of [8], one finds out that stability is now ruled by an algebraic verion of the complex of disordered arcs, whose connectivity we compute.

References

[1] S. K. Boldsen, Improved homological stability for the mapping class group with integral or twisted coefficients, Mathematische Zeitschrift 270 (2012), no. 1-2, 297–329.

[2] R. Charney. A generalization of a theorem of Vogtmann. In Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985), volume 44, pages 107–125, 1987.

[3] J. Harer, Stability of the homology of the mapping class groups of orientable surfaces, Annals of mathematics 121 (1985), no. 2, 215–249 (eng).

[4] O. Harr, M. Vistrup and N. Wahl. Disordered arcs and Harer stability, Higher Structures 8 (2024), no.1, 193–223.
[5] M. Krannich, Homological stability of topological moduli spaces, Geom. Topol. 23 (2019), no. 5, 2397–2474.
[6] B. Mirzaii and W. van der Kallen. Homology stability for unitary groups. Doc. Math., 7:143–166, 2002.
[7] O. Randal-Williams, Resolutions of moduli spaces and homological stability, Journal of the Euro- pean Mathematical Society 18 (2016), no. 1, 1–81.
[8] O. Randal-Williams and N. Wahl. Homological stability for automorphism groups, Advances in Mathematics 318 (2017), 534–626.
[9] I. Sierra and N. Wahl. Homological stability for symplectic groups via algebraic arc complexes. Preprint 2024.

Morse boundaries of 3-manifold groups

Stefanie Zbinden

The Morse boundary, as introduced in [2], is a generalization of the Gromov boundary and captures hyperbolic directions of non-hyperbolic groups. Unlike the Gromov boundary, the Morse boundary has an elusive topology which is in general neither compact nor metrizable. In this talk we first introduce the Morse boundary and highlight its significance. We then give an overview of what is known about the topology of the Morse boundary. In particular, we show that we can determine the Morse boundary for all 3-manifold groups (see [3]).

The Gromov boundary of a hyperbolic space is the collection of all its geodesic rays, which are identified if they have bounded Hausdorff distance. One can equip the Gromov boundary with a metric which makes it metrizable and compact and the Gromov boundary is a quasi-isometry invariant. The Morse boundary (see [2]) is an attempt to generalize the Gromov boundary for non-hyperbolic spaces. In order to retain that the boundary is a quasi-isometry invariant one only considers geodesics which are "hyperbolic like", or more precisely, which are Morse, that is, they satisfy the Morse property.

Unlike the Gromov boundary, the Morse boundary is in general neither metrizable nor compact, and hence understanding its topology is very tricky. In spite of this, in [1], they manage to describe the Morse boundary for several groups, including right angled Artin groups and fundamental groups of finite volume cusped hyperbolic 3-manifolds.

As a next step, one can ask the following. If we understand the Morse boundary of two groups A and B, what do we know about the Morse boundary of the free product $A * B$? This question was answered in [4], where it is shown that the Morse boundary of a free product $A * B$ is completely determined by the Morse boundary of its factors A and B.

Finally, to understand the Morse boundary of 3-manifold groups, it suffices to understand the Morse boundary of fundamental groups of prime manifolds and how the Morse boundary behaves with respect to certain graph of groups such as free products (see [3]). In particular, one can show that the Morse boundary of the fundamental group of any closed free manifold has one of 9 homeomorphism types.

References

[1] R. Charney, M. Cordes, and A. Sisto, *Complete topological descriptions of certain Morse boundaries*, arXiv preprint arXiv:1908.03542 (2019).
[2] M. Cordes *Morse boundaries of proper geodesic metric spaces*, Groups, Geometry, and Dynamics **11.4** (2017), 1281–1306.
[3] S. Zbinden, *Morse boundaries of 3-manifold groups*, Transactions of the American Mathematical Society (2024).
[4] S.. Zbinden, *Morse boundaries of graphs of groups with finite edge groups*, Journal of Group Theory **26.5** (2023), 969-1002.

Ext in functor categories and stable cohomology of $\mathrm{Aut}(F_n)$

Gregory Arone

Motivated by the work of Vespa [2], we study polynomial functors from the category of (finitely generated, free) groups to abelian groups. We present a homotopy theoretic method for calculating Ext groups between polynomial functors. The method is based on changing the setting from groups to simplicial groups and then from simplicial groups to topological spaces. It enables us to substantially extend the range of what can be calculated. In particular, we can calculate torsion in the Ext groups, about which very little had been known. We will discuss some applications to the stable cohomology of $\mathrm{Aut}(F_n)$, based on a theorem of Djament [1].

References

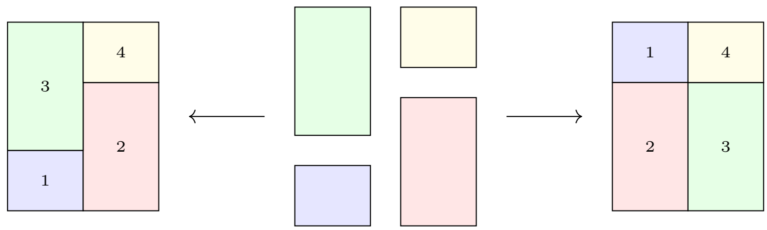
[1] A. Djament, *Décomposition de Hodge pour l'homologie stable des groupes d'automorphismes des groupes libres*, Compos. Math.155(2019), no.9, 1794–1844.
[2] C. Vespa, *Extensions between functors from free groups*, Bull. Lond. Math. Soc. 50 (2018), no. 3, 401–419.

Scissors automorphism groups and their homology

Alexander Kupers
(joint work with Ezekiel Lemann, Cary Malkiewich, Jeremy Miller, Robin J. Sroka)

A *polytope* P in n-dimensional Euclidean geometry is a finite union of n-simplices, which in turn are defined as convex hulls of $n+1$ points in general position. Two polytopes P and Q are *scissors congruent* if P can be cut into finitely many polytopes, and these can be rearranged by isometries to form Q. Hilbert's third problem asks for a classification of polytopes and complementary to this, we study the group $\mathrm{Aut}_{E^n}(P)$ of scissors automorphisms of a fixed polytope P; ways to cut a polytope and reassemble it from the pieces, up to refinement.

Example. Below we illustrate a scissors automorphism of a two-dimensional rectangle: we cut it into four pieces (here rectangles, in general polytopes), move them by isometries (here translations, in general isometries), and reassemble it to the rectangle:



Our goal is to establish "homological stability" for these groups and then compute their "stable homology"; these results will appear in a forthcoming paper [1]. Firstly, using the homological stability machinery of Randal-Williams–Wahl [4] we prove that the homology of the scissor automorphism group is in fact independent of the polytope:

Theorem. *For any two nonempty n-dimensional polytopes P and Q, there is an isomorphism*

$$H_*(\mathrm{Aut}_{E^n}(P)) \cong H_*(\mathrm{Aut}_{E^n}(Q)).$$

Secondly, we interpret these homology groups in terms of the Zakharevich's algebraic K-theory spectrum $K(\mathcal{E}^n)$ for the assembler $\mathcal{E}^n$ of n-dimensional Euclidean polytopes [6]:

Theorem. *For a nonempty n-dimensional polytope P, there is an isomorpism*

$$H_*(\mathrm{Aut}_{E^n}(P)) \cong H_*(\Omega_0^\infty K(\mathcal{E}^n)).$$

This is a justification for Zakharevich's construction being a good one; it relates to scissors automorphism groups as the algebraic K-theory spectra of rings relate to general linear groups.

The spectra $K(\mathcal{E}^n)$ are computationally accessible through the models in terms of Thom spectra of Tits buildings provided by Malkiewich [3]. For example, we can use this to compute the homology of the scissors automorphism groups of two-dimensional polytope, e.g. a rectangle:

Example. For a nonempty 2-dimensional polytope P, we have

$$H_*(\mathrm{Aut}_{E^2}(P)) \cong \Lambda^*\left(\bigoplus_{p+2q\geq 1} H_p(O(2);\Lambda_{\mathbb{Q}}^{2q+2}(\mathbb{R}^2)^t)[p+2q]\right),$$

where the discrete group $O(2)$ acts on $(\mathbb{R}^2)^t$ through a combination of its standard action and the determinant. Note the reduced homology consists of rational vector spaces, which is a general feature.

Our techniques extend to other geometries, other isometry groups, and restricted collections of polytopes. This allows us to recover influential results of Szymik–Wahl [5] and Li [2], which say that the Brin–Thompson groups are acyclic, as well as to study interval and rectangle exchange transformation groups. These groups have previously been studied in dynamics and geometric group theory.

References

[1] A. Kupers, E. Lemann, C. Malkiewich, J. Miller, and R. Sroka, *Scissors automorphism groups and their homology*, 2024, in preparation.

[2] X. Li, *Ample groupoids, topological full groups, algebraic K-theory spectra and infinite loop spaces*, 2022, arXiv:2209.08087.

[3] C. Malkiewich, *Scissors congruence K-theory is a Thom spectrum*, 2022, arXiv:2210.08082.

[4] O. Randal-Williams and N. Wahl, *Homological stability for automorphism groups*, Adv. Math. **318** (2017), 534–626.

[5] M. Szymik and N. Wahl, *The homology of the Higman-Thompson groups*, Invent. Math. **216** (2019), no. 2, 445–518.

[6] I. Zakharevich, *The K-theory of assemblers*, Adv. Math. **304** (2017), 1176–1218.

Cables of the figure-eight knot

Sungkyung Kang

(joint work with Irving Dai, Abhishek Mallick, JungHwan Park, Matthew Stoffregen, Masaki Taniguchi)

In 1980, Kawauchi asked in his unpublished manuscript [6] whether the $(2,1)$-cable of the figure-eight knot is *slice*, i.e. bounds a smoothly embedded disk in the 4-ball. This problem has attracted considerable attention due to its connection to the slice-ribbon conjecture. Explicitly, Miyazaki [11], using a result of Casson and Gordon [3], showed that if K is a fibered, negative amphichiral knot with irreducible Alexander polynomial, then the $(2n,1)$-cable of K is not (homotopy) ribbon for any $n \neq 0$. On the other hand, these knots are known to be algebraically slice [6] and rationally slice [7]. While such cables are generally believed not to be slice, the fact that no argument has appeared in the literature has left open the possibility that these generate counterexamples to the slice-ribbon conjecture.

Since the figure-eight knot 4_1 is the simplest possible fibered negative-amphichiral knot with irreduccible Alexander polynomial, we first consider the sliceness of $(4_1)_{2,1}$. We use the following fact as the obstruction to its smooth sliceness: *if a knot K is smoothly slice, then its branched double cover $\Sigma(K)$ bounds a rational homology ball W, together with a spin structure $\mathfrak{s}$, such that the deck transformation τ on $\Sigma(K)$ extends to a diffeomorphism $\tilde{\tau}$ of W that satisfies $f^*\mathfrak{s} = \mathfrak{s}$.*

This obstruction can be carried out by observing the action of τ on the involutive Heegaard Floer homology of $(\Sigma(K), \mathfrak{s})$, where $\mathfrak{s}$ here denotes the unique spin structure on $\Sigma(K)$. (For simplicity, we will denote all spin structures as $\mathfrak{s}$, so that we do not have to create too much symbols) In particular, we consider the chain endomorphisms

$$\tau, \iota : CF^-(\Sigma(K)) \to CF^-(\Sigma(K)),$$

where ι denotes the involutive action defined in [5]. The obstruction discussed above then can be translated as follows: *if K is slice, then there exists a U-nontorsion homology class in $HF^-(\Sigma(K))$ of absolute grading 0, which is invariant under the actions of τ and ι.* Using the Montesinos trick

$$\Sigma((4_1)_{2,1}) \simeq S^3_{+1}(4_1\sharp(4_1)^r),$$

one can *almost* compute $HF^-(\Sigma((4_1)_{2,1}))$ and the actions of τ and ι, from which we can deduce that the desired invariant homology class does not exist. Therefore we deduce (ref:paper1) that $(4_1)_{2,1}$ is not smoothly slice. This can also be generalized in a way that we can replace 4_1 by "half" of all knots that are torsion in the smooth knot concordance group.

Unfortunately, the same technique cannot be used to prove the non-sliceness of $(4_1)_{2n,1}$ for $n > 1$; this is because the Montesinos trick cannot be used anymore. Our starting point here is the existence of a smooth concordance S of homology class $(1, 3)$ in $2\mathbb{C}P^2$, from the 0-framed 4_1 to the (-10)-framed unknot, discovered by Aceto-Castro-Miller-Park-Stipsicz [1] in order to give a new proof of the non-sliceness of $(4_1)_{2,1}$. By cabling S, we get a smooth concordance S_n of homology class $(2n, 6n)$ in $2\mathbb{C}P^2$ from $(4_1)_{2n,1}$ to $T_{2n,1-20n}$. Hence, if $(4_1)_{2n,1}$ were slice, we would get a smooth disk D_n of the same homology class in $2\mathbb{C}P^2$ whose outgoing boundary is $T_{2n,1-20n}$.

To obstruct the existence of D_n, we use real Frøyshov invariant, $\bar{\delta}_R$ and $\underline{\delta}_R$, defined by Konno-Miyazawa-Taniguchi [8]. They satisfy certain cobordism inequalities; in our case, the inequality we get from D_n is

$$-\frac{1}{8} \leq \bar{\delta}_R(T_{2n,1-20n}) = -\underline{\delta}_R(T_{2n,20n-1}).$$

To compute $\underline{\delta}_R(T_{2n,20n-1})$, we use the lattice homotopy technique developed by Dai, Sasahira, and Stoffregen [4] (based on Néméthi's work [12] on lattice homology) to show that there exists a $\mathbb{Z}_2$-equivariant map

$$\mathcal{T} : \mathcal{H}(\Gamma, \mathfrak{s}) \to SWF(\Sigma(T_{2n,20n-1}), \mathfrak{s}),$$

where Γ is an almost rational (starshaped) negative-definite plumbing graph of $\Sigma(T_{2n,20n-1}) = \Sigma(2, 2n, 20n-1)$ in which the deck transformation action can be realized as the "swapping" of two identical legs, $\mathfrak{s}$ denotes the unique spin structure on $\Sigma(T_{2n,20n-1})$, and $\mathcal{H}$ denotes the lattice homotopy type of $(\Gamma, \mathfrak{s})$; note that $\mathbb{Z}_2$ is generated by $I = j \circ \tau$. Since all spectra involved are finite $\mathbb{Z}_2$-spectra and we only use mod 2 cohomology in order to define real Frøyshov invariants, we can apply the Sullivan conjecture [2, 3, 10] to show that $\mathcal{T}$ induces an isomorphism between mod 2 coefficient cohomology of $\mathbb{Z}_2$-fixed point spectra.

This allows us to deduce that

$$\underline{\delta}_R(T_{2n,20n-1}) = \bar{\delta}_R(T_{2n,20n-1}) = \frac{1}{2}\bar{\mu}(\Sigma(2, 2n, 20n-1)) = \frac{9}{8},$$

which would imply

$$-\frac{1}{8} \leq -\underline{\delta}_R(T_{2n,20n-1}) = -\frac{9}{8},$$

a contradiction. Therefore we were able to conclude in our work [9] that $(4_1)_{2n,1}$ is not smoothly slice. The same technique was also applied to show that $(4_1)_{2n,1}$ does not bound a normally immersed disk in B^4 which only has negative double points as its singularities.

References

[1] Paolo Aceto, Nickolas A. Castro, Maggie Miller, JungHwan Park, and András Stipsicz, *Slice obstructions from genus bounds in definite 4-manifolds*, preprint arXiv:2303.10587 (2023).

[2] Gunnar Carlsson, *Equivariant stable homotopy and Sullivan's conjecture*, Invent. Math. **103** (1991), no. 3, 497–525.

[3] William Dwyer, Haynes Miller, and Joseph Neisendorfer, *Fibrewise completion and unstable Adams spectral sequences*, Israel J. Math. **66** (1989), no. 1–3, 160–178.

[4] Irving Dai, Hirofumi Sasahira, and Matthew Stoffregen, *Lattice homology and Seiberg–Witten–Floer spectra*, preprint arXiv:2309.01253 (2023).

[5] Kristen Hendricks and Ciprian Manolescu, *Involutive Heegaard Floer homology*, Duke Math. J. **166** (2017), no. 7, 1211–1299

[6] Akio Kawauchi, *The (1,2)-cable of the figure eight knot is rationally slice*, unpublished (1980).

[7] Akio Kawauchi, *Rational-slice knots via strongly negative-amphicheiral knots*, Commun. Math. Res. **25** (2009), no. 2, 177–192.

[8] Hokuto Konno, Jin Miyazawa, and Masaki Taniguchi, *Involutions, links, and Floer cohomologies*, preprint arXiv:2304.01115 (2023). To appear in Journal of Topology.

[9] Sungkyung Kang, JungHwan Park, and Masaki Taniguchi, *Cables of the figure-eight knot via real Frøyshov invariants*, preprint arXiv:2405.09295 (2024).

[10] Jean Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire*, Inst. Hautes Études Sci. Publ. Math. **75** (1992), 135–244. With an appendix by Michel Zisman.

[11] Katura Miyazaki, *Nonsimple, ribbon fibered knots*, Trans. Amer. Math. Soc. **341** (1994), no. 1, 1–44.

[12] András Némethi, *On the Ozsváth–Szabó invariant of negative definite plumbed 3-manifolds*, Geom. Topol. **9** (2005), 991–1042.

Is the geography of Heegaard Floer homology restricted or is the L-space conjecture false?

Antonio Alfieri

A foliation of a three-manifold Y is a decomposition of Y into a disjoint union of immersed surfaces (leaves) that locally looks like the decomposition of $\mathbb{R}^3$ into horizontal planes. A foliation is called *taut* if there is a simple closed curve transverse to the leaves of the foliation, and intersecting every single leaf at least in one point. While every three-manifold admits a foliation, taut foliations do not always exist. Indeed, taut foliations impose strong topological restrictions on the underlying three-manifold .

In the 1980s Gabai showed that three-manifolds with $b_1 > 0$ always have a taut foliation. Deciding what rational homology spheres ($b_1 = 0$) admit a taut foliation is one of the largest open questions of three-manifold topology today. In recent times a conjecture answering the riddle started consolidating.

Conjecture (Boyer, Gordon and Watson, Juhász: the L-space conjecture). *For an irreducible rational homology sphere Y the following statements are equivalent:*

(1) *Y does not support a taut foliation,*
(2) *Y is an L-space – i.e. the Ozsváth-Szabó Heegaard Floer module $\mathrm{HF}^-(Y)$ is torsion free,*
(3) *$\pi_1(Y)$ has no left-ordering; that is an ordering $<$ such that: $a < b$ iff $ga < gb$.*

The L-space conjecture is known to be true for graph manifolds, some families of knot surgeries, and some branched coverings . The only part of the conjecture which is known to be true in general is the following: if a rational homology sphere Y admits a taut foliation then the Heegaard Floer group $\mathrm{HF}^-(Y)$ has non-trivial torsion. Recently Lin proved the following refinement of this result.

Theorem (Lin). *Suppose Y admits a taut foliation. Then the Heegaard Floer module $\mathrm{HF}^-(Y)$ contains a copy of $\mathbb{F}[U]/U$ as a $\mathbb{F}[U]$-summand.*

This statement puts us at a bifurcation: either the L-space conjecture is false, or the geography of the Heegaard Floer invariant is restricted, that is: not all modules can show up as torsion module of the Heegaard Floer homology group of some rational homology sphere.

In my recent paper with Binns we used Heegaard Floer surgery formulae to prove that a broad class of rational homology spheres satisfy the restriction imposed by Lin's theorem. Thus we presented evidence for the following conjecture.

Conjecture. *The Heegaard Floer module $\mathrm{HF}^-(Y)$ satisfies the Lin geography restriction for all rational homology spheres Y, that is either $\mathrm{HF}^-(Y)$ is torsion free or it contains a copy of $\mathbb{F}[U]/U$ as summand.*

Note that for a large class of graph manifolds the Lin geography restriction was first verified by Bodnar, and Plamenevskaya.

Further results. Toward the end of my talk I suggested that Heegaard Floer homology could satisfy an even stronger geography restriction than the one that would be prescribed by Lin's theorem in accordance with the L-space conjecture. I shall expand on that and make a precise statement here.

Definition. An $\mathbb{F}[U]$-module M *satisfies the strong geography restriction* if it contains a direct summand of the form

$$\mathbb{F}[U]/U^\ell \oplus \mathbb{F}[U]/U^{\ell-1} \oplus \cdots \oplus \mathbb{F}[U]/U^2 \oplus \mathbb{F}[U]/U$$

where $\ell = \min\{\ell \geq 0 : U^\ell \cdot M_{\mathrm{red}} = 0\}$.

The following theorem was proved in

Theorem. *If Y is a rational homology sphere that can be obtained performing either surgery on a knot in S^3, or large surgery on a link in S^3 then the Heegaard Floer module $\mathrm{HF}^-(Y)$ satisfies the strong geography restriction.*

Note that this Theorem holds true in the more general case of null-homologous knots and links in any L-space. Given the above results it is natural to ask the following question.

Question. *Does* $\mathrm{HF}^-(Y)$ *satisfy the strong geography restriction for every rational homology sphere* Y*?*

Of course, given that every rational homology sphere is integral surgery on *some* link, removing the hypothesis that Y is *large* surgery on a link in the statement of the previous Theorem would provide an affirmative answer to the Question. Indeed, we know many examples of links – for instance torus links, and connected sums of Hopf links – whose small surgeries satisfy at least the Lin geography restriction.

The stable chain rule for orthogonal calculus

Connor Malin
(joint work with Niall Taggart)

Goodwillie calculus was introduced by Goodwillie [4] in order to study functors arising in stable and unstable homotopy theory. Orthogonal calculus was introduced by Weiss [5] in order to study covariant functors which arise out of geometry. These two type of functor calculus share many similarities including similar notions of derivatives.

In a series of papers [1, 2, 3], Arone-Ching built the foundation for the study of Goodwillie calculus via operad actions on the sequences of derivatives. In particular, they studied the relation between functor composition and composition of derivatives relative to the Lie operad. This is the notion of a chain rule in functor calculus.

In this talk, we describe how to transport some of these results to orthogonal calculus. The primary difficulty is that one no longer expects an operad to act on the derivatives, but rather a category which is "Koszul dual" to the category of vector spaces and surjections.

We first produce such a theory of categorical Koszul duality, and then describe how it interacts with orthogonal calculus. In particular, by differentiating the Yoneda lemma, we are able to show the the derivatives in orthogonal calculus are acted upon by this category $K(\mathrm{OEpi})$, a result which parallels the appearance of the Lie operad in Goodwillie calculus.

If Koszul duality is sufficiently symmetric monoidal with respect to a product called Day convolution, it is possible to formally deduce both a product rule, for the pointwise smash product of functors, and a chain rule for the derivatives of a composite

$$\mathrm{Vect}_{\mathbb{R}} \to \mathrm{Spec} \to \mathrm{Spec}$$

which necessarily involves both the Goodwillie and the Weiss derivatives.

References

[1] G. Arone, M. Ching, 2011. "Operads and chain rules for the calculus of functors." *Astérisque.* no. 338.

[2] G. Arone, M. Ching, 2015. "A classification of Taylor towers of functors of spaces and spectra." *Adv. Math.* 272. 471–552.

[3] G. Arone, M. Ching, 2016. "Cross-effects and the classification of Taylor towers." *Geom. Topol.* no. 3. 1445–1537.

[4] T. Goodwillie, 2003. "Calculus. III. Taylor series." *Geom. Topol.* 7. 645–711.

[5] M. Weiss, 1995. "Orthogonal calculus." *Trans. Amer. Math. Soc.* 347(1995), no.10, 3743–3796.

Diffeomorphisms of 4-manifolds from graspers

Danica Kosanović

Recently the question of computing smooth mapping class groups of 4-manifolds – groups $\pi_0 \operatorname{Diff}_\partial(M)$ of isotopy classes of diffeomorphisms rel boundary – has attracted a lot of attention. Nevertheless, there is no single compact 4-manifold for which this group is known.

Inspired by *clasper surgery* for classical knots, Watanabe [15] constructed classes in homotopy groups $\pi_n \operatorname{Diff}_\partial(\mathcal{D}^4)$, including a single, so-called *theta class* $\mathsf{wat}(\Theta)$ in degree $n = 0$. He showed that for many $n \geq 1$ these classes are nontrivial, but the question of nontriviality of $\pi_0 \operatorname{Diff}_\partial(\mathcal{D}^4)$ remained open.

Budney and Gabai [3] found an infinite set of linearly independent classes in the abelian groups $\pi_0 \operatorname{Diff}_\partial(\mathcal{D}^3 \times \mathbb{S}^1)$ and $\pi_0 \operatorname{Diff}_\partial(\mathbb{S}^3 \times \mathbb{S}^1)$. Moreover, they gave a general recipe for constructing diffeomorphisms of 4-manifolds, called *barbell implantations* $_w$. Another work of Watanabe [16] followed, also giving infinitely many elements in $\pi_0 \operatorname{Diff}_\partial(\mathcal{D}^3 \times \mathbb{S}^1)$, as well as in $\pi_0 \operatorname{Diff}_\partial(\Sigma \times \mathbb{S}^1)$, where Σ is the Poincaré homology 3-sphere (these classes are variants of $\mathsf{wat}(\Theta)$).

Gay [6] constructed an infinite list of candidate classes in $\pi_0 \operatorname{Diff}_\partial(\mathcal{D}^4)$ called *Montesinos twin twists*, but together with Hartman [7] they showed that this list reduces up to isotopy to at most one nontrivial element $\overline{G}(\nu T_c)$, which is 2-torsion.

Moreover, Gay [6] (using Cerf theory) and later Krannich–Kupers [12] (using results of Quinn and Kreck) give a general procedure for constructing classes in $\pi_0 \operatorname{Diff}_\partial(M)$, which they show exhausts the whole group in the case $M = \mathcal{D}^4$. Similar constructions – which we propose to call *parameterised surgery* – have been used elsewhere, for example in [14, 4, 3].

In [8] we study the following version: for any smooth 4-manifold M and a framed embedded 2-sphere $\nu S \colon \nu\mathbb{S}^2 = \mathbb{S}^2 \times \mathcal{D}^2 \hookrightarrow M$, *parameterised surgery of index one* is the map

$$(1) \quad \mathsf{ps}_{\nu S} \colon \pi_1(\operatorname{Emb}(\nu\mathbb{S}^1, M_{\nu S}); \nu c) \xrightarrow{\delta_{\nu c}} \pi_0 \operatorname{Diff}_\partial(M \setminus \nu S) \xrightarrow{\cup \operatorname{Id}_{\nu S}} \pi_0 \operatorname{Diff}_\partial(M).$$

Here $M_{\nu S} \coloneqq (M \setminus \nu S) \cup_{\partial \nu S} \nu c$ is the surgery on νS, for $\nu c \cong \mathbb{S}^1 \times \mathcal{D}^3$. The map $\delta_{\nu c}$ is given by ambient isotopy extension: lift a loop of framed $\mathbb{S}^1 \hookrightarrow M_{\nu S}$ based at νc to a path of diffeomorphisms of $M_{\nu S}$, and restrict the endpoint diffeomorphism to the complement of νc (which it fixes, by construction). In other words, $\delta_{\nu c}$ is the *circle pushing*. The map $\cup \mathrm{Id}_{\nu S}$ in (1) is the extension by the identity over νS.

If S is unknotted (bounds an embedded 3-ball), then $M_{\nu S} \cong M \# \mathbb{S}^3 \times \mathbb{S}^1$, and we use the notation

$$\mathsf{ps}\colon \pi_1(\mathrm{Emb}(\nu\mathbb{S}^1, M\#\mathbb{S}^3\times\mathbb{S}^1); \nu c) \longrightarrow \pi_0 \,\mathrm{Diff}_\partial(M). \tag{2}$$

In our recent preprint [8] we explore connections between all mentioned constructions of diffeomorphisms, using the maps $\mathsf{ps}_{\nu S}$ and knotted families of circles constructed using *graspers* in our previous work [11, 10]. We explicitly relate graspers to Watanabe's and Budney–Gabai's classes in arbitrary 4-manifolds. All of them are then related to Gay's twists in $\mathbb{S}^4$, depicted in Figure 1.

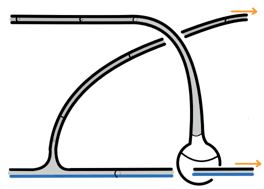


FIGURE 1. The embedded torus $T_c\colon \mathbb{S}^1\times\mathbb{S}^1 \hookrightarrow \mathbb{S}^1\times\mathbb{S}^3$ is the connect-sum of a thin torus containing $c = \mathbb{S}^1\times\{pt\}$ (the blue line), with a meridian sphere for c, along a guiding arc going around $[c] = 1 \in \mathbb{Z} \cong \pi_1(\mathbb{S}^1\times\mathbb{S}^3)$. Roughly speaking, the Gay twist $\overline{G}(\nu T_c)$ does Dehn twists on the curves that guide the movement of c around T_c. To define it precisely use the Dehn twist on $a\times[0,1]$ times the identity on $c\times m_T$ on the normal bundle of the circle bundle $\nu\partial(\nu T_c) \cong a\times c\times m_T\times[0,1]$; then surger c out.

Let us briefly explain graspers of degree one, giving elements

$$\mathfrak{r}(h) \in \pi_1(\mathrm{Emb}(\nu\mathbb{S}^1, X); \nu c),$$

for a 4-manifold X, $h\in\pi_1 X$, and the basepoint $\mathfrak{r}(h)_0 = \mathfrak{r}(h)_1 = c\colon \mathbb{S}^1 \hookrightarrow X$, represented in Figure 2(ii) by the horizontal line union the point at infinity. A grasper can be viewed as the union of two small unknotted meridian spheres m_0, m_1 to c at two points $p_0, p_1 \in c$, connected by a bar. This bar followed by $c|^{-1}_{[p_0,p_1]}$ represents h. The family $\mathfrak{r}(h)_s\colon \mathbb{S}^1 \hookrightarrow X$ for $s\in[0,1]$ takes a piece of c near p_0, drags it along the bar, and then swings it around m_1, before going back. One can easily extend this to framed embeddings. Let us mention that homotopy groups of embeddings of (framed) circles have been studied recently in [1, 5, 13, 9].

In particular, for $M = \mathbb{S}^4$ we show that *almost all* existing constructions of diffeomorphisms reduce to a single 2-torsion class, the parameterised surgery map from (2) applied to the class $\mathfrak{r}([c])$.

Theorem ([8])**.** *For $M = \mathbb{S}^4$ the image of the parameterized surgery map* ps*, as well as any barbell implantation, consists of at most of one class, at most 2-torsion:*

$$\mathsf{ps}\circ\mathfrak{r}([c]) = \mathsf{wat}(\Theta) = \mathsf{impl}({}_{yx}) = \overline{G}(\nu T_c),$$

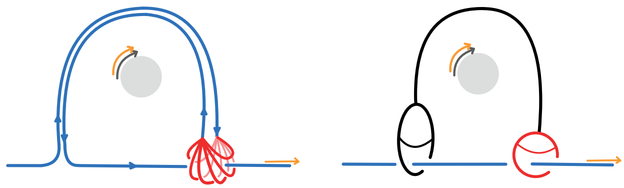


FIGURE 2. (i) The grasper family $\mathfrak{t}(h)$ is obtained by connect-summing the tip of the long finger into each of the arcs foliating a meridian sphere to c. (ii) The schematic depiction of $\mathfrak{t}(h)$.

The proof first expresses the Watanabe class $\mathsf{wat}(\Theta)$ in terms of ps, as in Figure 3(i) (this is similar to the work done by Botvinnik and Watanabe in [2]), and then shows that this family is isotopic to $\mathfrak{t}([c])$. And similarly for the Budney–Gabai barbell implantation $\mathsf{impl}(_w)$ for the bar word $w = yx$ from Figure 3(ii).

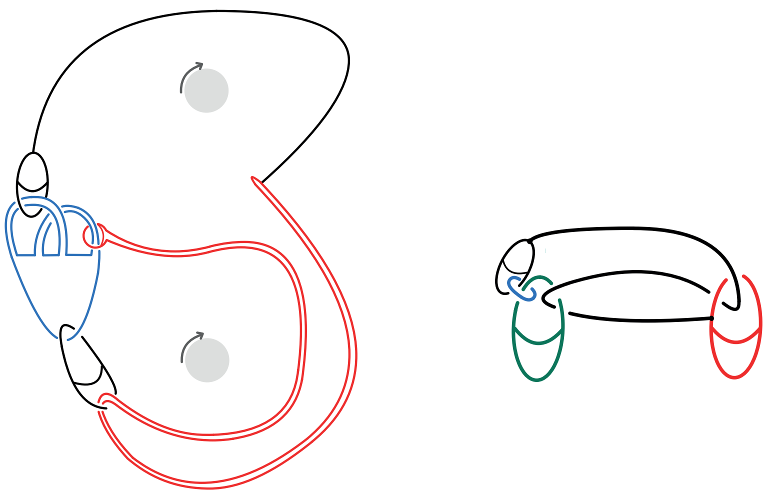


FIGURE 3. Parameterised surgery on depicted families of circles gives: (i) Watanabe's $\mathsf{wat}(\Theta)$ and (ii) Budney–Gabai's $\mathsf{impl}(_{yx})$.

It remains open if the class from the theorem is in fact trivial.

REFERENCES

[1] G. Arone, M. Szymik, 2020. "Spaces of Knotted Circles and Exotic Smooth Structures." *Canadian Journal of Mathematics*, 1–23. `https://doi.org/10.4153/S0008414X2000067X`.

[2] B. Botvinnik, T. Watanabe, 2023. "Families of Diffeomorphisms and Concordances Detected by Trivalent Graphs." *J. Topol.* 16 (1): 207–33. `https://doi.org/10.1112/topo.12283`.

[3] R. Budney, D. Gabai, 2019. "Knotted 3-Balls in S^4." `https://arxiv.org/abs/1912.09029`.

[4] S. E. Cappell, J. L. Shaneson, 1971. "On Four Dimensional Surgery and Applications." *Comment. Math. Helv.* 46: 500–528. `https://doi.org/10.1007/BF02566862`.

[5] D. Gabai, 2021. "Self-Referential Discs and the Light Bulb Lemma." *Comment. Math. Helv.* 96 (3): 483–513. `https://doi.org/10.4171/cmh/518`.
[6] D. T. Gay, 2021. "Diffeomorphisms of the 4-Sphere, Cerf Theory and Montesinos Twins." `https://arxiv.org/abs/2102.12890`.
[7] D. T. Gay, D. Hartman, 2022. "Relations Amongst Twists Along Montesinos Twins in the 4-Sphere." `https://arxiv.org/abs/2206.02265`.
[8] D. Kosanović, 2024a. "Diffeomorphisms of 4-Manifolds from Degree One Graspers." `https://arxiv.org/abs/2405.05822`.
[9] ———, 2024b. "On Fundamental Groups of Spaces of (Framed) Embeddings of the Circle in 4-Manifolds." `https://arxiv.org/abs/2407.06923`.
[10] ———, 2024c. "On Homotopy Groups of Spaces of Embeddings of an Arc or a Circle: The Dax Invariant." *Trans. Amer. Math. Soc.* 377 (2): 775–805. `https://doi.org/10.1090/tran/8805`.
[11] D. Kosanović, P. Teichner, 2021. "A Space Level Light Bulb Theorem in All Dimensions." arXiv. `https://arxiv.org/abs/2105.13032`. To appear in *Comment. Math. Helv.*
[12] M. Krannich, A. Kupers, 2022. "On Torelli Groups and Dehn Twists of Smooth 4-Manifolds." `https://arxiv.org/abs/2105.08904`.
[13] S. Moriya, 2024. "Models for Knot Spaces and Atiyah Duality." *Algebraic & Geometric Topology* 24 (1): 183–250. `https://doi.org/10.2140/agt.2024.24.183`.
[14] C. T. C. Wall, 1964. "Diffeomorphisms of 4-Manifolds." *J. London Math. Soc.* 39: 131–40. `https://doi.org/10.1112/jlms/s1-39.1.131`.
[15] T. Watanabe, 2018. "Some Exotic Nontrivial Elements of the Rational Homotopy Groups of Diff(S^4)." `https://arxiv.org/abs/1812.02448`.
[16] ———, 2020. "Theta-Graph and Diffeomorphisms of Some 4-Manifolds." `https://arxiv.org/abs/2005.09545`.

Scissors congruence and a conjecture of Goncharov

Inna Zakharevich

Hilbert's Third Problem [6] asks the following question: do there exist two polyhedra with the same volume which cannot be decomposed into finitely many piecewise-congruent pieces? More precisely, given two polyhedra P and Q, with the same volume, is it possible to write $P = \bigcup_{i=1}^n P_i$, $Q = \bigcup_{i=1}^n Q_i$, with $P_i \cong Q_i$ and $\text{measure}(P_i \cap P_j) = \text{measure}(Q_i \cap Q_j) = 0$? Two polyhedra for which such a relationship holds are called *scissors congruent*. This question was answered in 1901 by Dehn, who showed[1] that there exists a function

$$D : \left\{ \begin{array}{c} \text{scissors} \\ \text{congruence} \\ \text{classes} \end{array} \right\} \to \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$$

which is nonzero on any regular tetrahedron and zero on any cube, thus proving that volume is not sufficient to distinguish scissors congruence classes. This function has since been called the *Dehn invariant*. In 1965 Sydler proved that there are no other independent invariants: volume and the Dehn invariant are sufficient to distinguish scissors congruence classes. (See references for a more detailed discussion and history.)

[1]Given that tensor products were only defined in 1938, he did not actually show this; however, in modern terms, his proof implies this statement directly.

However, the question of how to distinguish scissors congruence classes in most other dimensions, and other geometries, remains open. In each new odd dimension, a new Dehn invariant appears, so that in the 5-dimensional case there are three known invariants (volume and two Dehn invariants), in the 7-dimensional case there are four (volume and three Dehn invariants), and so on. To fully understand and analyze these invariants, we switch to a group-theoretic analysis: instead of considering the set of scissors congruence classes, we consider the *scissors congruence group.*

Let X^n be an n-dimensional geometry (such as Euclidean, spherical, or hyberbolic). Then we define

$$\mathcal{P}(X^n) = \begin{matrix} \text{free ab. gp} \\ \text{on polytopes in } X^n \end{matrix} \Big/ \begin{matrix} [P]=[Q] & \text{if} P \cong Q \\ [P]+[Q]=[P\cup Q] & \text{if measure}(P\cap Q)=0 \end{matrix}.$$

In terms of these scissors congruence groups, we can now write Dehn invariants as homomorphisms

$$D_k : \mathcal{P}(X^n) \to \mathcal{P}(X^k) \otimes \tilde{\mathcal{P}}(S^{n-k-1}),$$

where $\tilde{\mathcal{P}}(S^{n-k-1})$ is a certain quotient of $\mathcal{P}(S^{n-k-1})$. (Whenever m is even, $\tilde{\mathcal{P}}(S^m) = 0$, which is why new Dehn invariants arise only at odd dimensions). The question of whether the volume and the Dehn invariants separate scissors congruence classes can now be rephrased in the following manner: is the volume homomorphism injective when restricted to $\bigcap_{k=1}^{n-1} \ker D_k$? Goncharov [5] conjectured an interesting manner of approaching this question: he showed that the Dehn invariants can be assembled into a chain complex $\mathcal{P}_*(X^n)$ consisting of scissors congruence groups, and conjectured that its homology is isomorphic to the filtered quotients of the γ-filtration on $K_*(\mathbf{C})$. Under this isomorphism the volume homomorphism becomes the Borel regulator, and the conjecture that it is injective becomes the question of whether the Borel regulator is injective (which is known in many cases, although not in the case of $\mathbf{C}$). (See Theorem 1.6 of Goncharov.)

In joint work with Jonathan Campbell [4], based off of work of Cathelineau [1, 2, 3], we construct a topological model of Goncharov's chain complex. In order to do this, we construct a space, $F_\bullet^{X^n}$, for any geometry X, which is acted upon by the isometry group of X. The space $F_\bullet^{X^n}$ has nonzero homology in exactly one degree $(n-1)$ and $H_0(\mathrm{Isom}(X^n); H_{n-1}(F_\bullet^{X^n})^\sigma) \cong \mathcal{P}(X^n)$. (Here the $\cdot^\sigma$ denotes that the action of the isometry group must be twisted by the determinant.) The Dehn invariant can be modeled via an equivariant map

$$D_k : F_\bullet^{X^n} \to \bigvee_{X^k \subseteq X^n} F_\bullet^{X^k} \,\tilde{\star}\, F_\bullet^{S^{n-k-1}}.$$

(Here, $\tilde{\star}$ is the reduced join of spaces.) After taking coinvariants via the isometry group of X^n, this models the Dehn invariant in the first nonzero homology degree. Goncharov's chain complex can be modeled as the total homotopy cofiber of a certain cube constructed via such Dehn invariants. As homotopy coinvariants and total homotopy cofibers commute, we are able to compute the homotopy type of the total homotopy cofiber of this cube *prior* to taking the homotopy cofiber and it turns out to be remarkably simple: a cube. This implies, in particular, that there is a spectral sequence whose E^1 page contains $\mathcal{P}_*(X^n)$ in the lowest nonzero portion, and which converges to the homology of the isometry group of X^n. This gives an alternate form of Goncharov's conjecture and constructs (via the edge homomorphism of the spectral sequence) a version of Goncharov's conjectured map: a homomorphism

$$H_{m+\lfloor \frac{n-1}{2} \rfloor}(\mathrm{Isom}(X); \mathbb{Q}^\sigma) \to H_m \mathcal{P}_*(X^n) \otimes \mathbb{Q}.$$

(Here, again, the $\cdot^\sigma$ indicates a twisting by the determinant.) In the case when $m = n - 1$ the homology on the right is the interesction of the kernels of Dehn invariants, and there is a Borel regulator which is defined on the group on the left (which in the case when X is hyperbolic can be shown to agree with the volume map on polytopes). If the horizontal morphism is an isomorphism (as Goncharov conjectures) and the Borel regulator is injective, this would imply the generalized Hilbert's third problem holds: that the volume and the Dehn invariant do, indeed, separate scissors congruence classes.

References

[1] J.-L. Cathelineau, *Scissors congruences and the bar and cobar constructions*, J. Pure Appl. Algebra, **181** (2003), 141–179.

[2] J.-L. Cathelineau, *Projective configurations, homology of orthogonal groups, and Milnor K-theory*, Duke Math. J., **121** (2004), 343–387.

[3] J.-L. Cathelineau, *Homology stability for orthogonal groups over algebraically closed fields*, Ann. Sci. Éc. Norm. Supér. (4), **40** (2007), 487–517.

[4] J. A. Campbell, I. Zakharevich, *Hilbert's third problem and a conjecture of Goncharov*, Adv. Math., **451** (2024), 109757.

[5] A. Goncharov, *Volumes of hyperbolic manifolds and mixed Tate motives*, J. Amer. Math. Soc., **12** (1999), 569–618.

[6] C. H. Sah, *Hilbert's third problem: scissors congruence*, Res. Notes Math., **33**, Pitman, Boston, Mass., (1979).

Pontryagin–Weiss classes

Manuel Krannich

(joint work with Alexander Kupers)

Pontryagin classes are certain characteristic classes of real vector bundles $\pi\colon E \to X$. There is one for every nonnegative integer $i \geq 0$, defined as the $2i$th Chern class of the complexification of the vector bundle up to a sign:

$$p_i(\pi) \coloneqq (-1)^i \cdot c_{2i}(\pi) \in \mathrm{H}^{4i}(X; \mathbf{Z}).$$

Two of their most important properties are:

(1) (Stability) $p_i(\pi) = p_i(\pi \oplus \mathbf{R})$.
(2) (Vanishing) $p_i(\pi) = 0$ for $i > \frac{\mathrm{rank}(\pi)}{2}$

One of the historically most significant applications of Pontryagin classes is that, when applied to tangent bundles, they give rise to invariants of smooth manifolds that can be used to distinguish smooth structures on a given topological manifold. This fundamentally uses that Pontryagin classes are *integral* characteristic classes: when considered as rational characteristic classes, their values on tangent bundles of manifolds M are independent of the smooth structure. The reason for this is a combination of two facts: firstly, by work of Novikov, Kirby–Siebenmann, Sullivan, and others, there is a unique extension of the rational Pontryagin classes $p_i(\pi) \in \mathrm{H}^{4i}(X;\mathbf{Q})$ to stable characteristic classes of $\mathbf{R}^d$*-bundles*—fibre bundles whose fibres are homeomorphic to $\mathbf{R}^d$ (without a fibrewise linear structure). Secondly, by results of Milnor, Kister, and Mazur, every topological d-manifold M has a *topological tangent bundle* which is an $\mathbf{R}^d$-bundle that agrees with the underlying $\mathbf{R}^d$-bundle of the usual tangent bundle if M comes with a smooth structure.

By construction, these more general rational Pontryagin classes $p_i(\pi) \in \mathrm{H}^{4i}(X;\mathbf{Q})$ that are defined for $\mathbf{R}^d$-bundles still satisfy the above stability property, but it was unclear for a long time whether they also satisfy the vanishing property (c.f. [Sul05, p. 210]) until Weiss [Wei21] proved—to the surprise of many—that they do not:

Theorem (Weiss)**.** *There are pairs of integers (i,d) with $i > \frac{d}{2}$ such that there exists an $\mathbf{R}^d$-bundle over a sphere $\pi\colon E \to S^{4i}$ with $p_i(\pi) \neq 0$.*

In my talk I presented the following strengthening of Weiss' result from [KK24]:

Theorem (Krannich–Kupers)**.** *For all pairs (i,d) with $d \geq 6$ and $i \geq 0$, there exists an $\mathbf{R}^d$-bundle over a sphere $\pi\colon E \to S^{4i}$ with $p_i(\pi) \neq 0$.*

Remark.

(1) Because of the stability property, the statement of the theorem for all $d \geq 6$ follows from the case $d = 6$ by taking products with trivial bundles.
(2) Closely related to the theorem, Galatius and Randal-Williams [GRW23] proved by a different approach that there are no universal relations between products of Pontryagin classes for $\mathbf{R}^d$-bundles when $d \geq 6$.

It can be shown to be equivalent to either of the following statements:

(1) The stabilisation map induced by taking products with Euclidean spaces
$$\mathrm{Top}(d) \longrightarrow \mathrm{Top} \coloneqq \mathrm{colim}_d \mathrm{Top}(d)$$
admits a rational section as long as $d \geq 6$. Here $\mathrm{Top}(d)$ is the topological group of homeomorphisms of $\mathbf{R}^d$ in the compact-open topology.
(2) Certain characteristic classes for smooth bundles with closed d-discs as fibres and a trivialisation of the boundary bundle which are defined in terms of Pontryagin classes and the Euler class (see [KRW21, 8.2.1]),
$$\mathrm{BDiff}_\partial(D^d) \longrightarrow \textstyle\prod_{i \geq \lfloor \frac{d}{2} \rfloor} K(4i-d-1,\mathbf{Q}),$$
can be detected for all $d \geq 6$ by bundles over spheres, i.e. the above map is surjective on rational homotopy groups.

Our strategy of proof is closely inspired by Weiss' original strategy, which relies on embedding calculus in the sense of Goodwillie–Weiss [Wei99, GW99] and Galatius–Randal-Williams' work on stable moduli spaces of manifolds [GRW17]. The most important additional ingredient in enhancing his strategy in order to prove our theorem is the following result on the space of derived automorphisms of the rationalised little d-discs operad $E_d^{\mathbf{Q}}$:

Theorem (Krannich–Kupers)**.** *For all $d \geq 2$, the double stabilisation map*

$$((-) \otimes \mathrm{id}_{E_2}^{\mathbf{Q}})\colon \mathrm{BAut}(E_{d-2}^{\mathbf{Q}}) \longrightarrow \mathrm{BAut}(E_d^{\mathbf{Q}})$$

induced by taking Boardman–Vogt tensor products with $E_2^{\mathbf{Q}}$, is nullhomotopic.

Remark. The above theorem is optimal in the following sense:

(1) The single stabilisation $\mathrm{BAut}(E_{d-1}^{\mathbf{Q}}) \longrightarrow \mathrm{BAut}(E_{d-2}^{\mathbf{Q}})$ is not null in general: it can be shown to be nontrivial on $\mathrm{H}^{4n}(-;\mathbf{Q})$ for $d = 2n+1$.

(2) The non-rationalised version of the double stabilisation map

$$\mathrm{BAut}(E_{d-2}) \to \mathrm{BAut}(E_{d-2})$$

is not null either: it factors the composition $\mathrm{BO}(d-2) \subset \mathrm{BO}(d) \to \mathrm{BG}(d) = \mathrm{BhAut}(S^{d-1})$ induced by block-inclusion of matrices and the usual $O(d)$-action on S^{d-1}. This composition induces the unstable J-homomorphism on homotopy groups, which is highly nontrivial.

References

[GRW17] S. Galatius and O. Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds. II*, Ann. of Math. (2) **186** (2017), no. 1, 127–204.

[GRW23] ———, *Algebraic independence of topological Pontryagin classes*, J. Reine Angew. Math. **802** (2023), 287–305.

[GW99] T. G. Goodwillie and M. Weiss, *Embeddings from the point of view of immersion theory. II*, Geom. Topol. **3** (1999), 103–118.

[KK24] M. Krannich and A. Kupers, *Pontryagin–Weiss classes and a rational decomposition of spaces of homeomorphisms*, 2024, in preparation.

[KRW21] M. Krannich and O. Randal-Williams, *Diffeomorphisms of discs and the second Weiss derivative of* BTop(−), 2021, arXiv:2109.03500.

[Sul05] D. P. Sullivan, *Geometric topology: localization, periodicity and Galois symmetry*, K-Monographs in Mathematics, vol. 8, Springer, Dordrecht, 2005, The 1970 MIT notes.

[Wei99] M. S. Weiss, *Embeddings from the point of view of immersion theory. I*, Geom. Topol. **3** (1999), 67–101.

[Wei21] M. S. Weiss, *Rational Pontryagin classes of Euclidean fiber bundles*, Geom. Topol. **25** (2021), no. 7, 3351–3424.

Infiniteness of 4-dimensional diffeomorphism groups

Hokuto Konno

Given a smooth manifold X, a basic question on its mapping class group $\pi_0(\mathrm{Diff}(X))$ is that whether this is finitely generated. If X is closed and simply-connected, and $\dim X \geq 5$, a classical result due to Sullivan [12] shows that $\pi_0(\mathrm{Diff}(X))$ is finitely generated (more strongly, finitely presented). In dimension ≤ 3, even dropping simple-connectivity, the mapping class groups are known to be finitely generated (again, more strongly, finitely presented). The following result proves the 4-dimensional analog of such finite generation fails:

Theorem (K. [8], independently also by Baraglia [1])**.** *There exist simply-connected closed smooth 4-manifolds X such that $\pi_0(\mathrm{Diff}(X))$ are not finitely generated.*

Example. Let $E(n)$ denote the simply-connected elliptic surface without multiple fiber of degree n. For example, $E(2) = K3$. Set $X = E(n)\#S^2 \times S^2$. Then $\pi_0(\mathrm{Diff}(X))$ is not finitely generated.

Remark. Quinn's result [10], together with a recent correction [5], implies that, for a simply-connected closed topological 4-manifold X, $\pi_0(\mathrm{Homeo}(X))$ is finitely generated. Thus the infinite generation exhibited in the previous theorem is a special phenomenon in the 4-dimensional smooth category.

Remark. If one drops simple-connectivity, it is known that the mapping class group can be infinitely generated in all dimensions ≥ 4. For example, Hatcher [6] proved that the tori T^n with $n \geq 5$ have infinitely generated mapping class groups. In dimension 4, recent results by Budney–Gabai [3] and Watanabe [13] showed that, for some non-simple-connected 4-manifolds (such as $S^1 \times S^3$), the mapping class groups are infinitely generated.

Remark. Ruberman [11] proved that the Torelli group of some simply-connected 4-manifolds can be infinitely generated. Note that the theorem does not follow from this fact, since an infinitely generated group might be embedded in a finitely generated group (such as the commutator subgroup of the rank 2 non-abelian free group).

The above Theorem is a consequence of the following more general result:

Theorem ([K. [8])**.** *For each $k > 0$, there exist simply-connected closed smooth 4-manifolds X such that $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ are not finitely generated.*

It is clear that $k = 1$ in above theorem recovers the original result.

Example. Given $k > 0$, set $X = E(n)\#_k S^2 \times S^2$. Then $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ are not finitely generated. More strongly, we proved in [8] that $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ contains a $(\mathbb{Z}/2)^\infty$-summand, which becomes trivial under the natural comparison map

$$H_k(B\mathrm{Diff}(X);\mathbb{Z}) \to H_k(B\mathrm{Homeo}(X);\mathbb{Z})$$

and the stabilization map for the connected sum with $S^2 \times S^2$.

Remark. It is interesting to compare this generalization with a result by Bustamante–Krannich–Kupers [4]. They proved that, for a closed smooth manifold X of even dimension ≥ 6 and with finite fundamental group, the homologies $H_k(B\mathrm{Diff}(X);\mathbb{Z})$ are finitely generated for all k.

This theorem is proven by introducing infinite series of characteristic classes of smooth families of 4-manifolds using Seiberg–Witten theory, which are variants of a construction given by the author [7] and Lin and the author [8]. The proof of the non-triviality of these characteristic classes uses a gluing formula for the families Seiberg–Witten invariant by Baraglia and the author [2].

References

[1] D. Baraglia, *On the mapping class groups of simply-connected smooth 4-manifolds*, arXiv:2310.18819.

[2] D. Baraglia, H. Konno, *A gluing formula for families Seiberg-Witten invariants*, Geom. Topol. **24** (2020), 1381–1456.

[3] R. Budney, D. Gabai, *Knotted 3-balls in* S^4, arXiv:1912.09029

[4] M. Bustamante, M. Krannich, A. Kupers, *Finiteness properties of automorphism spaces of manifolds with finite fundamental group*, Math. Ann. **388** (2024), 3321–3371.

[5] D. Gabai, D. Gay, D. Hartman, V. Krushkal and M. Powell, *Pseudo-isotopies of simply connected 4-manifolds*, arXiv:2311.11196.

[6] A. Hatcher, *Concordance spaces, higher simple-homotopy theory, and applications*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1, pp. 3–21.

[7] H. Konno, *Characteristic classes via 4-dimensional gauge theory*, Geom. Topol. **25** (2021), 711–773.

[8] H. Konno, J. Lin, *Homological instability for moduli spaces of smooth 4-manifolds*, arXiv:2211.03043.

[9] H. Konno, *The homology of moduli spaces of 4-manifolds may be infinitely generated*, arXiv:2310.18695.

[10] F. Quinn, *Isotopy of* 4*-manifolds*, J. Differential Geom. **24**, 343–372.

[11] D. Ruberman, *A polynomial invariant of diffeomorphisms of 4-manifolds*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), 473–488.

[12] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269–331.

[13] T. Watanabe, *Theta-graph and diffeomorphisms of some 4-manifolds*, arXiv:2005.09545.

The stable homology of Hurwitz spaces and Cohen–Lenstra moments

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(joint work with Aaron Landesman)

The Cohen–Lenstra heuristics [1] predict the distribution of the odd part of class groups of quadratic fields, and have been one of the driving conjectures in arithmetic statistics over the last four decades.

One of the primary approaches to determining this distribution is to compute its moments, by which we mean the average number of surjections from the class group of an extension to an odd order abelian group H, K ranges over imaginary quadratic fields or real quadratic fields.

The moment for $H = \mathbb{Z}/3\mathbb{Z}$ was computed over $\mathbb{Q}$ by Davenport and Heilbronn [2] and the analog over function fields has been verified by Datskovsky and Wright in [3]. In the function field case over $\mathbb{F}_q(t)$, work of Ellenberg–Venkatesh–Westerland (EVW) proved a weak version of the Cohen–Lenstra heuristics in [5], where one takes a limit as q approaches ∞. However, despite this, no other moments over any function field have been computed or even proven to exist.

The approach of EVW is to use the Grothendieck–Lefschetz trace formula and comparison results in étale cohomology to relate the problem to understanding the homology of certain Hurwitz spaces. Given a group G, and a union of conjugacy classes c, the Hurwitz space $\mathrm{Hur}_n^{G,c}$ is defined to be the homotopy quotient of the set c^n by the action of the braid group B_n determined by the fact that the twist acts on c^2 by sending $x, y \mapsto xyx^{-1}, x$. For each element $x \in c$, there is a stabilization map $\mathrm{Hur}_n^{G,c} \xrightarrow{\alpha_x} \mathrm{Hur}_{n+1}^{G,c}$, coming from the map $c^n \to c^{n+1}$ obtained by concatenating x. EVW showed that for certain G, the homology of the spaces $\mathrm{Hur}_n^{G,c}$ stabilize as n approaches ∞. Using this in the case $G = H \rtimes \mathbb{Z}/2\mathbb{Z}$ where the action is the negation action, and $c \subset G$ is the conjugacy class of order two elements, they prove their limiting version of the Cohen–Lenstra heuristics. From now on, we assume that G, c is of this form.

We compute the stable homology of the Hurwitz spaces for these G, allowing us to compute the H moments of the Cohen–Lenstra distribution when the finite field is large enough. In particular, we prove:

Theorem. *Suppose H is a finite abelian group of odd order. Let q be an odd prime with $\gcd(|H|, q(q-1)) = 1$. There is an integer C, depending only on H, so that if $q > C$ and $i \in \{0, 1\}$,*

$$\lim_{\substack{n\to\infty \\ n\equiv i \bmod 2}} \frac{\sum_{K\in\mathcal{MH}_{n,q}} |\mathrm{Surj}(\mathrm{Cl}(\mathscr{O}_K), H)|}{\sum_{K\in\mathcal{MH}_{n,q}} 1} = \begin{cases} 1 & \text{if } i = 1 \\ \frac{1}{|H|} & \text{if } i = 0. \end{cases} \tag{1}$$

The following is our computation of stable homology:

Theorem. *For G and c as above, there are constants I and J depending only on G so that for $n > iI + J$ and any connected component $Z \subset \mathrm{Hur}_n^{G,c}$, the map $H_i(Z; \mathbb{Q}) \to H_i(BB_n; \mathbb{Q})$ is an isomorphism.*

Here BB_n is the classifying space of the braid group, or the space of configurations of n unordered points in $\mathbb{C}$, and its rational homology for $n \geq 2$ is $\mathbb{Q}$ in degrees $0, 1$, and vanishes in all other degrees.

EVW also attempted to compute the stable homology of these Hurwitz spaces in [4], but a serious error was found by Oscar Randal-Williams, who also wrote an article explaining EVW's work [6].

In my talk I outlined our computation of stable homology above in the case $G = S_3$ and c is the conjugacy class of transpositions.

References

[1] Cohen, H. and Lenstra, H., *Heuristics on class groups of number fields*, Number Theory Noordwijkerhout 1983 (1984), 33–62.

[2] Davenport, H. and Heilbronn, H., *On the Density of Discriminants of Cubic Fields. II*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences (1971), 405–420.

[3] Datskovsky, Boris and Wright, David J., *Density of discriminants of cubic extensions*, J. Reine Angew. Math., 116–138.

[4] Ellenberg, Jordan S and Venkatesh, Akshay and Westerland, Craig, *Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields, II*, arXiv preprint arXiv:1212.0923v2, 2012

[5] Ellenberg, Jordan S and Venkatesh, Akshay and Westerland, Craig, *Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields*, Ann. of Math. (2), 2016 729–786

[6] Randal-Williams, Oscar, *Homology of Hurwitz spaces and the Cohen-Lenstra heuristic for function fields [after Ellenberg, Venkatesh, and Westerland]*, Astérisque, 2020 469–497

Motivic Milnor formulas

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(joint work with Weinan Lin)

1. Background

Morel and Voevodsky's work [5] laid the foundation of motivic stable homotopy theory. Roughly, motivic stable homotopy theory is the stable homotopy theory for schemes. On the one hand, one can use homotopy theoretical tools to study algebro-geometric objects; on the other hand, one can recover classical results from motivic information. Many classical stable homotopy theory constructions and results have their motivic analogs. Work in Dugger–Isaksen [1] studies the motivic version of Adams spectral sequence, which computes the motivic stable homotopy groups of the sphere.

Classically, one can build an algorithm and use computer programs to compute the E_2-page of the classical mod 2 Adams spectral sequence, $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$. Milnor in [4] discussed the Steenrod algebra and its dual, defined the Milnor basis for the Steenrod algebra, and gave the product formula for the Milnor basis. The Adams E_2-page is the cohomology of Steenrod algebra, and with Milnor's product formula, the computation is purely algebraic.

A natural question to ask is if one can do the same in the motivic case. The first step is to compute the motivic product formula for the motivic Milnor basis. In previous work of Kylling [3], the author worked out the product formula in some specific cases. It turns out that the motivic computation is much more complex, for several reasons.

(1) The motivic cohomology of a point can be complicated, and contains extra gradings.
(2) The motivic dual Steenrod algebra is a Hopf algebroid, therefore there is an asymmetry in the action of the unit map.
(3) The motivic dual Steenrod algebra is not free polynomial.

2. Main results

In this work, we compute explicit formulas for taking products using the motivic Milnor basis. For simplification, we work with base field $\mathbb{R}$.

2.1. **The Hopf algebroid structure.** In this subsection, we recall some results from [6, 7].

The motivic Steenrod algebra is

$$A_{**} = \mathbb{M}_2[\xi_j, \tau_i, j \geq 1, i \geq 0]/\tau_i^2 = \rho\tau_{i+1} + \tau\xi_{i+1} + \rho\tau_0\xi_{i+1},$$

where $\mathbb{M}_2 = \mathbb{F}_2[\tau, \rho]$ denotes the motivic cohomology of a point. The dual Steenrod algebra forms a Hopf algebroid; in particular, it has
(1) the right unit action on τ: $\eta_R(\tau) = \tau + \rho\tau_0$;
(2) the coproduct map:

$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \tau_i,$$

$$\psi(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i;$$

(3) and the conjugation map: χ.

Here we adopt the convention $\xi_0 = 1$.

Definition 1. Let Seq denote the set of all sequences $S = (s_0, s_1, s_2, ...)$ with only finitely many nonzero entries, and let Seq_1 be the subset of all such sequences with entries either 0 or 1.

For sequences $E \in Seq_1$ and $R \in Seq$, we use $Q(E), P(R)$ to denote the duals of

$$\tau(E) = \tau_0^{e_0}\tau_1^{e_1}\cdots,\ \xi(R) = \xi_1^{r_1}\xi_2^{r_2}\cdots.$$

All such $Q(E)P(R)$ form a basis for the motivic Steenrod algebra; we refer to this basis as the motivic Milnor basis.

2.2. **Formula for Milnor basis.**

We work out the formula in three steps.

(1) Change to the conjugate generators
(2) Compute a simplification formula for monomials that contain powers of τ_i, and use it to rewrite the coproduct formula.
(3) Use the standard method and the formula in the previous step.

2.2.1. *Step 1.* We switch to conjugate generators $\chi\tau_i$ and $\chi\xi_i$ without changing notations. Using the new convention and the right unit formula, the relation in the dual Steenrod algebra now writes

$$\tau_i^2 = \tau\xi_{i+1} + \rho\tau_{i+1}.$$

And the coproduct formulas write:

$$\psi(\tau_k) = 1 \otimes \tau_k + \sum_{i=0}^{k} \tau_i \otimes \xi_{k-i}^{2^i},$$

$$\psi(\xi_k) = \sum_{i=0}^{k} \xi_i \otimes \xi_{k-i}^{2^i}.$$

2.2.2. *Step 2.* We compute a simplification formula for monomials that contain powers of τ_i.

Notations

Let R and S be two sequences. We introduce the following notation.

(1) $|R| = \Sigma r_i 2^i$.
(2) $\ell(R) = \Sigma r_i$.
(3)
$$c(R,S) = \prod_{n\geq 0} \binom{\lfloor \sum_{i=0}^{n-1} 2^{i-n}(r_i - s_i)\rfloor}{s_n}.$$

Here we take $\binom{m}{n} = 0$ if $m < n$.

Theorem 1. For $R \in Seq$, we have

$$\tau(R) = \sum_{\substack{S\in Seq, E\in Seq_1 \\ |E|+|S|=|R|}} c(R,S)\tau^{\ell(S)}\rho^{\ell(R)-\ell(E)-2l(S)}\tau(E)\xi(S). \tag{1}$$

Proof. It is not hard to observe that if $\tau(E)\xi(R)$ appears in the summand, then we must have

$$|R| + |E| = |S|.$$

And the powers of τ and ρ can be determined by comparing degrees. Therefore, tt remains to determine the coefficient of $\tau(E)\xi(S)$.

When we start from a single term $\tau(R)$, we can create a binary tree whose root node is labeled by $\tau(R)$, and a node labeled by $\tau(E)\xi(S)$ has two child nodes labeled by

$$\tau(e_0, \cdots, e_{n-1}, e_n - 2, e_{n+1} + 1, e_{n+2}, \dots) \cdot \xi(S)$$

and

$$\tau(e_0, \cdots, e_{n-1}, e_n - 2, e_{n+1}, \dots) \cdot \xi(s_0, \dots, s_n, s_{n+1} + 1, s_{n+2}, \dots)$$

if E contains some $e_n > 1$. Otherwise $\tau(E)\xi(S)$ is a leaf node. The binary tree encodes the process of rewriting, and we always rewrite from the the left-most entry $e_i \geq 2$.

It is not difficult to see that the coefficient of $\tau(E)\xi(S)$ in (1) should be the number of occurrences of $\tau(E)\xi(S)$ in the leaf nodes.

We compute by reverse induction. Suppose n is the index of the rightmost entry in R. Then a node $\tau(E)\xi(S)$ is a descendant of

$$\tau(e_0, \dots, e_{n-2}, \sum_{i=0}^{n-2} 2^{i-n+1}(r_i - e_i - s_i) + r_{n-1} - s_{n-1}, \dots)\xi(s_0, s_1, \dots, s_{n-1}, 0, \dots),$$

where $e_i \in \{0, 1\}$ for $i \leq n-2$ by the rewriting order. Therefore, we have

$$\sum_{i=0}^{n-2} 2^{i-n+1}(r_i - e_i - s_i) + r_{n-1} - s_{n-1} = \lfloor \sum_{i=0}^{n-1} 2^{i-n+1}(r_i - s_i)\rfloor.$$

Such a node has

$$\binom{\lfloor(\lfloor\sum_{i=0}^{n-1} 2^{i-n+1}(r_i - s_i)\rfloor/2\rfloor}{s_n} = \binom{\lfloor\sum_{i=0}^{n-1} 2^{i-n}(r_i - s_i)\rfloor}{s_n}$$

descendants that are labeled by $\tau(E)\xi(S)$.

By reverse induction, we can conclude the coefficient equals $c(R, S)$ in (1). □

Recall the following notations from Milnor's work, with modification. Let X be a matrix

$$\begin{matrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ x_{1,0} & x_{1,1} & x_{1,2} & \\ \vdots & & & \end{matrix},$$

and let R, R' be in Seq. Define $T(X)_r = \sum x_{r-i,i}, R(X)_r = \sum 2^i x_{i,r}, S(X)_r = \sum x_{r,i}$, $b(X) = \Pi_i(x_{i,0}, x_{i-1,1}, \ldots, x_{0,i})$, and $b(R, R') = \Pi_i \binom{r_i}{r'_i}$.

Using the formula in Theorem 1, we deduce the following product formula.

Corollary 1. For any $E \in Seq_1, T \in Seq$, we have

$$\psi(\tau(E)\xi(T)) = \sum_{\substack{T(Y)=T \\ D'+D''=E, T(X)=D'' \\ R' \in Seq, E' \in Seq_1 \\ |R'|+|E'|=|S(X)|}} \begin{matrix} b(E, D')b(Y)b(X)c(S(X), R') \\ \tau^{\ell(R')}\rho^{\ell(S(X))-\ell(E')-2\ell(R')} \\ \tau(E')\xi(R' + S(Y)) \otimes \tau(D')\xi(R(X + Y)) \end{matrix}.$$

Proof. Using Theorem 1 and the coproduct formulas for $\tau(E)$ and $\xi(T)$ deduced similarly as in [4], we can compute:

$$\begin{aligned} &\psi(\tau(E)\xi(T)) \\ = &\sum_{\substack{T(Y)=T \\ D'+D''=E, T(X)=D''}} b(E, D')b(Y)b(X)\tau(S(X))\xi(S(Y)) \otimes \tau(D')\xi(R(X + Y)) \\ = &\sum_{\substack{T(Y)=T \\ D'+D''=E, T(X)=D'' \\ R' \in Seq, E' \in Seq_1 \\ |R'|+|E'|=|S(X)|}} \begin{matrix} b(E, D')b(Y)b(X)c(S(X), R') \\ \tau^{\ell(R')}\rho^{\ell(S(X))-\ell(E')-2\ell(R')} \\ \tau(E')\xi(R' + S(Y)) \otimes \tau(D')\xi(R(X + Y)) \end{matrix}. \end{aligned}$$

□

2.2.3. *Step 3.* With the coproduct formula for $\tau(E)\xi(T)$, we can compute the following.

Theorem 2. For any $E_1, E_2 \in Seq_1$ and $R_1, R_2 \in Seq$, we have

$$Q(E_1)P(R_1) \cdot Q(E_2)P(R_2)$$

$$= \sum_{\substack{R' \in Seq \\ R'+S(Y)=R_1 \\ R(X+Y)=R_2 \\ |R'|+|E_1|=|S(X)| \\ T(X)+E_2 \in Seq_1}} \begin{array}{l} b(T(X)+E_2, E_2)b(Y)b(X)c(S(X), R') \\ \tau^{\ell(R')}\rho^{\ell(S(X))-\ell(E_1)-2\ell(R')} \\ Q(T(X)+E_2)P(T(Y)) \end{array}.$$

Proof. This follows from the coproduct formula in Corollary 1 and the definition of the product in the dual. □

References

[1] D. Dugger, D. Isaksen. *The motivic Adams spectral sequence*, Geometry & Topology **14.2** (2010): 967–1014.

[2] D. Isaksen, G. Wang, Z. Xu. *Stable homotopy groups of spheres: from dimension 0 to 90*, Publications mathématiques de l'IHÉS **137.1** (2023): 107–243.

[3] J. I. Kylling. *Recursive formulas for the motivic Milnor basis.* New York journal of mathematics **23** (2017): 49–58.

[4] J. Milnor, *The Steenrod algebra and its dual*, Annals of Mathematics **67(1)** (1958), 150–171.

[5] F. Morel, V. Voevodsky *A^1-homotopy theory of schemes*, Publications Mathématiques de l'IHÉS, **90** (1999), 45–143.

[6] V. Voevodsky. *Reduced power operations in motivic cohomology*, Publications Mathématiques de l'IHÉS **98** (2003): 1–57.

[7] V. Voevodsky. *Motivic cohomology with $\mathbb{Z}/2$-coefficients*, Publications Mathématiques de l'IHÉS **98** (2003): 59–104.

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