Mathematisches Forschungsinstitut Oberwolfach

Report No. 32/2024

DOI: 10.4171/OWR/2024/32

Algebraic Geometry: Wall Crossing and Moduli Spaces, Varieties and Derived Categories

Organized by
Daniel Huybrechts, Bonn
Giulia Saccà, New York
Richard Thomas, London
Chenyang Xu, Princeton

7 July – 12 July 2024

Abstract. The workshop addressed a broad range of subjects in algebraic geometry. Recent results on moduli spaces of various types (of sheaves, complexes, varieties) played a prominent role in many of the talks, as well as derived categories of coherent sheaves. A number of talks were devoted to the geometry of special varieties (Calabi–Yau, hyperkähler) and to the geometry of the Hitchin system.

Mathematics Subject Classification (2020): 14B05, 14B07, 14C15, 14C17, 14C25, 14D06, 14D20, 14D22, 14D23, 14E05, 14E08, 14E30, 14F08, 14H10, 14J17, 14J28, 14J30, 14J32, 14J33, 14J45, 14N35, 14T90, 32A20, 32E10, 32Q20, 32Q25, 35J96, 53A15, 53D37.

License: Unless otherwise noted, the content of this report is licensed under CC BY SA 4.0.

Introduction by the Organizers

The workshop attracted leading specialists as well as early career researchers from all over the world. The number of participants affiliated to German institutions (7 out of 48) was lower than expected. We had originally planned to compensate late cancellations by early career researchers based in Germany, but unexpectedly we had essentially no cancellations at all. The number of female participants (13 out of 48, ever so slightly more than 25 % and stabilising the percentage of the last instalment) was only marginally higher than last time, but for the first time the networking effect among them was visible and is expected to have a long term effect.

The format of the workshop remained unchanged: We had 21 long talks (50 min) and 7 short talks (7min with 3min for questions) in the gong show on Tuesday night. All first time participants had been encouraged to use the gong show to present their findings to the community which led to more interaction and to a more relaxed atmosphere. Much more than usually, graduate students dared to ask questions during and after talks. There was no need for (or interest in) hybrid participation.

The timely nature of the workshop is also attested by the number of papers by participants that were published around the time of (or even during) the workshop. The list includes:

- A. Abasheva, *Shafarevich–Tate groups of holomorphic Lagrangian fibrations II*, arXiv:2407.09178.
- N. Addington, B. Tighe, *On the Torelli problem for Calabi–Yau 3-folds, and the integral cohomology of the Fermat quintic*, arXiv:2407.11176.
- I. Barros, P. Beri, L. Flapan, B. Williams, *Cones of Noether–Lefschetz divisors and moduli spaces of hyperkähler manifolds*, arXiv:2407.07622.
- H. Blum, Y. Liu, *Good moduli spaces for boundary polarized Calabi–Yau surface pairs*, arXiv:2407.00850.
- D. Maulik, J. Shen, Q. Yin, *Algebraic cycles and Hitchin systems*, arXiv:2407.05177.
- K. Oguiso, *Automorphisms of Calabi–Yau threefolds from algebraic dynamics and the second Chern class*, arXiv:2407.17297.
- K. O'Grady, *Modular sheaves with many moduli*, arXiv:2407.1810.

To name some highlights, the talks of Bottini (a first time participant) and of O'Grady concerning moduli spaces of sheaves on hyperkähler manifolds generated a lot of interest in the audience and triggered further discussions during (and presumably after) the workshop. Another highlight was the talk by Shen reporting on his long term project with Maulik and Yin concerning the Hitchin system. The audience also appreciated very much the talk by Zhuang who made his recent results on boundedness K-stable Fano singularities (joint with Xu) accessible to the wide audience of the workshop.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows".

Workshop: Algebraic Geometry: Wall Crossing and Moduli Spaces, Varieties and Derived Categories

Table of Contents

Abstracts

Boundedness of singularities and discreteness of local volumes

Ziquan Zhuang
(joint work with Chenyang Xu)

The purpose of this talk is to explain a local version of the following theorem, proved by Jiang [2] (later reproved by Xu and myself [8] using a different method):

Theorem 1. *For any positive integer n and any positive real number ε, the set of K-semistable Fano varieties of dimension n and volume $\geq \varepsilon$ is bounded.*

Here, the volume of a Fano variety X is defined as $(-K_X)^{\dim X}$ (i.e. the anti-canonical degree). We say a set of varieties is bounded if there exists a family $\mathcal{X} \to B$ of varieties over a finite type base such that every member of the given set appears as a fiber. In the global K-stability theory, this boundedness theorem is the first step in the construction of a projective moduli space (the so-called K-moduli space) parametrizing Kähler–Einstein Fano varieties of dimension n and volume at least ε.

The local analog of Fano varieties are widely known as the Kawamata log terminal (klt) singularities. In fact, orbifold cones over klt Fano varieties are klt singularities, and general klt singularities are the $\mathbb{Q}$-Gorenstein deformation of such orbifold cone singularities. We are interested in the local analog of the boundedness theorem above, and for this we need to define the correct notion of K-semistability and volume for klt singularities.

For Fano varieties, K-semistability is an algebraic condition with close ties with the existence of Kähler–Einstein metrics. This suggests that the local theory should at least include the cones over Kähler–Einstein Fano varieties. By the Calabi ansatz, these cones admit Ricci flat Kähler cone metrics. A general klt singularity has no cone structure, but the Stable Degeneration Theorem (proved in a series of works [1, 4, 5, 7, 8, 9]) states that every klt singularity has a volume preserving isotrivial degeneration to a uniquely determined Fano cone singularity that underlies a Ricci flat Kähler cone metric (a Fano cone singularity is a klt singularity $x \in X$ with a nontrivial torus action such that x is in the closure of every orbit). One can also introduce K-stability notions for Fano cone singularities and there is a Yau-Tian-Donaldson type correspondence: a Fano cone singularity has a Ricci flat Kähler cone metric if and only if it is K-polystable. From this perspective, it is quite natural to replace K-semistable Fano varieties by K-semistable Fano cone singularities in order to formulate a local boundedness statement.

As for the volume, Chi Li has introduced in [3] an interesting invariant that is later called the local volume of klt singularities. This invariant plays a central role in the local theory of K-stability. Denote by $\widehat{\mathrm{vol}}(x, X)$ the local volume of the singularity $x \in X$. It has the following nice property [6]: if $X = C(V, -K_V)$ is the cone over a K-semistable Fano variety (with respect to the anti-canonical polarization) and $x \in X$ is the vertex, then

$$\widehat{\mathrm{vol}}(x, X) = (-K_V)^{\dim V},$$

i.e. in this case the local volume equals the global volume of the Fano variety. If $x \in X$ is a singularity that appears on the Gromov–Hausdorff limit of Kähler–Einstein Fano manifolds, it is also known [6] that the local volume of $x \in X$ can be interpreted as the volume density of the singular Kähler–Einstein metric on X.

The main result (joint with Chenyang Xu) of the talk is the following. It is the local analog of the boundedness result for K-semistable Fano varieties.

Theorem 2. *For any positive integer n and any positive real number ε, the set of K-semistable Fano cone singularities of dimension n and local volume at least ε is bounded.*

This local boundedness result has several applications. Denote by $\widehat{\mathrm{Vol}}_n$ the set of all possible local volumes of klt singularities of dimension n. We would like to know how this set looks like. It is known [9] that $\widehat{\mathrm{Vol}}_n \subseteq \overline{\mathbb{Q}}$ but as soon as $n \geq 3$ we have $\widehat{\mathrm{Vol}}_n \not\subseteq \mathbb{Q}$ (see e.g. [1]). This latter feature makes the local theory very different from the global one. Our first application is that 0 is the only accumulation point of $\widehat{\mathrm{Vol}}_n$; in other words it is discrete away from 0.

As another application, we show that the local volume of a klt singularity controls most classical invariants of the singularity. This means that for any $\varepsilon > 0$ there exists some constant $M > 0$ depending only on ε such that any klt singularity $x \in X$ with local volume $\widehat{\mathrm{vol}}(x, X) \geq \varepsilon$ has embedded dimension (or multiplicity, minimal log discrepancy etc.) at most M.

References

[1] Harold Blum, *Existence of valuations with smallest normalized volume*, Compos. Math. **154** (2018), no. 4, 820–849.

[2] Chen Jiang, *Boundedness of $\mathbb{Q}$-Fano varieties with degrees and alpha-invariants bounded from below*, Ann. Sci. Éc. Norm. Supér. **53** (2020), no. 5, 1235–1248.

[3] Chi Li, *Minimizing normalized volumes of valuations*, Math. Z. **289** (2018), no. 1-2, 491–513.

[4] Chi Li, Xiaowei Wang, and Chenyang Xu, *Algebraicity of the metric tangent cones and equivariant K-stability*, J. Amer. Math. Soc. **34** (2021), no. 4, 1175–1214.

[5] Chi Li and Chenyang Xu, *Stability of Valuations: Higher Rational Rank*, Peking Math. J. **1** (2018), no. 1, 1–79.

[6] Chi Li and Chenyang Xu, *Stability of valuations and Kollár components*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 8, 2573–2627.

[7] Chenyang Xu, *A minimizing valuation is quasi-monomial*, Ann. of Math. (2) **191** (2020), no. 3, 1003–1030.

[8] Chenyang Xu and Ziquan Zhuang, *Uniqueness of the minimizer of the normalized volume function*, Camb. J. Math. **9** (2021), no. 1, 149–176.

[9] Chenyang Xu and Ziquan Zhuang, *Stable degenerations of singularities*, arXiv:2205.10915.

Relations of Kuznetsov components of a Gushel-Mukai variety and its hyperplane sections

Soheyla Feyzbakhsh
(joint work with Hanfei Guo, Zhiyu Liu, Shizhuo Zhang)

Let X be a Gushel–Mukai variety of dimension $n \geq 3$, which is a quadric section of the projective cone over the Grassmannian $\mathrm{Gr}(2,5)$. The Kuznetsov component of X, denoted by $\mathrm{Ku}(X)$, is the right orthogonal complement of a collection of rigid bundles on X:

$$\mathcal{D}^b(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{U}_X^\vee, \cdots, \mathcal{O}_X(n-3), \mathcal{U}_X^\vee(n-3) \rangle,$$

where $\mathcal{U}_X$ is the pull-back of the tautological subbundle on $\mathrm{Gr}(2,5)$.

- If $n = 3$ or 5, then $\mathrm{Ku}(X)$ is an Enriques category, i.e. its Serre functor is of the form $T_X \circ [2]$ for a non-trivial involution T_X of $\mathrm{Ku}(X)$. The numerical Grothendieck group of $\mathrm{Ku}(X)$ is of rank 2, generated by classes λ_1 and λ_2, and up to $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action, there is a unique Serre-invariant Bridgeland stability condition on $\mathrm{Ku}(X)$.
- If $n = 4$ or 6, then $\mathrm{Ku}(X)$ is a K3 category, i.e. its Serre functor is $[2]$. When X is very general, the numerical Grothendieck group of $\mathrm{Ku}(X)$ is of rank 2, generated by classes Λ_1 and Λ_2, and up to $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action, there is a unique Bridgeland stability condition on $\mathrm{Ku}(X)$.

Let $j\colon Y \hookrightarrow X$ be a smooth hyperplane section of a GM variety X. The restriction of the derived pull-back to $\mathrm{Ku}(Y)$ yields the functor $j^*\colon \mathrm{Ku}(X) \to \mathrm{Ku}(Y)$. However, the image of the push-forward does not always lie in the Kuznetsov component, so we need to project it into $\mathrm{Ku}(X)$. This gives us the functor $\mathrm{pr}_X \circ j_*\colon \mathrm{Ku}(Y) \to \mathrm{Ku}(X)$. The main result discussed in the talk is the preservation of stability under both these functors.

Theorem 1. *Let X be a GM variety of dimension $n \geq 4$ and $j\colon Y \hookrightarrow X$ be a smooth hyperplane section. Let σ_Y and σ_X be Serre-invariant stability conditions on $\mathrm{Ku}(Y)$ and $\mathrm{Ku}(X)$, respectively. We additionally assume that whichever of X and Y has even dimensionality is considered to be very general.*

(1) *An object $E \in \mathrm{Ku}(Y)$ is σ_Y-semistable if and only if $\mathrm{pr}_X(j_*E) \in \mathrm{Ku}(X)$ is σ_X-semistable.*
(2) *An object $F \in \mathrm{Ku}(X)$ is σ_X-semistable if and only if $j^*F \in \mathrm{Ku}(Y)$ is σ_Y-semistable.*

Now let X be a very general GM fourfold. We denote by $M^X_{\sigma_X}(a,b)$ the moduli space that parameterizes S-equivalence classes of σ_X-semistableobjects of class $a\Lambda_1 + b\Lambda_2$ in $\mathrm{Ku}(X)$. As a result of Theorem 1, we obtain the following two results:

- For any pair of coprime integers a, b, there is a Lagrangian family of $M^X_{\sigma_X}(a,b)$ over an open dense subset of $|\mathcal{O}_X(H)|$.
- The functor pr_X induces a rational map

$$X \dashrightarrow M^X_{\sigma_X}(1,2)$$

which sends the structure sheaf at a general point to its projection to the Kuznetsov component. Moreover, the map is generically an embedding.

Shafarevich–Tate twists of the LSV fibration

Evgeny Shinder
(joint work with Yajnaseni Dutta, Dominique Mattei)

1. Introduction to Shafarevich–Tate twists

1.1. **Twisting elliptic K3 surfaces.** Let $p\colon S \to \mathbb{P}^1$ be a complex elliptic K3 surface with a section. The set of smooth points $\mathcal{J} \subset S$ of p has a natural structure of a group scheme and the Shafarevich–Tate group is defined as

$$Ш(S/\mathbb{P}^1) := H^1(\mathbb{P}^1, \mathcal{J}).$$

Here we consider $\mathcal{J}$ as a sheaf of sections of the group scheme, in the analytic topology. A similar definition can be given for the étale topology.

For every class $\alpha \in Ш(S)$ we can construct, by regluing, a K3 surface $S_\alpha \to \mathbb{P}^1$, which has no sections unless $\alpha = 0$.

Theorem 1 (Ogg–Shafarevich theory)**.** *We have a natural isomorphism*

$$Ш(S/\mathbb{P}^1) \simeq \mathrm{Br}(S) := H^2(S, \mathcal{O}^*).$$

Here again the cohomology group on the right is taken in the analytic topology. From the exponential exact sequence we get an isomorphism

$$H^2(S, \mathcal{O}^*) \simeq H^2(S, \mathcal{O})/\mathrm{Im}(H^2(S, \mathbb{Z}) \to H^2(S, \mathcal{O})) = \mathbb{C}/\mathbb{Z}^{22-\rho}$$

where ρ is the Picard rank of S.

Sketch of Proof for the Ogg–Shafarevich theory. One can show that

$$Ш(S/\mathbb{P}^1) = H^1(\mathbb{P}^1, \mathcal{J}) \simeq H^1(\mathbb{P}^1, R^1p_*(\mathcal{O}^*)) \simeq \mathrm{Br}(S).$$

The last isomorphism follows immediately using the Leray spectral sequence. □

1.2. **Twisting Lagrangian fibrations.** Let M be a compact Kähler manifold which is irreducible holomorphic symplectic, i.e. M admits a nondegenerate complex 2-form σ and $\pi_1(M)$ is trivial. Such irreducible holomorphic symplectic manifolds are also called hyperkähler. Let $\dim(M) = 2n$.

Let $f\colon M \to \mathbb{P}^n$ be a Lagrangian fibration. This means that σ restricts trivially to general fibers of f and it follows that general fibers of f are abelian varieties. On many levels Lagrangian fibrations behave like elliptic K3 surfaces, and we'd like to know how to *define* and to *compute* the Shafarevich–Tate group of M. This works well when the fibers of f are irreducible which we assume from now on.

Theorem 2 (Arinkin–Fedorov [AF16])**.** *Let $f\colon M \to \mathbb{P}^n$ be a Lagrangian fibration with a section. Assume that fibers of f are irreducible. Then the smooth locus $\mathcal{J} \subset M$ has a natural structure of an abelian group scheme. Furthermore, the Lie algebra of $\mathcal{J}$ is isomorphic to $\Omega^1_{\mathbb{P}^n}$.*

The group scheme $\mathcal{J}$ is also isomorphic to the identity component of the automorphism group scheme $\mathrm{Aut}^0_{M/\mathbb{P}^n}$ which has been studied by Markushevich [Mar96] and Abasheva–Rogov [AR21]. By analogy with the elliptic K3 surface case, and following Markman [Mar14] and Abasheva–Rogov [AR21] we define

$$Ш(M/\mathbb{P}^n) := H^1(\mathbb{P}^n, \mathcal{J}).$$

For every $\alpha \in Ш(M/\mathbb{P}^n)$ there is a space M_α constructed by regluing, and M_α is also an irreducible holomorphic symplectic manifold with a Lagrangian fibration without a section. Under some mild conditions, M_α is Kähler. The manifold M_α is projective iff the class α is torsion.

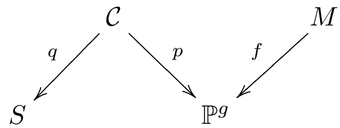
This is a great theoretical framework, however, there seems to be, at present, no unified computation for the group $Ш(M/\mathbb{P}^n)$ in terms of M.

1.3. **Example: Beauville–Mukai system.** Let S be a K3 surface, which we for simplicity assume to have Picard rank one. Let $h \in \mathrm{Pic}(S)$ be the ample generator and $h^2 = 2g-2$. Let $\mathcal{C} \subset S \times \mathbb{P}^g$ be the universal curve in the linear system $|h| \simeq \mathbb{P}^g$.

Let $M = M(0, h, 1-g)$ be the moduli space of semistable torsion sheaves with the first Chern class h and Euler characteristic $1-g$. A general point in M corresponds to a degree 0 line bundle on a smooth curve from the linear system $|h|$.

It is well known that M is a projective irreducible holomorphic symplectic manifold of type $\mathrm{K3}^{[g]}$ which admits a Lagrangian fibration $f\colon M \to \mathbb{P}^g$ given by the support of the sheaf. It admits a section given by the trivial bundle on each fiber. This Lagrangian fibration is called a Beauville–Mukai system. By assumption, p and f have irreducible fibers.

The Arinkin–Fedorov group scheme $\mathcal{J}$ of M is isomorphic to $\mathrm{Pic}^0_{\mathcal{C}/\mathbb{P}^g}$. We have the following diagram



where f is the Lagrangian fibration, p is a flat morphism of relative dimension 1 and q is a projective bundle. Similarly to the elliptic K3 surface case it is important to consider another group scheme $\widetilde{\mathcal{J}} := \mathrm{Pic}_{\mathcal{C}/\mathbb{P}^g} = R^1 p_*(\mathcal{O}^*)$ so that pushing forward the exponential exact sequence from $\mathcal{C}$ to $\mathbb{P}^g$ gives two exact sequences

$$0 \to \Lambda \to \Omega^1_{\mathbb{P}^g} \to \mathcal{J} \to 0, \qquad \Lambda := R^1 p_*(\mathbb{Z})$$
$$0 \to \mathcal{J} \to \widetilde{\mathcal{J}} \to R^2 p_*(\mathbb{Z}) \to 0.$$

The point is that we would like to compute $Ш(M/\mathbb{P}^g) = H^1(\mathcal{J})$, however from the Leray spectral sequence for p, we have an easier access to $H^1(\widetilde{\mathcal{J}})$.

Theorem 3 (Markman [Mar14]). *We have natural isomorphisms*

$$H^0(\widetilde{\mathcal{J}}) \simeq \mathrm{Pic}(S), \quad H^1(\widetilde{\mathcal{J}}) \simeq \mathrm{Br}(S) = \mathrm{Coker}(H^2(S,\mathbb{Z}) \to H^{0,2}(S))$$
$$H^0(\mathcal{J}) \simeq h^\perp \cap \mathrm{Pic}(S) = 0, \quad Ш(M/\mathbb{P}^g) = H^1(\mathcal{J}) \simeq \mathrm{Coker}(h^\perp \to H^{0,2}(S))$$

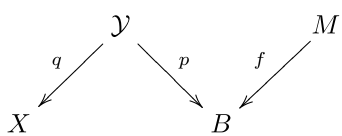
and an exact sequence

$$0 \to \mathbb{Z}/(2g-2) \to Ш(M/\mathbb{P}^g) \to \mathrm{Br}(S) \to 0.$$

The first term in the exact sequence should be interpreted as a degree twist, with the corresponding twists isomorphic to moduli spaces $M(0, h, \chi)$, $\chi \in \mathbb{Z}/(2g-2)$. The recent work of Huybrechts–Mattei [HM23] shows that the last term in the exact sequence corresponds to moduli spaces of Brauer-twisted sheaves parametrized by $\alpha \in \mathrm{Br}(S)$. Finally we note that $Ш(M/\mathbb{P}^g)$ is *not* the Brauer group of M like in the elliptic K3 surface case, but it is Brauer group of one of the twists, namely by a result of Mattei–Meinsma [MM24], $Ш(M/\mathbb{P}^g) \simeq \mathrm{Br}(M(0, h, 0))$.

2. Twisting the LSV fibration

2.1. **Laza–Saccà–Voisin (LSV) fibration.** Let $X \subset \mathbb{P}^5$ be a general cubic fourfold, and consider the hyperplane sections Y_b parameterized by points $b \in B := (\mathbb{P}^5)^\vee$. These fit into the universal hyperplane section $\mathcal{Y} \subset X \times B$. It was an idea by Donagi–Markman, further developed by others that the relative intermediate Jacobian fibration of $\mathcal{Y}$ over the smooth hyperplane sections should be compactified to an irreducible holomorphic symplectic variety. This has been achieved by Laza–Saccà–Voisin [LSV17], see [Sac23] for a construction without genericity assumptions on X. The outcome is a diagram similar to the Beauville–Mukai case



Here M is a projective irreducible holomorphic symplectic variety of OG10 type. The morphism f is a Lagrangian fibration with the general fiber being the intermediate jacobian of the corresponding smooth cubic threefold, that is a 5-dimensional principally polarized abelian variety. It is important to note how K3 surface S in the Beauville–Mukai system is replaced by the cubic fourfold X, and we will keep this analogy when computing the Shafarevich–Tate group of M.

Parameter count shows that the LSV construction parametrizes a codimension two locus in the moduli space of nonpolarized irreducible holomorphic symplectic manifolds of type OG10. Adding Shafarevich–Tate twists will allow us parametrizing a codimension one locus in the moduli.

2.2. **Main results: twists of the LSV.** Assuming that X is general, fibers of p and f will be irreducible, thus we can consider the Arinkin–Fedorov group scheme of M.

Theorem 4 (Dutta–Mattei–S.). *The sheaf of sections of the Arinkin–Fedorov group scheme $\mathcal{J}$ for M/B fits into the following exact sequences*

(1) $0 \to \Lambda \to \Omega^1_B \to \mathcal{J} \to 0$ *with* $\Lambda = R^3 p_*(\mathbb{Z})$.
(2) $0 \to \mathcal{J} \to \widetilde{\mathcal{J}} \to \mathbb{Z} \to 0$ *with* $\widetilde{\mathcal{J}} = R^4 p_*(\mathbb{Z}(2)_D)$.

Here $\mathbb{Z}(2)_D$ is the second Deligne complex $[\mathbb{Z} \to \mathcal{O} \to \Omega^1]$ which replaces the first Deligne complex $\mathbb{Z}(1)_D = [\mathbb{Z} \to \mathcal{O}] \simeq \mathcal{O}^*[-1]$, in the case of Beauville–Mukai system. Deligne cohomology produces a natural cycle class map for algebraic cycles; it was an idea of Voisin to use Deligne cohomology to incorporate degree twists in the LSV construction.

Similarly to the Beauville–Mukai system case, cohomologies of $\widetilde{\mathcal{J}}$ are accessible using the (quite complicated) Leray spectral sequence for p and we can compute

$$H^0(\widetilde{\mathcal{J}}) \simeq H^{2,2}(X, \mathbb{Z}), \quad H^1(\widetilde{\mathcal{J}}) \simeq \mathrm{Br}^2(X) := \mathrm{Coker}(H^4(X, \mathbb{Z}) \to H^{1,3}(X)).$$

Then using the Theorem we can deduce that

$$H^0(\mathcal{J}) \simeq (h^2)^\perp \cap H^{2,2}(X, \mathbb{Z}), \quad Ш(M/B) = H^1(\mathcal{J}) \simeq \mathrm{Coker}((h^2)^\perp \to H^{1,3}(X)).$$

Just like in the Beauville–Mukai system case we have an exact sequence

$$\mathbb{Z}/3 \to Ш(M/B) \to \mathrm{Br}^2(X) \to 0.$$

Here the twist corresponding to $\mathbb{Z}/3$ has been earlier constructed by Voisin [Voi18] as the relative intermediate Jacobian for degree 1 cycles of dimension one. The first map is injective iff divisibility of h^2 in $H^{2,2}(X, \mathbb{Z})$ equals three which is the case for very general X.

Finally we note that $\mathrm{Br}^2(X)$ is isomorphic to the Brauer group $\mathrm{Br}(F(X))$ of the Fano variety of lines, thus if M is represented as a moduli space of sheaves on $F(X)$ (which is work in progress by Bottini, reported in this workshop), then the Shafarevich–Tate twists should be parametrizing the corresponding moduli spaces of twisted sheaves.

References

[AF16] Dima Arinkin and Roman Fedorov. Partial Fourier-Mukai transform for integrable systems with applications to Hitchin fibration. *Duke Math. J.* **165**(15), pp. 2991–3042, 2016.

[AR21] Anna Abasheva and Vasily Rogov. Shafarevich–Tate groups of holomorphic Lagrangian fibrations, 2021, http://arxiv.org/abs/2112.10921.

[HM23] Daniel Huybrechts and Dominique Mattei. The special brauer group and twisted picard varieties, 2023, http://arxiv.org/abs/2310.04032.

[LSV17] Radu Laza, Giulia Saccà, and Claire Voisin. A hyperKähler compactification of the intermediate Jacobian fibration associated with a cubic 4-fold. *Acta Math.* **218**(1), pp. 55–135, 2017.

[Mar96] D. Markushevich. Lagrangian families of Jacobians of genus 2 curves. volume 82, pages 3268–3284. 1996. Algebraic geometry, 5.

[Mar14] Eyal Markman. Lagrangian fibrations of holomorphic-symplectic varieties of K3[n]-type. In *Algebraic and Complex Geometry*, pages 241–283, Cham, 2014. Springer International Publishing.

[MM24] Dominique Mattei and Reinder Meinsma. Obstruction classes for moduli spaces of sheaves and Lagrangian fibrations, 2024, http://arxiv.org/abs/2404.16652.

[Sac23] Giulia Saccà. Birational geometry of the intermediate Jacobian fibration of a cubic fourfold (appendix by Claire Voisin). *Geom. Topol.* **27**(4), pp. 1479–1538, 2023.

[Voi18] Claire Voisin. Hyper-Kähler compactification of the intermediate Jacobian fibration of a cubic fourfold: the twisted case. In *Local and global methods in algebraic geometry. Conference in honor of Lawrence Ein's 60th birthday, University of Illinois at Chicago, IL, USA, May 12–15, 2016. Proceedings*, pages 341–355. Providence, RI: American Mathematical Society (AMS), 2018.

The K-moduli of a family of conic bundles

Lena Ji

(joint work with Kristin DeVleming, Patrick Kennedy-Hunt, Ming Hao Quek)

Constructing moduli spaces of varieties is a central problem in algebraic geometry. A key example is the moduli space of smooth genus g curves, which Deligne–Mumford compactified using stable genus g curves. For Fano varieties, the theory of K-stability has established the existence of a compact moduli space parametrizing K-polystable Fano varieties of fixed dimension and anticanonical volume. However, it remains a challenge to determine these moduli spaces for explicit families of Fano varieties. For Fano varieties of dimension ≤ 2, the K-moduli spaces are understood, but much less is known for Fanos of higher dimensions. In general, this problem is difficult because it requires describing all possible K-polystable degenerations of a given class of Fano varieties. For example, certain families of Fano hypersurfaces have the property that all K-polystable limits must themselves be hypersurfaces [13], but for other families of Fano hypersurfaces this property fails, see e.g. [2].

In dimension 3, the deformation families of smooth Fano threefolds were classified by Mori–Mukai, and it is known by [4] which of these families contain K-polystable members. For some low-dimensional families of Fano threefolds, the

K-moduli spaces are understood, see e.g. [1, 8, 6], but much less is known about higher-dimensional families.

In [9], we describe the (compact) K-moduli space $\overline{M_{2.18}}$ of Fano threefolds in family №2.18 from the Mori–Mukai classification. This family is 6-dimensional, and smooth Fano threefolds in this family were shown to be K-stable by [7]. These threefolds are given as a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ branched along a smooth $(2,2)$-divisor R. We study $\overline{M_{2.18}}$ by studying the K-moduli spaces $\overline{M_c}$ of log Fano pairs $(\mathbb{P}^1 \times \mathbb{P}^2, cR)$ for $(2,2)$-divisors R and coefficients $c \in (0, \frac{1}{2}] \cap \mathbb{Q}$.

Theorem 1. *Let $c_1 \approx 0.472$ be the irrational number defined as the smallest root of the polynomial $10c^3 - 34c^2 + 35c - 10$. The following hold for $c \in (0, \frac{1}{2}] \cap \mathbb{Q}$:*

(1) *For $c < c_1$, the K-moduli space $\overline{M_c}$ is isomorphic to the GIT moduli space $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(2,2)|//\mathrm{SL}_2 \times \mathrm{SL}_3$ of $(2,2)$-divisors in $\mathbb{P}^1_{[t_0:t_1]} \times \mathbb{P}^2_{[y_0:y_1:y_2]}$.*

(2) *For $c > c_1$, the K-moduli spaces $\overline{M_c} \cong \overline{M_{1/2}}$ are isomorphic, and there is a bijective morphism $\overline{M_{1/2}} \to \overline{M_{2.18}}$.*

(3) *There is a birational wall-crossing morphism $\overline{M_{c_1+\epsilon}} \to \overline{M_{c_1-\epsilon}}$ contracting a divisor $E_{\mathfrak{n}}$ to a point corresponding to the unique non-normal, irreducible GIT polystable $(2,2)$-surface R_3, which is defined by $t_0t_1y_1^2 + (t_0y_2 + t_1y_0)^2$.*

In the "proportional" setting, Ascher–DeVleming–Liu established a wall-crossing framework in K-moduli [5]. They studied log Fano pairs (X, cD) with D proportional to the anticanonical divisor (i.e. $D \sim_{\mathbb{Q}} -rK_X$ for some $r \in \mathbb{Q}_{>0}$), and proved in this setting that, as the coefficient c varies, there are only finitely values at which the K-stability conditions change, and furthermore these occur at rational numbers. Theorem 1 is not in the proportional setting, and we exhibit an irrational wall at c_1.

Furthermore, we give explicit descriptions of the K-polystable log Fano pairs $(\mathbb{P}^1 \times \mathbb{P}^2, cR)$ for $c \in (0, \frac{1}{2}] \cap \mathbb{Q}$. For $c < c_1$, this involves describing the GIT polystable $(2,2)$-divisors in $\mathbb{P}^1 \times \mathbb{P}^2$. For $c > c_1$, we show that the exceptional divisor $E_{\mathfrak{n}}$ of the wall-crossing morphism in Theorem 1(3) parametrizes surfaces on a certain non-$\mathbb{Q}$-factorial nodal threefold $X_{\mathfrak{n}}$, and we describe $E_{\mathfrak{n}}$ as a certain GIT moduli space. As a consequence of our explicit descriptions, we show the following properties of Fano threefolds in $\overline{M_{2.18}}$:

Theorem 2. *In the table below, the left-hand column lists properties of smooth Fano threefolds in family №2.18, and the right-hand column lists properties of their K-polystable degenerations.*

Smooth Fano 3-folds in family №2.18	*K-polystable Fano 3-folds in $\overline{M_{2.18}}$*
• *Smooth*	• *Gorenstein terminal singularities*
• *Double cover of $\mathbb{P}^1 \times \mathbb{P}^2$*	• *Double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ or $X_{\mathfrak{n}}$*
• *Standard conic bundle over $\mathbb{P}^2$ with quartic discriminant curve*	• *Possibly non-flat conic bundle over $\mathbb{P}^2$ with quartic discriminant curve*
• *Quadric surface bundle over $\mathbb{P}^1$*	• *The double covers of $X_{\mathfrak{n}}$ do not have a del Pezzo fibration structure*

In particular, all K-polystable degenerations preserve a conic bundle structure.

To prove Theorems 1 and 2, we need to establish the K-stability of certain log Fano pairs $(\mathbb{P}^1 \times \mathbb{P}^2, cR)$ and $(X_{\mathfrak{n}}, cR)$. We use the valuative criterion [10, 12], which states that if (X, D) is a log Fano pair and the local δ-invariant

$$\delta_p(X, D) := \inf_{E/X,\, p \in C_X(E)} \frac{A_{X,D}(E)}{S_{X,D}(E)}$$

is greater than 1 for every $p \in X$, then (X, D) is K-stable. In the above, the infimum is taken over all prime divisors E over X whose centers on X contain p, $A_{X,D}(E)$ is the log discrepancy, and

$$S_{X,D}(E) = \frac{1}{\operatorname{vol}(-K_X - D)} \int_0^\infty \operatorname{vol}(f^*(-K_X - D) - uE)\, du$$

where $f \colon \widetilde{X} \to X$ is a birational morphism with $E \subset \widetilde{X}$. In order to compute lower bounds for $\delta_p(X, D)$, we use the Abban–Zhuang method of admissible flags [3]. This is an inductive method that can give lower bounds on $\delta_p(X, D)$ in a finite computation. For example, if E is a plt-type divisor over X whose center contains p, $f \colon \widetilde{X} \to X$ is a plt blow-up extracting E, and $V_{\bullet\bullet}$ is the refinement of the filtration of $f^*(-K_X - D)$ by E, then by [3]

$$\delta_p(X, D) \geq \min\left\{ \frac{A_{X,D}(E)}{S_{X,D}(E)},\ \inf_{q \in E} \delta_q(E, D_E; V_{\bullet\bullet}) \right\}. \tag{3}$$

The Abban–Zhuang method can then be applied again, for each closed $q \in E$, to give lower bounds on $\delta_q(E, D_E; V_{\bullet\bullet})$ using a further refinement of the filtration. In the case when $\dim X = 3$, Fujita gives formulas for the involved quantities in terms of Zariski decompositions on (a birational model of) the surface E [11].

A large difficulty in [9] is to find an appropriate plt-type divisor E to apply the inequality (3). If E is not chosen carefully, the right-hand side of (3) will frequently be less than 1 and hence will not give a useful lower bound to show K-stability. In our work, the choice of plt-type divisor E is highly sensitive to the singularities of R at p, and to the fibers of the projections $R \to \mathbb{P}^1$ and $R \to \mathbb{P}^2$ containing p.

The irrational number c_1 in Theorem 1 comes from computing

$$\frac{A_{\mathbb{P}^1 \times \mathbb{P}^2, cR_3}(E_3)}{S_{\mathbb{P}^1 \times \mathbb{P}^2, cR_3}(E_3)} \tag{4}$$

where R_3 is the non-normal GIT polystable surface in Theorem 1(3), and E_3 is the exceptional divisor of the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ at the singular locus of R_3. We show that $(\mathbb{P}^1 \times \mathbb{P}^2, cR_3)$ is K-polystable if and only if $c < c_1$; for $c > c_1$ the quantity (4) is strictly less than 1, so the pair becomes K-unstable. We find the replacement nodal threefold $X_{\mathfrak{n}}$ that appears after the wall by performing the birational transformations associated to the Nakayama–Zariski decompositions used in the computation of $S_{\mathbb{P}^1 \times \mathbb{P}^2, cR_3}(E_3)$.

References

[1] H. Abban, I. Cheltsov, E. Denisova, E. Etxabarri-Alberdi, A. Kaloghiros, D. Jiao, J. Martinez-Garcia, and T. Papazachariou, *One-dimensional components in the K-moduli of smooth Fano 3-folds*, preprint arXiv:2309.12518 (2023).
[2] H. Abban, I. Cheltsov, A. Kasprzyk, Y. Liu, and A. Petracci, *On K-moduli of quartic threefolds*, preprint arXiv:2210.14781 (2024).
[3] H. Abban and Z. Zhuang, *K-stability of Fano varieties via admissible flags*, Forum Math. Pi **10** (2022), Paper No. e15, 43 pp.
[4] C. Araujo, A. Castravet, I. Cheltsov, K. Fujita, A. Kaloghiros, J. Martinez-Garcia, C. Shramov, H. Süß, and N. Viswanathan, *The Calabi problem for Fano threefolds*, London Math. Soc. Lecture Note Ser., 485 Cambridge University Press, Cambridge, 2023. vii+441 pp.
[5] K. Ascher, K. DeVleming, and Y. Liu, *Wall crossing for K-moduli spaces of plane curves*, Proc. Lond. Math. Soc. (3) **128** (2024), no.6, Paper No. e12615, 113 pp.
[6] I. Cheltsov, T. Duarte Guerreiro, K. Fujita, I. Krylov, and J. Martinez-Garcia, *K-stability of Casagrande-Druel varieties*, preprint arXiv:2309.12522 (2023).
[7] I. Cheltsov, K. Fujita, T. Kishimoto, and J. Park, *K-stable Fano 3-folds in the families 2.18 and 3.4* , preprint arXiv:2304.11334 (2023).
[8] I. Cheltsov and A. Thompson, *K-moduli of Fano threefolds in family 3.10*, preprint arXiv:2309.12524 (2023).
[9] K. DeVleming, L. Ji, P. Kennedy-Hunt and M. Quek, *The K-moduli space of a family of conic bundle threefolds*, preprint arXiv:2403.09557 (2024).
[10] K. Fujita, *A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties*, J. Reine Angew. Math. **751** (2019), 309–338.
[11] K. Fujita, *On K-stability for Fano threefolds of rank 3 and degree 28*, Int. Math. Res. Not. IMRN(2023), no.15, 12601–12784.
[12] C. Li, *K-semistability is equivariant volume minimization*, Duke Math. J. 166(2017), no.16, 3147–3218.
[13] Y. Liu and C. Xu, *K-stability of cubic threefolds*, Duke Math. J. **168** (2019), no. 11, 2029–2073.

An algebro-geometric version of the Poincaré conjecture

Mirko Mauri
(joint work with Joaquín Moraga)

Dual complexes are CW-complexes encoding the combinatorial data of how the irreducible components of a simple normal crossing divisor intersect. The overall goal is to comprehend how the topology of dual complexes is constrained by the properties of the algebraic varieties from which they stem.

Definition 1. *Let X be a smooth projective variety, and $\Delta_X = \sum_{i \in I} \Delta_i$ be a simple normal crossing divisor. Then the* dual complex *of (X, Δ_X), denoted $\mathcal{D}(\Delta_X)$, is a CW-complex whose vertices are in correspondence with the irreducible components Δ_i of Δ_X and whose k-faces correspond to the strata of Δ_X of codimension $k+1$, i.e., the irreducible components of the intersections $\Delta_{i_0} \cap \ldots \cap \Delta_{i_k}$.*

Dual complexes naturally appear in the classification of singularities and in mirror symmetry; see [1, 2, 3, 4, 5]. According to mirror symmetry, Calabi–Yau varieties, i.e., varieties with a global algebraic volume form, should come in pairs

$(Y, \hat{Y})$, and one should reconstruct $\hat{Y}$ by dualising a special Lagrangian torus fibration $f: Y \to B$, called SYZ fibration.

Kontsevich and Soibelman conjectured that B should be a manifold (or at least an orbifold). Equivalently, B should be locally modelled on a ball, alias a cone over a sphere (or a finite quotient of a sphere). This sphere is expected to have a combinatorial interpretation: it should be the dual complex $\mathcal{D}(\Delta_X)$ of a *log Calabi–Yau* pair (X, Δ_X), in brief logCY, i.e., $K_X + \Delta_X \sim_{\mathbb{Q}} 0$. Kollár and Xu asked whether this is actually a general phenomenon of logCY pairs; see [3, Question 4].

Conjecture 2 (Algebro-geometric version of the Poincaré conjecture)**.** *The dual complex of a logCY pair is homeomorphic to a finite quotient of a sphere.*

More informally, a combinatorial Calabi–Yau variety should be a finite quotient of a sphere. Conjecture 2 should be regarded as an algebro-geometric version of the Poincaré conjecture. Indeed, the logCY condition $K_X + \Delta_X \sim 0$ is equivalent to the cohomological assumption $h^0(X, K_X + \Delta_X) = h^0(X, -(K_X + \Delta_X)) = 1$. Conjecture 2 requires that this cohomological datum should prescribe a spherical homeomorphism type for the dual complex, in the same way as for the classical Poincaré conjecture. Moreover, the classical Poincaré conjecture is a key ingredient in [3] and [6].

Conjecture 2 is known up to dimension 5 due to the work of Kollár and Xu [3]. In higher dimension, there exists only partial evidence:

(1) the conjecture holds if (X, Δ_X) is endowed with the structure of Mori fiber space $(X, \Delta_X) \to Z$ with $\dim Z \leq 2$ or $\rho(Z) \leq 2$; see [7].
(2) $\mathcal{D}(X, \Delta_X)$ has the rational homology of the sphere or of the point; see [3, Theorem 2.(2)]. On the other hand, the torsion of the integral homology remains quite mysterious!
(3) The fundamental group of $\mathcal{D}(X, \Delta_X)$ is finite, and the fundamental groups of $\mathcal{D}(X, \Delta_X)$ in any fixed dimension form a finite set; see [8, Corollary 1].
(4) $\mathcal{D}(X, \Delta_X)$ is orientable if and only if $K_X + \Delta_X \sim 0$; see [9, Corollary 4].

In [6], we provide new evidence to support Conjecture 2.

Theorem 3. *If (X, Δ_X) is a $\mathbb{Q}$-factorial logCY pair, the sum of the coefficients of Δ_X is greater than the Picard number of X, i.e., $|\Delta_X| > \rho(X)$, and $\mathcal{D}(X, \Delta_X)$ is a PL-manifold, then $\mathcal{D}(X, \Delta_X)$ is homeomorphic to a sphere or a disk.*

If $K_X + \Delta_X \sim 0$, the logCY condition grants that $\mathcal{D}(X, \Delta_X)$ is a rational homology sphere; see [3, §4]. Even if $\mathcal{D}(X, \Delta_X)$ is a simply-connected PL-manifold, it is not a priori clear that it is a sphere. As a counterexample, consider for instance the 5-dimensional Wu manifold $SU(3)/SO(3)$. By the classical Poincaré conjecture, the only obstruction is the torsion of the integral homology, which vanishes under our algebro-geometric assumption $|\Delta_X| > \rho(X)$.

The case of main interest is when the divisor Δ_X has a stratum of dimension zero. Up to birational transformation, we can always reduce to $|\Delta_X| \geq \rho(X)$; see [6, Theorem 1.4]. Hence, Theorem 3 states that logCY pairs with interesting dual complexes (neither a sphere nor a disk) must satisfy $|\Delta_X| = \rho(X)$. We expect that these logCY pairs are rather special. For instance, we show that their effective cone is generated by the irreducible components Δ_i; see [6, Theorem 1.19].

References

[1] János Kollár, *Singularities of the minimal model program*, Cambridge University Press, Cambridge, 2013. Cambridge Tracts in Mathematics, Vol. 200. Pages: 370.

[2] Tommaso de Fernex, János Kollár, and Chenyang Xu, *The dual complex of singularities*, Adv. Stud. Pure Math. **74** (2017), 103–130.

[3] János Kollár and Chenyang Xu, *The dual complex of Calabi–Yau pairs*, Invent. Math. **205** (2016), no. 3, 527–557.

[4] Maxim Kontsevich and Yan Soibelman, *Affine structures and non-Archimedean analytic spaces*, In: *The unity of mathematics*, Progr. Math., Vol. 244, Birkhäuser Boston, Boston, MA, 2006, 321–385.

[5] Johannes Nicaise, Chenyang Xu, and Tony Yue Yu, *The non-archimedean SYZ fibration*, Compos. Math. **155** (2019), no. 5, 953–972.

[6] Mirko Mauri and Joaquín Moraga, *Birational complexity and dual complexes*, 2024.

[7] Mirko Mauri, *The dual complex of log Calabi–Yau pairs on Mori fibre spaces*, Adv. Math. **364** (2020), 107009, 28.

[8] Joaquín Moraga, *On a toroidalization for klt singularities*, arXiv:2106.15019 [math.AG], 2021.

[9] Stefano Filipazzi, Mirko Mauri, and Joaquín Moraga, *The index of coregularity zero Calabi–Yau pairs*, To appear in Algebra & Number Theory. arXiv:2209.02925 [math.AG], 2022.

Non-commutative abelian surfaces and generalised Kummer varieties

Arend Bayer

(joint work with Laura Pertusi, Alex Perry, Xiaolei Zhao)

The derived category of an abelian surface A has a six-dimensional space of deformations; moreover, based on general principles, one should expect to get *algebraic families* of their categories over four-dimensional bases. Generalised Kummer varieties (GKV) are hyperkähler varieties arising from moduli spaces of stable sheaves on abelian surfaces. Polarised GKVs have four-dimensional moduli spaces, yet arise in the above manner only over three- dimensional subvarieties. I present a construction that addresses both issues. We construct families of categories that are deformations of Db(A) over a 4- dimensional algebraic base. Via Bridgeland stability conditions one can obtain every general polarised GKV, for every possible polarisation type of GKVs, as a moduli space of stable objects.

Algebraic cycles and Hitchin systems

Junliang Shen

(joint work with Davesh Maulik, Qizheng Yin)

The topology of the Hitchin system has been intensively studied over decades. It shares rich connections to other subjects, such as the Langlands program, mirror symmetry, non-abelian Hodge theory, and enumerative geometry. The goal of my talk is to discuss motivic aspects of the Hitchin system; the upshot is that, using tools from algebraic geometry and representation theory, we are able to construct many interesting algebraic cycles over the Hitchin moduli space, which govern most topological theories mentioned above. On the one hand, these algebraic cycles allow us to prove several *standard conjectures* for the Hitchin system, like the relative Lefschetz standard conjecture on the algebraicity of the inverse of the Lefschetz operator, and the motivic decomposition conjecture of Corti–Hanamura on lifting the sheaf-theoretic decomposition theorem of Beilinson, Bernstein, Deligne, and Gabber. On the other hand, the motivic theories provide algebraic explanations on mysterious cohomological structures for the Hitchin system, including the P=W conjecture concerning the non-abelian Hodge theory, the χ-independence phenomenon, *etc.* In the following we discuss some aspects of the motivic theory for the Hitchin system in more details.

Let C be a compact Riemann surface of genus $g \geq 2$; let $n > 0, d$ be two coprime integers. We denote by $M_{n,d}$ the moduli space of stable Higgs bundles

$$(\mathcal{E}, \theta), \quad \theta : \mathcal{E} \to \mathcal{E} \otimes \omega_C; \quad \mathrm{rank}(\mathcal{E}) = n, \ \deg(\mathcal{E}) = d.$$

Here the stability is with respect to the slope $\mu := \deg/\mathrm{rank}$. The moduli space $M_{n,d}$ is not compact, but it admits a proper Lagrangian fibration

$$h_d : M_{n,d} \to B_n := \bigoplus_{i=1}^{n} H^0(C, \omega_C^{\otimes i}), \quad (\mathcal{E}, \theta) \mapsto \mathrm{char.poly}(\theta)$$

calculating the characteristic polynomials of the Higgs fields θ. The Hitchin system combines the geometry of the moduli of stable vector bundles (an irreducible component in the central fiber $h_d^{-1}(0_{B_n})$) and the geometry of Jacobian varieties (a general fiber).

The first theorem we present here shows that the decomposition theorem for the Hitchin system $h_d : M_{n,d} \to B_n$ is motivic in a strong sense.

Theorem 1 ([6]). *We have the following results concerning the Hitchin system.*

(a) *(Motivic Decomposition.) There exists a decomposition of the relative diagonal cycle into orthogonal projectors:*

$$[\Delta_{M_{n,d}/B_n}] = \sum_i \mathfrak{r}_i, \qquad \mathfrak{r}_i \circ \mathfrak{r}_i = \mathfrak{r}_i, \quad \mathfrak{r}_i \circ \mathfrak{r}_j = 0, \quad i \neq j,$$

which induces a sheaf-theoretical decomposition in the sense of Beilinson, Bernstein, Deligne, and Gabber. Here $[\Delta_{M_{n,d}/B_n}]$ and $\mathfrak{r}_i$ lie in the rational Chow group of the relative product $M_{n,d} \times_{B_n} M_{n,d}$, and the composition is taken as relative correspondences.

(b). *(Projectors vs Fourier transform.) The projectors $\mathfrak{r}_i$ are (essentially) expressed in terms of the Fourier transform cycle given by the Arinkin sheaf* [1]. *In particular, they detect the tautological classes on the moduli space $M_{n,d}$.*

The part (a) of the theorem confirms Corti–Hanamura's motivic decomposition conjecture for the Hitchin system. Combining (a) and (b) also yields a proof of the P=W conjecture of de Cataldo–Hausel–Migliorini concerning the non-abelian

Hodge theory. Recall that the P=W conjecture states that the perverse filtration associated with the Hitchin system, passing through the non-abelian Hodge correspondence, is matched with the weight filtration associated with the mixed Hodge structure of the character variety. The conjecture was first proven by Maulik–Shen [4] and Hausel–Mellit–Minets–Schiffmann [2] independently via different methods. Both proofs prove P=W by reducing it to proving a P=C match concerning two different cohomological structures for the (same!) Hitchin moduli space $M_{n,d}$ — the perverse filtration P and the Chern filtration C induced by the tautological classes. The interaction between P and C can be seen naturally via Theorem 1: the projectors $\mathfrak{r}_i$ detects the perverse filtration by (a) and govern the tautological classes by (b). We refer to [5, Section 5.4] and [6, Proof of Theorem 0.3] for details. The proof of the P=W conjecture along this path suggests that the interaction between P and C is motivic, and it can be viewed as an extension of the Beauville decomposition for abelian fibrations with singular fibers.

Our second result concerns the χ-independence phenomenon for the Hitchin system. In order to state the result, we need the language of relative Chow motives. We rewrite Theorem 1 (a) as a decomposition for the relative Chow motives over B_n:

$$h\left(M_{n,d}\right) = \bigoplus_i h_i(M_{n,d}) \in \mathrm{CHM}(B_n) \tag{2}$$

with

$$h(M_{n,d}) = (M_{n,d}, [\Delta_{M_{n,d}/B_n}], 0), \quad h_i(M_{n,d}) = (M_{n,d}, \mathfrak{r}_i, 0).$$

Theorem 3 ([6]). *Assume that d, d' are coprime to n. We have for any $i \in \mathbb{Z}$ an isomorphism of relative Chow motives given by (2):*

$$h_i(M_{n,d}) \simeq h_i(M_{n,d'}) \in \mathrm{CHM}(B_n).$$

In particular, the relative motives of the Hitchin system are degree - independent:

$$h(M_{n,d}) = h(M_{n,d'}) \in \mathrm{CHM}(B_n). \tag{4}$$

The moduli space of Higgs bundles can be viewed as the moduli space of 1-dimensional stable sheaves properly supported on the surface T^*C in the curve class $n[C]$, and (4) implies that, in the coprime setting, the decomposition theorem is independent on the choice of the Euler characteristic $\chi\,(= \deg + (1-g)n)$ of the sheaves in the strongest sense. Pushing to a point, (4) recovers a theorem of Hoskins and Pepin Lehalleur [3].

We conclude with discussing ideas of the proofs. Our proofs for both Theorems 1 and 3 concern constructing desired algebraic cycles on the (quite singular) spaces

$$M_{n,d} \times_{B_n} M_{n,d}, \quad M_{n,d} \times_{B_n} M_{n,d'}. \tag{5}$$

There are two different geometric sources of producing algebraic cycles on the spaces like (5) which we used. The first is to view the Hitchin system as an abelian fibration with singular fibers; then Arinkin's extended Fourier-Mukai duality provides natural algebraic cycles. The second is to view $M_{n,d}$ as a global analog of a Lie algebra, where interesting correspondences can be constructed via the Springer theory. Finally, in order to show that the cycles we obtained are "correct" ones which give the desired homological realization, Ngô's support theorem and nearby/vanishing cycles play a crucial role.

References

[1] D. Arinkin, *Autoduality of compactified Jacobians for curves with plane singularities,* J. Algebraic Geom. 22 (2013), no. 2, 363–388.

[2] T. Hausel, A. Mellit, A. Minets, and O. Schiffmann, *$P=W$ via $\mathcal{H}_2$,* arXiv:2209.05429v1.

[3] V. Hoskins, S. Pepin Lehalleur, *Motivic mirror symmetry and χ-independence for Higgs bundles in arbitrary characteristic,* arXiv:2205.15393.

[4] D. Maulik and J. Shen, *The $P=W$ conjecture for* GL_n*,* Ann. of Math., to appear.

[5] D. Maulik, J. Shen, Q. Yin, *Perverse filtrations and Fourier transforms,* arXiv:2308.13160.

[6] D. Maulik, J. Shen, Q. Yin, *Algebraic cycles and Hitchin systems,* arXiv:2407.05177.

On the cohomological dimension of fields of meromorphic functions

Olivier Benoist

The function field $\mathbb{C}(X)$ of an integral algebraic variety of dimension n over $\mathbb{C}$ is a finitely generated field of transcendence degree n over $\mathbb{C}$. Its arithmetic properties impact the geometry of X and are therefore important to investigate.

The complex-analytic analogues of these fields are the fields $\mathcal{M}(S)$ of meromorphic functions on a connected normal complex space S of dimension n. They are not always interesting: they may be reduced to the field $\mathbb{C}$ of constants even if $n > 0$. To ensure that meromorphic functions on S are abundant, one must restrict the class of complex spaces under consideration. Two cases of interest are projective varieties and Stein spaces. If S is projective, then all meromorphic functions on S are algebraic, and one is reduced to the above-mentioned case of the algebraic function fields $\mathbb{C}(X)$.

We henceforth consider the case of *Stein spaces* (analytic analogues of affine varieties, characterized by the vanishing of higher cohomology groups of coherent sheaves). Examples of Stein spaces include all closed complex subspaces of $\mathbb{C}^n$. Their fields of meromorphic functions are much bigger than algebraic function fields, and correspondingly harder to study. A typical example is $\mathcal{M}(\mathbb{C}^n)$: the fraction field of the ring of convergent power series on $\mathbb{C}^n$.

Let F be a field and let $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$ be its absolute Galois group. A basic arithmetic invariant of F is the *cohomological dimension* $\mathrm{cd}(F)$ of F: the largest integer n such that there exists a finite Γ_F-module M with $H^n(\Gamma_F, M) \neq 0$ (or $+\infty$ if there is no upper bound on these integers). It is a measure the arithmetic complexity of F (since Galois cohomology largely controls, for instance, quadratic forms, Brauer classes and torsors under linear algebraic groups over F).

Let X be an integral algebraic variety of dimension n over $\mathbb{C}$. It is a consequence of Tsen's theorem that the field $F = \mathbb{C}(X)$ of rational functions on X has cohomological dimension n (see [8, II.4, Proposition 11]).

The corresponding result in Stein geometry is open in general.

Question 1. *Let S be a connected normal Stein space of dimension n. Is it true that* $\mathrm{cd}(\mathcal{M}(S)) = n$*?*

Question 1 has a positive answer in dimension 1 (i.e. for open Riemann surfaces), thanks to an argument attributed to M. Artin by Gurlanick [5]. Our first main result is a positive answer to Question 1 in dimension 2 (see [2]).

Theorem 2. *Let S be a connected normal Stein space of dimension 2. Then* $\mathrm{cd}(\mathcal{M}(S)) = 2$.

Theorem 2 is already new for $S = \mathbb{C}^2$. We do not know how to extend it in dimensions ≥ 3.

Our second main result answers in all dimensions a variant of Question 1, for germs of meromorphic functions along Stein compact subsets. Recall that a compact subset K in a Stein space S is said to be *Stein* if it admits a basis of Stein open neighborhoods. In this situation, we let $\mathcal{O}(K)$ denote the ring of germs of holomorphic functions in a neighborhood of K (which depends on the germ of S along K). When S is normal and K is connected, the ring $\mathcal{O}(K)$ is an integral domain, and we let $\mathcal{M}(K) := \mathrm{Frac}(\mathcal{O}(K))$ denote the field of germs of meromorphic functions in a neighborhood of K. The next theorem is proved in [1, Theorem 0.4].

Theorem 3. *Let S be a connected normal Stein space of dimension n and let $K \subset S$ be a connected Stein compact subset. Then* $\mathrm{cd}(\mathcal{M}(K)) = n$.

Theorem 3 is already new when $S = \mathbb{C}^n$ and $K \subset S$ is the closed unit ball.

Let us now illustrate the kind of consequences one can draw from Theorems 2 and 3 (and their proofs) by stating a concrete application.

Let F be a field with Brauer group $\mathrm{Br}(F)$. Two important invariants of a class $\alpha \in \mathrm{Br}(F)$ are its period $\mathrm{per}(\alpha)$ (its order in the group $\mathrm{Br}(F)$) and its index $\mathrm{ind}(\alpha)$ (the smallest degree of a finite field extension of F splitting α). The period-index problem aims at controlling the index of a class $\alpha \in \mathrm{Br}(F)$ when its period is known. Its difficulty increases with the arithmetic complexity of F. If F is of cohomological dimension 2, it is often reasonable to expect that $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$ for all $\alpha \in \mathrm{Br}(F)$. This may fail (see Merkurjev's counterexamples [7, §3]), but it holds for many fields of geometric or arithmetic interest, such as function fields of complex algebraic surfaces (de Jong's period-index theorem [4]). Our next result covers the case of fields of meromorphic functions on Stein surfaces.

Theorem 4. *Let S be a connected normal Stein surface. For any $\alpha \in \mathrm{Br}(\mathcal{M}(S))$, one has* $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$.

As the conclusion of Theorem 4 would not hold for all fields of cohomological dimension 2, Theorem 4 is not an immediate consequence of Theorem 2. It is rather an application of the way its proof controls the Galois cohomology of $\mathcal{M}(S)$.

Let us now explain the strategy of the proofs of Theorems 2 and 3. The basic idea is to exploit the weak Lefschetz theorem of Hamm [6]: a Stein space of dimension n has the homotopy type of a CW-complex of dimension $\leq n$, and hence its singular cohomology groups vanish in degree $> n$.

To deduce Theorem 2 and 3, which are vanishing theorems for Galois cohomology, from this vanishing result for singular cohomology, one must bridge the gap between singular and Galois cohomology. The standard tool to do so is étale cohomology. On the one hand, Galois cohomology is a particular case of étale cohomology. On the other hand, M. Artin's comparison theorem shows that the étale cohomology of a complex algebraic variety computes the singular cohomology of its analytification. The only missing piece to complete the proof is therefore an extension of Artin's comparison theorem in Stein geometry (more precisely, in relative algebraic geometry over a base Stein space).

Let S be a Stein space. Let $\mathcal{O}(S)$ be its ring of holomorphic functions. Let X be an $\mathcal{O}(S)$-scheme of finite presentation. Following Bingener [3], we associate with X its analytification $X^{\mathrm{an}} \to S$ which is a complex space over S. In addition, one can associate with any constructible étale sheaf $\mathbb{L}$ on X its analytification $\mathbb{L}^{\mathrm{an}}$ which is a sheaf on X^{an}. Then, for all $k \geq 0$, there is a comparison morphism

$$H^k_{\text{ét}}(X, \mathbb{L}) \to H^k(X^{\mathrm{an}}, \mathbb{L}^{\mathrm{an}}). \tag{5}$$

Ideally, our comparison theorem in Stein geometry would state that the morphisms (5) are always isomorphisms. Unfortunately, this hope is too optimistic. In fact, the morphisms (5) may fail to be injective, or surjective, already when $k = 0$.

Instead, we prove two weaker comparison theorems. The first one is a generic comparison theorem when S has dimension ≤ 2

Proposition 6. *Let S be a reduced Stein space of dimension ≤ 2. Let X be an $\mathcal{O}(S)$-scheme of finite presentation. Let $\mathbb{L}$ be a constructible étale sheaf on X. If one lets $a \in \mathcal{O}(S)$ run over all non-zero divisors, the comparison morphisms*

$$\operatorname*{colim}_a H^k_{\text{ét}}(X_{\mathcal{O}(S)[\frac{1}{a}]}, \mathbb{L}) \to \operatorname*{colim}_a H^k((X_{\mathcal{O}(S)[\frac{1}{a}]})^{\mathrm{an}}, \mathbb{L}^{\mathrm{an}}) \tag{7}$$

are isomorphisms for all $k \geq 0$.

The second one is a comparison theorem over a Stein compact set.

Proposition 8. *Let S be a Stein space. Let X be an $\mathcal{O}(S)$-scheme of finite presentation. Let $\mathbb{L}$ be a constructible étale sheaf on X. If one lets U run over all Stein open neighborhoods of a Stein compact subset K of S, the comparison morphisms*

$$\operatorname*{colim}_{K \subset U} H^k_{\text{ét}}(X_{\mathcal{O}(U)}, \mathbb{L}) \to \operatorname*{colim}_{K \subset U} H^k((X_{\mathcal{O}(U)})^{\mathrm{an}}, \mathbb{L}^{\mathrm{an}})$$

are isomorphisms for all $k \geq 0$.

Propositions 6 and 8 respectively imply Theorems 2 and 3. We do not know how to remove the hypothesis that S has dimension ≤ 2 in Proposition 6. Doing so would yield a positive answer to Question 1.

References

[1] O. Benoist, *Étale cohomology of algebraic varieties over Stein compacta*, preprint 2023, arXiv:2305.06054.

[2] O. Benoist, *On the field of meromorphic functions on a Stein surface*, in preparation.

[3] J. Bingener, *Schemata über Steinschen Algebren*, Schr. Math. Inst. Univ. Münster (2) vol. 10, 1976.

[4] A. J. de Jong, *The period-index problem for the Brauer group of an algebraic surface*, Duke Math. J. **123** (2004), no. 1, 71–94.
[5] R. M. Guralnick, *Matrices and representations over rings of analytic functions and other one-dimensional rings*, Visiting scholars' lectures–1987 (Lubbock, TX), Texas Tech Univ. Math. Ser. vol. 15, Texas Tech Univ., Lubbock, TX, 1988, pp. 15–35.
[6] H. A. Hamm, *Zur Homotopietyp Steinscher Räume*, J. Reine Angew. Math. **338** (1983), 121–135.
[7] A. S. Merkurjev, *Simple algebras and quadratic forms*, Izv. Akad. Nauk. SSR Ser. Mat. **55** (1991), no. 1, 218–224.
[8] J.-P. Serre *Cohomologie galoisienne*, fifth ed., Lecture Notes in Math. vol. 5, Springer-Verlag, Berlin, 1994.

The Calabi problem for Fano threefolds

Kento Fujita
(joint work with Carolina Araujo, Ana-Maria Castravet, Ivan Cheltsov, Anne-Sophie Kaloghiros, Jesus Martinez-Garcia, Constantin Shramov, Hendrik Süß, Nivedita Viswanathan)

Let X be an n-dimensional Fano manifold over the complex number field $\mathbb{C}$. The *Calabi problem* asks that whether X admits a *Kähler–Einstein metric* or not. Recently, it has been known that the problem is equivalent to the *K-polystability* of X, which is purely algebro-geometric. We consider not only K-polystability but also uniform K-stability and K-semistability of X, which are variants of K-polystability. We know the following implications:

$$\text{uniformly K-stable} \Rightarrow \text{K-polystable} \Rightarrow \text{K-semistable}.$$

When $n = 1$, then $X \simeq \mathbb{P}^1$. Thus the Calabi problem for the X is trivially affirmative, since $\mathbb{P}^n$ admits the Fubini–Study metric and is Kähler–Einstein. When $n = 2$, i.e., when X is a del Pezzo surface, then it is classically known that:

$$X \text{ is K-polystable} \Leftrightarrow X \text{ is isomorphic to neither } \mathrm{Bl}_p\,\mathbb{P}^2 \text{ nor } \mathrm{Bl}_{p,q}\,\mathbb{P}^2.$$

Our interest is the case $n = 3$. It is known by Iskovskikh, Mori and Mukai that, there are exactly 105 irreducible families

$$\pi_1 \colon \mathcal{X}_1 \to S_1, \dots, \pi_{105} \colon \mathcal{X}_{105} \to S_{105}$$

of isomorphism classes of smooth Fano threefolds. This means, each π_i is a smooth projective morphism with the relative dimension 3 and $-K_{\mathcal{X}_i/S_i}$ π_i-ample, any isomorphism class of a Fano threefold can be realized as a fiber of some $1 \leq i \leq 105$, each S_i is an irreducible variety, and all two fibers in different bases S_i and S_j cannot be deformed to each other among the category of smooth Fano threefolds. We note that, for each $1 \leq i \leq 105$, the choice of the base space S_i is not unique. In the book [2], we consider the following question:

$$\text{For each } 1 \leq i \leq 105, \text{ is } (\mathcal{X}_i)_s \text{ K-polystable for a general } s \in S_i?$$

The question makes sense: by Blum–Liu–Xu's result, the locus

$$\{s \in S_i \mid (\mathcal{X}_i)_s \text{ is K-polystable}\}$$

of closed points is a constructible set in the Zariski topology. Moreover, it turned out that the answer to the question does not depend on the choice of the base S_i for any $1 \leq i \leq 105$. In fact, we found that the K-polystable locus in S_i is either empty or Zariski dense for each $1 \leq i \leq 105$. Thus, we do not need to worry about the well-definedness of the above question. Here is one of the main results of our book:

Theorem 1 ([2])**.** *Exactly* 78/105 *(resp.,* 51/105*,* 79/105*) general members are K-polystable (resp., uniformly K-stable, K-semistable).*

In my talk in Oberwolfach, I explained an effective technique for showing the K-polystability of given Fano manifolds. The technique consists of 4 numbers of theories:

(1) G-equivariant version [7] for a valuative criterion [4, 6] of K-stability,
(2) an estimation via the local α_G-invariant,
(3) Abban–Zhuang's method [1], and
(4) the theory of quasi-log subadjunction [3, 5].

I explained the above theories and demonstrated how to apply the techniques to Fano manifolds, by showing the K-polystability of the blowup X of $\mathbb{P}^3$ along the *Bring curve*, the non-hyperelliptic curve of genus 4 with the largest automorphism group.

References

[1] H. Abban and Z. Zhuang, K-stability of Fano varieties via admissible flags, Forum Math. Pi **10** (2022), Paper No. e15, 43 pp.

[2] C. Araujo, A-M. Castravet, I. Cheltsov, K. Fujita, A-S. Kaloghiros, J. Martinez-Garcia, C. Shramov, H. Süss and N. Viswanathan, The Calabi problem for Fano threefolds, London Mathematical Society Lecture Note Series, **485**. Cambridge University Press, Cambridge, 2023.

[3] O. Fujino, *Cone theorem and Mori hyperbolicity*, arXiv::2102.11986v1.

[4] K. Fujita, A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties, J. Reine Angew. Math. **751** (2019), 309–338.

[5] K. Fujita, *On Fano threefolds of degree* 22 *after Cheltsov and Shramov*, Birational geometry, Kähler–Einstein metrics and degenerations, 249–262. Springer Proc. Math. Stat., **409**, Springer, Cham, 2023.

[6] C. Li, K-semistability is equivariant volume minimization, Duke Math. J. **166** (2017), no. 16, 3147–3218.

[7] Z. Zhuang, *Optimal destabilizing centers and equivariant K-stability*, Invent. Math. **226** (2021), no. 1, 195–223.

Moduli of sheaves on HK varieties of type $K3^{[2]}$

Kieran G. O'Grady

Moduli spaces of semistable sheaves on (complex) smooth projective symplectic surfaces, i.e. $K3$ or abelian surfaces, have played an important rôle in Geometry, see [2, 3, 4, 7, 5, 8, 6]. Such moduli spaces give (eventually after suitable desingularizations) all known deformation classes of Irreducible Symplectic Varieties

(ISV's), also known as hyperkähler (HK) manifolds (for us the two names are synonymous even though hyperkähler could refer to an ISV together with a choice of a Kähler class or of a Ricci-flat metric). Actually the first examples of ISV's of type OG10 and type OG6 were constructed as crepant desingularizations of singular moduli spaces of sheaves, and for a long time no other explicit construction of varieties in those deformation classes was known. Question: do analogous results hold for moduli spaces of semistable sheaves on higher dimensional hyperkähler (HK) manifolds? We present results that go towards giving a positive answer.

Let (X,h) be a polarized HK variety (polarizations are always primitive) of type $K3^{[2]}$, and let

$$w=(r,l,s)\in \mathbb{N}_{+}\oplus \mathrm{NS}(X)\oplus H^{2,2}_{\mathbb{Z}}(X). \tag{1}$$

Let $M_w(X,h)$ be the moduli space of h slope stable vector bundles F such that

$$w(F):=(r(F),c_1(F),\Delta(F))=w, \tag{2}$$

where $\Delta(F):=-2r(F)\,\mathrm{ch}_2(F)+\mathrm{ch}_1(F)^2$ is the discriminant of F. Then $M_w(X,h)$ is a scheme of finite type over $\mathbb{C}$ by a classical result [1] .

For $a,r_1\in\mathbb{N}_{+}$ we let

$$w:=ar_1\left(2r_1,\frac{2}{\mathrm{div}(h)}h,\frac{ar_1^3c_2(X)}{3}\right). \tag{3}$$

Theorem 4. *Let (X,h) be a polarized HK variety of type $K3^{[2]}$. Let r_1 be a positive integer, even if $\mathrm{div}(h)=1$ and odd if $\mathrm{div}(h)=2$. Let a be a positive integer greater than 1 and such that $r_1|2a$. Suppose in addition that*

$$q_X(h)\equiv\begin{cases}-2 \pmod{2r_1} & \text{if } r_1\equiv 0 \pmod 4,\\ -2r_1-8 \pmod{8r_1} & \text{if } r_1\equiv 1 \pmod 4,\\ -r_1-2 \pmod{2r_1} & \text{if } r_1\equiv 2 \pmod 4,\\ 2r_1-8 \pmod{8r_1} & \text{if } r_1\equiv 3 \pmod 4.\end{cases} \tag{5}$$

Then the moduli space $M_w(X,h)$ is non empty, and for (X,h) general it has an irreducible component of dimension $2a^2+2$.

Theorem 6. *Let (X,h) be a general polarized HK variety of type $K3^{[2]}$ with $\mathrm{div}(h)=1$ and $q_X(h)\equiv 0 \pmod 8$, and let*

$$w:=\left(8,4h,\frac{16c_2(X)}{3}\right). \tag{7}$$

The moduli space $M_w(X,h)$ contains a connected component $M_w(X,h)^\bullet$ which is a HK variety of type $K3^{[2]}$. Let $L^\bullet$ be the restriction to $M_w(X,h)^\bullet$ (eventually divided by a suitable natural number so that it becomes primitive) of the GIT polarization of $M_w(X,h)$. Then $\mathrm{div}(L^\bullet)=1$, $q_X(L^\bullet)=\frac{1}{4}q_X(h)$ and the rational map

$$\begin{array}{ccc}\mathcal{K}^1_{8d} & \dashrightarrow & \mathcal{K}^1_{2d}\\ [(X,h)] & \mapsto & [(M_w(X,h)^\bullet,L^\bullet)]\end{array} \tag{8}$$

is dominant of finite degree.

References

[1] Masaki Maruyama, *On boundedness of families of torsion free sheaves*, J. Math. Kyoto Univ., **21** (1981), 673–701.

[2] Shigeru Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math., **77** (1984), 101–116.

[3] Shigeru Mukai, *On the moduli space of bundles on K3 surfaces. I*, Vector bundles on algebraic varieties (Bombay, 1984), Tata Inst. Fund. Res. Stud. Math., **11** (1987), 341–413.

[4] Kieran G. O'Grady, *The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface*, J. Algebraic Geom. **6** (1997), 599–644.

[5] Kieran G. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, J. Reine Angew. Math. **512** (1999), 49–117.

[6] Kieran G. O'Grady, *A new six-dimensional irreducible symplectic variety*, J. Algebraic Geom. **12** (2003), 435–505.

[7] Kota Yoshioka, *Some examples of Mukai's reflections on K3 surfaces*, J. Reine Angew. Math. **515** (1999), 97–123.

[8] Kota Yoshioka, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann. **321** (2001), 817–884.

Gong talk
A less strange duality

Svetlana Makarova

Strange duality, also known as the rank-level duality, can be understood in algebraic geometry as an isomorphism between the space of global sections, and the dual of such, of two natural line bundles on moduli spaces of sheaves. So far, this phenomenon has been studied largely on a case by case basis. This talk is an attempt to present an abstract construction that brings together known cases.

The philosophy that I suggest is the following. Fix an abelian k-linear category $\mathcal{C}$, where k is a field, and $\mathcal{C}$ is assumed to be Ext-finite. Denoting $\mathcal{A} := \mathrm{Ind}(\mathcal{C})$, we further assume that $\mathcal{A}$ is locally noetherian and adically complete. Let $\mathcal{M}$ denote the moduli stack of f.p. objects in $\mathcal{A}$, as described in §7 of [1]. It follows from E3.1 in [2] that $\mathcal{M}$ is algebraic and l.f.t. Given two algebraic K-theory classes $\alpha, \beta \in \mathrm{K}_0(\mathcal{C})$, we can consider the stacks of objects of these classes, denoted $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$. We denote by $\mathcal{A}_{\mathcal{M}_\alpha}$ the base change of $\mathcal{A}$ to $\mathcal{M}_\alpha$, by $p^*\colon \mathcal{A} \to \mathcal{A}_{\mathcal{M}_\alpha}$ the natural pullback functor, and by $\mathcal{U}_\alpha$ the universal object in $\mathcal{A}_{\mathcal{M}_\alpha}$; and similarly for β. We then define the *determinantal line bundles* $\Theta^\alpha := \det \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{A}_{\mathcal{M}_\beta}}(p^*\alpha, \mathcal{U}_\beta)^\vee$ and $\Theta_\beta := \det \mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{A}_{\mathcal{M}_\alpha}}(\mathcal{U}_\alpha, p^*\beta)^\vee$. A line bundle gives rise to a notion of stability on points of a stack. From now on, we assume that the Euler characteristic $\langle \alpha, \beta \rangle = 0$. Then one can check that Θ^α- and Θ_β-stability correspond to appropriate notions of King's stability for abelian categories. The next observation is the following. If $\forall V \in \mathcal{M}_\alpha^{\Theta_\beta\text{-ss}}$ and $\forall W \in \mathcal{M}_\beta^{\Theta^\alpha\text{-ss}}$: $\mathrm{Ext}^{>1}(V, W) = 0$, then there is a natural section σ of $\Theta_\beta \boxtimes \Theta^\alpha$. The strange duality morphism is induced by σ and is denoted $\mathrm{SD}\colon \mathrm{H}^0(\mathcal{M}_\alpha^{\Theta_\beta\text{-ss}}, \Theta_\beta)^\vee \to \mathrm{H}^0(\mathcal{M}_\beta^{\Theta^\alpha\text{-ss}}, \Theta^\alpha)$.

In [3], the authors prove that SD is an isomorphism for the category $\mathcal{C}$ of finite-dimensional representations of an acyclic quiver. However, SD fails to be an isomorphism for $\mathcal{C} = \mathrm{Coh}(C)$, where C is a curve, because the dimensions differ. A possible remedy could be taking one of the vectors from topological $\mathrm{K}_0^{\mathrm{t}}(\mathcal{C})$. Indeed, in [4], the authors prove that SD is an isomorphism over curves, with $\alpha \in \mathrm{K}_0(C)$ and $\beta \in \mathrm{K}_0^{\mathrm{t}}(C)$.

References

[1] J. Alper, D. Halpern-Leistner, J. Heinloth, *Existence of moduli spaces for algebraic stacks*, Invent. Math **234** (2023), 949–1038.

[2] M. Artin, J. Zhang, *Abstract Hilbert schemes*, Algebr. Represent. Theory **4(4)** (2001), 305–394.

[3] H. Derksen, J. Weyman, *Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients*, J. Amer. Math. Soc. **13** (2000), 467–479.

[4] A. Marian, D. Oprea, *The level-rank duality for non-abelian theta functions*, Invent. Math. **168** (2007), 225–247.

Gong talk

Bounding Brauer groups using moduli spaces

Emma Brakkee

(joint work with D. Bragg, A. Várilly-Alvarado)

Let X be a K3 surface over a number field K. The algebraic Brauer group of X is $\mathrm{Br}_1(X) := \ker(\mathrm{Br}(X) \to \mathrm{Br}(X_{\overline{K}}))$. We consider the quotient

$$\mathrm{Br}'(X) := \mathrm{Br}(X)/\mathrm{Br}_1(X).$$

By [6, Theorem 1.2], the group $\mathrm{Br}'(X)$ is finite. Conjectures by, among others, Várilly–Alvarado (see e.g. [5]) predict that the size of $\mathrm{Br}'(X)$ is uniformly bounded. We study the following question:

Question 1. *Let p be a prime number. Is there an upper bound for the size of the p-primary torsion $\mathrm{Br}'(X)[p^\infty]$, for all K3 surfaces X over K?*

It follows from [3, Theorem 1.2.1] that this is true when restricting to K3 surfaces in one-dimensional families. We aim to give a different proof of this result for specific one-dimensional families of K3 surfaces, namely, K3 surfaces with a polarization by a fixed lattice (see e.g. [4]).

Theorem 2. *Let M be a lattice of signature $(1, 18)$. There exists a constant B depending on K, M and p, such that*

$$|\mathrm{Br}'(X)[p^\infty]| \le B$$

for all K3 surfaces X over K admitting an ample polarization $M \hookrightarrow \mathrm{Pic}(X_{\overline{K}})$.

The main idea of the proof is the following. Generalizing [2] and [1], we construct moduli spaces $\mathcal{F}_M^{p^r}$ parametrizing K3 surfaces with an M-polarization and (for very general points) a Brauer class of order p^r. We show that $\mathcal{F}_M^{p^r}(\mathbb{C})$ is a finite union of curves which when r is large, have genus bigger than 1 and hence finitely many K-points. Roughly, these correspond to K3 surfaces $X_1, \ldots, X_m$ over K, and the bound B is $\max\{|\mathrm{Br}'(X_i)[p^\infty]|\}_i$.

References

[1] D. Bragg, *Lifts of twisted K3 surfaces to characteristic 0*, Int. Math. Res. Not. **2023** (2023), no. 5, 4337–4407

[2] E. Brakkee, *Moduli spaces of twisted K3 surfaces and cubic fourfolds*, Math. Ann. **377** (2020), no. 3-4, 1453–1479

[3] A. Cadoret and F. Charles, *A remark on uniform boundedness for Brauer groups*, Algebr. Geom. **7** (2020), no. 5, 512–522

[4] I.V. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces*, J. Math. Sci. **81** (1996), no. 3, 2599–2630

[5] M. Orr, A.N. Skorobogatov and Y.G. Zarhin, *On uniformity conjectures for abelian varieties and K3 surfaces*, Amer. J. Math. **143** (2021), no. 6, 1665–1702

[6] A.N. Skorobogatov, and Y.G. Zarhin, *A finiteness theorem for the Brauer group of abelian varieties and K3 surfaces*, J. Algebraic Geom. **17** (2008), no. 3, 481–502

Gong talk
Moduli of sheaves and deformation to the normal cone

Yifan Zhao

Given a scheme X and a closed subscheme Y of X, denote by I the ideal sheaf of Y. The deformation of X to the normal cone of Y in X, is defined to be

$$\mathcal{X} := Spec_X(\oplus_{n\in\mathbb{Z}} t^n I^{-n}),$$

where we set I^i to be $\mathcal{O}_X$ if $i \leq 0$. The inclusion

$$\mathbb{C}[t] \to \oplus_{n\in\mathbb{Z}} t^n I^{-n}$$

induces a surjective morphism $\mathcal{X} \to \mathbb{C}$ and makes $\mathcal{X}$ a flat $\mathbb{C}$ family of schemes.

In [1], the authors considered the case of X being a $K3$ surface S and Y being a smooth integral curve C in S. They studied the corresponding flat $\mathbb{C}$ family $\mathcal{S} := Spec_S(\oplus_{n\in\mathbb{Z}} t^n I^{-n})$ of quasi-projective surfaces and the relative moduli space (constructed in [2]) of compactly-supported one-dimensional semi-stable sheaves on the $\mathbb{C}$ family $\mathcal{S}$. Here we ask the first Chern class of these sheaves to be $n[C]$ for some positive integer n and that no such sheaf is strictly semi-stable.

Theorem 1 ([1])**.** *There exists a flat $\mathbb{C}$ family of varieties: $\mathcal{M}_{\mathcal{S}/\mathbb{C}} \to \mathbb{C}$, with the generic fibre over $t \neq 0 \in \mathbb{C}$ isomorphic to M_S: the moduli of one-dimensional sheaves on S, and the special fibre over $0 \in \mathbb{C}$ isomorphic to M_{T^*C}: the moduli of compactly-supported one-dimensional sheaves on the cotangent bundle T^*C.*

Moreover, the map $Supp : \mathcal{M}_{\mathcal{S}/\mathbb{C}} \to Hilb_{\mathcal{S}/\mathbb{C}}(nC)$, sending a sheaf F to its support in the relative Hilbert scheme $Hilb_{\mathcal{S}/\mathbb{C}}(nC)$, restricts respectively to the Beauville–Mukai system and the Hitchin system over the generic and special fibres.

Suppose M_S and M_{T^*C} are smooth, our main result relates $\mathcal{M}_{\mathcal{S}/\mathbb{C}}$ to another deformation to the normal cone:

Theorem 2. *Let M_C be the closed subvariety of M_S consisting of stable sheaves supported scheme-theoretically on C. The deformation of M_S to the normal cone of M_C in M_S is isomorphic to an open dense subset of the relative moduli $\mathcal{M}_{\mathcal{S}/\mathbb{C}}$.*

References

[1] R. Donagi, L. Ein and R. Lazarsfeld, *Nilpotent cones and sheaves on K3 surfaces*, Birational algebraic geometry, Contemp. Math., 207 AMS(1996), 51–61.

[2] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 47–129.

Gong talk
Quotients of Hitchin systems

Riccardo Carini
(joint work with Roland Abuaf)

Given a smooth curve C of genus $g \geq 2$, the moduli space of semistable GL(2)-Higgs bundles of degree zero $M_{2,0}$ and that of semistable SL(2)-Higgs bundles $M_{2,\mathcal{O}_C}$ are instances of *singular* holomorphic symplectic varieties in the sense of Beauville [2]. Via the spectral correspondence [8, 3], Higgs bundles can be regarded as one-dimensional sheaves on the symplectic surface T^*C. As a consequence, it turns out these moduli spaces look very much like moduli spaces of Gieseker-semistable sheaves on compact symplectic surfaces (see also [4]). For instance, we have the following classification [9] – to be compared to the case of M_{2v} where v is a primitive Mukai vector with $v^2 = 2g-2$ on a K3 surface:

- When $g = 2$, $M_{2,0}$ and $M_{2,\mathcal{O}_C}$ are singular symplectic varieties of dimension 10 and 6 with a crepant resolution given by blowing-up the reduced singular locus (compare with [7, 6]).
- When $g \geq 3$ then $M_{2,0}$ and $M_{2,\mathcal{O}_C}$ are singular symplectic varieties with no crepant resolution (compare with [5]).

When C is hyperelliptic of genus 3, we show we can get a crepant resolution of such moduli spaces after passing to a quotient by a symplectic involution, thus producing examples of holomorphic symplectic manifolds from singular moduli spaces with terminal singularities [1]:

Theorem 1. *Let C be a genus 3 hyperelliptic curve and denote by $\sigma \in \mathrm{Aut}(C)$ the hyperelliptic involution. Then the assignment*

$$(E, \phi) \mapsto (\sigma^* E^*, -\sigma^* \phi^t) \tag{2}$$

defines a symplectic involution on $M_{2,0}$ preserving $M_{2,\mathcal{O}_C}$. The quotients of $M_{2,0}$ and $M_{2,\mathcal{O}_C}$ by (2) are singular symplectic varieties of dimension 18 and 12 with a crepant resolution given by blowing-up the reduced singular locus.

References

[1] R. Abuaf and R. Carini, *Holomorphic symplectic manifolds from semistable Higgs bundles*, arXiv:2406.00395 [math.AG], 2024.

[2] A. Beauville, *Symplectic singularities*, Invent. Math. **139** (2000), no. 3, 541–549.

[3] A. Beauville, M. S. Narasimhan, and S. Ramanan, *Spectral curves and the generalised theta divisor*, J. Reine Angew. Math. **398** (1989), 169–179.

[4] R. Donagi, L. Ein, and R. Lazarsfeld, *Nilpotent cones and sheaves on K3 surfaces*, In: *Birational algebraic geometry (Baltimore, MD, 1996)*, Contemp. Math., Vol. 207, Amer. Math. Soc., Providence, RI, 1997, 51–61.
[5] D. Kaledin, M. Lehn, and C. Sorger, *Singular symplectic moduli spaces*, Invent. Math. **164** (2006), no. 3, 591–614.
[6] K. G. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, J. Reine Angew. Math. **512** (1999), 49–117.
[7] K. G. O'Grady, *A new six-dimensional irreducible symplectic variety*, J. Algebraic Geom. **12** (2003), no. 3, 435–505.
[8] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. II*, Inst. Hautes Études Sci. Publ. Math. **80** (1994), 5–79.
[9] A. Tirelli, *Symplectic resolutions for Higgs moduli spaces*, Proc. Amer. Math. Soc. **147** (2019), no. 4, 1399–1412.

Gong talk
Flop, flop and scrolls

Lisa Marquand
(joint work with Corey Brooke, Sarah Frei, Xuqiang Qin)

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. The Fano variety of lines $F(X)$ is a hyperkähler manifold of dimension four. The locus of cubics containing a rational normal cubic scroll contains many geometrically interesting loci, such as cubics containing an involution (see [3]), and were first studied in [2]. In [1], we study the birational geometry of $F(X)$ when the cubic X contains two non-homologous rational normal cubic scrolls. Any pair of cubic scrolls spanning different hyperplane sections of X is either **syzygetic** or **non-syzygetic**, meaning they intersect in three or one points, respectively. Further, such a cubic fourfold is conjecturely irrational [3, Theorem 1.2], and one could hope this is reflected in the birational geometry of the associated hyperkählers.

We enumerate and identify the non-isomorphic hyperkähler birational models of $F(X)$ in both the syzygetic and non-syzygetic case. In both, $F(X)$ has Picard rank three, and we identify the movable and ample cones of $F(X)$.

We prove the following result in the syzygetic case:

Theorem 1. *Let X be a general cubic fourfold containing a syzygetic pair of cubic scrolls, and let $F := F(X)$. Then F has* ***five*** *isomorphism classes of birational hyperkähler models, represented by the following: F itself and four non isomorphic Mukai flops of F, each corresponding to a Lagrangian plane in F. Moreover, each of the four Mukai flops of F can be realized as a double EPW sextic.*

A similar result holds in the non-syzygetic case. Further, we obtain the following structural result for the group of birational transformations $\mathrm{Bir}(F)$:

Theorem 2. *Let X be a general cubic fourfold containing a syzygetic or non-syzygetic pair of cubic scrolls, and let $F := F(X)$. Then:*

(1) *The infinite order group $\mathrm{Bir}(F)$ is generated by the covering involutions of the double EPW sextics obtained as Mukai flops of F.*

(2) *In particular, in the syzygetic case we have:*

$$\mathrm{Bir}(F) \cong \langle a, b, c, d \rangle / \langle a^2, b^2, c^2, d^2 \rangle.$$

References

[1] C. Brooke, S. Frei, L. Marquand, X. Qin, *Birational geometry of Fano varieties of lines on cubic fourfolds containing pairs of cubic scrolls*, (2024), https://arxiv.org/abs/2407.18904
[2] B. Hassett, Y. Tschinkel, *Flops on holomorphic symplectic fourfolds and determinantal cubic hypersurfaces*, J. Inst. Math. Jussieu, **9** (2010) 125–153
[3] L. Marquand, *Cubic fourfolds with an involution*, Trans. Amer. Math. Soc., **376** (2023), 1373–1406.

Gong talk

Shafarevich–Tate groups of Lagrangian fibrations

Anna Abasheva

Let $\pi\colon X \to B$ be a Lagrangian fibration on an irreducible holomorphic symplectic manifold X. In [1] we defined the *Shafarevich–Tate group* Ш of π. It parametrizes *Shafarevich–Tate twists* of X, namely, fibrations $\pi^\phi\colon X^\phi \to B$ isomorphic to $\pi\colon X \to B$ locally over B.

We showed in [1] that the Shafarevich–Tate group is an extension

$$0 \to Ш^0 \to Ш \to Ш/Ш^0 \to 0,$$

where $Ш^0$ is the connected component of unity of Ш, and $Ш/Ш^0$ satisfies:

$$(Ш/Ш^0) \otimes \mathbb{Q} \simeq H^2(B, R^1\pi_*\mathbb{Q})$$

Question 1. *When is a Shafarevich–Tate twist of a hyperkähler manifold Kähler?*

I partially answer this question in my upcoming work.

Theorem 2. *Pick a class $\phi \in Ш$ such that $r\phi$ lies in $Ш^0$ for some positive integer r. Then the twist X^ϕ is Kähler.*

For the next theorem note that the differential d_2 on the second page of the Leray spectral sequence of $\mathbb{Q}_X$ maps $H^0(R^2\pi_*\mathbb{Q})$ to $H^2(R^1\pi_*\mathbb{Q}) \simeq (Ш/Ш^0)\otimes\mathbb{Q}$. Denote by $\overline{\phi}$ the image of ϕ in $H^2(R^1\pi_*\mathbb{Q})$.

Theorem 3. (1) *If $\overline{\phi} \in \operatorname{im} d_2$, then $b_2(X^\phi) = b_2(X)$ and there is a cohomology class $h \in H^2(X^\phi)$ which restricts to an ample class on a smooth fiber.*

(2) *If $\overline{\phi} \notin \operatorname{im} d_2$, then $b_2(X^\phi) = b_2(X) - 1$. In this case all cohomology classes $h \in H^2(X^\phi)$ restrict trivially to a smooth fiber. In particular, X^ϕ is not Kähler.*

References

[1] A. Abasheva, V.Rogov *Shafarevich–Tate groups of holomorphic Lagrangian fibrations*, arXiv:2112.10921 (2021).

Gong talk

Stability conditions on surfaces from contractions of many rational curves

Nicolás Vilches Reyes

The existence of stability conditions is an exciting and difficult problem. On dimension two, it was proven by Arcara and Bertram that the derived category of a smooth surface admits plenty of them. More precisely, they produce a collection $\{\sigma_{\beta,\omega}\}$ of stability conditions depending in numerical classes β, ω, where ω is ample. All of them lie in the *geometric chamber* of the stability manifold.

A natural question is to produce more stability conditions for some surfaces, by studying degenerations of the examples above. This was done in [2, 3], by looking at the pullback of an ample class by a map $f\colon S \to T$ contracting a single rational curve. In particular, either T is smooth or admits a single $1/n(1,1)$ singularity.

The goal of this talk is to discuss a work in progress where stability conditions are produced for T mildly singular. This includes canonical singularities and many cyclic quotient singularities, possibly many of them. More importantly, the techniques are different to the previous known results, by noting that the Arcara–Bertram construction admits a common (twisted) heart. This should help us get a better understanding of the boundary of the geometric chamber.

References

[1] D. Arcara, A. Bertram, *Bridgeland-stable moduli spaces for K-trivial surfaces*, J. Eur. Math. Soc. (JEMS) **1** (2013), 1–38, MR 2998828.

[2] Y. Toda, *Stability conditions and extremal contractions*, Math. Ann. **2** (2013), 631–685, MR 3096520.

[3] R. Tramel, B. Xia, *Bridgeland stability conditions on surfaces with curves of negative self-intersection*, Adv. Geom. **3** (2022), 383–408, MR 4453506.

A modular construction of OG10

Alessio Bottini

Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold. Its variety of lines $X = F(Y)$ is a hyper-Kähler manifold of type $\mathrm{K3}^{[2]}$. The locus of lines contained in a hyperplane section Y_H is a Lagrangian surface $F(Y_H) \subset F(Y)$. If Y is taken general, all these surfaces are integral.

Let M be the moduli space of semistable sheaves on X parametrising torsion-free sheaves of rank 1 and degree 0 supported on the Lagrangian surfaces $F(Y_H)$. Let M_0 be the irreducible component containing those sheaves which are locally free on their support. It is equipped with a support morphism $M_0 \to |\mathcal{O}_Y(1)|$, whose fiber over a hyperplane section Y_H is the compactified Picard scheme $\overline{\mathrm{Pic}}^0(F(Y_H))$.

It is not difficult to see that a line bundle on $F(Y_H)$ is a smooth point of M, and that M has a natural symplectic structure near these points. We wish to prove that this holds true even for non locally free sheaves, more precisely:

Theorem 1. *M_0 is a connected component of M, and it is a smooth hyper-Kähler manifold of type OG10. Moreover, there exists another hyper-Kähler manifold X' of type $\mathrm{K3}^{[2]}$, together with a derived equivalence*

$$\Phi : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(X', \theta), \quad \theta \in \mathrm{Br}(X'),$$

mapping M_0 isomorphically to a moduli space parametrising only slope stable (twisted) vector bundles on X'.

From Lagrangians to vector bundles To find X' and the derived equivalence Φ, we take X to be very general in certain Noether–Lefschetz divisors. One can find an associated polarized K3 surface (S, H) of degree 2, and we let X' be the relative compactified Picard scheme of the linear system $|H|$. The variety of lines X is equipped with a Lagrangian fibration, which is a Tate–Shafarevich twist of the support morphism $X' \to |H|$, as explained in [O'G22, Proposition B.4].

Arinkin [Ari13] constructs a relative Poincaré sheaf $\mathcal{P} \in \mathrm{D}^b(X' \times_{|H|} X')$, which gives an autoequivalence $\Phi : \mathrm{D}^b(X') \xrightarrow{\sim} \mathrm{D}^b(X')$. Our equivalence $\Phi : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(X', \theta)$ is given by a deformation of the Poincaré sheaf, and the Brauer class $\theta \in \mathrm{Br}(X')$ represents exactly the obstruction to reglue the Poincaré sheaf along the Tate–Shafarevich twist. By the same proof of [Bot22, Proposition 1.2], we get the following.

Theorem 2. *The equivalence Φ sends Cohen–Macaulay sheaves on $F(Y)$ to locally free twisted sheaves on X'.*

Therefore, to prove the first part of Theorem 1, it suffices to show that all sheaves in M_0 are Cohen–Macaulay. For this, the key idea is to "reduce to curves". One shows that there is a regular embedding

$$F(Y_H) \subset \overline{\mathrm{Prym}}(C/D) \subset \overline{\mathrm{Pic}^0}(\overline{\mathrm{Pic}^0}(C)),$$

where $C \to D$ is an étale $2:1$ cover of integral curves with planar singularities. This generalizes to the singular case the Albanese embedding of $F(Y_H)$, see [Huy23, Section 5.3]. Lastly, pulling back along this embedding gives a surjection

$$\overline{\mathrm{Pic}^0}(\overline{\mathrm{Pic}^0}(C)) \to \overline{\mathrm{Pic}^0}(F(Y_H)),$$

which preserves the property of being CM. We conclude using that torsion free sheaves on compactfied Jacobians of planar curves are CM by [Ari13].

The symplectic form. As mentioned before, it is not difficult to see that M is smooth and symplectic near line bundles on $F(Y_H)$. Surprisingly, in our case the symplectic form exists, at least at the level of tangent spaces, even if (a priori) the point is not smooth. This relies on the following result, which follows from [Ver96, Theorem 4.2A].

Theorem 3. *Let X be a HK manifold of dimension $2n$, and denote by $\sigma \in H^{2,0}(X)$ the symplectic form. Let E be a projectively hyperholomorphic locally free sheaf. Then the pairing*

$$\mathrm{Ext}^1(E,E) \times \mathrm{Ext}^1(E,E) \to \mathbb{C}, \quad (a,b) \mapsto \mathrm{Tr}(a \circ b \cup \overline{\sigma}^{n-1})$$

is symplectic.

A vector bundle E on a HK manifold is projectively hyperholomorphic if it is slope stable and satisfies a certain numerical condition on the discriminant. They were first introduced by Verbitsky [Ver96], who studied properties of their moduli spaces, but only recently, thanks to the works of O'Grady [O'G22], Markman [Mar23] and Beckmann [Bec22], new methods arose to construct examples.

Namely, one can "categorify" the notion of hyperholomorphic bundles, to that of atomic object in $\mathrm{D}^b(X)$. Atomicity is a numerical condition invariant under derived equivalences, easy to check on sheaves supported on Lagrangians. In our case, it turns out that $\mathcal{O}_{F(Y_H)} \in \mathrm{D}^b(X)$ is indeed an atomic object, hence the same holds for its image under Φ.

An atomic vector bundle, at least on a HK of type $\mathrm{K3}^{[2]}$, is projectively hyperholomorphic provided it is slope stable. Moreover, results of O'Grady show that slope stability for atomic bundles behaves formally as slope stability on surfaces, Using this, we are able to show that all sheaves in M_0 are mapped under Φ to slope stable (twisted) locally free sheaves.

Smoothness of the moduli space. Consider the universal family $\mathcal{E} \to X \times M_0$, which is a locally free twisted sheaf. From Grothendieck–Verdier duality relative to the second projection we get

$$(*) \qquad \mathsf{R}\pi_{M,*}(\mathcal{E}nd(\mathcal{E}))[4] \cong (\mathsf{R}\pi_{M,*}(\mathcal{E}nd(\mathcal{E})))^\vee.$$

This isomorphism combined with the symplectic form

$$- \cup \overline{\sigma} : \mathsf{R}\pi_{M,*}(\mathcal{E}nd(\mathcal{E})) \to \mathsf{R}\pi_{M,*}(\mathcal{E}nd(\mathcal{E}))[2]$$

put enough restriction on the moduli space to prove smoothness.

Unfortunately, at the moment the proof heavily relies on the geometry of the situation, particularly on [LSV17], and more precisely on the smoothness of the relative compactified Jacobian for a family of curves [FGvS99]. In any case, at last in principle one can hope that a more profound understanding the interaction between the symplectic form and (*) allows for a proof which does not require a geometric input.

References

[Ari13] Arinkin, D., Autoduality of compactified Jacobians for curves with plane singularities, *J. Algebraic Geom.* **22** (2013), 363–388.

[Bec22] Beckmann, T., Atomic objects on hyper-Kähler manifolds, eprint `arXiv:2202.01184`.

[Bot22] Bottini, A., Towards a modular construction of OG10, to appear in *Comp. Math.* (2024).

[FGvS99] Fantechi, B., Göttsche, L., van Straten, D., Euler number of the compactified Jacobian and multiplicity of rational curves, *J. Algebraic Geom.* **8** (1999), 115–133.

[Huy23] Huybrechts, D., The geometry of cubic hypersurfaces, *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 2023.

[LSV17] Laza, R., Saccà, G., Voisin, C., A hyper-Kähler compactification of the intermediate Jacobian fibration associated with a cubic 4-fold, *J. Acta Math.* **218** (2017), 55–135.

[Mar23] Markman, E., Stable vector bundles on a hyper-Kähler manifold with a rank 1 obstruction map are modular, to appear in *Kyoto J. Math.* `arXiv:2107.13991`.

[O'G22] O'Grady, K., Modular sheaves on hyperkähler varieties, *Algebr. Geom.* **9** (2022), 1–38.

[Tae21] Taelman, L., Derived equivalences of hyperkähler varieties, *Geom. Topol.*, **27** (2023), 2649–2693.

[Ver96] Verbitsky, M., Hyperholomorphic bundles over a hyper-Kähler manifold, *J. Algebraic Geom.*, **5** (1996), 633–669.

Automorphisms of Calabi–Yau threefolds from algebraic dynamics and the second Chern class

Keiji Oguiso

Throughout this report, X is a normal projective variety of dimension d over $\mathbf{C}$ and $f \in \operatorname{Bir} X$. This report consists of more speculations, rather than definite results, on automorphisms of strict Calabi–Yau threefolds in view of the following question after Dinh, Sibony and D.-Q. Zhang (See [6], [7] and [8] and references therein for background, variants and relevant known results):

Question 1. *Find many examples of smooth X admitting a* primitive *biregular automorphisms of* positive entropy*. (See below for definitions of some terms.)*

Definition 2. *Let H be a very ample Cartier divisor on X. Then, the pth dynamical degree of $f \in \operatorname{Bir} X$ is defined by*

$$d_p(f) := \lim_{n\to\infty}((f^n)^*(H^p).H^{d-p})_X^{1/n} := \lim_{n\to\infty}(q_n^*(H^p).p_n^*(H^{d-p}))_{X_n}^{1/n} \geq 1,$$

where $p_n : X_n \to X$ is a birational morphism from a smooth projective variety X_n such that $q_n : X_n \to X$ is a morphism and $f^n = q_n \circ p_n^{-1}$ (Hironaka's resolution). Let $\pi : X \dashrightarrow B$ be a dominant rational map to a normal projective variety B with $f_B \in \operatorname{Bir} B$ such that $f_B \circ \pi = \pi \circ f$. In this case, the relative pth dynamical degree is defined by

$$d_p(f|\pi) = \lim_{n\to\infty}(f^n(H^p).H^{d-e}.\pi^*(H_B^e))_X^{1/n} \geq 1,$$

where $e := \dim B$ and H_B is a very ample Cartier divisor on B.

Both $d_p(f)$ and $d_p(f|\pi)$ are well defined respectively by Dinh-Sibony [3] and Dinh-Nguyên [2] (see also [13]). There are shown the existence of the limit, independence of H and H_B, $d_{p+1}(f)d_{p-1}(f) \leq d_p(f)^2$ (concavity), birational invariance of $d_p(f)$ and the product formula: $d_p(f) := \operatorname{Max}_k\{d_k(f_B)d_{p-k}(f|\pi)\}$. In particular $d_p(f) = d_p(f_B)$ if $e = d$, i.e., if π is generically finite (thus recovers also the birational invariance).

When X is smooth and $f \in \operatorname{Aut} X$, the entropy of f is $\operatorname{Max}_p\{\log d_p(f)\}$ (Gromov–Yomdin) and it is positive if and only if $d_1(f) > 1$ (by concavity), $(f^n)^* = (f^*)^n$ (obvious) and $d_p(f)$ is the spectral radius of $f^*|_{N^p(X)}$ also the one on $H^{p,p}(X, \mathbf{R})$ (an elementary fact on a linear map $f \in \mathrm{GL}(V, \mathbf{R})$ preserving a strictly convex closed cone $C \subset V$ with non-empty interior).

Definition 3. *$f \in \mathrm{Bir}\, X$ is imprimitive, if there are a dominant rational map $\pi : X \dashrightarrow B$ to a normal projective variety B with $0 < e := \dim B < d$ and $f_B \in \mathrm{Bir}\, B$ such that $\pi \circ f = f_B \circ \pi$. We call f primitive if f is not imprimitive.*

This definition is due to De-Qi Zhang [15]. Using the product formula, we readily obtain the following useful criteria (See eg. [11], also [4] for other criteria):

Proposition 4. *Let $f \in \mathrm{Bir}\,(X)$. Then f is primitive if either $d_1(f) > 1$ and $\dim X = 2$, $d_1(f) \neq d_2(f)$ and $\dim X = 3$, or $d_1(f) > d_2(f)$ (in any dimension).*

By [15] (See also [6]), X with primitive $f \in \mathrm{Bir}\, X$ is birational to either a smooth rationally connected variety, an abelian variety or a weak Calabi–Yau variety, i.e., a minimal projective variety with numerivally trivial canonical class and vanishing irregularity, under the assumption that minimal model program with abundance works in dimension d, which is expected to be true (and true if $d \leq 3$).

When $d = 2$, X in Question 1 is then birational to either $\mathbf{P}^2$, an abelian surface, a K3 surface or an Enriques surface. In each case, there found many interesting examples (See [1], [12] and references therein). When $d = 3$, a strict Calabi–Yau threefold (SCY3 for short), which is a *simply connected smooth projective threefold with trivial canonical line bundle*, is one of the most interesting tagert threefolds for Question 1. However, by now, we have only two such SCY3 ([11] for X_3 and [9, Lemma 4.6] for X_7; it is easy to check $d_2(f) > d_1(f)$ for $f \in \mathrm{Aut}\,(X_7)$ there):

Theorem 5. *Two rigid SCY3 X_k ($k = 3, 7$) have a primitive $f_k \in \mathrm{Aut}\,(X_k)$ of positive entropy. (See [10] for explicit definitions of X_k).*

We have the following intrinsic characterization of X_k ([10]):

Definition 6. *Let X be a SCY3. A surjective morphism $\varphi : X \to B$ to a normal projective variety B with $\dim B > 0$ with connected fiber is called a c_2-contraction if $(c_2(X), \varphi^* H_B) = 0$ for an ample Caritier divisor H_B on B, or equivalently, if $(c_2(X), \varphi^* N^1(B)) = 0$ (by the psuedo-effectivity of $c_2(X)$). Note that c_2-contraction φ is never an isomorphism (as $c_2(X) \neq 0$ on $N^1(X)$ by the simply-connectedness of X). We call a c_2-contraction $\varphi_0 : X \to B_0$ maximal if φ_0 factors holomorphically through any c_2-contraction $\varphi : X \to B$.*

Theorem 7. *If a SCY3 X has a c_2-contraction, then X has a maximal c_2-contraction $\varphi_0 : X \to B_0$. A SCY3 X has a birational c_2-contraction φ if and only if X is isomorphic to one of X_k ($k = 3, 7$). Also φ is maximal in this case.*

See [10] for proof and details. Here partial vanishing of the second Chen class $c_2(X_k)$ on $\overline{\mathrm{Amp}\,(X_k)} \setminus \{0\}$ is natural for SCY3 X of $|\mathrm{Aut}\, X| = \infty$:

Proposition 8. *$c_2(X)^{\perp} \cap \overline{\mathrm{Amp}\,(X)}(\mathbf{R}) \setminus \{0\} \neq \emptyset$ for a SCY3 X of $|\mathrm{Aut}\,(X)| = \infty$.*

This is due to Wilson [14]. To my best knowledge, there is no known example such that $c_2(X)^{\perp} \cap \overline{\mathrm{Amp}\,(X)}(\mathbf{R}) \setminus \{0\} \neq \emptyset$ but $c_2(X)^{\perp} \cap \overline{\mathrm{Amp}\,(X)}(\mathbf{Q}) \setminus \{0\} = \emptyset$. So, it may be reasonable to ask:

Question 9. *Is* $c_2(X)^\perp \cap \overline{\mathrm{Amp}\,(X)}(\mathbf{Q}) \setminus \{0\} \neq \emptyset$ *for a SCY3* X *of* $|\mathrm{Aut}\,(X)| = \infty$*?*

There is also a notoriously difficult open conjecture (see eg. [5]):

Conjecture 10. *Any nef integral divsior on a SCY3 is semi-ample.*

Proposition 11. *If both Question 9 and Conjecture 10 are affirmative, then there is no SCY3* X *with primitive* $f \in \mathrm{Aut}\, X$ *of positive entropy other than* X_k *(*$k = 3$ *and* 7*).*

Indeed, since X has a primitive $f \in \mathrm{Aut}\, X$ of positive entropy, X has a non-zero nef integral divisor D with $(D.c_2(X)) = 0$ by Question 9. Then $|mD|$ for some $m > 0$ gives a c_2-contraction $X \to B$ by Conjecture 10. Thus X has a maximal c_2-contraction $\varphi_0 : X \to B_0$ by Theorem 7. By definition of maximality, we have a group homomorphism $\mathrm{Aut}\, X \to \mathrm{Aut}\, B_0$ being equivariant with respect to φ_0. Since $f \in \mathrm{Aut}\, X$ is primitive, it follows that dim $B_0 = 3$. Thus by Theorem 7, X must be isomorphic to one of X_k ($k = 3$, 7).

In view of Proposition 11, it is very interesting to answer the following:

Question 12. *Is there a SCY3 with primitive automorphism of positive entropy other than* X_k *(*$k = 3$ *and* 7*)?*

References

[1] Blanc, J., Cantat, S., *Dynamical degrees of birational transformations of projective surfaces*, J. Amer. Math. Soc. **29** (2016) 415–471.

[2] Dinh, T.-C., Nguyên V.-A., *Comparison of dynamical degrees for semiconjugate meromorphic maps*, Comment. Math. Helv. **86** (2011) 817–840.

[3] T.-C. Dinh and N. Sibony, *Une borne supérieure pour l'entropie topologique d'une application rationnelle*, Ann. of Math., **161** (2005), 1637–1644.

[4] Kaur, I., Prendergast-Smith, A., *Remark on a theorem of Oguiso*, Ann. Univ. Ferrara Sez. VII Sci. Mat.**70** (2024) 859–869.

[5] Lazić, V., Oguiso, K., Peternell, Th., *The Morrison-Kawamata cone conjecture and abundance on Ricci flat manifolds.* Adv. Lect. Math. **42** (2018) 157–185, Int. Press.

[6] Oguiso, K., *Some aspects of explicit birational geometry inspired by complex dynamics*, Proceedings of the International Congress of Mathematicians, Seoul 2014. Vol. II, 695–721, Kyung Moon Sa, Seoul, 2014.

[7] Oguiso, K., *Pisot units, Salem numbers, and higher dimensional projective manifolds with primitive automorphisms of positive entropy*, IMRN (2019) 1373–1400.

[8] Oguiso, K., *Fibered Calabi–Yau threefolds with relative automorphisms of positive entropy and c_2-contractions*, under revision (arXiv:2401.04298).

[9] Oguiso, K., *Endomorphisms of a variety of Ueno type and Kawaguchi-Silverman Conjecture*, to appear in International J. Math. (arXiv:2401.04386).

[10] Oguiso, K., Sakurai, J., *Calabi–Yau threefolds of quotient type*, Asian J. Math. **5** (2001) 43–77.

[11] Oguiso, K., Truong T.-T., *Explicit examples of rational and Calabi–Yau threefolds with primitive automorphisms of positive entropy*, Kodaira Centennial issue, J. Math. Sci. Univ. Tokyo **22** (2015) 361–385.

[12] Ogusi, K., Yu, X., *Minimum positive entropy of complex Enriques surface automorphisms*, Duke Math. J. **169** (2020) 3565–3606.

[13] Truong, T.-T., *Relative dynamical degrees of correspondences over a field of arbitrary characteristic*, J. Reine Angew. Math. **758** (2020) 139–182.

[14] Wilson, P. M., *The role of c_2 in Calabi–Yau classification – a preliminary survey*, Mirror symmetry, II, AMS/IP Stud. Adv. Math., vol. 1, Amer. Math. Soc., Providence, RI, 1997, pp. 381–392.
[15] Zhang, D.-Q., *Dynamics of automorphisms on projective complex manifolds*, J. Differential Geom. **82** (2009) 691–722.

Higher dimensional moduli spaces on Kuznetsov components of Fano threefolds

Laura Pertusi

(joint work with Chunyi Li, Yinbang Lin, Xiaolei Zhao)

Stability conditions on the Kuznetsov component of a Fano threefold of Picard rank 1, index 1 or 2 have been constructed by Bayer, Lahoz, Macrì and Stellari, making possible to study moduli spaces of semistable objects and their geometric properties. Although small dimensional examples of moduli spaces are well-understood and are related to classical moduli spaces of stable sheaves on the threefold, the higher dimensional ones are more mysterious.

In this talk, we will show a non-emptiness result for these moduli spaces. Then we will focus on the case of cubic threefolds. When the dimension of the moduli space with respect to a primitive numerical class is larger than 5, we show that the Abel-Jacobi map from the moduli space to the intermediate Jacobian is surjective with connected fibers, and its general fiber is a smooth Fano variety with primitive canonical divisor. When the dimension is sufficiently large, we further show that the general fibers are stably birational to each other.

1. Kuznetsov components of Fano threefolds and non-emptiness of moduli spaces

Let X be a Fano threefold of Picard rank 1. The anticanonical divisor satisfies $K_X = -i_X H$, where H is an ample generator of $\mathrm{Pic}(X)$ and i_X is called the index of X. Note that $i_X \leq \dim(X) + 1 = 4$. By [5] if $i_X = 4$, then X is a projective space, while if $i_X = 3$, then X is a quadric hypersurface in $\mathbb{P}^4$. In the case of index 1 and 2, a complete classification has been given in [6, 7] in terms of the degree H^3. As an example, if X has index 2 and degree 3, then X is a cubic threefold.

From the derived categories viewpoint, if X has index 2, then the bounded derived category of coherent sheaves on X has a semiorthogonal decomposition of the form

$$\mathrm{D}^b(X) = \langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{O}_X(H) \rangle,$$

where $\mathrm{Ku}(X)$ is known as the Kuznetsov component of X. If X has index 1 and degree ≥ 10, then the Kuznetsov component is defined as the right orthogonal complement $\mathrm{Ku}(X) := \langle \mathcal{E}, \mathcal{O}_X \rangle^\perp$, where $\mathcal{E}$ is an exceptional vector bundle on X constructed in [1, 3], while when the degree is ≤ 8, we set $\mathrm{Ku}(X) := \langle \mathcal{O}_X \rangle^\perp$.

Stability conditions have been constructed on Kuznetsov components of Fano threefolds of index 1 and 2 in [1]. We denote by σ a stability condition on $\mathrm{Ku}(X)$ in the same $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$-orbit of the constructed one. By the general theory in [2], given a numerical character $v \in \mathrm{K}_{\mathrm{num}}(\mathrm{Ku}(X))$, there is a good moduli space $M_\sigma(\mathrm{Ku}(X), v)$ parametrizing σ-semistable objects with character v in $\mathrm{Ku}(X)$, having the structure of a proper algebraic space.

Despite of many results, especially in low dimensions, general structure theorems for these moduli spaces are still missing. In particular, the non-emptiness of these moduli spaces was only known for small dimensional examples, or when the Kuznetsov component is in fact equivalent to the bounded derived category of a curve of genus ≥ 2 (this happens precisely when X has index 2, degree 4, or index 1, degree 12, 16, 18). In the other cases, the main difficulty is that the categories $\mathrm{Ku}(X)$ arising from Fano threefolds within a given deformation class are typically not equivalent to the derived category of a smooth projective variety. This prevents the application of any specialization argument to reduce to a known geometric case, such as done for cubic fourfolds.

In our first result, we settle the non-emptiness problem of the moduli spaces for Fano threefolds of index 2, and index 1, degree ≥ 10.

Theorem 1. *Let X be a Fano threefold with Picard rank 1, index 1, degree $10 \leq d \leq 18$, or index 2, degree $d \leq 4$. Then for every nonzero character $v \in \mathrm{K}_{\mathrm{num}}(\mathrm{Ku}(X))$, the moduli space $M_\sigma(\mathrm{Ku}(X), v)$ of σ-semistable objects of class v is non-empty.*

The actual new cases covered by Theorem 1 are those of cubic threefolds (index 2, degree 3), Fano threefolds with index 1 and degree 14, and double Veronese cones (index 2, degree 1). For quartic double solids and Gushel–Mukai threefolds (index 2, degree 2, and index 1, degree 10) this result was proved in [8] with different methods which apply in the context of Enriques categories. Theorem 1 provides a unified argument for the proof of the non-emptiness of the moduli spaces for all the cases.

Note also that if X has index 1, degree 22 (or index 2, degree 5), the component $\mathrm{Ku}(X)$ is equivalent to the bounded derived category of finite dimensional representations of the Kronecker quiver with three arrows. The remaining cases (index 1, degree 2, 4, 6, 8) seem to require a different strategy.

Theorem 1 is deduced from a general non-emptiness criterion, which is of independent interest. We develop an inductive argument that effectively reduces the question to checking the statement in the low dimensional cases, where the moduli spaces are related to low degree curves on Fano threefolds, and the non-emptiness can be verified directly.

2. Moduli spaces and cubic threefolds

Once the non-emptiness is settled, the next step is to investigate the geometric properties of the moduli spaces. Here we focus on the case of cubic threefolds.

Let Y_3 be a cubic threefold and $\mathrm{Ku}(Y_3)$ be its Kuznetsov component. For every $v \in \mathrm{K}_{\mathrm{num}}(\mathrm{Ku}(Y_3))$, set $M_\sigma(v) := M_\sigma(\mathrm{Ku}(Y_3), v)$ for simplicity. After choosing a base point F_0, we can define the Abel–Jacobi map by the (cycle-theoretic) second

Chern class:

$$\Phi_v\colon M_\sigma(v) \to J(Y_3) : F \mapsto c_2(F) - c_2(F_0).$$

For every $c \in J(Y_3)$, we denote by $M_\sigma(v,c) := \Phi_v^{-1}(c)$ the fiber of the Abel–Jacobi map. First, we note the following properties which can be obtained by standard techniques.

Theorem 2. *Let $v \in \mathrm{K}_{\mathrm{num}}(\mathrm{Ku}(Y_3))$ be a primitive character. Then the moduli space $M_\sigma(v)$ is smooth irreducible projective of the expected dimension $1-\chi(v,v)$.*

Our main results are the following.

Theorem 3. *For every $v \in \mathrm{K}_{\mathrm{num}}(\mathrm{Ku}(Y_3))$ the moduli space $M_\sigma(v)$ is irreducible.*

Theorem 4. *Let $v \in \mathrm{K}_{\mathrm{num}}(\mathrm{Ku}(Y_3))$ be a primitive character.*

(1) *If $\dim M_\sigma(v) > 5$, then Φ_v is surjective with connected fibers. For a general point $c \in J(Y_3)$, the fiber $M_\sigma(v,c)$ is a smooth Fano variety with primitive canonical divisor class.*

(2) *Assume that v, w are primitive characters such that $\dim M_\sigma(v) \geq 23$ and $\dim M_\sigma(w) \geq 23$. Then for general points $c, c' \in J(Y_3)$, the fibers $M_\sigma(v,c)$ and $M_\sigma(w,c')$ are stably birational equivalent.*

As a consequence of Theorem 4 we obtain the following statement.

Corollary 5. *For every primitive $v \in \mathrm{K}_{\mathrm{num}}(\mathrm{Ku}(Y_3))$ with $\chi(v,v) \leq -4$, the maximal rationally connected quotient of $M_\sigma(v)$ is the intermediate Jacobian $J(Y_3)$.*

In a different direction, we use our techniques to prove [4, Conjecture A.1]. Recall that if Y_4 is a cubic fourfold, then its Kuznetsov component $\mathrm{Ku}(Y_4)$ has a stability conditions σ_4 constructed in [1].

Theorem 6 ([4], Conjecture A.1)**.** *Let Y_4 be a very general cubic fourfold, and let $j\colon Y_3 \to Y_4$ be a smooth hyperplane section. Then for every primitive character $v \in \mathrm{K}_{\mathrm{num}}(\mathrm{Ku}(Y_3))$, there exists a non-empty open subset $U_v \subset M^s_\sigma(\mathrm{Ku}(Y_3), v)$ such that for every $E \in U_v$, the projection in $\mathrm{Ku}(Y_4)$ of j_*E is σ_4-stable.*

Combined with [4, Theorem A.4], Theorem 6 provides the construction of Lagrangian subvarieties inside hyperkähler manifolds arising as moduli spaces of stable objects in $\mathrm{Ku}(Y_4)$. This result could have potential applications to the construction of atomic objects supported on Lagrangian subvarieties.

References

[1] A. Bayer, M. Lahoz, E. Macrì, P. Stellari, *Stability conditions on Kuznetsov components*, Ann. Sci. Éc. Norm. Supér. (4) **56** (2023), 517–570.

[2] A. Bayer, M. Lahoz, E. Macrì, H. Nuer, A. Perry, P. Stellari, *Stability conditions in families*, Publ. Math. Inst. Hautes Études Sci. **133** (2021), 157–325.

[3] A. Bayer, A. Kuznetsov, E. Macrì, *Mukai bundles on Fano threefolds*, arXiv:2402.07154.

[4] S. Feyzbakhsh, H. Guo, Z. Liu, S. Zhang, *Lagrangian families of Bridgeland moduli spaces from Gushel-Mukai fourfolds and double EPW cubes*, arXiv:2404.11598.

[5] T. Fujita, *On the Structure of Polarized Varieties with Δ-Genera Zero*, J. Fac. Sci. Univ. Tokyo, Sect. IA 22, 103–115 (1975).

[6] V.A. Iskovskih, *Anticanonical models of three-dimensional algebraic varieties*, Current problems in mathematics, Vol.12 (Russian), Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow (1979), 59–157, 239 (loose errata).
[7] S. Mukai, H. Umemura, *Minimal rational threefolds*, Algebraic geometry (Tokyo/Kyoto, 1982), Vol.1016, Lecture Notes in Math., Springer, Berlin (1983), 490–518.
[8] A. Perry, L. Pertusi, X. Zhao, *Moduli spaces of stable objects in Enriques categories*, arXiv:2305.10702.

Quasi-BPS categories for K3 surfaces

Tudor Pădurariu
(joint work with Yukinobu Toda)

The talk was based on the preprint [5]. The paper is concerned with the study of derived categories of coherent sheaves on the moduli stack of semistable sheaves on a K3 surface.

Let S be a K3 surface, let σ be a generic stability condition, let v_0 be a Mukai vector, and let $g := \frac{\langle v_0, v_0\rangle + 2}{2}$. If v is primitive, then the moduli space $M^\sigma_S(v_0)$ of σ-semistable sheaves of support v_0 is an irreducible holomorphic symplectic (IHS) variety, deformation equivalent to the Hilbert scheme of g points on S. One may attempt to construct more IHS varieties starting from moduli of semistable sheaves with *non-primitive* support. For example, one may attempt to find symplectic (alternatively, crepant) resolutions

$$\widetilde{M} \to M^\sigma_S(v)$$

of the good moduli space for $v = dv_0$, where v_0 is primitive and $d \geq 2$. However, a theorem of Kaledin–Lehn–Sorger [4] says that, with the exception of the 10 dimensional example constructed by O'Grady (which corresponds to $d = g = 2$), there are no such resolutions.

One can then wonder whether there are crepant resolutions of $M^\sigma_S(v)$ in the categorical sense. The existence of such resolutions might be important for a couple of reasons: it highlights the importance of using derived categories to construct objects, close to actual geometric spaces, and with some desired properties which do not hold for geometric spaces; and it may actually provide new examples of (categorical or geometric) IHS varieties. In [5], we investigate this problem starting from the category of coherent sheaves on the *stack* $\mathfrak{M}^\sigma_S(v)$ of σ-semistable sheaves of support v. We construct categories

$$\mathbb{T}^\sigma_S(v)_w \subset D^b\left(\mathfrak{M}^\sigma_S(v)\right)_w \tag{1}$$

that indeed behave like categorical (twisted) crepant resolutions for $M^\sigma_S(v)$ for all $v = dv_0$ with $g \geq 2$. However, the spaces we construct are expected to be derived equivalent to categories of twisted sheaves on $M^\sigma_S(v'_0)$ for v'_0 a *primitive* Mukai vector, so they do not provide new examples of IHS varieties.

When attempting to construct a categorical resolution of singularities of a singular variety M, one usually starts with a geometric resolution

$$\widetilde{M} \to M$$

and then attempts to construct a smaller, categorical resolution as an admissible category

$$\mathbb{T} \subset D^b(\widetilde{M}).$$

Our process for constructing (1) is similar, but instead of an actual resolution of singularities, we start with the singular stack $\mathfrak{M}^\sigma_S(v)$. There are two reasons for this choice: first, even if $\mathfrak{M}^\sigma_S(v)$ is a singular stack, its cohomology behaves like that of a smooth projective variety (i.e. it exhibits *purity*, see [1]); and second, there are many techniques to study categories of coherent sheaves on a stack, which go under the name of "window categories" (and due to Segal, Halpern-Leistner, Ballard–Favero–Katzarkov, Špenko–Van den Bergh, and others). The following is the first main result of our paper, and it can be also used to give an inductive definition of the categories (1).

Theorem 2. *There is a semiorthogonal decomposition*

$$D^b\left(\mathfrak{M}^\sigma_S(v)\right) = \left\langle \otimes_{i=1}^k \mathbb{T}^\sigma_S(d_i v_0)_{w_i+(g-1)d_i(\sum_{i>j} d_j - \sum_{i<j} d_j)} \right\rangle, \tag{3}$$

where the right hand side is after all partitions $(d_i)_{i=1}^k$ *of* d *and all weights* $(w_i)_{i=1}^k \in \mathbb{Z}^k$ *such that*

$$\frac{w_1}{d_1} < \cdots < \frac{w_k}{d_k}.$$

Each functor $\otimes_{i=1}^k \mathbb{T}^\sigma_S(d_i v_0)_{w_i+(g-1)d_i(\sum_{i>j} d_j - \sum_{i<j} d_j)} \to D^b\left(\mathfrak{M}^\sigma_S(v)\right)$ *is induced by a push-pull functor from the stack of extension and is fully-faithful.*

The above semiorthogonal decomposition can be compared with decompositions of the Borel-Moore homology of $\mathfrak{M}^\sigma_S(v)$ in terms of the BPS cohomology $H^\bullet_{\mathrm{BPS}}(M^\sigma_S(v))$, due to Davison–Hennecart–Schelegel Mejia [2]. The BPS cohomology is a refinement of the BPS invariants of sheaves on the Calabi–Yau threefold $S\times\mathbb{C}$, which are fundamental enumerative invariants related to Donaldson-Thomas and Gromov–Witten theories. Conjecturally, one expects

$$H^\bullet_{\mathrm{BPS}}(M^\sigma_S(v)) \cong H^\bullet(\mathrm{Hilb}(S, g')), \tag{4}$$

where $g' := \frac{\langle v,v\rangle+2}{2}$. We can then compute the topological K-theory of the categories (1) in terms of BPS cohomologies, and thus conjecturally in terms of the cohomology of Hilbert scheme of points on S.

The following is our main theorem. First, we can eliminate a copy of the exterior algebra from the (derived) stacks $\mathfrak{M}^\sigma_S(v)$, which allows us to construct the smaller reduced categories $\mathbb{T}^\sigma_S(v)^{\mathrm{red}}_w$. We show that *some* of the categories $\mathbb{T}^\sigma_S(v)^{\mathrm{red}}_w$, namely those for which w is coprime with d, or alternatively those whose topological K-theory is isomorphic to BPS cohomology, behave analogously to crepant resolutions of singularities of $M^\sigma_S(v)$. The last part is a version of a theorem of Halpern-Leistner [3] for a non-primitive Mukai vector.

Theorem 5. *Suppose that* $g \geq 2$, $\sigma, \sigma' \in \mathrm{Stab}(S)$ *are generic stability conditions, and* w *is coprime to* dv. *Then:*

(i) (smoothness and properness): *the category* $\mathbb{T}^\sigma_S(v)^{\mathrm{red}}_w$ *is smooth and proper;*

(ii) (étale locally trivial Serre functor): *the Serre functor* $S_{\mathbb{T}}$ *of* $\mathbb{T}^{\sigma}_{S}(v)^{\mathrm{red}}_{w}$ *is trivial étale locally on* $M^{\sigma}_{S}(v)$*;*

(iii) (wall-crossing equivalence): *there is an equivalence*

$$\mathbb{T}^{\sigma}_{S}(v)^{\mathrm{red}}_{w} \simeq \mathbb{T}^{\sigma'}_{S}(v)^{\mathrm{red}}_{w}.$$

We do not expect the categories $\mathbb{T}^{\sigma}_{S}(v)_{w}$ to produce new examples of IHS varieties, but we believe they are related to known examples of IHS varieties by interesting derived equivalences, more precisely by compactifications of the Fourier-Mukai derived equivalence between a (relative) abelian variety and its dual. We will make a precise such conjecture in future work. A simplified form is the following, which can be also seen as a categorical version of the conjecture (4):

Conjecture 6. *For any* $g \geq 0$ *and any* $w \in \mathbb{Z}$ *coprime with* v*, the category* $\mathbb{T}_{S}(v)^{\mathrm{red}}_{w}$ *is deformation equivalent to* $D^{b}(\mathrm{Hilb}(S, g'))$*, where* $g' := \frac{\langle v,v\rangle+2}{2}$.

References

[1] B. Davison, *Purity and 2-Calabi-Yau categories*, arXiv:2106.07692.

[2] B. Davison, L. Hennecart, and S. Schlegel Mejia, *BPS Lie algebras for totally negative 2-Calabi-Yau categories and nonabelian Hodge theory for stacks*, arXiv:2212.07668.

[3] D. Halpern-Leistner, *Derived Θ-stratifications and the D-equivalence conjecture*, arXiv:2010.01127.

[4] D. Kaledin, M. Lehn, and Ch. Sorger, *Singular symplectic moduli spaces*, Invent. Math. **164** (2006), 591–614.

[5] T. Pădurariu and Y. Toda, *Quasi-BPS categories for K3 surfaces*, arXiv:2309.08437.

The categorical origins of equivariant quantum cohomology

Ed Segal

This talk is nominally about equivariant Fukaya categories, but in fact will contain almost no symplectic geometry. Instead it will be a discussion of the categorical (and 2-categorical) structures that are expected to appear in equivariant Floer theory and equivariant mirror symmetry. It is based on some ongoing discussions with Yanki Lekili and Dan Pomerleano, and heavily inspired by Teleman's 2014 ICM address [2].

Suppose we have a symplectic manifold X with a mirror variety $\check{X}$, or mirror Landau-Ginzburg model $(\check{X}, \check{W})$. Homological mirror symmetry is the statement that the Fukaya category $\mathrm{Fuk}(X)$ is equivalent to the derived category $D^{b}(\check{X})$, or to the category of matrix factorizations $\mathrm{MF}(\check{X}, \check{W})$. To pass from here to more traditional closed-string mirror symmety we take Hochschild cohomology, since it is known (in some situations at least) that $\mathrm{HH}^{\bullet}(\mathrm{Fuk}(X))$ is equal to the quantum cohomology $\mathrm{QH}^{\bullet}(X)$.[1] On the mirror side we get $\mathrm{HH}^{\bullet}(D^{b}(\check{X}))$, which is roughly the cohomology of X, or $\mathrm{HH}^{\bullet}(\mathrm{MF}(\check{X}, \check{W}))$, which is roughly the Jacobi ring of W.

[1] If X is non-compact we mean the wrapped Fukaya category, and we should use symplectic cohomology instead of quantum cohomology.

Now suppose we have a Lie group G and a Hamiltonian G-action on X. Equivariant quantum cohomology $\mathrm{QH}^\bullet_G(X)$ has been fairly well studied. Can we fill in the other parts of this story? *I.e.*

(1) Is there a G-equivariant Fukaya category $\mathrm{Fuk}_G(X)$?
(2) If so, is $\mathrm{HH}^\bullet(\mathrm{Fuk}_G(X))$ equal to $\mathrm{QH}^\bullet_G(X)$?
(3) What is the mirror structure?

The aim of the talk is to show that the answers to (1) and (2) are 'no', or at least 'it's a bit more complicated than that'.

We'll discuss only the case $G = U(1)$. Finite groups are not interesting for this story, and non-abelian groups are too hard.

The first puzzle is understanding the mirror data to the $U(1)$ action, or equivalently understanding how the $U(1)$-action manifests itself on the Fukaya category. If two Lagrangians are Hamiltonian-isotopic then they are isomorphic in $\mathrm{Fuk}(X)$, so at first sight the $U(1)$ action does nothing at all to the Fukaya category; moving an object L under $\theta \in U(1)$ just produces an isomorphic object $\theta(L)$.

We solve this puzzle by viewing $\mathrm{Fuk}(X)$ as a topological category, or ∞-category. So there is a space of objects $Ob_{\mathrm{Fuk}(X)}$, and two objects are isomorphic iff they are connected by a path in this space.[2] In particular, loops in this space are automorphisms:

$$\pi_1(Ob_{\mathrm{Fuk}(X)}, L) = \mathrm{Aut}(L)$$

So the orbit of L under the $U(1)$ action corresponds to an automorphism σ_L of L. We have this for all objects L, so we have a natural automorphism σ of the identity functor on $\mathrm{Fuk}(X)$, or equivalently an invertible element $\sigma \in \mathrm{HH}^0(\mathrm{Fuk}(X))$. This is the *Seidel element*.

On the mirror, we must have a corresponding $\breve{\sigma}$ in $\mathrm{HH}^0(D^b(\breve{X})) = \mathrm{H}^0(\mathcal{O}_X)$, *i.e.* a map $\breve{\sigma} : \breve{X} \to \mathbb{C}^*$. This is the mirror data to the $U(1)$ action. Teleman calls it a *topological* $U(1)$ *action* since it is the topology of $U(1)$ that matters, more than the group structure [2].

With this data both $\mathrm{Fuk}(X)$ and $D^b(\breve{X})$, or $\mathrm{MF}(\breve{X}, \breve{W})$, carry an action of the ring $\mathbb{C}[s^{\pm 1}]$. So it is tempting to guess that "$U(1)$-equivariant homological mirror symmetry" is the statement that the categories are equivalent relative to $\mathbb{C}[s^{\pm 1}]$. However, this cannot be the whole story, as the following example demonstrates.

Consider $X = \mathbb{C}^2$ with its standard (real) symplectic form, and the usual $U(1)$ action by rescaling. The mirror is $\breve{X} = (\mathbb{C}^*)^2$ with $\breve{W} = X + Y$. The mirror to the Seidel element[3] is $\breve{\sigma} = XY$. Then $\breve{W}$ has no critical points so $\mathrm{MF}(\breve{X}, \breve{W}) \cong 0$, matching the fact that $\mathrm{Fuk}(X) \cong 0$. So the two categories are indeed equivalent, and the equivalence is relative to $\mathbb{C}[s^{\pm 1}]$. But the statement is vacuous.

However, suppose we take a fibre of $\breve{\sigma}$ at some value $\lambda \in \mathbb{C}^*$. We get $\breve{X}_\lambda \cong \mathbb{C}^*$ with the superpotential $\breve{W}_\lambda = X + \lambda/X$. This is the mirror to $\mathbb{P}^1$, which is of course

[2] This space is really a spectrum, constructed from Floer homotopy theory.

[3] Since $\breve{X}$ is just the dual torus to the torus in X, the 1-parameter subgroup σ corresponds to the character $\breve{\sigma}$.

the symplectic quotient $X//U(1)$. The categories $\mathrm{Fuk}(\mathbb{P}^1)$ and $\mathrm{MF}(\check{X}_\lambda, \check{W}_\lambda)$ are equivalent, and non-zero.

From this example we learn two important points:

- Taking the fibres of $\check{\sigma}$ on the B-side corresponds to taking symplectic quotients on the A-side.
- The categories $\mathrm{Fuk}(X)$ or $\mathrm{MF}(\check{X}, \check{W})$ do not determine all the data we care about, even with their $\mathbb{C}[s^{\pm 1}]$ structures.

The first point is very interesting but we won't pursue it here,[4] we'll focus on the second point.

The solution, again according to Teleman [2], is to replace $\mathbb{C}^*$ with its cotangent bundle $T^*\mathbb{C}^*$, *i.e.* the ring $\mathbb{C}[s^{\pm 1}, t]$. This is a holomorphic symplectic manifold, albeit a completely trivial example of one. But for any holomorphic symplectic Y there is believed to be a 3d topological field theory which is a sigma model with target Y; this is *Rozansky-Witten theory.* And by the yoga of the Cobordism Hypothesis this corresponds to some nice 2-category constructed from Y. We call this conjectural 2-category $\mathrm{KRS}(Y)$, after Kapustin-Rozansky-Saulina [1].

The simplest objects in $\mathrm{KRS}(Y)$ should be holomorphic Lagrangians $L \subset Y$. Some more complicated objects will be such an L together with a 'category $\mathcal{C}$ which is linear over L', *i.e.* a holomorphic sheaf of categories over L. This is an analogue of what we do in the Fukaya category, when we equip a real Lagrangian with a local system. One way to produce such a sheaf $\mathcal{C}$ is to have another variety X and a map $X \to L$, then $\mathcal{C} = D^b(X)$ is linear over L. More generally we could take $\mathcal{C} = \mathrm{MF}(X, W)$ for some superpotential. The 'trivial sheaf' is $D^b(L)$ itself.

We won't discuss the morphisms in $\mathrm{KRS}(Y)$, except to say that the 1-morphisms between $(L_1, \mathcal{C}_1)$ and $(L_2, \mathcal{C}_2)$ should be some category which is linear over the intersection $L_1 \cap L_2$.

The Lagrangians, with trivial sheaves, should span the whole 2-category. So a general object in $\mathrm{KRS}(Y)$ can be described, Yoneda style, as an operation on Lagrangians

$$\mathcal{D} : L \to \mathrm{Hom}(L, \mathcal{D})$$

sending each L to a category linear over L.

Now we return to the case $Y = T^*\mathbb{C}^*$. Some obvious Lagrangians are the zero section $\{t = 0\}$ and the cotangent fibres $\{s = \lambda\}$. We claim that, given either:

(1) a variety $\check{X}$ or LG model $(\check{X}, \check{W})$, together with a map $\check{\sigma} : \check{X} \to \mathbb{C}^*$, or
(2) a symplectic X with a Hamiltonian $U(1)$-action,

there are corresponding objects of $\mathrm{KRS}(T^*\mathbb{C}^*)$, which we denote $\mathcal{D}_{\check{X}}, \mathcal{D}_{\check{X},\check{W}}$ and $\mathcal{F}_X$. On the obvious Lagrangians these take the following values:[5]

[4] It includes the question of how to match up the values of $\check{\sigma}$ with the values of the moment map on the A-side.

[5] The categories in the bottom row are naturally linear over $\mathbb{C}[t]$, the first two by the evident $t \in \mathrm{HH}^2(\check{X}_\lambda)$, and the last by $t = c_1$ of the $U(1)$ bundle.

	$\mathcal{D}_{\check{X}}$	$\mathcal{D}_{\check{X},\check{W}}$	$\mathcal{F}_X$
$\{t=0\}$	$D^b(\check{X})$	$\mathrm{MF}(\check{X},\check{W})$	$\mathrm{Fuk}(X)$
$\{s=\lambda\}$	$D^b(\check{X}_\lambda)$	$\mathrm{MF}(\check{X}_\lambda,\check{W}_\lambda)$	$\mathrm{Fuk}(X/\!/U(1))$

And the correct statement of $U(1)$-equivariant homological mirror symmetry is that $\mathcal{F}_X = \mathcal{D}_{\check{X},\check{W}}$.

Finally, for any object $\mathcal{D}$ in any 2-category we have an analogue of Hochschild cohomology: it's the endomorphisms of $1_{\mathcal{D}}$ where $1_{\mathcal{D}}$ is the unit in the monoidal category $\mathrm{End}(\mathcal{D})$. In the 2-category **Cat** this is exactly Hoschild cohomology. Applied to the object $\mathcal{F}_X$ in $\mathrm{KRS}(T^*\mathbb{C}^*)$ we claim it should produce the equivariant quantum cohomology of X.

On the mirror we make the further ansatz that

$$\mathrm{End}(\mathcal{D}_{\check{X},\check{W}}) = \mathrm{MF}(\check{X}\times_{\mathbb{C}^*}\check{X},\, \check{W}_2-\check{W}_1)$$

and then the endomorphisms of the unit are not hard to compute. The results match with $\mathrm{QH}^\bullet_{U(1)}(X)$ in toric examples.

References

[1] A. Kapustin, L. Rozansky, N. Saulina, *Three-dimensional topological field theory and symplectic algebraic geometry I*, Nuclear Physics B, 816 (3), 2009, 295-355. arXiv:0810.5415
[2] C. Teleman, *Gauge theory and mirror symmetry*, Proc. IC - Seoul 2014. Vol. II, 1309–1332. arXiv:1404.6305

Cones of Noether-Lefschetz divisors and moduli spaces of hyperkähler manifolds

Laure Flapan
(joint work with Ignacio Barros, Pietro Beri, Brandon Williams)

For the moduli space $\mathcal{F}_{2d}$ of quasi-polarized K3 surfaces of degree $2d$, as well as other moduli spaces of polarized hyperkähler manifolds, the most natural source of divisors is Noether–Lefschetz divisors. A very general point $(S,H)\in\mathcal{F}_{2d}$ has Picard group $\mathrm{Pic}(S)=\mathbb{Z}H$ and so the locus in $\mathcal{F}_{2d}$ where $\rho(S)\geq 2$ is a countable union of divisors, called Noether–Lefschetz divisors (or NL divisors). Concretely, a Noether–Lefschetz divisor $\mathcal{D}_{h,a}$ on $\mathcal{F}_{2d}$ is the reduced divisor obtained by taking the closure of the locus of points $(S,H)\in\mathcal{F}_{2d}$ for which there exists a class $\beta\in\mathrm{Pic}(S)$, not proportional to H, with $\beta^2=2h-2$ and $\beta.H=a$.

Heegner divisors generalize Noether–Lefschetz divisors to arbitrary orthogonal modular varieties $\mathcal{D}/\Gamma$ by viewing Noether–Lefschetz divisors as images of hyperplane arrangements in $\mathcal{D}$ under the modular projection $\pi:\mathcal{D}\longrightarrow\mathcal{D}/\Gamma$. More precisely, let Λ be an even lattice of signature $(2,n)$. Consider the bilinear form

$\langle \cdot, \cdot \rangle$ on Λ (which extends to $\Lambda_{\mathbb{C}}$) and $\mathcal{D}_\Lambda$ the Type IV domain given by one of the two components (exchanged by complex conjugation) of

$$\left\{ [Z] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \,\middle|\, \langle Z, Z \rangle = 0, \langle Z, \overline{Z} \rangle > 0 \right\}.$$

Let Γ be a finite index subgroup of $\widetilde{\mathrm{O}}^+(\Lambda)$, the group of orientation-preserving isomorphisms of Λ which act trivially on the discriminant group $D(\Lambda_{2d}) = \Lambda_{2d}^\vee/\Lambda$. We then consider the quotient $\mathcal{D}_\Lambda/\Gamma$. The fundamental example to have in mind is when

$$\Lambda = \Lambda_{2d} = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \mathbb{Z}\ell, \text{ with } \langle \ell, \ell \rangle = -2d$$

and $\Gamma = \widetilde{\mathrm{O}}^+(\Lambda_{2d})$. Then $\mathcal{F}_{2d} = \mathcal{D}_{\Lambda_{2d}}/\widetilde{\mathrm{O}}^+(\Lambda_{2d})$ is a coarse moduli space for primitively quasi-polarized K3 surfaces of degree $2d$.

For $v \in \Lambda_{\mathbb{Q}}$, one considers the hyperplane section $D_v = v^\perp \cap \mathcal{D}_\Lambda$. Using the quadratic form $Q(v) = \frac{\langle v,v \rangle}{2}$, for a fixed class $\mu + \Lambda \in D(\Lambda)$ and $m \in Q(\mu) + \mathbb{Z}$ non-positive, the cycle $\sum_{\substack{v \in \mu+\Lambda \\ Q(v)=m}} D_v$ is Γ-invariant and descends to a $\mathbb{Q}$-Cartier divisor $H_{m,\mu}$ on $\mathcal{D}_\Lambda/\Gamma$, called a *Heegner divisor*. In general, $H_{m,\mu}$ is neither reduced, nor irreducible and its irreducible components, called *primitive Heegner divisors*, are denoted $P_{\Delta,\delta}$. In the K3 case, Heegner divisors are related to Noether–Lefschetz divisors via $\mathcal{D}_{h,a} = H_{-m,\mu}$ if $d \not| a$ and $\mathcal{D}_{h,a} = \frac{1}{2} H_{-m,\mu}$ if $d \mid a$, where $m = \frac{a^2}{4d} - (h-1)$ and $\mu = a\ell_*$ for $\ell_* = \frac{\ell}{2d} \in D(\Lambda_{2d})$ the standard generator [8, Lemma 3].

Maulik–Pandharipande conjectured [8, Conjecture 3] that the rational Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$ is generated by Noether–Lefschetz divisors $\mathcal{D}_{h,a}$. Bergeron–Li–Millson–Moeglin [1] and Bruinier–Zuffetti [6] proved a generalization of Maulik–Pandharipande's conjecture, showing that, under the assumption that Λ splits off two copies of the hyperbolic plane and that $n \geq 3$, the Picard group with rational coefficients $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda_{2d}))$ of any orthogonal modular variety $\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda_{2d})$ is generated by Heegner divisors. The rank of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{D}_\Lambda/\Gamma)$ was computed by Bruinier in [3].

An important invariant of an algebraic variety X is its cone of pseudo-effective divisors $\overline{\mathrm{Eff}}(X)$, which governs much of the birational geometry of X. The cone $\overline{\mathrm{Eff}}(X)$ is defined as the closure of the cone of effective $\mathbb{R}$-divisors on X. In general, it can be quite difficult to determine when $\overline{\mathrm{Eff}}(X)$ is finitely-generated, let alone compute it explicitly.

In the case of an orthogonal modular variety $\mathcal{D}_\Lambda/\Gamma$, a natural subcone of $\overline{\mathrm{Eff}}(\mathcal{D}_\Lambda/\Gamma)$ is the NL-cone $\mathrm{Eff}^{NL}(\mathcal{D}_\Lambda/\Gamma)$ of effective $\mathbb{R}$-linear combinations of primitive Heegner divisors on $\mathcal{D}_\Lambda/\Gamma$. The NL-cone contains the subcone $\mathrm{Eff}^{H}(\mathcal{D}_\Lambda/\Gamma)$ generated by the (non-primitive) Heegner divisors on $\mathcal{D}_\Lambda/\Gamma$.

Bruinier–Möller [5] showed that for $X = \mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)$ with Λ of signature $(2,n)$ with $n \geq 3$ and splitting off two copies of the hyperbolic plane, the cone $\mathrm{Eff}^{NL}(X)$ is always polyhedral.

Here we examine the question of computing $\mathrm{Eff}^{NL}(X)$ for X under the same assumptions. Our first result is the following.

Theorem 1. *Let Λ be an even lattice of signature $(2,n)$ with $n \geq 3$ splitting off two copies of the hyperbolic plane and $X = \mathcal{D}_\Lambda / \widetilde{\mathrm{O}}^+(\Lambda)$ its modular variety. We produce an explicit bound Ω, depending on numerics of the lattice Λ, such that the cone $\mathrm{Eff}^{NL}(X)$ is generated by all $P_{-\Delta,\delta}$ with $0 \leq \Delta \leq \Omega$.*

Theorem 1 together with its implementation in Sage package [11] enables the computation of $\mathrm{Eff}^{NL}(X)$ given any such Λ.

The proof of Theorem 1 relies on the relationship between Heegner divisors on X and vector-valued modular forms with respect to the Weil representation for Λ. In [5] the polyhedrality of the NL-cone is established by showing that the Hodge class λ lies on the interior of the NL-cone, and the rays $P_{\Delta,\delta}\mathbb{Q}_{\geq 0}$ converge to $\lambda\mathbb{Q}_{\geq 0}$ as Δ grows. Establishing a concrete list of generators of $\mathrm{Eff}^{NL}(X)$ amounts to making the convergence rate explicit which translates into bounding explicitly the growth of the coefficients of the relevant vector-valued modular forms. For vector-valued cusp forms of half-integer weight, despite the considerable literature on bounds for the growth of Fourier coefficients, we are unaware of a general bound with explicit constants. Using Poincaré series and Kloosterman sums we derive weak, yet explicit, bounds that suffice for our purposes.

Uniruledness results. We consider the period domain partial compactifications $\mathcal{M}^\gamma_{\mathrm{OG6},2d}$ and $\mathcal{M}^\gamma_{\mathrm{Kum}_n,2d}$ of the moduli spaces $\left(\mathcal{M}^\gamma_{\mathrm{OG6},2d}\right)^\circ$ and $\left(\mathcal{M}^\gamma_{\mathrm{Kum}_n,2d}\right)^\circ$ parameterizing primitively polarized hyperkähler sixfolds of OG6-type respectively $2n$-folds of Kum_n-type with a primitive polarization of degree $2d$ and divisibility γ. We remark that the moduli space $\mathcal{M}^\gamma_{\mathrm{OG6},2d}$ is always irreducible and in the case $\gamma = 2$ it is non-empty only when $d \equiv -1, -2 \mod 4$. Similarly, setting $d = 1$ and $\gamma \in \{1, 2\}$, the moduli space $\mathcal{M}^\gamma_{\mathrm{Kum}_n,2}$ is irreducible and in the case $\gamma = 2$ its nonempty only when $n \equiv 2 \mod 4$.

We establish the following uniruledness results:

Theorem 2. *The moduli space $\mathcal{M}^\gamma_{\mathrm{OG6},2d}$ is uniruled in the following cases*

(i) *when $\gamma = 1$ for $d \leq 12$,*
(ii) *when $\gamma = 2$ for $t \leq 10$ and $t = 12$ with $d = 4t - 1$,*
(iii) *when $\gamma = 2$ for $t \leq 9$ and $t = 11, 13$ with $d = 4t - 2$.*

The moduli spaces $\mathcal{M}^1_{\mathrm{Kum}_n,2}$ and $\mathcal{M}^2_{\mathrm{Kum}_n,2}$ are uniruled in the following cases:

(i) *when $\gamma = 1$ for $n \leq 15$ and $n = 17, 20$,*
(ii) *when $\gamma = 2$ for $t \leq 11$ and $t = 13, 15, 17, 19$, where $n = 4t - 2$.*

Our approach to uniruledness is inspired by [10]. The idea is to express the canonical class of (a smoothing of a toroidal compactification of) such a quotient $\mathcal{M} = \mathcal{D}_\Lambda / \widetilde{\mathrm{O}}^+(\Lambda)$ in terms of Heegner divisors $H_{m,\mu}$. One then uses formulas of Kudla [7] and Bruinier–Kuss [4] expressing the intersection of these Heegner divisors with the power $\lambda^{\dim \mathcal{M} - 1}$ of the Hodge class λ in terms of coefficients of an Eisenstein series in order to show that the intersection of the canonical class with $\lambda^{\dim \mathcal{M} - 1}$ is negative. Since $\lambda^{\dim \mathcal{M} - 1}$ is a covering curve (in particular nef) the canonical class is then not pseudo-effective and uniruledness follows from [9, 2].

References

[1] N. Bergeron, Z. Li, J. Millson, and C. Moeglin, *The Noether-Lefschetz conjecture and generalizations*, Invent. Math. **208**, no. 2 (2017), 501–552
[2] S. Bouksom, J.-P. Demailly, M. Păun, T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22**, no. 2 (2013), 201–248
[3] J.H. Bruinier, *On the rank of Picard groups of modular varieties attached to orthogonal groups*, Compositio Math. **133**, no. 1 (2002), 49–63
[4] J.H. Bruinier and M. Kuss, *Eisenstein series attached to lattices and modular forms on orthogonal groups*, Manuscripta Math. **102**, no 4 (2001), 443–459.
[5] J.H. Bruinier and M. Möller, *Cones of Heegner divisors*, J. Algebraic Geom. **28**, no. 3 (2019), 497–517.
[6] J.H. Bruinier and R. Zuffetti, *Lefschetz decompositions of Kudla-Millson theta functions* (2024), arXiv.2406.19921.
[7] S. Kudla, *Integrals of Borcherds forms*, Compositio Math. **137**, no. 3 (2003), 293–349.
[8] D. Maulik and R. Pandharipande, *Gromov–Witten theory and Noether-Lefschetz theory*, A celebration of algebraic geometry, Clay Math. Proc., Amer. Math. Soc.**18** (2013), 469–507.
[9] Y. Miyaoka and S. Mori, *A numerical criterion for uniruledness*, Annals of Math. **124**, no. 1 (1986), 65–69
[10] A. Peterson, *Modular forms on the moduli space of polarised K3 surfaces*, PhD thesis, University of Amsterdam (2015).
[11] B. Williams, *Heegner Cones*, Available at https://github.com/btw-47/heegner-cones.

Fano varieties with torsion in $H^3(X, \mathbb{Z})$

John Christian Ottem
(joint work with Jørgen Vold Rennemo)

For a smooth complex projective variety X, the torsion subgroup of the third integral cohomology group, $\operatorname{Tors} H^3(X, \mathbb{Z})$, is an important stable birational invariant of X. The invariant was introduced by Artin and Mumford [1] in order to show that certain conic bundles are irrational. Specifically, their examples are unirational, but not rational because $\operatorname{Tors} H^3(X, \mathbb{Z}) \neq 0$, whereas any rational variety has no torsion in $H^3(X, \mathbb{Z})$.

Constructing examples with non-trivial torsion in $H^3(X, \mathbb{Z})$ is generally a challenging problem, especially if one imposes that X should satisfy some geometric requirements, such as being rationally connected or Fano. Most constructions utilize the relation to the Brauer group; if X is rationally connected, then $\operatorname{Tors} H^3(X, \mathbb{Z}) = \operatorname{Br}(X)$, which classifies projective space fibrations modulo projectivized vector bundles.

Beauville [2] asked whether there is a *Fano variety* with non-trivial torsion in $H^3(X, \mathbb{Z})$. In dimension 1 and 2, all Fano varieties are rational. In dimension 3, there is a classification into 105 types, and inspecting the list shows that all Fano 3-folds have torsion free $H^3(X, \mathbb{Z})$.

The main focus of the talk was to explain the following theorem.

Theorem 1 (O.-Rennemo). *In any even dimension ≥ 4, there exist Fano varieties of Picard number 1 with* $\operatorname{Tors} H^3(X, \mathbb{Z}) = \mathbb{Z}/2$.

The Fano varieties arise from a geometric construction involving certain rank loci of symmetric matrices. Below, we will describe a special case, which also relates to work by Hosono–Takagi [6] (who studied double symmetroid Calabi-Yau 3-folds).

Construction. Let $\mathbb{P}^{14} = \mathbb{P}(S^2V^\vee)$ denote the space of quadrics in $V = \mathbb{C}^5$. For $r = 1, \ldots, 4$, we define the closed subvariety $Z_r \subset \mathbb{P}^{14}$ as the set of quadrics of rank at most r. In other words, Z_r is the zero locus of the $(r+1)\times(r+1)$ minors of the generic symmetric 5×5 matrix of variables. Concretely,

- Z_4 is a quintic hypersurface (dimension 13),
- Z_3 is a subvariety of degree 20 (dimension 11),
- $Z_2 = \mathrm{Sym}^2(\mathbb{P}^4)$ (dimension 8),
- Z_1 is the second Veronese embedding of $\mathbb{P}^4$ (dimension 4).

These varieties are highly singular, as $\mathrm{sing}(Z_r) = Z_{r-1}$ for each $r = 2, 3, 4$.

The hypersurface Z_4 parameterizes rank 4 quadrics in $\mathbb{P}^4$, such as $Q = x_0x_3 - x_1x_2$. These are cones over smooth quadric surfaces in $\mathbb{P}^3$, and hence they contain two disjoint families of 2-planes. Consider the incidence variety

$$U = \left\{ \text{ pairs } ([L],[Q]) \text{ where } L \subset Q \text{ is a 2-plane on } Q \right\} \subset \mathrm{Gr}(3,V) \times \mathbb{P}(S^2V^\vee).$$

The first projection is a projective bundle over $\mathrm{Gr}(3,V)$, as the fibres are the projective space of all quadrics containing a given 2-plane.

Note that if $(L,Q) \in U$, then $\mathrm{rank}\, Q \leq 4$, because Q contains a 2-plane. Therefore, the second projection maps into Z_4. We define W_4 via the Stein factorization

$$U \xrightarrow{\ \eta\ } W_4 \xrightarrow{\ \tau\ } Z_4$$

Then τ is finite of degree 2, and η is generically a $\mathbb{P}^1$-bundle.

Conveniently, W_4 has rather mild singularities:

$$\mathrm{sing}\, W_4 = \tau^{-1}(Z_2),$$

which has dimension 8. Moreover, W_4 is a Fano variety with $K_{W_4} = \tau^* K_{Z_4} = -10H$, where $H = \tau^*\mathcal{O}_{\mathbb{P}^{14}}(1)$. If we now choose 9 general divisors $H_i \in |H|$, then

$$X = W_4 \cap H_1 \cap \ldots \cap H_9$$

is a smooth Fano 4-fold with $-K_X = H$.

The degree 3 torsion class. The 4-fold X admits a torsion class in $H^3(X,\mathbb{Z})$ which can be described geometrically as follows. The morphism $\eta\colon U \to W_4$ restricts to a $\mathbb{P}^1$-fibration $U_X \to X$ and a corresponding 2-torsion class $\sigma \in H^3(X,\mathbb{Z})$. To demonstrate that the class is non-zero, we show that η does not admit a rational section, following a computation of [6]. Regarding U as a projective bundle over $\mathrm{Gr}(3,V)$, we have $U = \mathbb{P}(E)$ for some vector bundle E on $\mathrm{Gr}(3,V)$. If $D \subset U$ is the closure of a rational section, then the divisor class of D is of the form $aL + b \cdot g$ where $a, b \in \mathbb{Z}$, $L = \mathcal{O}(1)$ and g is the pullback of Plücker polarization. As Z_4 has degree 5, we have $D \cdot L^{13} = 10$. On the other hand, using the Chern classes of E, we compute that $D \cdot L^{13} = -20b$, contradicting the fact that b is an integer. This shows that the Brauer group of the smooth locus of W_4 is non-trivial. Applying a generalized Lefschetz theorem, we conclude that $H^3(X, \mathbb{Z})$ has a non-trivial torsion element.

Remark 2. *The derived category of X has a description*

$$D^b(X) = \langle D^b(S), E_1, \ldots, E_4 \rangle$$

where E_i are exceptional objects, and S is a complete intersection of six (1,1)-divisors in $\mathbb{P}^4 \times \mathbb{P}^4$. This allows us to compute many of the invariants of X.

Higher dimensions. The construction generalizes by taking $V = \mathbb{C}^n$ for $n \geq 5$, and we get similar double covers $\tau\colon W_4 \to Z_4$ of dimension $4n - 7$. The singular locus of W_4 is given by $\tau^{-1}(Z_2)$, which has dimension $2n - 2$. A general complete intersection

$$X = W_4 \cap H_1 \cap \ldots \cap H_{2n-1} \tag{3}$$

is a smooth Fano manifold of index 1 of dimension $2n - 6$ with $H^3(X, \mathbb{Z}) = \mathbb{Z}/2$.

Application. The main motivation of Theorem 1 comes from studying a birational invariant related to the two *coniveau filtrations*, as introduced in [3]. Let $N^1H^l(X, \mathbb{Z}) \subset H^l(X, \mathbb{Z})$ denote the subgroup of classes supported on proper subvarieties $Y \subset X$ and $\widetilde{N}^1H^l(X, \mathbb{Z})$ the subgroup generated by $\widetilde{j}_*\beta$ where $\widetilde{j}$ is a composition $\widetilde{Y} \xrightarrow{\pi} Y \hookrightarrow X$ where π is a desingularization. The quotient

$$N^1H^l(X, \mathbb{Z})/\widetilde{N}^1H^l(X, \mathbb{Z})$$

is an interesting stable birational invariant. Voisin [8] asked whether there is a rationally connected variety where this invariant is non-zero. Using our geometric construction, we show that the answer is affirmative:

Theorem 4 (O.-Rennemo)**.** *In any dimension ≥ 6, there is a Fano variety X where $N^1H^3(X, \mathbb{Z}) = \mathbb{Z}/2$ and $\widetilde{N}^1H^3(X, \mathbb{Z}) = 0$.*

By results of Bloch–Srinivas and Colliot-Thélène–Voisin [5], for any rationally connected variety X we have $N^1H^l(X, \mathbb{Z}) = H^l(X, \mathbb{Z})$ for all $l > 0$. On the other hand, there is the following criterion, proved in [3]:

Proposition 5. *If $\sigma \in N^1H^3(X, \mathbb{Z})$, then $\bar{\sigma}^2 = 0 \in H^6(X, \mathbb{Z}/2)$.*

To conclude, we need to show that the 2-torsion class $\sigma \in H^3(X, \mathbb{Z})$ for the varieties appearing in (3) have a non-zero square modulo 2. To do this, we use an alternative description of Z_4 and W_4, in terms of GIT quotients

$$\begin{aligned} Z_4 &= \mathrm{Hom}(\mathbb{C}^5, \mathbb{C}^4)/\!\!/\mathrm{GO}(4) \\ W_4 &= \mathrm{Hom}(\mathbb{C}^5, \mathbb{C}^4)/\!\!/\mathrm{GO}(4)^\circ. \end{aligned}$$

Here $\mathrm{GO}(4)$ is the *orthogonal similtude group*, i.e., the elements $g \in \mathrm{GL}(4)$ preserving the inner product up to scaling, and $\mathrm{GO}(4)^\circ$ is the connected component of the identity. This allows us to view W_4 as an "algebraic approximation" to $\mathrm{BGO}(4)^\circ$. Using topological arguments, we compute the relevant part of the cohomology ring of W_4 and thereby X to conclude that indeed $\bar{\sigma}^2 \neq 0 \in H^6(X, \mathbb{Z}/2)$.

References

[1] M. Artin, and D. Mumford. "Some elementary examples of unirational varieties which are not rational." Proceedings of the London Mathematical Society 3.1 (1972): 75–95.
[2] abx. "Does there exist a Fano variety with torsion in H^3?", Mathoverflow, `https://mathoverflow.net/q/180514`
[3] O. Benoist, Olivier, and J. C. Ottem. "Two coniveau filtrations." Duke Mathematical Journal 170, no. 12 (2021): 2719–2753.
[4] S. Bloch, and V. Srinivas. "Remarks on correspondences and algebraic cycles." American Journal of Mathematics 105, no. 5 (1983): 1235–1253.
[5] J.-L. Colliot-Thélène, and C. Voisin. "Cohomologie non ramifiée et conjecture de Hodge entière". Duke Math. J. 161(5): 735–80
[6] S. Hosono, and H. Takagi. "Determinantal quintics and mirror symmetry of Reye congruences." Communications in Mathematical Physics 329 (2014): 1171–1218.
[7] J. C. Ottem and J. V. Rennemo. "Fano varieties with torsion in the third cohomology group." Journal für die reine und angewandte Mathematik (Crelles Journal) (2024).
[8] C. Voisin. "On the coniveau of rationally connected varieties". Geometry & Topology, 26 (2022): 2731–2772.

Moduli of boundary polarized CY surface pairs

Harold Blum
(joint work with Yuchen Liu)

From the perspective of the Minimal Model Program, the building blocks of algebraic varieties are canonically polarized varieties, Calabi–Yau varieties, and Fano varieties. While the theories of KSBA stability and K-stability give projective moduli theories for canonically polarized varieties and Fano varieties, the moduli theory of Calabi–Yau varieties is less well understood.

In [6], we construct a moduli theory for Calabi–Yau pairs of low dimension that interpolates between the KSBA and K-stability approaches. To have such a theory, we consider boundary polarized CY surface pairs (X, D), which are slc pairs (X, D) such that $K_X + D \sim_{\mathbb{Q}} 0$, D is ample, and $\dim X = 2$. Examples of such pairs arise in the study of Fano and CY varieties in low dimension.

In special cases, both KSBA and K-moduli theory can be applied to construct compactifications of the moduli of such pairs. Indeed, if (X, D) is a klt boundary polarized CY pair, then

- $(X, (1+\varepsilon)D)$ is a KSBA stable canonically polarized pair and
- $(X, (1-\varepsilon)D)$ is a K-semistable log Fano pair

for $0 < \varepsilon \ll 1$. Thus the general KSBA and K-moduli machinery can be used to construct compactifications M^{KSBA} and M^{K} of the moduli of such pairs with fixed numerical invariants [4, 7, 8, 11]. These compactifications have been explicitly described in many interesting cases including pairs of the form $(\mathbb{P}^2, \frac{3}{d}C)$ where $C \subset \mathbb{P}^2$ is a smooth degree $d \leq 6$ curve [2, 4, 9], pairs $(X, \frac{1}{9}(l_1 + \cdots l_{27}))$ where $X \subset \mathbb{P}^3$ is a smooth cubic surface and $l_1, \ldots, l_{27}$ are its lines [10, 12], and pairs that arise as quotients of K3 surfaces with a non-symplectic automorphism [1, 2, 3].

Ideally, there would be a moduli space of CY pairs that interpolates between M^{K} and M^{KSBA} via wall crossing. A first step toward this goal was achieved in

[5], which constructs the moduli stack of boundary polarized CY pairs as a locally finite type algebraic stack and proves various properties of the stack.

A fundamental difficulty towards constructing a moduli space for the above stack is that the irreducible components of the stack are often not of finite type. A simple example of this failure is given by the unbounded family of toric pairs

$$(\mathbb{P}(a^2, b^2, c^2), \{xyz = 0\})$$

with $a^2 + b^2 + c^2 = 3abc$. These pairs all arise as limits of pairs of the form $(\mathbb{P}^2, C)$ where $C \subset \mathbb{P}^2$ is a smooth cubic curve. Despite this issue, [6] constructs the desired moduli space.

To state the main result, fix a family of boundary polarized CY surface pairs $(\mathcal{X}, \mathcal{D}) \to T$ over a finite type scheme T.

- Let $\mathcal{M}^{\mathrm{CY}}$ denote the seminormalization of the closure of T in the moduli stack of boundary polarized CY pairs. Concretely, the $\mathbb{C}$-points of $\mathcal{M}^{\mathrm{CY}}$ are boundary polarized CY pairs that are degenerations of the fibers of the original family $(\mathcal{X}, \mathcal{D}) \to T$.
- Let $\mathcal{M}^{\mathrm{K}}$ and $\mathcal{M}^{\mathrm{KSBA}}$ denote the open substacks of $\mathcal{M}^{\mathrm{CY}}$ defined using K-stability and KSBA stability for the perturbed pairs. Write M^{K} and M^{KSBA} for the projective good moduli spaces of these substacks.

The following result constructs a moduli space for $\mathcal{M}^{\mathrm{CY}}$.

Theorem 1 ([6])**.** *There exists a surjective morphism to a projective scheme*

$$\mathcal{M}^{\mathrm{CY}} \to M^{\mathrm{CY}}$$

satisfying the following properties:

(1) $M^{\mathrm{CY}}(\mathbb{C})$ *parametrizes S-equivalence classes of pairs in* $\mathcal{M}^{\mathrm{CY}}(\mathbb{C})$.

(2) *There are natural morphisms*

$$M^{\mathrm{K}} \to M^{\mathrm{CY}} \leftarrow M^{\mathrm{KSBA}}.$$

(3) *The Hodge line bundle on* M^{CY} *is ample.*

The result was previous proven when the general fiber of $\mathcal{X} \to T$ is isomorphic to $\mathbb{P}^2$ in [5] using properties of slc degenerations of $\mathbb{P}^2$ that do not hold for arbitrary del Pezzo surfaces.

In statement (1), we say that two boundary polarized CY pairs (X, D) and (X', D') are *S-equivalent* if they admit isotrivial degenerations

$$(X, D) \rightsquigarrow (X_0, D_0) \leftsquigarrow (X', D')$$

to a common boundary polarized CY pair (X_0, D_0). While the set of pairs $\mathcal{M}^{\mathrm{CY}}(\mathbb{C})$ can be unbounded, the theorem implies that the pairs are bounded modulo S-equivalence.

The moduli space M^{CY} may be viewed as an analog of the Baily–Borel compactification of the moduli space of K3 surfaces or abelian varieties in this setting. Indeed, the Hodge line bundle is ample on the moduli space and the boundary of M^{CY} parametrizes lower dimensional CY pairs with discrete data. In fact, in the special case of boundary polarized CY surface pairs that arise as quotients of K3 surfaces with a non-symplectic involution, the CY moduli space agrees with the corresponding Baily–Borel compactification up to normalization [6].

References

[1] V. Alexeev and P. Engel, *Compactifications of moduli spaces of K3 surfaces with a non-symplectic involution*. arXiv:2208.10383 (2022).

[2] V. Alexeev, P. Engel, and A. Thompson, *Stable pair compactification of moduli of K3 surfaces of degree 2*. J. Reine Angew. Math. **799** (2023), 1–56.

[3] V. Alexeev, P. Engel, and C. Han. *Compact moduli of K3 surfaces with a nonsymplectic automorphism* Trans. Amer. Math. Soc. Ser. B **11** (2024), 144–163.

[4] K. Ascher, K. DeVleming, and Y. Liu. *Wall crossing for K-moduli spaces of plane curves.* Proc. Lond. Math. Soc. **128** (2024), Paper No. e12615.

[5] K. Ascher, D. Bejleri, H. Blum, K. DeVleming, G. Inchiostro, Y. Liu, and X. Wang, *Moduli of boundary polarized Calabi–Yau pairs* arXiv:2307.06522 (2023).

[6] H. Blum and Y. Liu, *Good moduli spaces for boundary polarized Calabi–Yau surface pairs.* arXiv:2407.00850 (2024).

[7] C. Birkar, *Geometry of polarised varieties*, Publ. Math. Inst. Hautes Études Sci. **137** (2023), 47–105.

[8] J. Kollár and C. Xu. *Moduli of polarized Calabi–Yau pairs* Acta Math. Sin. **36** (2020), no.6, 631–637.

[9] P. Hacking, Compact moduli of plane curves. *Duke Math. J.*, **124** (2004), 213–257.

[10] P. Hacking, S. Keel, J. Tevelev. *Stable pair, tropical, and log canonical compactifications of moduli spaces of del Pezzo surfaces.* Invent. Math. **178** (2009), 173–227.

[11] C. Zhou. *On wall-crossing for K-stability.* Adv. Math. **413** (2023), Paper No. 108857.

[12] J. Zhao. *Compactifications of moduli of del Pezzo surfaces via line arrangement and K-stability.* Canad. J. Math. (2023).

A Monge–Ampère equation with applications to the SYZ conjecture

Enrica Mazzon

Mirror symmetry is a fast-moving research area at the boundary between mathematics and theoretical physics. Originated from observations in string theory, it suggests the existence of a duality between Calabi–Yau (CY) manifolds, complex manifolds with a nowhere vanishing holomorphic form of maximal degree. It predicts that every CY manifold X has a mirror partner $\check{X}$, such that the complex geometry of $\check{X}$ is equivalent to the symplectic geometry of X, in some appropriate sense, and vice versa.

Various approaches have been developed to find a rigorous definition of a mirror pair $(X, \check{X})$, and methods to construct mirror partners; a geometric explanation was proposed by Strominger, Yau and Zaslow (SYZ) in [12]. In its current formulation, the SYZ conjecture concerns CY manifolds in certain degenerating families rather than individual manifolds. More precisely, consider a projective family $(X_t)_t$ of CY varieties of dimension n over a punctured disk, such that the family is maximally degenerate, i.e. the monodromy operator on the degree n cohomology of X_t has a Jordan block of maximal size, that is $n+1$.

Conjecture 1 (SYZ conjecture)**.** *For all sufficiently small t, X_t admits a fibration $\pi : X_t \to B$, whose fibres are special Lagrangian tori, away from a locus Δ of* *codimension* 2 *in* B*. Moreover, the mirror partner* $\check{X}_t$ *of* X_t *is obtained by dualizing the special Lagrangian toric fibres of* π *and by suitably compactifying the resulting space.*

While some examples of special Lagrangian torus fibrations can be produced, dealing with the general case seems very difficult. The insight of Kontsevich and Soibelman is to replace the above conjecture by an analogous one in the non-archimedean world, and to interpret the latter as an asymptotic limit of the complex phenomenon when $t \to 0$.

More precisely, one can associate to the degenerating family $X = (X_t)_t$ the Berkovich non-archimedean space X^{an}, whose points are valuations defined locally on X. Given a degeneration $\mathcal{X}$ of X, we say that $\mathcal{X}$ is snc (respectively dlt) if the pair $(\mathcal{X}, \mathcal{X}_0)$ is strict normal crossing (respectively divisorially log terminal), where $\mathcal{X}_0$ is fiber over $t = 0$; see [6] for more details. Given any snc or dlt degeneration, the dual intersection complex $\mathcal{D}(\mathcal{X}_0)$ is a simplicial complex encoding the combinatorics of the multiple intersections of the components of $\mathcal{X}_0$. It admits a canonical embedding in X^{an}, whose image is called the skeleton of $\mathcal{X}$ and denoted $\mathrm{Sk}(\mathcal{X})$, and a retraction $\rho_{\mathcal{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathcal{X})$. Among various degenerations, minimal (in the sense of MMP) dlt models $\mathcal{X}$ of X determine a canonical skeleton $\mathrm{Sk}(X) = \mathrm{Sk}(\mathcal{X})$, called the essential skeleton of X and independent of the choice of the minimal model; see [10, 11] for more details. The essential skeleton and the retractions $\rho_{\mathcal{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathcal{X})$ are of particular relevance in the non-archimedean approach to the SYZ conjecture, as we will see in the sequel.

By the celebrated Yau theorem, X_t carries a unique Kähler form ω_t such that $[\omega_t] \in c_1(L_t)$ and $\omega_t^n = C_t \Omega_t \wedge \overline{\Omega}_t$ for a constant C_t. Finding such form ω_t boils down to solving an equation, called complex Monge–Ampère equation. By the works [4, 3], non-archimedean Monge–Ampère equations can be defined on X^{an} as well, and solved for any measure supported on a subset $\mathcal{D}(\mathcal{X}_0)$ of X^{an}. In particular, let Ψ be the solution to

$$\mathrm{MA}_{\mathrm{NA}}(\Psi) = d\mu_{\mathrm{Sk}(X)},$$

where $\mathrm{MA}_{\mathrm{NA}}$ denotes the non-archimedean Monge–Ampère operator, and $d\mu_{\mathrm{Sk}(X)}$ the Lebesgue measure on $\mathrm{Sk}(X)$. Up to fixing a reference metric, we can think of Ψ as a function on X^{an}. In [9] Li reduced the SYZ conjecture to a conjecture in non-archimedean geometry about Monge–Ampère metrics

Theorem 2 ([9])**.** *Let* X *be a maximally degenerate family of Calabi–Yau varieties. If there exists a degeneration* $\mathcal{X}$ *of* X *such that the solution* Ψ *is invariant with respect to the retraction* $\rho_{\mathcal{X}}$*, i.e.*

$$\Psi = \Psi \circ \rho_{\mathcal{X}} \quad \text{on} \quad \rho_{\mathcal{X}}^{-1}(\mathrm{Int}(\tau)),$$

over the interior of any maximal face τ *of* $\mathcal{D}(\mathcal{X}_0)$*, then an SYZ fibration exists on a large region* $U_t \subseteq X_t$

This approach reduces the construction of SYZ fibrations to a property in non-archimedean geometry. In [5], Hultgren, Jonsson, McCleerey and I provided first evidence for such conjecture, when X is not one-dimensional or an abelian variety. More precisely, let $X = \{z_0 z_1 \dots z_{n+1} + tf(z) = 0\} \subset \mathbb{P}^{n+1}$ be a family of Calabi–Yau hypersurfaces where f is a generic polynomial of degree $n+2$. In this case, the essential skeleton $\mathrm{Sk}(X)$ is a sphere and can be identified with the boundary $A := \partial\Delta$ of the standard unit simplex Δ in $\mathbb{R}^{n+1}$.

Theorem 3 ([5])**.** *If ν is a symmetric measure on A, the solution to $\mathrm{MA}_{\mathrm{NA}}(\cdot) = \nu$ is the restriction of a symmetric toric metric on $\mathcal{O}_{\mathbb{P}^{n+1}}(n+2)^{\mathrm{an}}$, and thus is determined by the restriction to A of a convex function on $\mathbb{R}^{n+1}$.*

Applying Theorem 3 to $\nu = d\mu_{\mathrm{Sk}(X)}$, we show that the characterization of the solution Ψ provided by the theorem is sufficient to prove the invariance property of Theorem 2. We conclude therefore that SYZ fibrations exist on large regions of Calabi–Yau hypersurfaces X.

In analogy to Yau's theorem, we study the regularity of the solution for suitably regular measures ν. More precisely, we equip the standard unit simplex $A = \mathrm{Sk}(X)$ with a singular special affine structure, in which the regular set A_{reg} is the union of the interiors of the n-dimensional faces together with the open stars of the vertices of A in the barycentric subdivision of A (the co-dimension of A_{reg} is 2). In [2] we prove the following result.

Theorem 4. *Let ν a symmetric measure on A of the form $\nu = f\, d\mu_A$, where μ_A is Lebesgue measure on A and $f \in C^\infty(A_{\mathrm{reg}})$ satisfies $0 < \inf f \leq \sup f < \infty$. Then the metric on A_{reg} induced by ϕ is C^∞, strictly convex, and uniformly $C^{1,\alpha}$ for some $\alpha > 0$.*

This effectively produces a Monge–Ampère metric outside a set of codimension 2, confirming a widespread expectation related to the SYZ conjecture.

References

[1] R. Andreasson and J. Hultgren. *Solvability of Monge-Ampère equations and tropical affine structures on reflexive polytopes* . arXiv:2303.05276.

[2] R. Andreasson, J. Hultgren, M. Jonsson, E. Mazzon and N. McCleerey. *Regularity of the solution to a real Monge–Ampère equation on the boundary of a simplex.* arXiv:2403.01620.

[3] S. Boucksom, C. Favre, and M. Jonsson. *Solution to a non-Archimedean Monge–Ampère equation.* Journal of the American Mathematical Society **3** (2015), 617–667.

[4] A. Chambert-Loir. *Mesures et équidistribution sur les espaces de Berkovich.* Journal für die Reine und Angewandte Mathematik **595** (2006), 215–235.

[5] J. Hultgren, M. Jonsson, E. Mazzon and N. McCleerey. *Tropical and non-Archimedean Monge–Ampère equations for a class of Calabi–Yau hypersurfaces.* Adv. Math. **439** (2024), 109494.

[6] J. Kollár. *Singularities of the minimal model program.* Cambridge University Press, Cambridge , 200 (2013).

[7] M. Kontsevich and Y. Soibelman. *Affine structures and non-Archimedean analytic spaces.* In *The unity of mathematics.* Progr. Math. 244, 321–385. Birkhäuser Boston, Boston, MA, 2006.

[8] Y. Li. *Metric SYZ conjecture for certain toric Fano hypersurfaces.* Cambridge J. Math. **12** (2024), 223–252.

[9] Y. Li. *Metric SYZ conjecture and non-archimedean geometry.* Duke Math. J. **172** (2023), 3227–3255.

[10] M. Mustaţă and J. Nicaise. *Weight functions on non-archimedean analytic spaces and the Kontsevich–Soibelman skeleton.* Algebr. Geom. **2** (2015), 365–404.
[11] J. Nicaise and C. Xu. *The essential skeleton of a degeneration of algebraic varieties.* American Journal of Mathematics **138** (2016), 1645–1667.
[12] A. Strominger, S.-T. Yau, E. Zaslov. *Mirror Symmetry is T-duality.* Nucl. Phys. B479, 243–259, 1996.

On the Torelli problem for Calabi–Yau 3-folds, and the integral cohomology of the Fermat quintic

Nicolas Addington
(joint work with Benjamin Tighe)

For me a *Calabi–Yau 3-fold* is a smooth complex projective 3-fold X with $\omega_X = \mathcal{O}_X$ and $b_1(X) = 0$, but I allow $\pi_1(X) \neq 0$.

There is no global Torelli theorem for Calabi–Yau 3-folds; the best counterexample is due independently to Ottem and Rennemo [12] and to Borisov, Căldăraru, and Perry [5]. Take the Grassmannian $\mathrm{Gr}(2,5)$ in its Plücker embedding in $\mathbb{P}^9$, and a matrix $A \in \mathrm{GL}_{10}(\mathbb{C})$, and set $X = \mathrm{Gr} \cap A(\mathrm{Gr})$ and $Y = \mathrm{Gr} \cap A^\top(\mathrm{Gr})$. If A is general then X and Y are not isomorphic, and therefore not birational because $\mathrm{Pic}(X) = \mathrm{Pic}(Y) = \mathbb{Z}$: a birational map between them would factor as a sequence of flops, but varieties with Picard number 1 do not admit any contractions. On the other hand, Kuznetsov and Perry [10] proved that $D^b(X) \cong D^b(Y)$, so $H^3(X,\mathbb{Z}) \cong H^3(Y,\mathbb{Z})$ as polarized Hodge structures by [2, footnote 8]. But perhaps one should hope for a derived Torelli theorem for Calabi–Yau 3-folds, as one has with K3 surfaces?

In general, for a pair of Calabi–Yau 3-folds X and Y, each of the following implies the next:

(i) X and Y are isomorphic.
(ii) X and Y are birational.
(iii) $D^b_{\mathrm{coh}}(X) \cong D^b_{\mathrm{coh}}(Y)$.
(iv) $H^3(X,\mathbb{Z})/\mathrm{torsion} \cong H^3(Y,\mathbb{Z})/\mathrm{torsion}$ as polarized Hodge structures.
(v) $H^3(X,\mathbb{Q}) \cong H^3(Y,\mathbb{Q})$ as polarized Hodge structures.

The implication (ii) ⇒ (iii) is due to Bridgeland [6], and (iii) ⇒ (iv), as already mentioned, is [2, footnote 8], with a fuller account given by Ottem and Rennemo in [12, Prop. 2.1]. The reverse implications, if they held, would be Torelli-type theorems, but they tend to fail. Counterexamples to (ii) ⇒ (i) were first given by Szendrői [15]. We have already seen the strongest counterexample to (iii) ⇒ (ii), but earlier counterexamples (with X and Y in different deformation classes) were given by Borisov and Căldăraru [4], Schnell [14], and Hosono and Takagi [9]. The latter two also demonstrate the need to kill torsion in (iv), as I observed in [1]. But (v) ⇒ (iv) holds generically for quintics, as Voisin explained in [17, Rmk. 0.3], so one might expect it to hold more broadly.

The implication (iv) ⇒ (iii) would be a derived Torelli theorem. No counterexample is known, but the following construction of Aspinwall and Morrison [3] was a promising candidate. Take the Dwork pencil of quintic 3-folds

$$Q_t = \{z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 5t \cdot z_0z_1z_2z_3z_4\} \subset \mathbb{CP}^4,$$

with the well-known Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & & 1 & & \\ 1 & 101 & & 101 & 1 \\ & & 1 & & \\ & & 1 & & \end{array}$$

To obtain the famous mirror family to the quintic, one first divides Q_t by a $(\mathbb{Z}/5)^3$ action generated by the automorphisms

$$\begin{aligned} (z_0 : z_1 : z_2 : z_3 : z_4) &\mapsto (z_0 : \xi z_1 : \xi^2 z_2 : \xi^3 z_3 : \xi^4 z_4), \\ (z_0 : z_1 : z_2 : z_3 : z_4) &\mapsto (z_0 : \xi z_1 : \xi^3 z_2 : \xi z_3 : z_4), \\ (z_0 : z_1 : z_2 : z_3 : z_4) &\mapsto (\xi z_0 : \xi^{-1} z_1 : z_2 : z_3 : z_4), \end{aligned}$$

where $\xi = e^{2\pi/5}$, then takes a crepant resolution of singularities, for which the most natural choice is Nakamura's $(\mathbb{Z}/5)^3$-Hilbert scheme of Q_t, and ends up with a Calabi–Yau 3-fold with $\pi_1 = 0$ and Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & & 101 & & \\ 1 & 1 & & 1 & 1 \\ & & 101 & & \\ & & 1. & & \end{array}$$

In a variation on this construction, Aspinwall and Morrison started by dividing Q_t by the first two automorphisms above, which give a $(\mathbb{Z}/5)^2$ acting freely apart from 50 points with stabilizer $\mathbb{Z}/5$, to get a 3-fold $\bar{Z}_t$ with 10 isolated singularities of type $\frac{1}{5}(1,1,3)$. Next, they took a crepant resolution of singularities to get a smooth Calabi–Yau 3-fold Z_t with $\pi_1 = 0$ and Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & & 21 & & \\ 1 & 1 & & 1 & 1 \\ & & 21 & & \\ & & 1. & & \end{array}$$

Finally, they divided Z_t by the automorphism that cyclically permutes the coordinates,

$$(z_0 : z_1 : z_2 : z_3 : z_4) \mapsto (z_1 : z_2 : z_3 : z_4 : z_0),$$

which is a free $\mathbb{Z}/5$ action, yielding a Calabi–Yau 3-fold Y_t and $\pi_1 = \mathbb{Z}/5$ and Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & & 5 & & \\ 1 & 1 & & 1 & 1 \\ & & 5 & & \\ & & 1. & & \end{array}$$

Aspinwall and Morrison gave a physics calculation of the genus-0 and genus-1 Gromov–Witten invariants of the mirror family, which suggested that Y_t and $Y_{\xi t}$ should have the same Hodge structure on H^3, but different derived categories of coherent sheaves. Szendrői [16] observed that the pullback $H^3(Y_t, \mathbb{Q}) \to H^3(Z_t, \mathbb{Q})$

is a Hodge isometry (up to multiplying the intersection form by 5), and that $Z_t \cong Z_{\xi t}$ via the map

$$(z_0 : z_1 : z_2 : z_3 : z_4) \mapsto (\xi^{-1} z_0 : z_1 : z_2 : z_3 : z_4),$$

so $H^3(Y_t, \mathbb{Q}) \cong H^3(Y_{\xi t}, \mathbb{Q})$. On the other hand, he proved that Y_t and $Y_{\xi t}$ are not isomorphic, because such an isomorphism would specialize to an automorphism of the central fiber Y_0 which he ruled out. He conjectured that Y_t and $Y_{\xi t}$ are not birational, and later that they are not derived equivalent.

Our initial hope was to upgrade to an integral Hodge isometry $H^3(Y_t, \mathbb{Z}) \cong H^3(Y_{\xi t}, \mathbb{Z})$ and then prove that $D^b(Y_t) \ncong D^b(Y_{\xi t})$; it seemed impractical to emulate Szendrői's argument by specializing to $t = 0$ and getting a complete description of the autoequivalence group of $D^b(Y_0)$, but we hoped to mathematicize Aspinwall and Morrison's genus-1 calculation using either the categorical enumerative invariants of Căldăraru and Tu [7], or the BCOV invariant of Fang, Lu, and Yoshikawa [8]. Instead, we managed to prove:

Theorem 1. *If t is very general, then $H^3(Y_t, \mathbb{Z}) \ncong H^3(Y_{\xi t}, \mathbb{Z})$, even as unpolarized Hodge structures. In particular, Y_t and $Y_{\xi t}$ are neither birational nor derived equivalent.*

So Aspinwall and Morrison's example, rather that showing that (iv) does not imply (iii), instead shows that (v) does not imply (iv) even generically. We still do not know whether (iv) implies (iii); we doubt it, but we do not know where to look for counterexamples.

The idea of the proof is that if t is very general, then the only Hodge isomorphism from $H^3(Y_t, \mathbb{Q})$ to $H^3(Y_{\xi t}, \mathbb{Q})$ is the one discussed above, but this not respect the integral structure. We delete the 50 points of Q_t at which $(\mathbb{Z}/5)^2$ does not act freely, as well as the 20 exceptional divisors of Z_t and the four exceptional divisors of Y_t, which does not change $H^3(-, \mathbb{Z})$ but leaves us with free quotients everywhere, so we can use several Cartan–Leray spectral sequences of the form $E_2^{p,q} = H^p(G, H^q(X)) \Rightarrow H^{p+q}(X/G)$. The boundary maps are subtle to analyze, and we require an explicit set of cocycles generating $H^3(Q_0, \mathbb{Z})$. One possibility is to take the Poincaré dual of the real locus $Q_0(\mathbb{R}) \cong \mathbb{RP}^3$ and argue that its $\mathrm{Aut}(Q_0)$-orbit spans $H^3(Q_0, \mathbb{Z})$; we thought we had discovered this, but later learned that it already appeared in the string theory literature [18, §5]. Another possibility is to follow Looijenga [11, §2] in using a more complicated S^3 in $Q_0 \setminus \{z_0 = 0\}$ constructed by Pham [13].

References

[1] N. Addington. The Brauer group is not a derived invariant. In *Brauer groups and obstruction problems*, volume 320 of *Progr. Math.*, pages 1–5. Birkhäuser/Springer, Cham, 2017. Also arXiv:1306.6538.

[2] N. Addington, W. Donovan, and C. Meachan. Moduli spaces of torsion sheaves on K3 surfaces and derived equivalences. *J. Lond. Math. Soc. (2)*, 93(3):846–865, 2016. Also arXiv:1507.02597.

[3] P. S. Aspinwall and D. R. Morrison. Chiral rings do not suffice: $N = (2, 2)$ theories with nonzero fundamental group. *Phys. Lett. B*, 334(1-2):79–86, 1994. Also hep-th/9406032.

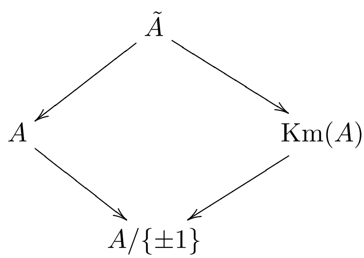
[4] L. Borisov and A. Căldăraru. The Pfaffian-Grassmannian derived equivalence. *J. Algebraic Geom.*, 18(2):201–222, 2009. Also math/0608404.
[5] L. A. Borisov, A. Căldăraru, and A. Perry. Intersections of two Grassmannians in $\mathbb{P}^9$. *J. Reine Angew. Math.*, 760:133–162, 2020. Also arXiv:1707.00534.
[6] T. Bridgeland. Flops and derived categories. *Invent. Math.*, 147(3):613–632, 2002. Also math/0009053.
[7] A. Căldăraru and J. Tu. Effective categorical enumerative invariants. Preprint, arXiv:2404.01499.
[8] H. Fang, Z. Lu, and K. Yoshikawa. Analytic torsion for Calabi-Yau threefolds. *J. Differential Geom.*, 80(2):175–259, 2008. Also math/0601411.
[9] S. Hosono and H. Takagi. Double quintic symmetroids, Reye congruences, and their derived equivalence. *J. Differential Geom.*, 104(3):443–497, 2016. Also arXiv:1302.5883.
[10] A. Kuznetsov and A. Perry. Categorical joins. *J. Amer. Math. Soc.*, 34(2):505–564, 2021. Also arXiv:1804.00144.
[11] E. Looijenga. Fermat varieties and the periods of some hypersurfaces. In *Algebraic and arithmetic structures of moduli spaces (Sapporo 2007)*, volume 58 of *Adv. Stud. Pure Math.*, pages 47–67. Math. Soc. Japan, Tokyo, 2010. Also arXiv:1005.1733.
[12] J. C. Ottem and J. V. Rennemo. A counterexample to the birational Torelli problem for Calabi-Yau threefolds. *J. Lond. Math. Soc. (2)*, 97(3):427–440, 2018. Also arXiv:1706.09952.
[13] F. Pham. Formules de Picard-Lefschetz généralisées et ramification des intégrales. *Bull. Soc. Math. France*, 93:333–367, 1965.
[14] C. Schnell. The fundamental group is not a derived invariant. In *Derived categories in algebraic geometry*, EMS Ser. Congr. Rep., pages 279–285. Eur. Math. Soc., Zürich, 2012. Also arXiv:1112.3586.
[15] B. Szendrői. Calabi–Yau threefolds with a curve of singularities and counterexamples to the Torelli problem. *Internat. J. Math.*, 11(3):449–459, 2000. Also math/9901078.
[16] B. Szendrői. On an example of Aspinwall and Morrison. *Proc. Amer. Math. Soc.*, 132(3):621–632, 2004. Also math/9911064.
[17] C. Voisin. Schiffer variations and the generic Torelli theorem for hypersurfaces. *Compos. Math.*, 158(1):89–122, 2022. Also arXiv:2004.09310.
[18] J. Walcher. Landau-Ginzburg models in real mirror symmetry. *Ann. Inst. Fourier (Grenoble)*, 61(7):2865–2883, 2011.

Hyper-Kummer construction

Lie Fu
(joint work with Salvatore Floccari)

Starting from a 2-dimensional complex torus A, the classical Kummer construction produces a K3 surface $S := \mathrm{Km}(A)$ as the crepant resolution of the quotient surface $A/\{\pm 1\}$. S contains 16 disjoint (-2)-curves, corresponding to the 16 2-torsion points of A. We have the following cartesian commutative diagram:

(1)



Kummer surfaces can be characterized among all K3 surfaces in various ways, and there are many relations between the geometry of S and the equivariant geometry of A with the $G = \mathbb{Z}/2$-action:

(i) A projective K3 surface S is Kummer, iff there is a primitive embedding of the Kummer lattice L_{Km} into the Néron–Severi lattice $\mathrm{NS}(S)$, iff its transcendental lattice $H^2_{\mathrm{tr}}(S, \mathbb{Z})$ can be primitively embedded into $U(2)^3$, iff S contains 16 disjoint (-2)-curves.

(ii) There is a derived equivalence $D^b(S) \cong D^b([A/G])$.

(iii) There is an isomorphism of Chow motives $h(S) \cong h_{orb}([A/G])$ as algebra objects, where h_{orb} stands for the orbifold motive.

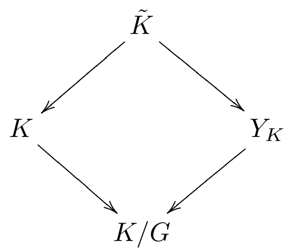
Starting from a 6-dimensional compact hyper-Kähler manifold of generalized Kummer type, there is an analog of the Kummer construction discovered by Floccari in [2], that we call *hyper-Kummer construction.*

Let us briefly summarize the hyper-Kummer construction. Let K be a compact hyper-Kähler manifold of generalized Kummer type. Let $\mathrm{Aut}_0(-)$ denote the group of automorphisms of a hyper-Kähler variety acting trivially on its second cohomology. Hassett and Tschinkel proved in [1] that Aut_0 is a deformation invariant and for a hyper-Kähler manifold K deformation equivalent to a generalized Kummer 6-fold, $\mathrm{Aut}_0(K) \cong (\mathbb{Z}/4)^4 \rtimes \mathbb{Z}/2$. Let G be the subgroup of $\mathrm{Aut}_0(K)$ generated by involutions whose fixed locus has a codimension-2 part. Then $G \cong (\mathbb{Z}/2)^5$. A main result in [2] says that blowing up the reduced singular locus of the quotient K/G yields a crepant resolution

$$Y_K \to K/G,$$

which is a hyper-Kähler manifold of $K3^{[3]}$-type. We call Y_K the *hyper-Kummer variety* associated to K. Alternatively, one can first blow-up K along the union of the fixed loci of elements in G. The G-action lifts to the blow-up and the quotient is nothing but Y_K. We have a cartesian diagram:

(2)



A hyper-Kähler manifold Y of $K3^{[3]}$-type is called *hyper-Kummer*, if there is a hyper-Kähler 6-fold K of Kummer type such that $Y \cong Y_K$.

In fact, there are also many more hyper-Kähler varieties of dimension 4 and 2 canonically arising from this situation: the fixed loci of various involutions in G as well as the crepant resolutions of their quotients by G. Without giving the full detailed definition, let me just say that there are

- sixteen 4-dimensional $K3^{[2]}$-type hyper-Kähler varieties $\{W_i\}_{i=1}^{16}$ arising as components of fixed loci in K, all isomorphic to each other,
- $\binom{16}{2}$ K3 surfaces $\{V_{i,j}\}_{1\leq i<j\leq 16}$ arising as components of fixed loci in K (falling into 15 isomorphism classes),
- for any $1 \leq i \leq 16$, a crepant resolution M_i of W_i/G,
- for any $1 \leq i < j \leq 16$, a crepant resolution $S_{i,j}$ of $V_{i,j}/G$.

Let us illustrate in the example when $K = K_3(A)$ for a 2-dimensional complex torus A. We have $\mathrm{Aut}_0(K) = A[4] \rtimes \{\pm 1\}$ and $G = A[2] \times \{\pm 1\}$. For any $\tau \in A[2]$, the fixed locus of $(\tau, -1) \in G$ has a 4-dimensional component, denoted by W_τ, which is hyper-Kähler of $K3^{[2]}$-type (in fact birational to $\mathrm{Km}(A)^{[2]}$). W_τ is equipped with the induced G-action, and blowing up the singular locus of the quotient W_τ/G gives rise to a crepant resolution

$$M_\tau \to W_\tau/G,$$

which is again a hyper-Kähler variety of $K3^{[2]}$-type (in fact, again birational to $\mathrm{Km}(A)^{[2]}$). Similarly, for any $\tau_1 \neq \tau_2 \in A[2]$, the K3 surface $V_{\tau_1,\tau_2} := W_{\tau_1} \cap W_{\tau_2}$ is a connected component of the fixed locus of the involution $(\tau_1 + \tau_2, 1) \in G$ (in fact isomorphic to $\mathrm{Km}(A)$). V_{τ_1,τ_2} is globally preserved by G and the minimal resolution of the quotient is a K3 surface S_{τ_1,τ_2}, again isomorphic to $\mathrm{Km}(A)$.

We first give a birational characterization of the hyper-Kummer varieties among projective hyper-Kähler varieties of $K3^{[3]}$-type:

Theorem 3 (Birational characterization)**.** *Let Y be a projective hyper-Kähler varieties of $K3^{[3]}$-type. The following are equivalent:*

- *Y is birationally hyper-Kummer: there exists a hyper-Kähler 6-fold K of generalized Kummer type, and Y is birational to Y_K;*
- *There is a primitive embedding of the Kummer lattice into the Néron–Severi lattice of Y,*

$$j\colon L_{\mathrm{Km}} \hookrightarrow \mathrm{NS}(Y),$$

such that $j(L_{\mathrm{Km}})^{\perp_{H^2(Y,\mathbb{Z})}}$ is isometric to $U(2)^3 \oplus \langle -4\rangle$.

Concerning the relation between the geometry of Y_K and the equivariant geometry of K with G-action, we prove the following results:

Theorem 4. *Let K be a hyper-Kähler 6-fold of generalized Kummer type. Let G be the abelian group $(\mathbb{Z}/2\mathbb{Z})^5$ acting on Y as before. Let Y_K be the associated hyper-Kummer variety. Then*

- *Y_K is isomorphic to the equivariant Hilbert scheme $G{-}\mathrm{Hilb}(K)$, and via this isomorphism $\tilde{K}$ is identified with the universal G-cluster.*
- *There is a derived equivalence $D^b(Y_K) \cong D^b([K/G])$.*
- *There is an isomorphism of rational Chow motives $h(Y_K) \cong h_{\mathrm{orb}}([K/G])$.*

As for the relation between the hyper-Kähler varieties K, Y_K and the lower-dimensional ones W_i, M_i, $V_{i,j}$ and $S_{i,j}$, we have

Theorem 5. *For any $i \neq j$, the pull-backs along natural rational maps induce the following Hodge isometries between transcendental lattices*

$$H^2_{\mathrm{tr}}(Y_K, \mathbb{Z}) \to H^2_{\mathrm{tr}}(M_i, \mathbb{Z}) \to H^2_{\mathrm{tr}}(S_{i,j}, \mathbb{Z}). \tag{6}$$

In particular, all $S_{i,j}$'s are (abstractly) isomorphic.

References

[1] B. Hassett and Y. Tschinkel *Hodge theory and Lagrangian planes on generalized Kummer fourfolds*, Mosc. Math. J. 13 (2013), no. 1, p. 33–56

[2] S. Floccari *Sixfolds of generalized Kummer type and K3 surfaces*, Compositio Mathematica, Volume 160, Issue 2, February 2024, pp. 388–410

Reporter: Riccardo Carini

Participants

Anna Abasheva
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Nicolas Addington
Department of Mathematics
University of Oregon
Eugene, OR 97403-1222
UNITED STATES

Prof. Dr. Arend Bayer
School of Mathematics
University of Edinburgh
James Clerk Maxwell Building
Peter Guthrie Tait Road
Edinburgh EH9 3FD
UNITED KINGDOM

Prof. Dr. Pieter Belmans
Université du Luxembourg
6, Avenue de la Fonte
4364 Esch-sur-Alzette
LUXEMBOURG

Dr. Olivier Benoist
CNRS – DMA
École Normale Supérieure
45 rue d'Ulm
75230 Paris Cedex 05
FRANCE

Prof. Dr. Harold Blum
Department of Mathematics
University of Utah
Salt Lake City, UT 84112
UNITED STATES

Dr. Alessio Bottini
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Emma Brakkee
Mathematisch Instituut
Universiteit Leiden
Postbus 9512
2300 RA Leiden
NETHERLANDS

Dr. Riccardo Carini
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Francois Charles
DMA
École Normale Supérieure
45, rue d'Ulm
75005 Paris
FRANCE

Prof. Dr. Giulio Codogni
Dipartimento di Matematica
Universita degli Studi di Roma II
Tor Vergata
Via della Ricerca Scientifica
00133 Roma
ITALY

Dr. Kristin DeVleming
University of Massachusetts Amherst
Department of Mathematics
710 N Pleasant St.
Amherst, MA 01003
UNITED STATES

Prof. Dr. Stéphane Druel
Institut Camille Jordan
Université Claude Bernard de Lyon 1
43, Bd. du 11 Novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Philip M. Engel
Hausdorff Institute
University of Bonn
02138 Bonn
GERMANY

Dr. Soheyla Feyzbakhsh
Department of Mathematics
Imperial College London
614 Huxley Building
South Kensington Campus
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Dr. Laure Flapan
Department of Mathematics
Michigan State University
619 Red Cedar Road
East Lansing, MI 48824
UNITED STATES

Prof. Dr. Lie Fu
I.R.M.A.
Université de Strasbourg
67084 Strasbourg Cedex
FRANCE

Dr. Kento Fujita
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-1, Toyonaka
Osaka 560-0043
JAPAN

Prof. Dr. Daniel Huybrechts
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Lena Ji
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Dr. Anne-Sophie Kaloghiros
Department of Mathematics and
Statistics
Brunel University
Kingston Lane
Uxbridge Middlesex UB8 3PH
UNITED KINGDOM

Prof. Dr. Emanuele Macri
Laboratoire de Mathématiques d'Orsay
Université Paris-Saclay
Bâtiment 307
Rue Michel Magat
91405 Orsay Cedex
FRANCE

Dr. Svetlana Makarova
Fakultät für Mathematik
Universität Duisburg-Essen
Essen 45127
GERMANY

Dr. Lisa Marquand
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street
New York, NY 10012-1110
UNITED STATES

Dr. Mirko Mauri
Centre de Mathématiques
École Polytechnique
Plateau de Palaiseau
91128 Palaiseau Cedex
FRANCE

Dr. Enrica Mazzon
Department of Mathematics
University of Regensburg
93040 Regensburg
GERMANY

Prof. Dr. Ben Moonen
Department of Mathematics
Radboud Universiteit Nijmegen
Postbus 9010, Postvak 59
6500 GL Nijmegen
NETHERLANDS

Prof. Dr. Kieran Gregory O'Grady
Dipartimento di Matematica
"Guido Castelnuovo"
Sapienza Università di Roma
Piazzale Aldo Moro, 5
00185 Roma
ITALY

Prof. Dr. Keiji Oguiso
Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. John Christian Ottem
Department of Mathematics
University of Oslo
Blindern
P.O. Box 1053
0316 Oslo
NORWAY

Dr. Tudor Pădurariu
Institut de Mathématiques de Jussieu
Case 247
Université de Paris VI
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Zsolt Patakfalvi
EPFL SB MATH CAG
MA C3 635
Station 8
École Polytechnique Fédérale de Lausanne
1015 Lausanne
SWITZERLAND

Prof. Dr. Alexander Perry
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor MI 48109
UNITED STATES

Dr. Laura Pertusi
Dipartimento di Matematica
"Federigo Enriques"
Università degli Studi di Milano
Via Cesare Saldini 50
20133 Milano
ITALY

Dr. Giulia Saccà
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Ed Segal
Department of Mathematics
University College London
Gower Street
London WC1E 6BT
UNITED KINGDOM

Prof. Dr. Junliang Shen
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Prof. Dr. Evgeny Shinder
School of Mathematics and Statistics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UNITED KINGDOM

Spela Spenko
Department of Mathematics
Université Libre de Bruxelles
1050 Bruxelles
BELGIUM

Prof. Dr. Paolo Stellari
Dipartimento di Matematica
"Federigo Enriques"
Università di Milano
Via Saldini 50
20133 Milano
ITALY

Dr. Richard P. W. Thomas
Department of Mathematics
Imperial College of Science and
Technology
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Michel Van den Bergh
Universiteit Hasselt
Martelarenlaan 42
Hasselt 3500
BELGIUM

Nicolás Vilches Reyes
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Chenyang Xu
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Prof. Dr. Qizheng Yin
Beijing International Center for
Mathematical Research (BICMR)
Peking University
5 Yiheyuan Road, Haidian
Beijing 100871
CHINA

Dr. Xiaolei Zhao
Department of Mathematics
University of California at Santa Barbara
South Hall
Santa Barbara, CA 93106
UNITED STATES

Yifan Zhao
Imperial College London
Department of Mathematics
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Dr. Ziquan Zhuang
Department of Mathematics
Johns Hopkins University
3400 N. Charles Street, Krieger Hall 404
Baltimore, MD 21218
UNITED STATES