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# Recent Developments in Dirichlet Form Theory and Related Fields

Organized by
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Abstract. Theory of Dirichlet forms is one of the main achievements in modern probability theory. It has numerous interactions with other areas of mathematics and sciences. The recent notable developments are its role in the study of Liouville Brownian motion, Gaussian free field, stochastic partial differential equations, stochastic analysis on metric measure spaces, and Markov processes in random environments.

The workshop brings together top experts in Dirichlet form theory, stochastic analysis and related fields, with the common theme of developing new foundational methods and their applications to specific areas of probability.

*Mathematics Subject Classification (2020):* 31C25, 60J25, 60H15, 46E36, 60K37.



## Introduction by the Organizers

The workshop *Recent Developments in Dirichlet Form Theory and Related Fields*, organized by Zhen-Qing Chen (Seattle), Michael Röckner (Bielefeld), Masayoshi Takeda (Osaka) and Anita Winter (Duisburg) was well attended with 41 on site participants from 9 different countries in Europe, Asia and North America. In addition, 9 virtual participants took part in the event. The workshop united persons with various backgrounds. Their interests and methods display exhilarating variety and resourcefulness, while all being directly or partly related to Dirichlet form theory with its vast scope in analysis and stochastic. The results shared and vividly discussed within a dynamic group of high profile scientists and promising young researchers cope with some of toady's most eminent and relevant problems in the field. Young researchers were given the opportunity to present their work, in particular during the session of short communications held on Tuesday afternoon.

A brief overview of covered topics conveys the following picture, with strong focus on stochastic analysis on general or infinite-dimensional state spaces, stochastic partial differential equations, non-local operators and Lévy noises, Markov processes in random environment and the theory of non-linear Dirichlet forms.

- *Theory of regular Dirichlet forms and Hunt processes;* comprising limit laws such as small deviations principle (M. Gordina); boundary traces of local Dirichlet forms (N. Kajino); boundary Harnack principle and Green function estimates for non-local operators (J.-M. Wang) as well as for Markov processes with jump kernels decaying at the boundary (P. Kim); and the Harnack inequality for weakly coupled non-local systems (X. Meng).
- *Time-changed Brownian motion with singular and/or random data;* including distorted Brownian motion with permeable sticky behaviour on sets of Lebesgue measure zero (M. Grothaus), heat kernel estimates for the Brox diffusion (J. Wang), Liouville Brownian motion and Liouville Cauchy process (T. Ooi).
- *Scaling limits in random environments;* primarily the study of ergodic random conductance models including the quenched local central limit theorem for long range random walks (T. Kigami), scaling limit of the harmonic crystal and Green's function (S. Andres), as well as the quenched invariance principle for non-symmetric random walks with cycle decomposition (J.-D. Deuschel); and also the study of the intrinsic metric for two-dimensional critical percolation clusters (J. Miller).
- *Gaussian free fields in discrete or continuum;* with their crucial role in some aforementioned contributions (S. Andres, T. Ooi) and the discussion of the discrete Gaussian free field on the infinite binary tree under a hard wall condition (L. Hartung).
- *Large particle systems and infinite-dimensional diffusions;* such as stochastic reaction-diffusion equations (W.-T. Fan); strongly correlated infinite particle systems (H. Osada); infinite Dyson Brownian motion with focus on Bakry–Émery estimates and dual flows (K. Suzuki); dynamical particles modeling networks of noisy neurons and their mean field limit (B. Hambly); stochastic quantization of the three-dimensional Edwards measure (S. Kusuoka) and quasi-regularity of Dirichlet forms on Wasserstein space (S. Wittmann).
- *Stochastic partial differential equations with Lévy noise;* including analysis of irreducibility for additive or multiplicative pure jump Lévy noise (T. Zhang); the construction of Hunt processes via the Lyapunov method with application to infinte dimensional Lévy driven Ornstein-Uhlenbeck processes (I. Cîmpean); and spatial asymptotic behaviors of fractional stochastic heat equations with Lévy white noise (Y. Shiozawa).
- *Analysis for second order SDE and FPK equations;* finding convergence rate of 1/2 for the Euler-Maruyama scheme for kinetic stochastic differential equations

with singular coefficients (Z. Hao) and refuting the parabolic Harnack inequality for the fractional Fokker-Planck-Kolmogorov equation (M. Kaßmann).

- *Theory of non-linear Dirichlet forms and Sobolev spaces;* including extended Dirichlet spaces and criticality theory (M. Schmidt); quasiregular mappings (L. Beznea) and Riesz transform (K. Kuwae) given a Dirichlet form on a Lusin measurable space; or diverse techniques to construct $p$-energy forms on metric spaces such as the Sierpiński carpet (M. Murugan), Cheeger spaces (P. Alonso Ruiz), as well as metric spaces possessing conductive homogeneity (J. Kigami) or weak monotonicity (R. Shimizu).
- Further related topics with high relevancy and a larger scope; such as non-symmetric perturbations of closed forms (T. Uemura); the Einstein relation connecting Hausdorff, local walk and spectral dimension (U. Freiberg); the geometric aspect of Navier-Stokes equations (S. Fang) and principles of probability (J. Swanson).

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## Workshop: Recent Developments in Dirichlet Form Theory and Related Fields

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# Abstracts

## Quenched local central limit theorem for random conductance models

Takashi Kumagai
(joint work with Xin Chen, Jian Wang)

Let $\{C_{x,y}(\omega) : x, y \in \mathbb{Z}^d\}$ be non-negative numbers with $C_{x,y}(\omega) = C_{y,x}(\omega)$ for all $x, y \in \mathbb{Z}^d$ (called the conductance) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A model of reversible random walk on random media with transition probability given by $P(x, y) := C_{x,y}/\sum_{z\in\mathbb{Z}^d} C_{x,z}$ for all $x, y \in \mathbb{Z}^d$ is called random conductance model (RCM). During last decades, there has been a lot of work concerning quenched invariance principle (QIP), namely the convergence of the associated scaled processes to Brownian motion for a.s. sample of random conductances, for the nearest neighbor random walk (NNRW) in RCMs (see [7, 15] for lecture notes in this area). In particular, the QIP was proved in [1] under general conditions on *i.i.d.* random conductances including supercritical Bernoulli percolation cluster. The quenched local central limit theorem (QLCLT) is proved by Barlow and Hambly ([4]) for the random walk on the supercritical Bernoulli percolation cluster and by Croydon and Hambly ([13]) for more general situations.

QIP and QLCLT for the NNRW on ergodic RCMs also attract lots of interests. Assume the following moment condition on the conductance:

$$C_{x,y}(\omega) \in L^p(\Omega; \mathbb{P}),\ \frac{1}{C_{x,y}(\omega)} \in L^q(\Omega; \mathbb{P}), \quad x, y \in \mathbb{Z}^d \text{ with } x \sim y \tag{1}$$

with $p, q \geq 1$. (Here we write $x \sim y$ when $x$ is the nearest neighbor of $y$.)

Under the moment condition $1/p + 1/q < 2/d$ in (1), Andres, Deuschel and Slowik [2] proved the QIP for the NNRW in ergodic RCMs. Bella and Schäffner [5] then improved the moment condition above into $1/p+1/q < 2/(d-1)$, which was illustrated to be nearly optimal for the everywhere sublinearty of the corrector by a counterexample given in [8, Theorem 2.5]. Under the same moment conditions, the QLCLT for the NNRW on ergodic RCMs was shown by Andres, Deuschel and Slowik [3] and Bella and Schäffner [6], respectively. QIP and QLCLT for the time-inhomogeneous NNRW in space-time ergodic random media have also been investigated recently.

For the long range random walk (LRRW) on ergodic RCMs, proving the QIP is harder due to the effects of long range jumps. For the LRRW with $L^2$ integrable jumping kernel in ergodic RCMs, Biskup, Chen, Kumagai and Wang [8] proved the QIP by introducing the time-change arguments as well as an idea of locally modifying environments. For the LRRW with heavy tails in the sense that the jumping kernel is not $L^2$ integrable, the limit process is no longer Brownian motion. The readers are referred to [9, 10, 12, 14] etc. for more details. We note that the simple random walk on the open cluster of long range percolation belongs to the class of RCMs with long range jumps.

Framework and main theorem. Let $\{\tau_x : x \in \mathbb{Z}^d\}$ be shift operators on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\tau_0 = id$ and $\tau_{x+y} = \tau_x \circ \tau_y$ for every $x, y \in \mathbb{Z}^d$. We assume that $d \geq 2$, and $\{\tau_x : x \in \mathbb{Z}^d\}$ is stationary and ergodic on $(\Omega, \mathcal{F}, \mathbb{P})$, namely

(i) $\mathbb{P}(\tau_x A) = \mathbb{P}(A)$ for every $A \in \mathcal{F}$ and $x \in \mathbb{Z}^d$;
(ii) if there exists $A \in \mathcal{F}$ such that $\tau_x A = A$ for every $x \in \mathbb{Z}^d$, then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$;
(iii) the map $(x, \omega) \mapsto \tau_x \omega$ is $\mathcal{B}(\mathbb{Z}^d) \times \mathcal{F}$ measurable.

Let $\{C_{x,y}(\omega) : x, y \in \mathbb{Z}^d\}$ be a class of non-negative symmetric random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$C_{x,y}(\omega) = C_{y,x}(\omega), \quad C_{x+z,y+z}(\omega) = C_{x,y}(\tau_z \omega), \quad x, y, z \in \mathbb{Z}^d, \ \omega \in \Omega.$$

For any fixed environment $\omega \in \Omega$, consider the operator $\mathcal{L}^\omega$ on $L^2(\mathbb{Z}^d; \lambda)$:

$$\mathcal{L}^\omega f(x) := \sum_{y \in \mathbb{Z}^d} \left(f(y) - f(x)\right) C_{x,y}(\omega), \quad f \in L^2(\mathbb{Z}^d; \lambda), \tag{2}$$

where $\lambda$ is the counting measure on $\mathbb{Z}^d$. For a.s. any fixed $\omega \in \Omega$, $\mathcal{L}^\omega$ is the infinitesimal generator for the variable speed random walk (VSRW), $(X_t^\omega)_{t \geq 0}$ corresponding to the conductance $\{C_{x,y}(\omega) : x, y \in \mathbb{Z}^d\}$. We note that when $C_{x,y}(\omega) = 0$ for a.s. $\omega \in \Omega$ and every $x, y \in \mathbb{Z}^d$ with $|x - y| > 1$, the above process $(X_t^\omega)_{t \geq 0}$ is the NNRW in RCMs.

For the LRRW we will consider the following moment condition

$$\tilde{\mu}(\omega) := \sum_{z \in \mathbb{Z}^d} |z|^2 C_{0,z}(\omega) \in L^p(\Omega; \mathbb{P}), \quad \tilde{\nu}(\omega) := \sum_{z \in \mathbb{Z}^d : |z| = 1} \frac{1}{C_{0,z}(\omega)} \in L^q(\Omega; \mathbb{P}) \tag{3}$$

instead of (1). The following assumption is made in [8].

**Assumption 1.** *The condition* (3) *holds with some* $p, q \in (1, +\infty]$ *such that*

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{d}.$$

In [8, Theorem 2.1], it is proved that under Assumption 1 the QIP holds for the LRRW $(X_t^\omega)_{t \geq 0}$ such that the limit process is Brownian motion with (constant) diffusion matrix $M = (M_{ij})_{1 \leq i,j \leq d}$ that is defined by

$$M_{ij} := \mathcal{E}\left[\sum_{z \in \mathbb{Z}^d} C_{0,z}\left(z_i + \chi_i(z)\right)\left(z_j + \chi_j(z)\right)\right], \quad 1 \leq i, j \leq d, \tag{4}$$

where $\chi = (\chi_1, \cdots, \chi_d) : \mathbb{Z}^d \times \Omega \to \mathbb{R}^d$ is the corrector.

In order to study the QLCLT, we need the following stronger assumption.

**Assumption 2.** *Suppose that* (3) *holds with some* $p, q \in (1, +\infty]$ *that satisfy*

$$\frac{1}{p} + \frac{1}{q} \leq \left(1 + \frac{1}{p}\right)\frac{1}{d}, \quad \frac{1}{p-1} + \frac{1}{q} < \frac{2}{d}, \tag{5}$$

*and that either the following condition holds*

$$\tilde{\mu}_{d+2}(\omega) := \sum_{z\in\mathbb{Z}^d} |z|^{d+2} C_{0,z}(\omega) \in L^p(\Omega;\mathbb{P}) \tag{6}$$

*or* $q = +\infty$ *(which is equivalent to* $\inf_{x,y\in\mathbb{Z}^d:|x-y|=1} C_{x,y} > 0$*).*

Let $p^\omega(t,x,y)$ be the heat kernel of the process $(X_t^\omega)_{t\geq 0}$, and write $k_M(t,x)$ for the Gaussian heat kernel with the diffusion matrix $M$ defined by (4). Our main theorem in [11] is the following.

**Theorem 3.** *Under Assumption* 2*, for any* $T_2 > T_1 > 0$ *and* $R > 1$*,*

$$\lim_{n\to\infty} \sup_{|x|\leq R} \sup_{t\in[T_1,T_2]} |n^d p^\omega(n^2 t, 0, [nx]) - k_M(t,x)| = 0,$$

*where* $[x] = ([x_1],[x_2],\cdots,[x_d])$ *for any* $x = (x_1,x_2,\cdots,x_d)\in\mathbb{R}^d$*.*

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## Construction of distorted Brownian motion with permeable sticky behaviour on sets with Lebesgue measure zero

Martin Grothaus
(joint work with Torben Fattler, Nathalie Steil)

The starting point is a gradient Dirichlet form with respect to $\varrho\lambda^d$ on $L^2(\mathbb{R}^d, \varrho\mu)$. Here $\lambda^d$ is the Lebesgue measure on $\mathbb{R}^d$, $\varrho$ a strictly positive density and $\mu$ puts weight on a set $A \subset \mathbb{R}^d$ with Lebesgue measure zero. We show that the Dirichlet form admits an associated stochastic process $X$. We derive an explicit representation of the corresponding generator if $A$ is a Lipschitz boundary. This representation together with the Fukushima decomposition identifies $X$ as a distorted Brownian motion with drift given by the logarithmic derivative of $\varrho$ in $\mathbb{R}^d \setminus A$. Furthermore, we prove $X$ to be irreducible and recurrent. Finally, via ergodicity we prove positive séjour time of $X$ on $A$. Hence we obtain a stochastic process $X$ with permeable sticky behaviour on $A$.

Details can be found in [1].

### References

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## Towards the scaling limit of the intrinsic metric for two-dimensional critical percolation clusters

Jason Miller
(joint work with Valeria Ambrosio, Yizheng Yuan)

We consider the conformal loop ensembles ($\mathrm{CLE}_\kappa$) in the regime $\kappa \in (4, 8)$, which is the range of parameter values where the loops intersect themselves, each other, and the domain boundary. We show that a natural approximation procedure to construct the chemical distance metric in the gasket of a ($\mathrm{CLE}_\kappa$), the set of points not surrounded by a loop, is tight and that every subsequential limit is a geodesic metric. We further show that the limit is unique by arguing that there is at most one metric on the ($\mathrm{CLE}_\kappa$) gasket satisfying some natural axioms. We conjecture that for $\kappa = 6$ this metric describes the scaling limit of the intrinsic metric for $2D$ percolation.

## Limit laws in metric measure spaces

Maria Gordina
(joint work with Marco Carfagnini, Alexander Teplyaev)

Suppose $(X, d)$ is a locally compact separable metric space, and $\mu$ is a Radon $\sigma$-finite measure on the Borel $\sigma$-algebra over $X$. We consider a regular Dirichlet form $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ on $L^2(X, \mu)$. We denote by $A$ the non-negative self-adjoint generator for the Dirichlet form $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$, and by $\{X_t\}_{t \geqslant 0}$ $X$-valued (Hunt) stochastic process with $X_0 = x_0 \in X$ a.s.

The main subject of this talk is based on several papers written with Marco Carfagnini, and one paper written with Marco Carfagnini and Alexander Teplyaev. We are interested in several limit laws for the stochastic process $X_t$ as described below. Let $|X_t| := d\left(X_t, x_0\right)$, $x_0 \in X$.

We say that $X_t$ satisfies a *small deviations principle* with rates $\alpha$ and $\beta$ if there exists a constant $c > 0$ such that

$$\lim_{\varepsilon \to 0} -\varepsilon^{\alpha} |\log \varepsilon|^{\beta} \log \mathbb{P}\left( \max_{0 \leqslant t \leqslant 1} |X_t| < \varepsilon \right) = c.$$

We say that $X_t$ satisfies *Chung's LIL* with rate $a > 0$ if there exists a constant $C > 0$ such that

$$\liminf_{t \to \infty} \left( \frac{\log \log t}{t} \right)^{a} \max_{0 \leqslant s \leqslant t} |X_s| = C \quad \text{a.s.}$$

We are also interested in finding the asymptotics for a continuous process $X_t$ given by an *Onsager-Machlup functional*

$$\mathbb{P}\left( \max_{0 \leqslant t \leqslant 1} d\left(X_t, \varphi(t)\right) < \varepsilon \right) \qquad \text{as } \varepsilon \to 0, \qquad \varphi \in W_{x_0}\left(\mathcal{X}\right)$$

Establishing such laws and finding not only parameters, but also identifying limits is closely related to the spectral properties of the infinitesimal generator $A$ restricted to a metric ball in $X$.

Recall that if $U$ is a Borel subset of $X$, then one can define the *restricted Dirichlet form* $(\mathcal{E}, \mathcal{D}_{\mathcal{E}}(U))$ by considering the domain to be the functions $\mathcal{D}_{\mathcal{E}}(U) \subset \mathcal{D}_{\mathcal{E}}$ that vanish outside of $U$. We denote by $A_U$ the negative self-adjoint operator on $L^2\left(U, \mu\right)$ corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}_{\mathcal{E}}(U))$. We can view $A_U$ as the operator corresponding to the Dirichlet boundary problems with zero boundary values. If $U$ is open and $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ is a regular Dirichlet form on $L^2(X, \mu)$, then $(\mathcal{E}, \mathcal{D}_{\mathcal{E}}(U))$ is a regular Dirichlet form on $L^2(U, \mu)$ as well.

**Proposition 1** (Carfagnini, Gordina, Teplyaev). *Suppose $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ is a regular Dirichlet form on $L^2\left(X, \mu\right)$. Let $U$ be an open set in $X$ such that $0 < \mu(U) < \infty$, and let $P_t^U$ be the semigroup associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}_{\mathcal{E}}\left(U\right))$ with the infinitesimal generator $A_U$.*

*If* ess $\sup_{(x,y) \in U \times U} p_t\left(x, y\right) < \infty$ *for some $t > 0$, then $A_U$ has a discrete spectrum.*

*If in addition there exists a $t_U > 0$ such that*

$$\operatorname*{ess\,sup}_{(x,y)\in U\times U} p_{t_U}(x,y) < \frac{1}{\mu(U)^2}, \tag{1}$$

*then the first eigenvalue $\lambda_1^U$ of $-A_U$ is strictly positive.*

We establish such spectral results under very mild conditions such as the Nash inequality. Note that no regularity of the boundary is assumed, as our applications are for metric balls where such regularity is often not known. J. Kigami proved this estimate in the case of self-similar p.c.f. fractals The preprint by Carfagnini, Gordina, Teplyaev gives more detailed estimates for more general ultracontractive cases and under more specific heat kernel bounds.

**Theorem 2** (Carfagnini, Gordina, Teplyaev)**.** *Suppose that the measure metric space $(X, d, \mu)$ and $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ is a regular Dirichlet form on $L^2(X, \mu)$ satisfying the usual assumptions. If in addition the Dirichlet from satisfies the Nash inequality, then for any open set $U$ of finite measure, the spectrum is discrete and eigenfunctions satisfy*

$$\|\varphi_n\|_{L^\infty} \leqslant c\lambda_n^\delta,$$

*where $c$ is a constant depending on $U$, $\alpha$, and $\beta$.*

Here $\delta = \frac{\alpha}{\beta}$, $(M, d, \mu)$ is Ahlfors $\alpha$-regular, $\beta$ is the time scaling exponent for a (distance-)self-similar process

$$d(X^x_{t\varepsilon}, x) \stackrel{d}{=} \varepsilon^{\frac{1}{\beta}} d(X^x_t, x).$$

As one of the applications, we prove the following small deviations principle.

**Theorem 3** (Carfagnini, Gordina, Teplyaev)**.** *Assume that $P_t^{B_1(x)}$ is irreducible for some $x \in X$ and the heat kernel $p_t^{B_1(x)}(x,y)$ exists for all $t$ and for all $x, y \in X$ and that*

$$p_t^{B_1(x)}(x,y) \leqslant c\, t^{-\frac{\alpha}{\beta}} \text{ for any } t, x, y,$$

*there exists a $t_0$ such that $p_{t_0}^{B_1(x)}(x,y)$ is continuous for $x, y \in X$.*

*Suppose that $X_t^x$ is self-similar, then the following small deviations principle holds.*

$$\lim_{\varepsilon\to 0} e^{\lambda_1 \frac{t}{\varepsilon^\beta}} \mathbb{P}^x \left( \sup_{0\leqslant s\leqslant t} d(X_s, x) < \varepsilon \right) = c_1\varphi_1(x),$$

*where $\lambda_1 > 0$ is the spectral gap of $\mathcal{L}^{B_1(x)}$ , $\varphi_1$ is the corresponding positive eigenfunction, $c_n := \int_{B_1(x)} \varphi_n(y)\mu(dy)$.*

These results first were proven for Carnot groups by Carfagnini-Gordina, and under different sets of assumptions hold for Riemannian manifolds with non-negative Ricci curvature, self-similar processes, not necessarily continuous; Brownian motion on fractals; Dirichlet forms on metric measure spaces under Sturm's assumptions (such as complete closed balls, doubling, weak Poincaré, PHI); group action on metric measure spaces. We also prove convergence of spectra under approximate dilations.

**Quenched invariance principle for random walks in random environments admitting a cycle decomposition**

Jean-Dominique Deuschel
(joint work with Martin Slowik, Weile Weng)

This paper deals with time-continuous random walk on $\mathbb{Z}^d$ having a finite but unbounded cycle decomposition. Random walks with finite cycle decomposition, also called centered walks (cf. [13]), are the discrete version of diffusions on $\mathbb{R}^d$ in divergence form generated by

$$\big(Lf\big)(x) \;=\; \operatorname{div}\big(c(x)\nabla f(x)\big),$$

where $c(x) = s(x)+a(x)$ with $s(x)^* = s(x)$ being the symmetric and $a^*(x) = -a(x)$ being the anti-symmetric part of the matrix $c(x) \in \mathbb{R}^{d\times d}$. Assuming uniform ellipticity and boundedness:

$$\lambda \;<\; s(x) \;<\; \Lambda \qquad \text{and} \qquad |a(x)| \;<\; B,$$

the celebrated results of Moser–De Giorgi–Nash–Aronson show parabolic and elliptic Harnack inequalities and Gaussian type estimates for the corresponding heat kernel, cf. [3], [14], [15], [6]. Moreover, Papanicolaou and Varadhan proved in [16] a quenched invariant principle (QIP), i.e. the almost sure convergence for the diffusively rescaled diffusion in ergodic environment, see also Fannjiang–Komorovski [9, 10, 5] for the unbounded case.

Random walks with finite cycle decomposition generated by

$$\big(Lf\big)(x) \;=\; \sum_y c(x,y)\,\big(f(y)-f(x)\big)$$

have jump rates $c(x,y) \geq 0$ given in terms of oriented cycles $\gamma = (x_0, x_1, \ldots, x_n = x_0)$ of length $\ell(\gamma) = n$ and weights $w(\gamma) \geq 0$,

$$c(x,y) \;=\; \sum_\gamma w(\gamma)\,\mathbb{1}_{(x,y)\in\gamma}.$$

The corresponding (non-symmetric) Dirichlet form is determined by

$$\mathcal{E}(f,g) \;=\; \sum_\gamma w(\gamma)\,\mathcal{E}_\gamma(f,g), \quad \text{where} \quad \mathcal{E}_\gamma(f,g) \;=\; \sum_{x_i\in\gamma} f(x_i)\big(g(x_i)-g(x_{i+1})\big).$$

In particular the counting measure is invariant, and the jump rates of the adjoint generator, $L^*$, are given in terms of the reversed cycles $\gamma^* = (x_n, x_{n-1}, \ldots, x_0 = x_n)$ with weight $w^*(\gamma^*) = w(\gamma)$. In the special case that all the cycles have length $\ell(\gamma) = 2$, then $\gamma = \gamma^*$, and the associated random walk is reversible with symmetric jump rates called conductances.

In a paper with Kumagai [8], we showed that diffusions in divergence form can be approximate by random walks on $\frac{1}{n}\mathbb{Z}^d$ those jump rates admits a cycle decomposition with bounded cycle length and bounded weights. With Koesters [7] the quenched invariant principle (QIP) is derived for uniformly elliptic random walks in ergodic random environments with bounded cycle decomposition and bounded weights. In the symmetric situation, i.e. dealing with cycles of length 2, the QIP has been proven for non-elliptic random conductance models under suitable moment conditions, cf. [2, 4].

The objective of the current paper is to obtain the QIP in the setting of unbounded cycle length, and non-uniform elliptic, i.e. unbounded and degenerate weights under appropriate moment conditions. In particular our result extends the QIP for the reversible random conductance model in degenerated ergodic environment to the non-symmetric case under similar moment conditions. When the cycle lengths are unbounded, the random walk does not satisfy the standard sector condition

$$\left|\mathcal{E}(f,g)\right|^2 \;\le\; \text{const}\cdot\mathcal{E}(f,f)\cdot\mathcal{E}(g,g).$$

which is usually assumed for the derivation of the invariance principle in non-symmetric situations, cf. [11].

Our proof of the QIP follows the general strategy of previous papers, e.g. [2] that deals with the symmetric case in the setting of non-uniform elliptic conductance model, that is, we first introduce harmonic coordinates and construct the corrector. Next, using PDE techniques adapted to the discrete setting we show the almost sure sub-linearity of the corrector. The main challenge here is to deal with and control the non-symmetric part of the generator. In particular, in view of the unboundedness of the cycle length and the loss of the sector condition, the construction of the corrector by means of the Lax-Milgram theorem necessitate a moment condition on the weighted length of cycles, which implies a bound on the $\mathcal{H}_{-1}$-norm of the corresponding drift which is a natural condition in the setting of double stochastic generators, cf. [12]. The next step is to control the corrector, here we adapt the analytical technique used in [1] of the De Giorgi iteration. While the Sobolev and weighted Poincaré inequalities relying on the symmetric part are identical to the reversible case, the corresponding energy estimate is, in view of the non-symmetry, more challenging. Here again the moment condition on the weighted length is implemented.

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## Stochastic analysis for strongly correlated, infinite particle systems

Hirofumi Osada

### 1. Introduction

A strongly correlated infinite particle system in Euclidean space is typically an infinite number of particles interacting through the Coulomb potential. A Coulomb random point field (RPF) in two-dimensional space is known as the Ginibre RPF only when the inverse temperature $\beta$ is 2. A more strongly correlated model is the set of the zero points of the planar Gaussian analytic function (GAF). These RPFs have different geometric properties from Gibbs and Poisson RPFs, reflecting their strong correlation. The equilibrium state of the unlabeled dynamics manifests as RPFs on $\mathbb{R}^d$, and the equilibrium state categorizes the systems.

There are three types of RPFs: (1) Potential type. (2) Kernel type. (3) Zero points of random analytic functions.

The examples of these systems are Coulomb and Riesz RPFs for (1), determinantal RPFs for (2), and the zero points of the planar Gaussian analytic function for (3).

We construct the unlabeled diffusions for these RPFs and represent the labeled dynamics as a unique, strong solution of infinite-dimensional SDEs (ISDE).

## 2. Coulomb and Riesz interacting Brownian motions

Let $\Psi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$, $d \geq 2$, be the $d$-dimensional Coulomb potential such that

$$\Psi(x) = \frac{1}{d-2}\frac{1}{|x|^{d-2}} \quad \text{for } d \geq 3, \quad \Psi(x) = -\log|x| \quad \text{for } d = 2. \tag{1}$$

The Coulomb RPF $\mu_{d,\beta}$ is a sub-sequential weak limit, with confining potential $\Phi_N$,

$$\mu_{d,\beta} = \lim_{N\to\infty} \frac{1}{\mathscr{Z}_N} \exp\{-\beta \sum_{i<j}^{N} \Psi(x_i - x_j) - \beta \sum_{k=1}^{N} \Phi_N(s_k)\} \prod_{k=1}^{N} ds_k. \tag{2}$$

We assume that $\Phi_N$ and $\nabla\Phi_N$ converge to $\Phi$ and $\nabla\Phi$ uniformly on each compact set.

Let $\mathscr{D}_{\bullet,\mathrm{b}}$ (resp. $\mathscr{D}_{\circ,\mathrm{b}}$) be the set of smooth (resp. local) function of the configuration space with bounded derivatives. Let $\mathscr{E}^{\mu_{d,\beta}}$ be the Dirichlet form on $L^2(\mu_{d,\beta})$ defined by

$$\mathscr{E}^{\mu_{d,\beta}}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g]\mu_{d,\beta}(d\mathsf{s}). \tag{3}$$

Here, $\mathsf{S}$ be the configuration space over $\mathbb{R}^d$ and $\mathbb{D}$ is the standard carré du champ on $\mathsf{S}$.

We call a measurable function $\mathfrak{l} : \mathsf{S} \to (\mathbb{R}^d)^{\mathbb{N}} \cup_{n=1}^{\infty} (\mathbb{R}^d)^n$ a label if $\mathsf{s} = \sum_i \delta_{\mathfrak{l}^i(\mathsf{s})}$. Let $\mathfrak{l} = (\mathfrak{l}^n)$ be a label such that $|\mathfrak{l}^i(\mathsf{s})| \leq |\mathfrak{l}^{i+1}(\mathsf{s})|$. Let $N_{\mathsf{n}}$ be an increasing sequence of natural numbers such that the limit in the right-hand side of (2) exists if we take $N = N_{\mathsf{n}}$.

**Theorem 1** (H.O.–S. Osada)**.** *For each $\beta > 0$, $(\mathscr{E}^{\mu_{d,\beta}}, \mathscr{D}_{\bullet,\mathrm{b}})$ is closable on $L^2(\mathsf{S}, \mu_{d,\beta})$. The closure of $(\mathscr{E}^{\mu_{d,\beta}}, \mathscr{D}_{\bullet}^{\mu_{d,\beta}})$ is a strongly local, quasi-regular Dirichlet form on $L^2(\mathsf{S}, \mu_{d,\beta})$. The labeled diffusion $\mathbf{X} = (X^i)_{i=1}^{\infty}$, given by the Dirichlet form, satisfies the ISDE:*

$$dX_t^i = dB_t^i + \beta\Big\{\nabla\Phi(X_t^i) + \lim_{\mathsf{n}\to\infty} \sum_{j=1, j\neq i}^{N_{\mathsf{n}}} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^d}\Big\}dt, \quad \mathbf{X}_0 = \mathbf{s}. \tag{Cln}$$

*ISDE* (Cln) *has a unique strong solution for $\mu_{d,\beta} \circ \mathfrak{l}^{-1}$-a.s. $\mathbf{s}$. If, in addition, $\mu_{d,\beta}$ satisfies*

$$\lim_{R\to\infty} \sum_i (1_{S_R}(x-y) - 1_{S_R}(y)) \frac{x - s^i}{|x - s^i|^d} = \nabla\Phi(x) \quad \textit{in } L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathsf{S}, \mu_{d,\beta}^{[1]}) \tag{4}$$

*and $\lim_{\mathsf{n}\to\infty} R_{\mathsf{n}}/|\mathfrak{l}^{N_{\mathsf{n}}}(\mathsf{s})| = 1$, then $\mathbf{X} = (X^i)$ is a unique strong solution of*

$$dX_t^i = dB_t^i + \beta\Big\{\lim_{\mathsf{n}\to\infty} \sum_{|X_t^i - X_t^j| < R_{\mathsf{n}}, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^d}\Big\}dt, \quad \mathbf{X}_0 = \mathbf{s}. \tag{Cln'}$$

*Here $\mu_{d,\beta}^{[1]}$ is the one-Campbell measure of $\mu_{d,\beta}$.*

We define the Riesz RPF $\nu_{d,\gamma,\beta}$ replacing $\Psi$ in (1) by the Riesz potential

$$\Psi(x) = \frac{1}{d-2+\gamma}\frac{1}{|x|^{d-2+\gamma}} \tag{5}$$

**Theorem 2** (H.O.–S. Osada). *Let $d \geq 1$. Then $(\mathscr{E}^{\nu_{d,\gamma,\beta}}, \mathscr{D}_{\bullet,\mathrm{b}})$ is closable on $L^2(\mathsf{S}, \nu_{d,\gamma,\beta})$. The closure of $(\mathscr{E}^{\nu_{d,\gamma,\beta}}, \mathscr{D}_{\bullet}^{\nu_{d,\gamma,\beta}})$ is a strongly local, quasi-regular Dirichlet form on $L^2(\mathsf{S}, \nu_{d,\gamma,\beta})$. The labeled diffusion $\mathbf{X} = (X^i)_{i=1}^{\infty}$, given by the Dirichlet form, satisfies*

$$dX_t^i = dB_t^i + \beta\Big\{\lim_{R\to\infty}\sum_{|X_t^i - X_t^j|<R, j\neq i}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{d+\gamma}}\Big\}dt \quad i \in \mathbb{N}. \tag{Rsz}$$

(Rsz) *has a unique strong solution for $\nu_{d,\gamma,\beta}\circ\mathfrak{l}^{-1}$-a.s. $\mathsf{s}$.*

## 3. Determinantal RPFs and the Gaussian analytic functions

Let $mdx$ be a Radon measure on $\mathbb{R}^d$. Let $K(x,y)$ be a determinantal kernel such that $K(x,y)$ is Hermitian symmetric on $L^2(\mathbb{R}^d, mdx)$ and locally of trace class satisfying $0 \leq \mathrm{spec}(K) \leq 1$. Then there exists a unique determinantal RPF $\mu_K$ given by $(K, mdx)$.

**Theorem 3** (H.O.-S. Osada). *$(\mathscr{E}^{\mu_K}, \mathscr{D}_{\bullet,\mathrm{b}})$ and $(\mathscr{E}^{\mu_K}, \mathscr{D}_{\circ,\mathrm{b}})$ are closable on $L^2(\mathsf{S}, \mu_K)$. The closure of $(\mathscr{E}^{\mu_K}, \mathscr{D}_{\circ,\mathrm{b}})$ on $L^2(\mathsf{S}, \mu_K)$ is a strongly local, quasi-regular Dirichlet form. We have the $\mu_K$-reversible diffusion properly associated with the Dirichlet form.*

Let $\{\zeta_k\}$ be a sequence of i.i.d. Gaussian random variable with mean free and unit variance in $\mathbb{C}$ defined on $(\Omega, P_{\mathscr{G}})$. We consider the Gaussian entire function

$$F_{\mathrm{planar}}(z) = \sum_{k=0}^{\infty}\frac{\zeta_k}{\sqrt{k!}}z^k. \tag{6}$$

The planar GAF $\mu_{\mathrm{pgaf}}$ is a RPF given by the zero points of $F_{\mathrm{planar}}$. $\mu_{\mathrm{pgaf}}$ is translation and rotation invariant. We identify $\mathbb{C}$ as $\mathbb{R}^2$ and regard $\mu_{\mathrm{pgaf}}$ as a RPF on $\mathbb{R}^2$. Let

$$e_k(\mathsf{s}) = \sum_{1\leq j_1<\cdots<j_k\leq n}\frac{1}{s^{j_1}\cdots s^{j_k}}\ (k \leq n), \quad e_k(\mathsf{s}) = 0\ (k > n), \tag{7}$$

$$\mathcal{G}_N(\mathsf{s}) = \frac{N+1}{\sum_{k=0}^{N}k!|e_k(\mathsf{s})|^2}, \quad \mathcal{H}_j(\mathsf{s}) = \sum_{k=0}^{\infty}k!e_{k+j+1}(\mathsf{s})\overline{e_k(\mathsf{s})}, \quad j \geq 1. \tag{8}$$

There exists a function $\mathcal{G}$ on $\mathsf{S}$ such that the random variable $\mathcal{G}(\mathsf{s}(\omega))$ and $\mathcal{H}^{[2]}$

$$\lim_{N\to\infty}\mathcal{G}_N(\mathsf{s}^N(\omega)) = \mathcal{G}(\mathsf{s}(\omega)) \text{ a.s.},$$

$$\mathcal{H}^{[2]}(x,z,\mathsf{s}) := \sum_{j=1}^{\infty}(-1)^j(x^j - z^j)\mathcal{H}_j(\delta_x + \delta_z + \mathsf{s}).$$

It is known that the random variable $\mathcal{G}(\mathsf{s}(\omega))$ has the $\chi^2$-distribution under $P_{\mathscr{G}}$.

Let $a = \mathcal{G}(\delta_x + \delta_z + \mathsf{s})$ and $b_j = \mathcal{H}_j(\delta_x + \delta_z + \mathsf{s})$. They are constant on each component of tail decomposition of $\mu_{\mathrm{pgaf}}$ and invariant under the dynamics. Let

$$\mathsf{d}^{\mu_{\mathrm{pgaf}}}(x,z,\mathsf{s}) =$$
$$2\frac{x-z}{|x-z|^2} + 2\lim_{R\to\infty}\sum_{\substack{|x-s^i|<R\\|z-s^i|<R}}\Big(\frac{x-s^i}{|x-s^i|^2} - \frac{z-s^i}{|z-s^i|^2}\Big) + a\sum_{j=1}^{\infty}(-1)^j(x^j - z^j)b_j.$$

**Theorem 4** (S. Ghosh, H. O., S. Osada, T. Shirai, K.A.Tran)**.** (i) *$(\mathscr{E}^{\mu_{\mathrm{pgaf}}}, \mathscr{D}_{\bullet,\mathrm{b}})$ is closable on $L^2(\mathsf{S}, \mu_{\mathrm{pgaf}})$. The closure of $(\mathscr{E}^{\mu_{\mathrm{pgaf}}}, \mathscr{D}_{\bullet,\mathrm{b}})$ on $L^2(\mathsf{S}, \mu_{\mathrm{pgaf}})$ is a strongly local, quasi-regular Dirichlet form and the associated $\mu_{\mathrm{pgaf}}$-reversible diffusion $\mathsf{X} = \sum_i \delta_{X^i}$ exists.*
(ii) *The labeled dynamics $\mathbf{X} = (X^i)$ of $\mathsf{X} = \sum_i \delta_{X^i}$ is a unique strong solution of*

$$X^i_t - X^{i+1}_t - (X^i_0 - X^{i+1}_0)$$
$$= B^i_t - B^{i+1}_t + \frac{1}{2}\int_0^t \mathsf{d}_{\mathrm{pgaf}}(X^i_u, X^{i+1}_u, \sum_{j\neq i}^{\infty}\delta_{X^j_u - X^{j+1}_u})du \quad (i \in \mathbb{N}).$$

Let $\mu_{\mathrm{hgaf}}$ be the zero points of the hyperbolic Gaussian analytic function.

$$F_{\mathrm{hyp}}(z) = \sum_{k=0}^{\infty}\frac{\zeta_k}{k!}z^k \tag{9}$$

**Theorem 5** (H.O.-S. Osada)**.** *The closure of $(\mathscr{E}^{\mu_{\mathrm{hgaf}}}, \mathscr{D}_{\circ,\mathrm{b}})$ on $L^2(\mathsf{S}, \mu_{\mathrm{hgaf}})$ is a strongly local, quasi-regular Dirichlet form and the associated $\mu_{\mathrm{hgaf}}$-reversible diffusion exists.*

## 4. Vanishing self-diffusion

We call $\mu$ $k$-decomposable with $\{\mathsf{S}^{\diamond}_m\}_{m=0}^{k}$ if, for $0 \le m \le k$, (1) $\mathsf{S}^{\diamond}_m \cap \mathsf{S}^{\diamond}_n = \emptyset$ for $m \neq n, 0 \le n \le k$, (2) $\mathsf{S}^{\diamond}_0 \subset \mathsf{S}^{\diamond}_m + \mathsf{S}_m$ , (3) $\mathsf{S}^{\diamond}_m \in \overline{\mathcal{B}(\mathsf{S})}^{\mu(\cdot\|\mathsf{x})}$ and $\mu(\mathsf{S}^{\diamond}_m\|\mathsf{x}) = 1$ for all $\mathsf{x} \in \mathsf{S}_m$.

**Theorem 6** (H.O.)**.** *Let $\mu$ be a translation invariant RPF on $\mathbb{R}^d$, $d \ge 2$. Assume that $\mu$ is one-decomposable. Then each tagged particle of the labeled stochastic dynamics given by $(\mathscr{E}^{\mu}, \underline{\mathscr{D}}^{\mu})$ on $L^2(\mathsf{S}, \mu)$ is sub-diffusive, that is, $\lim_{\varepsilon\to 0}\varepsilon X^i_{t/\varepsilon^2} = 0$ in $\mu$-probability.*

**Theorem 7.** (i) *Tagged particles of two-dimensional Coulomb interacting Brownian motions in $\mathbb{R}^2$ is sub-diffusive for each $\beta > 0$ such that the $\beta$-Coulomb RPF is number rigid.*
(ii) *Tagged particles of the planar GAF dynamics are sub-diffusive.*

**Sobolev spaces on metric spaces**

JUN KIGAMI

Classically, Sobolev spaces $W^{1,p}$ on smooth spaces have been defined through the gradient of smooth functions. For example, in the case of $\mathbb{R}^n$, define the $p$-energy form $\mathcal{E}_p(u,v)$ as

$$\mathcal{E}_p(u,v) = \int (\nabla u, \nabla v)|\nabla u|^{p-2}dx = -\int \Delta_p u \cdot v dx,$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian. Note that

$$\mathcal{E}_p(u,u) = \int |\nabla u|^p dx.$$

Then the $(1,p)$-Sobolev space $W^{1,p}(\mathbb{R}^n)$ is given by

$$W^{1,p}(\mathbb{R}^n) = \{f | f \in L^p(\mathbb{R}^n), \int |\nabla f|^p dx < \infty\}.$$

In this talk, we are concerned with the case where the space is not smooth like fractals, in particular self-similar sets as below. In fact, from 1990's, analysis

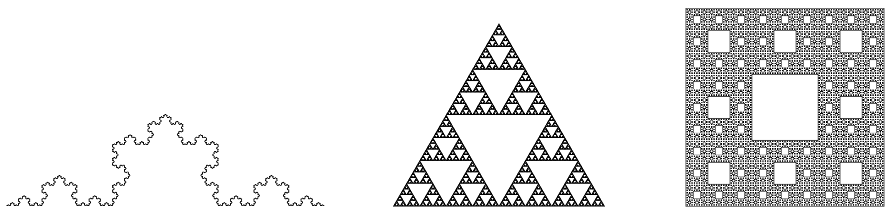

FIGURE 1. von Koch curve, Sierpinski gasket and Sierpinski carpet

on fractals has been developed. In this direction, we first constructed diffusion processes and/or Laplacians and then studied a space-time asymptotic behavior of them. Through the theory of Dirichlet forms, this study corresponds to that of $(1,2)$-Sobolev spaces.

Another direction is to use the local Lipschitz constant $\overline{\nabla} u(x)$ defined by

$$\overline{\nabla} u(x) = \limsup_{r \downarrow 0} \sup_{y \in B_d(x,r)} \frac{|u(x) - u(y)|}{r},$$

where $u$ is a Lipschitz continuous function on a metric space $(X,d)$ and $B_d(x,r) = \{y | y \in X, d(x,y) < r\}$ for $x \in X$ and $r > 0$, as a substitute of the gradient of smooth fuctions. This idea was employed by Hajłasz [2], Cheeger [1] and Shanmugalingam [5] from 1990's and has been developed extensively since. Now this theory is considered as the standard theory of Sobolev spaces on metric spaces.

So, why bother? Why don't you apply this theory to fractals. However, Kajino-Murugan [3] has been revealed that this theory based on Lipschitz functions can not be applied to well-known self-similar sets like the Vicsek set and the Sierpinski carpets. More precisely the domains of the Dirichlet forms associated with the "Brownian motions" on those self-similar set are not within the scope of the theory.

So, we need a new approach to construct Sobolev spaces on fractals. Our naive idea is based on the following fact on the Sobolev spaces on the unit interval. For $n \in \mathbb{N}$, and $f : [0,1] \to \mathbb{R}$, define

$$\mathcal{E}_p^n(f) = \sum_{i=1}^{2^n} \left| f\Big(\frac{i-1}{2^n}\Big) - f\Big(\frac{i}{2^n}\Big) \right|^p,$$

where $f$ is smooth or more generally $f \in W^{1,p}([0,1])$. Then we see that

$$(2^{p-1})^n \mathcal{E}_p^n(f) \underset{n\to\infty}{\longrightarrow} \int_0^1 |\nabla f|^p dx,$$

where $\nabla f$ is the derivative of $f$. This says if $f \in W^{1,p}(\mathbb{R}^n)$, then $(2^{p-1})^n \mathcal{E}_p^n(f)$ converges as $n \to \infty$. One can reverse, however, the direction as well and obtain

$$\mathbb{W}^{1,p}([0,1]) = \{f | f \in C([0,1]), (2^{p-1})^n \mathcal{E}_p^n(f) \text{ converges as } n \to \infty\}.$$

So how can we generalize this fact to the case of a general compact metric space $(X,d)$. A rough ides is as follows: Let $\{(T_n, E_n)\}_{n\geq 0}$ be a sequence of finite graphs, where $T_n$ is the vertices and $E_n$ is the edges, which approximating the metric space $(X,d)$. For $f : T_n \to \mathbb{R}$, define the discrete $p$-energy of $f$ on the graph $(T_n, E_n)$ by

$$\mathcal{E}_p^n(f) = \frac{1}{2} \sum_{(x,y)\in E_n} |f(x) - f(y)|^p$$

Now the problem is that whether or not there exists a proper constant $\sigma_p$ such that the space

$$\{f | f \in L^p(X,\mu), (\sigma_p)^n \mathcal{E}_p^n(P_n f) \text{ "converges" as } n \to \infty\},$$

where $\mu$ is a Borel regular measure on $(X,d)$ and $P_n f$ is a suitable projection of $f$ to a function from $T_n$ to $\mathbb{R}$, is rich enough to be called a "Sobolev space". The point is that **we no longer pursue generalization of the notion of differential any more**. More precisely we have two main issues:

1. What kind of metric spaces does a proper scaling constant $\sigma_p$ exist?
2. What are examples of such spaces?

Our answer to the first question is the notion of **$p$-conductive homogeneity** introduced in [4]. Regarding the second question, this strategy can be applied to the Sierpinski carpet by Shimizu [6] and the nested fractals by [4]. Moreover a new class of self-similar sets called polygon-based locally symmetric self-similar sets has been introduced as examples of spaces having conductive homogeneity by a joint work with Yuka Ota of Kyoto University.

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## Sobolev spaces and energy measures on the Sierpiński carpet

Mathav Murugan
(joint work with Ryosuke Shimizu)

Motivated by extending analysis on Euclidean spaces to possibly non-smooth settings, there were several works in the nineties attempting to a define the notion of a first order Sobolev space on metric measure spaces [7, 6, 15]. The most widely accepted among these definitions are the works of Cheeger and Shanmugalingam [6, 15]. The Sobolev spaces in [6, 15] had different definitions but turned out to be equivalent in a very general setting [1]. These definitions are based on the notion of upper gradient introduced by Heinonen and Koskela [8]. However it is known that the notion of gradient in this theory (minimal $p$-weak upper gradient) vanishes identically on the Sierpinski carpet and hence leads to the Sobolev space being the Lebesgue space.

Let $K$ denote the Sierpinski carpet equipped with Euclidean metric $d$ and the Hausdorff measure $m$ normalized so that $m(K) = 1$. Then the minimal $p$-weak upper gradient for any $p \in (1, \infty)$ for any function in $L^p(K, m)$ on $(K, d)$ is identically zero. Hence the Sobolev space $N^{1,p}(K, d, m)$ coincides with the Lebesgue space $L^p(K, m)$ and does not lead to an interesting space. Following an old idea of Kusuoka and Zhou (see also [16, 11]), we use rescaled discrete energies to construct our Sobolev space. For any $p \in (1, \infty)$, we construct a *Sobolev space* $\mathcal{F}_p$, a *p-energy* $\mathcal{E}_p : \mathcal{F}_p \to \mathbb{R}$ and *energy measure* $\Gamma_p : \mathcal{F}_p \to \mathcal{M}_+(K)$, where $\mathcal{M}_+(K)$ denotes the set of Borel measures of $K$. We think of $\mathcal{F}_p$ as the analogue of $W^{1,p}(\mathbb{R}^n)$ Sobolev space, $\mathcal{E}_p(f)$ as the analogue of $\int_{\mathbb{R}^n} |\nabla f|^p (x)\, dx$ and $\Gamma_p(f)$ as the analogue of the Borel measure $A \mapsto \int_A |\nabla f|^p (x)\, dx$, where $\nabla f$ denotes the distributional gradient of a function in $W^{1,p}(\mathbb{R}^n)$.

To describe this construction, let $\{F_i : K \to K | 1 \le i \le 8\}$ denote the similitudes generating the carpet $K$. Set $S := 1, ..., 8$, $F_w := F_{w_1} \circ \cdots \circ F_{w_n}$ for any $n \in \mathbb{N} \cup \{0\}$ and any $w = w_1 \cdots W_n \in S^n$ (with $S^0 = \{\emptyset\}$ and $F_\emptyset = \mathrm{Id}_K$). We define a sequence of (undirected, simple) graphs $G_n = (V_n, E_n)$ that approximate $(K, d)$ as follows. The vertex set $V_n = S^n$ and two distinct vertices $v, w \in V_n$ are adjacent if and only if $F_v w(K) \cap F_w(K) \neq \emptyset$. For a function $f \in L^1(K, m)$, we define its approximation $M_n f \in R^{V_n}$ as $M_n f(v) = \frac{1}{m(F_v(K))} \int_{F_w(K)} f\, dm$ and the *discrete energy* of $M_n f$ is

$$\mathcal{E}_p^{G_n}(M_n f) = \sum_{\{v,w\} \in E_n} |M_n f(v) - M_n f(w)|^2 .$$

The following theorem describes our Sobolev space and energy measures constructed in [14].

**Theorem 1.** *Then there exists $\rho(p) \in (0,\infty)$ such that the normed linear space $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ defined by*

$$\mathcal{F}_p := \left\{ f \in L^p(K,m) \;\middle|\; \int_K |f|^p \, dm + \sup_{n\in\mathbb{N}} \rho(p)^n \mathcal{E}_p^{\mathbb{G}_n}(M_n f) < \infty \right\},$$

*and*

$$|f|_{\mathcal{F}_p} := \left( \sup_{n\in\mathbb{N}} \rho(p)^n \mathcal{E}_p^{\mathbb{G}_n}(M_n f) \right)^{1/p}, \quad \|f\|_{\mathcal{F}_p} := \|f\|_{L^p(m)} + |f|_{\mathcal{F}_p},$$

*satisfies the following properties.*

(i) $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ *is a reflexive separable Banach space.*

(ii) (Regularity) $\mathcal{F}_p \cap C(K)$ *is a dense subspace in the Banach spaces* $(\mathcal{F}_p, \|\cdot\|_{\mathcal{F}_p})$ *and* $(C(K), \|\cdot\|_\infty)$.

*Furthermore, there exist $C \geq 1$ and $\mathcal{E}_p \colon \mathcal{F}_p \to [0,\infty)$ satisfying the following:*

(iii) $\mathcal{E}_p(\,\cdot\,)^{1/p}$ *is a uniformly convex semi-norm satisfying* $C^{-1}|f|_{\mathcal{F}_p} \leq \mathcal{E}_p(f)^{1/p} \leq C|f|_{\mathcal{F}_p}$ *for all* $f \in \mathcal{F}_p$.

(iv) (Lipschitz contractivity) *For any $f \in \mathcal{F}_p$ and 1-Lipschitz map $\varphi \in C(\mathbb{R})$, we have $\varphi \circ f \in \mathcal{F}_p$ and $\mathcal{E}_p(\varphi \circ f) \leq \mathcal{E}_p(f)$.*

(v) (Poincaré inequality) *It holds that*

$$\|f - f_K\|_{L^p(m)}^p \leq C\mathcal{E}_p(f) \quad \text{for all } f \in \mathcal{F}_p,$$

*where $f_K := \fint_K f \, dm$ is the $m$-average of $f$. In particular,*

$$\{f \in \mathcal{F}_p : \mathcal{E}_p(f) = 0\} = \{f \in L^p(K,m) : f \text{ is constant } m\text{-a.e.}\}.$$

(vi) (Self-similarity) *For any $f \in \mathcal{F}_p$, we have $f \circ F_i \in \mathcal{F}_p$ for all $i \in S$ and*

$$\mathcal{E}_p(f) = \rho(p) \sum_{i\in S} \mathcal{E}_p(f \circ F_i).$$

*Furthermore, $\mathcal{F}_p \cap C(K) = \{f \in C(K) \mid f \circ F_i \in \mathcal{F}_p \text{ for all } i \in S\}$.*

(vii) (Symmetry) *Let $D_4$ denote the dihedral group of isometries of $K$. For any $f \in \mathcal{F}_p$ and $\Phi \in D_4$, we have $f \circ \Phi \in \mathcal{F}_p$ and $\mathcal{E}_p(f \circ \Phi) = \mathcal{E}_p(f)$.*

*There exists a family of Borel finite measures $\{\Gamma_p\langle f\rangle\}_{f\in\mathcal{F}_p}$ on $K$ satisfying the following:*

*(1) For any $f \in \mathcal{F}_p$, we have $\Gamma_p\langle f\rangle(K) = \mathcal{E}_p(f)$ and*

$$\Gamma_p\langle f\rangle(F_w(K)) = \rho(p)^n \mathcal{E}_p(f \circ F_w) \quad \text{for all } w \in S^n, n \in \mathbb{N}.$$

*(2)* (Self-similarity) *For any $n \in \mathbb{N}$ and $f \in \mathcal{F}_p$,*

$$\Gamma_p\langle f\rangle = \rho(p)^n \sum_{w\in S^n} (F_w)_*\bigl(\Gamma_p\langle f \circ F_w\rangle\bigr).$$

*(3)* (Chain rule and strong locality) *For any* $\Psi \in C^1(\mathbb{R})$ *and* $f \in \mathcal{F}_p \cap C(K)$,

$$\Gamma_p\langle \Psi \circ f\rangle(dx) = |\Psi'(f(x))|^p \, \Gamma_p\langle f\rangle(dx).$$

*If* $f, g \in \mathcal{F}_p \cap C(K)$ *and* $A \in \mathcal{B}(K)$ *satisfy* $(f-g)\big|_A = a \cdot \mathbf{1}A$ *for some* $a \in \mathbb{R}$, *then* $\Gamma_p\langle f\rangle(A) = \Gamma_p\langle g\rangle(A)$.

Cao, Chen and Kumagai [5] show that the Sobolev space $\mathcal{F}_p \subset C(K)$ if and only if $p$ is strictly larger than the Ahlfors regular conformal dimension. When $p = 2$ this leads to a Dirichlet form as shown in [13]. Another important motivation for our construction of energy measures and Sobolev space is the attainment problem for Ahlfors regular conformal dimension. This seeks for optimal metrics in the conformal gauge and the corresponding Ahlfors-regular measures that minimize the Hausdorff dimension. We show that any optimal measure is necessarily a bounded perturbation of energy measure (see [14, Theorem 1.8]). Therefore to know the existence of optimal metrics and measures, we need a better understanding of the energy measures. The existence of optimal metrics and measures in this context is a long-standing problem. Similar attainment problems have important consequences in geometric group theory; for example to Cannon's conjecture; see [2, 12, 3, 4] for further details and background.

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## The branching random walk subject to a hard wall constraint

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(joint work with Maximilian Fels, Oren Louidor)

The discrete Gaussian free field (DGFF) on the infinite binary tree $\mathbb{T}$ rooted at 0 (with Dirichlet boundary conditions) is a centered Gaussian process $h = (h(x) : x \in \mathbb{T})$ with covariances given by

$$\mathrm{E}\big[h(x)h(y)\big] = \frac{1}{2}\big(|x| + |y|) - \mathrm{d}_{\mathbb{T}}(x,y)\big) = |x \wedge y|\,. \tag{1}$$

Above $\mathrm{d}_{\mathbb{T}}$ is the graph distance on the tree, $|x| := \mathrm{d}_{\mathbb{T}}(x,0)$ is the depth of $x \in \mathbb{T}$ and $x \wedge y$ denotes the deepest common ancestor of $x, y \in \mathbb{T}$. We shall often also write $[x]_k$ for the ancestor of $x$ in generation $k \leq |x|$. Alternatively, $h$ can be seen as a branching random walk (BRW) with fixed binary branching and standard Gaussian steps, in which case we shall use the term generation instead of depth.

Denoting by $\mathbb{T}_n$, resp. $\mathbb{L}_n$, the sub-graph of $\mathbb{T}$ which includes all vertices at depth at most, resp. equal, to $n \geq 0$, the goal is to study the realization by the field of the event

$$\Omega_n^+ := \big\{h(x) \geq 0 :\ x \in \mathbb{L}_n\big\}\,. \tag{2}$$

The motivation for studying this event comes from the area of random surfaces such as the two-dimensional discrete Gaussian free field. We obtain a remarkably precise description of the conditional law and the conditional field. The conditioning leads to an upward shift of the whole field. We obtain sharp estimates on this upward shift (up to o(1) terms). We show that the properly rescaled maximum converges to a Gumbel distribution (without a random shift!), and the rescaled minimum is exponentially distributed. Our results including a detailed proof outline can be found in [1, 2]. We now describe some of our results in more detail. Let

$$\Omega_n(u) := \Big\{\min_{\mathbb{L}_n} h \geq -m_n + u\Big\} \quad ; \qquad u \in \mathbb{R}, \tag{3}$$

where $m_n = c_0 n - \frac{3}{2c_0}\log(n)$ with $c_0 = \sqrt{2\log(2)}$.

We obtain precise estimates on the probability of this event for a wide range of $u$, including a precise control on the derivative.

**Theorem 1.** *There exists a bounded function $\theta : [0,1) \to \mathbb{R}$ such that for $u \in \mathbb{R}$, $n \geq 1$,*

$$-\log \mathrm{P}\big(\Omega_n(u)\big) = \frac{\big(u^+ - c_0 \log_2(u \vee 1)\big)^2}{2} + \theta_{[u]_2} u + o(u^+)\,, \tag{4}$$

*where* $o(u)/u \to 0$ *as* $u \to \infty$ *uniformly in* $n$ *such that* $u \leq 2^{\sqrt{n}}$. *Moreover,*

$$-\frac{\mathrm{d}}{\mathrm{u}} \log \mathrm{P}\big(\Omega_n(u)\big) = u^+ - c_0 \log_2(u \vee 1) + O(1)\,, \tag{5}$$

*where the* $O(1)$ *term is bounded uniformly in* $n$ *and* $u$ *such that* $u \leq 2^{\sqrt{n}}$.

The most likely way for the branching random walk to achieve the event $\Omega_n^+$ is the follwowing. The first

$$l_n \equiv \lfloor \log_2(n) \rfloor \tag{6}$$

are used to go up to $m_{n'}$ with $n' = n - l_n$ and afterwards the branching random walks starting from there and running for $n'$ generations perform ordinary branching random walls and are only subject to a mild conditioning.

We denote the law of the filed under conditioning on $\Omega_n^+$ by $\mathrm{P}_n^+$, its expectation by $\mathrm{E}_n^+$ and the covariance by $\mathrm{Cov}_n^+$. A key tool to understanding the behaviour of the random field under the conditioning is the following covariance estimate. It shows that under the conditioning the values of the field are more independent than without.

**Theorem 2.** *There exists* $c > 0$ *such that for all* $n \geq 1$ *and* $x, y \in \mathbb{T}_n$ *with* $|x|, |y| \geq l_n$,

$$\mathrm{Cov}_n^+\left(h(x), h(y)\right) = \begin{cases} |x \wedge y| - l_n + O(1) & |x \wedge y| \geq l_n\,, \\ O\big(\mathrm{e}^{-c(l_n - |x \wedge y|)}\big) & |x \wedge y| < l_n\,. \end{cases} \tag{7}$$

**Theorem 3.** *There exists an explicit constant* $c_1 \in \mathbb{R}$ *such that with*

$$m_n^+ := 2m_{n'} + c_0^{-1} \log_2 n + c_0^{-1} \log\log_2 n + \theta_{[u]_2} + c_1\,, \tag{8}$$

*where* $\theta_{[u]_2}$ *is again some term depending on* $\log_2(n) - \lfloor \log_2(n) \rfloor$ *and* $c_0$ *os as before. It holds for all* $u \in \mathbb{R}$ *that*

$$\mathrm{P}_n^+\Big(\max_{x \in \mathbb{L}_n} h(x) - m_n^+ \leq u\Big) \underset{n \to \infty}{\longrightarrow} \mathrm{e}^{-\ \mathrm{e}^{-c_0 u}}\,. \tag{9}$$

*In particular, under* $\mathrm{P}_n^+$,

$$\max_{x \in \mathbb{L}_n} h(x) - m_n^+ \overset{\mathrm{d}}{\underset{n \to \infty}{\longrightarrow}} \mathrm{Gumbel}(\mathrm{c}_0)\,. \tag{10}$$

**Theorem 4.** *There is a* $\kappa_{[n]_2}$ *depending only on* $\log_2(n) - \lfloor \log_2(n) \rfloor$ *(and of which we have a more precise description) such that, for all* $u \geq 0$,

$$\mathrm{P}_n^+\big(n\kappa_{[n]_2} \min_{x \in \mathbb{L}_n} h(x) \leq u\big) \underset{n \to \infty}{\longrightarrow} 1 - \mathrm{e}^{-u}\,. \tag{11}$$

*In particular, under* $\mathrm{P}_n^+$,

$$n\kappa_{[n]_2} \min_{x \in \mathbb{L}_n} h(x) \underset{n \to \infty}{\Longrightarrow} \mathrm{Exp}(1)\,. \tag{12}$$

**Corollary 5.** *For any* $(\mu_n)_{n \geq 1}$, *with* $\mu_n \in \mathbb{R}^{\mathbb{L}_n}$, *the law of* $h$ *under* $\mathrm{P}_n$ *and the law of* $h + \mu_n$ *under* $\mathrm{P}_n^+$ *are asymptotically mutually singular with respect to each other.*

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## Quenched and annealed heat kernel estimates for Brox diffusions


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(joint work with Xin Chen)


Brox diffusion $X := (X_t)_{t\geq 0}$ is the solution of the following one-dimensional stochastic differential equation (SDE)

$$dX_t = -\frac{1}{2}\dot{W}(X_t)\,dt + d\beta_t, \tag{1}$$

where $(\beta(t))_{t\geq 0}$ is a one-dimensional standard Brownian motion, $(W(x))_{x\in\mathbb{R}}$ is a two-sided one-dimensional standard Brownian motion that is independent of $(\beta(t))_{t\geq 0}$, and $\dot{W}(x)$ denotes the formal derivative of $W(x)$. This model was first introduced in [2] by Brox as a continuous analogue of Sinai's random walk (see [4]), with the motivation that when studying the process $X$ we can exploit the scaling properties of $(\beta(t))_{t\geq 0}$ and $(W(x))_{x\in\mathbb{R}}$. Due to the singularity of the drift $\dot{W}(x)$, the SDE (1) above can not be solved by the standard theory for neither the strong solution nor the weak solution of SDEs. Actually, Brox's construction (see [2, Section 1]) is based on the time and space transformations as in the Itô-McKean construction of Feller-diffusion, which will be recalled below.

The Brox diffusion $X$ can be reviewed as a Feller-diffusion process on $\mathbb{R}$ with the generator of Feller's canonical form

$$\frac{e^{W(x)}}{2}\frac{d}{dx}\left(e^{-W(x)}\frac{d}{dx}\right).$$

Thanks to the Itô-McKean construction of Feller-diffusion process form a Brownian motion via the scale-transformation and the time-change, the Brox diffusion $X$ can be explicitly given by

$$X(t) = S^{-1}(B(T^{-1}(t))), \quad t\geq 0 \tag{2}$$

with

$$T(t) = \int_0^t \exp(-2W(S^{-1}(B(s))))\,ds, \quad S(x) = \int_0^x e^{W(z)}\,dz,$$

where $B := (B(t))_{t\geq 0}$ is a one-dimensional standard Brownian motion starting from the origin on some probability space. As we will see, the Brox's construction is crucial for the proofs in our paper. Later, it is proved in [3, Theorems 2.5 and 2.5] that for any Brownian motion $B$, independent of $W$, the representation (2) is a weak solution of the SDE (1); and that for any given Brownian motion $(\beta_t)_{t\geq 0}$ we can construct a particular Brownian motion $B$, independent of $W$, such that the representation (2) is a unique strong solution of the SDE (1).

With the representation (2), Brox in [2] proved that for large $t$, a typical value of $X_t$, which is far smaller than $t^{1/2}$, the magnitude order of a standard Brownian motion in a nonrandom environment. The averaged speed (in the so-called annealed setting) of $X_t$ is of order of $(\log t)^2$, which is surprisingly slow. Therefore, the Brox diffusion $X$ describes a Brownian motion moving in a random medium, and it will possess anomalous behaviors. After the work of Brox [2] there have been a number of papers devoted to the study of the Brox diffusion $X$.

The purpose of our paper is to establish heat kernel estimates for the Brox diffusion $X$. Throughout this paper, we use $\mathbb{E}$ and $\mathbb{P}$ to represent the (annealed) expectation and the probability for the environment $W$, while $\mathbf{E}$ and $\mathbf{P}$ are used to denote the (quenched) expectation and the probability for the randomness induced by the environment $W$. $\Omega$ is used to denote the probability space, where the process $W$ is contained, and $\omega$ means an element in $\Omega$. At the same time, we will omit the variable $\omega$ from time to time if no confusion is caused.

First, we have the following quenched estimates of the heat kernel $p^X(t,x,y)$.

**Theorem 1.** *There exist positive random variables $C_i(\omega)$, $i=1,\cdots,6$, such that for every $x,y\in\mathbb{R}$, $t>0$ and almost all $\omega\in\Omega$,*

$$\begin{aligned} &p^X(t,x,y)\\ &\geq C_1(\omega)t^{-1/2}\exp\left(-\frac{C_2(\omega)|x-y|^2}{t}\right)e^{W(y,\omega)}\exp\left(-C_3(\omega)t[\log(2+|x|+|y|)]^2\right)\end{aligned}$$

*and*

$$\begin{aligned} &p^X(t,x,y)\\ &\leq C_4(\omega)t^{-1/2}\exp\left(-\frac{C_5(\omega)|x-y|^2}{t}\right)e^{W(y,\omega)}\exp\left(C_6(\omega)t[\log(2+|x|+|y|)]^2\right).\end{aligned}$$

As a consequence of Theorem 1, we have the following statement about quenched estimates of $p^X(t,0,x)$ for small time.

**Corollary 2.** *For any $T_0>0$, there exist positive random variables $C_i(\omega)$, $7\leq i\leq 10$, such that such that for every $x\in\mathbb{R}$, $0<t\leq T_0$ and almost all $\omega\in\Omega$,*

$$C_7(\omega)t^{-1/2}\exp\left(-\frac{C_8(\omega)|x|^2}{t}\right)\leq p^X(t,0,x)\leq C_9(\omega)t^{-1/2}\exp\left(-\frac{C_{10}(\omega)|x|^2}{t}\right).$$

*Proof.* For every $c_1>0$ and $T_0>0$, there exists a positive constant $c_2$ such that for all $x\in\mathbb{R}$ and $t\in(0,T_0]$,

$$\frac{|x|^2}{t}\geq -c_2+c_1t[\log(2+|x|)]^2.$$

Combining this with Theorem 1, we can obtain the desired conclusion immediately. $\square$

Let $p(t,x,y)$ be the heat kernel of the Brox diffusion $X$ with respect to the Lebesgue measure. Then, for any $x,y\in\mathbb{R}$ and $t>0$,

$$p(t,x,y)=p^X(t,x,y)e^{-W(y)}. \tag{3}$$

The following result is devoted to annealed estimates of $p(t,x,x)$ for large time.

**Theorem 3.** *There exists constants $T_0, C_1 \geq 1$ such that for all $x \in \mathbb{R}$ and $t \geq T_0$,*

$$\frac{1}{C_1(\log^2 t)(\log\log t)^{11}} \leq \mathbb{E}\left[p(t,x,x)\right] \leq \frac{C_1(\log\log t)^4}{\log^2 t}.$$

The proofs of two theorems above are based on Brox's construction via the scale-transformation and the time-change and the theory of resistance forms for strongly recurrent Markov processes (see [1]) as well as the relation between the heat kernel of Brox diffusion $X$ and the time-change process involved in (2), which will be given in details below.

According to Brox's construction above, formally $S(x)$ is the scale function of the Brox diffusion $X$, and $T(t)$ is a positive continuous additive functional of Brownian motion $B$. Thus, $Y := (B(T^{-1}(t)))_{t\geq 0}$ is a time-change of $B$, which is a $\mu_Y$-symmetric strong Markov process on $\mathbb{R}$ with $\mu_Y(dx) = \exp(-2W(S^{-1}(x)))\,dx$. Recall that the Brox diffusion $X$ is a $\mu_X$-symmetric Markov process on $\mathbb{R}$ with $\mu_X(dx) = \exp(-W(x)))\,dx$. Denote by $p^X(t,x,y)$ (resp. $p^Y(t,x,y)$) the heat kernel of the Brox process $X$ with respect to $\mu_X$ (resp. the time-change process $Y$ with respect to $\mu_Y$). Then, by $X(t) = S^{-1}(Y(t))$, for any $f \in C_b(\mathbb{R})$, $t > 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}
\int_{\mathbb{R}} p^X(t,x,y)f(y)\,\mu_X(dy) &= \mathbf{E}(f(X(t))|X(0)=x)\\
&= \mathbf{E}(f(S^{-1}(Y(t)))|Y(0)=S(x))\\
&= \int_{\mathbb{R}} p^Y(t,S(x),y)f(S^{-1}(y))\,\mu_Y(dy)\\
&= \int_{\mathbb{R}} p^Y(t,S(x),y)f(S^{-1}(y))\exp(-2W(S^{-1}(y)))\,dy\\
&= \int_{\mathbb{R}} p^Y(t,S(x),S(y))f(y)e^{-W(y)}\,dy\\
&= \int_{\mathbb{R}} p^Y(t,S(x),S(y))f(y)\,\mu_X(dy),
\end{aligned}$$

which implies that for any $t > 0$ and $x, y \in \mathbb{R}$,

$$p^X(t,x,y) = p^Y(t,S(x),S(y)).$$

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**Scaling limit of the harmonic crystal with random conductances**

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(joint work with Martin Slowik, Anna-Lisa Sokol)

We study random Gaussian fields on $\mathbb{Z}^d$ with covariances given by the Green's function of a random walk among random conductances. For $d \geq 2$ let $(\mathbb{Z}^d, E_d)$ be the $d$-dimensional Euclidean lattice with edge set $E_d := \{\{x, y\} \subset \mathbb{Z}^d : |x-y| = 1\}$. Let $(\Omega, \mathcal{F}) := ([0,\infty)^{E_d}, \mathcal{B}([0,\infty))^{\otimes E_d})$ be a measurable space equipped with the Borel-$\sigma$-algebra. Furthermore, for any $\omega = \{\omega(e), e \in E_d\} \in \Omega$, we refer to $\omega(e)$ as the conductance of the edge $e$, and we call two vertices $x, y \in \mathbb{Z}^d$ adjacent if $\{x, y\} \in E_d$; we then write $x \sim y$.

**Assumption 1.** *(i) The law $\mathbb{P}$ is stationary and ergodic with respect to space shifts of $\mathbb{Z}^d$.*
*(ii) There exist $p, q \in [1,\infty]$ satisfying $1/p + 1/q < 2/d$ such that for any $e \in E_d$,*

$$\mathbb{E}\big[\omega(e)^p\big] \; < \; \infty \qquad \text{and} \qquad \mathbb{E}\big[\omega(e)^{-q}\big] \; < \; \infty.$$

We now introduce the inhomogeneous harmonic crystal.

**Definition 2** (Inhomogeneous DGFF)**.** *For any $\omega \in \Omega$ and any finite $\Lambda \subset \mathbb{Z}^d$, the* inhomogeneous discrete Gaussian free field *is the Gaussian process $\varphi^\Lambda = \big(\varphi^\Lambda_x : x \in \mathbb{Z}^d\big)$ with law $\mathbf{P}^\omega$ given by*

$$\mathbf{P}^\omega\big[\varphi^\Lambda \in A\big] \; = \; \frac{1}{Z_\Lambda} \int_A e^{-\frac{1}{2}\sum_{\{x,y\}\in E_d} \omega(\{x,y\})(\varphi_x - \varphi_y)^2} \prod_{x\in\Lambda} d\varphi_x \prod_{x\in\Lambda^c} \delta_0\big[d\varphi_x\big],$$

*for any measurable $A \subset \mathbb{R}^\Lambda$ where $\delta_0$ denotes the Dirac measure at 0 and $Z_\Lambda$ is a normalization constant.*

The field $\varphi^\Lambda$ as in Definition 2 is a multivariate Gaussian process with mean and covariance given by

$$\text{(1)} \qquad \mathbf{E}^\omega\big[\varphi^\Lambda_x\big] \; = \; 0 \quad \text{and} \quad \mathbf{E}^\omega\big[\varphi^\Lambda_x\, \varphi^\Lambda_y\big] \; = \; g^\omega_\Lambda(x,y), \qquad x, y \in \Lambda,$$

where $g^\omega_\Lambda(\cdot,\cdot)$ denote the Green's function on $\Lambda$ associated with the operator $\mathcal{L}^\omega$ acting on bounded functions $f\colon \mathbb{Z}^d \to \mathbb{R}$ as

$$\text{(2)} \qquad \big(\mathcal{L}^\omega f\big)(x) \; = \; \sum_{y\sim x} \omega(\{x,y\})\, \big(f(y) - f(x)\big).$$

That is, $g^\omega_\Lambda(\cdot,\cdot)$ is the solution of the Poisson equation

$$\begin{cases} \big(\mathcal{L}^\omega g^\omega_\Lambda(\cdot,y)\big)(x) \; = \; -\delta_y(x), & x \in \Lambda, \\ \qquad\quad g^\omega_\Lambda(x,y) \; = \; 0, & x \in \Lambda^c. \end{cases}$$

Next we introduce the associated random walk in random environment, which is well-known under the label random conductance model (RCM). For any realization $\omega \in \Omega$ consider the continuous-time Markov chain $X \equiv (X_t : t \geq 0)$ on $\mathbb{Z}^d$ with generator $\mathcal{L}^\omega$ defined in (2). When visiting a vertex $x \in \mathbb{Z}^d$, the random walk $X$ waits at $x$ an exponential time with mean $1/\mu^\omega(x)$ and then it jumps to a vertex $y \sim x$ with probability $\omega(\{x,y\})/\mu^\omega(x)$, where $\mu^\omega(x) := \sum_{y\sim x} \omega(x,y)$. Since the law of the waiting time depends on the location of the walk, this walk is often called the variable speed walk (VSRW) in the literature. We denote by $P_x^\omega$ the quenched law of the process $X$ starting at $x \in \mathbb{Z}^d$, and the corresponding expectation by $E_x^\omega$. For any path $w : [0, \infty) \to \mathbb{R}^d$, and any subset $\Lambda \subset \mathbb{R}^d$ we define the first exit time from $\Lambda$ by

$$\tau_\Lambda(w) := \inf\bigl\{t \geq 0 : w_t \notin \Lambda\bigr\}.$$

Then, for any finite $\Lambda \subset \mathbb{Z}^d$, the Green's function $g_\Lambda^\omega(x, y)$ describes the expected amount of time that the VSRW $X$ spends in $y$ when starting in $x$ before exiting $\Lambda$, i.e.

$$g_\Lambda^\omega(x, y) \;=\; E_x^\omega\biggl[\int_0^{\tau_\Lambda} \mathbb{1}_{\{X_t = y\}}\, dt\biggr] \;=\; \int_0^\infty P_x^\omega\bigl[X_t = y,\, t < \tau_\Lambda\bigr]\, dt, \qquad x, y \in \mathbb{Z}^d.$$

**Theorem 3** (QFCLT [1])**.** *Suppose Assumption 1 holds. Set* $X_t^n := n^{-1} X_{tn^2}$ *for any* $n \in \mathbb{N}$ *and* $t \geq 0$*. Then, for* $\mathbb{P}$*-a.e.* $\omega$*, the process* $X^n \equiv \bigl(X_t^n\bigr)_{t \geq 0}$*, converges (under* $P_0^\omega$*) in law towards a Brownian motion on* $\mathbb{R}^d$ *with a deterministic non-degenerate covariance matrix* $\Sigma^2$*.*

For any domain $\Lambda \subset \mathbb{R}^d$, let $k_t^{\Sigma,\Lambda}(x, y)$ denote the heat kernel of the Brownian motion with covariance matrix $\Sigma^2$ killed upon exiting the domain $\Lambda$. More precisely, $k^{\Sigma,\Lambda} = k_t^{\Sigma,\Lambda}(x, y) : (0, \infty) \times \Lambda \times \Lambda \to [0, \infty)$ is the jointly continuous function such that $P_x^\Sigma[W_t^\Lambda \in dy] = k_t^{\Sigma,\Lambda}(x, y)\, dy$ for $t > 0$ and $x \in \Lambda$, where $W^\Lambda$ is coordinate process killed upon exiting $\Lambda$. The associated Green's function is given by

$$g_\Lambda^\Sigma(x, y) := \int_0^\infty k_t^{\Sigma,\Lambda}(x, y)\, dt,$$

From now on we will fix an open bounded domain $D \subset \mathbb{R}^d$, and we will assume that its boundary points a regular in the following sense.

**Definition 4.** *We call a point* $z \in \partial D$ strongly regular *if* $P_z^\Sigma[\tau_{\bar{D}}(W) = 0] = 1$*. We say that* $D$ *is strongly regular if every point* $z \in \partial D$ *is strongly regular.*

**Theorem 5** ([2])**.** *Let* $d \geq 2$ *and* $D \subset \mathbb{R}^d$ *be a bounded, strongly regular domain. Suppose that Assumption 1 holds. For any bounded, measurable function* $f \colon D \to \mathbb{R}$*, let*

$$\varphi^{D_n}(f) := n^{d/2-1} \int_D f(x)\, \varphi_{\lfloor nx \rfloor}^{D_n}\, dx$$

*with* $D_n := nD \cap \mathbb{Z}^d$*,* $n \in \mathbb{N}$*. Then, for* $\mathbb{P}$*-a.e.* $\omega$*, under* $\mathbf{P}^\omega$*,*

$$\varphi^{D_n}(f) \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}(0, \sigma_\Sigma^2(f)),$$

*where* $\Sigma$ *is as in Theorem 3 and*

$$\sigma_\Sigma^2(f) := \int_{D \times D} f(x) f(y)\, g_D^\Sigma(x, y)\, dx\, dy.$$

The corresponding result for the homogeneous DGFF with constant conductances on $\mathbb{Z}^d$ can be found in [3]. More recently, a similar scaling limit for inhomogeneous DGFFs on convex or smooth domains or with periodic boundary conditions have been obtained in [5] for uniformly elliptic conductances with finite range dependence. Level-set percolation for inhomogeneous DGFFs with uniformly elliptic ergodic conductances has been studied in [4].

As our second main result we present a quenched local limit theorem for the Green's function, stating that under diffusive scaling the Green's function of the killed VSRW $X$ converges uniformly on compact sets to the Green kernel of the killed Brownian motion with covariance matrix $\Sigma^2$.

**Theorem 6** ([2]). *Let $d \geq 2$ and $D \subset \mathbb{R}^d$ be a bounded, strongly regular domain. Suppose Assumptions 1 holds. For any $0 < \varepsilon < \delta$, set*

$$K_{\varepsilon,\delta} := \big\{(x,y) \in D \times D : \operatorname{dist}(x, \partial D) \wedge \operatorname{dist}(y, \partial D) \geq \delta, |x-y| \geq \varepsilon\big\}.$$

*Then, for $\mathbb{P}$-a.e. $\omega$,*

$$\lim_{n\to\infty} \sup_{(x,y)\in K_{\varepsilon,\delta}} \left| n^{d-2} g^{\omega}_{D_n}(\lfloor nx \rfloor, \lfloor ny \rfloor) - g^{\Sigma}_{D}(x,y)\right| = 0,$$

*where $D_n := nD \cap \mathbb{Z}^d$, $n \in \mathbb{N}$ and $\Sigma$ is as in Theorem 3.*

In view of (1), Theorem 6 immediately implies the following scaling limit for the covariances of the inhomogeneous DGFF.

**Corollary 7** ([2]). *Under the assumptions of Theorem 6, for any $0 < \varepsilon < \delta$ and for $\mathbb{P}$-a.e. $\omega$,*

$$\lim_{n\to\infty} n^{d-2} \mathbf{Cov}^{\omega}\left[\varphi^{D_n}_{\lfloor nx \rfloor}, \varphi^{D_n}_{\lfloor ny \rfloor}\right] \;=\; g^{\Sigma}_{D}(x,y),$$

*uniformly in $(x,y) \in K_{\varepsilon,\delta}$.*

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**Quantitative approximation of kinetic stochastic differential equations: from discrete to continuum**

Zimo Hao
(joint work with Khoa Lê, Chengcheng Ling)

We consider the weak and strong Euler-Maruyama (EM) scheme for the kinetic type SDEs (also known as second order SDEs) with singular coefficients. We show that when the drift admits the least conditions according to the state of the art so that the system is well-posed in the weak or strong sense, the EM scheme converges with rate 1/2 in the weak or strong sense correspondingly.

**Liouville Brownian motion and Liouoville Cauchy process**

Takumu Ooi

In this talk, we present some results from [9] and ongoing work concerning properties for Liouville Cauchy process. For $\lambda > 0$, let $g_\lambda$ be a $\lambda$-order Green's function of Brownian motion on $\mathbb{R}^2$, and $\mathcal{S}(\mathbb{R}^2)$ be the Schwartz space on $\mathbb{R}^2$. A centred Gaussian field $X = \{X_f\}_{f\in\mathcal{S}(\mathbb{R}^2)}$ with covariance kernel $\pi g_\lambda$ is called the massive Gaussian free field (GFF). GFF can be realized as a Gaussian field associated with a Dirichlet form for Brownian motion on $\mathbb{R}^2$, and some properties for Gaussian fields associated with Dirichlet forms are studied in [10, 5, 8]. For $\gamma > 0$ and a Gaussian field $X$, a random measure $d\mu$“ = ” $\exp(\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}(X(x)^2))dx$ is called Gaussian multiplicative chaos (GMC). Since, in general, Gaussian field is not a random function but a random distribution, this is a formal definition. However, rigorous constructions of GMC and properties concerning convergence of GMC are studied in [7, 11] for example. Liouville Brownian motion (LBM) is the time-changed Brownian motion on $\mathbb{R}^2$ by GMC for GFF. See [6, 3, 1] for example.

In [9], we considered a sufficient condition of convergence of time-changed processes by GMC. In this report, we state some examples of the main result in [9]. Let $Z^n$ be a continuous-time simple symmetric random walk on $\frac{1}{\sqrt{n}}\mathbb{Z}^2$, $g^n_\lambda$ be a $\lambda$-order Green's function of $Z^n$ on $\frac{1}{\sqrt{n}}\mathbb{Z}^2$, and $X^n$ be a centred Gaussian field on $\frac{1}{\sqrt{n}}\mathbb{Z}^2$ whose covariance kernel is $n\pi g^n_\lambda$. We define $\mu^n$ as GMC for $X^n$ with a parameter $\gamma < 2$. Then, by Donsker's invariance principle, $Z^n$ converges weakly to Brownian motion $Z^\infty$ on $\mathbb{R}^2$ with local uniform topology, by local central limit theorem, $ng^n$ converges to the $\lambda$-order Green's function of Brownian motion on $\mathbb{R}^2$, and $\mu^n$ converges weakly to GMC for GFF with vague topology by using [11]. There exists PCAFs $A^n$ corresponding to $\mu^n$, so we set $\hat{Z}^n$ the time-changed process of $Z^n$ by $A^n$. We call $\hat{Z}^n$ Liouville simple random walk on $\frac{1}{\sqrt{n}}\mathbb{Z}^2$. Then the following holds.

**Theorem 1** ([9, §6.2]). *For $\gamma < \sqrt{2}$ and any starting point $x \in \mathbb{R}^2$, Liouville simple random walk $\hat{Z}^n$ under $\mathbb{P}^{X^n} \otimes \mathbb{P}^{Z^n}_x$ converges weakly to LBM $\hat{Z}^\infty$ under $\mathbb{P}^{X^\infty} \otimes \mathbb{P}^{Z^\infty}_x$ with local uniform topology as $n \to \infty$.*

Next, we define Liouville Cauchy process on $\mathbb{R}$. Let $C$ be a symmetric Cauchy process (1-stable process) on $\mathbb{R}$, $Y$ be a centred Gaussian field on $\mathbb{R}$ whose covariance kernel is a $\lambda$-order Green's function of $C$ times $\pi$. We can define $d\nu$ as GMC for $Y$ with a parameter $\gamma < \sqrt{2}$. Set $\check{C}$ as the time-changed process of $C$ by $\nu$. We call $\check{C}$ Liouville Cauchy process (LCP) on $\mathbb{R}$. LCP on the circle is constructed by [2] with some type of a trace field of GFF instead of $Y$. However there is no essential difference between our definition and that in [2] because both Gaussian fields have covariance kernels logarithmically diverging. We remark that, on one dimensional case, the Green's function of Brownian motion does not diverges logarithmically, but that of Cauchy process does. So we consider not LBM on $\mathbb{R}$ but LCP on $\mathbb{R}$.

By considering the boundary theory for Dirichlet forms (see [4], for example), LCP is a trace process of LBM in the following sense.

**Proposition 2.** *LCP is a time-changed process of LBM by $\nu \otimes \delta_0$.*

As another example as the result of [9], we have the following.

**Theorem 3** ([9, §6.1])**.** *For $\gamma < 1$ and any $x \in \mathbb{R}$, Liouville $\alpha$-stable process on $\mathbb{R}$ converges weakly to LCP on $\mathbb{R}$ with $J_1$-top. as $\alpha \searrow 1$.*

By using the strong Feller property and the symmetry, LCP has a Borel measurable heat kernel $\check{p}(t,x,y)$. By using properties of PCAF in [9], comparing Green's functions and using Faber-Krahn's inequality and Nash's inequality, we have some upper estimate. Similarly to [1], we have some lower estimate. These are incomplete but non-trivial estimate for $\check{p}$.

**Proposition 4.** (1) *For $\gamma < 1$, $\nu$-a.e.$x$, bounded $U \subset \mathbb{R}$, there exists $C = C_{Y,\gamma,x,U} > 0$ such that, for any $y \in U$ and small $t$, $\check{p}(t,x,y) \leq Ce^t t^{-1} \log(t^{-1})$.*
(2) *For $\gamma < \sqrt{2}$, large $\eta$ and $\nu$-a.e. $x$, there exists $C = C_{Y,\gamma,\eta,|x|} > 0$ such that, for small $t$, $\check{p}(t,x,x) \geq Ct^{-1}(\log(t^{-1}))^{-\eta}$.*

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## Harnack inequality for weakly coupled non-local systems

Xiangqian Meng
(joint work with Zhen-Qing Chen)

In this talk, we consider a weakly coupled system of non-local operators which contain both diffusion part with uniformly elliptic diffusion matrices and bounded drift vectors and the jump part with relatively general jump kernels. We use the two-sided scale-invariant Green function estimation to prove the scale-invariant Harnack inequality for this weakly coupled non-local systems. In the case where the switching rate matrix is strictly irreducible, the scale-invariant full rank Harnack inequality is proved. Our approach is mainly probabilistic.

## The extended Dirichlet space and criticality theory for nonlinear Dirichlet forms

Marcel Schmidt
(joint work with Ian Zimmermann)

Nonlinear Dirichlet forms were introduced by Cipriani and Grillo [3] as those lower semicontinuous convex functionals $\mathcal{E}\colon L^2(X,\mu) \to [0,\infty]$ whose induced (in general nonlinear) semigroup is order preserving and extends by monotonicity to a contraction on $L^\infty(X,\mu)$. They show that - as in the case of classical Dirichlet forms - these properties are related to the compatibility of $\mathcal{E}$ with certain normal contractions. Recently, in [4, 2], this characterization was extended to more general normal contractions and in [8] the following very symmetric version of the compatibility with normal contractions was obtained: $\mathcal{E}$ is a nonlinear Dirichlet form if and only if for all normal contractions $C\colon \mathbb{R} \to \mathbb{R}$ and all $f, g \in L^2(X,\mu)$ it satisfies

$$\mathcal{E}(f + Cg) + \mathcal{E}(f - Cg) \leq \mathcal{E}(f + g) + \mathcal{E}(f - g).$$

Typical examples are energy functionals of $p$-Laplacians, where $p$ need not be a constant but can be a function. More precisely, for some open domain $\Omega$ and measurable $p\colon \Omega \to [1,\infty)$ the functional

$$\mathcal{D}_p\colon L^2(\Omega) \to [0,\infty], \quad \mathcal{D}_p(f) = \begin{cases} \int_\Omega \frac{1}{p(x)} |\nabla f(x)|^{p(x)} dx & \text{if } f \in W^{1,1}_{\mathrm{loc}}(\Omega) \\ \infty & \text{else} \end{cases}$$

is a nonlinear Dirichlet form (as are its restrictions to certain smaller effective domains). Other examples are Cheeger energies on metric measure spaces, which are not necessarily assumed to be infinitesimally Hilbertian. Many more examples are described in [3, 4].

While nonlinear Dirichlet forms have been applied successfully, a lot of the basic theory available for classical Dirichlet forms is missing in their context. Motivated by this lack and by possible applications to optimal Hardy inequalities, we develop criticality theory for a large class of nonlinear Dirichlet forms. A main tool for these considerations is the extended Dirichlet space, whose existence we show along the way.

To do so we need four standing assumptions, which are all optimal in some sense. The least restrictive is the *symmetry* of $\mathcal{E}$, i.e., we assume $\mathcal{E}(f) = \mathcal{E}(-f)$ for all $f \in L^2(X,\mu)$. It is satisfied by all interesting examples. As a very weak form of linearity we assume the $\Delta_2$-*condition* (this name is borrowed from the theory of Orlicz spaces), namely that there exists $K > 0$ such that $\mathcal{E}(2f) \leq K\mathcal{E}(f)$ for all $f \in L^2(X,\mu)$. Under this condition the effective domain $D(\mathcal{E}) = \{f \in L^2(X,\mu) \mid \mathcal{E}(f) < \infty\}$ is a vector space. For stating the next assumptions we introduce the *Luxemburg seminorm* of $\mathcal{E}$ given by

$$\|\cdot\|_L \colon D(\mathcal{E}) \to [0,\infty), \quad \|f\|_L = \inf\{\lambda > 0 \mid \mathcal{E}(\lambda^{-1} f) \leq 1\}.$$

In order to transfer some of the Hilbert space arguments available for classical Dirichlet forms to the nonlinear situation our third standing assumption is that $(D(\mathcal{E}), \|\cdot\|_L)$ is *reflexive* in the sense that $D(\mathcal{E})$ is norm dense in the bidual of $(D(\mathcal{E}), \|\cdot\|_L)$. The fourth and last assumption is that $D(\mathcal{E}) \cap L^1(X,\mu)$ is $\|\cdot\|_L$-dense in $D(\mathcal{E})$. It is always satisfied for classical Dirichlet forms but need not hold for general nonlinear Dirichlet forms. For the functional $\mathcal{D}_p$ all four assumptions are satisfied if $\inf_{x\in\Omega} p(x) > 1$ and $\sup_{x\in\Omega} p(x) < \infty$.

As for classical Dirichlet forms we define the *extended Dirichlet space* $D(\mathcal{E}_e)$ as the set of those functions $f \in L^0(X,\mu)$ for which there exists an $\mathcal{E}$-Cauchy sequence $(f_n)$ in $D(\mathcal{E})$ with $f_n \to f$ locally in measure. Such a sequence is called *approximating sequence* for $f$. Our first main result is the following.

**Theorem** (Existence of the extended Dirichlet form)**.** *Under our four standing assumptions the functional* $\mathcal{E}_e \colon L^0(X,\mu) \to [0,\infty]$

$$\mathcal{E}_e(f) = \begin{cases} \lim\limits_{n\to\infty} \mathcal{E}(f_n) & \textit{if } (f_n) \textit{ is an approximating sequence for } f \\ \infty & \textit{if } f \textit{ has no approximating sequence} \end{cases}$$

*is well-defined and lower semicontinuous with respect to local convergence in measure.*

This extends classical results by Silverstein [11] and Schmuland [10] to the nonlinear setting but the proof requires new ideas. Even the existence of $\lim_{n\to\infty} \mathcal{E}(f_n)$ for approximating sequences is non-trivial in the nonlinear situation.

The $\Delta_2$-condition extends to $\mathcal{E}_e$ and hence $D(\mathcal{E}_e)$ is a vector space and the Luxemburg seminorm of $\mathcal{E}_e$ is well-defined on $D(\mathcal{E}_e)$ and denoted by $\|\cdot\|_{L,e}$. Moreover, we denote by $G_\alpha = (\alpha + \partial\mathcal{E})^{-1}$, $\alpha > 0$, the resolvent induced by the subgradient $\partial\mathcal{E}$ of $\mathcal{E}$.

**Theorem** (Characterization subcriticality (transience))**.** *Under our four standing assumptions the following assertions are equivalent.*

*(i) There exists* $g\colon X \to (0,\infty)$ *such that* $Gg = \lim_{\alpha\to 0+} G_\alpha g < \infty$ *a.e.*
*(ii) There exists* $h\colon X \to (0,\infty)$ *such that*

$$\int_X |f|hd\mu \le \|f\|_{L,e}, \quad f \in D(\mathcal{E}_e).$$

*(iii)* $\ker \mathcal{E}_e = \{0\}$.
*(iv)* $(D(\mathcal{E}_e), \|\cdot\|_{L,e})$ *is a reflexive Banach space.*
*(v) For one/all* $1 \le p < \infty$*, one/all integrable* $w\colon X \to (0,\infty)$ *there exists a decreasing* $\alpha\colon (0,\infty) \to (0,\infty)$ *such that*

$$\left(\int_X |f|^p w d\mu\right)^{1/p} \le \alpha(r)\|f\|_{L,e} + r\|f\|_\infty, \quad r > 0,\ f \in D(\mathcal{E}_e) \cap L^\infty(X,\mu).$$

Assertions (i) - (iv) appear in the textbook characterization of transience for classical Dirichlet forms, cf. [6]. The main observation here is that the inequalities in (ii) and (v) have to be formulated with respect to the Luxemburg seminorm (instead of powers of $\mathcal{E}_e$) and that other than in the classical situation $Gg < \infty$ need not hold for all $g \in L^1(X,\mu)_+$. Assertion (v) is a weak Hardy inequality and its relation to subcriticality was first observed in [9]. As a direct application of this theorem we obtain the existence of equilibrium potentials and hence a potential theory recovering recent results of [5, 1, 7]. For criticality (recurrence) the following characterization is the same as for classical Dirichlet forms.

**Theorem** (Characterization criticality (recurrence))**.** *Additionally to our standing assumptions assume that* $\partial\mathcal{E}(0) = \{0\}$*. The following assertions are equivalent.*

*(i)* $\mathcal{E}_e(1) = 0$.
*(ii) There exists* $(f_n)$ *in* $D(\mathcal{E})$ *with* $f_n \to 1$ *locally in measure and* $\mathcal{E}(f_n) \to 0$.
*(iii) For all* $g\colon X \to [0,\infty)$ *we have* $Gg = \lim_{\alpha\to 0+} G_\alpha g = 0$ *or* $\infty$ *a.s.*

A measurable set $A \subseteq X$ is called *invariant* if $\mathcal{E}(1_A f) \le \mathcal{E}(f)$ for all $f \in L^2(X,\mu)$. Moreover, $\mathcal{E}$ is called *irreducible* if every invariant set is either null or co-null. Irreducibility implies $\ker \mathcal{E}_e \subseteq \mathbb{R}\cdot 1$ and we obtain:

**Theorem** (Dichotomy of criticality and subcriticality)**.** *Under our four standing assumptions and* $\partial\mathcal{E}(0) = \{0\}$ *the nonlinear Dirichlet form* $\mathcal{E}$ *is either critical or subcritical.*

In summary, we obtain criticality theory for a relatively large class of nonlinear Dirichlet forms that is quite similar to the one for classical Dirichlet forms. However, due to the lack of linearity and the lack of representation theorems there are subtle differences in the statements and many proofs need different ideas compared to the classical situation.

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## Spatial asymptotic behaviors of fractional stochastic heat equations driven by additive Lévy white noise

Yuichi Shiozawa
(joint work with Jian Wang)

In this talk, we are concerned with the spatial asymptotic behavior of the solution to the fractional stochastic heat equation with Lévy white noise, which is formally given by

$$(1)\quad \frac{\partial X}{\partial t}(t,x) = -(-\Delta)^{\alpha/2}X(t,x) + \dot{\Lambda}(t,x) \ \ ((t,x) \in (0,\infty)\times\mathbb{R}^d), \qquad X(0,x) = 0 \ (x \in \mathbb{R}).$$

Here $\alpha \in (0,2)$ and $\dot{\Lambda}(t,x)$ is the time-space derivative of the Lévy white noise. For a fixed time $t>0$, we here focus on the following two problems:

(a) (Spatial asymptotic behavior) To determine the growth rate of $\sup_{x\in\mathbb{R}^d,\,|x|\le r} X(t,x)$ as $r\to\infty$.
(b) (Attainability problem) To clarify if we can attain $\sup_{x\in\mathbb{R}^d,\,|x|\le r} X(t,x)$ over $\mathbb{Z}^d$, or to determine the growth rate of $\sup_{x\in\mathbb{Z}^d,\,|x|\le r} X(t,x)$ as $r\to\infty$.

These problems are studied by Chong-Kevei (2022) for $\alpha=2$. Our result in this talk is a generalization of Chong-Kevei (2022) to $\alpha\in(0,2)$. Regarding the index $\alpha$ as the parameter, we can see how $\alpha$ affects the size of the supremum of the mild solution.

We first clarify the meaning of the solution to (1). Let $\lambda(\mathrm{d}z)$ be a Borel measure on $(0,\infty)$ such that $\int_{(0,\infty)}(1\wedge z^2)\,\lambda(\mathrm{d}z)<\infty$. Let $\mu=\mu(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}z)$ be the Poisson random measure on $(0,\infty)\times\mathbb{R}^d\times(0,\infty)$ associated with intensity measure $\nu(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}z)=\mathrm{d}s\,\mathrm{d}y\,\lambda(\mathrm{d}z)$. Here $\mathrm{d}s$ and $\mathrm{d}y$ are Lebesgue measures on $\mathcal{B}((0,\infty))$

and $\mathcal{B}(\mathbb{R}^d)$, respectively. A Poisson point $(s,y,z) \in (0,\infty)\times\mathbb{R}^d\times(0,\infty)$ designates the time-space-height. We then define the Lévy time-space white noise on $\mathcal{B}((0,\infty))\otimes\mathcal{B}(\mathbb{R}^d)$ by

$$\Lambda(\mathrm{d}s\,\mathrm{d}y) = \int_{(0,1]} z\,(\mu-\nu)(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}z) + \int_{(1,\infty)} z\,\mu(\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}z).$$

We formulate the solution to the equation (1) as the mild solution:

$$X(t,x) = \int_{(0,t]\times\mathbb{R}^d} p_{t-s}(x-y)\,\Lambda(\mathrm{d}s\,\mathrm{d}y) \ \ ((t,x)\in(0,\infty)\times\mathbb{R}^d).$$

Here $p_t(x)$ $((t,x)\in(0,\infty)\times\mathbb{R}^d)$ is the heat kernel associated with $-(-\Delta)^{\alpha/2}$.

We next present our result. For simplicity, we assume that the measure $\lambda$ satisfies

$$\lambda((0,1]) = 0$$

and for some $\beta > d/(d+\alpha)$,

$$\lambda((r,\infty)) = \frac{1}{r^\beta} \quad (r>1).$$

The condition $\beta > d/(d+\alpha)$ is necessary and sufficient for the mild solution $X(t,x)$ to be finite almost surely. This condition is peculiar for $\alpha\in(0,2)$; we see by Chong-Kevei (2022) that for $\alpha=2$, the mild solution $X(t,x)$ is finite almost surely for any $\beta>0$.

**Theorem 1.** *Let $f:(1,\infty)\to(0,\infty)$ be a nondecreasing function.*

(1) *The following dichotomy holds*:

$$\limsup_{r\to\infty}\frac{\sup_{x\in\mathbb{R}^d,\,|x|\le r} X(t,x)}{f(r)} = 0 \ \textit{a.s. or} \ = \infty \ \textit{a.s.}$$

*according as*

$$\begin{cases} \displaystyle\int_1^\infty \frac{r^{d-1}}{f(r)^{(\alpha/d)\wedge\beta}}\,\mathrm{d}r < \infty \ \textit{ or } \ = \infty & \left(\beta \ne \dfrac{\alpha}{d}\right), \\ \displaystyle\int_1^\infty \frac{r^{d-1}}{f(r)^{\alpha/d}}\log f(r)\,\mathrm{d}r < \infty \ \textit{ or } \ = \infty & \left(\beta = \dfrac{\alpha}{d}\right). \end{cases}$$

(2) *The following dichotomy holds*:

$$\limsup_{r\to\infty}\frac{\sup_{x\in\mathbb{Z}^d,\,|x|\le r} X(t,x)}{f(r)} = 0 \ \textit{a.s. or} \ = \infty \ \textit{a.s.}$$

*according as*

$$\begin{cases} \displaystyle\int_1^\infty \frac{r^{d-1}}{f(r)^{(1+\alpha/d)\wedge\beta}}\,\mathrm{d}r < \infty \ \textit{ or } \ = \infty & \left(\beta \ne 1+\dfrac{\alpha}{d}\right), \\ \displaystyle\int_1^\infty \frac{r^{d-1}}{f(r)^{1+\alpha/d}}\log f(r)\,\mathrm{d}r < \infty \ \textit{ or } \ = \infty & \left(\beta = 1+\dfrac{\alpha}{d}\right). \end{cases}$$

We note that the size of $X(t,x)$ is contributed by the Poisson point whose time-space point is close to $(t,x)$. Moreover, if $\beta$ is small, then there are infinitely many Poisson points in which the corresponding heights take large values. Theorem 1 says that, in order to find the local maximum of $X(t,x)$, it is enough to take the supremum over $\mathbb{Z}^d$.

We finally present an example describing the difference of the supremums of $X(t,x)$ over $\mathbb{R}^d$ and $\mathbb{Z}^d$, respectively.

**Example 2.** Let $\beta = \dfrac{\alpha}{d}$ and $p \geq 0$.

(i) If $p > 2d/\alpha$, then almost surely,

$$\lim_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^d,\,|x|\leq r} X(t,x)}{r^{d^2/\alpha}(\log r)^p} = \lim_{r\to\infty} \frac{\sup_{x\in\mathbb{Z}^d,\,|x|\leq r} X(t,x)}{r^{d^2/\alpha}(\log r)^p} = 0.$$

(ii) If $d/\alpha < p \leq 2d/\alpha$, then almost surely,

$$\limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^d,\,|x|\leq r} X(t,x)}{r^{d^2/\alpha}(\log r)^p} = \infty$$

and

$$\lim_{r\to\infty} \frac{\sup_{x\in\mathbb{Z}^d,\,|x|\leq r} X(t,x)}{r^{d^2/\alpha}(\log r)^p} = 0.$$

(iii) If $0 \leq p \leq d/\alpha$, then almost surely,

$$\limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{R}^d,\,|x|\leq r} X(t,x)}{r^{d^2/\alpha}(\log r)^p} = \limsup_{r\to\infty} \frac{\sup_{x\in\mathbb{Z}^d,\,|x|\leq r} X(t,x)}{r^{d^2/\alpha}(\log r)^p} = \infty.$$

## Construction of Korevaar–Schoen $p$-energy forms and associated $p$-energy measures

Ryosuke Shimizu

(joint work with Naotaka Kajino)

The purpose of this talk is to describe the main results in [5], which provides constructions of good $p$-energy forms as subsequential pointwise limits of Besov-type $p$-energy functionals given by

$$E_p^{\beta_p}(u,r) := \int_K \fint_{B_d(x,r)} \frac{|u(x)-u(y)|^p}{r^{\beta_p}}\, m(dy)m(dx), \quad u \in L^p(K,m),$$

where $(K,d)$ is a locally compact separable metric space, $m$ is a Radon measure on $K$ with full topological support, $p \in (1,\infty)$, $r \in (0,\infty)$ and $\beta_p := p\alpha_p \in (0,\infty)$ where $\alpha_p$ is the $L^p$ *critical Besov exponent* [2, Definition 4.1]. The resulting $p$-energy forms are often called the *Korevaar–Schoen $p$-energy forms* in the literature. The following theorem shows the existence of such nice $p$-energy forms under the assumption called the *weak monotonicity*, $(\mathrm{WM})_p$.

**Theorem 1** (Korevaar–Schoen $p$-energy form; [5]). *Assume that there exist $\beta_p \in (0,\infty)$ and $C \in (0,\infty)$ such that*

$$(\mathrm{WM})_p \qquad \bigl(|u|_{\mathcal{F}_p}\bigr)^p := \sup_{r\in(0,\infty)} E_p^{\beta_p}(u,r) \le C \liminf_{r\downarrow 0} E_p^{\beta_p}(u,r)$$

*for any $u \in \mathcal{F}_p := \{f \in L^p(K,m) \mid \sup_{r\in(0,\infty)} E_p^{\beta_p}(f,r) < \infty\}$.*

*(a)* [2, Theorem 4.4] *$\mathcal{F}_p$ equipped with the norm $\|\cdot\|_{L^p(K,m)} + |\cdot|_{\mathcal{F}_p}$ is a reflexive and separable Banach space.*

*(b) Any $\{\tilde{r}_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$ with $\tilde{r}_n \to 0$ has a subsequence $\{r_n\}_{n\in\mathbb{N}}$ such that the following limit exists in $[0,\infty)$ for any $u \in \mathcal{F}_p$:*

$$\mathcal{E}_p(u) := \lim_{n\to\infty} E_p^{\beta_p}(u,r_n).$$

*Furthermore, $\mathcal{E}_p(\,\cdot\,)^{1/p}$ is comparable to $|\cdot|_{\mathcal{F}_p}$.*

*(c) For any $u,v \in \mathcal{F}_p$,*

$$\begin{aligned}\mathcal{E}_p(u;v) &:= \frac{1}{p}\,\frac{d}{dt}\mathcal{E}_p(u+tv)\biggr|_{t=0} \\ &= \lim_{n\to\infty}\int_K \fint_{B_d(x,r_n)} \frac{\mathrm{sgn}\bigl(u(x)-u(y)\bigr)\,|u(x)-u(y)|^{p-1}\,\bigl(v(x)-v(y)\bigr)}{r_n^{\beta_p}} \\ &\qquad\qquad m(dy)m(dx).\end{aligned}$$

*Furthermore, there exists $C_p \in (0,\infty)$ determined solely and explicitly by $p$ such that for any $u_1,u_2,v \in \mathcal{F}_p$,*

$$(1)\quad |\mathcal{E}_p(u_1;v) - \mathcal{E}_p(u_2;v)| \le C_p\left[\max_{i\in\{1,2\}}\mathcal{E}_p(u_i)\right]^{\frac{(p-2)^+}{p}} \mathcal{E}_p(u_1-u_2)^{\frac{(p-1)\wedge 1}{p}}\mathcal{E}_p(v)^{\frac{1}{p}}.$$

*(d)* (Strong locality) *Let $u_1,u_2,v \in \mathcal{F}_p$. If $\mathrm{supp}_m[u_1 - a_1\mathbf{1}_K] \cap \mathrm{supp}_m[u_2 - a_2\mathbf{1}_K] = \emptyset$ and either $\mathrm{supp}_m[u_1 - a_1\mathbf{1}_K]$ or $\mathrm{supp}_m[u_2 - a_2\mathbf{1}_K]$ is compact for some $a_1,a_2 \in \mathbb{R}$, then*

$$\begin{aligned}\mathcal{E}_p(u_1+u_2+v) + \mathcal{E}_p(v) &= \mathcal{E}_p(u_1+v) + \mathcal{E}_p(u_2+v),\\ \mathcal{E}_p(u_1+u_2;v) &= \mathcal{E}_p(u_1;v) + \mathcal{E}_p(u_2;v).\end{aligned}$$

*(e)* (Function-wise generalized $p$-contraction property) *Let $n_1,n_2 \in \mathbb{N}$, $q_1 \in (0,p]$, $q_2 \in [p,\infty]$, $\boldsymbol{u} = (u_1,\dots,u_{n_1}) \in \mathcal{F}_p^{n_1}$ and $\boldsymbol{v} = (v_1,\dots,v_{n_2}) \in L^p(K,m)^{n_2}$. If*

$$\|\boldsymbol{v}(x) - \boldsymbol{v}(y)\|_{\ell^{q_2}} \le \|\boldsymbol{u}(x) - \boldsymbol{u}(y)\|_{\ell^{q_1}} \quad \text{for } m\times m\text{-a.e. } (x,y) \in K\times K,$$

*then $\boldsymbol{v} \in \mathcal{F}_p^{n_2}$ and*

$$\left\|\bigl(\mathcal{E}_p(v_l)^{1/p}\bigr)_{l=1}^{n_2}\right\|_{\ell^{q_2}} \le \left\|\bigl(\mathcal{E}_p(u_k)^{1/p}\bigr)_{k=1}^{n_1}\right\|_{\ell^{q_1}}.$$

It is not easy to show $(\mathrm{WM})_p$ in general. This key condition $(\mathrm{WM})_p$ was verified for *Cheeger spaces* in [2, Theorem 5.1] (see also [1] for another approach using the notion of $\Gamma$-*convergence* to construct Korevaar–Schoen type $p$-energy forms on Cheeger spaces), and for the Sierpiński carpet in [7, 8]. In our work [5], we have shown that $(\mathrm{WM})_p$ holds in two general frameworks: the first one is based on the notion of *p-conductive homogeneity* [6], and the second one is based on the work [3], which focuses on the case of *post-critically finite self-similar sets*. Therefore, it turns out that Theorem 1 is applicable to many "homogeneous" spaces including fractals. Moreover, if $K$ is a self-similar set and the *pre-self-similarity condition* (see [7, Theorem 8.12]) holds, then we can construct $\mathcal{E}_p^{\mathrm{ss}}\colon \mathcal{F}_p \to [0,\infty)$ such that $(\mathcal{E}_p^{\mathrm{ss}}, \mathcal{F}_p)$ is a *self-similar p-energy form*, $\mathcal{E}_p^{\mathrm{ss}}(\,\cdot\,)^{1/p}$ is comparable to $|\cdot|_{\mathcal{F}_p}$ and $\mathcal{E}_p^{\mathrm{ss}}$ satisfies (1), d and e in Theorem 1; see [5, Subsections 5.4 and 6.2] for details.

The second main result in [5] provides a natural way to obtain the $p$-energy measure $\Gamma_p\langle u\rangle$ associated with $(\mathcal{E}_p, \mathcal{F}_p)$ given in Theorem 1. This measure plays the role of $|\nabla u|^p\, dx$ in the classical setting of $\mathbb{R}^d$. Such an analogue of "$|\nabla u|^p\, dx$" associated with a self-similar $p$-energy form constructed in [3, 6, 7] has been defined with the help of the self-similarity in the previous studies (see, e.g., [7, Section 9]), and establishing a way to define $p$-energy measures without using the self-similarity is an open problem [7, Problem 12.5]. However, the construction in Theorem 1 allows us to employ an approach relying on the Riesz–Markov–Kakutani representation theorem. This approach can be regarded as a generalization of the definition of energy measures in the theory of symmetric regular Dirichlet forms.

**Theorem 2** (Korevaar–Schoen $p$-energy measure; [5])**.** *Assume* $(\mathrm{WM})_p$ *and that* $\mathcal{F}_p \cap C_c(K)$ *is dense in* $C_c(K)$ *with respect to the uniform norm.*

*(a) For any* $u \in \mathcal{F}_p \cap C_b(K)$*, there exists a unique Radon measure* $\Gamma_p\langle u\rangle$ *on* $K$ *with* $\Gamma_p\langle u\rangle(K) \le \mathcal{E}_p(u)$ *such that*

$$\int_K \varphi\, d\Gamma_p\langle u\rangle = \mathcal{E}_p(u; u\varphi) - \left(\frac{p-1}{p}\right)^{p-1} \mathcal{E}_p\bigl(|u|^{\frac{p}{p-1}}\,; \varphi\bigr) \quad \text{for any } \varphi \in \mathcal{F}_p \cap C_c(K).$$

*Furthermore, for any* $u, v \in \mathcal{F}_p \cap C_b(K)$ *and any Borel set* $A$ *of* $K$*, the derivative* $\Gamma_p\langle u; v\rangle(A) := \frac{1}{p}\frac{d}{dt}\Gamma_p\langle u + tv\rangle(A)\big|_{t=0}$ *exists and is a signed Borel measure on* $K$ *as a function of* $A$*.*

*(b)* (Chain rule) *If* $n \in \mathbb{N}$*,* $u \in \mathcal{F}_p \cap C_b(K)$*,* $\boldsymbol{v} = (v_1, \ldots, v_n) \in \bigl(\mathcal{F} \cap C_b(K)\bigr)^n$*,* $\Phi \in C^1(\mathbb{R})$*,* $\Psi \in C^1(\mathbb{R}^n)$ *and* $\Phi(0) = \Psi(0) = 0$*, then* $\Phi(u), \Psi(\boldsymbol{v}) \in \mathcal{F}_p \cap C_b(K)$ *and*

$$d\Gamma_p\langle \Phi(u); \Psi(\boldsymbol{v})\rangle = \sum_{k=1}^{n} \mathrm{sgn}\bigl(\Phi'(u)\bigr)\, |\Phi'(u)|^{p-1}\, \partial_k \Psi(\boldsymbol{v})\, d\Gamma_p\langle u; v_k\rangle.$$

*Here* $\partial_k \Psi$ *is the first-order partial derivative of* $\Psi$ *in the* $k$*-th coordinate.*

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## The Principles of Probability: From Formal Logic to Measure Theory to the Principle of Indifference

Jason Swanson

In this work, we develop a formal system of inductive logic. It uses an infinitary language that allows for countable conjunctions and disjunctions. It is based on a set of nine syntactic rules of inductive inference, and contains classical first-order logic as a special case. We also provide natural, probabilistic semantics, and prove both $\sigma$-compactness and completeness.

We show that the whole of modern, measure-theoretic probability theory is properly embedded in this system of inductive logic. The semantic models of inductive logic are probability measures on sets of structures. (Structures are the semantic models of finitary, deductive logic.) Moreover, any probability space, together with a set of its random variables, can be mapped to such a model in a way that gives each outcome, event, and random variable a logical interpretation. This embedding, however, is proper. There are probabilistic ideas that are expressible in this system of logic which cannot be formulated in a classical measure-theoretic probability model.

One such idea is the principle of indifference, a heuristic notion originating with Laplace. Roughly speaking, it says that if we are "equally ignorant" about two possibilities, then we should assign them the same probability. The principle of indifference has no rigorous formulation in modern probability theory. It exists only as a heuristic. Moreover, its use has a history of being problematic and prone to apparent paradoxes. Within inductive logic, however, we provide a rigorous formulation of this principle, and illustrate its use through a number of typical examples.

Many of the ideas in inductive logic have counterparts in measure theory. The principle of indifference, however, does not. Its formulation requires the structure of inductive logic, both its syntactic structure and the semantic structures embedded in its models. As such, it exemplifies the fact that inductive logic is a strictly broader theory of probability than any that is based on measure theory alone.

The totality of this work is presently available in monograph form on the author's home page at

https://math.swansonsite.com/wp-content/uploads/2024/09/principles.pdf.

## The stability of the domain of a lower bounded closed form under non-symmetric perturbation

Toshihiro Uemura

The celebrated KLMN (Kato-Lions-Lax-Milgram-Nelson) theorem plays an important role in the theory of operators, the PDEs and the Mathematical Physics, which states a stability of perturbation of self-adjoint operators ([6, §X.2]). The theorem says the following: Let $(A, \mathcal{D}(A))$ be a self-adjoint operator which is bounded below and $(\mathcal{E}_A, \mathcal{F}_A)$ the associated symmetric (lower bounded) closed form on a Hilbert space $\mathcal{H}$. Let $\eta$ be a symmetric bilinear form on $\mathcal{F}$ and suppose there exist $0 < a < 1$ and $b \geq 0$ such that for all $u \in \mathcal{D}(A)$,

$$\big|\eta(u,u)\big| \leq a(Au,u) + b\|u\|^2 = a\mathcal{E}_A(u,u) + b\|u\|^2. \tag{1}$$

Then there exists a unique self-adjoint operator $(B, \mathcal{D}(B))$ with $\mathcal{F}_B = \mathcal{F}_A$ which is also bounded below and $\mathcal{E}_B(u,v) = \mathcal{E}_A(u,v) + \eta(u,v)$ hold for $u \in \mathcal{F}_B = \mathcal{F}_A$.

In this talk, we consider the stability of the domain of non-symmetric perturbation of a semi-bounded closed operator which generates a strongly continuous $C_0$-semigroup. More specific, let $(\mathcal{E}, \mathcal{F})$ be a lower bounded closed form on a real Hilbert space $\mathcal{H}$. Then it admits a strongly continuous $C_0$-semigroup on $\mathcal{H}$ ([4, 5]). Assume that there exists a form core $\mathcal{C}$ of $(\mathcal{E}, \mathcal{F})$ and we are given a bilinear form on $\mathcal{C} \times \mathcal{C}$ which is not necessarily symmetric. We are interested in the following:

(i) Is there a strongly continuous $C_0$-semigroup on $\mathcal{H}$ associated wtih $\mathcal{Q} := \mathcal{E} + \eta$?
(ii) If so, does the domain of the perturbed form $\mathcal{Q}$ be the same as $\mathcal{F}$?

Before answering the question, note that there exists $\delta \geq 0$ such that $\mathcal{E}_\delta(u,u) := \mathcal{E}(u,u) + \delta(u,u) \geq 0$ holds for any $u \in \mathcal{F}$ because $(\mathcal{E}, \mathcal{F})$ is lower bounded. Then the following is an answer and the main theorem:

**Theorem.** Assume there exist $0 < a < 1$ and $b \geq 0$ such that for $u, v \in \mathcal{C}$,

$$|\eta(u,v)| \leq a\sqrt{\mathcal{E}_\delta(u,u)}\sqrt{\mathcal{E}_\delta(v,v)} + b\Big(\sqrt{\mathcal{E}_\delta(u,u)}\|v\| + \|v\|\sqrt{\mathcal{E}_\delta(v,v)} + \|u\|\|v\|\Big). \tag{2}$$

Then there exists a strongly continuous $C_0$-semigroups on $\mathcal{H}$ associated with the bilinear form $\mathcal{Q} := \mathcal{E} + \eta$ with $\mathcal{F}$ as its domain. In other words, $(\mathcal{Q}, \mathcal{F})$ is a lower bounded closed form on $\mathcal{H}$ (see [5, 4]).

**Remark 1.** (I) Plugging $v = u$ in (2), the following holds for $0 < (a <)a' < 1$,

$$(3) \quad \big|\eta(u,u)\big| \le a\mathcal{E}_\delta(u,u) + b\big(2\sqrt{\mathcal{E}_\delta(u,u)}\|u\| + \|u\|^2\big) \le a'\mathcal{E}_\delta(u,u) + b'\|u\|^2$$
$$= a'\mathcal{E}(u,u) + \big(a'\delta + b'\big)\|u\|^2, \quad u \in \mathcal{C},$$

where $b' := b^2/(a'-a) + b \ge 0$ (*c.f.* (1)).
(II) Z.-Q. Chen commented that the condition (2) is relaxed to more weaker condition as follows: there exist $0 < a < 1$, $\lambda \ge \delta$ and $K \ge 1$ such that for any $u, v \in \mathcal{C}$,

$$(4) \qquad \big|\eta(u,u)\big| \le a\mathcal{E}_\lambda(u,u) \quad \text{and} \quad \big|\eta(u,v)\big| \le K\sqrt{\mathcal{E}_\lambda(u,u)}\sqrt{\mathcal{E}_\lambda(v,v)}.$$

Indeed, (2) implies (4) and the result in the theorem holds true even under (4). But, we keep the assumption (2) in this note because it can be easily checked in some examples.

**Corollary.** Consider the case that $\mathcal{H} = L^2(E;m)$ for a locally compact separable metric space and $m$ a positive Radon measure on $E$ with full support. Assume further that $\mathcal{C} \subset \mathcal{F} \cap C_0(E)$ and the Markov property holds for $\mathcal{Q}$:

$$\mathcal{Q}(u^\#, u - u^\#) \ge 0,\ u \in \mathcal{F},\ \text{where } u^\# := (0 \wedge u) \vee 1 \text{ is the unit contraction of } u.$$

Then the pair $(\mathcal{Q}, \mathcal{F})$ is a lower bounded semi-Dirichlet form on $L^2(E;m)$. Here $C_0(E)$ is the set of continuous functions on $E$ with compact support.

**Example.** (diffusion operator perturbed by nonlocal operator) Let $D \subset \mathbb{R}^d$ $(d \ge 3)$ be an open set. Let $A(x) = (a_{ij}(x))$ be a symmetric $d$-square matrix valued measurable function on $D$ and $dk = k(dx)$ a positive Radon measure on $D$ so that
(D) $\exists\, \alpha, \beta > 0$ s.t. $\alpha|\xi|^2 \le A(x)\xi \cdot \xi \le \beta|\xi|^2,\ x \in D,\ \xi \in \mathbb{R}^d$;
(K) $\exists\, C > 0$ s.t. $\displaystyle\int_D u(x)^2 k(dx) \le C\Big(\int_D |\nabla u(x)|^2 dx + \int_D u(x)^2 dx\Big) =: C\big(\mathbb{D}(u,u) + \|u\|^2\big)$,
$u \in C_0^\infty(D)$, the set of smooth functions on $D$ with compact support.
Then it is known in [2] that the following bilinear form

$$\mathcal{E}(u,v) := \int_D A(x)\nabla u(x) \cdot \nabla v(x) dx + \int_D u(x)v(x)k(dx), \quad u \in C_0^\infty(D)$$

is a closable Markovian form on $L^2(D) := L^2(D;dx)$ and the closure $(\mathcal{E}, \mathcal{F})$ is a regular local symmetric Dirichlet form on $L^2(D)$. Moreover the domain $\mathcal{F}$ coincides with the Sobolev space $W_0^{1,2}(D)$ and the following holds:

$$(5) \qquad \alpha\mathbb{D}(u,u) \le \int_D A(x)\nabla u(x) \cdot \nabla u(x) dx \le \mathcal{E}(u,u), \quad u \in W_0^{1,2}(D).$$

Now take a measurable function $J(x,h) = J(x,-h) : D \times (\mathbb{R}^d \setminus \{0\}) \to [0,\infty)$ satisfying the following conditions:
(J) (J1) (big jumps) $\exists\, \varphi_1 : \{1 \le |h|\} \to [0,\infty)$ s.t.

$$\sup_{x \in D} J(x,h) \le \varphi_1(h),\ |h| \ge 1 \text{ and } \int_{|h| \ge 1} \varphi_1(h) dh < \infty;$$

(J2) (small jumps) $\exists\, \varphi_2 : \{0 < |h| < 1\} \to [0,\infty)$ s.t.

$$\sup_{x\in D} J(x,h) \le \varphi_2(h),\ 0<|h|<1 \ \text{ and } \int_{0<|h|<1} |h|^2\varphi_2(h)dh<\infty;$$

(J3) (small jumps – anti-symmetric part) $\exists\,\varphi_3 : \{0<|h|<1\}\to[0,\infty)$ s.t.

$$\sup_{\substack{x\in D\\ x+h\in D}} \big|J(x,h)-J(x+h,h)\big| \le \varphi_3(h),\ 0<|h|<1 \ \text{ and } \int_{0<|h|<1} |h|\varphi_3(h)dh<\infty.$$

Consider a bilinear form $\eta$:

$$\eta(u,v) := -\lim_{n\to\infty} \iint_{\substack{x\in D, x+h\in D\\ |h|>1/n}} \big(u(x+h)-u(x)\big)v(x)J(x,h)dxdh. \tag{6}$$

Under the condition (J), the limit in (6) converges absolutely and the limits has the following expression for $u,v\in C_0^\infty(D)$:

$$\eta(u,v) = \frac{1}{2}\iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u(x+h)-u(x)\big)\big(v(x+h)-v(x)\big)\Big(J(x,h)+J(x+h,h)\Big)dhdx$$
$$+\frac{1}{2}\iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u(x+h)-u(x)\big)v(x)\Big(J(x,h)-J(x+h,h)\Big)dhdx.$$

Moreover, we can show the following inequality: for any $\varepsilon>0$, there exists $c>0$ such that for any $u,v\in C_0^\infty(D)$,

$$|\eta(u,v)| \le \varepsilon\sqrt{\mathbb{D}(u,u)}\sqrt{\mathbb{D}(v,v)} + c\big(\sqrt{\mathbb{D}(u,u)}\|v\| + \|u\|\|v\|\big).$$

Combining this with (5), we can conclude that the pair $(\mathcal{Q}:=\mathcal{E}+\eta, W_0^{1,2}(D))$ is a regular lower bounded semi-Dirichlet form on $L^2(D)$ according to Theorem and Corollary.

**Remark 2.** Consider the case that $D=\mathbb{R}^d$ and $J(x,h)=C|h|^{-d-\alpha(x)}$ for some constant $C>0$ and a measurable function $\alpha:\mathbb{R}^d\to\mathbb{R}$.

(I) (stable-like jump diffusion process) Assume there exist $\alpha_1$, $\alpha_2>0$ s.t. $0<\alpha_1\le\alpha(x)\le\alpha_2<2, x\in\mathbb{R}^d$ and

$$\int_{0<|h|<1} \beta(h)\big(-\log|h|\big)|h|^{1-d-\alpha_2}dh<\infty, \tag{7}$$

where $\beta(h):=\sup_x\big|\alpha(x)-\alpha(x+h)\big|$, $0<|h|<1$. Then (J) holds for $J(x,h)$. Hence $(\mathcal{Q}=\mathcal{E}+\eta, W_0^{1,2}(\mathbb{R}^d))$ becomes a regular lower bounded semi-Dirichlet form on $L^2(\mathbb{R}^d)$.

(II) (stable-like process) In [7], Schilling and Wang gave a similar result without diffusion part (i.e., $A=0$ and $k=0$) under the following condition (see also [1, 3, 8]):

$$\int_{0<|h|<1} \beta(h)^2\big(-\log|h|\big)^2|h|^{-d-\alpha_2}dh<\infty. \tag{8}$$

Namely, they showed that under the conditions $0<\alpha_1(x)\le\alpha(x)\le\alpha_2<2, x\in\mathbb{R}^d$ and (8),

$$\eta(u,v) := C\iint_{h\neq 0}\frac{\big(u(x+h)-u(x)\big)\big(v(x+h)-v(x)\big)}{|h|^{d+\alpha(x)}}dxdh + \frac{C}{2}\iint_{h\neq 0}\big(u(x+h)-u(x)\big)v(x)\Big(\frac{1}{|h|^{d+\alpha(x)}}-\frac{1}{|h|^{d+\alpha(x+h)}}\Big)dxdh$$

itself induces a regular lower bounded semi-Dirichlet form on $L^2(\mathbb{R}^d)$ but whose domain may be bigger than $W_0^{1,2}(\mathbb{R}^d)$. Note that (8) implies (7).

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## Some new Markov processes on Wasserstein space


Simon Wittmann

(joint work with Panpan Ren, Feng-Yu Wang)


To study diffusion processes on the $p$-Wasserstein space $\mathscr{P}_p(X)$ for $p\in[1,\infty)$ over a separable, reflexive Banach space $X$, we present a handy criterion on the quasi-regularity of Dirichlet forms in $L^2(\mathscr{P}_p(X),\Lambda)$ with reference probability $\Lambda$ on $\mathscr{P}_p(X)$, see [2, Thm. 2.1]. It is formulated in terms of an upper bound condition with the uniform norm of the intrinsic derivative. Let

$$\mathscr{F}C_b^1(\mathscr{P}(X)) := \big\{\mathscr{P}(X)\ni\mu\mapsto g(\mu(\psi_1),\cdots,\mu(\psi_n)) \;:\; n\in\mathbb{N},\,\psi_i\in\mathscr{F}C_b^1(X),\,g\in C_b^1(\mathbb{R}^n)\big\}.$$

The intrinsic derivative of a cylindrical function $u\in\mathscr{F}C_b^1(\mathscr{P}(X))$ as above reads by definition

$$Du(\mu,x) := \sum_{i=1}^n(\partial_i g)(\mu(\psi_1),\cdots,\mu(\psi_n))\nabla\psi_i(x)\in X^*,\quad (\mu,x)\in\mathscr{P}(X)\times X.$$

**Theorem 1.** *Let $p \in [1, \infty)$ and $\Lambda$ be a Borel probability measure on $\mathscr{P}_p(X)$. We set $p^* := \frac{p}{p-1} \in (1, \infty]$. The following is sufficient for a Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ in $L^2(\mathscr{P}_p(X), \Lambda)$ to be quasi-regular:*

*$(\mathscr{D}(\mathscr{E}), \mathscr{E}_1^{1/2})$ has a dense subspace consisting of quasi-continuous functions and there exists a constant $C \in (0, \infty)$ such that for all $u \in \mathscr{F}C_b^1(\mathscr{P}(X))$ it holds*

$$u \in \mathscr{D}(\mathscr{E}), \quad \mathscr{E}(u, u) \leq C \sup_{\mu \in \mathscr{P}_p} \|Du(\mu, \cdot)\|^2_{L^{p^*}(X \to X^*, \mu)}.$$

The condition is easy to check in relevant applications. The Ornstein-Uhlenbeck type Dirichlet form on $\mathscr{P}_2(H)$ in [2, Sect. 4] with a separable Hilbert space $H$, as first introduced for $X = \mathbb{R}^d$ in [1], is an important example: We choose a transport regular measure $\lambda \in \mathscr{P}_2(H)$. The map

$$\Psi : L^2(H \to H, \lambda) \ni \phi \mapsto \lambda \circ \phi^{-1} \in \mathscr{P}_2(H)$$

is contractive and surjective. Let $(A, \mathscr{D}(A))$ be a strictly positive-definite, self-adjoint linear operator in $L^2(H \to H, \lambda)$ with pure point spectrum. We denote its eigenvalues in increasing order with multiplicities by $\{\alpha_n\}_{n \in \mathbb{N}}$ and assume that

$$\sum_{n=1}^{\infty} \alpha_n^{-1} < \infty.$$

There is a unique Gaussian measure $G_{\lambda,Q}$ on $L^2(H \to H, \lambda)$ with covariance operator $Q = A^{-1}$ and mean $id_H$, the identity map on $H$. Then,

$$N_{\lambda,Q} := G_{\lambda,Q} \circ \Psi^{-1}$$

yields a measure on $\mathscr{P}_2(H)$ with full topological support.

**Definition 2.** *Let $f : \mathscr{P}_2(H) \to \mathbb{R}$ be continuous and $T_\mu := L^2(H \to H, \mu)$ for $\mu \in \mathscr{P}_2(H)$.*

*(i) $f$ is called intrinsically differentiable if for every $\mu \in \mathscr{P}_2(H)$ the map*

$$T_\mu \ni \phi \mapsto D_\phi f(\mu) := \lim_{\varepsilon \to 0} \frac{f(\mu \circ (id_H + \varepsilon\phi)^{-1}) - f(\mu)}{\varepsilon}$$

*is a bounded linear functional. The intrinsic derivative of $f$ at $\mu$ is defined as the unique element $Df(\mu) \in T_\mu$ such that*

$$D_\phi f(\mu) = \langle Df(\mu), \phi \rangle_{T_\mu}, \quad \phi \in T_\mu.$$

*(ii) For intrinsically differentiable $f$ we write $f \in C^1(\mathscr{P}_2(H))$ if additionally*

$$\lim_{\|\phi\|_{T_\mu} \downarrow 0} \frac{|f(\mu \circ (id_H + \phi)^{-1}) - f(\mu) - D_\phi f(\mu)|}{\|\phi\|_{T_\mu}} = 0, \quad \mu \in \mathscr{P}_2(H),$$

*and $Df$ has a continuous version $\mathscr{P}_2(H) \times H \to H$ in the sense that there exists a continuous map $g : \mathscr{P}_2(H) \times H \to H$ such that $g(\mu, \cdot)$ is a $\mu$-version of $Df(\mu)$ for each $\mu \in \mathscr{P}_2(H)$.*

*(iii) We write* $f \in C_b^1(\mathscr{P}_2(H))$ *if* $f \in C^1(\mathscr{P}_2(H))$ *with*

$$\sup_{\mu \in \mathscr{P}_2(H)} \Big( |f(\mu)| + \|Df(\mu)\|_{L^\infty(H \to H, \mu)} \Big) < \infty.$$

**Theorem 3.** *(1) The bilinear form*

$$\mathscr{E}(u,v) := \int_{\mathscr{P}_2(H)} \langle Du(\mu), Dv(\mu) \rangle_{T_\mu} \, \mathrm{d}N_{\lambda,Q}(\mu), \quad u, v \in C_b^1(\mathscr{P}_2(H)),$$

*is a closable pre-Dirichlet form in* $L^2(\mathscr{P}_2(H), N_{\lambda,Q})$ *and its closure* $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ *is a quasi-regular, local Dirichlet form.*

*(2) The Markov semigroup* $(P_t)_{t \geq 0}$ *associated to* $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ *is absolutely continuous, i.e.*

$$P_t f(\nu) = \int_{\mathscr{P}_2(H)} f(\mu) p_t(\mu, \nu) \, \mathrm{d}N_{\lambda,Q}(\mu), \quad \nu \in \mathscr{P}_2(H), \, t \geq 0,$$

*for bounded measurable* $f : \mathscr{P}_2(H) \to \mathbb{R}$ *with density* $p_t : \mathscr{P}_2(H) \times \mathscr{P}_2(H) \to \mathbb{R}$ *satisfying*

$$\int p_t(\mu,\nu)^2 \, \mathrm{d}N_{\lambda,Q}(\mu) \, \mathrm{d}N_{\lambda,Q}(\nu) \leq \prod_{n \in \mathbb{N}} \Big( 1 + \frac{2\mathrm{e}^{-2\alpha_n t}}{(2\alpha_n t) \wedge 1} \Big) < \infty, \quad t > 0.$$

*(3) The log-Sobolev inequality*

$$N_{\lambda,Q}(u^2 \log u^2) \leq \frac{2}{\alpha_1} \mathscr{E}(u,u), \quad u \in \mathscr{D}(\mathscr{E}), \; N_{\lambda,Q}(u^2) = 1,$$

*holds true.*

As an outlook, we briefly discuss a complementary result derived in [3]. It concerns the class of finite (resp. probability) measures absolutely continuous with respect to a $\sigma$-finite Radon measure $\lambda$ on a Polish space $X$, denoted by $\mathbb{M}_\lambda^{\mathrm{ac}}(X)$ and $\mathscr{P}_\lambda^{\mathrm{ac}}(X)$. We present a criterion on the quasi-regularity of general (non-local) Dirichlet forms in terms of upper bound conditions given by the uniform $(L^1 + L^\infty)$-norm of the extrinsic derivative, see [3, Thm. 2.2]. As application, we obtain a class of general type Markov processes on $\mathbb{M}_\lambda^{\mathrm{ac}}(X)$ and $\mathscr{P}_\lambda^{\mathrm{ac}}(X)$ via quasi-regularity of the Dirichlet forms in [3, Sect. 3] containing the diffusion, jump and killing terms. Moreover, stochastic extrinsic derivative flows on $\mathbb{M}_\lambda^{\mathrm{ac}}(\mathbb{R}^d)$ and $\mathscr{P}_\lambda^{\mathrm{ac}}(X)$ are studied by giving martingale solutions to SDEs with a versatile class of drifts, which include the extrinsic derivative of entropy functionals, see Section [3, Sect. 4].

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**Nonlinear Dirichlet forms associated with quasiregular mappings**

Lucian Beznea
(joint work with Camelia Beznea, Michael Röckner)

We present a general procedure of constructing nonlinear Dirichlet forms in the sense introduced by Petra van Beusekom in [Beu], starting from a strongly local, regular, Dirichlet form (cf. [FOT]), admitting a carré du champ operator (see e.g. [BH]). We describe the nonlinear form associated with a quasiregular mapping.

If $(\mathcal{E}, \mathcal{D})$ is a symmetric, regular, strongly local Dirichlet form on $L^2(X, m)$, admitting a carré du champ operator $\Gamma$, and $p > 1$ is a real number, then one can define a nonlinear form $\mathcal{E}^p$ by the formula

$$\mathcal{E}^p(u, v) = \int_X \Gamma(u)^{\frac{p-2}{2}} \Gamma(u, v) dm,$$

where $u$, $v$ belong to an appropriate subspace of the domain $\mathcal{D}$. We show that $\mathcal{E}^p$ is a nonlinear Dirichlet form in the sense introduced by P. van Beusekom. We then construct the associated Choquet capacity, starting with compacts, following the approach from [HKM] for the $p$-Laplace operator. As a particular case we obtain the nonlinear form associated with the $p$-Laplace operator on $W_0^{1,p}$.

An independently achieved result on the $p$-energy forms and the induced capacity is contained in [K]. However, our $p$-form is closer to the classical situation since we succeeded to express the $p$-form $\mathcal{E}^p$ by means of a gradient operator and in this way, we emphasised that $\mathcal{E}^p$ is a generalisation of the $p$-form associated with the $p$-Laplace operator.

In the last four decades results from nonlinear potential theory have been used in the study of quasiconformal and quasiregular mappings; cf. [BI], [IM], and [HKM]. Using the above procedure, for each $n$-dimensional quasiregular mapping $f$ we construct a nonlinear Dirichlet form $\mathcal{E}^n$ $(p = n)$ such that the components of $f$ become harmonic functions with respect to $\mathcal{E}^n$. This statement should be compared with the results from the monograph [HKM], where to a quasiregular mapping it is associated a different structure, namely a nonlinear harmonic space. Finally, we obtain Caccioppoli type inequalities in the intrinsic metric induced by $\mathcal{E}$ (cf. [St1] and [St2]), for harmonic functions with respect to the form $\mathcal{E}^p$. We apply the obtained Caccioppoli inequalities to the quasiregular mapping and we discuss the connections with the results from [BI].

For convenience we assumed that $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form on a locally compact separable metric space $X$. However, the results from this section may be extended to the quasi-regular case (cf. [MR]), on a general Lusin topological space $X$. In addition, by [BBR] we even can drop the quasi-regularity hypothesis, it is enough to start with a measurable structure only, in the sense that $X$ is merely a Lusin measurable space

This abstract is based on the joint work [BeBeRö] with Camelia Beznea (Bucharest) and Michael Röckner (Bielefeld).

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# Stochastic waves on metric graphs and their genealogies

Wai-Tong (Louis) Fan

Stochastic reaction-diffusion equations are important models in mathematics and in applied sciences such as spatial population genetics and ecology. These equations arise as the scaling limit of discrete systems such as interacting particle models, and are robust against model perturbation. In this talk, I will discuss methods to compute the probability of extinction, the quasi-stationary distribution, the asymptotic speed and other long-time behaviors for stochastic reaction-diffusion equations of Fisher-KPP type. Importantly, we consider these equations on general metric graphs that flexibly parametrize the underlying space. This enables us to not only bypass the ill-posedness issue of these equations in higher dimensions, but also assess the impact of space and stochasticity on the coexistence and the genealogies of interacting populations.

## Markov processes with jump kernels decaying at the boundary

Panki Kim

(joint work with Soobin Cho, Renming Song, Zoran Vondraček)

In this talk, we discuss pure-jump Markov processes on smooth open sets whose jumping kernels vanishing at the boundary and part processes obtained by killing at the boundary or (and) by killing via the killing potential. The killing potential may be subcritical or critical. This work can be viewed as developing a general theory for non-local singular operators whose kernel vanishing at the boundary. Due to the possible degeneracy at the boundary, such operators are, in a certain sense, not uniformly elliptic. These operators cover the restricted, censored and spectral Laplacians in smooth open sets and much more. The main results are the boundary Harnack principle and its possible failure, and sharp two-sided Green function estimates.

This talk is based on [1] and can be viewed as the generalization of main results of [2]–[4]. Here, we give a simplified version of the main results in [1]. See [1] for the full generality.

We say that $D$ is a $C^{1,1}$ open set with characteristics $(\widehat{R}, \Lambda)$, if for each $Q \in \partial D$, there exist a $C^{1,1}$ function $\Psi = \Psi^Q : \mathbb{R}^{d-1} \to \mathbb{R}$ with

$$\Psi(\widetilde{0}) = |\nabla\Psi(\widetilde{0})| = 0 \quad \text{and} \quad |\nabla\Psi(\widetilde{y}) - \nabla\Psi(\widetilde{z})| \le \Lambda|\widetilde{y} - \widetilde{z}| \ \text{ for all } \ \widetilde{y}, \widetilde{z} \in \mathbb{R}^{d-1},$$

and an orthonormal coordinate system $\mathrm{CS}_Q$ with origin at $Q$ such that

$$B_D(Q, \widehat{R}) := B(Q, \widehat{R}) \cap D = \left\{ y = (\widetilde{y}, y_d) \in B(0, \widehat{R}) \text{ in } \mathrm{CS}_Q : y_d > \Psi(\widetilde{y}) \right\}.$$

From now on we assume that $D \subset \mathbb{R}^d$ is a $C^{1,1}$ open set with characteristics $(\widehat{R}, \Lambda)$, $d \ge 2$ and $\alpha \in (0, 2)$.

We consider the bilinear form

$$\mathcal{E}^0(u, v) := \frac{1}{2} \int_{D \times D} (u(x) - u(y))(v(x) - v(y)) \frac{\mathcal{B}(x, y)}{|x - y|^{d+\alpha}} dx dy,$$

where $\mathcal{B} : D \times D \to (0, \infty)$ is a Borel function satisfying following **(Sym)**, **(Ho)** and **(Es)**:

**(Sym)** $\mathcal{B}(x, y) = \mathcal{B}(y, x)$ for all $x, y \in D$.

**(Ho)** If $\alpha \ge 1$, then there exist constants $\theta_0 > \alpha - 1$ and $C > 0$ such that

$$|\mathcal{B}(x, x) - \mathcal{B}(x, y)| \le C\Big(\frac{|x - y|}{d(x) \wedge d(y) \wedge \widehat{R}}\Big)^{\theta_0} \quad \text{for all } x, y \in D.$$

**(Es)** Let $\beta_1, \beta_2, \beta_3, \beta_4 \ge 0$ be such that $\beta_1 > 0$ if $\beta_3 > 0$, and $\beta_2 > 0$ if $\beta_4 > 0$. There exist comparison constants such that for all $x, y \in D$,

$$\mathcal{B}(x, y) \asymp \Phi_1\Big(\frac{d(x) \wedge d(y)}{|x - y|}\Big) \Phi_2\Big(\frac{d(x) \vee d(y)}{|x - y|}\Big) \ell\Big(\frac{d(x) \wedge d(y)}{(d(x) \vee d(y)) \wedge |x - y|}\Big)$$

where $\Phi_1(r) = (r \wedge 1)^{\beta_1}$, $\Phi_2(r) = (r \wedge 1)^{\beta_2}(\log(1 + 1/(r \wedge 1)))^{\beta_4}$ and $\ell(r) = (\log(1 + 1/(r \wedge 1)))^{\beta_3}$.

Here is our assumption on the killing potential $\kappa$:
**(K)** There exist constants $\eta_0 > 0$ and $C, \widehat{C} \geq 0$ such that for all $x \in D$,

$$\begin{cases} |\kappa(x) - \widehat{C}\mathcal{B}(x,x)d(x)^{-\alpha}| \leq Cd(x)^{-\alpha+\eta_0} & \text{if } d(x) < 1, \\ \kappa(x) \leq C & \text{if } d(x) \geq 1. \end{cases}$$

When $\alpha \leq 1$, we further assume that $\widehat{C} > 0$.

Let $\mathcal{F}^0$ be the closure of $\text{Lip}_c(D)$ in $L^2(D)$ under $\mathcal{E}^0_1$. With $\kappa \geq 0$, we consider a symmetric form $(\mathcal{E}, \mathcal{F})$ defined by

$$\mathcal{E}(u,v) = \mathcal{E}^0(u,v) + \int_D u(x)v(x)\kappa(x)dx, \quad \mathcal{F} = \widetilde{\mathcal{F}}^0 \cap L^2(D, \kappa(x)dx),$$

where $\widetilde{\mathcal{F}}^0$ is the family of all $\mathcal{E}^0_1$-quasi-continuous functions in $\mathcal{F}^0$. Then $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(D)$. Let $X$ be the Hunt process associated with $(\mathcal{E}, \mathcal{F})$.

For $a \in \mathbb{R}$, let $\mathbb{H}_a = \{(\widetilde{y}, y_d) \in \mathbb{R}^d : y_d > a\}$, and denote $\mathbb{H}_0$ by $\mathbb{H}$. Flattening the boundary of $D$ is a common way of proving certain results for non-local operators (or part processes) in $C^{1,1}$ open sets, and amounts to setting up an orthonormal coordinate system at a boundary point of the $C^{1,1}$ opens set, and ingeniously using the results known for the half-space in the local coordinate system. The flattening of the boundary method does not work directly – the function $\mathcal{B}$ (and thus the jump kernel) is connected with distances of the points to the boundary of $D$, while its counterpart in the case of the half-space $\mathbb{H}$ should be defined in terms of the distances of the points to the boundary of $\mathbb{H}$. When one flattens the boundary of $D$, distance to the boundary changes. Thus, flattening destroys structure of the function $\mathcal{B}$, in terms of distances to the boundary one and can not connect to the half-space case directly. We address this challenge by introducing the following assumption **(B)**.

For $Q \in \partial D$, $\nu \in (0,1]$ and $r \in (0, \widehat{R}/4]$, we introduce the set

$$E^Q_\nu(r) = \left\{ y = (\widetilde{y}, y_d) \text{ in } \text{CS}_Q : |\widetilde{y}| < r/4,\, 4r^{-\nu}|\widetilde{y}|^{1+\nu} < y_d < r/2 \right\}.$$

**(B)** There exist constants $\nu \in (0,1]$, $\theta_1, \theta_2, C > 0$, and a non-negative Borel function $\mathbf{F}_0$ on $\mathbb{H}_{-1}$ such that for any $Q \in \partial D$ and $x, y \in E^Q_\nu(\widehat{R}/8)$ with $x = (\widetilde{x}, x_d)$ in $\text{CS}_Q$,

$$\begin{aligned} &\left|\mathcal{B}(x,y) - \mathcal{B}(x,x)\mathbf{F}_0((y-x)/x_d)\right| + \left|\mathcal{B}(x,y) - \mathcal{B}(y,y)\mathbf{F}_0((y-x)/x_d)\right| \\ &\leq C\left(\frac{d(x) \vee d(y) \vee |x-y|}{d(x) \wedge d(y) \wedge |x-y|}\right)^{\theta_1} \left(d(x) \vee d(y) \vee |x-y|\right)^{\theta_2}. \end{aligned}$$

Under the assumption **(B)**, we define a function $\mathbf{F}$ on $\mathbb{H}_{-1}$ by

$$\mathbf{F}(y) = \frac{\mathbf{F}_0(y) + \mathbf{F}_0(-y/(1+y_d))}{2}, \qquad y = (\widetilde{y}, y_d) \in \mathbb{H}_{-1}.$$

The function $\mathbf{F}$ above and $q \in [(\alpha-1)_+, \alpha+\beta_1)$, we associate a constant $\bar{C}(\alpha, q, \mathbf{F})$ defined by

$$\bar{C}(\alpha, q, \mathbf{F}) = \int_{\mathbb{R}^{d-1}} \int_0^1 \frac{(s^q-1)(1-s^{\alpha-1-q})}{(1-s)^{1+\alpha}} \frac{\mathbf{F}\big(((s-1)\widetilde{u}, s-1)\big)}{(|\widetilde{u}|^2+1)^{(d+\alpha)/2}} ds d\widetilde{u}.$$

Cf., [5]. We have $\lim_{q\to\alpha+\beta_1} \bar{C}(\alpha, q, \mathbf{F}) = \infty$ and $\bar{C}(\alpha, (\alpha-1)_+, \mathbf{F}) = 0$. Moreover, $q \mapsto \bar{C}(\alpha, q, \mathbf{F})$ is a well-defined strictly increasing continuous function on $[(\alpha-1)_+, \alpha+\beta_1)$. Therefore, when $\widehat{C} > 0$, there exists a unique constant $p \in ((\alpha-1)_+, \alpha+\beta_1)$ such that

$$\widehat{C} = \bar{C}(\alpha, p, \mathbf{F}).$$

We take $p = \alpha - 1$ if $\widehat{C} = 0$.

Here are main results. We assume that $\mathcal{B} : D \times D \to (0, \infty)$ is a Borel function satisfying **(Sym)**, **(Ho)**, **(Es)** and **(B)** and $\kappa$ satisfies **(K)**.

**Theorem 1** (Boundary Harnack principle)
Suppose also that $p < \alpha + (\beta_1 \wedge \beta_2)$. Then for any $Q \in \partial D$, $0 < r \le \widehat{R}$, and any non-negative Borel function $f$ in $D$ which is harmonic in $D \cap B(Q, r)$ with respect to $X$ and vanishes continuously on $\partial D \cap B(Q, r)$, we have

$$\frac{f(x)}{d(x)^p} \asymp \frac{f(y)}{d(y)^p} \quad \text{for } x, y \in D \cap B(Q, r/2),$$

where the comparison constant is independent of $Q, r$ and $f$.

**Theorem 2** (Failure of Boundary Harnack principle) Assume that $\alpha + \beta_2 \le p < \alpha + \beta_1$. Then the inhomogeneous non-scale-invariant boundary Harnack principle is not valid for $X$.

**Theorem 3** (Green function estimates) The process $X$ admits a Green function $G : D \times D \to [0, \infty]$ such that $G(x, \cdot)$ is continuous in $D \setminus \{x\}$ and regular harmonic with respect to $X$ in $D \setminus B(x, \epsilon)$ for any $\epsilon > 0$.

Suppose that $D$ is a bounded $C^{1,1}$ open set.
(1) If $p \in [(\alpha-1)_+, \alpha + \frac{1}{2}[\beta_1 + (\beta_1 \wedge \beta_2)]) \cap (0, \infty)$, then on $D \times D$,

$$\begin{aligned} G(x, y) &\asymp \frac{1}{|x-y|^{d-\alpha}} \left( \frac{d(x)}{|x-y|} \wedge 1 \right)^p \left( \frac{d(y)}{|x-y|} \wedge 1 \right)^p \\ &= \frac{1}{|x-y|^{d-\alpha}} \left( \frac{d(x) \wedge d(y)}{|x-y|} \wedge 1 \right)^p \left( \frac{d(x) \vee d(y)}{|x-y|} \wedge 1 \right)^p. \end{aligned}$$

(2) If $\beta_1 > \beta_2$ and $p = \alpha + \frac{\beta_1+\beta_2}{2}$, then on $D \times D$,

$$G(x,y) \asymp \frac{1}{|x-y|^{d-\alpha}} \left(\frac{d(x) \wedge d(y)}{|x-y|} \wedge 1\right)^p \left(\frac{d(x) \vee d(y)}{|x-y|} \wedge 1\right)^p \times \log^{\beta_4+1}\left(1 + \frac{|x-y|}{(d(x) \vee d(y)) \wedge |x-y|}\right).$$

(3) If $\beta_1 > \beta_2$ and $p \in (\alpha + \frac{\beta_1+\beta_2}{2}, \alpha + \beta_1)$, then on $D \times D$,

$$G(x,y) \asymp \frac{1}{|x-y|^{d-\alpha}} \left(\frac{d(x) \wedge d(y)}{|x-y|} \wedge 1\right)^p \left(\frac{d(x) \vee d(y)}{|x-y|} \wedge 1\right)^{2\alpha-p+\beta_1+\beta_2} \times \log^{\beta_4}\left(1 + \frac{|x-y|}{(d(x) \vee d(y)) \wedge |x-y|}\right).$$

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# Heat kernel estimates for boundary traces of reflected diffusions on uniform domains


Naotaka Kajino

(joint work with Mathav Murugan)


This talk is aimed at presenting the main results of [7] on boundary trace processes of reflected diffusions on uniform domains, in the general setting of a strongly local regular symmetric Dirichlet space satisfying sub-Gaussian heat kernel estimates. Our main results consist of: (1) matching two-sided estimates and the volume doubling property of the harmonic measure; (2) the Doob–Naïm formula identifying the Dirichlet form of the boundary trace process as the pure-jump Dirichlet form with an explicit jump kernel; and (3) two-sided stable-like heat kernel estimates for the boundary trace process. Similar results (except the exact equality in Theorem 6) in a slightly more general framework have been obtained independently in [2].

Throughout this article, we fix a *metric measure Dirichlet (MMD) space* $\mathcal{D}$, i.e., the triple $\mathcal{D} = (\mathcal{X}, d, m, \mathcal{E}, \mathcal{F})$ of a (locally compact separable) metric space $(\mathcal{X}, d)$ such that $B(x,r) := \{y \in \mathcal{X} \mid d(x,y) < r\}$ has compact closure in $\mathcal{X}$ for any $(x,r) \in \mathcal{X} \times (0,\infty)$, a Radon measure $m$ on $\mathcal{X}$ with full support, and a strongly local regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}, m)$; see [6, Section 1.1] for the definition of the notion of strongly local regular symmetric Dirichlet form.

**Definition 1.** (1) *We say that* $\mathcal{D}$ *satisfies* VD *if and only if* $m(B(x,2r)) \le c_{\mathrm{v}} m(B(x,r))$ *for any* $(x,r) \in \mathcal{X} \times (0,\infty)$ *for some* $c_{\mathrm{v}} \in (0,\infty)$.

(2) *Let* $\beta \in (1,\infty)$. *We say that* $\mathcal{D}$ *satisfies* HKE($\beta$) *if and only if* $(\mathcal{X}, m, \mathcal{E}, \mathcal{F})$ *has a continuous heat kernel* $p = p_t(x,y)\colon (0,\infty)\times\mathcal{X}\times\mathcal{X} \to [0,\infty)$ *and there exist* $c_1, c_2, c_3, c_4 \in (0,\infty)$ *such that for any* $(t,x,y) \in (0,\infty)\times\mathcal{X}\times\mathcal{X}$,

$$\text{HKE}(\beta) \qquad \frac{c_1 \mathbf{1}_{[0,c_2]}\big(d(x,y)^\beta/t\big)}{m(B(x,t^{1/\beta}))} \le p_t(x,y) \le \frac{c_3 \exp\big(-c_4\big(d(x,y)^\beta/t\big)^{\frac{1}{\beta-1}}\big)}{m(B(x,t^{1/\beta}))}.$$

**Definition 2.** *We say that an open subset* $U$ *of* $\mathcal{X}$ *is a uniform domain in* $(\mathcal{X},d)$ *if and only if* $\emptyset \neq U \neq \mathcal{X}$ *and there exist* $c_U, C_U \in (0,\infty)$ *such that the following holds: for any* $x,y \in U$ *there exists a continuous map* $\gamma\colon [0,1] \to U$ *such that* $\gamma(0)=x$, $\gamma(1)=y$, $\mathrm{diam}(\gamma([0,1])) := \sup_{z_1,z_2\in\gamma([0,1])} d(z_1,z_2) \le C_U d(x,y)$ *and*

$$\delta_U(z) := \inf_{w\in\mathcal{X}\setminus U} d(z,w) \ge c_U \min\{d(x,z), d(y,z)\} \quad \textit{for any } z \in \gamma([0,1]).$$

In the rest of this article, we fix $\beta \in (1,\infty)$, assume that $\mathcal{D}$ satisfies VD and HKE($\beta$), and fix a uniform domain $U$ in $(\mathcal{X},d)$. In this setting, (the MMD space associated with) a canonical reflected diffusion on $U$ can be constructed as follows. Recall that $\mathcal{F}_e$ denotes the extended Dirichlet space of $(\mathcal{X}, m, \mathcal{E}, \mathcal{F})$; see, e.g., [4, Definition 1.1.4 and Theorem 1.1.5]. Let $\widetilde{u}$ denote an ($\mathcal{E}$-q.e. unique) $\mathcal{E}$-quasi-continuous $m$-version of $u \in \mathcal{F}_e$ (see [6, Section 2.1] for details), and let $\Gamma(u,v)$ denote the mutual $\mathcal{E}$-energy measure of $u,v \in \mathcal{F}_e$ as defined in [6, (3.2.15)].

**Theorem 3** ([9]; see also [7, Theorem 2.16])**.** *Let* $\overline{U}$ *be the closure of* $U$ *in* $\mathcal{X}$, *and define* $\mathcal{F}(U) \subset L^2(\overline{U}, m|_{\overline{U}})$ *and* $\mathcal{E}^{\mathrm{ref}}\colon \mathcal{E}(U)\times\mathcal{E}(U) \to \mathbb{R}$ *by* $\mathcal{F}(U) := \{\widetilde{u}|_{\overline{U}} \mid u \in \mathcal{F}\}$ *and* $\mathcal{E}^{\mathrm{ref}}(\widetilde{u}|_{\overline{U}}, \widetilde{v}|_{\overline{U}}) := \Gamma(u,v)(U)$, *where any two functions defined* $\mathcal{E}$*-q.e. on* $U$ *and equal* $\mathcal{E}$*-q.e. on* $U$ *are identified. Then* $\mathcal{D}^{\mathrm{ref}} := (\overline{U}, d, m|_{\overline{U}}, \mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ *is a MMD space satisfying* VD *and* HKE($\beta$), *and a subset* $A$ *of* $\overline{U}$ *has capacity zero with respect to* $\mathcal{D}^{\mathrm{ref}}$ *if and only if* $A$ *has capacity zero with respect to* $\mathcal{D}$. *Moreover,* $\mathcal{F}(U)_e = \{\widetilde{u}|_{\overline{U}} \mid u \in \mathcal{F}_e\}$, *and* $\widetilde{u}|_{\overline{U}}$ *is* $\mathcal{E}^{\mathrm{ref}}$*-quasi-continuous for any* $u \in \mathcal{F}_e$.

Our main results require also the following condition, which guarantees that the boundary $\partial U := \overline{U} \setminus U$ of $U$ is "uniformly thick" in the potential-theoretic sense.

**Definition 4** ([7, Definition 4.1])**.** *We say that* $U$ *satisfies the capacity density condition, abbreviated as* CDC, *if and only if there exist* $A_0 \in (8K,\infty)$ *(, where* $K \in (1,\infty)$ *is such that* $(\mathcal{X},d)$ *is* $K$*-relatively ball connected; see* [7, Definition 2.26-(b)]*) and* $A_1, C \in (1,\infty)$ *such that for any* $\xi \in \partial U$ *and any* $R \in (0, \mathrm{diam}(U)/A_1)$,

$$\text{CDC} \qquad \mathrm{Cap}_{B(\xi,A_0R)}(B(\xi,R)) \le C\,\mathrm{Cap}_{B(\xi,A_0R)}(B(\xi,R)\setminus U).$$

In the rest of this article, we assume that $U$ satisfies CDC. Our identification of the boundary trace Dirichlet form of $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$ and its stable-like heat kernel estimates are stated in terms of the harmonic measure $\omega^U_{x_0}$ of $U$, and as our first main result we state matching two-sided estimates on $\omega^U_{x_0}$ in the following theorem, which partially extends [1, Lemmas 3.5 and 3.6]. Let $X = (\{X_t\}_{t\in[0,\infty)}, \{\mathbb{P}_x\}_{x\in\mathcal{X}})$ be a diffusion on $\mathcal{X}$ such that $\mathbb{P}_x(X_t \in dy) = p_t(x,y)\,m(dy)$ for any $(t,x) \in (0,\infty)\times\mathcal{X}$, which exists by VD, HKE($\beta$) and the strong locality of $(\mathcal{E},\mathcal{F})$ (see [7, Proposition 2.18]). Then the $\mathcal{E}$*-harmonic measure* of $U$ with base point $x_0 \in U$ is defined by $\omega^U_{x_0}(dy) := \mathbb{P}_{x_0}(X_{\tau_U} \in dy,\, \tau_U < \infty)$, where $\tau_U := \inf\{t \in [0,\infty) \mid X_t \notin U\}$ $(\inf \emptyset := \infty)$, so that $\omega^U_{x_0}(\mathcal{X} \setminus \partial U) = 0$ by the sample-path continuity of $X$.

**Theorem 5** ([7, Theorem 4.6 and Corollary 4.7])**.** *There exist $C, A \in (1,\infty)$ such that for any $\xi \in \partial U$, any $x_0 \in U$ and any $r \in (0, d(\xi, x_0)/A)$,*

$$C^{-1} g_U(x_0, \xi_r) m(B(\xi,r)) r^{-\beta} \le \omega^U_{x_0}(B(\xi,r) \cap \partial U) \le C g_U(x_0,\xi_r) m(B(\xi,r)) r^{-\beta},$$
$$\omega^U_{x_0}(B(\xi,r) \cap \partial U) \le C \omega^U_{x_0}(B(\xi, r/2) \cap \partial U),$$

*where $g_U$ denotes the Green function on $U$ with respect to $\mathcal{D}$ and $\xi_r$ is any element of $U$ satisfying $d(\xi,\xi_r) = r$ and $\delta_U(\xi_r) > \frac{1}{2} c_U r$* (recall Definition 2)*. In particular, the topological support* $\operatorname{supp}_{\mathcal{X}}[\omega^U_{x_0}]$ *of $\omega^U_{x_0}$ in $\mathcal{X}$ is $\partial U$.*

Next, as our second main result, we have the following identification of the boundary trace Dirichlet form $\mathcal{E}^{\mathrm{ref}}|_{\partial U}$ of $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}(U))$. For each $u \in \mathcal{F}_e$, we define $H_{\partial U}\widetilde{u}$ as the $\mathcal{E}$*-harmonic extension* of $\widetilde{u}|_{\partial U}$ to $\overline{U}$, i.e., $(H_{\partial U}\widetilde{u})(x) := \widetilde{u}(x)$ for $x \in \partial U$ and $(H_{\partial U}\widetilde{u})(x) := \int_{\partial U} \widetilde{u}\, d\omega^U_x$ for $x \in U$ with $\int_{\partial U} |\widetilde{u}|\, d\omega^U_x < \infty$, so that $H_{\partial U}\widetilde{u}$ is an $\mathcal{E}^{\mathrm{ref}}$-quasi-continuous $m|_{\overline{U}}$-version of an element of $\mathcal{F}(U)_e$ by Theorem 3 and [4, Theorem 3.4.8]. We write $(A)^2_{\mathrm{od}} := (A \times A) \setminus \{(x,x) \mid x \in A\}$ for a set $A$.

**Theorem 6** (Doob–Naïm formula; [7, Propositions 3.14, 5.7 and Theorem 5.8])**.** *Define $\mathcal{E}^{\mathrm{ref}}|_{\partial U}(\widetilde{u}|_{\partial U}, \widetilde{v}|_{\partial U}) := \mathcal{E}^{\mathrm{ref}}(H_{\partial U}\widetilde{u}, H_{\partial U}\widetilde{v})$ for $u, v \in \mathcal{F}_e$, and let $x_0 \in U$. Then $\mathcal{E}^{\mathrm{ref}}|_{\partial U}(\widetilde{u}|_{\partial U}, \widetilde{u}|_{\partial U}) = \frac{1}{2}\int_{(\partial U)^2_{\mathrm{od}}} (\widetilde{u}(\xi) - \widetilde{u}(\eta))^2\, \Theta^U_{x_0}(\xi,\eta)\, d\omega^U_{x_0}(\xi)\, d\omega^U_{x_0}(\eta)$ for any $u \in \mathcal{F}_e$, where $\Theta^U_{x_0} \colon (\overline{U} \setminus \{x_0\})^2_{\mathrm{od}} \to (0,\infty)$ is the unique $\mathbb{R}$-valued continuous function on $(\overline{U} \setminus \{x_0\})^2_{\mathrm{od}}$ such that $\Theta^U_{x_0}(x,y) = \frac{g_U(x,y)}{g_U(x_0,x) g_U(x_0,y)}$ for any $(x,y) \in (U \setminus \{x_0\})^2_{\mathrm{od}}$, and called the* Naïm kernel *of $U$ with base point $x_0$.*

The existence of the continuous extension of $\Theta^U_{x_0}$ is a consequence of the *boundary Harnack principle (BHP)* due to [3]. While there is a well-established identification of traces of regular symmetric Dirichlet forms in terms of *Feller and supplementary Feller measures* due to [5] and [4, Sections 5.4–5.7], Theorem 6 gives yet another identification of $\mathcal{E}^{\mathrm{ref}}|_{\partial U}$, and we have proved it by a direct calculation of $\mathcal{E}^{\mathrm{ref}}|_{\partial U}$ based on a method in [8, p. 389, Proof of Proposition] for the strongly local part and on VD of $(\partial U, d, \omega^U_{x_0})$ from Theorem 5 for the jump part.

Lastly, we state our third main result on stable-like heat kernel estimates for $\mathcal{E}^{\mathrm{ref}}|_{\partial U}$ (Theorem 8 below). There is a version of this result for the case where $\operatorname{diam}(U) = \infty$, but here we assume $\operatorname{diam}(U) < \infty$ for simplicity of the presentation; see [7, Subsection 4.3 and Section 5] for the precise statement for the case where $\operatorname{diam}(U) = \infty$. Theorem 8 requires the following lemma. We fix $x_0 \in \partial U$.

**Lemma 7.** *Assume that* $\operatorname{diam}(U) < \infty$*. Then there exists $\Phi \colon \partial U \times [0,\infty) \to [0,\infty)$ such that $\Phi(\xi,\cdot) \colon [0,\infty) \to [0,\infty)$ is a homeomorphism for any $\xi \in \partial U$ and*

$$C^{-1} g_U(x_0,\xi_r) \le \Phi(\xi, r) \le C g_U(x_0, \xi_r) \quad \textit{for any } (\xi, r) \in (\partial U) \times (0, \operatorname{diam}(U)/A)$$

*for some $C, A \in (1,\infty)$.* (We set $\Phi^{-1}(\xi,\cdot) := (\Phi(\xi,\cdot))^{-1}$ for each $\xi \in \partial U$.)

**Theorem 8** ([7, Theorem 5.13]). *Assume that* $\operatorname{diam}(U) < \infty$*, and that* $(\partial U, d)$ *is* uniformly perfect*, i.e., either* $B(\xi, r) \supset \partial U$ *or* $\partial U \cap B(\xi, r) \setminus B(\xi, \delta r) \neq \emptyset$ *for any* $(\xi, r) \in (\partial U) \times (0, \infty)$ *for some* $\delta \in (0, 1)$*. Then the* boundary trace Dirichlet space $\bigl(\partial U, d, \omega^U_{x_0}, \mathcal{E}^{\mathrm{ref}}|_{\partial U}, \{\widetilde{u}|_{\partial U} \mid u \in \mathcal{F}_e\} \cap L^2(\partial U, \omega^U_{x_0})\bigr)$ *has a continuous heat kernel* $\check{p}^{\mathrm{ref}} = \check{p}^{\mathrm{ref}}_t(\xi, \eta)$*, and* $C^{-1} \leq \check{p}^{\mathrm{ref}}_t(\xi, \eta) / \bigl(\frac{1}{\mu(B(\xi, \Phi^{-1}(\xi, t)))} \wedge \frac{t}{\mu(B(\xi, d(\xi, \eta)))\Phi(\xi, d(\xi, \eta))}\bigr) \leq C$ *for any* $(t, \xi, \eta) \in (0, \infty) \times (\partial U) \times (\partial U)$ *for some* $C \in (1, \infty)$*.*

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# Irreducibility of SPDEs driven by pure jump noise

Tusheng Zhang

(joint work with Jian Wang, Hao Yang, Jianliang Zhai)

The irreducibility is fundamental for the study of ergodicity of stochastic dynamical systems. In the literature, there are very few results on the irreducibility of stochastic partial differential equations (SPDEs) and stochastic differential equations (SDEs) driven by pure jump noise. The existing methods on this topic are basically along the same lines as that for the Gaussian case. They heavily rely on the fact that the driving noises are additive type and more or less in the class of stable processes. The use of such methods to deal with the case of other types of additive pure jump noises appears to be unclear, let alone the case of multiplicative noises.

In this paper, we develop a new, effective method to obtain the irreducibility of SPDEs and SDEs driven by multiplicative pure jump noise. The conditions placed on the coefficients and the driving noise are very mild and in some sense they are necessary and sufficient. This leads to not only significantly improving all of the results in the literature, but also to new irreducibility results of a much larger class of equations driven by pure jump noise with much weaker requirements than those treatable by the known methods. As a result, we are able to apply the main results to SPDEs with locally monotone coefficients, SPDEs/SDEs with singular coefficients, nonlinear Schrödinger equations, Euler equations etc. We emphasize that under our setting the driving noises could be compound Poisson processes, even allowed to be infinite dimensional. It is somehow surprising.

Let $H$ be a topological space with Borel $\sigma$-field $\mathcal{B}(H)$, and let $\mathbb{X} := \{X^x(t), t \geq 0; x \in H\}$ be an $H$-valued Markov process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathbb{X}$ is said to be irreducible in $H$ if for each $t > 0$ and $x \in H$

$$\mathbb{P}(X^x(t) \in B) > 0 \quad \text{for any non-empty open set } B.$$

For the sake of clarity, we consider here stochastic partial differential equations driven by additive noise. Let

$$V \subset H \simeq H^* \subset V^*$$

be a Gelfand triple, i.e., $\big(H, \langle \cdot, \cdot \rangle_H\big)$ is a separable Hilbert space and identified with its dual space $H^*$ by the Riesz isomorphism, $V$ is a reflexive Banach space that is continuously and densely embedded into $H$. If ${}_{V^*}\langle \cdot, \cdot \rangle_V$ denotes the dualization between $V$ and its dual space $V^*$, then it follows that

$${}_{V^*}\langle u, v \rangle_V = \langle u, v \rangle_H, \quad u \in H,\ v \in V.$$

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$, be a filtered probability space satisfying the usual conditions.

Now we consider the following stochastic partial differential equations driven by additive pure jump noise:

$$\begin{aligned} dX(t) &= \mathcal{A}(X(t))dt + dL(t), \\ X(0) &= x, \end{aligned} \tag{1}$$

where $\mathcal{A} : V \to V^*$ is a measurable mapping, $L(t), t \geq 0$ is a $H$-valued pure jump Levy process with jumping measure $\nu$.

**Assumption 1.** *For any $x \in H$, there exists a unique global solution $X^x = \big(X^x(t)\big)_{t \geq 0}$ to (1) and $\{X^x, x \in H\}$ forms a strong Markov process.*

For any $x, y \in H$, $\eta > 0$, define $\mathbb{F}$-stopping time

$$\tau^\eta_{x,y} = \inf\{t \geq 0 : X^x(t) \notin B(y, \eta)\}. \tag{2}$$

**Assumption 2.** *For any $h \in H$, there exists $\eta_h > 0$ such that, for any $\eta \in (0, \eta_h]$, there exist $(\epsilon, t) = (\epsilon(h, \eta), t(h, \eta)) \in (0, \frac{\eta}{2}] \times (0, \infty)$ satisfying,*

$$\inf_{\tilde{h} \in B(h,\epsilon)} \mathbb{P}(\tau^{\eta}_{\tilde{h},h} \geq t) > 0.$$

Now we introduce the conditions on the jumping measure of the driving noise of the equation (1), which basically says that for any $\hbar, y \in H$, the neighbourhoods of $y$ can be reached from $\hbar$ through a finite number of choosing jumps.

**Assumption 3.** *For any $h \in H$ and $\eta_h > 0$, there exist $n \in \mathbb{N}$, a sequence of strict positive numbers $\eta_1,\ \eta_2, \cdots, \eta_n$, and $a_1,\ a_2, \cdots,\ a_n \in H \setminus \{0\}$, such that $0 \notin \overline{B(a_i, \eta_i)}$, $\nu\big(B(a_i, \eta_i)\big) > 0$ , $i = 1, ..., n$, and that $\sum_{i=1}^n B(a_i, \eta_i) := \{\sum_{i=1}^n h_i : h_i \in B(a_i, \eta_i), 1 \leq i \leq n\} \subset B(h, \eta_h)$.*

One of the main results of the paper reads as

**Theorem 4.** *Suppose Assumptions 1, 2 and 3 hold. Then the Markov process formed by the solution $\{X^x, x \in H\}$ of equation (1) is irreducible in $H$.*

**Applications**. As a result, we are able to apply the main results to SPDEs with locally monotone coefficients, SPDEs/SDEs with singular coefficients, nonlinear Schrödinger equations, Euler equations etc. We emphasize that under our setting the driving noises could be compound Poisson processes.

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## Boundary Harnack principle for non-local operators on metric measure spaces

Jie-Ming Wang

(joint work with Zhen-Qing Chen)

In this paper, a necessary and sufficient condition is obtained for the scale invariant boundary Harnack principle (BHP in abbreviation) for a large class of Hunt processes on metric measure spaces that are in weak duality with another Hunt process. We next consider a discontinuous subordinate Brownian motion with Gaussian component $X_t = W_{S_t}$ in $\mathbb{R}^d$ for which the Lévy density of the subordinator $S$ satisfies some mild comparability condition. We show that the scale invariant BHP holds for the subordinate Brownian motion $X$ in any Lipschitz domain satisfying the interior cone condition with common angle $\theta \in (\cos^{-1}(1/\sqrt{d}), \pi)$, but fails in any truncated circular cone with angle $\theta \leq \cos^{-1}(1/\sqrt{d})$, a Lipschitz domain whose Lipschitz constant is larger than or equal to $1/\sqrt{d-1}$.

**Riesz transforms for Dirichlet spaces tamed by distributional curvature lower bounds**

Kazuhiro Kuwae
(joint work with Syota Esaki , Zi Jian Xu)

Let $(M,\tau)$ be a Hausdorff topological space, which is a Lusin space, endowed with a $\sigma$-finite Borel measure $\mathfrak{m}$ on $M$ with full topological support. Let $(\mathscr{E},D(\mathscr{E}))$ be a quasi-regular symmetric strongly local Dirichlet space on $L^2(M;\mathfrak{m})$ and $(P_t)_{t\geq 0}$ the associated symmetric sub-Markovian strongly continuous semigroup on $L^2(M;\mathfrak{m})$. Then there exists an $\mathfrak{m}$-symmetric special standard process $\mathbf{X}=(\Omega,X_t,\mathbb{P}_x)$ associated with $(\mathscr{E},D(\mathscr{E}))$. The $\mathfrak{m}$-symmetry of $\mathbf{X}$, $(P_t)_{t\geq 0}$ can be extended to a strongly continuous contraction semigroup on $L^p(M;\mathfrak{m})$ for $p\in[1,+\infty[$. Denote by $(D(\Delta),\Delta)$ its generator on $L^2(M;\mathfrak{m})$. Denote by $\dot{D}(\mathscr{E})_{\rm loc}$, the space of functions locally in $D(\mathscr{E})$ in the broad sense. For $u\in\dot{D}(\mathscr{E})_{\rm loc}$ and an $\mathscr{E}$-nest $\{G_n\}$ of $\mathscr{E}$-quasi-open sets satisfying $u|_{G_n}\in D(\mathscr{E})|_{G_n}$, we write $\{G_n\}\in\Xi(u)$. It is known that $D(\mathscr{E})\subset\dot{D}(\mathscr{E})_{\rm loc}$.

It is known that for $u,v\in D(\mathscr{E})\cap L^\infty(M;\mathfrak{m})$ there exists a unique signed finite Borel measure $\mu_{\langle u,v\rangle}$ on $M$ such that

$$2\int_M \tilde{f}\mathrm{d}\mu_{\langle u,v\rangle}=\mathscr{E}(uf,v)+\mathscr{E}(vf,u)-\mathscr{E}(uv,f)\quad\text{for}\quad u,v\in D(\mathscr{E})\cap L^\infty(M;\mathfrak{m}).$$

We set $\mu_{\langle f\rangle}:=\mu_{\langle f,f\rangle}$ for $f\in D(\mathscr{E})\cap L^\infty(M;\mathfrak{m})$. Moreover, for $f,g\in D(\mathscr{E})$, there exists a signed finite measure $\mu_{\langle f,g\rangle}$ on $M$ such that $\mathscr{E}(f,g)=\mu_{\langle f,g\rangle}(M)$, hence $\mathscr{E}(f,f)=\mu_{\langle f\rangle}(M)$. We assume $(\mathscr{E},D(\mathscr{E}))$ admits a carré-du-champ $\Gamma$.

We can extend the singed smooth $\mu_{\langle f,g\rangle}$ or carré-du-champ $\Gamma(f,g)$ from $f,g\in D(\mathscr{E})$ to $f,g\in\dot{D}(\mathscr{E})_{\rm loc}$ by polarization. In this case, $\mu_{\langle f,g\rangle}$ (resp. $\Gamma(f,g)$) is no longer a finite signed measure (resp. an $\mathfrak{m}$-integrable function). For $u\in\dot{D}(\mathscr{E})_{\rm loc}$ and an $\mathscr{E}$-nest $\{G_n\}$ of $\mathscr{E}$-quasi-open sets satisfying $u|_{G_n}\in D(\mathscr{E})|_{G_n}$, then we can define $\mathscr{E}(u,v)$ for $v\in\bigcup_{n=1}^\infty D(\mathscr{E})_{G_n}$ by

$$\mathscr{E}(u,v)=\mu_{\langle u,v\rangle}(M).$$

Here $D(\mathscr{E})_{G_n}:=\{u\in D(\mathscr{E})\mid\tilde{u}=0\ \mathscr{E}\text{-q.e. on } G_n^c\}$.

Let $\kappa$ be a signed smooth measure with its Jordan-Hahn decomposition $\kappa=\kappa^+-\kappa^-$. We assume that $\kappa^+$ is a smooth measure of Dynkin class ($\kappa^+\in S_D(\mathbf{X})$ in short) and $2\kappa^-$ is a smooth measure of extended Kato class ($2\kappa^-\in S_{EK}(\mathbf{X})$ in short). More precisely, $\nu\in S_D(\mathbf{X})$ (resp. $\nu\in S_{EK}(\mathbf{X})$) if and only if $\nu\in S(\mathbf{X})$ and $\mathfrak{m}\text{-}\sup_{x\in M}\mathbb{E}_x[A_t^\nu]<\infty$ for any/some $t>0$ (resp. $\lim_{t\to 0}\mathfrak{m}\text{-}\sup_{x\in M}\mathbb{E}_x[A_t^\nu]<1$). Here $S(\mathbf{X})$ denotes the family of smooth measures with respect to $\mathbf{X}$ and $\mathfrak{m}\text{-}\sup_{x\in M}f(x)$ denotes the $\mathfrak{m}$-essentially supremum for a function $f$ on $M$. For $\nu\in S(\mathbf{X})$ and set $U_\alpha\nu(x):=\mathbb{E}_x\left[\int_0^\infty e^{-\alpha t}\mathrm{d}A_t^\nu\right]$ with its $\mathfrak{m}$-essentially supremum $\|U_\alpha\nu\|_\infty:=\mathfrak{m}\text{-}\sup_{x\in M}U_\alpha\nu(x)$, $\|U_\alpha\nu\|_\infty<\infty$ for some/any $\alpha>0$ (resp. $\lim_{\alpha\to\infty}\|U_\alpha\nu\|_\infty=0$, $\lim_{\alpha\to\infty}\|U_\alpha\nu\|_\infty<1$) if and only if $\nu\in S_D(\mathbf{X})$ (resp. $\nu\in S_{EK}(\mathbf{X})$, $\nu\in S_K(\mathbf{X})$). For $\nu\in S_D(\mathbf{X})$, the following inequality holds

$$\int_M \tilde{f}^2 \mathrm{d}\nu \leq \|U_\alpha \nu\|_\infty \mathscr{E}_\alpha(f,f), \quad f \in D(\mathscr{E}) \tag{1}$$

which is called the Stollmann-Voigt's inequality. Here $\mathscr{E}_\alpha(f,f) := \mathscr{E}_\alpha(f,f) + \alpha\|f\|^2_{L^2(M;\mathfrak{m})}$. Then, the quadratic form

$$\mathscr{E}^{2\kappa}(f) := \mathscr{E}(f) + \langle 2\kappa, \tilde{f}^2\rangle$$

with finiteness domain $D(\mathscr{E}^{2\kappa}) = D(\mathscr{E})$ is closed, lower semi-bounded, moreover, there exists $\alpha_0 > 0$ and $C > 0$ such that

$$C^{-1}\mathscr{E}_1(f,f) \leq \mathscr{E}^{2\kappa}_{\alpha_0}(f,f) \leq C\mathscr{E}_1(f,f) \quad \text{for all} \quad f \in D(\mathscr{E}^{2\kappa}) = D(\mathscr{E}) \tag{2}$$

by (1). The Feynman–Kac semigroup $(p_t^{2\kappa})_{t\geq 0}$ and it coincides with the strongly continuous semigroup $(P_t^{2\kappa})_{t\geq 0}$ on $L^2(M;\mathfrak{m})$ associated with $(\mathscr{E}^{2\kappa}, D(\mathscr{E}^{2\kappa}))$. Here $A_t^{q\kappa}$ is a continuous additive functional (CAF in short) associated with the signed smooth measure $2\kappa$ under Revuz correspondence. Under $\kappa^- \in S_K(\mathbf{X})$ and $p \in [1,+\infty]$, $(p_t^\kappa)_{t\geq 0}$ can be extended to be a bounded operator on $L^p(M;\mathfrak{m})$ denoted by $P_t^\kappa$ such that there exist finite constants $C = C(\kappa) > 0, C_\kappa \geq 0$ depending only on $\kappa^-$ such that for every $t \geq 0$

$$\|P_t^\kappa\|_{p,p} \leq Ce^{C_\kappa t}. \tag{3}$$

Here $C = 1$ under $\kappa^- = 0$. $C_\kappa \geq 0$ can be taken to be 0 under $\kappa^- = 0$.

Let $\Delta^{2\kappa}$ be the $L^2$-generator associated with $(\mathscr{E}^{2\kappa}, D(\mathscr{E}))$ called the *Schrödinger operator* with potential $2\kappa$. Formally, $\Delta^{2\kappa}$ can be understood as "$\Delta^{2\kappa} = \Delta - 2\kappa$", where $\Delta$ is the $L^2$-generator associated with $(\mathscr{E}, D(\mathscr{E}))$.

**Definition 1 (2-Bakry–Émery condition).** Suppose that $\kappa^+ \in S_D(\mathbf{X})$, $2\kappa^- \in S_{EK}(\mathbf{X})$. We say that $(M, \mathscr{E}, \mathfrak{m})$ or simply $M$ satisfies the 2-Bakry–Émery condition, briefly $\mathsf{BE}_2(\kappa,\infty)$, if for every $f \in D(\Delta)$ with $\Delta f \in D(\mathscr{E})$ and every nonnegative $\phi \in D(\Delta^{2\kappa})$ with $\Delta^{2\kappa}\phi \in L^\infty(M;\mathfrak{m})$ with $\phi \in L^\infty(M;\mathfrak{m})$, we have

$$\frac{1}{2}\int_M \Delta^{2\kappa}\phi\Gamma(f)\mathrm{d}\mathfrak{m} - \int_M \phi\Gamma(f,\Delta f)\mathrm{d}\mathfrak{m} \geq 0.$$

**Assumption 2.** We assume that $M$ satisfies $\mathsf{BE}_2(\kappa,\infty)$ condition for a given signed smooth measure $\kappa$ with $\kappa^+ \in S_D(\mathbf{X})$ and $2\kappa^- \in S_{EK}(\mathbf{X})$.

Under Assumption 2, we say that $(M, \mathscr{E}, \mathfrak{m})$ or simply $M$ is *tamed*. In fact, under $\kappa^+ \in S_D(\mathbf{X})$ and $2\kappa^- \in S_{EK}(\mathbf{X})$, the condition $\mathsf{BE}_2(\kappa,\infty)$ is *equivalent* to that the heat flow $(P_t)_{t\geq 0}$ satisfies

$$\sqrt{\Gamma(P_t f)} \leq P_t^\kappa\sqrt{\Gamma(f)} \quad \mathfrak{m}\text{-a.e. for any } f \in D(\mathscr{E}) \quad \text{and} \quad t \geq 0. \tag{4}$$

The inequality (4) plays a crucial role in our paper. Note that our condition $\kappa^+ \in S_D(\mathbf{X})$, $\kappa^- \in S_{EK}(\mathbf{X})$ (resp. $\kappa^+ \in S_D(\mathbf{X})$, $2\kappa^- \in S_{EK}(\mathbf{X})$) is stronger than the 1-moderate (resp. 2-moderate) condition treated in [2] for the definition of tamed space. The $\mathfrak{m}$-symmetric Markov process $\mathbf{X}$ treated in our paper may not be conservative in general. Hereafter, we consider the following quantities:

$$C_{\gamma,\delta} := \sup_{t\geq 0} e^{-\gamma t}\|p_t^{\delta\kappa}1\|_{L^\infty(M;\mathfrak{m})} \in ]0,+\infty]$$

for $\gamma \geq 0$ and $\delta \geq 0$ and

$$D_{p,\alpha,\kappa} := \left\| \mathbb{E}_\cdot \left[ \int_0^\infty e^{-p\alpha s - pA_s^\kappa} \mathrm{d}A_s^{(\alpha\mathfrak{m}+\kappa)^-} \right] \right\|_{L^\infty(M;\mathfrak{m})} \in [0,+\infty]$$

for $\alpha \geq 0$ and $p \in ]1,+\infty[$. Note that $C_{\gamma,\delta} \leq 1$ and $D_{p,\alpha,\kappa} = 0$ when $\kappa^- = 0$.

**Main Results.** Our main theorem under Assumption 2 is the following:

**Theorem 3.** *Let $p \in ]1,+\infty[$ and $\alpha \geq 0$. For $f \in L^p(M;\mathfrak{m}) \cap L^2(M;\mathfrak{m})$, we define the Riesz operator $R_\alpha(\Delta)$ by*

$$R_\alpha(\Delta)f := \Gamma((\alpha - \Delta)^{-\frac{1}{2}}f)^{\frac{1}{2}}.$$

(1) *Assume $\alpha > 0$ and $\kappa^- \in S_K(\mathbf{X})$. Suppose $p \in [2,+\infty[$, or $p \in ]1,2]$ and $\kappa^- = R\mathfrak{m}$ with $R \geq 0$. Then, $R_\alpha(\Delta)$ can be extended to a (non-linear) bounded operator on $L^p(M;\mathfrak{m})$. The operator norm $\|R_\alpha(\Delta)\|_{p,p}$ depends only on $\kappa, p$ and $\alpha$.*

(2) *Assume $\alpha = 0$, $\kappa^- \in S_K(\mathbf{X})$ and $C_{0,\beta} < \infty$ for some $\beta > p$ and $D_{q,0,\kappa} < \infty$ for $q := p/(p-1)$. Suppose $p \in [2,+\infty[$, or $p \in ]1,2]$ and $\kappa^- = 0$. Then $R_0(\Delta)$ can be extended to a (non-linear) bounded operator on $L^p(M;\mathfrak{m})$. In particular, if $\kappa^- = 0$, then for all $p \in ]1,+\infty[$, $R_0(\Delta)$ can be extended to a (non-linear) bounded operator on $L^p(M;\mathfrak{m})$.*

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## Dyson Brownian motion as a gradient flow

Kohei Suzuki

**Infinite Dyson Brownian motion.** We are interested in the interacting particle system solving the following formal stochastic differential equation of infinitely many particles in $\mathbb{R}$:

$$\text{(DBM)} \qquad \mathrm{d}X_t^k = \frac{\beta}{2} \lim_{r\to\infty} \sum_{\substack{i:i\neq k \\ |X_t^k - X_t^i|<r}} \frac{1}{X_t^k - X_t^i}\,\mathrm{d}t + \mathrm{d}B_t^k, \quad k \in \mathbb{N}\ ,$$

where $(B_t^k : t \geq 0, k \in \mathbb{N})$ is the sequence of infinitely many independent Brownian motions on $\mathbb{R}$. The solution $\mathbb{X}_t = (X_t)_{k\in\mathbb{N}}$ to (DBM) is called *infinite Dyson Brownian motion*, which has a particular importance in relation to the random matrix theory. The existence and the uniqueness of strong/weak solutions to (DBM) have been intensively studied, e.g., in [Dys62, Spo87, NF98, KT10, Osa96, Osa12, Osa13, Tsa16]. In [Osa12] for $\beta = 1, 2, 4$ and later in [Tsa16] for $\beta \geq 1$, the existence and the pathwise uniqueness of the strong solution to (DBM) have been proven under suitable choice of initial conditions. By the map $(x_i)_{i\in\mathbb{N}} \mapsto \sum_{i=1}^\infty \delta_{x_i}$, we can think of $\mathbb{X}_t$ as a diffusion process on *the configuration spcae* $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}(\mathbb{R})$ over $\mathbb{R}$ (i.e., the space of locally finite point measures on $\mathbb{R}$) endowed with the vague topology $\tau_{\mathrm{v}}$ (i.e., the topology induced by the duality of compactly supported continuous functions in $\mathbb{R}$). This diffusion process on $\boldsymbol{\Upsilon}$ is called *unlabelled solution* to (DBM) and denoted by $\mathsf{X}_t$. For $\beta = 1, 2, 4$, the solution $\mathsf{X}_t$ has been identified with the diffusion process associated with a certain Dirichlet form whose symmetrising measure $\mu$ is the $\mathsf{sine}_\beta$ ensemble, see [Osa12, Thm. 24] and [Tsa16, §8].

The main result of this report is based on [Suz22, Suz24a, Suz24b]:

**Theorem 1.** *Let $\beta > 0$. There exists a symmetric local Dirichlet form $(\mathcal{E}^{\boldsymbol{\Upsilon},\mu}, \mathcal{D}(\mathcal{E}^{\boldsymbol{\Upsilon},\mu}))$ with the square field operator $\Gamma^{\boldsymbol{\Upsilon}}$ and the symmetrising measure $\mu = \mathsf{sine}_\beta$ such that*

*(a) the Bakry–Émery gradient estimate $\mathsf{BE}(0,\infty)$ holds: Namely, for the $L^2$-semigroup $\{T_t^{\boldsymbol{\Upsilon},\mu}\}_{t>0}$ associated with $(\mathcal{E}^{\boldsymbol{\Upsilon},\mu}, \mathcal{D}(\mathcal{E}^{\boldsymbol{\Upsilon},\mu}))$,*

$$\Gamma^{\boldsymbol{\Upsilon}}\left(T_t^{\boldsymbol{\Upsilon},\mu} u\right) \leq T_t^{\boldsymbol{\Upsilon},\mu}\Gamma^{\boldsymbol{\Upsilon}}(u)\ , \qquad u \in \mathcal{D}(\mathcal{E}^{\boldsymbol{\Upsilon},\mu}) \quad t > 0\ ,$$

*where $T_t^{\boldsymbol{\Upsilon},\mu}$ is the $L^2$-semigroup associated with $(\mathcal{E}^{\boldsymbol{\Upsilon},\mu}, \mathcal{D}(\mathcal{E}^{\boldsymbol{\Upsilon},\mu}))$. In the case $\beta = 2$, the curvature lower bound $K = 0$ is optimal.*

*(b) $(\mathcal{E}^{\boldsymbol{\Upsilon},\mu}, \mathcal{D}(\mathcal{E}^{\boldsymbol{\Upsilon},\mu}))$ is irreducible for $\beta = 2$, which is equivalent to say*[1]

$$T_t^{\boldsymbol{\Upsilon},\mu} u \xrightarrow{t\to\infty} \int_{\boldsymbol{\Upsilon}} u\,\mathrm{d}\mu\ , \qquad u \in L^2(\mu)\ .$$

*Furthermore, the spectral gap does not exists.*

[1] The irreducibility has been proved independently also by Osada-Osada'23

(c) $(\mathcal{E}^{\Upsilon,\mu}, \mathcal{D}(\mathcal{E}^{\Upsilon,\mu}))$ *is quasi-regular and properly associated with the unlabelled solution to* (DBM) *for* $\beta = 1, 2, 4$

(d) *There exists a closed subspace* $\mathcal{F} \subset \mathcal{D}(\mathcal{E}^{\Upsilon,\mu})$ *so that* $(\mathcal{E}^{\Upsilon,\mu}, \mathcal{F})$ *is quasi-regular for* $\beta > 0$.

*Dyson Brownian motions as a gradient flow.* By Thm. 1, we can show that the dual flow of the infinite Dyson Brownian motions is the gradient flow in the space of probability measures on $\Upsilon$ in terms of the Boltzmann-Shannon entropy $\mathsf{Ent}_\mu(\nu) = \int_\Upsilon \rho \log \rho \, \mathrm{d}\mu$ associated with $\mu = \mathsf{sine}_\beta$ for $\beta > 0$ and a Benamou–Brenier-like extended distance $\mathsf{W}_\mathcal{E}$. Let $\mathcal{P}(\Upsilon)$ be the space of all Borel probability measures in $\Upsilon$ and $\mathcal{P}_\mu(\Upsilon) = \{\nu \in \mathcal{P}(\Upsilon) : \nu \ll \mu\}$. For $\nu, \sigma \in \mathcal{P}_\mu(\Upsilon)$, we define $\mathsf{W}_\mathcal{E}$ as

$$\mathsf{W}_\mathcal{E}(\nu,\sigma)^2 := \inf\left\{ \int_0^1 \|\rho_t'\|^2 \, \mathrm{d}t : (\rho_t) \in \mathsf{CI}(\mathcal{E}^{\Upsilon,\mu}) \ , \ \nu = \rho_0 \cdot \mu \ , \ \sigma = \rho_1 \cdot \mu \right\} ,$$

where $(\rho_t) \in \mathsf{CI}(\mathcal{E}^{\Upsilon,\mu})$ satisfies a *continuity inequality*, and $\|\rho_t'\|$ is the *modulus of verocity.* If there is no $(\rho_t) \in \mathsf{CI}(\mathcal{E}^{\Upsilon,\mu})$ connecting $\nu$ and $\sigma$, we define $\mathsf{W}_\mathcal{E}(\nu,\sigma) = +\infty$. Let $\mathcal{D}(\mathsf{Ent}_\mu) := \{\nu \in \mathcal{P}(\Upsilon) : \mathsf{Ent}_\mu(\nu) < \infty\}$ be the domain of $\mathsf{Ent}_\mu$. Let $t \mapsto \mathcal{T}_t^{\Upsilon,\mu}\nu$ be the dual flow of $T_t^{\Upsilon,\mu}$ defined as

$$\mathcal{T}_t^{\Upsilon,\mu}\nu := (T_t^{\Upsilon,\mu}\rho) \cdot \mu \ , \qquad \nu = \rho \cdot \mu \in \mathcal{P}(\Upsilon) \ .$$

**Corollary 2.** *Let* $\mu$ *be the* $\mathsf{sine}_\beta$ *ensemble with* $\beta > 0$.

- *Evolutional variation inequality: For every* $\nu, \sigma \in \mathcal{D}(\mathsf{Ent}_\mu)$ *with* $\mathsf{W}_\mathcal{E}(\nu,\sigma) < \infty$, *the curve* $t \mapsto \mathcal{T}_t^{\Upsilon,\mu}\sigma \in (\mathcal{P}(\Upsilon), \mathsf{W}_\mathcal{E})$ *is locally absolutely continuous,* $\mathsf{Ent}_\mu(\mathcal{T}_t^{\Upsilon,\mu}\sigma) < \infty$, $\mathsf{W}_\mathcal{E}(\mathcal{T}_t^{\Upsilon,\mu}\sigma, \nu) < \infty$ *for every* $t > 0$

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}\mathsf{W}_\mathcal{E}\big(\mathcal{T}_t^{\Upsilon,\mu}\sigma, \nu\big)^2 \le \mathsf{Ent}_\mu(\nu) - \mathsf{Ent}_\mu(\mathcal{T}_t^{\Upsilon,\mu}\sigma) \ , \qquad t > 0 \ .$$

- *Geodesic convexity: The space* $(\mathcal{D}(\mathsf{Ent}_\mu), \mathsf{W}_\mathcal{E})$ *is an extended geodesic metric space. Namely, for every pair* $\nu, \sigma \in \mathcal{D}(\mathsf{Ent}_\mu)$ *with* $\mathsf{W}_\mathcal{E}(\nu,\sigma) < \infty$, *there exists* $\mathsf{W}_\mathcal{E}$*-Lipschitz curve* $\nu_\cdot : [0,1] \to (\mathcal{D}(\mathsf{Ent}_\mu), \mathsf{W}_\mathcal{E})$ *so that*

$$\nu_0 = \nu \ , \qquad \nu_1 = \sigma \ , \qquad \mathsf{W}_\mathcal{E}(\nu_t, \nu_s) = |t-s| \mathsf{W}_\mathcal{E}(\nu,\sigma) \ , \qquad s, t \in [0,1] \ .$$

- *Gradient flow: The dual flow* $\big(\mathcal{T}_t^{\Upsilon,\mu}\nu_0\big)_{t>0}$ *is the unique solution to the* $\mathsf{W}_\mathcal{E}$*-gradient flow of* $\mathsf{Ent}_\mu$ *starting at* $\nu_0$. *Namely, for any* $\nu_0 \in \mathcal{D}(\mathsf{Ent}_\mu)$, *the curve* $[0,\infty) \ni t \mapsto \nu_t = \mathcal{T}_t^{\Upsilon,\mu}\nu_0 \in \mathcal{D}(\mathsf{Ent}_\mu)$ *is the unique solution to the energy equality starting at* $\nu_0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{Ent}_\mu(\nu_t) = -|\dot\nu_t|^2 = -|\mathsf{D}^-_{\mathsf{W}_\mathcal{E}}\mathsf{Ent}_\mu|^2(\nu_t) \quad \textit{a.e. } t > 0 \ .$$

*Here,* $|\dot\nu_t| := \lim_{s\to t} \frac{\mathsf{W}_\mathcal{E}(\nu_s,\nu_t)}{|s-t|}$ *is the metric speed of* $\nu_t$ *and*

$$|\mathsf{D}^-_{\mathsf{W}_\mathcal{E}}\mathsf{Ent}_\mu|(\nu) := \begin{cases} \limsup\limits_{\sigma\to\nu} \dfrac{(u(\sigma)-u(\nu))^-}{\mathsf{W}_\mathcal{E}(\sigma,\nu)} & \textit{if } \nu \textit{ is not isolated,} \\ 0 & \textit{otherwise .} \end{cases}$$

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## Networks of Noisy Neurons


Ben Hambly

(joint work with Aldair Petronilia, Christoph Reisinger, Andreas Søjmark)


We consider a simple particle system model for a large network of integrate-and-fire neurons. The integrate-and-fire model for a neuron considers just the membrane potential of the neuron with its evolution determined by external and internal inputs, as well as noise. When the potential reaches a threshold the neuron fires, producing a spike which transmits to other neurons that are connected and causes the original neuron's membrane potential to reset to its resting potential. In a network of such neurons, which we take to be the complete graph, a spike from one neuron can cause other neurons to spike and we will be interested in the overall effect on the network.

The approach we take is to consider the empirical measure of the membrane potentials of particles, specifying their threshold to be 0 and the reset point to be -1. Then by taking a mean field limit we can look at the evolution of a measure valued process which captures the behaviour of the system as a whole. A key point is that, when a neuron fires, it transmits its spike instantaneously to all others, with their potential increasing by $\alpha/N$, which could cause other neurons to fire. This may lead to a cascading effect where a proportion of all the neurons fire at the same time, thus there could potentially be jumps in the empirical measure.

A simple version of this model has the particles absorbed when they hit 0 and when a particle hits 0, all other particles jump toward 0. Such systems have been analysed and a global uniqueness theorem proved under a so called physical jump condition in the case where the particles move according to Brownian motion [2].

For the networks of integrate-and-fire neurons the first paper considering a probabilistic model like this was [1]. For small enough $\alpha$ there is a continuous solution for all time, while if $\alpha$ is large enough, there exists a solution satisfying a physical jump condition which balances the mass hitting the threshold with the size of the jump in the system. We introduce a more general set up and start taking into account other features of real neurons: the process of transmitting the spike to other neurons is not instantaneous; there is often common noise, with many neurons subject to the same external noise; and there is a refractory period immediately after firing, during which the neuron cannot fire again. By incorporating all these features we consider a more general model, where the membrane potential of $i$-th particle, $X^i$, for $i = 1, \dots, N$, has dynamics

$$\begin{cases} dX_t^i = b(t, X_t^i, \nu_t^N, \mathfrak{f}_t^N) I_{\{X_t^i<0\}} dt + \sigma(t,\, X_t^i) \sqrt{1-\rho^2(t, \nu_t^N, \mathfrak{f}_t^N)} I_{\{X_t^i<0\}} dW_t^i \\ \qquad\qquad + \sigma(t,\, X_t^i) \rho(t, \nu_t^N, \mathfrak{f}_t^N) I_{\{X_t^i<0\}} dW_t^0 - d \sum_{k\geq 1} \xi_k^i I_{[0,t]}(\tau_k^i + \varsigma_k^i) \\ \tau_k^i = \inf\{t > \tau_{k-1}^i + \varsigma_{k-1}^i \ :\ X_{t-}^i \geq 0\}, \qquad \tau_0^i = 0, \\ J_t^i = \sum_{k\geq 1} I_{[0,t]}(\tau_k^i),\ J_t^{D,i} = \sum_{k\geq 1} I_{[0,t]}(\tau_k^i + \varsigma_k^i), \\ F_t^N = \frac{1}{N} \sum_{i=1}^N J_t^i,\ F_t^{D,N} = \frac{1}{N} \sum_{i=1}^N J_t^{D,i}, \\ \nu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i} I_{\{X_t^i<0\}}, \\ \mathfrak{f}_t^N = \frac{d}{dt} \left( \mathfrak{K} * F^N \right)_t = \int_0^t \mathfrak{K}'(t-s) F_s^N ds. \end{cases}$$

The random variable $\varsigma_k^i$ represents the length of the refractory period of the neuron and has probability density $r$. The term $\xi_k^i$ captures the inherent stochasticity in the reset potential of the neuron. The empirical distribution of the system, denoted by $\nu_t^N$, gives the distribution of the neurons that are not in their refractory period. The variable $J_t^i$ represents the number of times that neuron $i$ has fired by time $t$, while $F_t^N$, which we call the firing function, captures the total number of firings that have occurred in the entire system by time $t$. The feedback is incorporated into the drift coefficient $b$ via the term $\mathfrak{f}_t^N$, which is the derivative of the firing function $F_t^N$, mollified by an appropriate kernel $\mathfrak{K}$. The Brownian driver $W^0$ is common to all particles, while the $W^i$ are independent Brownian motions capturing the idiosyncratic noise for the individual neurons. Finally $\sigma$ is the diffusion coefficient and $\rho$ is the correlation between idiosyncratic and common noise.

This model has some features which make it better behaved as the mollification of the transmission of the spike leads to more regularity. Our main result is to establish the existence and uniqueness for the mean field limit of this particle system. Firstly the existence theorem can be (loosely) stated as.

**Theorem 1.** *(Limit SPDE) Under natural assumptions $\{(\nu^N, F^N, W^0)\}_N$ is tight on $(D_{\mathcal{S}'}, M_1) \times (D_{\mathbb{R}}, M_1) \times (\mathcal{C}_{\mathbb{R}}, \|\cdot\|_\infty)$. Given each limit point, $(\nu, F, W^0)$, $\nu$ is a cadlag $\mathbf{M}_{\leq 1}(\mathbb{R}_-)$-valued process with probability 1. Moreover, $\nu$ obeys, with* *probability* 1*, the limit SPDE*

$$\begin{aligned} d\langle \nu_t,\, \phi\rangle =& \langle \nu_t,\, b(t,\cdot,\nu_t,\mathfrak{f}_t)\partial_x\phi\rangle dt + \frac{1}{2}\langle \nu_t, \sigma^2(t,\cdot)\partial_{xx}\phi\rangle dt \\ & + \langle \nu_t,\, \sigma(t,\cdot)\rho(t,\nu_t,\mathfrak{f}_t)\partial_x\phi\rangle dW_t^0 + \mathbb{E}\left[\phi(-\xi)\right] d\int_0^t r(t-s)F_s ds \\ & - \phi(0)dF_t, \\ \nu_0 \sim^d & V_0(x)dx \end{aligned}$$

*where* $\phi \in \mathcal{S}$ *and* $\mathfrak{f} = (\mathfrak{K}' * F)$*.*

It can be shown that any limit pair $(\nu, F)$ is sufficiently regular in that it satisfies a range of natural conditions such as $\nu$ is supported on $\mathbb{R}_-$, the measure has exponential decay at infinity and polynomial decay near zero. Also the firing function $F$ is increasing and satisfies the mass balance $\nu_t(-\infty,0) + F_t - \int_0^t r(t-s)F_s ds = 1$. We prove that this system has a unique solution within this class of sufficiently regular solutions and under natural assumptions on the coefficients. There is also a McKean-Vlasov representation for the solution, conditional on the common noise.

**Theorem 2.** *Let* $(\nu, F, W^0)$ *be the unique strong solution to the limit SPDE. Then, for any Brownian motion* $W$ *and random variables* $\{\xi_k\}_k$ *and* $\{\varsigma_k\}_k$*, all mutually independent and independent of* $X_0$ *and* $W^0$*, we have*

$$\nu_t = \mathbb{P}\left[\, X_t \in \cdot, X_t < 0 |\, W^0\right],$$

*where* $X_t$ *is the unique solution to the conditional McKean–Vlasov diffusion*

$$(1)\quad \begin{cases} dX_t = b(t,X_t,\nu_t,\mathfrak{f}_t)I_{\{X_t<0\}}dt + \sigma(t,X_t)\sqrt{1-\rho^2(t,\nu_t,\mathfrak{f}_t)}I_{\{X_t<0\}}dW_t \\ \qquad +\sigma(t,X_t)\rho(t,\nu_t,\mathfrak{f}_t)I_{\{X_t<0\}}dW_t^0 - d\sum_{k\geq 1}\xi_k I_{[0,t]}(\tau_k+\varsigma_k), \\ \tau_k = \inf\{t > \tau_{k-1}+\varsigma_{k-1} \;:\; X_{t-}\geq 0\}, \qquad \tau_0 = 0, \\ F_t = \sum_{k\geq 1}\mathbb{P}\left[\,\tau_k \leq t |\, W^0\right],\ \mathfrak{f}_t = \frac{d}{dt}\left(\mathfrak{K} * F\right)_t = \int_0^t \mathfrak{K}'(t-s)F_s ds. \end{cases}$$

Finally we can try to return to a more general version of the original model of [1]. Using ideas from [3] we can then take a further limit as we let the refractory period and the transmission delay go to zero. That is we rescale the kernels $\mathfrak{K}$ and $r$ so that they converge to delta functions to try to recover the instantaneous transmission model in a more general setting than that of [1]. The result is an existence theorem for a scaling limit but it only provides an upper bound on the size of jumps that can occur in the firing function. Under stronger conditions on the coefficients, in particular drift depending on time and space, diffusion depending on time and constant $\alpha$ parameter, we can establish the existence of a generalized neuron model under a physical jump condition. These results will shortly be available on the ArXiv.

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## Einstein relation on metric measure spaces


Uta Freiberg

(joint work with Fabian Burghart)


Many physical phenomena proceed in or on irregular objects which are often modeled by fractal sets. Using the model case of the Sierpinski gasket, the notions of Hausdorff, spectral and walk dimension are introduced in a survey style. These characteristic numbers of the fractal are essential for the Einstein relation, expressing the interaction of geometric, analytic and stochastic aspects of a set. Herby, the spectral dimension denotes the double of the exponent in the leading term of the eigenvalue counting function of the (natural) Laplacian, while the walk dimension denotes the power of the radius $R$ has to equipped with in order to get the mean exit time of the (natural) Brownian motion from a ball of radius $R$.

It turns out that in the case of Sierpinski gasket the numbers of Hausdorff, spectral and walk dimension Ñ denoted by $d_H$, $d_S$ and $d_W$ Ñ take the values

$$d_H = \frac{\ln 3}{\ln 2}, d_S = \frac{\ln 9}{\ln 5} \text{ and } d_W = \frac{\ln 5}{\ln 2}.$$

So, the Einstein relation

$$\frac{d_H}{d_S} = \frac{d_W}{2}$$

is satisfied. As it also holds for domains in the Euclidean space $\mathbb{R}^n$ – thanks to $d_H = d_S = n$ and $d_W = 2$ – one could think that it holds for any set. However, on the 2–dimensional comb Einstein relation fails (we have $d_H = 2, d_S = 3/2$ and $d_W = 2$ on the comb), and so we start seeking for other counterexamples.

We review the Einstein relation, which connects the Hausdorff, local walk and spectral dimensions on a space, in the abstract setting of a metric measure space equipped with a suitable operator. This requires some twists compared to the usual definitions from fractal geometry. The main result establishes the invariance of the three involved notions of fractal dimension under bi-Lipschitz continuous isomorphisms between mm-spaces and explains, more generally, how the transport of the analytic and stochastic structure behind the Einstein relation works. While any homeomorphism suffices for this transport of structure, non-Lipschitz maps distort the Hausdorff and the local walk dimension in different ways. To illustrate this, we take a look at Hölder regular transformations and how they influence the local walk dimension and prove some partial results concerning the Einstein relation on graphs of fractional Brownian motions. We conclude by giving a short list of further questions that may help building a general theory of the Einstein relation. For further reading, we refer to [1].

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## Construction of Hunt processes by the Lyapunov method and applications to generalized Mehler semigroups

Iulian Cîmpean

(joint work with Lucian Beznea, Michael Röckner)

This abstract is based on [3].

Generalized Mehler semigroups have been intensively studied e.g. by [8], [4], [7], [10], [6], [9], [15], [11], [14], [1] [16], [13], [2], and this is just a short list. They arise as transition functions for infinite dimensional generalizations of Lévy-driven Ornstein-Uhlenbeck processes, which are a role model of infinite dimensional processes. These processes are governed by the linear SDE

$$dX^x(t) = AX^x(t)dt + dZ(t), \quad t > 0, X^x(0) = x \in H, \tag{1}$$

where $A$ is a (possibly unbounded) linear operator on a general Hilbert space $H$ which generates a $C_0$-semigroup, whilst $Z$ is a general (cylindrical) Lévy noise on $H$.

It is known that in general, generalized Mehler semigroups may not correspond to càdlàg (or even càd) Markov processes with values in $H$ endowed with the norm topology. Here, we deal with the problem of characterizing those generalized Mehler semigroups that do correspond to càdlàg Markov processes, which is highly non-trivial. More precisely, we shall provide answers to the problem of existence of càdlàg Markov processes associated with such semigroups, which has been open for some time (see e.g. [5], [15, pg. 99], [14, Question 4, pg. 723], or [1, pg. 40]).

Recall that SDEs as above with general Lévy noise have been studied by starting from the generalized Mehler semigroup e.g. in [4], [7], [10] in the time-homogeneous case, or by [9] and [13] for the time-inhomogeneous case. Fundamentally, it was shown in [7, Theorem 5.3] that the generalized Mehler semigroup associated with the SDE (1) can be represented by a càdlàg Markov process, but on a larger space $E$ in which $H$ can be Hilbert-Schmidt embedded. As mentioned, in general, it is not possible to take $E = H$. In fact, it was first shown in [5, Theorem 2.1] that if $Z(t) = \sum_{n\geq 1} \beta_n Z^{(n)}(t)e_n, t \geq 0$, where $\{e_n\}_{n\geq 1} \subset D(A^*)$ is an orthonormal basis in a Hilbert space $H$, $A$ generates a $C_0$-semigroup on $H$, whilst $\left(Z^{(n)}\right)_{n\geq 1}$ are iid Lévy processes on $\mathbb{R}$ with non-zero jump intensity, then the following negative result holds: *if* $(\beta_n)_{n\geq 1}$ *does not converge to* 0*, then with probability* 1*, the trajectories* *of $X$ have no point $t \in [0,\infty)$ at which the left or right limit exists in $H$.* It was shown later on in [15] that if $\left(Z^{(n)}\right)_{n\geq 1}$ are iid $\alpha$-stable Lévy processes on $\mathbb{R}$ with $\alpha \in (0,2)$, then $\sum\limits_{n\geq 1} \beta_n^\alpha < \infty$ is a sufficient condition to ensure that $X$ (and in fact $Z$) has càdlàg paths in $H$. In [11] it was proved that the above condition is also necessary, and this characterization has been extended in [12] for $\alpha$-semi-stable diagonal Lévy noises. Further steps have been achieved in [14, Theorem 3.1], showing that if the Lévy noise takes values in the domain of some convenient (negative) power of $-A$ and some further moment bounds for the Lévy measure are satisfied, then $X$ has a càdlàg modification in $H$ even if the noise $Z$ lives merely on a larger space; however, this result does not cover the diagonal $\alpha$-stable case from [11], as explained in [14, Remark 3.5]. It is worth to mention that these results have been obtained by stochastic analysis tools like maximal inequalities for the norm of the stochastic convolution, whilst our approach relies on the construction of some convenient Lyapunov function which plays the role of the norm function, but it is fundamentally more flexible, in particular it is allowed to take infinite values on a set which is polar.

*To summarize, in general situations, the question if, and under which conditions, a Markov process associated to* (1) *can be constructed having càdlàg paths in the original space $H$, has remained fundamentally open. In this work we address it from a potential theoretic perspective, starting from the generalized Mehler semigroup.*

On brief, our approach is to reconsider the *càdlàg problem* for generalized Mehler semigroups as a particular case of the much broader problem of constructing Hunt (hence càdlàg and quasi-left continuous) processes from a given Markov semigroup. One main difficulty in constructing Hunt Markov processes associated with the generalized Mehler semigroup corresponding to (1) is simply that bounded sets in infinite dimensions are not compact with respect to the norm topology. Thus, at a first stage, our approach is to search for conditions that ensure that $X$ has cadlag paths with respect to the weak topology, but this is also problematic since the weak topology in infinite dimensions is not metrizable, and this condition is fundamental for constructing Hunt Markov processes by the general theory existing in the literature. Having these in mind, a consistent part of this work is devoted to prove that starting from a Markov semigroup on a general (possibly non-metrizable) state space, the existence of a suitable Lyapunov function with relatively compact sub/sup-sets in conjunction with a local Feller-type regularity of the resolvent are sufficient to ensure the existence of an associated Hunt Markov process. Based on such a general result, we prove that under natural assumptions, the generalized Mehler semigroup associated with (1) is the transition function of a Hunt Markov process that lives on the original Hilbert space $H$ endowed with the norm topology; essentially, we first prove the result for the weak topology, and then we lift it to the norm topology by a general argument that we also develop in this work. As an application, we give explicit conditions under which the generalized

Mehler semigroup associated with the SPDE

$$dX^x(t) = \Delta_0 X^x(t)dt + dZ(t), \quad t > 0, X^x(0) = x \in L^2(D), \quad D \subset \mathbb{R}^d, \tag{2}$$

is the transition function of a Hunt Markov process on the original space $L^2(D)$ with respect to the norm topology; in this example, $Z(t), t \geq 0$, is a (non-diagonal and cylindrical) Lévy noise on $L^2(D)$ whose characteristic exponent $\lambda : L^2(D) \to \mathbb{C}$ is given by

$$\lambda(\xi) := \|\Sigma_1\xi\|^2_{L^2(D)} + \|\Sigma_2\xi\|^\alpha_{L^2(D)}, \quad \xi \in L^2(D), \quad \alpha \in (0,2), \tag{3}$$

whilst $\Sigma_1$ and $\Sigma_2$ are positive definite bounded linear operators on $L^2(D)$. Some further fine regularity properties of the constructed Markov processes are obtained as byproducts of our potential theoretic approach.

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## The De Giorgi-Moser theory for non-local (kinetic) equations

Moritz Kassmann

The fractional Kolmogorov equation

$$\partial_t u + (-\Delta_v)^s u + v \cdot \nabla_x u = 0$$

for a function $u : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ can be seen as a linearized model of the Boltzmann equation. This viewpoint uses results from Luis Silvestre (Comm. Math. Physics, 2016). Recently, there have been published preprints stating the parabolic Harnack inequality holds for kinetic equations including the fractional Kolmogorov equation. In the talk I present a counterexample showing that the parabolic Harnack inequality fails. The counterexample transfers ideas from a paper by R. Bass and Z.-Q. Chen (2006) to an analysis framework and adopts it to the aforementioned equation.

## Who is your $p$-energy when your world is not smooth?

Patricia Alonso Ruiz
(joint work with Fabrice Baudoin)

What does it mean that your world is not smooth? In this talk, your world is a metric measure space $(X, d, m)$ without a classical differential structure. Yet you would like to study nonlinear Dirichlet forms analogue to the Euclidean

$$\mathcal{E}_p(f, g) = \int_{\mathbb{R}^n} |\nabla f|^{p-2} \langle \nabla f, \nabla g \rangle \, dx$$

and its associated $p$-energy functional

$$\mathcal{E}_p(f) = \int_{\mathbb{R}^n} |\nabla f|^p dx. \tag{1}$$

Naturally you wonder

*how could I construct a $p$-energy if there is no classical $\nabla f$ in my world?*

Thankfully you are not alone in your question and many people have been thinking about it in the past years. During this workshop we have seen several ways to carry out this construction under different kinds of "non-smoothness" in your world. For example:

- ∘ If your space admits a *(weak) upper gradient* structure, you may replace $\nabla f$ with the (weak) upper gradient of $f$, see e.g. [2].
- ∘ If your space is equipped with a strongly regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(X,m)$ that admits a *Carré du Champ* operator $\Gamma$, you may replace $\nabla f$ with $\sqrt{\Gamma(f)}$, see e.g. [3].
- ∘ If your space is locally compact and is equipped with a Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(X,m)$ with energy dominant measure $\mu$, you may replace $\nabla f$ with $\sqrt{\Gamma_\mu(f)}$, where $\Gamma_\mu(f)$ denotes the Radon Nykodym derivative of the energy measure of $f$ with respect to $\mu$, see [8].
- ∘ If your space fits in the framework of *conductive homogeneity* [6] or in that of self-similar p-energies in [7], you may approximate the whole expression in (1) by sums over certain partitions.

In this talk you learn a further approach that relies solely on the metric measure structure of your world, when that is a *Cheeger space*. The construction is based on seminal work by Korevaar and Schoen in [5] and the construction of diffusion on $d$-sets by Kumagai and Sturm in [9].

Is your world $(X, d, m)$ a Cheeger space? For us it will be so if $(X, d, m)$ is a locally compact metric measure space whose underlying measure $m$ is doubling, and the space admits the following $(p, p)$-Poincaré inequality with respect to Lipschitz functions

$$\int_{B(x,R)} |f - f_{B(x,R)}|^p dm \leq CR^p \int_{B(x,\Lambda R)} |\mathrm{Lip} f|^p dm, \tag{2}$$

for some uniform constants $C >$ and $\Lambda > 1$, where $⨍_{B(x,R)} f$ denotes the average of $f$ over the ball $B(x,R)$ and

$$\mathrm{Lip} f(y) := \limsup_{\delta > 0} \sup_{d(y,z)<\delta} \frac{|f(z) - f(y)|}{\delta}.$$

**"Theorem":** In a Cheeger space $(X, d, m)$ one is able to construct a $p$-energy analogue to (1) using the *Korevaar-Schoen* $p$-energy functionals

$$E_{p,r}(f) := \frac{1}{r^p} \int_X ⨍_{B(x,r)} |f(x) - f(y)|^p dm(y)\, dm(x)$$

and the construction roughly works as follows:

**Step 1.** You notice that $L^2(X,m)$ is separable and therefore any sequence $\{E_{p,r_n}\}_{n\geq 1}$ will have a $\Gamma$-convergent subsequence. From now own, you will only work with that subsequence, but still denote it by $\{E_{p,r_n}\}_{n\geq 1}$ for simplicity.

Which sequence do you start with? We will address that question a few steps later.

**Step 2.** You set

$$
\begin{aligned}
&\mathcal{E}_p(f) := \Gamma\!-\lim_{n\to\infty} E_{p,r_n}(f),\\
(3)\qquad &\mathcal{F}_p := \Big\{ f \in L^p(X,m)\colon \sup_{r>0} E_{p,r}(f) < \infty \Big\}.
\end{aligned}
$$

Why is this definition of the domain fine? This will also become clear in a few steps.

**Step 3.** The functional $\mathcal{E}_p$ will give reise to a nonlinear form by setting

$$\mathcal{E}_p(f,g) = \frac{1}{p}\lim_{t\to 0}\frac{\mathcal{E}_p(f+tg)-\mathcal{E}_p(f)}{t}.$$

Of course you need to make sure that this limit actually exists! That may be achieved by exploiting the Taylor expression of $|\cdot|^p$ and the possibility to approximate $\mathcal{E}_p(f,g)$ by

$$(4)\quad \frac{1}{r_n}\int_X \fint_{B(x,r_n)} |f_n(x)-f_n(y)|^{p-2}(f_n(x)-f_n(y))(g_n(x)-g_n(y))\,dm(y)\,dm(x)$$

for $f_n \to f$ and any $g_n \to g$ converging strongly in $L^p(X,m)$.

**Step 4.** You look again at (4) and realize that $\mathcal{E}_p(f,g)=0$ as long as $g$ is constant in a neighborhood of $\mathrm{supp} f$. That means, the form is *local*.

**Step 5.** After some computations you also realize that the functional (3) satisfies the following *weak monotonicity property*: For any sequence $r_n \downarrow 0$ there exists a constant $C>0$ such that

$$(5)\qquad \sup_{r>0} E_{p,r}(f) \le C \liminf_{n\to\infty} E_{p,r_n}(f_n)$$

for any $f_n \to f$ converging strongly in $L^p(X,m)$. As a consequence, $\sup_{r>0} E_{p,r}(f)$ is comparable to $\mathcal{E}_p(f)$, which in particular will allow you to deduce using results collected in [10] that $\mathrm{Lip}_{\mathrm{loc}}(X)\cap C_c(X)$ is dense in $\mathcal{F}_p$ with respect to

$$(\mathcal{E}_p(\cdot) + \|\cdot\|_{L^p}^p)^{1/p}.$$

From (5) and the definition of Γ-convergence you can now deduce that the definition of $\mathcal{F}_p$ in (3) is fine. Also, the sequence you start with corresponds to the sequence $r_n \downarrow 0$ from the weak monotonicity property.

**Step 6.** You smile because you found a nice way to construct a $p$-energy.

**Remark.** If in addition your world $(X,d,m)$ is compact, you will be able to upgrade the Γ-limit (3) to be a *Mosco* limit since it will be possible to prove that the (sub)sequence $\{E_{p,r_n}\}_{n\ge 1}$ is *asymptotically compact* in the same spirit as [11].

**Remark.** Following the *localization method* from the theory of Γ-convergence, see [4], you can also construct a Radon ($p$-energy) measure $\Gamma_p(f)$ for each $f\in\mathcal{F}_p$. That measure satisfies certain good properties, including a $(p,p)$-Poincaré inequality like (2) but substituting the integral on the right hand side by $\int_{B(x,\Lambda R)} d\Gamma_p(f)$. Moreover, these $p$-energy measures are absolutely continuous with respect to the underlying measure $m$.

This talk is based on joint work with Fabrice Baudoin and more details can be read in [1].

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# Geometric aspect of Navier-Stokes equations

Shizan Fang

(joint work with Zhonglin Qian)

We consider the Navier-Stokes equation on a Riemannian manifold with the Ricci curvature bounded below. In stochastic analysis, a non-degenerate diffusion process on a Riemannian manifold was obtained by rolling Brownian motion with respect to a suitable metric compatible linear connection, which was introduced by N. Ikeda and S. Watanabe about 40 years ago. To each velocity, a solution of the Navier-Stokes equation, we associate such a connection and compute the related time-dependent Ricci curvature, which allow us to obtain a link with the strain tensor and the helicity density in a simple formula in the case of dimension 3.

**Three-dimensional polymer measure with selfinteractions and the stochastic quantization**

Seiichiro Kusuoka

(joint work with Sergio Albeverio, Song Liang, Makoto Nakashima)

Edwards [Edw65] introduced the following measure on the path spaces.

$$\mu_{\mathrm{Pol}}(d\omega) = N_\lambda^{-1} \exp\left(-\lambda J_{0,1}(\omega)\right) \mu_{\mathrm{W}}(d\omega),$$

where $\lambda > 0$,

$$J_{0,1}(\omega) = \int_0^1 \int_s^1 \delta_0(\omega_t - \omega_s) dt ds,$$

$\delta_0$ is the delta-function (distribution), $\mu_{\mathrm{W}}$ is the $d$-dimensional Wiener measure and $N_\lambda$ is a normalizing constant. Mathematically we have a problem to define $J_{0,1}(\omega)$, because there exists the composition of the distribution $\delta_0$ and the random variable $\omega_t - \omega_s$ in the definition. The functional $J_{0,1}(\omega)$ of a path $\omega$ is called the self-intersection local time.

Edwards introduced the polymer measure as an example of the failure of the Einstein law. Later, Symanzik [Sym69] obtained the polymer measure $\mu_{\mathrm{Pol}}$ of the case $d = 3$ by a formal transformation of the $\phi_3^4$-measure, which appears in the quantum field theory. The transformation is not mathematically rigorous. But, there are a lot of similarities between these two models. For example, we do not need any renormalization for the construction of $\mu_{\mathrm{Pol}}$ in the case of $d = 1$, and we need renormalizations for $d = 2, 3$. Moreover, the asymptotics of the renormalization constants coincides with each other. Hence, the relation between the two models are strongly expected.

Here we consider the case that $d = 3$. The first construction of $\mu_{\mathrm{Pol}}$ is given by Westwater [Wes80, Wes82]. Later, Bolthausen [Bol93] obtained a simpler construction of $\mu_{\mathrm{Pol}}$, as follows. For $\varepsilon, a > 0$ let

$$J_{0,1}^{\varepsilon,a}(\omega) := \int_0^1 \int_{(s+\varepsilon)\wedge 1}^1 p_a(\omega_t - \omega_s) dt ds.$$

We first show the almost-sure convergence as $a \downarrow 0$. Denote $\lim_{a\downarrow 0} J_{0,1}^{\varepsilon,a}$ by $J_{0,1}^{\varepsilon}$. Then, for sufficiently small $\lambda > 0$ we prove the weak convergence of the measures

$$\mu_{\mathrm{Pol}}(d\omega) = \lim_{\varepsilon\downarrow 0} \frac{1}{N_\lambda^\varepsilon} \exp\left(-\lambda J_{0,1}^\varepsilon(\omega) + \lambda\kappa_1(\varepsilon) - \lambda^2\kappa_2(\varepsilon)\right) \mu_{\mathrm{W}}(d\omega) \tag{1}$$

where

$$\kappa_1(\varepsilon) := \int_\varepsilon^1 p_t(0)dt = \frac{2}{(2\pi)^{3/2}} \cdot \left(\frac{1}{\sqrt{\varepsilon}} - 1\right),$$

$$\begin{aligned}\kappa_2(\varepsilon) &:= \frac{1}{(2\pi)^3} \int_0^1 ds_1 \int_{s_1}^1 ds_2 \int_{s_2}^1 ds_3 \mathbf{1}_{\varepsilon \le s_2} \mathbf{1}_{\varepsilon \le s_3 - s_1} \\ &\qquad\qquad \times \frac{1}{[s_1(s_2-s_1) + s_1(s_3-s_2) + (s_3-s_2)(s_2-s_1)]^{\frac{3}{2}}} \\ &= -\frac{1}{(2\pi)^2} \log \varepsilon + O_\varepsilon(1),\end{aligned}$$

and $N_\lambda^\varepsilon$ is the normalizing constant. We remark that after a lot of calculations we see that $N_\lambda^\varepsilon$ converges to a positive real number as $\varepsilon \downarrow 0$, if $\lambda$ small. This implies that the renormalizations by $\kappa_1(\varepsilon)$ and $\kappa_2(\varepsilon)$ have suitable asymptotics. For the proof of the convergence in (1) we need a lot of calculations. See [Bol93] for details.

As the next issue, the stochastic quantization of $\mu_{\mathrm{Pol}}$ by the Dirichlet form theory is considered by [ARZ96]. Let

$$\begin{aligned} B &:= \{f \in C([0,1];\mathbb{R}^3); f(0) = 0\} \\ H &:= \left\{h \in B; h \text{ is absolutely continuous and } |h|_H^2 := \int_0^1 |h'(t)|^2 dt < \infty\right\} \\ \mathrm{Cyl}_b^\infty &:= \{f(\langle l_1, \cdot\rangle, \dots, \langle l_m, \cdot\rangle); m \in \mathbb{N},\ f \in C_b^\infty(\mathbb{R}^m),\ l_1, \dots, l_m \in B^*\}.\end{aligned}$$

For each $f \in \mathrm{Cyl}_b^\infty$ define $\nabla f(\omega) \in H$ by

$$\langle \nabla f(\omega), h\rangle_H = \frac{d}{ds} f(\omega + sh)\bigg|_{s=0}, \quad h \in H.$$

Consider the bilinear form $\mathcal{E}$ defined by

$$\mathcal{E}(f,g) := \int_B \langle \nabla f, \nabla g\rangle_H \mu_{\mathrm{Pol}}(d\omega), \quad f, g \in \mathrm{Cyl}_b^\infty. \tag{2}$$

The strategy to obtain the Dirichlet form associated with $\mu_{\mathrm{Pol}}$ is showing the closability of $(\mathcal{E}, \mathrm{Cyl}_b^\infty)$ on $L^2(B, \mu_{\mathrm{Pol}})$ and taking the closure. In the case that $d = 2$, the Dirichlet form associated with the polymer measure with selfinteractions was obtained in [AHRZ99]. On the other hand, unfortunately the work [ARZ96] by Albeverio, Röckner and Zhou is an unfinished preprint, because of Zhou's early death. Similarly to the $\phi^4$-quantum field model, the three-dimensional case is much more difficult than the two-dimensional case, because of the singularity of the heat kernel in small time. Thus, the construction of the Dirichlet form associated with $\mu_{\mathrm{Pol}}$ has remained as an open problem for a long time.

In our recent work [AKLN23], we constructed the Dirichlet form associated with $\mu_{\mathrm{Pol}}$ by following the strategy in [ARZ96]. This means that we gave proofs of the unfinished parts of [ARZ96] by modifying their arguments. Furthermore, as in [ARZ96], from the general theory of Dirichlet forms we obtain the quasi-regularity and local property of the Dirichlet form. Hence, we obtain a diffusion process associated to the Dirichlet form. The most difficult part of the proofs is obtaining the existence of a continuous version of the Radon-Nikodym derivative $\frac{d\mu_{\mathrm{Pol}}(\cdot+sh)}{d\mu_{\mathrm{Pol}}}$ in $s$ for all $h \in H \cap W^{2,\infty}([0,1];\mathbb{R}^3)$, which is a sufficient condition in [AR90] for the closability of the bilinear form (2). To obtain the sufficient condition we extend Rosen's method by a lot of delicate estimates. This step is necessary to adjust Rosen's method to the case that $d = 3$, and is very different from previous works.

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