Mathematisches Forschungsinstitut Oberwolfach

Report No. 33/2024

DOI: 10.4171/OWR/2024/33

Arithmetic Geometry

Organized by
Bhargav Bhatt, Princeton
Ana Caraiani, London
Gerd Faltings, Bonn
Peter Scholze, Bonn

14 July – 19 July 2024

Abstract. Arithmetic geometry is at the interface between algebraic geometry and number theory, and studies schemes over the ring of integers of number fields, or their p-adic completions. The talks covered a wide range of topics including the categorical Langlands program, Shimura varieties, complex and p-adic Hodge theory, homotopy theory, and Diophantine geometry.

Mathematics Subject Classification (2020): 11-XX, 14-XX.

Introduction by the Organizers

The workshop *Arithmetic geometry*, organized by Bhargav Bhatt (Princeton), Ana Caraiani (London), Gerd Faltings (Bonn) and Peter Scholze (Bonn), was well attended by over 45 participants from various backgrounds. It covered a wide range of topics in number theory and algebraic geometry, with some focus on p-adic questions.

One major theme at the workshop was the categorical local Langlands program over l-adic fields, in the formulation of Fargues–Scholze. This featured heavily in the talks of Hamann, Hansen, Koshikawa and Zhang. Hansen talked about a strategy to prove the categorical local Langlands conjecture of Fargues–Scholze for many groups. He sketched how these ideas can be used to give a complete proof in the case of GL_2. Koshikawa talked about a generalization of Fargues's conjecture on the existence of eigensheaves to cover the case of A-parameters and A-packets. The approach to this is via the spectral side of the conjectured equivalence, where one can construct generalized eigensheaves inspired by the geometric approach of Adams–Barbasch–Vogan for real groups. Hamann discussed Eisenstein series functors over the stack of G-bundles on the Fargues-Fontaine curve. The talk fits into the context of a longer-term project that aims to understand these functors, which geometrize the classical notion of parabolic induction. The focus in this talk was on establishing geometric analogues of more classical properties of parabolic induction. Finally, Zhang talked about aspects related to global Shimura varieties, namely the construction of Igusa stacks for Shimura varieties of Hodge type. She discussed applications to the Mantovan product formula, Eichler–Shimura relations and torsion vanishing results.

The theme of Shimura varieties recurred in the talks of Lee, Shankar and Sweeting. Lee talked about constructing integral models for the action of the spherical Hecke algebra by correspondences on Shimura varieties of abelian type. This has applications to a conjecture of Fakhruddin and Pilloni in the setting of coherent cohomology. Ananth Shankar introduced the notion of special point on the characteristic p fiber (in the sense of Bakker–Shankar–Tsimerman, for p sufficiently large) of a Shimura variety of exceptional type. This in turn leads to a well-behaved notion of (μ-)ordinary locus. Sweeting discussed certain Tate classes on the product of a Siegel modular threefold and a modular curve, which arise from endoscopy and which she showed arise from algebraic cycles in the globally generic case. In non-generic cases she showed that the Tate classes arise from Hodge cycles.

Another important theme was the p-adic Simpson correspondence, with talks by Andreatta and Heuer. Improving upon the previous work of Faltings, Heuer was able to prove a general p-adic Simpson correspondence for all proper smooth rigid-analytic varieties over an algebraically closed complete extension C of $\mathbb{Q}_p$. In Heuer's formulation, one gets an equivalence between v-vector bundles (also known as "generalized representations") and Higgs bundles, depending on the choice of a lift to B_{dR}^+/ξ^2 and an exponential for C. There remains the question of which Higgs bundles correspond to actual representations, i.e. local systems. The naive expectation is that these are semistable Higgs bundles with vanishing Chern classes. Andreatta explained a result that this expectation is too optimistic, even in the case of trivial Higgs field.

Anschütz explained how one can get a 6-functor formalism with $\mathbb{Q}_p$-coefficients for rigid-analytic varieties, by using nuclear modules on the Fargues–Fontaine curve. An application of these results is towards duality theorems for the pro-étale $\mathbb{Q}_p$-cohomology, which forms a Banach–Colmez space.

In a similar direction, Rodríguez Camargo defined analytic de Rham stacks, which make it possible to talk about (analytic) D-modules even on spaces without differentials, such as perfectoid spaces. Again, this yields a 6-functor formalism, and this has applications to locally analytic representations of p-adic groups via Beilinson–Bernstein localization, and to the theory of p-adic automorphic forms.

Groechenig explained how to extend the theory of p-adic integration to certain algebraic stacks, with applications to enumerative questions. The explicit computation involves some interesting formulas, involving plethystic logarithms.

The cohomology theories used in p-adic geometry are usually not $\mathbb{A}^1$-invariant. Thus, an extension of Voevodsky's theory of motives to the non-$\mathbb{A}^1$-invariant context is desired, and Iwasa gave a very compelling answer. Using it, one can unconditionally define an integral variant of crystalline cohomology (which in the presence of a good compactification agrees with log-crystalline cohomology).

Also of motivic nature was Efimov's talk, who explained the rigidity of the stable ∞-category of localizing motives. This makes it possible to define refined version of (negative or topological) cyclic homology, taking values in nuclear modules over certain E_∞-rings. Efimov explained the computations which naturally lead to overconvergent theories.

Dimitrov explained the proof of irrationality of $L(2, \chi_3)$, the first result of this type since Apéry's proof of irrationality of $\zeta(3)$. The proof makes use of novel arithmetic holonomy bounds.

Esnault explained an application of the theory of ℓ-adic companions to the geometry of the space of local systems on a complex variety, in particular to the question of integral points, leading to new restrictions on fundamental groups of complex varieties.

Gee discussed what the reductions of crystalline Galois representations of fixed Hodge–Tate weights can look like by formulating and answering an analogous question for crystalline Breuil–Kisin modules.

Finally, Kisin considered the number of isomorphism classes of abelian varieties with bounded Faltings height in a given isogeny class considered over an algebraic closure of a fixed number field. Assuming the Mumford–Tate conjecture for the abelian varieties in this isogeny class, he explained a proof that the number of isomorphism classes is finite.

During the conference, many active discussions took place. As just one example, discussions between Faltings and Heuer led to an improved understanding of the situation, namely the choice of certain base points in the formulation of Heuer's result is actually not necessary.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, “US Junior Oberwolfach Fellows”.

Workshop: Arithmetic Geometry

Table of Contents

Abstracts

Diophantine aspects of the Betti moduli space

Hélène Esnault
(joint work with Johan de Jong)

For X a smooth quasi-projective variety over the field of complex numbers $\mathbb{C}$, and r a natural number ≥ 1, one defines the coarse Betti moduli space $M_B(X,r)$ of semi-simple local systems of rank r. It is constructed as the GIT quotient by GL_r of the moduli space $M_B^{\square}(X,r) = \mathrm{Hom}(\pi_1(X(\mathbb{C}),x), \mathsf{GL}_r)$, framed by the choice of the base point x, where $\pi_1(X(\mathbb{C}),x)$ is the topological fundamental group based at x. Then $M_B(X,r)$ is defined over $\mathrm{Spec}(\mathbb{Z})$. As $\pi_1(X(\mathbb{C}),x)$ is finitely presented, in particular finitely generated, $M_B(X,r)$ is of finite type over $\mathbb{Z}$. We can decorate $M_B(X,r)$ by fixing a natural number $\delta \geq 1$ and requesting the local systems to have determinant of order dividing δ. This yields a closed subscheme $M_B(X,r;\delta) \hookrightarrow M_B(X,r)$ over $\mathrm{Spec}(\mathbb{Z})$ depending only on the topological invariant $\pi_1(X(\mathbb{C}))$, the isomorphism class of $\pi_1(X(\mathbb{C}),x)$. We can also fix a good compatification $X \hookrightarrow \bar{X}$ of X and request the local systems to have quasi-unipotent monodromies at infinity with finite order eigenvalues, say $\lambda_{ij}, j = 1,\ldots,r$ and i indexing the components of $\bar{X} \setminus X$. This yields the closed embedding $M_B(X,r;\delta,\lambda_{ij}) \hookrightarrow M_B(X,r;\delta)$ which is now no longer a topological invariant but depends on the variety X, as the elements of $\pi_1(X(\mathbb{C}))$ corresponding to loops around $\bar{X} \setminus X$ depend on X.

A general problem, posed notably by Sarnak, is to understand when those moduli spaces admit an integral -that is $\bar{\mathbb{Z}}$-valued- point. The scheme $M_B(X,r;\delta)$ admits an integral point, the trivial local system. So we refine problem by asking when $M_B(X,r;\delta)$ or $M_B(X,r)$ admits an integral point which over $\bar{\mathbb{Q}}$ is irreducible. We can also extend the problem to $M_B(X,r;\delta,\lambda_{ij})$ for which it is no longer a property of $\pi_1(X(\mathbb{C}))$ solely.

A corollary of our main theorem [2, Theorem 1.1] says the following.

Theorem 1. *Assume the set $M_B(X,r;\delta,\lambda_{ij})(\mathbb{C})$ is non-empty and consists of irreducible $\mathbb{C}$-local systems. If $M_B(X,r;\delta,\lambda_{ij})$ and $M_B(X,r;\delta,\lambda_{ij}) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}$ are irreducible, then $M_B(X,r;\delta,\lambda_{ij})$ has a $\bar{\mathbb{Z}}$-local system (which over $\bar{\mathbb{Q}}$ is then irreducible).*

As $\pi_1(X(\mathbb{C}),x)$ is finitely generated, the $\bar{\mathbb{Z}}$-local systems in the conclusion are defined over a number ring $\mathcal{O}$ and the underlying representation $\rho\colon \pi_1(X(\mathbb{C}),x) \to \mathsf{GL}_r(\mathcal{O})$, localized at any finite place v of $\mathcal{O}$, factors through the étale fundamental group, yielding an ℓ-adic local system on X, where ℓ is the residual characteristic of v. Rumely's theorem, see [11, Theorem 1] and [10, Theorem 1.7], asserts that the conclusion is equivalent to X having a rank r absolutely irreducible ℓ-adic local system with decoration (δ,λ_{ij}) for all prime numbers ℓ. This formulation is the one we prove in general without assumption on the geometry of the Betti moduli space, except for the existence of an irreducible rank r complex local system with decoration (δ, λ_{ij}).

Theorem 2. *Assume the set $M_B(X, r; \delta, \lambda_{ij})(\mathbb{C})$ contains an irreducible local system. Then for any prime number ℓ, $M_B(X, r; \delta, \lambda_{ij})$ has a $\bar{\mathbb{Z}}_\ell$-local system which over $\bar{\mathbb{Q}}_\ell$ is irreducible.*

Again we remark that to have a $\bar{\mathbb{Z}}_\ell$-local system of rank r with the decorations (δ, λ_{ij}) is equivalent to having an ℓ-adic local system of rank r with the decorations (δ, λ_{ij}).

On the other hand, an application of de Jong's conjecture [1] proved by Gaitsgory [8] in general (for $p \geq 3$) is the following proposition.

Proposition 3 ([5])**.**

$$\cup_{\lambda_{ij} \in \mu_\infty} M_B(X, r; \delta, \lambda_{ij}) \subset M_B(X, r; \delta)$$

is Zariski dense.

This enables one to transpose the statement of Theorem 2, which depends on X, to the right hand side, which depends only on $\pi_1(X(\mathbb{C}))$.

Definition 4 ([2])**.** A finitely presented group π is *weakly integral* if whenever there is an irreducible representation $\rho\colon \pi \to \mathsf{GL}_r(\mathbb{C})$ with determinant of finite order dividing $\delta \in \mathbb{N}_{\geq 1}$, then for all prime numbers ℓ, there is a representation

$$\rho_\ell\colon \pi \to \mathsf{GL}_r(\bar{\mathbb{Z}}_\ell)$$

with determinant of order dividing $\delta \in \mathbb{N}_{\geq 1}$ which is irreducible over $\bar{\mathbb{Q}}_\ell$.

Theorem 5. *If X is smooth quasi-projective over $\mathbb{C}$, then $\pi_1(X(\mathbb{C}))$ is weakly integral.*

As there are finitely presented groups which are not weakly integral -the first example was constructed by Breuillard- Theorem 5 yields an *obstruction* for a finitely presented group to be the topological fundamental group of a smooth complex quasi-projective variety. See [2], [7] for details.

The proof of Theorem 2 relies on two main building blocks. Construct using algebraic geometry and de Jong's conjecture [1], the proof of which relies on the geometric Langlands program, a tame arithmetic ℓ-adic local system over a mod p reduction $X_{\bar{\mathbb{F}}_p}$ of X, for p large, with the decoration (δ, λ_{ij}). The second ingredient consists in using Deligne's companions in a way initiated in [4], to produce on $X_{\bar{\mathbb{F}}_p}$ ℓ'-adic tame local systems of the same rank with the same decorations (δ, λ_{ij}). The proof of the existence of the companions relies on the Langlands program on $X_{\bar{\mathbb{F}}_p}$ when X has dimension 1 as proved by L. Lafforgue [9]. In general in higher dimension it has been completed by Drinfeld [3]. See [6] for a broader discussion and references in there. Finally go back to the topological fundamental group via Grothendieck's specialization homomorphism.

References

[1] de Jong, J.: *A conjecture on arithmetic fundamental groups*, Israel J. Math. **12** (2001), 61–84.

[2] de Jong, J., Esnault, H.: *Integrality of the Betti moduli space*, Transactions of the AMS **377** (1) (2024),431–448.

[3] Drinfeld, V.: *On a conjecture of Deligne*, Volume dedicated to the memory of I. M. Gelfand, Moscow Math. J. **12** (2012) no. 3.

[4] Esnault, H., Groechenig, M.: *Cohomologically rigid connections and integrality*, Selecta Mathematica **24** (5) (2018), 4279–4292.

[5] Esnault, H., Kerz, M.: *Local systems with quasi-unipotent monodromy at infinity are dense*, Israel J. Math. **257** (2023) no1, 251–262.

[6] Esnault, H.: *Lectures on Local Systems in Algebraic-Arithmetic Geometry*, Springer Lecture Notes **2337** (2023), i–vii+ 96 pp.

[7] Esnault, H.: *Some Arithmetic Properties of Complex Local Systems*, Notices American Math. Soc. **70** (2023) no11, 1793–1801.

[8] Gaitsgory, D.: *On de Jong's conjecture*, Israel J. Math. **157** (2007), 155–191.

[9] Lafforgue, L.: *Chtoucas de Drinfeld et correspondance de Langlands*, Inventiones math. **147** (2002), no. 1, 1–241.

[10] Moret-Bailly, L.: *Groupes de Picard et problèmes de Skolem*, Ann. Sc. ÉNS 4o série **22** (1989), 161–179.

[11] Rumely, R.: *Arithmetic over the ring of all algebraic integrers*, J. reine angewandte Mathematik **368** (1986), 127–133.

A 6-functor formalism with values in solid quasi-coherent sheaves on the Fargues–Fontaine curve

Johannes Anschütz

(joint work with Arthur-César Le Bras, Lucas Mann)

Let p be a prime. This talk presented a cohomology theory with $\mathbb{Z}_p$- or $\mathbb{Q}_p$-coefficients in p-adic geometry. The construction of this formalism is motivated by potential applications in p-adic versions of Fargues' program ([3]) We start by recalling the ℓ-adic case where $\ell \neq p$ is another prime. Let Perfd be the category of perfectoid spaces over $\mathbb{Z}_p$, and $\mathrm{Perf}_{\mathbb{F}_p}$ the full subcategory of perfectoid spaces over $\mathbb{F}_p$. We equip both with the v-topology. In [5] Scholze has made the powerful observation that small v-stacks on $\mathrm{Perf}_{\mathbb{F}_p}$ allow a reasonable geometry. Most notably there exists a 6-functor formalism $Y \mapsto D_{ét}(Y, \mathbb{Z}/\ell^n)$ calculating étale cohomology. Here, $D_{ét}(Y, \mathbb{Z}/\ell^n)$ is defined by v-descent: If $Y = S$ is a strictly totally disconnected perfectoid space, then $D_{ét}(Y, \mathbb{Z}/\ell^n) \cong D(S_{ét}, \mathbb{Z}/\ell^n)$ is just the usual derived category of étale sheaves on S, and this category is shown by Scholze to satisfy v-descent.

The formalism $D_{ét}(-, \mathbb{Z}/\ell^n)$ can be applied to several interesting small v-stacks.

(1) For an adic space X over $\mathbb{Z}_p$ let $X^\diamond$ be the small v-sheaf sending $S \in \mathrm{Perf}_{\mathbb{F}_p}$ to the set $\{(S^\sharp \in \mathrm{Perfd}, \iota\colon (S^\sharp)^\flat \cong S, \alpha\colon S^\sharp \to X\}/\text{isom.}$ of isomorphism classes of untilts of S over X. In particular, $\mathrm{Perf}_{\mathbb{F}_p}/\mathrm{Spd}(\mathbb{Z}_p) \cong \mathrm{Perfd}$ (using the notation $\mathrm{Spd}(-) = \mathrm{Spa}(-)^\diamond$). If X is analytic, then $D^+_{ét}(X, \mathbb{Z}/\ell^n) \cong D^+(X_{ét}, \mathbb{Z}/\ell^n)$.

(2) Scholze proves the non-trivial result that $D_{ét}(\mathrm{Spd}(\overline{\mathbb{F}}_p), \mathbb{Z}/\ell^n) \cong D(\mathbb{Z}/\ell^n)$.

(3) Let G be a reductive group over $\mathbb{Q}_p$, and let Bun_G be the small v-stack over $\overline{\mathbb{F}}_p$ of G-torsors on the Fargues–Fontaine curve X_S. Here the Fargues–Fontaine curve is constructed as follows: If $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid with pseudo-uniformizer $\pi \in R$, then

$$\mathcal{Y}_{[0,\infty),S} := \mathrm{Spa}(W(R^+)) \setminus V([\pi]),$$

and

$$\mathcal{Y}_{(0,\infty),S} := \mathcal{Y}_{[0,\infty),S} \setminus V(p),\ X_S := \mathcal{Y}_{(0,\infty),S}/\varphi^{\mathbb{Z}},$$

where φ is induced by the Frobenius on R^+.

The inclusion $j_1\colon [\mathrm{Spa}(\overline{\mathbb{F}}_p)/G(\mathbb{Q}_p)] \to \mathrm{Bun}_G$ of the open substack of trivial G-torsors induces a fully faithful embedding

$$j_{1,!}\colon D_{ét}([\mathrm{Spa}(\overline{\mathbb{F}}_p)/G(\mathbb{Q}_p)], \mathbb{Z}/\ell^n) \cong D(\mathrm{Rep}^{\infty}_{\mathbb{Z}/\ell^n} G(\mathbb{Q}_p)) \to D_{ét}(\mathrm{Bun}_G, \mathbb{Z}/\ell^n),$$

where $\mathrm{Rep}^{\infty}_{\mathbb{Z}/\ell^n} G(\mathbb{Q}_p)$ is the abelian category of smooth $\mathbb{Z}/\ell^n$-representations of $G(\mathbb{Q}_p)$. The embedding $j_{1,!}$ lies at the heart of Fargues' geometrization program for the local Langlands correspondence.

(4) Another central object in Fargues' program is the small v-sheaf $\mathrm{Div}^1 \cong \mathrm{Spd}(\mathbb{Q}_p)/\varphi^{\mathbb{Z}}$. Base changed to $\overline{\mathbb{F}}_p$ its $D_{ét}(-, \mathbb{Z}/\ell^n)$ identifies with smooth $\mathbb{Z}/\ell^n$-representations of the Weil group $W_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$.

The aim of this project is to construct a p-adic analogue of $Y \mapsto D_{ét}(Y, \mathbb{Z}/\ell^n)$. With mod p coefficients a reasonable candidate was constructed by Mann in [4].

Theorem 1 (Mann)**.** *Fix a perfectoid space $S = \mathrm{Spa}(R, R^+)$ and a pseudo-uniformizer $\pi \in R$.*

(1) There exists a unique hypercomplete v-sheaf $T \in \mathrm{Perfd}/S \mapsto D^a_{\square}(\mathcal{O}^+_T/\pi)$ such that $D^a_{\square}(\mathcal{O}^+_T/\pi) = D^a_{\square}(A^+/\pi)$ if $T = \mathrm{Spa}(A, A^+)$ is totally disconnected.

(2) $T \mapsto D^a_{\square}(\mathcal{O}^+_T/\pi)$ extends to a 6-functor formalism on small v-stacks with good properties, e.g., if $S = \mathrm{Spd}(\mathbb{C}_p)$, then $\mathrm{Spa}(\mathbb{C}_p\langle T^{\pm 1}\rangle)^{\diamond} \to S$ is cohomologically smooth.

(3) If $\pi | p$, then $T \mapsto D^a_{\square}(\mathcal{O}^+_T/\pi)^{\varphi}$ calculates $\mathbb{F}_p$-cohomology and there exists a fully faithful functor

$$\mathrm{RH}\colon D_{ét}(T, \mathbb{F}_p)^{\text{overconvergent}} \to D^a_{\square}(\mathcal{O}^+_T/\pi)^{\varphi}.$$

Here, φ denotes the Frobenius and $(-)^{\varphi}$ the category of φ-modules.

Notably Theorem 1 implies Poincaré duality for $\mathbb{F}_p$-cohomology on rigid-analytic varieties over $\mathbb{C}_p$. The notation $(-)^a$ refers to almost mathematics, and the notation $D_{\square}$ to solid modules in the sense of Clausen and Scholze ([6]). In our project we extend now Theorem 1 to $\mathcal{O}^+$-modules on arbitrary perfectoid spaces. This has among others the advantage of removing the necessity of a perfectoid base S as in Theorem 1.

Theorem 2.

(1) *There exists a unique hypercomplete v-sheaf* $T \in \mathrm{Perfd} \mapsto D^a_{\hat{\square}}(\mathcal{O}_T^+)$ *such that* $D^a_{\hat{\square}}(\mathcal{O}_T^+) = D^a_{\hat{\square}}(A^+)$ *for* $T = \mathrm{Spa}(A, A^+)$ *if there exists a morphism* $f\colon T^\flat \to T_0$ *of finite* dimtrg *to a totally disconnected perfectoid space* T_0*.*

(2) $T \mapsto D^a_{\hat{\square}}(\mathcal{O}_T^+)$ *extends to a* 6*-functor formalism on small v-stacks with good properties.*

Similar statements hold for $D_{\hat{\square}}(\mathcal{O}_T) := \mathrm{Mod}_{\mathcal{O}_T}(D^a_{\hat{\square}}(\mathcal{O}_T^+))$*.*

We note that the respective φ-module categories still calculate $\mathbb{F}_p$-cohomology. Here, the "$\hat{\square}$" refers to a slight, but important modification of the category of solid modules. Namely, $D_{\hat{\square}}(A^+) := \mathrm{Ind}(\mathcal{C}_{A^+})$ with $\mathcal{C}_{A^+} \subseteq D_{\square}(A^+)$ the full subcategory spanned by completions of compact objects. This formalism applies to any adic ring instead of A^+. Given a stably uniform adic space Z we let $D_{\hat{\square}}(Z)$ be the analytic descent of the $U = \mathrm{Spa}(B, B^+) \to \mathrm{Mod}_B D_{\hat{\square}}(B^+)$ for $U \subseteq Z$ open and affinoid.

For $\mathbb{Z}_p$-coefficients we use Theorem 2 to prove the following main theorem.

Theorem 3.

(1) *There exists a unique hypercomplete v-sheaf* $S \in \mathrm{Perf}_{\mathbb{F}_p} \to D_{[0,\infty)}(S)$ *such that for any* S *with a morphism* $g\colon S \to S_0$ *of finite* dimtrg *to a totally disconnected perfectoid space* S_0*, we have* $D_{[0,\infty)}(S) \cong D_{\hat{\square}}(\mathcal{Y}_{[0,\infty)})$*.*

(2) $S \mapsto D_{[0,\infty)}(S)$ *extends to a* 6*-functor formalism on small v-stacks.*

(3) *If* $f\colon Y' \to Y$ *is cohomologically smooth for* $D^a_{\square}(\mathcal{O}^+_{(-)}/\pi)$ *(over some implicit base), then* f *is cohomologically smooth for* $D_{[0,\infty)}(-)$ *if it is* $!$*-able for* $D_{[0,\infty)}(-)$*.*

(4) *There exists a fully faithful functor*

$$\mathrm{RH}_{\mathbb{Z}_p}\colon D_{\mathrm{nuc}}(Y, \mathbb{Z}_p) \to D_{[0,\infty)}(Y/\varphi^{\mathbb{Z}}).$$

In particular, $D_{[0,\infty)}((-)/\varphi^{\mathbb{Z}})$ *calculates pro-étale cohomology with* $\mathbb{Z}_p$*-coefficients.*

We make some remarks.

(1) Replacing $\mathcal{Y}_{S,[0,\infty)}$ by $\mathcal{Y}_{S,(0,\infty)}$ yields a formalism $D_{(0,\infty)}(-)$ with similar properties, e.g., there exists a fully faithful functor

$$\mathrm{RH}_{\mathbb{Q}_p}\colon D_{\mathrm{nuc}}(Y, \mathbb{Q}_p) \to D_{(0,\infty)}(Y/\varphi^{\mathbb{Z}}),$$

i.e., $D_{(0,\infty)}((-)/\varphi^{\mathbb{Z}})$ calculates pro-étale cohomology with $\mathbb{Q}_p$-coefficients.

(2) $\mathrm{RH}_{\mathbb{Z}_p}$ induces an equivalence of dualizable objects, but $\mathrm{RH}_{\mathbb{Q}_p}$ not.

(3) If S has a morphism of finite dimtrg to a totally disconnected perfectoid space S_0, then $D_{(0,\infty)}(S/\varphi^{\mathbb{Z}}) \cong D_{\hat{\square}}(X_S)$.

(4) The category $D_{\mathrm{nuc}}(Y, \mathbb{Z}_p)$ is a category of "overconvergent $\mathbb{Z}_p$-sheaves" satisfying v-descent. For example, if $Y = S$ is strictly totally disconnected, then $D_{\mathrm{nuc}}(S, \mathbb{Z}_p) \cong D_{\mathrm{nuc}}(C(S, \mathbb{Z}_p))$.

Theorem 3 implies Poincaré duality for pro-étale cohomology with $\mathbb{Z}_p$- and $\mathbb{Q}_p$-coefficients. Another consequence is the following: If $g\colon X' \to X$ is a proper, smooth morphism of rigid-analytic varieties over $\mathbb{C}_p$, then for $f = g^\diamond\colon X'^{,\diamond} \to X^\diamond$ the pushforward $f_*\colon D_{(0,\infty)}(X'^{,\diamond}/\varphi^{\mathbb{Z}}) \to D_{(0,\infty)}(X^\diamond/\varphi^{\mathbb{Z}})$ preserves dualizable objects (aka perfect complexes by a result of Andreychev [1]). This has a funny application: Let $G/\mathbb{Q}_p$ be a reductive group. Then there exists a natural tensor functor from finite dimensional algebraic representations of G to dualizable objects in $D_{(0,\infty)}(\mathrm{Bun}_G)$. Composing with a minuscule Hecke operator (if it exists) yields a functor to dualizable objects in $D_{(0,\infty)}(\mathrm{Bun}_G \times \mathrm{Div}^1)$. This generalizes the fact that the pro-étale cohomology of the Lubin–Tate local system on $\mathbb{P}^1_{\mathbb{C}_p}$ is naturally underlying a finite complex of Banach–Colmez spaces.

Coming back to the examples of the beginning, we note the following theorem of Zillinger $(+\varepsilon)$.

Theorem 4 (Zillinger$+\varepsilon$)**.**

(1) $D_{(0,\infty)}(\mathrm{Spd}(\mathbb{F}_p)) \cong D_{\square}(\mathrm{AnSpec}(\mathbb{Q}_p))$

(2) $D_{(0,\infty)}([\mathrm{Spd}\mathbb{F}_p/G(\mathbb{Q}_p)]) \cong D_{\square}(\mathrm{AnSpec}(\mathbb{Q}_p)/G(\mathbb{Q}_p))$

In particular, the (rather wild) category $D_{\square}(\mathrm{AnSpec}(\mathbb{Q}_p)/G(\mathbb{Q}_p))$ of "continuous $G(\mathbb{Q}_p)$-representations" embeds into $D_{(0,\infty)}(\mathrm{Bun}_G)$. In contrast, $D_{(0,\infty)}(\mathrm{Spd}(\mathbb{F}_p)/\varphi^{\mathbb{Z}})$ is the derived category of solid isocrystals, which shows that it was a good decision to base the formalism on $\mathcal{Y}_{S,(0,\infty)}$ or $\mathcal{Y}_{S,[0,\infty)}$ instead of X_S. The value of $D_{[0,\infty)}(-)$ on $\mathrm{Spd}(\mathbb{F}_p)$ is more mysterious than $D_{\square}(\mathbb{Z}_p)$. We can also note that $D_{(0,\infty)}(\mathrm{Div}^1)$ yields a derived category of "continuous $\mathrm{Gal}_{\mathbb{Q}_p}$-equivariant vector bundles on the Fargues–Fontaine curve $X_{\mathbb{C}_p^\flat}$" as discussed in the work of Hellmann–Hernandez–Schraen on an analytic Emerton–Gee stack ([2]). It seems to be an interesting question to analyze to which extend the results in [3] can be transfered to p-adic coefficients using the formalisms $D_{[0,\infty)}(-)$ or $D_{(0,\infty)}(-)$.

References

[1] Grigory Andreychev. Pseudocoherent and perfect complexes and vector bundles on analytic adic spaces. 2021. Preprint. Available at https://arxiv.org/abs/2105.12591.

[2] Matthew Emerton, Toby Gee, and Eugen Hellmann. An introduction to the categorical p-adic Langlands program. *arXiv preprint arXiv:2210.01404*, 2022.

[3] Laurent Fargues and Peter Scholze. Geometrization of the local Langlands correspondence. 2021. https://arxiv.org/abs/2102.13459.

[4] Lucas Mann. A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry. 2022. Preprint. Available at https://arxiv.org/pdf/2206.02022.pdf.

[5] Peter Scholze. Étale cohomology of diamonds. https://www.math.uni-bonn.de/people/scholze/EtCohDiamonds.pdf, 2018.

[6] Peter Scholze. Lectures on Condensed Mathematics, 2019. https://www.math.uni-bonn.de/people/scholze/Condensed.pdf

The Reduction modulo p of Crystalline Breuil–Kisin Modules

Toby Gee
(joint work with Mark Kisin)

Our motivation is the following question (which is in turn motivated by questions about congruences between modular forms of different weights): Let $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \longrightarrow GL_n(\overline{\mathbf{Q}}_p)$ be a crystalline Galois representation, of Hodge–Tate weights $h_1 \geq h_2 \geq \cdots \geq h_n$. Then what are the possibilities for $\overline{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \longrightarrow GL_n(\mathbf{F}_p)$, the (semisimplified) reduction modulo p?

Write I_p for the inertia subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. There is a simple classification of the absolutely irreducible representations $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \longrightarrow GL_d(\mathbf{F}_p)$, and using this one can read off from $\overline{\rho}|_{I_p}$ a multiset of n "inertial weights".

In particular, if $\overline{\rho}$ is a direct sum of characters, then

$$\overline{\rho}|_{I_p} \cong \oplus_{i=1}^{n} \omega^{m_i}$$

where ω is the mod p cyclotomic character (of order $p-1$), and the inertial weights of $\overline{\rho}$ are by definition $m_1, \ldots m_n$. If $h_1, \ldots, h_n \in [0, p]$, then a theorem of Gee–Liu–Savitt [GLS14] shows that after possibly reordering, we can take $m_i = h_i$ (note that the m_i are only well-defined modulo $p-1$).

However, if the h_i are not all contained in an interval of length p, this need no longer hold. For example, Berger–Breuil [BB05] showed that if $n = 0$ and $h_1, h_2 = 0, h$ with $p+1 \leq h \leq 2p-1$, then $\overline{\rho}|_{I_p} \cong 1 \oplus \omega^h$ (in which case we can again take $m_i = h_i$), or alternatively $\overline{\rho}|_{I_p} \cong \omega \oplus \omega^{h-1}$; and both of these possibilities can occur for each such h.

Our first main result is the following.

Theorem 1. *If $\overline{\rho}$ is a direct sum of characters, then we can choose integers $m_1 \geq \cdots \geq m_n$ such that $\overline{\rho}|_{I_p} \cong \oplus_{i=1}^{n} \omega^{m_i}$, and in addition:*

(1) for each $1 \leq k \leq n$, we have

$$\sum_{i=1}^{k} h_i \geq \sum_{i=1}^{k} m_i,$$

with equality for $i = n$, and

(2) there is a permutation $\sigma \in S_n$ such that $m_i \equiv h_{\sigma(i)} \pmod{p}$ for all i.

Here the first condition was already guaranteed by a theorem of Levin–Wang-Erickson [LWE20], but the second is new.

For example, in the context of the result of Berger–Breuil, we see that the two possibilities for $\{m_1, m_2\}$ permitted by Theorem 1 are $\{h, 0\}$ and $\{p, h-p\}$, which is consistent with the results of [BB05] because $\omega^p \oplus \omega^{h-p} = \omega \oplus \omega^{h-1}$. Furthermore, using the Breuil–Mézard conjecture, one can show that Theorem 1 is best possible for $n = 2$ (for any choice of h_1, h_2). It is unclear whether or not to expect that it is best possible for $n > 2$.

We deduce Theorem 1 from a similar statement for Breuil–Kisin modules, which we now state. After twisting by a power of the cyclotomic character, we may suppose that the h_i are non-negative. Then by a theorem of Kisin [Kis06], we may associate to ρ a Breuil–Kisin module. By definition, the reduction modulo p of this Breuil–Kisin module is a free $\mathbf{F}_p[[u]]$-module $\mathfrak{M}$ of rank n, together with an $\mathbf{F}_p[[u]]$-module homomorphism

$$\Phi_{\mathfrak{M}} : \mathbf{F}_p[[u]] \otimes_{\varphi, \mathbf{F}_p[[u]]} \mathfrak{M} \to \mathfrak{M},$$

whose cokernel has finite $\mathbf{F}_p$-dimension (where the tensor product is with respect to the Frobenius endomorphism $x \mapsto x^p$ of $\mathbf{F}_p[[u]]$).

Let $r_1 \geq \cdots \geq r_h$ be the non-negative integers such that

$$\mathrm{coker}\Phi_{\mathfrak{M}} \cong \oplus_{i=1}^n \mathbf{F}_p[u]/u^{r_i}.$$

Theorem 2.

(1) For each $1 \leq k \leq n$, we have

$$\sum_{i=1}^k h_i \geq \sum_{i=1}^k r_i,$$

with equality for $i = n$, and

(2) there is a permutation $\sigma \in S_n$ such that $r_i \equiv h_{\sigma(i)} \pmod p$ for all i.

Again, the first part of the theorem was already guaranteed by the work of Levin–Wang-Erickson [LWE20], but the second part is new. The proof uses the theory of prismatic F-gauges due to Bhatt–Lurie [Bha22]; the integers r_i and h_i both admit interpretations in terms of the dimensions of the graded pieces of filtrations on $\mathfrak{M}$, and the theorem ultimately follows from a consideration of graded modules over the Weyl algebra $\mathbf{F}_p[x, \frac{d}{dx}]$, using an argument that we learned from Jacob Lurie. Finally, we deduce Theorem 1 from Theorem 2 by reducing to the case that the Breuil–Kisin module $\mathfrak{M}$ is semi-simple.

References

[BB05] Laurent Berger and Christophe Breuil, *Sur la réduction des représentations cristallines de dimension 2 en poids moyens*, 2005.

[Bha22] Bhargav Bhatt, *Prismatic F-gauges.*

[GLS14] Toby Gee, Tong Liu, and David Savitt, *The Buzzard-Diamond-Jarvis conjecture for unitary groups*, J. Amer. Math. Soc. **27** (2014), no. 2, 389–435.

[Kis06] Mark Kisin, *Crystalline representations and F-crystals*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 459–496.

[LWE20] Brandon Levin and Carl Wang-Erickson, *A Harder-Narasimhan theory for Kisin modules*, Algebr. Geom. **7** (2020), no. 6, 645–695.

Techniques of projective bundle homotopies

Ryomei Iwasa
(joint work with Toni Annala, Marc Hoyois)

I report my joint project with Toni Annala and Marc Hoyois to innovate motivic homotopy theory beyond $\mathbb{A}^1$-homotopy invariance. The references are [AI, AHI, AHI2]. While $\mathbb{A}^1$-homotopy theory has had great success in introducing homotopical approach to algebraic geometry, it has the obvious drawback that it cannot capture non-$\mathbb{A}^1$-homotopy invariants such as crystalline cohomology, syntomic cohomology, and algebraic K-theory of non-regular schemes. We have succeeded in enlarging $\mathbb{A}^1$-invariant motivic stable homotopy theory so that all relevant cohomology theories are representable while maintaining a certain homotopy invariant called the *projective bundle homotopy invariance*. More precisely, we constructed an ∞-category MS_S of *motivic spectra* over a scheme S, and we proved that the projective bundle homotopy invariance holds in MS_S. Below I explain its construction and formulate this homotopy invariance.

We consider the ∞-category $\mathrm{Sh}(\mathrm{Sm}_S;\mathrm{Sp})$ of Zariski sheaves of spectra on the category Sm_S of smooth S-schemes, and consider its full subcategory $\mathrm{Sh}_{\mathrm{ebe}}(\mathrm{Sm}_S;\mathrm{Sp})$ consisting of those sheaves that satisfy *elementary blowup excision*, i.e., send the blowup square

$$\begin{array}{ccc} \mathbb{P}^{n-1}_X & \longrightarrow & \mathrm{Bl}_{\{0\}_X}\mathbb{A}^n_X \\ \downarrow & & \downarrow \\ \{0\}_X & \longrightarrow & \mathbb{A}^n_X \end{array}$$

to a cartesian square of spectra for $X \in \mathrm{Sm}_S$ and $n \geq 2$. The key point is that $\mathbb{P}^n/\mathbb{P}^{n-1}$ is equivalent to $(\mathbb{P}^1)^{\otimes n}$ up to elementary blowup excision. Then the ∞-category MS_S of motivic spectra over S is defined to be the formal inversion of the pointed projective line $\mathbb{P}^1$ in $\mathrm{Sh}_{\mathrm{ebe}}(\mathrm{Sm}_S;\mathrm{Sp})$ as a presentably symmetric monoidal ∞-category;

$$\mathrm{MS}_S := \mathrm{Sh}_{\mathrm{ebe}}(\mathrm{Sm}_S;\mathrm{Sp})[(\mathbb{P}^1)^{-1}].$$

It is equipped with a symmetric monoidal functor

$$\Sigma^{\infty}_{\mathbb{P}^1}(-)_+ \colon \mathrm{Sm}_S \to \mathrm{MS}_S.$$

To put the definition in a slightly deferent way, MS_S together with $\Sigma^{\infty}_{\mathbb{P}^1}(-)_+$ is universally characterized by the three axioms: Zariski descent, elementary blowup excision, and $\mathbb{P}^1$-invertibility. Intuitively, the last two conditions guarantee that the motivic stable homotopy type of the projective space $\mathbb{P}^n$ is “correct”.

Every motivic spectrum $E \in \mathrm{MS}_S$ defines a bigraded cohomology theory by the formula

$$E^{p,q}(X) := \pi_{2q-p}\,\mathrm{map}(\Sigma^{-q}_{\mathbb{P}^1}\Sigma^{\infty}_{\mathbb{P}^1}X_+, E)$$

for $X \in \mathrm{Sm}_S$. Then a huge variety of cohomology theories of schemes are representable by motivic spectra in this way, including crystalline cohomology, prismatic cohomology, and syntomic cohomology.

The following is the projective bundle homotopy invariance in MS_S.

Theorem ([AHI, Theorem 4.1])**.** *Let $\mathcal{E}$ be a finite locally sheaf on $X \in \mathrm{Sm}_S$ and s a section of $\mathbb{V}(\mathcal{E}) \to X$. Then there is a homotopy $h(s)$ in MS_S between the two composites*

$$X \underset{0}{\overset{s}{\rightrightarrows}} \mathbb{V}(\mathcal{E}) \longrightarrow \mathbb{P}(\mathcal{E} \oplus \mathcal{O}).$$

The homotopy $h(s)$ is functorial in $(S, X, \mathcal{E}, s)$ and is the identity when $s = 0$.

This theorem allows us to do "homotopy theory" in algebraic geometry while keeping the affine line $\mathbb{A}^1$ non-contractible. For example, using projective bundle homotopies, we proved an equivalence $\mathrm{Grss}_n \simeq \mathrm{BGL}_n$ in MS_S ([AHI, Theorem 5.1]). Another nice use of projective bundle homotopies is a construction of Gysin map due to Longke Tang. For a finite locally free sheaf $\mathcal{E}$ on $X \in \mathrm{Sm}_S$, we define the *Thom spectrum* $\mathrm{Th}_X(\mathcal{E})$ of $\mathcal{E}$ to be $\mathbb{P}(\mathcal{E} \oplus \mathcal{O})/\mathbb{P}(\mathcal{E})$. Then, for a closed immersion $Z \to X$ between smooth S-schemes, Tang constructed a *Gysin map*

$$\mathrm{gys}\colon X_+ \to \mathrm{Th}_Z(\mathcal{N}_{Z/X})$$

in $\mathrm{MS}_S^{\mathrm{Nis}}$, the Nisnavich version of MS_S. His Gysin map plays a crucial role in the Atiyah duality established in [AHI2]. Note that the Thom spectrum $\mathrm{Th}_X(\mathcal{E})$ is tensor-invertible in MS_X, and thus $\mathrm{Th}_X(-\mathcal{E})$ makes sense. Then our version of Atiyah duality is as follows.

Theorem ([AHI2, Corollary 5.15])**.** *Every smooth projective S-scheme X is dualizable in $\mathrm{MS}_S^{\mathrm{Nis}}$ with the dual $\mathrm{Th}_X(-\Omega_{X/S})$.*

I describe some applications of Atiyah duality. We rewrite the notation as $\mathrm{MS}_S = \mathrm{MS}_S^{\mathrm{Nis}}$. Consider the full subcategory $\mathrm{MS}_S^{\mathbb{A}^1}$ of MS_S spanned by $\mathbb{A}^1$-invariant motivic spectra; which is exactly Voevodsky's category of motivic spectra. We denote the unit object in $\mathrm{MS}_S^{\mathbb{A}^1}$ by $\mathbf{1}_{\mathbb{A}^1}$ and consider $\mathbf{1}_{\mathbb{A}^1}$-modules;

$$\mathrm{Mod}_{\mathbf{1}_{\mathbb{A}^1}}(\mathrm{MS}_S) \hookrightarrow \mathrm{MS}_S^{\mathbb{A}^1}.$$

This inclusion has both left and right adjoints by abstract nonsense. We think about the right adjoint, denoted by $E \mapsto E^\dagger$ and called the $\mathbb{A}^1$-*colocalization*.

Let E be a $\mathbf{1}_{\mathbb{A}^1}$-module in MS_S. For a dualizable object A in MS_S, we have $E^\dagger(A) \simeq E(A)$, where $E(A)$ denotes the mapping spectrum $\mathrm{map}(A, E)$ in MS_S and same for $E^\dagger(A)$. Then it follows from Atiyah duality that the $\mathbb{A}^1$-colocalization of E does not change the value on smooth projective S-schemes. Furthermore, if $U \in \mathrm{Sm}_S$ has a strict normal crossings compactification (X, D), then $E^\dagger(U)$ can be regarded as the *logarithmic E-cohomology* of (X, D).

Let us take crystalline cohomology as an example. For a perfect field k of characteristic $p > 0$, we have a motivic spectrum $\mathrm{H}W(k)^{\mathrm{crys}}$ over k that represents crystalline cohomology. It is naturally a $\mathbf{1}_{\mathbb{A}^1}$-module and thus its $\mathbb{A}^1$-colocalization $\mathrm{H}W(k)^{\mathrm{crys},\dagger}$ makes sense. Then:

(i) $\mathrm{H}W(k)^{\mathrm{crys},\dagger}[1/p]$ represents Berthelot's rigid cohomology.
(ii) If $U \in \mathrm{Sm}_k$ has a strict normal crossings compactification (X, D), then $\mathrm{H}W(k)^{\mathrm{crys},\dagger}(U)$ is identified with the existing logarithmic crystalline cohomology of (X, D).

In particular, it implies that the logarithmic crystalline cohomology is independent of the choice of compactification. This has been a fundamental open question since the late 80's.

References

[AI] T. Annala, R. Iwasa, *Motivic spectra and universality of* K*-theory*. Preprint, arXiv:2204.03434.

[AHI] T. Annala, M. Hoyois, R. Iwasa, *Algebraic cobordism and a Conner–Floyd isomorphism for algebraic* K*-theory*. To appear in JAMS.

[AHI2] T. Annala, M. Hoyois, R. Iwasa, *Atiyah duality for motivic spectra*. Preprint, arXiv:2403.01561.

The p-adic Simpson correspondence via moduli spaces

Ben Heuer

In this talk, we discuss recent developments in the moduli-theoretic formulation of p-adic non-abelian Hodge theory in geometric and arithmetic setups.

Let C be a complete algebraically closed extension of $\mathbb{Q}_p$. Let X be a connected smooth proper rigid space over C. Scholze associates to X the v-site X_v of the diamond $X^\diamondsuit$ [14], whose underlying category Perf_C is given by perfectoid spaces over C . It is equipped with a natural structure sheaf $\mathcal{O}_{X_v}$. Our goal is to answer:

Question 1. *Can we describe the category* $\mathrm{VB}(X_v)$ *of* ***v-vector bundles****, i.e. finite locally free sheaves on* $(X_v, \mathcal{O}_{X_v})$*, in terms of more classical objects on* X_{an}*?*

By a Theorem of Kedlaya–Liu, v-vector bundles are equivalent to vector bundles on the pro-étale site $(X_{\text{proét}}, \widehat{\mathcal{O}}_{X_{\text{proét}}})$ of [13]. There is moreover a pullback functor

$$\mathrm{VB}(X_{\mathrm{an}}) \to \mathrm{VB}(X_v)$$

which is fully faithful, but it turns out that it is not essentially surjective outside of trivial cases. Let us give an example of how v-vector bundles arise "in nature":

Example 2. *For any* $\mathbb{Q}_p$*-local system* $\mathbb{L}$ *on* X*, the sheaf* $\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}$ *is a v-vector bundle on* X*. More generally, this works for* C*-local systems (appropriately defined). As a consequence, there is for any choice of base point* $x \in X(C)$ *a fully faithful functor*

$$\mathrm{Rep}_C(\pi_1(X, x)) \hookrightarrow \mathrm{VB}(X_v)$$

where $\mathrm{Rep}_C(\pi_1(X, x))$ *is the category of continuous representations of the étale fundamental group of* X *on finite dimensional* C*-vector spaces.*

1. The p-adic Simpson correspondence

Our first main result describes v-vector bundles in terms of the following more classical objects of Hodge theory:

Definition 3. *A Higgs bundle on* X *is a pair* (E, θ) *where*

- E *is a vector bundle on* X_{an}*,*

- $\theta \in \mathrm{End}(E) \otimes \Omega^1_X(-1)$ *is a Higgs field, i.e. it induces a morphism of* $\mathcal{O}_X$*-algebras* $\mathrm{Sym}\,\Omega^{1\vee}_X(1) \to \mathrm{End}(E)$*. Here* (-1) *and* (1) *denote Tate twists.*

Theorem 4 (p-adic Simpson correspondence, [5])**.** *Choices of a* $B_2 := \mathbb{B}^+_{\mathrm{dR}}/\xi^2$*-lift* $\mathbb{X}$ *of* X *and of an exponential* $C \to 1 + \mathfrak{m}_C$ *induce an exact tensor equivalence*

$$S : \mathrm{VB}(X_v) \leftrightarrow \{\textit{Higgs bundles on } X\}.$$

For any v-vector bundle V *on* X*, we moreover have a comparison of cohomologies*

$$R\Gamma_v(X, V) = R\Gamma_{\mathrm{Dol}}(X, S(V)).$$

This was first proved by Faltings when X is a curve [3]. Our proof in [5] hinges on analytic moduli spaces of pro-étale invertible sheaves on spectral varieties.

In the case of $V = \mathcal{O}$, the cohomological comparison recovers the Hodge–Tate decomposition of Faltings and Scholze via the Primitive Comparison Theorem [13]. The perspective that the p-adic Simpson correspondence generalises this decomposition to more general coefficient systems is one reason why this subject is also called "p-adic non-abelian Hodge theory".

Question 5. *Under the equivalence* S *of Theorem 4, which Higgs bundles correspond to representations via the embedding described in Example 2?*

This is expected to be a very difficult question. The answer is known in the case of line bundles [6] and abeloid varieties [8], where it is already quite subtle. In fact, the answer in these cases is in terms of moduli spaces of v-line bundles.

2. Moduli spaces in the case of curves (joint with Daxin Xu)

Following [7], one can define moduli stacks on Perf_C for either side of the p-adic Simpson correspondence, which turn out to be small v-stacks: Let $n \in \mathbb{N}$ and set

$$\mathcal{B}un_{v,n} : \mathrm{Perf}_C \to \mathrm{Grpds}, \quad S \mapsto \{\text{v-vector bundles on } X \times S \text{ of rank } n\},$$

$$\mathcal{H}iggs_n : \mathrm{Perf}_C \to \mathrm{Grpds}, \quad S \mapsto \{\text{Higgs bundles on } X \times S \text{ of rank } n\}.$$

Assume now that X is a smooth projective curve, then in joint work with Xu [9], we show that $\mathcal{B}un_{v,n}$ is an étale twist of $\mathcal{H}iggs_n$. To make this more precise, let

$$\mathcal{A}_n := \textstyle\bigoplus_{i=1}^n H^0(X, \Omega_X^{\otimes i}(-i)) \otimes \mathbb{G}_a$$

be the Hitchin base of rank n, then we have the classical Hitchin fibration

$$\mathcal{H} : \mathcal{H}iggs_n \to \mathcal{A}_n.$$

In [7], we explain that there is an analogous morphism

$$\widetilde{\mathcal{H}} : \mathcal{B}un_{v,n} \to \mathcal{A}_n.$$

One way to define this is by sending any v-vector bundle to the characteristic polynomial of the canonical Higgs field of Pan and Rodríguez Camargo [12].

Let now $Z \to X \times \mathcal{A}_n$ be the universal spectral curve and let $\mathcal{P}$ be the étale Picard stack of the relative curve $Z \to \mathcal{A}_n$. Then by the theory of "abelianization", $\mathcal{P}$ acts naturally on $\mathcal{H}$, and it turns out that there is an analogous action on $\widetilde{\mathcal{H}}$.

Theorem 6 ([9]). *There is a canonical $\mathcal{P}$-torsor $\mathcal{L}$ and a canonical and natural equivalence of v-stacks*

$$\mathcal{B}un_{n,v} = \mathcal{H}iggs_n \times^{\mathcal{P}} \mathcal{L}.$$

Choices of a B_2-lift of X and an exponential induce a splitting $\mathcal{A}_n(C) \to \mathcal{L}(C)$ which induces a homeomorphism $|\mathcal{B}un_{v,n}(C)| \cong |\mathcal{H}iggs_n(C)|$.

Thus the choices in Theorem 4 admit a geometric interpretation in terms of moduli spaces, namely they trivialise a twist on C-points.

3. The arithmetic non-abelian Hodge correspondence

We now switch to an arithmetic setup, namely let X be a proper smooth rigid space over $\mathbb{Q}_p$.

Question 7. *Can we describe the category* $\mathrm{VB}(X_v)$ *in this case?*

This question was first studied by Liu–Zhu for local systems [10], then by Tsuji [15], He [4], Min–Wang [11] and in joint work with Anschütz–Le Bras [2]. But in fact, one could say that work on this question already starts with the work of Sen, as the following example demonstrates:

Example 8. *Let $X = \mathrm{Spa}(\mathbb{Q}_p)$ and set $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$. Then any v-vector bundle is trivialised by the cover $\mathrm{Spa}(\mathbb{C}_p) \to \mathrm{Spa}(\mathbb{Q}_p)$, a $G_{\mathbb{Q}_p} := \mathrm{Gal}(\mathbb{C}_p|\mathbb{Q}_p)$-torsor on X_v. Hence v-vector bundles on $\mathrm{Spa}(\mathbb{Q}_p)$ are equivalent to continuous finite dimensional semi-linear $\mathbb{C}_p$-representations of $G_{\mathbb{Q}_p}$. These are described by Sen theory.*

In general, one can describe v-vector bundles in terms of the following class of objects, which incorporates the data of both Higgs bundles and Sen modules:

Definition 9. *A Higgs–Sen bundle on X is a triple (E, θ, ϕ) where*

- *E is a vector bundle on X,*
- *$\theta : E \to E \otimes \Omega_X$ is a Higgs field,*
- *$\phi : E \to E$ is an $\mathcal{O}_X$-linear morphism,*

subject to the condition that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\ \theta\ } & E \otimes \Omega \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \phi\otimes\mathrm{id}+\mathrm{id}} \\ E & \xrightarrow{\ \theta\ } & E \otimes \Omega. \end{array}$$

Note the additional $+\mathrm{id}$ *on the right (a shadow of the Tate twist in Definition 3).*

Like in the geometric setup, one can now define moduli stacks for these objects, but we need to take a different test category: Let $\mathrm{Rig}^{\mathrm{sm}}_{\mathbb{Q}_p,\text{ét}}$ be the site of smooth rigid spaces over $\mathbb{Q}_p$ with the étale topology. Then for any $n \in \mathbb{N}$, we set

$$\mathcal{B}un_{v,n} : \mathrm{Rig}^{\mathrm{sm}}_{\mathbb{Q}_p,\text{ét}} \to \mathrm{Grpds}, \quad S \mapsto \{\text{v-vector bundles on } X \times S \text{ of rank } n\},$$

$$\mathcal{H}igSen_n : \mathrm{Rig}^{\mathrm{sm}}_{\mathbb{Q}_p,\text{ét}} \to \mathrm{Grpds}, \quad S \mapsto \{\text{Higgs–Sen bundles on } X \times S \text{ of rank } n\}.$$

Once again, it turns out that these admit natural "Hitchin maps"

$$\mathcal{H} : \mathcal{H}igSen_n \to \mathbb{A}^n, \quad \widetilde{\mathcal{H}} : \mathcal{B}un_{n,v} \to \mathbb{A}^n$$

where the first is given by sending (E, θ, ϕ) to the characteristic polynomial of ϕ.

Let $Z \to X \times \mathbb{A}^n$ be the spectral variety defined by interpreting the coordinates of $\mathbb{A}^n$ as coefficients of a characteristic polynomial. Like before, we define $\mathcal{P}$ to be the relative Picard stack of $Z \to \mathbb{A}^n$. It turns out that there are natural actions of $\mathcal{P}$ on $\mathcal{B}un_{v,n}$ and $\mathcal{H}igSen_n$. We can now state our third main theorem:

Theorem 10 (work in progress)**.** *There is a canonical $\mathcal{P}$-torsor $\mathcal{L}$ and a canonical equivalence*

$$\mathcal{B}un_{v,n} = \mathcal{H}igSen_n \times^{\mathcal{P}} \mathcal{L}.$$

Moreover, there is a canonical splitting of $\mathcal{L} \to \mathbb{A}^n$ over the "small" locus of $\mathbb{A}^n$.

The base case of $X = \mathrm{Spa}(\mathbb{Q}_p)$ thus gives a geometrisation of Sen theory.

References

[1] A. Abbes, M. Gros, and T. Tsuji. *The p-adic Simpson correspondence*, volume 193 of *Annals of Mathematics Studies*. Princeton University Press, 2016.

[2] J. Anschütz, B. Heuer, and A.-C. Le Bras. The small p-adic Simpson correspondence in terms of moduli spaces. *Preprint, arXiv:2312.07554*, 2023.

[3] G. Faltings. A p-adic Simpson correspondence. *Adv. Math.*, 198(2):847–862, 2005.

[4] T. He. Sen operators and Lie algebras arising from Galois representations over p-adic varieties. *Preprint, arXiv:2208.07519*, 2022.

[5] B. Heuer. A p-adic Simpson correspondence for smooth proper rigid spaces. *Preprint, arXiv.org:2307.01303*, 2023.

[6] B. Heuer. A geometric p-adic Simpson correspondence in rank one. *Compositio Mathematica*, 160(7):1433–1466, 2024.

[7] B. Heuer. Moduli spaces in p-adic non-abelian Hodge theory. *Preprint, arXiv:2207.13819*, 2022.

[8] B. Heuer, L. Mann, and A. Werner. The p-adic Corlette–Simpson correspondence for abeloids. *Math. Ann.*, 385:1639–1676, 2023.

[9] B. Heuer and D. Xu. p-adic non-abelian Hodge theory for curves via moduli stacks. *Preprint, arXiv.org:2402.01365*, 2024.

[10] R. Liu and X. Zhu. Rigidity and a Riemann-Hilbert correspondence for p-adic local systems. *Invent. Math.*, 207(1):291–343, 2017.

[11] Y. Min and Y. Wang. p-adic Simpson correpondence via prismatic crystals. *Preprint, arXiv:2201.08030*, 2022.

[12] J. E. Rodríguez Camargo. Geometric Sen theory over rigid analytic spaces. *Preprint, arXiv:2205.02016*, 2022.

[13] P. Scholze. p-adic Hodge theory for rigid-analytic varieties. *Forum of Mathematics, Pi*, 1, 2013.

[14] P. Scholze. Étale cohomology of diamonds. *Preprint, arXiv:1709.07343*, 2018.

[15] T. Tsuji. Notes on the local p-adic Simpson correspondence. *Math. Ann.*, 371(1-2):795–881, 2018.

On a p-adic version of Narasimhan and Seshadri's theorem

Fabrizio Andreatta

Let R be a complete discrete valuation ring, with fraction field K of characteristic 0, uniformizer π and residue field $k = R/(\pi) \cong \overline{\mathbf{F}}_p$. Fix an algebraic closure $\overline{K}$ of K and let $\overline{R}$ be the integral closure of R in $\overline{K}$. Let $\mathcal{O}_{\mathbf{C}_p}$ be the p-adic completion of $\overline{R}$ and $\mathbf{C}_p := \mathcal{O}_{\mathbf{C}_p}[p^{-1}]$. Let $\mathcal{C} \to \mathrm{Spec}(R)$ be a smooth proper morphism of relative dimension 1 with geometrically connected fibers, of genus $g \geq 1$, and generic fiber $\mathcal{C}_K$. Let $\mathcal{C}_{\overline{K}}$ be the base change of $\mathcal{C}_K$ to $\overline{K}$ and let $\overline{x}$ be $\overline{K}$-valued point of $\mathcal{C}$.

Work of G. Faltings [Fa] associates to continuous representations $\rho\colon \pi_1\big(\mathcal{C}_{\overline{K}}, \overline{x}\big) \to \mathrm{GL}_r(\mathbf{C}_p)$ of the geometric fundamental group of $\mathcal{C}_{\overline{K}}$ a Higgs bundle (E_ρ, θ_ρ). Here, E_ρ is a vector bundle of rank r on $\mathcal{C}_{\mathbf{C}_p}$ and $\theta_\rho \in \mathrm{Hom}\big(E_\rho, E_\rho \otimes_{\mathcal{O}_{\mathcal{C}_{\mathbf{C}_p}}} \Omega^1_{\mathcal{C}_{\mathbf{C}_p}/\mathbf{C}_p}\big)$ (the Higgs field). This is the so called *p-adic Simpson correspondence*. We refer the reader to the work of A. Abbes, M. Gros and T. Tsuji [AGT] for a complete and detailed account of Faltings' approach, developing the necessary foundations, and to work of B. Heuer [He] for a different and independent approach using vector bundles on Scholze's pro-étale and v-sites. Motivated by the result of Narasimhan and Seshadri [NS], in §5 of his paper Faltings remarks that any Higgs bundle constructed in this way is semistable of degree 0 and asks whether the converse is true; see also [Xu, Conjecture 1.1.8] or [He, Question 1.3]:

Question: *Is every semistable Higgs bundle over $\mathcal{C}_{\mathbf{C}_p}$ of degree* 0 *in the image of the p-adic Simpson correspondence?*

Thanks to work of Faltings [Fa], C. Deninger and A. Werner [DW] and of D. Xu [Xu] the evidence supporting a positive answer are the cases $r = 1$ and arbitrary genus and $g = 0$ or $g = 1$ and arbitrary rank r; see [Xu, Thm. 1.1.7]. In this paper, we answer this question for Higgs bundles with trivial Higgs field:

Theorem: Assume that $p > r(r-1)(g-1)$, that $g \geq 2$ and $r \geq 2$. Let F be a locally free sheaf on $\mathcal{C} \otimes_R \mathcal{O}_{\mathbf{C}_p}$ such that $F_k := F \otimes_{\mathcal{O}_{\mathbf{C}_p}} k$ is a stable sheaf of $\mathcal{O}_{\mathcal{C}_k}$-modules of degree 0. Then, its generic fiber $F_{\mathbf{C}_p}$ is in the image of the p-adic Simpson correspondence if and only if F_k is strongly semistable.

Recall that a vector bundle on $\mathcal{C}_k$ is called strongly semistable if it is semistable and its pull-back by any positive power of the absolute Frobenius on $\mathcal{C}_k$ is again semistable.

The Theorem provides a negative answer to Faltings' question already for $r = 2$, $g \geq 2$ and $p > 2g - 2$ thanks to work of Joshi and Pauli [JP].

Strategy of proof: The starting point of our analysis are the works of Deninger-Werner [DW] and of Xu [Xu]. The latter relates the work of Deninger-Werner and that of Faltings, see [Xu, Thm. 1.1.6 & 1.1.7], characterizing in this way the vector bundles on $\mathcal{C}_{\mathbf{C}_p}$ (with 0 Higgs field) that are in the image of the Simpson correspondence as those having potentially strongly semistable reduction. We can then rephrase our theorem as follows:

Theorem: Suppose that $p > r(r-1)(g-1)$, that $g \geq 2$ and $r \geq 2$. Let F be a locally free sheaf on $\mathcal{C} \otimes_R \mathcal{O}_{\mathbf{C}_p}$ such that F_k is stable of degree 0. Then, $F_{\mathbf{C}_p}$ has potentially strongly semistable reduction if and only if F_k is strongly semistable.

The first ingredient in our proof of this result is Raynaud's theory of reduction of Galois covers [Ra]. The second ingredient is Seshadri's theory of semistable sheaves on semistable curves over k. The third key input is the study of subbundles of the push-forward of stable vector bundles under a (power of) Frobenius on curves in positive characteristic p using the bound on p in the Theorem and work of X. Sun [Su] and refinements of K. Joshi and C. Pauly [JP].

References

[AGT] A. Abbes, M. Gros, T. Tsuji, *The p-adic Simpson correspondence*, Annals of Mathematics Studies **193** (2016).

[DW] C. Deninger, A. Werner, *Vector bundles on p-adic curves and parallel transport*, Annales scientifiques ENS **38** (2005), 553–597.

[Fa] G. Faltings, *A p-adic Simpson correspondence*, Advances in Math. **198** (2005), 847–862.

[He] B. Heuer, *A p-adic Simpson correspondence for smooth proper rigid spaces.* Preprint, arXiv:2307.01303, 2023.

[JP] K. Joshi, C. Pauly, *Hitchin-Mochizuki morphism, opers and Frobenius-destabilized vector bundles over curves*, Advances in Math. **274** (2015), 39–75.

[NS] M. S. Narasimhan, C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. **82** (1965), 540–567.

[Ra] M. Raynaud, *Spécialisation des revêtements en charactéristique $p > 0$*, Ann. Scient. ENS **32** (1999), 87–126.

[Se] C. S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque, **96** (1982).

[Su] X. Sun, *Direct images of bundles under Frobenius morphism*, Invent. Math. **173** (2008), 427–447.

[Xu] D. Xu, *Parallel transport for Higgs bundles over p-adic curves*, with an appendix joint with Tongmu He. Preprint (2022), https://arxiv.org/abs/2201.06697

p-isogenies with G-structure

Si Ying Lee

(joint work with Keerthi Madapusi)

This is a project which is currently work in progress.

1. Motivation

Let p be a prime. Let $\mathrm{Sh}_K(G, X)$ be an abelian type Shimura variety which is of hyperspecial level at p, with reflex field E. Let v be a prime above p. Following the work of Kisin [Kis10], we have a smooth integral model $\mathscr{S}_K(G, X)$ over O_{E_v} of $\mathrm{Sh}_K(G, X)$.

Let $\mathcal{H}(G)$ denote the spherical Hecke algebra at p. Given a double coset $1_{G(\mathbb{Z}_p)g_pG(\mathbb{Z}_p))} \in \mathcal{H}(G)$, one can consider $\mathrm{Sh}_{g_pKg_p^{-1}\cap K}(G, X)$, with its two projection maps to $\mathrm{Sh}_K(G, X)$. This algebraic correspondence gives rise to an action of $\mathcal{H}(G)$ on various cohomology groups of $\mathrm{Sh}_K(G, X)$.

Observe that in the case where (G, X) is of Hodge type, after choosing a Hodge embedding, $\mathrm{Sh}_{g_p K g_p^{-1} \cap K}(G, X)$ can be interpreted as the space of isogenies of abelian varieties with p-power degree which preserve additional structure, and whose map on the p-adic Tate modules is given (up to choice of trivialization of the Tate module) by g_p.

One can try to extend this description integrally, to consider various stacks of p-isogenies with G-structure which are defined over O_{E_v}. One key motivation for doing so is that such a construction would provide a geometric formalism for the integral action of the Hecke algebra at p. Such integral Hecke actions have been conjectured in various instances by others, and we mention two here: The conjecture of Fakhruddin-Pilloni [FP23] on the Hecke action on integral coherent cohomology of $\mathscr{S}_K(G, X)$, and the conjecture of Li-Rapoport-Zhang [LRZ23] on the Hecke action on Gillet-Soulé K-groups of Rapoport-Zink spaces. In both instances we also observe that in order for such a stack of p-isogenies to give rise to an action of the Hecke algebra, we require that the projection maps are both local complete intersection (lci) morphisms.

2. Construction

Our construction is local, and we first construct spaces of isogenies over the stack of (n-truncated) (G, μ)-apertures, as defined by Gardner-Madapusi-Mathew [GMM24] following the work of Drinfeld [Dri23] for $n = 1$. We briefly review the definition of an n-trucated (G, μ)-aperture here.

Following the work of Bhatt-Lurie and Drinfeld, for any p-complete commutative ring R, we may define the syntomification R^{syn} whose coherent cohomology computes the (p-adic) syntomic cohomology of R, as well as the Nygaard filtered prismatization $R^{\mathcal{N}}$. By construction, for all R, we have a map

$$x_{dR}^{\mathcal{N}} : \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \hookrightarrow R^{\mathcal{N}}.$$

For any positive integer n, we can consider the derived base change of R over $\mathbb{Z}/p^n\mathbb{Z}$, denoted by $R/^{\mathbb{L}}p^n$. We then define an n-truncated (G, μ)-aperture over R to be a G-torsor over $(R/^{\mathbb{L}}p^n)^{\mathcal{N}}$ whose pullback via $x_{dR}^{\mathcal{N}}$ to $B\mathbb{G}_m \times \operatorname{Spec} \kappa$ for any geometric point of $\operatorname{Spec} R/^{\mathbb{L}}p^n$ is isomorphic to the canonical G-torsor induced by μ.

Let $\mathrm{BT}_n^{G,\mu}$ denote the moduli stack of n-truncated (G, μ)-apertures. Over this stack, using m-isogeny models, we can construct stacks $\mathrm{Isog}_{Z,n}$ equipped with m projection maps to $\mathrm{BT}_n^{G,\mu}$.

Definition 2.1. *An n-isogeny model is a triple (Z, j, n) where Z is a separated finite type $\mathbb{Z}_p$-scheme with non-empty generic fiber and an action of G^{n+1} such that j is an isomorphism*

$$j : G_{\mathbb{Q}_p}^n \xrightarrow{\sim} Z_{\mathbb{Q}_p},$$

which is equivariant for the action of $G_{\mathbb{Q}_p}^{n+1}$, where the action on the source is the right action of $G_{\mathbb{Q}_p}^{n+1}$ given by

$$(h_0, \ldots, h_{n-1}) \cdot (g_0, \ldots, g_n) = (g_0^{-1} h_0 g_1, g_1^{-1} h_1 g_2, \ldots, g_{n-1}^{-1} h_{n-1} g_n).$$

We have the following key example of a 1-isogeny model, for $G = \mathrm{GL_n}$:

Example 2.2. *For all $r \in \mathbb{Z}$, let $Z_n^r \to \operatorname{Spec} \mathbb{Z}_p$ be the scheme given by*

$$Z_n^r(C) = \{(A_0, A_1) \in \mathrm{Mat}_n(C)^2 : \; A_0A_1 = A_1A_0 = p^r I_n\}.$$

The action of GL_n^2 is given by

$$(A_0, A_1) \cdot (g_0, g_1) = (g_0^{-1}A_0g_1, g_1^{-1}A_1g_0).$$

Now, given another integer $s \in \mathbb{Z}$, we obtain an isomorphism

$$j_s : Z^r_{n,\mathbb{Q}_p} \xrightarrow{\simeq} \mathrm{GL}_{n,\mathbb{Q}_p}$$
$$(A_0, A_1) \mapsto p^s A_0.$$

Observe now that if (Z_1, j_1, m_1) and (Z_2, j_2, m_2) are isogeny models for G then so is $(Z_1 \times Z_2, j_1 \times j_2, m_1 + m_2)$, with the action of $G^{m_1+m_2+1}$ being obtained via the map

$$G^{m_1+m_2+1} \to G^{m_1+1} \times G^{m_2+1}$$
$$(g_1, \ldots, g_{m_1+m_2+1}) \mapsto ((g_1, \ldots, g_{m_1+1}), (g_{m_1+1}, \ldots, g_{m_1+m_2+1})).$$

In particular, observe that from the example of a 1-isogeny model for GL_n, for $\underline{r} = (r_0, \ldots, r_{m-1})$ and $\underline{s} = (s_0, \ldots, s_{m-1})$, we have an m-isogeny model given by

$$(Z_n^{r_0} \times \ldots \times Z_n^{r_{m-1}}, j_{s_0} \times \ldots \times j_{s_{m-1}}, m).$$

We also observe that the free abelian group on the set of isogeny models forms a ring, with the multiplicative structure given by taking

$$(Z_1, j_1, m_1) \cdot (Z_2, j_2, m_2) = (Z_1 \times Z_2, j_1 \times j_2, m_1 + m_2).$$

We denote this by $\mathcal{H}_{geom}(G)$. This should be thought of as a geometric realization of the spherical Hecke algebra, but we note that this multiplicative structure is not commutative.

We can relate $\mathcal{H}_{geom}(G)$ with $\mathcal{H}(G)$, as follows. We can associate to an m-isogeny model an element of the spherical Hecke algebra. Observe that we have maps

$$Z(\mathbb{Z}_p) \hookrightarrow Z(\mathbb{Q}_p) \underset{j}{\xrightarrow{\sim}} G(\mathbb{Q}_p)^m \xrightarrow{\text{mult}} G(\mathbb{Q}_p),$$

and thus for any m-isogeny model (Z, j, m) we can consider the element

$$\tilde{\varphi}_{(Z,j,m)} := \mathrm{mult}_* 1_{j(Z(\mathbb{Z}_p))}.$$

One can show that this defines a surjective map

$$\tilde{\varphi} : \mathcal{H}_{geom}(G) \to \mathcal{H}(G).$$

Now, we can use the construction of [GMM24] to attach to a smooth m-isogeny model Z a stack $\mathcal{X}_Z := (\mathcal{X}_Z^{\diamond}, X_Z^{\circ})$ over $\mathbb{Z}_p^{\mathrm{syn}}$, from which we construct the stack $\mathrm{Isog}_{Z,n}$. More generally, we only need $(\mathcal{X}_Z^{\diamond}, X_Z^{\circ})$ to be 1-bounded as defined in [GMM24, §4.8]. From $\mathcal{X}_Z$ one can construct a moduli functor Isog_Z. Roughly speaking, for any p-complete commutative ring R, this is the functor of sections of $\mathcal{X}_Z^{\diamond}$ whose pullback via $x_{dR}^{\mathcal{N}}$ is given by a section of X_Z°. The 1-bounded condition is a condition on the deformation theory, and roughly implies that the weights of an associated cotangent complex are not less than -1.

Now, taking limits over n, we can define

$$\mathrm{BT}_{\infty}^{\mathrm{G},\mu} := \varprojlim_n \mathrm{BT}_n^{G,\mu} \qquad \mathrm{Isog}_{Z,\infty} := \varprojlim_n \mathrm{Isog}_{Z,n}.$$

Moreover, by construction, we have $m+1$ projection maps $\pi_i : \mathrm{Isog}_{Z,\infty} \to \mathrm{BT}_{\infty}^{\mathrm{G},\mu}$.

We first recall the following theorem of Imai-Kato-Youcis [IKY23] and Madapusi [Mad22] (revised version), which should be viewed as syntomic realization map:

Theorem 2.3. *We have a formally étale map*

$$\widehat{\mathscr{S}_K(G,X)} \to \mathrm{BT}_{\infty}^{\mathrm{G},\mu}.$$

Now, we may for each π_i, form the following (derived) Cartesian square:

$$\begin{array}{ccc} \widehat{p-\mathrm{Isog}}_{Z,i} & \longrightarrow & \widehat{\mathscr{S}_K(G,X)} \\ \downarrow & & \downarrow \\ \mathrm{Isog}_{Z,\infty} & \xrightarrow{\pi_i} & \mathrm{BT}_{\infty}^{G,\mu}. \end{array}$$

We want to algebrize the above derived formal scheme $\widehat{p-\mathrm{Isog}}_{Z,i}$: to do this, exactly as in the main construction in [Mad22], if the projection maps π_i are proper, it suffices to glue this formal scheme to a scheme over the generic fiber, which is determined by $j(Z(\mathbb{Z}_p))$. As a consequence, we have the following construction of the global space of isogenies attached to Z, with the following properties:

Theorem 2.4. *For all isogeny models (Z,j,m) which are 1-bounded, with proper projection maps, and all $i=0,\dots,m$, there is a derived scheme $p-\mathrm{Isog}_{Z,i}$ over O_{E_v} together with a quasi-smooth map*

$$p-\mathrm{Isog}_{Z,i} \to \mathscr{S}_K(G,X),$$

such that

(i) If $m=1$, then $p-\mathrm{Isog}_{Z,i}$ is classical, flat over O_{E_v}, and a local complete intersection;

(ii) We have isomorphisms $p-\mathrm{Isog}_{Z,i} \simeq p-\mathrm{Isog}_{Z,j}$ for all $i \neq j$, and thus for all i,j, the following diagram is Cartesian:

$$\begin{array}{ccccc} \widehat{\mathscr{S}_K(G,X)} & \longleftarrow & \widehat{p-\mathrm{Isog}}_{Z,i} & \longrightarrow & \widehat{\mathscr{S}_K(G,X)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{BT}_{\infty}^{G,\mu} & \xleftarrow{\pi_j} & \mathrm{Isog}_{Z,\infty} & \xrightarrow{\pi_i} & \mathrm{BT}_{\infty}^{G,\mu}. \end{array}$$

In order to show (i), we do a dimension count to show that the underlying classical scheme of $p-\mathrm{Isog}_{Z,i}$ has the correct expected dimension, from which being lci and flat easily follow. To show (ii), we reduce to the case where $m=1$, from which we can deduce the existence of the isomorphism from flatness.

Note that in the case $m = 1$, for unitary PEL Shimura varieties at a split prime, the example of the 1-isogeny models for GL_n given above should recover the open and closed subspace of $p - \mathrm{Isog}$ parametrizing p-power isogenies with degree r and whose inverse has denominator bounded by p^s. Here, we denote by $p - \mathrm{Isog}$ the space of isogenies with p-power degree respecting endomorphism, polarization and prime-to-p level structure, as defined by Wedhorn [Wed00]. The spaces of p-isogenies we construct here should be considered as generalizations of various connected components of $p - \mathrm{Isog}$. In the case of $m > 1$, what we construct should be considered as generalizations of m-iterated derived fiber products of $p - \mathrm{Isog}$.

We expect that this construction will allow us to construct the Hecke actions as conjectured by Fakhruddin-Pilloni and Li-Rapoport-Zhang. Namely, we expect that since the various spaces are lci, we can construct cohomological correspondences attached to various isogeny models, and get an action of $\mathcal{H}_{geom}(G)$ on various invariants. We believe that in both this situations, this action factors through the map $\tilde{\varphi}$, inducing an action of $\mathcal{H}(G)$.

References

[Dri23] Vladimir Drinfeld. On Shimurian generalizations of the stack $BT_1 \otimes F_p$. 2023.

[FP23] Najmuddin Fakhruddin and Vincent Pilloni. Hecke operators and the coherent cohomology of Shimura varieties. *J. Inst. Math. Jussieu*, 22(1):1–69, 2023.

[GMM24] Zachary Gardner, Keerthi Madapusi, and Akhil Mathew. Moduli of truncated prismatic (G, μ)-displays and conjectures of Drinfeld, 2024.

[IKY23] Naoki Imai, Hiroki Kato, and Alex Youcis. The prismatic realization functor for Shimura varieties of abelian type. 2023.

[Kis10] Mark Kisin. Integral models for Shimura varieties of abelian type. *J. Amer. Math. Soc.*, 23(4):967–1012, 2010.

[LRZ23] Chao Li, Michael Rapoport, and Wei Zhang. Arithmetic Fundamental Lemma for the spherical Hecke algebra. 2023.

[Mad22] Keerthi Madapusi. Derived special cycles on Shimura varieties. December 2022.

[Wed00] Torsten Wedhorn. Congruence relations on some Shimura varieties. *J. Reine Angew. Math.*, 524:43–71, 2000.

Some new applications of G-functions to arithmetic geometry.

Vesselin Dimitrov

(joint work with Frank Calegari and Yunqing Tang)

André's refinement of the Siegel–Shidlovsky theorem on special values of E-functions can be expressed like a statement in the language of commutative algebra:

Theorem 1. *The ring of all E-functions with $\mathbf{Q}$-coefficients generates a (countably infinite)* ***free*** *submodule of $\mathbf{Q}((x))$ over $\mathbf{Q}[x, 1/x]$.*

My talk attempted to address a robust, but partly conjectural, G-functions counterpart (in a restricted form attached to a pseudoconcave formal-analytic arithmetic surface $\widetilde{\mathcal{V}}$ in the sense of Bost) of this type of statement, as an algebraization problem on integrable connections over $\widetilde{\mathcal{V}}^{\mathrm{alg}} = \mathrm{Spec}(\mathcal{O}(\widetilde{\mathcal{V}}))$.

This is the subject of arithmetic holonomy bounds, both rational and integral, currently being developed in collaboration with Frank Calegari and Yunqing Tang with inspiration from the work of Bost and Charles. One application is to a $\mathbf{Q}$-linear independence proof of 1, $\zeta(2)$, $L(2, \chi_{-3})$ whereas here I reported instead on some applications to arithmetic geometry proper, such as the following "integral converse theorem" and its own consequences:

Theorem 2. *For Dirichlet series with (almost) integer coefficients, the classical (Hecke–Weil)* GL_2*-converse theorem holds without any character twists.*

The categorical local Langlands program

David Hansen
(joint work with Lucas Mann)

This talk is a report on joint work in progress with Lucas Mann. Our goal is to formulate a program to prove the categorical local Langlands conjecture (CLLC) of Fargues–Scholze [3] for many groups.

We begin by briefly recalling the setup for this conjecture. Fix a finite extension $E/\mathbf{Q}_p$ and a connected reductive quasisplit group G/E. Fix also a prime $\ell \neq p$. On the automorphic side, the main geometric player is the stack Bun_G of G-bundles on the Fargues–Fontaine curve. This behaves like a smooth Artin stack of dimension zero. Moreover, it has a stratification indexed by the Kottwitz set $B(G)$ whose strata Bun_G^b are essentially the classifying stacks of the locally profinite groups $G_b(E)$. Here G_b is an inner form of a Levi in G.

With specific technical effort, Fargues–Scholze defined a category $D(\mathrm{Bun}_G) := D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$ of constructible ℓ-adic sheaves on Bun_G. Similar categories are defined for each stratum, which satisfy equivalences $D(\mathrm{Bun}_G^b) \cong D(G_b(E), \overline{\mathbf{Q}_\ell})$ where the right-hand side denotes the derived category of the usual category of smooth $G_b(E)$-representations. There are then some obvious functors

$$D(G_b(E), \overline{\mathbf{Q}_\ell}) \underset{i_b^*}{\overset{i_{b!}}{\rightleftarrows}} D(\mathrm{Bun}_G)$$

relating sheaves on Bun_G with representations of the groups G_b, and in fact $D(\mathrm{Bun}_G)$ is semi-orthogonally decomposed into the categories $D(G_b(E), \overline{\mathbf{Q}_\ell})$. We note in particular that for $b = 1$, $G_1 = G$, and $i_{1!}$ embeds smooth $G(E)$-representations fully faithfully into sheaves on Bun_G.

On the spectral side, the main player is the stack Par_G of ℓ-adically continuous L-parameters $\phi : W_E \to {}^L G(\overline{\mathbf{Q}_\ell})$. It is a little subtle to make this notion precise, but after pinning down its meaning, this turns out to be a very reasonable space: by independent works of Fargues–Scholze, Zhu, Hellmann, and Dat–Helm–Kurinczuk–Moss, we know that Par_G is a reduced Artin stack which is a global lci of pure dimension zero over $\mathrm{Spec} \overline{\mathbf{Q}_\ell}$, and each connected component is the quotient of an affine variety by a reductive group action. Moreover, Par_G comes with a canonical map $\mathrm{Par}_G \to B\hat{G}$.

There are then two closely related sheaf categories on the spectral side: the usual quasicoherent derived category $\mathrm{QCoh}(\mathrm{Par}_G)$, and the slightly larger category of *ind-coherent* sheaves $\mathrm{IndCoh}(\mathrm{Par}_G)$. These are related by a pair of adjoint functors

$$\mathrm{QCoh}(\mathrm{Par}_G) \overset{\Xi}{\hookrightarrow} \mathrm{IndCoh}(\mathrm{Par}_G) \overset{\Psi}{\twoheadrightarrow} \mathrm{QCoh}(\mathrm{Par}_G).$$

We note that Ψ is an equivalence on the "obvious" copies of Coh contained in its source and target, but this fails very badly for Ξ.[1]

A priori, these two sides are unrelated. However, Fargues–Scholze constructed a canonical $\otimes$-action of $\mathrm{QCoh}(\mathrm{Par}_G)$ on $D(\mathrm{Bun}_G)$, usually called "the spectral action". Given $\mathcal{F} \in \mathrm{QCoh}(\mathrm{Par}_G)$ and $A \in D(\mathrm{Bun}_G)$, we write $\mathcal{F} * A$ for the object obtained by acting via $\mathcal{F}$ on A. Very roughly, this action is normalized by the requirement that $V * (-) = T_V(-)$, where on the left $V \in \mathrm{Rep}\hat{G}$ is regarded as a vector bundle on $B\hat{G}$ and then pulled back to a vector bundle on Par_G, and on the right T_V denotes a Hecke operator acting on sheaves on Bun_G, constructed via a suitable form of geometric Satake.

To state the categorical conjecture, we need one more piece of data, namely a choice of Whittaker datum. This is a pair (B, ψ) where $B = TN \subset G$ is a Borel and $\psi : N(E) \to \overline{\mathbf{Q}_\ell}^\times$ is a nondegenerate character. From this we form the Whittaker representation $W_\psi = c - \mathrm{ind}_{N(E)}^{G(E)} \psi$, where $c - \mathrm{ind}$ denotes smooth induction with compact support. Via the functor $i_{1!}$, we extend this to a sheaf $i_{1!} W_\psi$ on Bun_G, and then consider the functor

$$\begin{aligned} a_\psi : \mathrm{QCoh}(\mathrm{Par}_G) &\to D(\mathrm{Bun}_G) \\ \mathcal{F} &\mapsto \mathcal{F} * i_{1!} W_\psi \end{aligned}$$

given by acting spectrally on this sheaf. We can now formulate the categorical local Langlands conjecture after Fargues–Scholze.

Conjecture 1. *The functor a_ψ is fully faithful, and extends to an equivalence of categories $\mathbf{L}_\psi : D(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}(\mathrm{Par}_G)$ such that the diagram*

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{Par}_G) & \xrightarrow{a_\psi} & D(\mathrm{Bun}_G) \\ & \searrow^{\Xi} & \downarrow{\scriptstyle \wr}\, \mathbf{L}_\psi \\ & & \mathrm{IndCoh}(\mathrm{Par}_G) \end{array}$$

commutes.

In fact, this conjecture can be sharpened quite a bit.

Proposition 2. *The equivalence $\mathbf{L}_\psi$ is unique if it exists, and it exists if and only if the functor c_ψ - the right adjoint of a_ψ - restricts to give an equivalence*

$$c_\psi : D(\mathrm{Bun}_G)^{\mathrm{cpct}} \overset{\sim}{\to} \mathrm{Coh}(\mathrm{Par}_G),$$

in which case $\mathbf{L}_\psi$ is simply the ind-completion of this equivalence.

[1] We write Coh where many people would write $D^{b,\mathrm{qc}}_{\mathrm{coh}}$. With this notation, it is literally true that $\mathrm{IndCoh} = \mathrm{Ind}(\mathrm{Coh})$ for Par_G.

Our goal is to prove this sharpened form of CLLC for many groups. For general groups, this is a hopeless task at present, because the conjecture simply carries too much information. However, for groups where we have a solid understanding of *classical* local Langlands, we are in much better shape.

Definition 3. A quasisplit group G is *well-understood* if there is a known $B(G)_{\mathrm{basic}}$ local Langlands correspondence for G and all its Levi subgroups, which satisfies some standard expected properties (finite fibers, Whittaker-normalized, expected parametrization of discrete L-packets and the endoscopic character identites for them), and which agrees up to semisimplification with the Fargues–Scholze construction of L-parameters.

This is quite a lot to demand, but actually many groups are well-understood at present, including GL_n (for any E), as well as GSp_4, SO_{2n+1}, and the unramified form of U_{2n+1} (all with some restrictions on E).

To make progress for this class of groups, we also need one more piece of control: we *assume* that the functor c_ψ is compatible with Eisenstein series (in a precise sense). In "classical" geometric Langlands, this is a recent result of Faergeman–Hayash. In the present setting, the case of $G = \mathrm{GL}_2$ was proved by Hamann, and the general case is work in progress of Hamann–DH–Mann. We will assume this compatibility in what follows, but we only need it as a black box. We can now state our first result.

Theorem 4. *The categorical local Langlands conjecture is true for* GL_2.

This is a specialization from much more general results.

Theorem 5. *Let G be any well-understood group, with a fixed choice of Whittaker datum.*

i. *There is a unique continuous functor* $\mathbf{L}_\psi : D(\mathrm{Bun}_G) \to \mathrm{IndCoh}(\mathrm{Par}_G)$ *preserving compact objects and making the diagram*

$$\begin{array}{ccc} D(\mathrm{Bun}_G) & \xrightarrow{\mathbf{L}_\psi} & \mathrm{IndCoh}(\mathrm{Par}_G) \\ & \searrow^{c_\psi} & \downarrow \Psi \\ & & \mathrm{QCoh}(\mathrm{Par}_G) \end{array}$$

commute. The functor $\mathbf{L}_\psi$ *is* $\mathrm{QCoh}(\mathrm{Par}_G)$*-linear and compatible with Eisenstein series.*

ii. *The functor* $\mathbf{L}_\psi$ *has a* $\mathrm{QCoh}(\mathrm{Par}_G)$*-linear continuous right adjoint* $\mathbf{R}_\psi : \mathrm{IndCoh}(\mathrm{Par}_G) \to D(\mathrm{Bun}_G)$ *compatible with constant terms, which also preserves compact objects.*

iii. *If a_ψ is fully faithful, then $\mathbf{R}_\psi$ is fully faithful.*

For GL_n, we can say much more.

Theorem 6. *Assume* $G = \mathrm{GL}_n$.

i. *The functor* $\mathbf{L}_\psi \circ i_{1!}$ *coincides with the fully faithful embedding* $D(\mathrm{GL}_n(E), \overline{\mathbf{Q}_\ell}) \to \mathrm{IndCoh}(\mathrm{Par}_{\mathrm{GL}_n})$ *constructed by Ben-Zvi–Chen–Helm–Nadler* [1].

ii. *The functors* a_ψ *and* $\mathbf{R}_\psi$ *are fully faithful.*
iii. *We have* $\mathbf{R}_\psi \circ \Xi = a_\psi$.
iv. *On compact sheaves, we have the duality compatibility* $\mathbf{D}_{\mathrm{tw.GS}} \circ \mathbf{L}_\psi = \mathbf{L}_{\psi^{-1}} \circ \mathbf{D}_{\mathrm{BZ}}$*, where* $\mathbf{D}_{\mathrm{tw.GS}}$ *denotes Chevally-twisted Grothendieck–Serre duality, and* $\mathbf{D}_{\mathrm{BZ}}$ *is the Bernstein–Zelevinsky ("miraculous") duality on* Bun_G.

We note that parts ii.-iv. depend crucially on part i.

For GL_n, this reduces the whole conjecture to the conservativity of $\mathbf{L}_\psi$. In "classical" geometric Langlands, this conservativity was a recent breakthrough of Faergeman–Raskin [2], but their microlocal techniques do not seem to adapt to our setting.

To go further, we import some idea from geometric Langlands theory "with restricted variation". Let $D(\mathrm{Bun}_G)_{\mathrm{fin}} \subset D(\mathrm{Bun}_G)$ denote the full subcategory of sheaves A such that

$$\sum_{b \in B(G), n \in \mathbf{Z}} \mathrm{length} H^n(i_b^* A) < +\infty.$$

Let $\mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}} \subset \mathrm{Coh}(\mathrm{Par}_G)$ denote the full subcategory of objects which are supported set-theoretically on finitely many closed fibers of the map from Par_G to its GIT quotient. It is easy to see that if the full CLLC is true, then it restricts to an equivalence $D(\mathrm{Bun}_G)_{\mathrm{fin}} \simeq \mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}}$. This also has a strong converse.

Theorem 7. *If G is well-understood, $\mathbf{L}_\psi$ and $\mathbf{R}_\psi$ restrict to an adjoint pair of functors between $D(\mathrm{Bun}_G)_{\mathrm{fin}}$ and $\mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}}$, and if either of those restricted functors is an equivalence, then the full CLLC is true for G.*

For GL_n, this reduces the whole conjecture to showing that the (fully faithful!) functor $\mathbf{R}_\psi : \mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}} \to D(\mathrm{Bun}_G)_{\mathrm{fin}}$ is essentially surjective. For GL_2, we are (barely) able to check this by hand, taking advantage of the compatible gradings on the source and target by semisimple L-parameters. Up to twist, the only parameters which cause difficulty are the trivial L-parameter, where we make use of Bezrukavnikov's theory of perverse coherent sheaves on the nilpotent cone, and the semisimplification of the Steinberg parameter, where we make heavy use of an exhaustive table of RHom's between explicit generating sheaves on the coherent side, which was computed independently by Bertoloni Meli and Koshikawa.

In the talk I had almost no time to discuss the proofs. Let me briefly mention some key new ideas here:

- Very strong finiteness theorems for spectral constant term functors.
- A new theory of "admissible" ind-coherent sheaves, which comes with its own intrinsic stability properties and duality functor.
- New duality theorems for the spectral action.
- A spectral analogue of the fact that "$\mathrm{ps} - \mathrm{id}_*$ annihilates antitempered D-modules".

Using these ingredients, we are able to give an *explicit formula* for $\mathbf{R}_\psi|_{\mathrm{Coh}(\mathrm{Par}_G)_{\mathrm{fin}}}$ purely in terms of the spectral action and various dualities on both sides. This is the crucial source of control in many of our results.

References

[1] D. Ben-Zvi, H. Chen, D. Helm and D. Nadler, *Coherent Springer theory and the categorical Deligne-Langlands correspondence*, Invent. Math. 235 (2024).

[2] J. Faergeman and S. Raskin, *Nonvanishing of geometric Whittaker coefficients for reductive groups*, preprint (2022).

[3] L. Fargues and P. Scholze, *Geometrization of the local Langlands correspondence*, preprint (2021).

A-parameters and eigensheaves

Teruhisa Koshikawa

(joint work with Alexander Bertoloni Meli)

Fix distinct prime numbers p, ℓ. Let E be a finite extension of $\mathbb{Q}_p$, and G a quasi-split reductive group over E.

The most refined form of the categorical local Langlands conjecture has been proposed in the work of Fargues–Scholze [5, Conjecture I.10.2]. Technically speaking, we also fix a square root of q (the cardinality of the residue field of E) in $\overline{\mathbb{Q}}_\ell$, and a Whittaker datum to normalize the conjectural equivalence.

Conjecture (Fargues–Scholze)**.** *There exists an equivalence*

$$D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell) \cong \mathrm{IndCoh}(Z^1(W_E, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G})$$

that interchanges $T_V(-)$ and $\underline{V} \otimes -$ for $V \in \mathrm{Rep}(\widehat{G})$, where T_V is the Hecke operator of Fargues and Scholze, and $\underline{V}$ is the associated vector bundle on $Z^1(W_E, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}$.

Before the joint work with Scholze, Fargues [4] has originated the geometrization of the local Langlands with his conjecture on Hecke eigensheaves for discrete L-parameters. Here is a slightly modified version of (a part of) the conjecture:

Conjecture (Fargues)**.** *Let ϕ be a discrete L-parameter of G. There exists $\mathcal{F}_\phi \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ satisfying the following properties.*

(1) $\mathcal{F}_\phi$ is an Hecke eigensheaf, i.e., for every $V \in \mathrm{Rep}(\widehat{G})$,

$$T_V(\mathcal{F}_\phi) \cong V \otimes_{\overline{\mathbb{Q}}_\ell} \mathcal{F}_\phi.$$

(2) $\mathcal{F}_\phi$ is perverse; see [8, Proposition 1.2.1] *for the definition of the perverse t-structure.*

(3) For any basic point $b \in B(G)$ and the open immersion $i_b \colon B\underline{G_b(E)} \hookrightarrow \mathrm{Bun}_G$,

$$i_b^* \mathcal{F}_\phi \cong \bigoplus_{\pi \in \Pi_\phi(G_b(E))} \pi^{\langle \pi, e \rangle},$$

where $\langle \pi, e \rangle$ is the dimension of the representation of the centralizer of ϕ corresponding to π under the local Langlands conjecture for extended pure inner forms.

(4) The pushforward along the Hodge–Tate period map from the perfectoid compact Shimura variety, or rather from the Igusa stack, is related to eigensheaves.

The property (4) is called the local-global compatibility in [4], and is inspired by results of Caraiani and Scholze. However, the original statement does not seem to be precise enough and it is somehow nontrivial to make a reasonable conjecture along this line (already in the above setup) that is consistent with conjectures of Arthur and Kottwitz; this was discussed intensively after the workshop.

In fact, it has been observed by several people that the above conjecture should hold in greater generalities. The optimal class seems to be the one of *generic* L-parameters in the sense that the monodromy of the corresponding Weil–Deligne parameter is maximal possible with the same semisimplification. See [8, Section 3] for some related discussion based on the categorical local Langlands conjecture and the generalized coherent Springer sheaves.

We propose that a variant of the conjecture should hold for A-parameters, also known as Arthur parameters:

Definition. *Fix an isomorphism* $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. *A map*

$$\psi \colon W_E \times \mathrm{SL}_2 \times \mathrm{SL}_2 \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

is an A-parameter with respect to ι if the restriction of ψ to W_E is an L-parameter, the restriction of ψ to $\mathrm{SL}_2 \times \mathrm{SL}_2$ is algebraic, and $\iota(\psi(W_E))$ is bounded.

In fact, one should work with a broader class of parameters independent of ι that includes all generic L-parameters. For the purpose of exposition, we restrict ourselves to the above setting.

We conjecture that a variant of eigensheaf exists for any A-parameter:

Conjecture. *Let ψ be an A-parameter with respect to ι. There exists $\mathcal{F}_\psi \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ satisfying the following properties.*

(1) For a given $V \in \mathrm{Rep}(\widehat{G})$, let $V \circ \psi \cong \bigoplus_{i \in \mathbb{Z}} V_i$ be the weight decomposition with respect to $\mathbb{G}_m$ in the second SL_2, where V_i is the weight i space. There exists an isomorphism

$$T_V(\mathcal{F}_\psi) \cong \bigoplus_i V_i \otimes_{\overline{\mathbb{Q}}_\ell} \mathcal{F}_\psi[-i].$$

(2) $\mathcal{F}_\psi$ is perverse.

(3) Assume G is semisimple, in which case G_b is a pure inner form of G for any basic b. There exists a decomposition

$$i_b^* \mathcal{F}_\psi \cong \bigoplus_{\pi \in \Pi_{\iota \circ \psi}(G_b(E))} \pi^{\langle \pi, e \rangle},$$

where the left hand side is regarded as a $\mathbb{C}$-representation via ι and $\Pi_{\iota\circ\psi}$ is the p-adic Adams–Barbasch–Vogan packet with parametrization discussed in [3], *and $\langle \pi, e \rangle$ is the dimension of the representation corresponding to π.*

(4) *Again, the pushforward along the Hodge–Tate period map from the perfectoid compact Shimura variety, or rather from the Igusa stack, is related to eigensheaves.*

We realized, not surprisingly, that the variant of eigenproperty (1) above is not completely new. See [6, 7] in a more classical context or more recent [2]. In particular, the modification of the eigenproperty in (1) is regarded in [2] as a special case of *shearing*, and we do so as well. Note that Property (3) makes sense only when we know some version of the classical local Langlands conjecture as the Adams–Barbasch–Vogan packets need such an input.

Here is one evidence towards the conjecture:

Theorem. *Let ψ be an A-parameter with respect to ι. There exists a nonzero ind-coherent sheaf $\mathcal{F}_\psi$ on $Z^1(W_E, \widehat{G})_{\overline{\mathbb{Q}}_\ell}/\widehat{G}$ satisfying*

$$\underline{V} \otimes \mathcal{F}_\psi \cong \bigoplus_i V_i \otimes_{\overline{\mathbb{Q}}_\ell} \mathcal{F}_\psi[-i].$$

In particular, the categorical local Langlands conjecture would imply that there exists a nonzero $\mathcal{F}_\psi \in D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ satisfying the property (1).

If ψ is tempered, i.e., trivial on the second SL_2, the construction of the above ind-coherent sheaf is not difficult, and it is essentially obtained from the regular representation of the centralizer of ψ; see [8, Section 3.1]. In general, our construction of these ind-coherent sheaves is very much inspired by [1], where the authors compare, in the unipotent case, the categorical local Langlands conjecture and the p-adic Adams–Barbasch–Vogan theory via the Koszul duality. This is also one reason why we think the property (3) should hold.

It would be possible to make sense of properties (2) and (3) in a suitable sense purely on the spectral side, and we are currently working on (formulating such statements and) verifying these properties. For instance, this can be checked for GL_2, and this example led us to the general conjecture.

Acknowledgements. TK was supported by JSPS KAKENHI Grant Number 24K16895.

References

[1] D. Ben-Zvi, H. Chen, D. Helm, D. Nadler, *Between Coherent and Constructible Local Langlands Correspondences*, preprint.
[2] D. Ben-Zvi, Y. Sakellaridis, A. Venkatesh, *Relative Langlands duality*, preprint.
[3] C. Cunningham, A. Fiori, A. Moussaoui, J. Mracek, B. Xu, Arthur packets for p-adic groups by way of microlocal vanishing cycles of perverse sheaves, with examples, Vol. 276, American Mathematical Society, 2022.
[4] L. Fargues, *Geometrization of the local Langlands correspondence: an overview*, preprint.
[5] L. Fargues, P. Scholze, *Geometrization of the local Langlands correspondence*, preprint.
[6] E. Frenkel, R. Langlands, B.C. Ngô, *Formule des Traces et Fonctorialité: le Début d'un Programme*, Ann. Sci. Math. Québec **34**, 199–243 (2010)
[7] E. Frenkel, B.C. Ngô, *Geometrization of Trace Formulas*, Bull. Math. Sci. 1 (2011), no. 1, 129–199.
[8] D. Hansen, *Beijing notes on the categorical local Langlands conjecture*, preprint.

Hodge type Igusa stacks and cohomology of Shimura varieties

Mingjia Zhang
(joint work with Patrick Daniels, Pol van Hoften, Dongryul Kim)

Igusa stacks are geometric objects akin to Shimura varieties. Scholze predicted their existence, motivated by questions related to local-global compatibility in the Langlands program. In [2], the author has constructed Igusa stacks for some PEL-type Shimura data. The current project [1] extends this to Hodge type Shimura data. This geometric result enables us to apply the geometric local Langlands correspondence of [3] to study the ℓ-adic cohomology of Shimura varieties.

More precisely, let (G, X) be a Hodge type Shimura datum with Hodge cocharacter μ and reflex field E. Fix a prime number p, a place v of E above p and set $E = \mathsf{E}_v$, $G = \mathsf{G}_{\mathbb{Q}_p}$. We consider the Shimura variety attached to (G, X) over E, i.e. a tower of algebraic varieties $\{\mathbf{Sh}_K\}_K$ for K running through neat compact open subgroups of $\mathsf{G}(\mathbb{A}_f)$. Fixing K^p, we denote by $\mathbf{Sh}_{K^p}$ the inverse limit $\lim_{K_p \to 1} \mathbf{Sh}_{K^pK_p}$. Let $\ell \neq p$ be a prime that does not divide the order of $\pi_0(Z_G)$, where Z_G is the center of G. We consider a torsion noetherian $\mathbb{Z}_\ell$-algebra Λ containing a fixed square root of p. Our main cohomological results are as follows:

Theorem 1 (Mantovan's product formula)**.** *There exists a filtration on $R\Gamma(\mathbf{Sh}_{K^p,\overline{E}}, \Lambda)$ by complexes of smooth representations of $G(\mathbb{Q}_p) \times W_E$, labeled by the Kottwitz set $B(G, \mu^{-1})$, whose graded pieces are given by*

$$R\Gamma(\mathrm{Ig}^b, \Lambda)^{\mathrm{op}} \otimes^{\mathbb{L}}_{\mathcal{C}_c(G_b(\mathbb{Q}_p),\Lambda)} R\Gamma_c(G, b, \mu).$$

Here Ig^b denotes a perfect Igusa variety over $\overline{\mathbb{F}}_p$; G_b is an inner form of a Levi of G determined by b, and up to shifts and twists, $R\Gamma_c(G, b, \mu)$ is the cohomology of a local Shimura variety attached to the datum (G, b, μ). This describes the structure of $R\Gamma(\mathbf{Sh}_{K^p,\overline{E}}, \Lambda)$ and extends results of [4, 6]. Compared to [5], we do not make extra assumptions on the Hodge type Shimura data.

Let $W_E \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the Weil group of E and fix the prime-to-p level K^p.

Theorem 2 (Eichler-Shimura relation)**.** *Assume G is unramified, and fix a Iwahoric in a hyperspecial subgroup $I_p \subset K_p \subset G(\mathbb{Q}_p)$. Let $\mathcal{H}_{K_p}$ be the corresponding spherical Hecke algebra with Λ coefficient. Then the inertia subgroup of W_E acts unipotently on $R\Gamma(\mathbf{Sh}_{I_pK^p,\overline{E}}, \Lambda)$, and for any lift σ of the Frobenius, the relation $H_\mu(\sigma) = 0$ holds, where $H_\mu(X) \in \mathcal{H}_{K_p}[X]$[1] is a renormalized Hecke polynomial.*

This result generalizes the classical Eichler-Shimura relation for modular curves, confirming Blasius-Rogawski's conjecture [9, Section 6] for Hodge type Shimura varieties and extends the work of [10]. It shows that at all unramified primes, the Frobenius eigenvalues are constraint to the roots of the Hecke polynomials, and thus provide much information about the cohomology of the Shimura variety (over E) as a global Galois representation. Note that most previous works only consider

[1] where $\mathcal{H}_{K_p}$ identifies with the center of $\mathcal{H}_{I_p}$ via Bernstein isomorphism

Shimura varieties at hyperspecial level. It is therefore a novelty that we establish this result at Iwahoric levels. Our proof is strongly influenced by [7, 8].

Theorem 3 ("Generic" part). *Assume G is unramified and "well-understood"*[2], *K_p is hyperspecial and $\mathbf{Sh}_K$ is compact. Then for a generic L-parameter ϕ, the isotypical part $R\Gamma(\mathbf{Sh}_{K,\overline{E}}, \Lambda)_\phi$*[3] *is concentrated in the middle degree.*

Here we use the notion of "generic" in the sense of [11], which is inspired by [12, 13, 14]. This result shows analogy to the phenomenon that tempered automorphic representations only occur in the middle degree in the cohomology of locally symmetric spaces attached to compact Shimura varieties, albeit with torsion coefficients. Our approach follows and extends [6].

All the above results are consequences of our main geometric result below. To our surprise, given this geometric input, the cohomological consequences follow formally from the setup of [3], with a minimal amount of extra effort. One therefore wonders how much more the categorical local Langlands correspondence can be exploited to provide information about the global Langlands correspondence, as a first approximation of which, about the cohomology of Shimura varieties.

We write Perf_v for the v-site of perfectoid spaces over $\overline{\mathbb{F}}_p$. The (good reduction locus of the) Shimura variety will be considered as a sheaf on Perf_v and denoted by $\mathbf{Sh}^{\diamondsuit,\circ}_{K^p}$. Let Bun_G be the stack of G-bundles on the Fargues-Fontaine curve with its Beauville-Laszlo map $BL : \mathrm{Gr}_G \to \mathrm{Bun}_G$, from the $\mathrm{B}^+_{\mathrm{dR}}$-affine Grassmannian for G. We restrict BL to the minuscule Schubert cell $\mathrm{Gr}_{G,\mu^{-1}}$.

Theorem 4. *There exists a functorial (in Shimura data) construction of an "Igusa stack" Igs_{K^p} on Perf_v, together with a Cartesian diagram*

$$\begin{array}{ccc} \mathbf{Sh}^{\diamondsuit,\circ}_{K^p} & \xrightarrow{\pi^{\diamondsuit}_{\mathrm{HT}}} & \mathrm{Gr}_{G,\mu^{-1}} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{BL}} \\ \mathrm{Igs}_{K^p} & \xrightarrow{\overline{\pi}_{\mathrm{HT}}} & \mathrm{Bun}_{G,\mu^{-1}}, \end{array}$$

where the top row is the Hodge-Tate period map. Furthermore, for all $\ell \neq p$, Igs_{K^p} is ℓ-cohomologically smooth of dimension 0 with constant dualizing sheaf.

Let us give some ideas how this is used to prove Theorem 1 to 3. Theorem 1 is direct: $\mathrm{Bun}_{G,\mu^{-1}}$ has a Newton stratification labeled by $B(G,\mu^{-1})$, and the theorem follows by identifying the stratum corresponding to $b \in B(G,\mu^{-1})$ of the Cartesian diagram. In particular, horizontally the fiber is the Igusa variety Ig^b while vertically the fiber is the local Shimura variety attached to (G, b, μ). For Theorem 2 and 3, we consider the complex

$$\mathcal{F} := R\overline{\pi}_{\mathrm{HT},*}\Lambda \in D(\mathrm{Bun}_{G,\mu^{-1}}, \Lambda).$$

It is not hard to check that there is a $G(\mathbb{Q}_p) \times W_E$-equivariant isomorphism

$$i_1^* T_\mu \mathcal{F}[-d](-\tfrac{d}{2}) \simeq R\Gamma(\mathbf{Sh}_{K^p,\overline{E}}, \Lambda),$$

[2]As defined in David Hansen's talk, see the corresponding abstract in this proceeding.
[3]Defined as in [6, Section 4].

where T_μ is the Hecke operator attached to μ, and $i_1 : \underline{BG(\mathbb{Q}_p)} \hookrightarrow \mathrm{Bun}_G$ is the open immersion of the neutral stratum. This reduces the problem of studying $R\Gamma(\mathbf{Sh}_{K^p,\overline{E}}, \Lambda)$ to studying $\mathcal{F}$ and the functor $i_1^* T_\mu$ individually. Indeed, Theorem 3 follows by showing $\mathcal{F}$ is perverse for a t-structure on $D(\mathrm{Bun}_{G,\mu^{-1}}, \Lambda)$ while $i_1^* T_\mu$ (whose target identifies with the derived category of smooth $G(\mathbb{Q}_p)$-representations with usual t-structure) is perverse t-exact on generic objects.

Let $X_{\widehat{G}}$ be the stack of L-parameters for G. Theorem 2 uses the spectral action of [3, Section X], a Λ-linear action of the ∞-category of perfect complexes on $X_{\widehat{G}}$ on $\mathcal{D}(\mathrm{Bun}_G, \Lambda)$. In particular, the Hecke operator T_μ acts as (the spectral action of) a vector bundle $\mathcal{V}_\mu$ on $X_{\widehat{G}}$. This leads to an action of the endomorphism algebra of $\mathcal{V}_\mu$ on $R\Gamma(\mathbf{Sh}_{K^p,\overline{E}}, \Lambda) = i_1^* T_\mu \mathcal{F}[-d](-\frac{d}{2})$. Since W_E acts on $\mathcal{V}_\mu$ via the universal L-parameter, through this it also acts on $R\Gamma(\mathbf{Sh}_{K^p,\overline{E}}, \Lambda)$. One can show that this spectral W_E-action coincides with the natural W_E-action coming from the structure map of $\mathbf{Sh}_{K^p}$ to $\mathrm{Spec} E$. Now any Frobenius lift, as an endomorphism of $\mathcal{V}_\mu$, has a characteritic polynomial with coefficients in the spectral Bernstein center. Passing to Iwahoric levels, the latter acts on $R\Gamma(\mathbf{Sh}_{K^p I_p}, \Lambda)$ through the Iwahoric-Hecke algebra, and this polynomial reduces to the Hecke polynomial. The theorem follows.

References

[1] P. Daniels, P. van Hoften, D. Kim and M. Zhang, *Igusa Stacks and the Cohomology of Shimura Varieties*, arXiv e-prints, arXiv:2408.01348, (2024).

[2] M. Zhang, *A PEL type Igusa stack and the p-adic Geometry of Shimura Varieties*, arXiv e-prints, arXiv: 2309.05152 (2023).

[3] L. Fargues and P. Scholze, *Geometrization of the local Langlands correspondence*, arXiv e-prints, arXiv:2102.13459v3 (2024).

[4] E. Mantovan, *On the cohomology of certain PEL-type Shimura varieties*, Duke Math. J,**129**, (2005), 573–610.

[5] P. Hamacher and W. Kim, *l-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients*, Math. Ann., **375**, (2019), 973–1044.

[6] L. Hamann and S. Lee, *Torsion Vanishing for Some Shimura Varieties*, arXiv e-prints, arXiv:2309.08705, (2023).

[7] L. Xiao and X. Zhu, *Cycles on Shimura varieties via geometric Satake*, arXiv e-prints, arXiv:1707.05700, (2017).

[8] T. Koshikawa, *Eichler–Shimura relations for local Shimura varieties*, arXiv e-prints, arXiv: 2106.10603, (2021).

[9] D. Blasius and J. Rogawski, *Zeta functions of Shimura varieties*,Motives (Seattle, WA, 1991),Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, **55, Part 2** (1994), pages 525–571.

[10] S-Y. Lee, *Eichler–Shimura Relations for Shimura Varieties of Hodge Type*, arXiv e-prints, arXiv:2006.11745, (2021).

[11] L. Hamann, *Geometric Eisenstein Series, Intertwining Operators, and Shin's Averaging Formula*, arXiv e-prints, arXiv:2209.08175, (2023).

[12] A. Caraiani and P. Scholze, Peter, *On the generic part of the cohomology of compact unitary Shimura varieties*,Ann. of Math. (2),**186**, (2017), pages 649–766.

[13] A. Caraiani and P. Scholze, Peter, *On the generic part of the cohomology of non-compact unitary Shimura varieties*, Ann. of Math. (2), **199, 2**, (2024), pages 483–590.

[14] T. Koshikawa, *On the generic part of the cohomology of local and global Shimura varieties*, arXiv e-prints, arXiv:2106.10602, (2021).

Special points on Shimura varieties defined over characteristic p and p-adic fields

Ananth Shankar
(joint work with Benjamin Bakker, Jacob Tsimerman)

$\mathcal{A}_g$ and Shimura varieties of abelian type. Let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties. A point $x \in \mathcal{A}_g$ is said to be special if the abelian variety A_x associated to x has CM, i.e. if its Endomorphism algebra contains a degree 2g CM algebra. This definition applies to points defined over fields of any characteristic. Grothendieck proves that every special point $x \in \mathcal{A}_g(\mathbb{C})$ is defined over $\overline{\mathbb{Q}}$, and such a point has (potentially) good reduction at all places. Further, an arbitrary $\overline{\mathbb{Q}}$-valued point of $\mathcal{A}_g$ is *not* special.

Moving to the case of positive characteristic, Tate in [12] proves that every abelian variety over a finite field is CM, and therefore every point $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ is a special point. There are also examples of abelian varieties whose minimal field of definition has positive transcendence degree over $\mathbb{F}_p$ that have CM – in other words, a word-to-word characteristic p analogue of Grothendieck's theorem is false. Such abelian varieties necessarily cannot admit CM lifts to characteristic zero. A natural question that arises from these phenomena is the following:

Question: Does every abelian variety over a finite field admit a CM lift to characteristic zero?

Serre-Tate show that the answer is **yes** for *ordinary* abelian varieties, and in fact ordinary abelian varieties admit *canonical* CM lifts. Work of Deuring shows that the answer remains **yes** for *almost ordinary*[1] abelian varieties. Honda-Tate show that the answer is **yes** but only up to isogeny, i.e. that an abelian variety over a finite field is isogenous to an abelian variety that admits a CM lift. All these results are also true at the level of principally polarized abelian varieties – i.e. at the level of points in $\mathcal{A}_g$.

However, the answer to the question in general is no! In [9], Oort shows the existence of abelian varieties over finite fields that do not admit CM lifts. In [2], Chai-Conrad-Oort prove stronger results, and demonstrate the existence of supersingular abelian varieties that do not admit CM lifts. In [6], the authors show that the set of points in $\mathcal{A}_g(\overline{\mathbb{F}}_p)$ that admit lifts to special points are contained in finitely many central leaves (see [10]) for the definition of central leaves.

These theorems all have analogues in the setting of Shimura varieties of Hodge (and abelian) type. Let S be a Shimura variety of Hodge[2] type and let p be a prime of good reduction for S. The notion of a point being special is the same as for $\mathcal{A}_g$, and therefore Tate's theorem shows that every $\overline{\mathbb{F}}_p$-point of S is special. The results about lifts to special points is by no means formal, as the results for $\mathcal{A}_g$ a-priori only yield that an $\overline{\mathbb{F}}_p$-point of S admits a CM lift to some point in

[1]A g-dimensional abelian variety A is said to be almost ordinary if the étale part of $A[p]$ has rank $g-1$.

[2]The same results hold for Shimura varieties of abelian type.

$\mathcal{A}_g$. However, the results of [8], [7], and [11] show $\overline{\mathbb{F}}_p$-valued points of S contained in the μ-ordinary locus (analogue of the ordinary locus) admit canonical lifts to special points. Kisin [5] proves a CM-lifting upto isogeny theorem for such Shimura varieties. The negative results of [9], [2], and [6] formally apply to this setting.

Along similar lines, work of Ito-Ito-Koshikawa [4] and Yang [13] independently show that non-supersingular K3 surfaces admit CM lifts to characteristic zero, and in [6], the authors show that only finitely many supersingular K3 surfaces admit CM lifts. These results make no reference to polarizations, and therefore do not follow from the case of $\mathcal{A}_g$.

Exceptional Shimura varieties. Let (G, X) denote a Shimura datum that is of exceptional type (i.e. not of abelian type) with reflex field E. Let S/E denote the Shimura variety associated to (G, X). For sufficiently large primes p, we may reduce S mod v where v is a place of E dividing p. A representation V of G gives rise to motivic data on S. Specifically, we obtain a filtered flat bundle (satisfying Griffiths transersality), ℓ-adic étale local systems, and a Fontaine-Laffaile module (by work of Esnault-Groechenig [3]) at large enough primes p.

It is a-priori unclear how to even define the notion of special points in positive characteristic. In characteristic zero, the definition of special points defined is Hodge theoretic and therefore doesn't generalize. As S is not known to carry families of varieties, Tate's theorem does not apply. The results of [9], [2] and [6] show that defining a point to be special if it is the mod p reduction of a special point in characteristic zero is not reasonable.

However, a complex point being special can be detected at the level of Hecke correspondences. To that end, we let τ_h denote the Hecke correspondence for some $h \in G(\mathbb{Q}_\ell)$. We make the following definition.

Definition 1. *Let K be a field and let $x \in S(K)$ be some point. Then x is special if there exists a finite set of points $\{x_1, x_2, \dots x_n\}$ and infinitely many primes ℓ and split maximal tori $T_\ell \subset G \otimes \mathbb{Q}_\ell$, such that $\tau_h(x) \cap \{x_1 \dots x_n\}$ is non-empty for every $h \in T_\ell(\mathbb{Q}_\ell)$.*

This definition is equivalent to the classical definitions in characteristic zero. The advantage of this definition is that it applies equally well in any characteristic. With this in hand, we prove the following results in [1].

Theorem 1 (Bakker-S-Tsimerman)**.** *Let p be a prime large enough prime.*

1. *The ℓ-adic Frobenius conjugacy class at every closed point $x \in S(\mathbb{F}_q)$ is semisimple. Further, every $\overline{\mathbb{F}}_p$-point of S is a special point.*
2. *The μ-ordinary locus of S* mod v *is non-empty.*
3. *Every μ-ordinary $\overline{\mathbb{F}}_p$-point of S admits a canonical lift to a characteristic zero special point of S.*
4. *Let x and y be ordinary points whose ℓ-adic Frobenii are conjugate. Then the canonical lifts of x and y give rise to isomorphic rational Hodge structures.*

The last part of this theorem should be regarded as an ordinary analogue of Tate's isogeny theorem.

References

[1] B. Bakker, A.N. Shankar, J. Tsimerman. *Integral canonical models of exceptional Shimura varieties.* arXiv:2405.12392.

[2] C-L. Chai, B. Conrad, F. Oort. *Complex multiplication and lifting problems*, Mathematical Surveys and Monographs, vol. 195, (2014).

[3] H. Esnault, M. Groechenig. *Rigid connections and F-isocrystals*, Acta Math. **225** (2020), no.1, 103–158.

[4] K. Ito, T. Ito, T. Koshikawa. *CM liftings of K3 surfaces over finite fields and their applications to the Tate conjecture*, Forum Math. Sigma **9** (2021), Paper No. e29.

[5] M. Kisin. mod p *points on Shimura varieties of abelian type.* J. Amer. Math. Soc. **30** (2017), no.3, 819–914.

[6] M. Kisin, Y.J.H. Lam, A.N. Shankar, P. Srinivasan. *Finiteness of reductions of Hecke orbits.*, J. Eur. Math. Soc., DOI 10.4171/JEMS/1395.

[7] B. Moonen. *Serre-Tate theory for moduli spaces of PEL type*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no.2, 223–269.

[8] R, Noot. *Models of Shimura varieties in mixed characteristic*, J. Algebraic Geom. **5** (1996), no.1, 187–207.

[9] F. Oort. *CM-liftings of abelian varieties*, J. Algebraic Geom. **1** (1992), no. 1, 131–146.

[10] F. Oort. *Foliations in moduli spaces of abelian varieties*, J. Amer. Math. Soc. **17** (2004), no. 2, 267–296.

[11] A.N. Shankar, R. Zhou. *Serre-Tate theory for Shimura varieties of Hodge type,* Math. Z. **297** (2021), no.3–4, 1249–1271.

[12] J. Tate. *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144.

[13] Z. Yang. *Isogenies between K3 surfaces over* $\overline{\mathbb{F}}_p$. Int. Math. Res. Not. (2022), no.6, 4407–4450.

Analytic prismatization

Juan Esteban Rodríguez Camargo

(joint work with Johannes Anschütz, Arthur-César le Bras and Peter Scholze)

Prismatic cohomology [5] is a unifying integral p-adic cohomology theory for p-adic formal schemes. In [6] and [4] Drinfeld and Bhatt-Lurie have introduced a geometrization of prismatic cohomology in the form of the prismatization of formal schemes. As consequence, the comparison between prismatic cohomology with other integral cohomology theories (eg. de Rham, crystalline, Hodge-Tate) is explained from geometric features of the prismatization.

Motivated from the integral theory of the prismatization, some recent developments on the p-adic Simpson correspondence [1, 2], the theory of solid locally analytic representations of p-adic Lie groups [9, 10] and its novel interactions with p-adic automorphic forms [8, 11], we have been pursuing a theory of analytic prismatization that is adapted to rigid geometry instead. In the process to realize this picture we work in the framework of condensed mathematics and analytic geometry of Clausen and Scholze.

Let C be a complete algebraically closed non-archimedean field over $\mathbb{Q}_p$ and X a smooth rigid space over C (or a more general adic space). Consider $Y_C = \mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}(O_C))\backslash V(p[p^\flat])$ the Y-curve attached to C. The (rational) analytic prismatization of X (relative to Y_C) is a suitable Frobenius equivariant analytic stack over Y_C

$$f : X^{\Delta\!\!\!\!\Delta/Y_C,an} \to Y_C.$$

The analytic prismatic cohomology (relative to Y_C) is nothing but the (derived) pusforward of the structural sheaf of $X^{\Delta\!\!\!\!\Delta/Y_C,an}$ along f.

In order to describe $X^{\Delta\!\!\!\!\Delta/Y_C,an}$ and explain its relationship with other cohomology theories we need three key constructions:

- Let $X^{\diamondsuit}$ be the diamond attached to X. Via descent from perfectoids one can construct an analytic stack $Y_{X^\diamond}$ over Y_C which informally can be understood as a family of Y-Fargues-Fontaine curves over $X^{\diamondsuit}$. The theory of solid quasi-coherent sheaves on $Y_{X^\diamond}$ is (essentially) the one constructed by Anschütz-le Bras-Mann in their forthcoming work on a six functor formalism for v-stacks with values in quasi-coherent sheaves over Fargues-Fontaine curves. Furthermore, they proved a Riemann-Hilbert correspondence for $\mathbb{Q}_p$-local systems and étale vector bundles in the Fargues-Fontaine curve $\mathrm{FF}_{X^\diamond} := Y_{X^\diamond}/\varphi$.

  It turns out that there is a natural map of analytic stacks over Y_C

  $$Y_{X^\diamond} \to X^{\Delta\!\!\!\!\Delta/Y_C,an}$$

- Given Z an adic space (or more general analytic stacks over $\mathbb{Q}_p$), in [12] it is defined the analytic de Rham stack $Z^{dR,an}$, an analytic analogue of Simpson's de Rham stack in algebraic geometry [13]. Quasi-coherent sheaves on the analytic de Rham stack of X, also called analytic D-modules, are an enhancement of the category of coadmissible D-cap modules of Ardakov-Wadsley [3]. The analytic de Rham stack has the virtue that even spaces without differentials (like perfectoid spaces or Fargues-Fontaine curves) have a good theory of analytic D-modules. For a general analytic stack Z one has a natural map towards its analytic de Rham stack

  $$Z \to Z^{dR,an}.$$

  For a morphism $Z \to W$ of analytic stacks one defines the relative de Rham stack of Z over W to be the pullback

  $$\begin{array}{ccc} Z^{dR/W,an} & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z^{dR,an} & \longrightarrow & W^{dR,an}. \end{array}$$

- Finally, the analytic Hodge-Tate stack of X is defined to be the pullback

  $$\begin{array}{ccc} X^{HT/C,an} & \longrightarrow & X^{\Delta\!\!\!\!\Delta/Y_C,an} \\ \downarrow & & \downarrow \\ \mathrm{AnSpec}(C) & \longrightarrow & Y_C. \end{array}$$

  The analytic Hodge-Tate stack has a natural map $X^{HT/C,an} \to X$, when X is a perfectoid space this map is an isomorphism.

The following theorem summarizes the main rational cohomology comparisons for X:

Theorem 1 (In progress)**.** *Let X be an adic space over $\mathbb{Q}_p$ and consider the diagram of Frobenius-equivariant analytic stacks over Y_C*

$$\begin{array}{ccccc} Y_{X^\diamond} & \xrightarrow{f} & X^{\mathbb{\Delta}/Y_C,an} & \xrightarrow{g} & (X^{\mathbb{\Delta}/Y_C,an})^{dR/Y_C,an} \\ & \searrow & \downarrow{\scriptstyle \pi} & \swarrow & \\ & & Y_C & & \end{array}$$

Let $x \in Y_C$ be the Hodge-Tate point corresponding to the divisor $HT: \mathrm{AnSpec}(C) \to Y_C$ and let $U = Y_C \backslash \{\varphi^{-n}(x) \colon n \in \mathbb{N}\}$.

The following hold:

(1) ***de Rham comparison.*** *We have an equivalence of analytic de Rham stacks*

$$(Y_{X^\diamond})^{dR/Y_C,an} \xrightarrow{\sim} (X^{\mathbb{\Delta}/Y_C,an})^{dR/Y_C,an}.$$

Moreover, on the open locus $U \subset Y_C$ the natural map

$$X^{\mathbb{\Delta}/Y_C,an}|_U \xrightarrow{\sim} (X^{\mathbb{\Delta}/Y_C,an})^{dR/Y_C,an}|_U$$

is an equivalence. In particular, the fiber at $\varphi(x) \in Y_C$ of π is isomorphic to the analytic de Rham stack $X^{dR/C,an}$.

(2) ***Hodge-Tate comparison.*** *Let X be a smooth rigid space over C and let $X^{HT/C,an}$ be its analytic Hodge-Tate stack. Let T_X be the tangent space of X and $T_X^\dagger \subset T_X$ the overconvergent neighbourhood of the zero section. Then the natural map $X^{HT/C,an} \to X$ is a gerbe banded by $T_X^\dagger(1)$ where* (1) *refers to a Tate twist.*

(3) ***Proétale comparison.*** *Let X be a smooth rigid space. Let $X^{\mathbb{\Delta}^{perf}/Y_C,an} = \varprojlim_\varphi X^{\mathbb{\Delta}/Y_C,an}$ be the perfection of the analytic prismatization and write $f' : Y_{X^\diamond} \to X^{\mathbb{\Delta}^{perf}/Y_C,an}$ for the induced map. Then the pullback $f'^* : D(X^{\mathbb{\Delta}^{perf}/Y_C,an}) \to D(Y_{X^\diamond})$ of solid quasi-coherent sheaves is fully faithful and gives rise an equivalence of vector bundles.*

To conclude we briefly mention some extensions of the analytic prismatization to a somehow integral analytic theory and the expected relationship with the analytic prismatization of Drinfeld and Bhatt-Lurie.

- We expect the equivalence of vector bundles of Theorem 1 (3) to also hold for perfect and pseudo-coherent complexes. The smooth hypothesis is sufficient though we do not know exactly for which class of rigid spaces the theorem holds.
- One can define an extension of the analytic prismatization to suitable analytic stacks with a pseudo-uniformizer over $\mathbb{Z}_p$, also taking values in analytic stacks over $\mathbb{Z}_p$. Some additional (expected) features are the following:

  - The Hodge-Tate stack admits an extension to characteristic p. This theory is an analytic analogue of the Hodge-Tate stack for schemes over $\mathbb{F}_p$ which is related to the Ogus-Vologodsky correspondence.
  - We expect this theory to also capture an incarnation of rigid cohomology when taking the generic fiber of the prismatization of a rigid space in characteristic p.
  - The mod p-fiber of the analytic prismatization of rigid spaces over $\mathbb{Q}_p$ is expected to compute (pro)étale cohomology modulo p. Furthermore, its theory of solid quasi-coherent sheaves should make appear the theory of locally analytic representations modulo p and decompleted (φ, Γ)-modules.
- Let I be the Hodge-Tate divisor. Given a p-adic formal scheme $\mathfrak{X}$, we expect the prismatization of Drinfeld and Bhatt-Lurie to be recovered from the (p, I)-adic completion of the analytic prismatization of $\mathfrak{X}$. In this way the analytic prismatization will be a decompletion of the prismatization.
- The analytic prismatization provides a way to implement the geometrization of the p-adic Langlands program for locally analytic representations following [7]. We expect this theory to give some light on the still mysterious properties of the correspondence.

References

[1] *J. Anschütz, B. Heuer, A-C. le Bras* , *v-vector bundles on p-adic fields and Sen theory via the Hodge-Tate stack*, Preprint, arXiv:2211.08470 [math.NT] (2022)

[2] *J. Anschütz, B. Heuer, A-C. le Bras* , *Hodge-Tate stacks and non-abelian p-adic Hodge theory of v-perfect complexes on rigid spaces*, Preprint, arXiv:2302.12747 [math.AG] (2023)

[3] *K. Ardakov* and *S. J. Wadsley*, *D-cap modules on rigid analytic spaces I*, J. Reine Angew. Math. 747, 221–275 (2019; Zbl 1439.14064)

[4] *B. Bhatt* and *J. Lurie*, *The prismatization of p-adic formal schemes*, Preprint, arXiv: 2201.06124 [math.AG] (2022)

[5] *B. Bhatt* and *P. Scholze*, *Prisms and prismatic cohomology*, Ann. Math. (2) 196, No. 3, 1135–1275 (2022; Zbl 07611906)

[6] *V. Drinfeld*, *Prismatization*, Sel. Math., New Ser. 30, No. 3, Paper No. 49, 150 p. (2024; Zbl 07852827)

[7] *L. Fargues* and *P. Scholze*, *Geometrization of the local Langlands correspondence*, Preprint, arXiv:2102.13459 [math.RT] (2021)

[8] *L. Pan*, *On locally analytic vectors of the completed cohomology of modular curves*, Forum Math. Pi 10, Paper No. e7, 82 p. (2022; Zbl 1497.11135)

[9] *J. Rodrigues Jacinto* and *J. E. Rodríguez Camargo*, *Solid locally analytic representations of p-adic Lie groups*, Represent. Theory 26, 962–1024 (2022; Zbl 1510.11118)

[10] *J. Rodrigues Jacinto* and *J. E. Rodríguez Camargo*, *Solid locally analytic representations*, Preprint, arXiv:2305.03162 [math.RT] (2023)

[11] *J. E. Rodríguez Camargo*, *Locally analytic completed cohomology*, Preprint, arXiv: 2209.01057 [math.NT] (2022)

[12] *J. E. Rodríguez Camargo*, *The analytic de Rham stack in rigid geometry*, Preprint, arXiv:2401.07738 [math.AG] (2024)

[13] *C. Simpson*, *Homotopy over the complex numbers and generalized de Rham cohomology*, Lect. Notes Pure Appl. Math. 179, 229–263 (1996; Zbl 0858.18008)

Eisenstein Series over Fargues-Fontaine curves

Linus Hamann
(joint work with David Hansen and Peter Scholze)

Let $\ell \neq p$ be distinct primes and G/F a quasi-split connected reductive group over a non-archimedean local field. We fix a Borel B with maximal torus T, and a parabolic $P \subset G$ which we assume to be standard with respect to the choice of Borel. We write M for the Levi factor and N for the unipotent radical. Given a smooth representation (π_M, V) of $M(F)$ on a $\overline{\mathbb{F}}_\ell$-vector space V, we recall that we can form the parabolic induction

$$i_P^G(\pi_M) := \{f : G(F) \to V | f(mng) = \delta_P^{1/2}(m)\pi(m)f(g)\},$$

where f is assumed to be locally constant and invariant under right translation by a compact open subgroup of $G(\mathbb{Q}_p)$, $\delta_P^{1/2}$ is the modulus character of P, $m \in M(F)$, $n \in N(F)$, and $g \in G(F)$. The vector space $i_P^G(\pi_M)$ has the structure of a smooth $G(F)$-representation.

Theorem 1. *The following is true.*

(1) The functor $i_P^G(-)$ takes admissible (resp. finitely generated and projective) smooth representations of $M(F)$ to admissible (resp. finitely generated and projective) smooth representations of $G(F)$.

(2) The functor $i_P^G(-)$ admits a left adjoint given by the Jacquet module r_P^G, as well as a right adjoint given by the Jacquet module of the opposite parabolic $r_{P^-}^G$.

(3) We have natural isomorphisms

$$\mathbb{D}_{\mathrm{coh}} i_P^G(-) \simeq i_{P^-}^G \mathbb{D}_{\mathrm{coh}}(-)$$

and

$$(r_P^G(-))^\vee \simeq r_{P^-}^G((-)^\vee),$$

where $(-)^\vee$ denotes smooth duality and $\mathbb{D}_{\mathrm{coh}}$ denotes Bernstein's cohomological duality functor (cf. [1, Definition 5.1]*).*

For an account of such results in very general coefficient systems (which in this level of generality was only proven very recently, by using the Fargues-Scholze correspondence and its properties!), we point the reader to [4].

We study the analogues of these foundational results in the context of the geometrization of the local Langlands correspondence of Fargues-Scholze [3]. To understand this, we have the following dictionary between classical smooth representation theory and the geometric local Langlands correspondence.

- The category of smooth representations of $G(F)$ on $\overline{\mathbb{F}}_\ell$-vector spaces is replaced by the category $\mathrm{D}(\mathrm{Bun}_G) := \mathrm{D}(\mathrm{Bun}_G, \overline{\mathbb{F}}_\ell)$ of étale $\overline{\mathbb{F}}_\ell$-sheaves on Bun_G.
- The subcategory of admissible representations is replaced by the full subcategory $\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_G) \subset \mathrm{D}(\mathrm{Bun}_G)$ of sheaves which are ULA over the point.

- The subcategory of finitely generated projective representations is replaced by the subcategory of compact objects $D^{\omega}(\mathrm{Bun}_G) \subset D(\mathrm{Bun}_G)$ inside the compactly generated category $D(\mathrm{Bun}_G)$.
- The smooth duality functor is replaced by Verdier duality denoted

$$\mathbb{D} : D^{\mathrm{ULA}}(\mathrm{Bun}_G) \xrightarrow{\simeq} D^{\mathrm{ULA}}(\mathrm{Bun}_G)^{\mathrm{op}}.$$

- The cohomological duality $\mathbb{D}_{\mathrm{coh}}$ Bernstein is replaced by the categorical Bernstein-Zelevinsky duality of Fargues- Scholze, denoted

$$\mathbb{D}_{\mathrm{BZ}} : D(\mathrm{Bun}_G)^{\omega} \xrightarrow{\simeq} D(\mathrm{Bun}_G)^{\omega,\mathrm{op}}.$$

One can recover the classical notions in smooth representation theory from their geometric analogues. In particular, the moduli stack Bun_G admits a open Harder-Narasimhan (abbv. HN) strata $j_{1_G} : \mathrm{Bun}_G^{1_G} \hookrightarrow \mathrm{Bun}_G$ corresponding to the locus defined by the trivial G-bundle. Moreover, one has an isomorphism $\mathrm{Bun}_G^{1_G} \simeq [*/\underline{G(F)}]$ between the neutral HN-strata and the classifying stack attached to $\underline{G(F)}$. This gives rise to an isomorphism $D(\mathrm{Bun}_G^{1_G}) \simeq D(G(F), \overline{\mathbb{F}}_\ell)$, where $D(G(F), \overline{\mathbb{F}}_\ell)$ denotes the left-complete derived category of smooth representations, as well as a fully faithful embedding

$$j_{1_G!}(-) : D(G(F), \overline{\mathbb{F}}_\ell) \hookrightarrow D(\mathrm{Bun}_G).$$

One can check that the above notions restrict to the classical ones when applied to this full subcategory.

It remains to explain what the parabolic induction functors $i_P^G(-)$ and Jacquet module functors $r_P^G(-)$ correspond to under this dictionary. This is given by studying the diagram

$$\mathrm{Bun}_M \xleftarrow{\mathfrak{q}_P} \mathrm{Bun}_P \xrightarrow{\mathfrak{p}_P} \mathrm{Bun}_G$$

induced by the diagram of groups

$$M \leftarrow P \rightarrow G.$$

The moduli stack Bun_P is ℓ-cohomologically smooth of some fixed pure ℓ-dimension after pulling back to a connected components of Bun_M. We denote the function given by this ℓ-dimension by $\dim(\mathrm{Bun}_P)$. We define

$$\mathrm{IC}_{\mathrm{Bun}_P} := \mathfrak{q}_P^*(\Delta_P^{1/2})[\dim(\mathrm{Bun}_P)],$$

where $\Delta_P^{1/2}$ is a sheaf given by a choice of square root of the modulus character. It is a Theorem [5, Theorem 1.5] that $\mathrm{IC}_{\mathrm{Bun}_P}^{\otimes 2}$ is the dualizing sheaf on Bun_P, which implies that $\mathrm{IC}_{\mathrm{Bun}_P}$ is Verdier self-dual on Bun_P. Using this, we then define the Eisenstein functors

$$\mathrm{nEis}_{P!}(-) := \mathfrak{p}_{P!}(\mathfrak{q}_P^*(-) \otimes \mathrm{IC}_{\mathrm{Bun}_P})$$

and

$$\mathrm{nEis}_{P*}(-) := \mathfrak{p}_{P*}(\mathfrak{q}_P^*(-) \otimes \mathrm{IC}_{\mathrm{Bun}_P}),$$

as well as the constant term functors

$$\mathrm{CT}_{P*}(-) := \mathfrak{q}_*(\mathfrak{p}^!(-) \otimes \mathrm{IC}_{\mathrm{Bun}_P}^{-1})$$

and

$$\mathrm{CT}_{P!}(-) := \mathfrak{q}_!(\mathfrak{p}^*(-) \otimes \mathrm{IC}_{\mathrm{Bun}_P}),$$

which satisfy the adjunction relationships

$$(\mathrm{CT}_{P!}, \mathrm{nEis}_{P*}) \text{ and } (\mathrm{nEis}_{P!}, \mathrm{CT}_{P*}).$$

These recover the classical Jacquet and parabolic induction functors after restricting to the locus defined by the trivial bundle. In particular, if j_{1_M} denotes the inclusion of the neutral HN-strata inside Bun_M, one easily verifies the following.

Lemma 2. *We have natural isomorphisms*

(1) $\mathrm{Eis}_{P!} j_{1_M !}(-) \simeq j_{1_G !} i_P^G(-)$,
(2) $\mathrm{Eis}_{P*} j_{1_M !}(-) \simeq j_{1_G *} i_P^G(-)$,
(3) $j_{1_M}^* \mathrm{CT}_{P!}(-) \simeq r_P^G j_{1_G}^*(-)$, *and*
(4) $j_{1_M}^* \mathrm{CT}_{P*}(-) \simeq r_{P-}^G j_{1_G}^*(-)$.

This motivates our main theorem which is the geometrization of the results discussed in Theorem 1.

Theorem 3. [2, Theorem 1.1.1] *For a parabolic $P \subset G$ with Levi factor M, the following is true.*

(1) The functor $\mathrm{Eis}_{P!}$ *preserves compact objects. If $A \in D(\mathrm{Bun}_M, \Lambda)$ is ULA and supported on finitely many connected components, then* $\mathrm{Eis}_{P!}(A)$ *and* $\mathrm{Eis}_{P*}(A)$ *are ULA.*
(2) The functor CT_{P*} *preserves ULA objects. If $A \in D(\mathrm{Bun}_G, \Lambda)$ is compact, then* $\mathrm{CT}_{P!}(A)|_{\mathrm{Bun}_M^\alpha}$ *is compact for all $\alpha \in \pi_0(\mathrm{Bun}_M)$.*
(3) There is a canonical isomorphism of functors $\mathrm{CT}_{P!} \cong \mathrm{CT}_{P^-*}$.
(4) We have the following duality isomorphisms:
  i. $\mathbb{D}_{\mathrm{BZ}} \mathrm{Eis}_{P!} \cong \mathrm{Eis}_{P^-!} \mathbb{D}_{\mathrm{BZ}}^M$ *on compact objects.*
  ii. $\mathbb{D}_{\mathrm{Verd}} \mathrm{Eis}_{P!} \cong \mathrm{Eis}_{P*} \mathbb{D}_{\mathrm{Verd}}^M$ *on all objects.*
  iii. $\mathbb{D}_{\mathrm{Verd}}^M \mathrm{CT}_{P!} \cong \mathrm{CT}_{P*} \mathbb{D}_{\mathrm{Verd}}$ *on all objects.*

References

[1] J. Bernstein, *Representations of p-adic groups*, Lecture Notes at Harvard University, (1992).
[2] L. Hamann, D. Hansen and P. Scholze, *Geometric Eisenstein Series I: Finiteness Theorems*, In preperation.
[3] L. Fargues and P. Scholze, *Geometrization of the local Langlands Correspondence*, arXiv:2102.13459 (2021).
[4] J.F Dat, D. Helm, R. Kurinczuk, and G. Moss *Finiteness for Hecke Algebras of p-adic groups*, arXiv. 2203.04929 (2022).
[5] L. Hamann and N. Imai, *Dualizing Complexes on the moduli of parabolic bundles*, arXiv: 2401.06342 (2024).

Tate classes and endoscopy for $\mathbf{GSp_4}$

Naomi Sweeting

Let g be a classical cuspidal eigenform of weight two for GL_2; then g has a Galois representation, constructed by Deligne as a quotient

$$H^1(X_1(N)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) \twoheadrightarrow \rho_g$$

for some sufficiently large level N. However, ρ_g also appears in the étale cohomology of many other Shimura varieties; this talk dealt in particular with the Shimura variety $S_K(\mathrm{GSp}_4)$, which is a three-dimensional moduli space of principally polarized abelian surfaces with some level structure determined by an open compact subgroup $K \subset \mathrm{GSp}_4(\mathbb{A}_f)$. Consider the decomposition of the étale cohomology

$$H^3_{\mathrm{et}}(S_K(\mathrm{GSp}_4)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) = \bigoplus \Pi_f^K \otimes \rho_{\Pi_f},$$

with Π_f the finite part of an automorphic representation of GSp_4 and ρ_{Π_f} a Galois representation. When it is nonzero, ρ_{Π_f} is typically four-dimensional and irreducible [1]. But for certain Π_f corresponding to endoscopic Yoshida lifts, ρ_{Π_f} will be a Tate twist of the two-dimensional representation ρ_g [3]; by Poincaré duality and the Kunneth formula, one deduces the existence of Galois-invariant étale cohomology classes in middle degree four on the product $S_K(\mathrm{GSp}_4) \times X_0(N)$.

The main question of this talk is when these classes arise from algebraic cycles, as predicted by the Tate conjecture. It turns out that a natural special cycle class on $S_K(\mathrm{GSp}_4)$ accounts for *some*, but not all, of the Galois-invariant classes: it only sees the ones corresponding to *globally generic* automorphic representations of GSp_4 [2]. The automorphic representations of GSp_4 are organized by the Langlands program into L-packets, each of which has a unique generic member, so the failure of the special cycle to generate all of the Tate classes of interest is closely related to the existence of nontrivial packet structure on GSp_4. In fact, an analogous result holds over totally real fields, and for Galois-invariant classes in étale cohomology with coefficients in certain automorphic local systems.

In the non-generic case, it is not known whether the Tate classes arise from algebraic cycles, because it is quite difficult to construct (or work with) algebraic cycles that are not special. However, one can at least show that all the Galois-invariant classes arise from Hodge classes under the Betti-étale comparison isomorphism [2]. These Hodge classes are constructing using non-tempered theta lifts on the group GSp_6. The strategy is inspired by the groundbreaking work of Ichino and Prasanna [4], which showed that Tate classes reflecting the Jacquet-Langlands transfer between inner forms of GL_2 also arise from Hodge classes. In the second half of the talk, I gave a schematic overview of the theory of theta lifting, which I then used to sketch a proof of the main results.

References

[1] R. Weissauer, *Four dimensional Galois representations*, Astérisque **302** (2005), 67–150.
[2] N. Sweeting, *Tate classes and endoscopy for GSp4 over totally real fields*, arXiv preprint 2211.10838 (2022).

[3] R. Weissauer, *Endoscopy for GSp (4) and the cohomology of Siegel modular threefolds* (2009), Springer.
[4] A. Ichino and K. Prasanna, *Hodge classes and the Jacquet-Langlands correspondence*, Forum of Mathematics, Pi **11** (2023), 1–135.

p-adic integrals and plethystic logarithms

Michael Groechenig
(joint work with Dimitri Wyss, Paul Ziegler)

Let F denote a non-archimedean local field of mixed characteristic and $\mathcal{O}_F$ its ring of integers with residue field $k = \mathbb{F}_q$. It was observed by Weil [9] that for a smooth $\mathcal{O}_F$-scheme there exists a canonical measure μ on $X(\mathcal{O}_F)$ with the property

$$\mu(\mathsf{sp}^{-1}(\bar{x})) = q^{-\dim_{\bar{x}} X_k},$$

where $\bar{x} \in X(k)$, $\mathsf{sp}\colon X(\mathcal{O}_F) \to X(k)$ denotes the specialisation map, and $\dim_{\bar{x}} X_k$ refers to the Krull dimension of the local ring $\mathcal{O}_{X_k,\bar{x}}$. The construction of this measure utilises that the set of F-points $X(F)$ is endowed with the structure of an F-analytic manifold, on which we have a well-defined integration theory of densities (= p-adic absolute values of top-degree differential forms).

This elementary construction can be of great use for certain point-counting problems over finite fields. Indeed, if $X/\mathcal{O}_F$ is smooth and of pure relative dimension d, we have

$$\mu(X(\mathcal{O}_F)) = \frac{|X(k)|}{q^d}.$$

That is, the volume of $X(\mathcal{O}_F)$ counts the numbers of k-points of the special fibre X_k. This insight was exploited by Batyrev [2] to show that birational Calabi-Yau varieties (assumed to be smooth and projective) have equal zeta function over finite fields, and thus, equal Betti numbers. As was noticed by Ito [7], the same strategy can be used to infer equality of Hodge numbers.

Analogues of Weil's canonical measure (henceforth referred to as p-adic integration) can also be defined for certain singular varieties. A well-understood case is given by coarse moduli spaces $X/\mathcal{O}_F$ of a smooth and tame DM-stack $\mathcal{X}/\mathcal{O}_F$. In this case, there exists a canonical measure μ_{orb}, a refined specialisation map $\mathsf{sp}\colon X(\mathcal{O}_F) \to I_{\widehat{\mu}}\mathcal{X}(k) = \mathsf{Maps}(B_k\widehat{\mu}, \mathcal{X}_k)$, which is defined almost everywhere, and allows us to state the following analogous relation to point-counting:

$$\mu_{\mathrm{orb}}(\mathsf{sp}^{-1}(\bar{y})) = \frac{q^{-w(\bar{y})}}{|\mathsf{Aut}_{I_{\widehat{\mu}}\mathcal{X}(k)}(\bar{y})|}.$$

The *weight function* w appearing above, takes values in the rationals and only depends on the tangential $\widehat{\mu}$-action associated to a point of $I_{\widehat{\mu}}\mathcal{X}(k)$.

This formula is closely related to the McKay correspondence, in the context of which a motivic analogue was first studied by Denef–Loeser [5] and Yasuda [10]. The p-adic formula above was established (under additional technical assumptions) in our previous work [6], and in the generality above by Angelinos [1] in his thesis.

As a consequence one obtains that the total volume $X(\mathcal{O}_F)$ can be expressed in terms of a weighted point-count of the *twisted inertia stack* $I_{\widehat{\mu}}\mathcal{X}$:

$$\mu_{\text{orb}}(X(\mathcal{O}_F)) = \sum_{\bar{y}\in I_{\widehat{\mu}}\mathcal{X}(k)} q^{-w(\bar{y})}.$$

In [6] this p-adic volume formula was applied to the moduli DM-stack of G-Higgs bundles (with additional assumptions to guarantee smoothness) $\widetilde{\mathcal{M}}_G^{\heartsuit}$. In this case, the twisted inertia stack can be shown to be a disjoint union $\bigsqcup_{^L H} \widetilde{\mathcal{M}}^{\heartsuit}$, where H ranges over the endoscopic groups of $^L G$. The p-adic volume of the coarse moduli space and related p-adic integrals can then be used to offer a proof of the fundamental lemma (as established in Ngô's [8]).

In ongoing work in progress we aim to understand what happens to Weil's formula in the presence of non-quotient singularities as they arise in moduli spaces of objects in abelian categories (e.g., Higgs bundles, coherent sheaves on projective varieties, quiver representations, etc). We have a satisfactory understanding of the analogue of Weil's formula for the case of moduli spaces of objects in hereditary abelian categories with symmetric Euler forms (satisfying additional assumptions to be omitted here).

As a first step, one may consider a linear quotient stack $\mathcal{M} = [U/\mathsf{GL}_N]$, where U is a smooth algebraic $\mathcal{O}_F$-space together with a map $\mathcal{M} \to M$ assumed to be an isomorphism over a dense-open $W \subset M$ with complement of codimension at least 2. In addition one may assume M to be an adequate moduli space of $\mathcal{M}$.

One can then show that there exists a canonical measure μ on $M(\mathcal{O}_F)$, for which we have an analogue of Weil's formula. Furthermore, if $\mathcal{M}$ is endowed with a $\mathbb{G}_m$-gerbe α, we can integrate the Hasse invariant of α with respect to this measure. If the map $\mathcal{M} \to M$ satisfies the so-called *cyclic lifting property*, which asserts that a map $\mathsf{Spec}(\mathcal{O}_F) \to M$ can be lifted to $\mathcal{M}$ after applying a root stack construction to the domain, then we obtain

$$\int e^{2\pi i \text{Hasse}(\alpha)} d\mu = -\lim_{T\to\infty} \sum_{N\geq 1} \Big(\sum_{\bar{y}\in I_{\mu_N}\mathcal{M}(k)} h_\alpha \cdot q^{-w(\bar{y})} \Big) T^N,$$

where $h_\alpha \colon I_{\mu_N}\mathcal{M}(k) \to \mu_N(\mathbb{C})$. It is an open problem to understand for which class of stacks the cyclic lifting property is satisfied. We know it to hold in all examples of interest.

The right-hand side can be understood more concretely under the additional assumption that $\mathcal{M}$ is the $\mathbb{G}_m$-rigidification of a moduli stack of objects in a hereditary symmetric abelian category (e.g., representations of a symmetric quiver). This part of our ongoing work in progress is now entirely combinatorial and takes place over finite base fields. By virtue of its construction as a $\mathbb{G}_m$-rigidification, $\mathcal{M}$ is endowed with a $\mathbb{G}_m$-gerbe α represented by the unrigidified stack of objects $\mathbb{M}$. This gerbe is also known as the *obstruction gerbe*, as it measures the obstruction to the existence of a universal family of objects on $\mathcal{M}$.

Given a k-linear abelian category $\mathcal{A}$ satisfying the aforementioned assumptions with a moduli stack of objects $\mathbb{M}$, we construct a λ-ring of counting functions

$CF(\mathbb{M})$. Although counting functions can be realised as functions on the set of isomorphism classes of $\mathbb{M}(\bar{k})$, their precise definition will be omitted here for brevity. We remark that at every k-point of $\mathbb{M}$ a counting function can be evaluated as an element of $\overline{\mathbb{Q}}^{\mathbb{N}}$. The ring structure is defined using convolution of counting functions with respect to the direct sum operation $\oplus$.

Our main result then expresses the counting function analogue of the right-hand side of our volume formula

$$\mathcal{F}\colon \bar{x} \mapsto \pm \mathbb{L}^{\frac{(\bar{x},\bar{x})+1}{2}} \cdot \lim_{T\to\infty} \sum_{N\geq 1} \Big(\sum_{\beta\colon \mu_N \to \mathsf{Aut}(\bar{x})} h_\alpha \cdot q^{-w(\beta)} \Big) T^N$$

as a *plethystic logarithm*

$$\frac{\mathcal{F}}{\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}} = \mathsf{Log}\Big(\frac{\mathbb{L}^{(\cdot,\cdot)/2}}{|\mathsf{Aut}(\cdot)|}\Big).$$

This relation is reminiscent of identities arising in Donaldson-Thomas theory (see the work of Davison–Meinhardt [4]). For the abelian categories considered here, it confirms the suggestion of work by Carocci–Orecchia–Wyss [3] that the p-adic integral of the Hasse invariant of the obstruction gerbe α should be related to BPS invariants.

References

[1] P. Angelinos, *p-adic Integration for Derived Equivalent Abstract Hitchin Systems*, PhD thesis, University of Toronto (2023).

[2] V. V. Batyrev, *Birational Calabi-Yau n-folds have equal Betti numbers*, L.M.S Ser. 1 **264** (1999), 1–12.

[3] F. Carocci, G. Orecchia, D. Wyss, *BPS invariants from p-adic integrals*, Compositio Mathematica, **160**(7) (2024), 1525–1550.

[4] B. Davison, S. Meinhardt, *Cohomological Donaldson–Thomas theory of a quiver with potential and quantum enveloping algebras*, Invent. math. **221** (2020), 777–871.

[5] J. Denef, F. Loeser, *Motivic integration, quotient singularities and the McKay correspondence*, Compos. Math. **131**(3) (2002), 267–290.

[6] M. Groechenig, D. Wyss, P. Ziegler, *Geometric stabilisation via p-adic integration*, JAMS **33**, Number 3 (2020), 807–873.

[7] T.Ito, *Stringy Hodge numbers and p-adic Hodge theory*, Compos. Math. **140**(6) (2004), 1499–1517.

[8] B. C. Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publications Mathématiques de l'IHÉS, **111**(1) (2010), 1–169.

[9] A. Weil, *Adeles and Algebraic Groups*, Springer **23**, Berlin (2012).

[10] T. Yasuda, *Motivic integration over Deligne–Mumford stacks*, Adv. Math. **207**(2) (2006), 707–761.

Refined variants of (topological) Hochschild homology

Alexander I. Efimov

In this talk I introduce the refined variants of (topological) Hochschild homology and give examples of their computation. As a special case, we recover the (2-periodized) rigid cohomology of smooth algebraic varieties over a perfect field of positive characteristic.

We first explain the general idea. Let k be a commutative ring. We consider the negative cyclic homology as a functor

$$HC^- : \mathrm{dgcat}^{\mathrm{tr}}_{\mathrm{k}} \to \mathrm{Mod}_{\hat{u}}\text{-}\mathrm{k}[[u]]. \tag{1}$$

Here the source is the ∞-category of small triangulated idempotent-complete dg (differential graded) categories over k. The variable u has cohomological degree 2, and the target of (1) is the category of u-complete dg modules over $\mathrm{k}[[u]]$.

The functor (1) is a finitary localizing invariant in the terminology of [BGT13]. This means that it takes short exact sequences of categories to exact triangles, and it commutes with filtered colimits. It follows that we can consider the negative cyclic homology as a functor from the category of localizing motives over k (the target of the universal finitary localizing invariant):

$$HC^- : \mathrm{Mot}^{\mathrm{loc}}_{\mathrm{k}} \to \mathrm{Mod}_{\hat{u}}\text{-}\mathrm{k}[[u]]. \tag{2}$$

The category $\mathrm{Mot}^{\mathrm{loc}}_{\mathrm{k}}$ has a natural symmetric monoidal structure. In the forthcoming paper [E1] we prove the following result.

Theorem 1. *(*[E1]*) The symmetric monoidal category* $\mathrm{Mot}^{\mathrm{loc}}_{\mathrm{k}}$ *is rigid (in the sense of Gaitsgory and Rozenblyum).*

For a precise definition of rigidity for large (presentable) monoidal categories we refer to [GR17, Definition 9.1.2]. Informally, Theorem 1 states that the category $\mathrm{Mot}^{\mathrm{loc}}_{\mathrm{k}}$ looks like ind-completion of a small rigid category (for small symmetric monoidal categories rigidity means that every object is dualizable).

Now, the target of the functor (2) is not rigid, because the unit object $\mathrm{k}[[u]]$ is not compact. We can replace it with its *rigidification*, i.e. a universal rigid presentable symmetric monoidal category with a symmetric monoidal (colimit-preserving) functor to $\mathrm{Mod}_{\hat{u}}$. This rigidification is given by the category of nuclear modules $\mathrm{Nuc}(\mathrm{k}[[u]])$ – a version of the category of nuclear solid modules defined in [CS20]. We obtain the refined negative cyclic homology functor

$$HC^{-,\mathrm{ref}} : \mathrm{Mot}^{\mathrm{loc}}_{\mathrm{k}} \to \mathrm{Nuc}(\mathrm{k}[[u]]).$$

More precisely, the category $\mathrm{Nuc}(\mathrm{k}[[u]])$ can be described as a full subcategory of the ind-completion $\mathrm{Ind}(\mathrm{Mod}_{\hat{u}}\text{-}\mathrm{k}[[u]])$, generated (via colimits) by the formal colimits of sequences of trace-class maps. We refer to the forthcoming paper [E2] for details on this version of the category of nuclear modules.

The following general result is the consequence of the proof of Theorem 1.

Proposition 2. *Let A be a proper (associative, unital) dg algebra over* k *(properness means that A is a perfect* k*-module). Choose a sequence of finitely presented dg* k*-algebras $A_1 \to A_2 \to \dots$ such that $A \cong \varinjlim_n A_n$. Then we have*

$$HC^{-,\mathrm{ref}}(A) \cong \text{“}\varinjlim_n\text{”} HC^-(A_n).$$

Using Proposition 2, we can compute the refined negative cyclic homology of the affine line, for simplicity in characteristic zero.

Proposition 3. *Suppose that* k *is a* $\mathbb{Q}$*-algebra. Then we have*

$$HC^{-,\mathrm{ref}}(\mathrm{k}[x]) \cong \mathrm{k}[[u]] \oplus \text{“}\bigoplus_{n\geq 1}\text{”}\mathrm{k}[1].$$

Proposition 3 allows to define the functor HP^{ref} on the category of $\mathbb{A}^1$-invariant localizing motives

$$HP^{\mathrm{ref}} : \mathrm{Mot}_{\mathrm{k}}^{\mathrm{loc},\mathbb{A}^1} \to \mathrm{Nuc}(\mathrm{k}((u))).$$

Here the category $\mathrm{Nuc}(\mathrm{k}((u)))$ is the quotient of $\mathrm{Nuc}(\mathrm{k}[[u]])$ by the full subcategory $\mathrm{Mod}_{u\text{-tors}}\text{-}\mathrm{k}[[u]]$.

Note that we have a fully faithful inclusion $\mathrm{Mod}\text{-}\mathrm{k}[[u]] \subset \mathrm{Nuc}(\mathrm{k}[[u]])$. Identifying usual $\mathrm{k}[[u]]$-modules with the corresponding nuclear modules, we have the following description of HP^{ref} for smooth algebraic varieties.

Theorem 4. *Suppose that* k *is a field of characteristic zero, and let* X *be a smooth algebraic variety over* k*. Then we have*

$$HP^{\mathrm{ref}}(\mathrm{Perf}(X)) \cong HP(\mathrm{Perf}(X))$$

– the usual 2*-periodized de Rham cohomology.*

The situation is more interesting for schemes over a perfect field k of characteristic $p > 0$. In this case we consider the topological negative cyclic homology as a functor

$$TC^{-} : \mathrm{dgcat}_{\mathrm{k}}^{\mathrm{tr}} \to \mathrm{Mod}_{\hat{u}}\text{-}TC^{-}(\mathrm{k}).$$

Here u is an element of $\pi_{-2}TC^{-}(\mathrm{k})$, where we use the well known identification

$$\pi_* TC^{-}(\mathrm{k}) \cong W(\mathrm{k})[u,v]/(uv-p),$$

see [NS18]. Arguing as above, we obtain the refined version of TC^{-} :

$$TC^{-,\mathrm{ref}} : \mathrm{dgcat}_{\mathrm{k}}^{\mathrm{tr}} \to \mathrm{Nuc}(TC^{-}(\mathrm{k})).$$

A more subtle computation for the affine line yields the functor on the category of $\mathbb{A}^1$-invariant localizing motives:

$$TP^{\mathrm{ref}} : \mathrm{Mot}_{\mathrm{k}}^{\mathrm{loc},\mathbb{A}^1} \to \mathrm{Nuc}(TP(\mathrm{k})[\frac{1}{p}]).$$

The dualizable objects of the latter category are simply the 2-periodic perfect dg vector spaces over the field $W(\mathrm{k})[\frac{1}{p}]$. We have the following result.

Theorem 5. *Let* k *be a perfect field of characteristic* p*. Let* X *be a smooth algebraic variety over* k*. Then we have*

$$TP^{\mathrm{ref}}(\mathrm{Perf}(X)) \cong \mathbf{R}\Gamma_{\mathrm{rig}}(X/W(\mathrm{k})[\frac{1}{p}])((u))$$

– the 2*-periodized rigid cohomology of* X*.*

We conclude with the following examples of computations, returning to the case of $\mathbb{Q}$-algebras.

Proposition 6. *Let* $\mathrm{k} = \mathbb{Q}[x]$*. Then the idempotent* E_∞*-algebra* $HC^{-,\mathrm{ref}}(\mathbb{Q}[x^{\pm 1}]/\mathbb{Q}[x]) \in \mathrm{Nuc}(\mathbb{Q}[x][[u]])$ *is identified with the algebra of overconvergent functions on the subset*

$$\bigcap_{n>0} \{|u| \leq |x|^n \neq 0\} \subset \mathrm{Spa}(\mathbb{Q}[x][[u]], \mathbb{Q}[x][[u]]).$$

Strictly speaking, we don't have an actual adic space (since u has cohomological degree 2), but the above geometric description still makes sense.

Proposition 7. *Let* $\mathrm{k} = \mathbb{Q}[x]$*, and we identify* $\mathbb{Q}$ *with* $\mathbb{Q}[x]/(x)$*. Then the object* $HP^{\mathrm{ref}}(\mathbb{Q}/\mathbb{Q}[x]) \in \mathrm{Nuc}(\mathbb{Q}[x]((u)))$ *is naturally an idempotent* E_∞*-algebra, which is identified with the algebra of overconvergent functions on the subset*

$$\bigcap_{\varepsilon>0} \{|x| \leq |u|^{1-\varepsilon}\} \subset \mathrm{Spa}(\mathbb{Q}[x]((u)), \mathbb{Q}[x][[u]])$$

References

[BGT13] A. Blumberg, D. Gepner, and G. Tabuada, "A universal characterization of higher algebraic K-theory". Geom. Topol. 17 (2013), no. 2, pp. 733-838.

[CS20] D. Clausen, P. Scholze. "Lectures on Analytic Geometry". Available at: https://www.math.uni-bonn.de/people/scholze/Analytic.pdf 2020.

[E1] A. I. Efimov, "Rigidity of the category of localizing motives", in preparation.

[E2] A. I. Efimov, "Localizing invariants of inverse limits", in preparation.

[GR17] D. Gaitsgory, N. Rozenblyum. "A study in derived algebraic geometry. Volume I: Correspondences and duality". Volume 221 of Math. Surv. Monogr. Providence, RI: American Mathematical Society (AMS), 2017.

[NS18] T. Nikolaus and P. Scholze, "On topological cyclic homology". Acta Mathematica, 221(2):203-409, 2018

Heights in the isogeny class of an abelian variety

Mark Kisin

(joint work with Lucia Mocz)

Let A be an abelian variety over a number field K, with algebraic closure $\bar{K}$. Assuming the Mumford-Tate conjecture for A, we show that the isogeny class of A over $\bar{K}$ contains only finitely many isomorphism classes of bounded Faltings height. As the Mumford-Tate conjecture is known for many abelian varieties, our theorem is unconditional in those cases.

There is a connection between this result, and the unramified Fontaine-Mazur conjecture: If G is the Mumford-Tate group of A, then after replacing K by a finite extension, the Galois action on the Tate module of A induces a representation $\varrho_p : \mathrm{Gal}(\bar{K}/K) \to G(\mathbb{Q}_p)$, by a result of Deligne. The Mumford-Tate conjecture says that ϱ_p has Zariski dense image. Let G_{Gal} denote the Zariski closure of ϱ_p, and let $G_{\mathrm{ur}} \subset G_{\mathrm{Gal}}$ be the smallest (normal) subgroup containing the images of all inertia subgroups at p.

What we use in the proof of the theorem is that $G_{\mathrm{ur}} = G_{\mathrm{Gal}}$, which follows from the Mumford-Tate conjecture. This statement is a special case of the unramified

Fontaine-Mazur conjecture (consider any faithful representation of $G_{\mathrm{Gal}}/G_{\mathrm{ur}}$). On the other hand, by combing the theorem with an argument of Faltings, one can *recover* a case of the unramified Mazur conjecture: Let K^{ur}/K be the maximal extension unramified at all primes $w|p$. Then the analogue of Faltings' theorem on the Tate conjecture for abelian varieties holds with K^{ur} in place of K.

The proof of the theorem uses arguments from p-adic Hodge theory. In particular, in the case where one considers a sequece of p-power isogenies, it uses the classification of p-divisible groups of Scholze-Weinstein.

Reporter: Bence Hevesi

Participants

Prof. Dr. Fabrizio Andreatta
Dipartimento di Matematica
"Federigo Enriques"
Università di Milano
Via C. Saldini, 50
20133 Milano
ITALY

Dr. Johannes Anschütz
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Ko Aoki
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Bhargav Bhatt
Institute for Advanced Study
1 Einstein Drive
Princeton NJ 08540
UNITED STATES

Dr. George Boxer
Imperial College London
Huxley Building, 180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Ana Caraiani
Imperial College London
Department of Mathematics
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Dr. Kęstutis Česnavičius
Institut de Mathématiques d'Orsay
Université Paris Sud (Paris XI)
Bât. 307
91405 Orsay Cedex
FRANCE

Dr. Dustin Clausen
IHÉS
Institut des Hautes Ètudes Scientifiques
Le Bois-Marie 35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Sanath Devalapurkar
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Dr. Vesselin Dimitrov
Department of Mathematics
California Institute of Technology
Pasadena, CA 91125
UNITED STATES

Dr. Alexander I. Efimov
Einstein Institute of Mathematics
Edmond J. Safra Campus
The Hebrew University of Jerusalem
Givat Ram
Jerusalem 91904
ISRAEL

Prof. Dr. Dr. h.c.mult. Hélène Esnault
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Joakim Færgeman
Department of Mathematics
Yale University
Box 208 283
New Haven CT 06511
UNITED STATES

Prof. Dr. Gerd Faltings
MPI Bonn
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Ofer Gabber
I.H.E.S.
Le Bois-Marie
Université Paris-Saclay
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Prof. Dr. Toby Gee
Department of Mathematics
Imperial College London
Huxley Building
180, Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Dr. Ian Gleason
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Michael Groechenig
Department of Mathematics
University of Toronto
40 St. George Street
Toronto ON M5S 2E4
CANADA

Dr. Linus Hamann
Department of Mathematics
Stanford University
Sloan Mathematical Center
Washington Road
San Francisco, CA 94301-1000
UNITED STATES

Prof. Dr. David Hansen
Department of Mathematics
National University of Singapore
Level 4, Block S17
10 Lower Kent Ridge Road
Singapore 119076
SINGAPORE

Dr. Ben Heuer
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Str. 6-8
60054 Frankfurt am Main
GERMANY

Bence Hevesi
Department of Mathematics
King's College London
Strand
London WC2R 2LS
UNITED KINGDOM

Dr. Ryomei Iwasa
Laboratoire de Rech. Informatique
CNRS, Université Paris Sud (Paris XI)
Centre d'Orsay, Bat. 490
91405 Orsay Cedex
FRANCE

Dr. Kalyani Kansal
Institute for Advanced Study
Theoretical Computer Science/
Discrete Math Group
Princeton, NJ 08540
UNITED STATES

Prof. Dr. Mark Kisin
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Prof. Dr. Bruno Klingler
Fachbereich Mathematik
Humboldt Universität Berlin
Unter den Linden 6
10099 Berlin
GERMANY

Dr. Teruhisha Koshikawa
Research Institute for Math. Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Dr. Arthur-César Le Bras
CNRS & Université de Strasbourg
7, rue René Descartes
67084 Strasbourg Cedex Cedex
FRANCE

Dr. Si Ying Lee
Stanford University
450 Jane Stanford Way Building 380
94304 Stanford
UNITED STATES

Lucas Mann
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

Dr. Akhil Mathew
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Dr. Kojiro Matsumoto
Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Wiesława Nizioł
CNRS/Sorbonne University
Paris Cedex
FRANCE

Prof. Dr. Lue Pan
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton NJ 08544-1000
UNITED STATES

Dr. Alexander Petrov
Institute for Advanced Study
1 Einstein Drive
Princeton, NJ 08540
UNITED STATES

Dr. Juan Esteban Rodriguez Camargo
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Tomer Schlank
Institute of Mathematics
The Hebrew University
Givat-Ram
Jerusalem 91904
ISRAEL

Prof. Dr. Peter Scholze
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Ananth Shankar
Department of Mathematics
Lunt Hall
Northwestern University
2033 Sheridan Road
Evanston, IL 60208-2730
UNITED STATES

Naomi Sweeting
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Longke Tang
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton NJ 08544-1000
UNITED STATES

Prof. Dr. Eva Viehmann
Fakultät für Mathematik
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Vadim Vologodsky
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Ferdinand Wagner
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Bogdan Zavyalov
School of Mathematics
Institute for Advanced Study
1 Einstein Drive
Princeton, NJ 08540
UNITED STATES

Dr. Mingjia Zhang
Princeton University
Fine Hall
08540 Princeton
UNITED STATES

Prof. Dr. Xinwen Zhu
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES