Mathematisches Forschungsinstitut Oberwolfach

Report No. 41/2024

DOI: 10.4171/OWR/2024/41

Riemann Surfaces: Random, Flat, and Hyperbolic Geometry

Organized by
Gaëtan Borot, Berlin
Samuel Grushevsky, Stony Brook
Martin Möller, Frankfurt

8 September – 13 September 2024

Abstract. This workshop brought together experts in Riemann surfaces from the point of view of hyperbolic geometry, experts on flat surfaces from the point of view of Teichmüller dynamics, and the combinatorics and probability community working on asymptotic properties, to focus on asymptotic behavior of Riemann surfaces and their combinatorial models at large genus.

Mathematics Subject Classification (2020): 32G15, 30F30, 60B99.

Introduction by the Organizers

The workshop *Riemann Surfaces: Random, Flat, and Hyperbolic Geometry*, was aimed at bringing together the communities working on metric and dynamical aspects of Riemann surfaces with the usually disjoint community working on random geometry. The workshop was comprised of 20 research talks by participants (50min + questions), and of a session of 5-minute presentations by early career researchers. A core topic of the workshop was the behaviour of surfaces as genus increases to infinity - both for hyperbolic surfaces and for flat surfaces, and also for related combinatorial models. The study of geometric properties was frequently combined with analyzing random and statistical aspects.

Random surfaces and large genus asymptotics. Both large genus aspects and random aspects may be grouped according to whether they are studied for hyperbolic or flat surfaces. Added value usually comes from comparing the two aspects, for example closed hyperbolic geodesics and saddle connections on flat surfaces. Additionally, probabilistic methods often provide a tool that allows for actual asymptotic solutions of enumerative problems.

On the hyperbolic side, a long-standing question is to determine the spectral gap of a closed Riemann surface, the first non-zero eigenvalue of the Laplacian. *Anantharaman* reported on a recent breakthrough, determining the optimal spectral gap with high probability, with respect to the Weil-Peterson measure on the moduli of Riemann surfaces of large genus. *Petri* used contructions from random graph theory to construct (hyperbolic) Riemann surfaces with large systoles. *Erlandsson* explained that long random multicurves equidistributed on the space of geodesic currents, with respect to the Thurston measure.

Louf provided a comparison of statistics of the lengths of simple closed curves in the hyperbolic setting with the setting of random graphs embedded in a Riemann surface. Again, the similarity between these settings becomes apparent in the large genus limit.

A square-tiled surface with many squares is a good approximation to a flat surface (a Riemann surface together with a holomorphic differential), and *Delecroix* explained the state of the art in determining the asymptotic shape of the statistics of the number of cylinders on a square-tiled surface in large genus. *Randecker* refined counting results for saddle connections with length bounds to distribution questions for the length statistics on flat surfaces of large genus. *Rafi* considered the large genus limits of flat surfaces as a whole, in the sense of Benjamini-Schramm.

The talk of *Budd* was complementary in the sense of constructing random surfaces, in fact disks from sides a random set approaching a two-dimensional Brownian bridge and analyzing the area statistics.

Flat surfaces. There are (still) many open questions on the geometry of flat surfaces and their moduli spaces. Many questions also remain on how to relate the flat geometry with the hyperbolic geometry on the underlying moduli of curves in a fixed genus. The focus is often on statistical aspects such as (equi)distribution of geodesics and their lengths.

A prime example in this direction, combining aspects of hyperbolic and flat geometry, was the talk of *Farre*. He presented a solution to Mirzakhani's question on the equidistribution of twist tori. These are defined using pants decomposition, i.e. hyperbolically, while the solution is by appyling the orhogeodesic foliation to build flat surfaces and using the breakthrough classification result for invariant measures on flat surfaces, by Eskin, Mirzakhani and Mohammadi.

Several talks addressed the central counting problems for saddle connections on flat surfaces and their asymptotics, the Siegel-Veech constants. *Aulicino* presented formulas for Siegel-Veech constants of flat surfaces that are cyclic covers. *Fairchild* related the count of pairs of saddle connections to moments of the Siegel-Veech transforms. *Masur* refined the problem to the count of saddle connections that avoid a given saddle connection.

Progress on the classification problems for invariant subvarieties of moduli spaces of flat surfaces was presented by *Winsor* for the closure of the leaves of isoperiodic forms.

The topology of the moduli space of flat surfaces was addressed in the talks of Chen and Gadre. *Chen* gave a stratification of these spaces by the singularity type of a certain contraction, in order to relate the topology in some cases to that of hyperplane complements. *Gadre* showed that geodesic flow trajectories capture the full fundamental group of those moduli spaces.

Intersection theory on the moduli spaces. Besides statistical questions, many problems on flat and hyperbolic Riemann surfaces are of enumerative nature and thus accesible through techniques relying on intersection theory.

Schmitt defined the logarithmic Chow ring as a generalization of the usual Chow ring of the moduli space of curves and showed how it can be used to capture enumerative invariants related to Hurwitz numbers.

The moduli space of cone surfaces should provide a bridge between the flat and the hyperbolic world, as explained by *Sauvaget*. The volumes of these moduli spaces can be computed by intersection theory, the Masur-Veech volumes in some range of angle parameters, and Weil-Petersson volumes in a complementary set of angle parameters.

Lewański gave an overview of the topological recursion and its multitude of applications, from Witten's conjecture to recent progress on enumerative problems, like JT gravity. Another application of topological recursion was presented by *Bouttier* to count maps (i.e. cellular embeddings of graphs) with tight boundaries.

A complementary talk of *Deroin* on Toledo invariants of quantum representations of the mapping class group provided examples of hyperbolic structures on lower genus moduli spaces of curves.

In a lively session packed with concise presentations the younger generation presented the following results, providing ample material for discussion during the subsequent afternoon free of talks.

Speaker	**Topic**
Sam Freedman	Veech Fibrations
Simon Barazer	Oriented ribbon graphs and acyclic decomposition
Kai Fu	Siegel-Veech measures
Mingkun Liu	Length spectra of random maps: a Teichmüller theory approach
Victor le Guilloux	Average counting of closed geodesics on hyperbolic surfaces
Nihar Gargava	Dense lattice packings
Sahana Vasudevan	Triangulated surfaces in moduli space fibrations

Edmond Covanov	Spectrum of pseudo-Anosov mapping classes
Miguel Prado	Veech Counting isoresidual differentials on the Riemann sphere
Vivian He	Counting curves intersecting a fixed measured lamination
Ivan Yakoulev	Counting functions for metric ribbon graphs
Riccardo Giannini translation surfaces	Monodromy kernel for some strata of

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows".

Workshop: Riemann Surfaces: Random, Flat, and Hyperbolic Geometry

Table of Contents

Abstracts

Spectral gap of random hyperbolic surfaces

Nalini Anantharaman
(joint work with Laura Monk)

Let X be a closed, connected, oriented surface of genus g, with a hyperbolic metric chosen at random according to the Weil–Petersson measure on the moduli space $\mathcal{M}_g$ of Riemannian metrics. This measure is known to be finite, of total mass V_g; we normalize it to be a probability measure, denoted by $\mathbb{P}_g^{\mathrm{WP}}$. The aim of this work is to establish asymptotic results true *with high probability*, i.e. with probability going to 1 in the large genus limit $g \longrightarrow +\infty$. In particular, let $\lambda_1 = \lambda_1(X)$ be the first non-zero eigenvalue of the Laplacian on X, known as its *spectral gap*. In [1], we proved that for $\alpha = \frac{1}{6}$ and $\varepsilon > 0$ arbitrary, we have

$$\mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \leq \frac{1}{4} - \alpha^2 - \varepsilon\right) \underset{g \longrightarrow +\infty}{\longrightarrow} 0. \tag{1}$$

In other words, we proved that, for any $\varepsilon > 0$, we have $\lambda_1 \geq \frac{2}{9} - \varepsilon$ with high probability. Two previous independent papers due to Wu–Xue [6] and Lipnowski–Wright [4] proved (1) for $\alpha = \frac{1}{4}$, i.e. that $\lambda_1 \geq \frac{3}{16} - \varepsilon$ with high probability.

In this talk we explain a strategy to prove (1) for arbitrarily small α. This allows to conclude that, for any $\varepsilon > 0$, we have $\lambda_1 \geq \frac{1}{4} - \varepsilon$ with high probability. In light of Huber's work [3] proving that $\limsup_g \sup_{X \in \mathcal{M}_g} \lambda_1(X) \leq \frac{1}{4}$, this means that typical hyperbolic surfaces of large genus have an almost optimal spectral gap. This is joint work with Laura Monk.

The starting point of our analysis is, without surprise, the trace formula proven by Selberg in [5]. It relates the spectrum of the Laplacian

$$\lambda_0(X) = 0 < \lambda_1(X) \leq \lambda_2(X) \leq \ldots \longrightarrow +\infty$$

on a closed connected oriented hyperbolic surface X of genus g, to the lengths of all its periodic geodesics. It reads, for a smooth even function $H : \mathbb{R} \to \mathbb{R}$,

$$\sum_{j=0}^{+\infty} \hat{H}(r_j(X)) = (g-1) \int_{\mathbb{R}} \hat{H}(r) \tanh(\pi r) r dr + \sum_{\gamma \in \mathcal{G}(X)} \sum_{k=1}^{+\infty} \frac{\ell(\gamma)\, H(k\ell(\gamma))}{2 \sinh\left(\frac{k\ell(\gamma)}{2}\right)} \tag{2}$$

where:

- for all j, $r_j(X) \in \mathbb{R} \cup i[-1/2, 1/2]$ is a solution of $\lambda_j(X) = \frac{1}{4} + r_j(X)^2$ – the left-hand side of the formula is thus called the *spectral side*;
- the Fourier transform $\hat{H}$ is defined by $\hat{H}(r) = \int_{\mathbb{R}} H(\ell)\, e^{-ir\ell} d\ell$;
- the first term on the right-hand side is the so-called *topological term*, referring to the fact that this term only depends on the genus g (in particular, when studying random hyperbolic surfaces of genus g, this term is deterministic);
- $\mathcal{G}(X)$ is the set of primitive oriented periodic geodesic on X and $\ell(\gamma)$ stands for the length of the smooth curve γ in X.

The sum over periodic geodesics is called the *geometric term* of the trace formula. In the formula, the integer k represents the number of times the primitive geodesic is run over: the sum $\sum_{k=2}^{+\infty}$ thus describes non-primitive periodic geodesics.

We draw the attention to the fact that non-simple geodesics appear in the geometric term. Dealing with non-simple closed geodesics in the Selberg trace formula is one of the core challenges faced when taking the average of the Selberg trace formula for random Weil–Petersson surfaces.

The Selberg trace formula holds for a class of "nice" functions H. For our purposes, we will only consider functions H of compact support, in which case both sums are absolutely convergent. More precisely, let $H : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a fixed smooth even function, with compact support $[-1, 1]$, such that $\hat{H}$ is non-negative on $\mathbb{R} \cup i[-\frac{1}{2}, \frac{1}{2}]$. For any $L \geq 1$, we shall apply the trace formula to $H_L(\ell) := H(\frac{\ell}{L})$. The function H_L has support $[-L, L]$, so that the geometric term only involves periodic geodesics of length $\ell(\gamma) \leq L$. Note that H_L still has non-negative Fourier transform. The parameter L will be taken to grow to $+\infty$ as $g \longrightarrow \infty$.

Eigenvalues $\lambda_j \leq \frac{1}{4}$ correspond to purely imaginary r_j, which are responsible for terms such that $\hat{H}_L(r_j)$ grows roughly like $e^{|r_j|L}$. For the first non-trivial eigenvalue, corresponding to $j = 1$, this is expressed in the following lemma:

Lemma 1 ([1, Lemma 3.10])**.** *Let $\alpha \in (0, \frac{1}{2})$. For any $0 < \varepsilon < \frac{1}{4} - \alpha^2$, there exists a constant $C_{\alpha,\varepsilon} > 0$ such that, for any hyperbolic surface X, any $L \geq 1$,*

$$\lambda_1(X) \leq \frac{1}{4} - \alpha^2 - \varepsilon \quad \Longrightarrow \quad \hat{H}_L(r_1(X)) \geq C_{\alpha,\varepsilon}\, e^{(\alpha+\varepsilon)L}.$$

The Selberg trace formula holds for any hyperbolic surface X, and if X is random, then both sides also become random. Lemma 1 provides us with the beginning of a strategy to prove probabilistic lower bounds on λ_1. Suppose we want to prove that $\lambda_1 \geq \frac{1}{4} - \alpha^2 - \varepsilon$ with high probability. First, we use Lemma 1 to write

$$\mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \leq \frac{1}{4} - \alpha^2 - \varepsilon\right) \leq \mathbb{P}_g^{\mathrm{WP}}\left(\hat{H}_L(r_1) \geq C_{\alpha,\varepsilon} e^{(\alpha+\varepsilon)L}\right).$$

Then, the Markov inequality yields

$$\mathbb{P}_g^{\mathrm{WP}}\left(\lambda_1 \leq \frac{1}{4} - \alpha^2 - \varepsilon\right) \leq \frac{\mathbb{E}_g^{\mathrm{WP}}\left[\hat{H}_L(r_1)\right]}{C_{\alpha,\varepsilon}\, e^{(\alpha+\varepsilon)L}}. \tag{3}$$

In order to imply (1), it is thus sufficient to prove that, for some choice of $L = L(g) \longrightarrow +\infty$,

$$\mathbb{E}_g^{\mathrm{WP}}\left[\hat{H}_L(r_1)\right] = o(e^{(\alpha+\varepsilon)L}), \tag{4}$$

where by $o(\,)$ we mean that the ratio between the left-hand side and right-hand side converges to 0 as $g \longrightarrow \infty$. It is in our interest to take L as small as possible, because we are summing over periodic geodesics of lengths in $[0, L]$, and the number of such geodesics is known to grow exponentially in L. At this stage of the discussion, $L(g)$ can grow arbitrarily slowly, but will be forced to grow at a certain rate in the coming lines.

To control the left-hand side of (4), we can use the Selberg trace formula (2) and the positivity of $\hat{H}_L$, to write

(5)
$$\mathbb{E}_g^{\mathrm{WP}}\left[\hat{H}_L(r_1)\right] \leq (g-1)\int_{\mathbb{R}} \hat{H}_L(r)\tanh(\pi r) r dr + \mathbb{E}_g^{\mathrm{WP}}\left[\sum_{\gamma\in\mathcal{G}(X)}\sum_{k=1}^{+\infty}\frac{\ell(\gamma)\, H_L(k\ell(\gamma))}{2\sinh\left(\frac{k\ell(\gamma)}{2}\right)}\right].$$

Let us acknowledge the necessary presence of the deterministic topological term, growing linearly in g on the right-hand side: if we are to prove (4), this term forces us to take $L \geq \frac{\log g}{\alpha+\varepsilon}$. Hence our discussion always takes place at logarithmic scales in g. The crux of the analysis lies in the choice of the multiplicative constant: the smaller α we aim at, the larger multiplicative constant we need. In particular, $\alpha = 0$ requires to take $L \geq \frac{\log g}{\varepsilon}$ for any arbitrary $\varepsilon > 0$. With this in mind, we take from now on $L = A \log g$ with A fixed.

In our proof we try to bound the right-hand side of (5), in which all the topologies of periodic geodesics appear. The key steps of the proof are the following.

- First, we define a notion of *volume functions* associated to arbitrary topologies of closed geodesics, extending the *volume polynomials* investigated by Mirzakhani for simple geodesics. We provide an expression and an asymptotic expansions in powers of g^{-1} for these volume functions.
- Then, we prove that the coefficients appearing in these expansions satisfy the *Friedman–Ramanujan* property, a newly defined notion related to on-average cancellations in the trace method. This statement yields the crucial argument to prove (1) and is the focus of an upcoming article.
- Finally, we explain the necessity to discard a set of "tangled surfaces" of small but non-zero probability : the exponential proliferation of topologies of closed geodesics in tangled surfaces is responsible for the failure of the naive trace method. We solve this issue by conditioning our argument on the set of *tangle-free surfaces*. This delicate step is made possible by a new kind of Moebius inversion formula [2].

References

[1] N. Anantharaman and L. Monk. Friedman–Ramanujan functions in random hyperbolic geometry and application to spectral gaps. *preprint*, (2023).

[2] N. Anantharaman and L. Monk. A Moebius inversion formula to discard tangled hyperbolic surfaces. (2023).

[3] H. Huber. Über den ersten Eigenwert des Laplace-Operators auf kompakten Riemannschen Flächen. *Commentarii mathematici Helvetici*, **49** (1974), 251–259.

[4] M. Lipnowski and A. Wright. Towards optimal spectral gaps in large genus. *arXiv:2103.07496*, (2021).

[5] A. Selberg. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *The Journal of the Indian Mathematical Society*, **20** (1956), 47–87.

[6] Y. Wu and Y. Xue. Random hyperbolic surfaces of large genus have first eigenvalues greater than 3/16-ϵ. *Geometric and Functional Analysis*, **32** (2022), 340–410.

Siegel–Veech Constants for Cyclic Covers of Generic Translation Surfaces

David Aulicino

(joint work with Aaron Calderon, Carlos Matheus, Nick Salter, and Martin Schmoll)

1. Introduction

It was found by [7, 5] that the number of cylinders on a translation surface grows quadratically with exact asymptotics. Define

$$N_{\mathrm{area}^s}((X,\omega),L) = \sum_{\substack{cyl\subseteq (X,\omega)\\ w(cyl)<L}} \frac{\mathrm{area}^s(cyl)}{\mathrm{area}^s(X,\omega)}.$$

In particular, it follows from [5] that the following limit exists for almost every (X,ω) in its ambient space $\mathcal{M}$ (i.e., $\mathrm{SL}_2(\mathbb{R})$-orbit closure).

$$c_{\mathrm{area}^s}(\mathcal{M}) = \lim_{L\to\infty} \frac{N_{\mathrm{area}^s}((X,\omega),L)}{\pi L^2}$$

We consider the following family of covers of translation surfaces. Let $(X,\omega)\in\mathcal{H}(\kappa)$ be a generic translation surface with $n\geq 2$ marked points $\Sigma\subset X$. Let $\mathcal{H}$ denote a component of $\mathcal{H}(\kappa)$. We consider branched covers of (X,ω) so that $\pi_1(X\setminus\Sigma,p_0)\to Sym(d)$ has image in a cyclic group $\mathbb{Z}/d\mathbb{Z}$. It follows from the Hurewicz theorem that covers are classified by elements of $H^1(X\setminus\Sigma;\mathbb{Z}/d\mathbb{Z})$.

From this, the mapping class group induces an action on $H^1(X\setminus\Sigma;\mathbb{Z}/d\mathbb{Z})$, and we denote by $\mathcal{M}(\alpha)$ the locus of all cyclic covers with branching specified by $\alpha\in H^1(X\setminus\Sigma;\mathbb{Z}/d\mathbb{Z})$. Equivalently, $\mathcal{M}(\alpha)$ is the $\mathrm{SL}_2(\mathbb{R})$-orbit closure of a cyclic cover of a generic surface in $\mathcal{H}(\kappa)$ with branching specified by α.

Let $d_i\in\mathbb{Z}/d\mathbb{Z}$ denote the local monodromy about $p_i\in\Sigma$. Then for all $p_i\in\Sigma$ consider the tuple $(d_1,\dots,d_n)$, which we denote by δ. Let $\mathcal{M}_\delta$ denote the locus of cyclic covers with branching specified by δ. If α is compatible with δ, we can consider the (not necessarily proper) subset $\mathcal{M}_\delta(\alpha)$.

Define d_{abs} to be the largest divisor of d relatively prime to $\gcd(d_1,\dots,d_n,d)$. Finally, define $d_{rel}=d/d_{abs}$.

Theorem 1. *For even degree cyclic covers of translation surfaces, the space $\mathcal{M}_\delta$ is not necessarily connected. If $\mathcal{H}$ is not a hyperelliptic component, then $\mathcal{M}_\delta$ has at most two components and there exists an invariant that classifies them. If $\mathcal{H}$ is hyperelliptic, then there are at most $g+1$ components, where g is the genus and there exists an invariant that classifies them.*

With this classification we compute explicit formulas for the ratio appearing in Theorem 2 below. For Theorem 2, we state the result in the case where there is one connected component.

Define the *Jordan totient function* to be the number of primitive m-tuples, which is given by the formula

$$\Phi_m(d) = d^m \prod_{p|d} \left(1 - \frac{1}{p^m}\right).$$

Theorem 2. *If $\mathcal{M}_\delta$ is connected, (which includes the cases where d is odd and genus is one), then for all $s \geq 0$,*

$$\frac{c_{area^s}(\mathcal{M}(\alpha))}{c_{area^s}(\mathcal{H})} = \frac{d_{rel}}{d^{s-1}\Phi_{2g}(d_{rel})} \left(\sum_{\mathfrak{d}|d_{rel}} \Phi_{2g-1}\left(\frac{d_{rel}}{\mathfrak{d}}\right) \frac{\Phi(\mathfrak{d})}{\mathfrak{d}^{4-2g-s}} \right) \cdot \left(\sum_{\mathfrak{d}|d_{abs}} \left(\frac{\Phi(\mathfrak{d})}{\mathfrak{d}^{3-s}} \right) \right),$$

where Φ is the Euler totient function.

We remark that this formula and its variations for the other components facilitate the explicit computation of $c_{area^s}(\mathcal{M})$ as follows. By [1], there is an explicit formula for $c_{area^s}(\mathcal{H}))$ in terms of $c_{cyl}(\mathcal{H})$. By [6], recursive formulas are given for computing $c_{cyl}(\mathcal{H})$ for all $\mathcal{H}$.

A remarkable invariance appears as a corollary of our main theorem. Regardless of which component $\mathcal{M}(\alpha)$ that we consider, the following formula holds.

Theorem 3. *Let $\mathcal{M}(\alpha)$ be a locus of degree d branched cyclic covers over $(X, \omega) \in \mathcal{H}$ with branching specified by α. Then*

$$\frac{c_{area^3}(\mathcal{M}(\alpha))}{c_{area^3}(\mathcal{H})} = \frac{1}{d}.$$

2. Proof Sketch and Ideas

There are two main parts to the proof. The first concerns classifying the connected components of loci of cyclic covers in $\mathcal{M}_\delta$. This involves extending work of [3] in the non-hyperelliptic case, and a separate argument is used to classify the components in the hyperelliptic case.

Furthermore, substantial very precise information is needed. The key to computing Siegel–Veech constants is the Siegel–Veech formula, and by [6], this can be accomplished by computing a ratio of volumes - namely that of the principal boundary to that of the entire space. For these volumes, we need to know the cardinality of the orbit of an element of $H^1(X \setminus \Sigma; \mathbb{Z}/d\mathbb{Z})$ under the induced action of the mapping class group, but what is much more involved is computing the cardinality of an orbit of such an element where the lift of a cylinder is specified.

With these cardinalities computed, we can proceed to computing the Siegel–Veech constants. We apply the Siegel–Veech formula and are able to derive a single closed formula for all possible cases. This formula is in terms of the quotients of the orbits with a fixed cylinder monodromy to that of the full orbit. To derive this general expression, we make very strong number theoretic assumptions on these quotients and justify a posteriori that they are always satisfied.

References

[1] M. Bauer and E. Goujard, *Geometry of periodic regions on flat surfaces and associated Siegel-Veech constants*, Geom. Dedicata **174** (2015), 203–233. MR 3303050

[2] A. Calderon and N. Salter, *Higher spin mapping class groups and strata of abelian differentials over Teichmüller space*, Adv. Math. **389** (2021), Paper No. 107926, 56. MR 4289049

[3] ———, *Relative homological representations of framed mapping class groups*, Bulletin of the London Mathematical Society **53** (2021), no. 1, 204–219.

[4] ———, *Framed mapping class groups and the monodromy of strata of abelian differentials*, J. Eur. Math. Soc. (JEMS) **25** (2023), no. 12, 4719–4790. MR 4662301

[5] A. Eskin and H. Masur, *Asymptotic formulas on flat surfaces*, Ergodic Theory Dynam. Systems **21** (2001), no. 2, 443–478. MR 1827113

[6] A. Eskin, H. Masur, and A. Zorich, *Moduli spaces of abelian differentials: the principal boundary, counting problems, and the Siegel-Veech constants*, Publ. Math. Inst. Hautes Études Sci. (2003), no. 97, 61–179. MR 2010740 (2005b:32029)

[7] ———, *Siegel measures*, Ann. of Math. (2) **148** (1998), no. 3, 895–944. MR 1670061

On the enumeration of maps with tight boundaries

Jérémie Bouttier

(joint work with Emmanuel Guitter, Grégory Miermont)

Maps, in the combinatorial sense, are cellular embeddings of graphs into surfaces, considered up to homeomorphism. In this talk we consider orientable maps, whose topology is characterized by a pair of integers (g, n), where g is the genus and n the number of boundaries. For a given topology we consider the problem of counting maps according to the distribution of their face degrees: this is a classical problem first investigated by Tutte, who solved it for genus $g = 0$ and even face degrees [1] (throughout the talk we restrict to maps with even face degrees for simplicity). The case of other topologies can be treated via the formalism of *topological recursion*, reviewed for instance in [2]. Precisely, it gives an effective way to compute the generating function $F^{(g)}_{L_1,\dots,L_n}$ of maps of genus g with n boundaries of lengths $L_1, \dots, L_n$, or equivalently the grand generating function

$$W^{(g)}_n(x_1, \dots, x_n) := \sum_{L_1,\dots,L_n \geq 1} \frac{F^{(g)}_{L_1,\dots,L_n}}{x_1^{L_1+1} \cdots x_n^{L_n+1}},$$

via a recursion on minus the Euler character $2g + n - 2$.

In our project, we are investigating the enumerative properties of maps with *tight* boundaries. Here, by tight boundary we mean that its contour is a path of minimal length in its homotopy class. Denoting by $T^{(g)}_{\ell_1,\dots,\ell_n}$ of maps of genus g with n tight boundaries of lengths $\ell_1, \dots, \ell_n$, we introduce the so-called *trumpet decomposition* [3] which gives a relation between $F^{(g)}_{L_1,\dots,L_n}$ and $T^{(g)}_{\ell_1,\dots,\ell_n}$. It gives a combinatorial interpretation to the so-called *Zhukovsky transformation*

$$x(z) = \gamma(z + z^{-1})$$

which plays a key role in [2]. Namely, we find that we have the series expansion

$$W_n^{(g)}(x(z_1),\dots,x(z_n))x'(z_1)\dots x'(z_n) = \left(-\frac{\partial}{\partial z_1}\right)\cdots\left(-\frac{\partial}{\partial z_n}\right)\sum_{\ell_1,\dots,\ell_n\geq 1}\frac{T^{(g)}_{\ell_1,\dots,\ell_n}}{(\gamma z_1)^{\ell_1}\cdots(\gamma z_n)^{\ell_n}}.$$

Here γ is a certain series which is related to the generating function of pointed rooted planar maps, corresponding to the “initial data” for the topological recursion.

As a byproduct, we find that $T^{(g)}_{\ell_1,\dots,\ell_n}$ is of the form $\gamma^{\ell_1+\cdots+\ell_n}$ times a polynomial of degree $3g+n-3$ is $\ell_1^2,\dots,\ell_n^2$. We also find recursions satisfied by these quantities as n varies, which have a nice combinatorial/geometric interpretation.

The second part of the talk is devoted to the case of pairs of pants $(g,n)=(0,3)$, for which the series $T^{(0)}_{\ell_1,\ell_2,\ell_3}$ is equal to $\gamma^{\ell_1+\ell_2+\ell_3}$ times a quantity independent of the boundary lengths. Such a simple formula calls for a combinatorial interpretation, and we provide a bijective proof [4]. Interestingly, it seems to be a discrete analogue of a classical construction in hyperbolic geometry, which consists in building a pair of pants (with its hyperbolic metric) by gluing a pair of ideal triangles with appropriate shifts. Whether our bijective approach can be extended to other topologies remains an intriguing open question.

References

[1] W.T. Tutte, *A census of slicings*, Canad. J. Math. **14** (1962) 708–722.
[2] B. Eynard, *Counting surfaces*, CRM Aisenstadt chair lectures, Birkhäuser, Basel (2016).
[3] J. Bouttier, E. Guitter and G. Miermont, *Enumeration of maps with tight boundaries and the Zhukovsky transformation*, arXiv:2406.13528 [math.CO].
[4] J. Bouttier, E. Guitter and G. Miermont, *Bijective enumeration of planar bipartite maps with three tight boundaries, or how to slice pairs of pants*, Annales Henri Lebesgue **5** (2022) 1035–1110, arXiv:2104.10084 [math.CO].

Uniform random flat disks

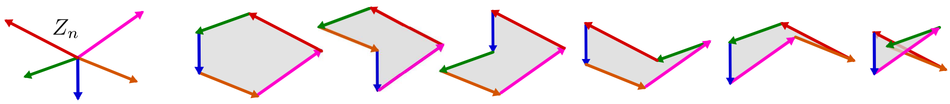
Timothy Budd

Recent developments in the physics literature in the context of two-dimensional quantum gravity, and Jackiw–Teitelboim (JT) gravity in particular, have led to natural questions concerning random flat metrics on the disk. Notably, Frank Ferrari recently asked [5, 6] whether there exist solvable discrete models of n-sided self-overlapping polygons possessing a continuous scaling limit as $n\to\infty$ towards a random flat metric on the disk with fractal boundary. A *self-overlapping polygon* here will be understood as a translation structure on the disk with a piecewise linear boundary consisting of n segments. Unless otherwise stated, there is no restriction on the interior angles of self-overlapping polygons, which may thus exceed 2π. We present two combinatorial families of self-overlapping polygons for which the enumeration problem is solved and we describe some statistical properties of uniformly sampled such disks, hinting at the existence of a universal scaling limit.

For the first family, we consider n-sided self-overlapping polygons with fixed *side set*, i.e. the set of n vectors in $\mathbb{R}^2 \setminus \{0\}$ corresponding to the boundary segments oriented in counterclockwise direction. We say an n-element set $Z_n \subset \mathbb{R}^2 \setminus \{0\}$ that sums to zero is *generic* if for any pair of non-empty disjoint subsets $U, V \subset Z_n$ such that $U \cup V \neq Z_n$ the sums of U and V are linearly independent.

Theorem 1 ([2]). *For $n \geq 3$ and a fixed generic set Z_n, the number of self-overlapping polygons with side set Z_n is $(n-2)!$.*

An example of the $(n-2)!$ self-overlapping polygons sharing a particular side set Z_n for $n = 5$ is



We provide two proofs for this simple result, both of which have their own advantage when it comes to studying more precises statistics. The first relies on an inclusion-exclusion argument involving the complex of convex diagonalizations [4] of a self-overlapping polygon. The second proof is based on an explicit bijection between self-overlapping polygons and certain walks in the plane with increment set Z_n. To be precise, assuming without loss of generality that $(x, 0) \in Z_n$ for some $x > 0$, we consider the walks whose first step is $(x, 0)$ and that stay in the upperhalf plane. By a simple cyclic permutation argument, the number of such walks is indeed $(n-2)!$.

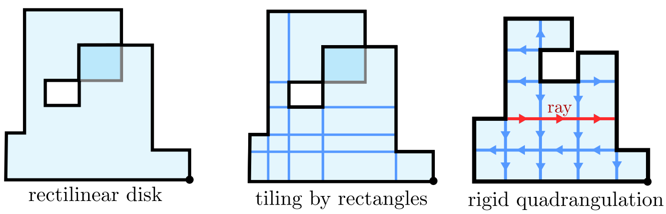
In light of this bijective result it is natural to consider a sequence of random sets $(Z_n)_{n\geq 3}$, such that the uniform random walk with increment set Z_n approaches a standard two-dimensional Brownian bridge. For concreteness, let Z_n be the set of increments of such a Brownian bridge sampled at n equally spaced times, which is almost surely generic. A natural statistic to consider is the area of the self-overlapping polygon, for which the inclusion-exclusion method provides the following result.

Theorem 2 ([2]). *The expectation value of the area of the uniform self-overlapping polygon with side set Z_n is asymptotic to $\frac{1}{2\pi}\log n + C + o_n(1)$ as $n \to \infty$ for a known constant $C = 0.0285\ldots$.*

Numerical experiments suggest that this random area, upon subtraction of $\frac{1}{2\pi}\log n$, converges in distribution as $n \to \infty$, but this remains an open problem.

When the side set is not generic and/or one imposes constraints on the interior angles, other enumeration methods are called for. An example is the class of *rectilinear disks*, in which the only allowed interior angles of the self-overlapping polygons are $\pi/2$ and $3\pi/2$. The moduli space of $2n$-sided rectilinear disks of fixed perimeter is $(2n-3)$-dimensional and carries a natural volume measure corresponding locally to the Lebesgue measure on $2n-3$ independent side lengths. A rectilinear disk carries a natural tiling by rectangles obtained by drawing for each

$3\pi/2$ corner the two rays inward that extend the adjacent sides. For almost every disk this tesselation has the combinatorial structure of a rigid quadrangulation with $2n$ corners. A *rigid quadrangulation* is a planar map in which all bounded faces are quadrangles, the outer face (called the boundary) is a simple face of arbitrary degree, 1, 2 or 3 quadrangles meet at every vertex on the boundary and exactly 4 quadrangles meet at every other vertex. Every ray, i.e. maximal path along non-boundary edges going straight at every inner vertex, is required to connect a pair of boundary vertices where 3, respectively 2, quadrangles meet.



The volume of the moduli space of rectilinear disks of fixed perimeter is directly related to the enumeration of rigid quadrangulations.

Theorem 3 ([3])**.** *The generating function of rigid quadrangulations (with a distinguished $\pi/2$ corner) is* $\frac{1}{4} - \frac{x}{2} - \frac{1}{4x}R(x) = x^2 + 5x^3 + 33x^4 + \cdots$*, where* $R(x)$ *is the power series solution to*

$$\sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k}^2 R(x)^{k+1} = x.$$

The proof relies on a bijection between rigid quadrangulations with $2n$ corners and rooted planar Eulerian orientations with $n-1$ edges (or colorful $\mathbb{Z}$-labeled rooted quadrangulations with $n-1$ faces). The enumeration of the latter class of maps was obtained by Bousquet-Mélou and Elvey Price in [1].

References

[1] M. Bousquet-Mélou, A. Elvey Price, *The generating function of planar Eulerian orientations*, J. Combin. Theory Ser. A **172** (2020) 105183.

[2] T. Budd, *Enumeration and statistics of flat disks*, in preparation.

[3] T. Budd, *Rectilinear disks and the enumeration of rigid quadrangulations*, in preparation.

[4] S. Devadoss, R. Shah, X. Shao, E. Winston, *Deformations of associahedra and visibility graphs*, Contributions to Discrete Mathematics **7** (2012), 68–81.

[5] F. Ferrari, *Jackiw-Teitelboim Gravity, Random Disks of Constant Curvature, Self-Overlapping Curves and Liouville* CFT_1, arXiv preprint arXiv:2402.08052

[6] F. Ferrari, *Random Disks of Constant Curvature: the Lattice Story*, arXiv preprint arXiv:2406.06875

A further stratification of strata of differentials

Dawei Chen
(joint work with Fei Yu)

Given a positive integral partition $\mu = (m_1, \ldots, m_n)$ of $2g-2$, let $\mathcal{H}(\mu)$ be the stratum of holomorphic differentials on smooth complex curves whose orders of zeros are prescribed by the signature μ. Up to scaling of $\mathbb{C}^*$, the stratum $\mathcal{H}(\mu)$ parameterizes pointed smooth curves $(C, z_1, \ldots, z_n)$ where $\sum_{i=1}^n m_i z_i$ is a canonical divisor of C.

For each $(C, z_1, \ldots, z_n) \in \mathcal{H}(\mu)$, we perform a μ-weighted blowup for the isotrivial family $C \times \mathbb{C}$ at the marked points z_i in the central fiber C_0, and then contract the proper transform of C_0. The resulting singularity s is an isolated Gorenstein curve singularity with n rational branches and a $\mathbb{C}^*$-action. We propose to further decompose the stratum $\mathcal{H}(\mu)$ according to the isomorphism classes of these singularities s. For example, by the work of Pinkham, when s is a monomial curve singularity, the corresponding locus picks up (C, z) where the Weierstrass semigroup of z in C is determined by the monomial exponents in the parameterization of s.

As an application, certain deformation spaces of such singularities can help us understand the topology of $\mathcal{H}(\mu)$ and, in particular, connect to the $K(\pi, 1)$ conjecture of Kontsevich–Zorich. Along this circle of ideas, several cases have been studied, including ADE singularities which correspond to some hyperelliptic and low-genus strata, due to Deligne, Looijenga–Mondello and Giannini, among others. As another application, we can systematically study this kind of singularities from the viewpoint of the log minimal model program for the Deligne–Mumford moduli space of curves, initially predicted by Alper–Fedorchuk–Smyth.

What we do not know (yet) about square-tiled surfaces in large genus?

Vincent Delecroix

We present several results and conjectures concerning the cylinder decomposition of square-tiled surfaces in large genus.

Introduction: square-tiled surfaces

Square-tiled surfaces, holonomy and k-differentials on Riemann surfaces. We consider quadrangulations of a compact surface S. In this text these are called *square-tiled surfaces* to emphasize that we consider the induced metric on the surface obtained by making each quadrilateral isometric to the unit square $[0,1] \times [0,1]$ in $\mathbb{R}^2$. The metric is everywhere flat except possibly at the vertices where there is a conical singularity.

The parallel transport of the metric induces a *holonomy* map $\pi_1(S \setminus V) \to \mathbb{Z}/4\mathbb{Z}$. This allows to distinguish three kinds of square-tiled surfaces

(1) *Abelian square-tiled surfaces* whose holonomy is trivial,
(2) *quadratic square-tiled surfaces* whose holonomy is $2\mathbb{Z}/4\mathbb{Z}$,
(3) *quartic square-tiled surfaces* whose holonomy is $\mathbb{Z}/4\mathbb{Z}$.

An example of each kind is provided in Figure 1.

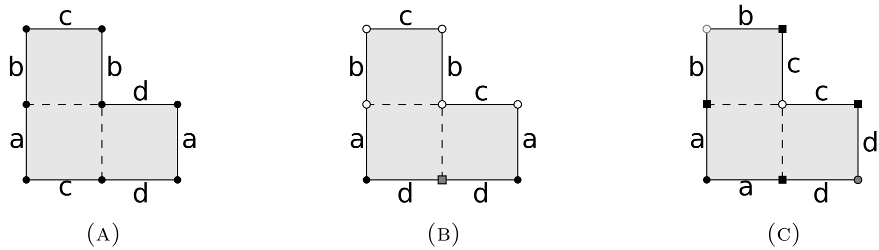


FIGURE 1. The three kinds of monodromies of quadrangulations. (A) An Abelian square-tiled surface in $\Omega^{(1)}\mathcal{M}_{2,1}(2)$. (B) A quadratic square-tiled surface in $\Omega^{(2)}\mathcal{M}_{1,3}(2,-1,-1)$. (C) A quartic square-tiled surface in $\Omega^{(4)}\mathcal{M}_{0,5}(2,-1,-3,-3,-3)$.

An Abelian, quadratic and quartic square-tiled surfaces give rise respectively to an Abelian, quadratic or quartic differentials on the underlying surface by considering $dz = dx + \sqrt{-1}dy$, dz^2 or dz^4 on each unit square. Hence the terminology.

The second important invariant of square-tiled surfaces is given by the degree of vertices or equivalently the angle of the conical singularities. Given the angles $2 \cdot \alpha_1 \cdot \pi, \ldots, 2 \cdot \alpha_n \cdot \pi$ of the conical singularities different from 2π we consider the following vector $\mu = (\mu_1, \ldots, \mu_n)$

(1) for an Abelian square-tiled surface $\mu_i = \frac{1}{4}(\alpha_i - 2)$,
(2) for a quadratic square-tiled surface $\mu_i = \frac{1}{2}(\alpha_i - 2)$,
(3) for a quartic square-tiled surface $\mu_i = \alpha_i - 2$.

The number μ_i corresponds to the degree of vanishing of the differential: in the neighborhood of the vertex corresponding to α_i, an Abelian, quadratic or quartic square-tiled differential can be written $z^{\mu_i} dz^k$ where k is respectively 1, 2 or 4. We denote by $\Omega^{(k)}\mathcal{M}_{g,n}(\mu)$ the moduli space of k-differentials with singularity pattern μ[1].

Let us emphasize that here we ignore vertices with angle 2π that gives rise to regular point for the differential. In particular, each stratum $\Omega^{(k)}(\mu)$ contains infinitely many (isomorphism classes of) square-tiled surfaces.

Note that a k-differential for $k = 2, 4$ admits a k-fold cover which is an Abelian differential. The resulting differential admits a $\mathbb{Z}/k\mathbb{Z}$ symmetry. See Figure 2.

[1]Note that for $k = 1$ the differentials are holomorphic. While for $k = 2$ we allow simple poles and for $k = 4$ we allow simple, double and triple poles. This can be phrased by saying that we only consider meromorphic k-differentials of finite area.

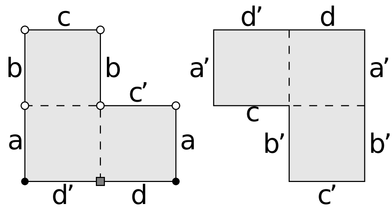


Figure 2. The covering of the surface from Figure 1b.

Cylinder decompositions. An Abelian differential admits a one-parameter family of vector fields v_θ: each direction θ makes (globally) sense in the surface. In particular, one has well-defined left, right, up and down directions.

The horizontal flow (associated to v_0) and the vertical flow (associated to $v_{\pi/2}$) are completely periodic: either trajectories hit a conical point in both past and future or it is periodic. In other words, the surface decomposes into cylinders (or beam of trajectories) separated by singular layers.

We associate a third invariant to an Abelian square-tiled surface: the multiset of areas of the cylinders normalized so that it sums to one. In Figure 1a both the horizontal and vertical cylinder decompositions gives the vector $A^{horiz}(S) = A^{vert}(S) = \frac{1}{3}\{1, 2\}$.

Note that by taking k-covering, one can define cylinders for k-differentials for quadratic and quartic differentials.

Problem 1: cylinder decompositions in large genus

We consider limits of square-tiled surfaces when the number of squares N and the genus g tend to infinity. In this first problem we first let N tend to infinity and then let g tend to infinity.

When letting N tend to infinity, we have the following result which is very close in spirit to Mirzakhani results on counting multicurves with respect to topological types.

Theorem 1 ([1]). *For each stratum $\Omega^{(k)}(\mu)$ of Abelian or quadratic differentials, we consider the multiset of cylinder areas $A^{horiz}(S)$ and $A^{vert}(S)$ as S run through the (finite) set of square-tiled surfaces with at most N squares endowed with the uniform measure*[2]*. As N tend to infinity, the joint distribution $(A^{horiz}(S), A^{vert}(S))$ converges to a product measure $\nu_{k,\mu} \otimes \nu_{k,\mu}$ on pairs of finite multisets.*

In a work in progress we describe a similar result for quartic differentials (though in this context there is a single vector $A(S)$ to consider).

The behaviour of $\nu_{k,\mu}$ as the genus tend to infinity is very intriguing. For now, we only have a complete answer for the principal strata of quadratic differentials $\Omega^{(2)}\mathcal{M}_{g,4g-4}(1^{4g-4})$.

[2]It is actually more natural to weight each square-tiled surface with the weight $1/|\operatorname{Aut}(S)|$ but it makes no difference when taking limits.

Theorem 2 ([3, 4, 5])**.** *When g tends to infinity, the measures $\nu_{2,(1^{4g-4})}$ describing the areas of cylinder decomposition in the principal stratum $\Omega^{(2)}\mathcal{M}_{g,4g-4}(1^{4g-4})$ satisfy the following*

(1) *the expectation of the length of the multiset is asymptotic to* $\log(g)/2$
(2) (macroscopic limit) *the large components behave as a Poisson-Dirichlet process with parameter* $1/2$
(3) (microscopic limit) *when multiplied by $4g-4$, the small components behave as a Poisson point process with intensity*

$$\frac{e^{-x}}{x}\sum_{n=1}^{\infty}(\cosh(x/n)-1) = \frac{e^{-x}}{x}\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(2n)!}\,x^{2n}$$

Conjectures generalizing Theorem 2 are stated for all strata of quadratic and Abelian differentials. Though, the proof crucially uses an explicit description of $\nu_{k,\mu}$ that only holds in the special case of $\Omega^{(2)}\mathcal{M}_{g,4g-4}(1^{4g-4})$, see [2].

Problem 2: cylinders of intermediate area

As stated in Theorem 2, in the large genus asymptotics we have a good understanding of macroscopic cylinders (the ones with an area proportional to the area of the surface) and the microscopic ones (the ones with an area proportional to the area of the surface divided by g). However, it seems delicate to understand the full spectrum of area: for example how many cylinders are there with renormalized area between $1/\sqrt{g}$ and $2/\sqrt{g}$?

Problem 3: diagonal limit $N = \lfloor \alpha g \rfloor$

In Theorem 2 the limit in N (the number of squares) is taken first and then the limit in g is considered. It is highly interesting to try to understand the behavior of random square-tiled surfaces with $N = \lfloor \alpha g \rfloor$ squares where α is some parameters and g tends to infinity.

References

[1] V. Delecroix, É. Goujard, P. Zograf, and A. Zorich. Enumeration of meanders and Masur-Veech volumes. *Forum Math. Pi*, **8** (2020), Id/No e4.
[2] V. Delecroix, É. Goujard, P. Zograf, and A. Zorich. Masur-Veech volumes, frequencies of simple closed geodesics, and intersection numbers of moduli spaces of curves. *Duke Math. J.*, **170** (2021), 2633–2718.
[3] V. Delecroix, É. Goujard, P. Zograf, and A. Zorich. Large genus asymptotic geometry of random square-tiled surfaces and of random multicurves. *Invent. Math.*, **230** (2022), 123–224.
[4] V. Delecroix and M. Liu. Length partition of random multicurves on large genus hyperbolic surfaces. arXiv:2202.10255, accepted in JEMS.
[5] V. Delecroix and M. Liu. Small components of random multicurves on large genus hyperbolic surfaces. in preparation.

Toledo Invariants of Quantum Representations of the Mapping Class Group

Bertrand Deroin

The moduli spaces $\mathcal{M}_{g,n}$ of genus g curves with n marked points, do not seem to have interesting geometric structures in general, nor their partial compactifications. However, some very interesting curiosities happen in particular cases. For example, the Torelli map sending a curve to its Jacobian provides the compact type partial compactification of $\mathcal{M}_2$ and $\mathcal{M}_3$ with a structure locally modelled on the Siegel spaces. Other examples have been studied by Deligne and Mostow: they build complex hyperbolic structures on certain partial compactifications of $\mathcal{M}_{g,n}$ when $g = 0$ and a finite list of n's, using hypergeometric integrals. In all these examples, a key role is played by the holonomy of the corresponding structure: a linear representation of the corresponding mapping class group $\mathrm{Mod}_{g,n}$.

The original motivation of this work is to investigate wether quantum representations might provide interesting new geometric structures on moduli spaces and/or their partial compactifications. Quantum representations are projective representations from the mapping class groups to the projective linear group of a vector space called the conformal block arising from a modular category. A consequence of the topological construction is that they take values in a projective pseudo-unitary group $\mathrm{PU}(p, q)$ for some integers p, q. We find the following examples:

For $(g, n) \in \{(0, 4), (1, 1), (0, 5), (1, 2), (1, 3), (2, 1)\}$, there exists a certain compactification $\overline{\mathcal{M}_{g,n}}^{\mathcal{E}}$ of $\mathcal{M}_{g,n}$ carrying a complex hyperbolic structure whose holonomy is the quantum representation associated to $SO(3)$ Quantum Field Theory of level 5.

The corresponding compactifications of $\mathcal{M}_{g,n}$ are obtained from Deligne-Mumford's one by taking the fifth root over the boundary, and by contracting the elliptic tail divisor. It turns out that this produces a smooth orbifold in general, unless in the case $(g, n) = (2, 0)$. The complex hyperbolic structure comes from classical uniformization of curves in the first two examples, and in the $(0, 5)$ and $(1, 2)$ cases was known to Hirzebruch/Deligne-Mostow and Livne respectively. The last two ones are genuinely new. We note that the natural forgetful map $\overline{\mathcal{M}_{1,3}}^{\mathcal{E}} \to \overline{\mathcal{M}_{1,2}}^{\mathcal{E}}$ produces a dominant morphism between complex hyperbolic manifolds of dimension 3 and 2, answering by the negative to a problem of Siu.

The proof consists in computing Toledo invariants of the Fibonacci quantum representations: we put this computation in a broader context, replacing them with any Hermitian modular functor and extending the Toledo invariant to a full series of cohomological invariants beginning with the signature $p - q$. We prove that these invariants satisfy the axioms of a Cohomological Field Theory and compute the R-matrix at first order (hence the usual Toledo invariants) in the case of the $SU2/SO3$-quantum representations at any level.

Long Random Multicurves

Viveka Erlandsson

Let S be an orientable finite-type surface of negative Euler characteristic, say of genus g with r punctures. A multicurve γ on S is a finite union of free homotopy classes of (always closed) curves, possibly with positive weights. If S is equipped with a hyperbolic metric then there is a unique geodesic in each homotopy class and we define the length of γ to be the (weighted) sum of the lengths of its components. The mapping class group $\mathrm{Map}(S) = \mathrm{Homeo}^+/\{\text{homotopy}\}$ acts on the set of multicurves in a natural way, and we say that two multicurves are of the *same type* if they lie in the same orbit.

Given a multicurve γ^o Mirzakhani obtained the asymptotic growth of the number of multicurves of type γ^o and of hyperbolic length at most L [10, 11]:

$$\#\{\gamma \text{ of type } \gamma^o \mid \ell(\gamma) \leq L\} \sim C \cdot L^{6g-6+2r} \tag{1}$$

where $C > 0$ is a constant depending on the hyperbolic metric and γ^o (see [10, 6] for more details on the constant) and $\sim$ denotes precise asymptotics, that is, the ratio of the two sides goes to 1 as $L \to \infty$.

Mirzakhani first proved (1) in the case that γ^o is simple in [10] and then, using very different methods, for general γ^o in [11]. In the simple case, she deduced it from the convergence of a certain family of measures on the space $\mathrm{ML}(S)$ of measured laminations (which can be viewed as the completion of weighted simple curves under a natural identification). In joint work with Juan Souto, and motivated by the ideas in [10] we gave a new proof of (1) using convergence of analogous measures, but now in a setting that also allows for non-simple curves.

Instead of $\mathrm{ML}(S)$ we consider the space $C(S)$ of geodesic currents, a nice topological space containing (under appropriate identifications) all multicurves as well as measured laminations (in fact, it can be obtained as the closure of weighted curves). We then consider the following family of measures on $C(S)$: given a multicurve γ^o and $L > 0$, set

$$m_L^{\gamma^o} = \frac{1}{L^{6g-6+2r}} \sum_{\gamma \text{ type } \gamma^o} \delta_{\frac{1}{L}\gamma}$$

where δ_p denotes the Dirac measure centered at p. We show in [6] that these converge to a multiple of a natural measure, the Thurston measure m_{Thu}:

Theorem 1. *For any multicurve γ^o on S there exists $c = c(\gamma_0)$ such that*

$$m_L^{\gamma_0} \to c \cdot m_{\mathrm{Thu}}$$

in the weak- topology on $C(S)$ as $L \to \infty$.*

Again, we refer to [10, 6] for more information on the constant c and we stress that Mirzakhani already proved Theorem 1 in [10] in the case γ^o is simple and viewing the measures on $\mathrm{ML}(S)$.

The asymptotics (1) follow easily from Theorem 1 by evaluating the measures at the unit ball with respect to the hyperbolic length function on $C(S)$. That this length function extends continuously from curves to currents is a result by Bonahon

[2]. In fact there are many length functions that extend nicely to currents (those induced by negatively curved metrics or Euclidean structures [12, 4], word length with respect to standard generators of $\pi_1(S)$ [5], extremal length with respect to conformal structures [9], to mention a few) and Theorem 1 implies that (1) also holds when the length is replaced by any such length. More precisely, to any $F : C(S) \to \mathbb{R}_{\geq 0}$ which is continuous, positive, and homogeneous in the sense that $F(c\lambda) = cF(\lambda)$ for all $c > 0$, which we refer to as a *nice length function.*

Being able to count multicurves of fixed type, ordered with respect to a nice length function, allows one to talk about a random multicurve: choose one uniformly at random among those of length at most L and let L go to infinity. What properties could you expect? For example, if we think of a multicurve $\gamma = \sum \gamma_i$ as a *labeled* multicurve $\vec{\gamma} = (\gamma_1, \dots \gamma_k)$ and take a nice length function F, how do the vectors $(F(\gamma_1), \dots, F(\gamma_k))$ distribute in $\mathbb{R}^k$? This question was studied by Mirzakhani for hyperbolic length and pair of pants decompositions [11] and was generalized both by Liu [8] and Arana-Herrera [1] to simple multicurves. In [7] we generalize their distribution results to any multicurve and to any nice length function. More precisely, writing $\mathbb{R}^k = \Delta_k \times \mathbb{R}_{\geq 0}$, where Δ_k is the standard $(k-1)$-simplex, and

$$\mathbb{L}_F(\gamma_1, \dots, \gamma_k) = \left(\frac{1}{\sum F(\gamma_i)} (F(\gamma_1), \dots, F(\gamma_k)), \sum F(\gamma_i) \right)$$

we consider the probability measures on $\Delta_k \times \mathbb{R}_{\geq 0}$ given by

$$\underline{m}(\vec{\gamma}^o, L, F) = \frac{1}{M(\vec{\gamma}^o, L, F)} \sum_{\vec{\gamma} \text{ type } \vec{\gamma}^o} \delta_{\mathbb{L}_F(\frac{1}{L}\gamma)}$$

where $M(\vec{\gamma}^o, L, F)$ denotes the number of labeled multicurves of type $\vec{\gamma}^o$ of F-length at most L and prove that:

Theorem 2. *For every labeled multicurve $\vec{\gamma}^o = (\gamma_1^o, \dots, \gamma_k^o)$ and any nice length function F,*

$$\underline{m}(\vec{\gamma}^o, L, F) \to p_{\vec{\gamma}^0} \otimes \left((6g - 6 + 2r) \cdot t^{6g-7+2r} \cdot dt \right)$$

as $L \to \infty$ in the weak- topology on $\Delta_k \times \mathbb{R}_{\geq 0}$, where dt is the standard Lebesgue measure and $p_{\vec{\gamma}^0}$ is a probability measure of Δ_k* independent *of F.*

The measure $p_{\vec{\gamma}^o}$ can in theory, and in some instances explicitly, be computed (see [7] for more details).

As mentioned above, Theorem 2 was already proved in [11, 8, 1] for simple multicurves and the hyperbolic metric, however, their methods involve studying the dynamics of the earthquake flow, while we instead generalize Theorem 1 to study the distribution of labeled multicurve inside the product space $C(S)^k = C(S) \times \cdots \times C(S)$ of so-called *k-currents*: for a labeled multicurve $\vec{\gamma}^o$ and $L > 0$ we study the measures

$$m(\vec{\gamma}^o, L) = \frac{1}{L^{6g-6+2r}} \sum_{\vec{\gamma} \text{ type } \vec{\gamma}^o} \delta_{\frac{1}{L}\vec{\gamma}}$$

on $C(S)^k$. Using similar arguments which prove convergence on $C(S)$ of the measures $m^L_{\gamma_0}$ we show that these measures converge on $C^k(S)$, which we then can use to deduce Theorem 2.

Theorem 3. *For any labeled multicurve $\vec{\gamma}^o \in C^k(S)$ there exists a measure $q_{\vec{\gamma}^o}$ on Δ_k such that*

$$m(\vec{\gamma}^o, L) \to \mathbb{D}_*(q_{\vec{\gamma}^o} \otimes m_{\mathrm{Thu}})$$

where $\mathbb{D} : \Delta_k \times \mathrm{ML}(S) \to C(S)^k$ is given by $\mathbb{D} : ((a_i), \lambda) \mapsto (a_1\lambda, \ldots, a_k\lambda)$ and the convergence is with respect to weak- convergence on $C(S)^k$.*

Again, the measure $q_{\vec{\gamma}^o}$ can in theory be computed and we refer to [7] for more details and a couple of examples where it is explicitly done. The measure $p_{\vec{\gamma}^o}$ in Theorem 2 is the probability measure corresponding to $q_{\vec{\gamma}^o}$, that is $p_{\vec{\gamma}^o} = q_{\vec{\gamma}^o}/||q_{\vec{\gamma}^o}||$.

We emphasize that the measure $p_{\vec{\gamma}^0}$ is independent of the length function F, and in particular, if we restrict to hyperbolic metrics independent on which point in Teichmüller space we consider. This, of course, was also the case in the earlier work on simple multicurves [11, 8, 1] which was very cleverly exploited by Liu and Delacroix in [3] to study large genus asymptotics. For example, they managed to answer questions such as: Given a random simple multicurve on a surface of genus g, what percentage of the total length does its longest component have as $g \to \infty$? It would be very interesting to study similar questions for general multicurves, that is, investigate how the measures $p_{\vec{\gamma}^0}$ (say, as we take the union of all $\vec{\gamma}^o$ with at most i self-intersections) in Theorem 2 vary with g. However, to be able to do so, these measures must first be much better understood.

References

[1] F. Arana-Herrera, *Counting hyperbolic multigeodesics with respect to the lengths of individual components and asymptotics of Weil-Petersson volumes*, Geom. Topol. **26** (2022).

[2] F. Bonahon, *The geometry of Teichmüller space via geodesic currents*, Invent. Math. **92** (1988).

[3] V. Delecroix and M. Liu, *Length partition of random multicurves on large genus hyperbolic surfaces*, to appear in JEMS, `arXiv:2202.10255`

[4] M. Duchin, C. J. Leininger, and K. Rafi, *Length spectra and degeneration of flat metrics*, Invent. Math. **182** (2010), 231–277.

[5] V. Erlandsson, *A remark on the word length in surface groups*, Trans. Amer. Math. Soc. **372** (2019).

[6] V. Erlandsson and J. Souto, *Mirzakhani's curve counting and geodesic currents*, Progr. Math., 345 Birkhäuser/Springer, 2022.

[7] V. Erlandsson and J. Souto, *On the distribution of the components of multicurves of given type*, `arXiv:2403.18544`

[8] M. Liu, *Length statistics of random multicurves on closed hyperbolic surfaces*, Groups Geom. Dyn. **16** (2022).

[9] D. Martínez-Granado and D. Thurston, *From curves to currents*, Forum Math. Sigma **9** (2021).

[10] M. Mirzakhani, *Growth of the number of simple closed geodesics on hyperbolic surfaces*, Ann. of Math. (2) **168** (2008).

[11] M. Mirzakkhani, *Counting Mapping Class group orbits on hyperbolic surfaces*, arXiv:1601.03342
[12] J.-P. Otal, *Le spectre marqué des longueurs des surfaces à courbure négative*, Ann. of Math. (2) **131** (1990), 151–162.

Siegel Veech Transforms: Using Second Moments to Count Pairs

Samantha Fairchild

Consider a translation surface (X, ω) where X is a compact Riemann surface and ω is a non-zero holomorphic 1-form. See for example Figure 1 where two Riemann surfaces are given by complex translations gluing a polygon in the plane, and the holomorphic one form is locally dz with a zero when the angle around a vertex is more than 2π. For more information see [7]. Let $G = \mathrm{SL}_2(\mathbf{R})$ act linearly in the plane. The *Veech group* $\Gamma \subset G$ stabilizes (X, ω). A *Veech surface* is a translation surface whose Veech group is a nonuniform lattice: a discrete subgroup so that G/Γ is not compact, but has finite volume.

We study the dynamics of the straight line flow on a translation surfaces (X, ω). Special trajectories of the flow are saddle connections, which start and end at zeroes of ω with no zeroes in between. Given a saddle connection γ, we can define it's holonomy vector $v_\gamma = \int_\gamma d\omega \in \mathbb{C}$. Define

$$\Lambda_\omega = \{v_\gamma | \gamma \text{ is a saddle connection}\}.$$

To understand the distribution, we can first ask about the growth of the function

$$N_\omega(R) = \#\{z \in \Lambda_\omega : |z| < R\}.$$

Following Masur's seminar bounds [8] showing the quadratic growth of $N_\omega(R)$, over 20 years, [5, 9, 11, 12] showed the exact growth rate of $N_\omega(R)$ with quantitative error terms. To better understand the distributions of Λ_ω, we now want to investigate how pairs of holonomy vectors interact.

Siegel–Veech transforms. To understand $N_\omega(R)$, a common technique is to use the Siegel–Veech transform: given h a bounded function with compact support on $\mathbf{C}$, the Siegel–Veech transform is a map on translation surfaces

$$h_{SV}(\omega) = \sum_{v \in \Lambda_\omega} h(v).$$

Notice that if h is the characteristic function of the ball of radius R, then $N_\omega(R) = h_{SV}(\omega)$.

We now define a more generalized Siegel–Veech transformed used to detect pairs. Given f a bounded function with compact support on $\mathbf{C}^2$, the Siegel–Veech theta-transform is

$$\Theta_f(\omega) = \sum_{\mathbf{x}, \mathbf{y} \in \Lambda_\omega} f(\mathbf{x}, \mathbf{y}).$$

Notice if $f(\mathbf{x}, \mathbf{y}) = h(\mathbf{x})h(\mathbf{y})$ for $h \in B_c(\mathbf{C})$, then $\Theta_f(\omega) = h_{SV}(\omega)^2$, thus Θ_f is a generalized second moment of the Siegel–Veech transform. The average value of h_{SV} was well studied by [11], and the key result given in [6, 2] is a mean value

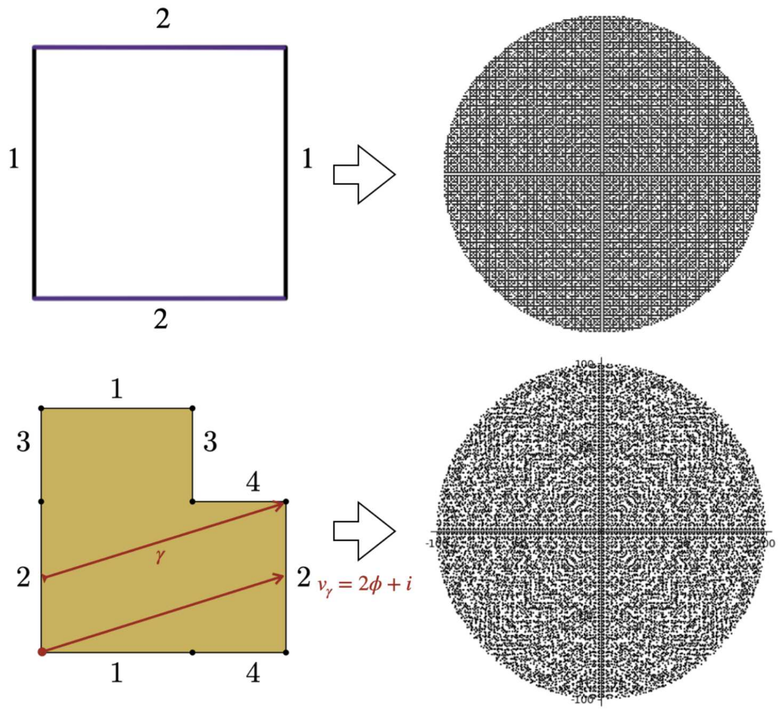


FIGURE 1. On the top left is the square torus where opposite sides are identified by translation, and below is the golden L whose side length is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. A saddle connection γ is a straight line flow starting and ending at vertex. The corresponding holonomy vector $v_\gamma \in \mathbb{C}$ keeps track of the horizontal and vertical displacement. The graphs on the right plot holonomy vectors corresponding to saddle connections of length at most 100.

formula for Θ_f when ω is a Veech surface, in which case Λ_ω is a finite disjiont union of orbits of Γ [10]. Hence one can hope to say more, and in particular to specify results to individual surfaces.

Results for Veech Surfaces. Instead of stating the mean value formula for Θ_f, we will focus on sharing the applications of the result for the case of Veech surfaces. If (X, ω) is a Veech surface, [10, 3] gives

$$N_\omega(R) = c_\omega R^2 + O\left(R^{2-\delta}\right), \tag{1}$$

where the power-saving $\delta = \delta(\Gamma)$ is explicit. The first application recovers a (nearly) optimal count for typical shapes.

Theorem 1 ([2])**.** *Let $\Omega \subset \mathbf{R}^2$ be a bounded Borel set that contains the origin and consider its dilates $\Omega_R = R \cdot \Omega$. Then for almost every*[1] *linear transformation g we have*

$$|A(\Lambda_\omega) \cap \Omega_R||\det(A)| = c_\omega |\Omega| R^2 + O\left(R^{2-\delta} \log^{3/2}(R)\right),$$

where $\delta = \delta(\Gamma)$ is as in 1 and $\varepsilon > 0$.

Weak uniform discreteness. For the second application, recall that a discrete planar set is η-uniformly discrete if $|\Lambda_\omega \cap B_\eta(\mathbf{x})| \leq 1$ for all $\mathbf{x} \in \mathbf{R}^2$. When Γ is non-arithmetic, [13] showed that Λ_ω is not uniformly discrete. The next theorem quantifies the failure of uniform discreteness when (X, ω) is a nonarithmetic Veech surface.

Theorem 2. *For any Veech surface, for each $\varepsilon > 0$ there is an $\eta > 0$ such that*

$$\limsup_{R \to \infty} \frac{|\{\mathbf{x} \in \Lambda_\omega \cap B_R : |\Lambda_\omega \cap B_\eta(\mathbf{x})| \geq 2\}|}{|\Lambda_\omega \cap B_R|} < \varepsilon.$$

This has a remarkable application proved in the appendix of [2].

Theorem 3. *For any Veech surface, for Lebesgue almost every pair $(\theta, \psi) \in S^1 \times S^1$, the translations flows in directions θ and ψ are disjoint, and thus not isomorphic.*

This provides the first family of surfaces other than branched covers of tori for which the flows in almost every pair of directions are not isomorphic. If Theorem 3 could be extended to all surfaces, we would even be able to recover the result of Chaika and Forni that there is a weakly mixing billiard in a polygon [4].

Counting pairs with bounded determinant. The same arguments also recover an upper bound on pairs of saddle connections with bounded determinant. For a vector $\mathbf{x} \in \mathbf{R}^2$, set

$$\mathcal{D}_{D,1}(\mathbf{x}) = \{\mathbf{y} \in \mathbf{R}^2 : |\mathbf{y}| \leq |\mathbf{x}| \text{ and } |\mathbf{x} \wedge \mathbf{y}| \leq D\}.$$

For a typical surface M, the second author with Athreya and Masur [1] showed that for $D > 0$ there is a *non-explicit* constant $C_D > 0$ so that

$$\lim_{R \to \infty} \frac{|\{(\mathbf{x}, \mathbf{y}) \in S_M \times S_M : \mathbf{x} \in B_R,\ \mathbf{y} \in \mathcal{D}_{D,1}(\mathbf{x})|}{R^2} = C_D.$$

Theorem 4. *Let M be a Veech surface. For any $D > 0$, there are constants C_M and c depending only on M so that*

$$\limsup_{R \to \infty} \frac{|\{(\mathbf{x}, \mathbf{y}) \in S_M \times S_M : \mathbf{x} \in B_R,\ \mathbf{y} \in \mathcal{D}_{D,1}(\mathbf{x})\}|}{R^2} \leq C_M(D + c).$$

Note we have two terms in the upper bound. This comes from the fact that when D is small, there are essentially only parallel pairs, and thus a constant multiple of the Siegel–Veech constant c_Γ will be dominating (cf. [1, Theorem 1.2]). However for D large, we have an upper bound which is asymptotically linear in D. There

[1] With respect to the Euclidean metric induced by the matrix representation of A.

is much we still don't know about the distribution of saddle connections, which we will study in future work.

References

[1] J. S. Athreya, S. Fairchild, and H. Masur. Counting pairs of saddle connections. *Adv. Math.*, **431**, (2023). Id/No 109233.
[2] C. Burrin, S. Fairchild, and J. Chaika. Pairs in discrete lattice orbits with applications to Veech surfaces. Preprint, arXiv:2211.14621 [math.DS] (2022).
[3] C. Burrin, A. Nevo, R. Rühr, and B. Weiss. Effective counting for discrete lattice orbits in the plane via Eisenstein series. *Enseign. Math.*, **66** (2020), 259–304.
[4] J. Chaika and G. Forni. Weakly Mixing Polygonal Billiards. *arxiv:2003.00890*, (2020).
[5] A. Eskin and H. Masur. Asymptotic formulas on flat surfaces. *Ergodic Theory Dynam. Systems*, **21**, (2001), 443–478.
[6] S. Fairchild. A higher moment formula for the Siegel-Veech transform over quotients by Hecke triangle groups. *Groups Geom. Dyn.*, **15**, (2021), 57–81.
[7] D. Massart. A short introduction to translation surfaces, Veech surfaces, and Teichmüller dynamics. In *Surveys in geometry I*, pages 343–388. Cham: Springer, 2022.
[8] H. Masur. Closed trajectories for quadratic differentials with an application to billiards. *Duke Math. J.*, **53**, (1986), 307–314.
[9] A. Nevo, R. Rühr, and B. Weiss. Effective counting on translation surfaces. *Adv. Math.*, **360**, (2020).
[10] W. A. Veech. Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Invent. Math.*, **97**, (1989), 553–583.
[11] W. A. Veech. Siegel measures. *Ann. of Math. (2)*, **148**, (1998), 895–944.
[12] Y. Vorobets. Periodic geodesics on translation surfaces. *arxiv:math/0307249*, 2003.
[13] C. Wu. Deloné property of the holonomy vectors of translation surfaces. *Israel J. Math.*, **214**, (2016), 733–740.

Equidistribution in moduli space and the orthogeodesic foliation

James Farre
(joint work with Aaron Calderon)

1. Twist tori

Given a pants decomposition $\mathcal{P}$ of a closed oriented surface S of genus $g \geq 2$ and a positive length vector $L \in \mathbb{R}^{\mathcal{P}}_{>0}$, we consider the *twist torus*

$$\mathbb{T}_{\mathcal{P}}(L) \subset \mathcal{M}_g$$

of hyperbolic metrics with pants of type $\mathcal{P}$ whose lengths are given by L. Then $\mathbb{T}_{\mathcal{P}}(L)$ is an immersed (finite quotient) of a torus of dimension $3g-3$, equipped with a natural homogeneous probability measure $\tau_{\mathcal{P}}(L)$ in the class of Lebesgue. For $t \to +\infty$, $\mathbb{T}_{\mathcal{P}}(e^{-t}L)$ exits the end of $\mathcal{M}_g$.

Question 1 (Mirzakhani)**.** How do twist tori $\mathbb{T}_{\mathcal{P}}(e^t L)$ distribute in $\mathcal{M}_g$ as $t \to \infty$?

With Aaron Calderon, we prove that twist tori equidistribute *on average* to a limiting measure that depends on $\mathcal{P}$ and L.

Theorem 2 ([4]). *Given L and $\mathcal{P}$, there is a Borel probability measure μ on $\mathcal{M}_g$ such that*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \tau_{\mathcal{P}}(e^t L)\, dt = \mu,$$

in the weak-$*$ topology on probability measures on $\mathcal{M}_g$.

Moreover, μ is the Mirzakhani measure *in the Lebesgue measure class if, for all curves $a, b, c \subset \mathcal{P}$ that bound a pair of pants, we have $L(a) + L(b) \neq L(c)$.*

The Mirzakhani measure is absolutely continuous with respect to the Weil–Petersson volume measure on $\mathcal{M}_g$ with density function

$$X \in \mathcal{M}_g \mapsto B(X) = \mu_{\text{Thurston}}(\{\ell_X(\lambda) \leq 1\}),$$

where μ_{Thurston} is the *Thurston measure* on the space $\mathcal{ML}$ of measured laminations, and $\ell_X : \mathcal{ML} \to \mathbb{R}_{>0}$ is the hyperbolic length function.

The strategy of our proof is to translate the question into the language of the $\mathsf{PSL}(2,\mathbb{R})$-action on strata of holomorphic quadratic differentials using the *orthogeodesic foliation* construction. We then use powerful tools from Teichmüller dynamics to tackle the translated problem. We prove that the orthogeodesic foliation construction and its inverse are *continuous enough* to again translate the solution back to the original setting and conclude.

2. The orthogeodesic foliation

To a hyperbolic metric $X \in \mathcal{T}(S)$ and a measured geodesic lamination $\lambda \in \mathcal{ML}$, the *orthogeodesic foliation* $\mathcal{O}_\lambda(X)$ is constructed geometrically from the support of λ and the hyperbolic geometry of X. The transverse measure on $\mathcal{O}_\lambda(X)$ measures "length along λ," and there are isolated singularities of $\mathcal{O}_\lambda(X)$ at points in the complementary components of λ in X that have more than one shortest path to a leaf of λ. See [2] for details of the construction. Thurston's construction of the *horocycle foliation* for maximal geodesic laminations coincides with ours (up to measure equivalence) in the case that the support of λ cuts X into ideal triangles.

Denote by $\mathcal{PT}_g$ the flat $\mathcal{ML}$-bundle over $\mathcal{T}_g$, the Teichmüller space of isotopy classes of hyperbolic metrics on S. The quotient by the mapping class group is $\mathcal{PM}_g$. Let $\mathcal{QT}_g$ be the complex vector bundle of holomorphic quadratic differentials over $\mathcal{T}_g$. Each such point corresponds to a singular flat metric on S with isolated singularities where there is excess angle a positive integer multiple of π and distinguished real and imaginary measured foliations whose leaves are vertical and horizontal, respectively.

Theorem 3 ([2]). *There is a (unique) mapping class group equivariant bijection*

$$\mathcal{O} : \mathcal{PT}_g \to \mathcal{QT}_g$$

such that $\mathcal{O}(X,\lambda)$ has imaginary foliation measure equivalent to λ and has real foliation isotopic to $\mathcal{O}_\lambda(X)$; it is an extension of Mirzakhani's map defined on pairs (X,λ), where λ is maximal [7].

The map $\mathcal{O}$ is not everywhere continuous (as was pointed out by Mirzakhani [7]), but the set of points which are points of continuity are fairly ubiquitous.

Theorem 4 ([3])**.** *Suppose $\lambda_n \to \lambda$ in the measure topology and in the Hausdorff topology on geodesic laminations. Then $\mathcal{O}(X_n, \lambda_n) \to \mathcal{O}(X, \lambda)$ as $n \to \infty$ in $\mathcal{QT}_g$ if and only if $X_n \to X \in \mathcal{T}_g$.*

Using this theorem together with establishing some conditions which convergence in the measure topology of measured laminations implies Hausdorff convergence of their supports, we obtain the following.

Theorem 5 ([3])**.** *For every stratum $\mathcal{Q}$ of unit area holomorphic quadratic differentials, every point without horizontal saddle connections is a point of continuity for $\mathcal{O}^{-1}|_{\mathcal{Q}}$.*

Lebesgue almost every point is a point of continuity for $\mathcal{O}$.

An elementary result from measure theory gives the following.

Corollary 6 ([3])**.** *Let $\mathcal{Q}$ be a stratum of unit area holomorphic quadratic differentials. Let ν_n be a sequence of Borel probability measures on $\mathcal{Q}$ that converge weak-$*$ to some ν giving zero mass to quadratic differentials with a horizontal saddle connection. Then the sequence of Borel probability measures $\mathcal{O}_*^{-1}\nu_n$ on $\mathcal{PM}_g$ converges weak-$*$ to $\mathcal{O}_*^{-1}\nu$.*

Let μ_n be Borel probability measures on $\mathcal{PM}_g$ that converge weak-$*$ to a measure μ in the class of Lebesgue. Then $\mathcal{O}_*\mu_n \to \mathcal{O}_*\mu$ on the principal stratum of quadratic differentials.

3. Sketch of proof of the main theorem

First, the twist torus $\mathbb{T}_{\mathcal{P}}(e^t L)$ lifts to a torus

$$\widehat{\mathbb{T}}_{\mathcal{P}}(e^t L) = (\mathbb{T}_{\mathcal{P}}(e^t L), \mathcal{P}/e^t \|L\|_1) \subset \mathcal{PM}_g,$$

i.e., every surface is equipped with the pants curves of $\mathcal{P}$, weighted so that their total hyperbolic length is 1. Then $\mathcal{O}(\widehat{\mathbb{T}}_{\mathcal{P}}(e^t L))$ is a *flat twist torus* in a stratum $\mathcal{Q}$ of unit are holomorphic quadratic differentials. The corresponding flat surfaces are glued together from horizontal cylinders coming from the curves of $\mathcal{P}$ with small height and long length, as $t \to \infty$. The stratum component can be read off of $\mathcal{P}$ and L; the condition that $L(a) + L(b) \neq L(c)$ for all curves $a, b, c \subset \mathcal{P}$ bounding pants ensures that the flat twist torus lands in the principal stratum.

The Lebesgue measure $\tau_{\mathcal{P}}(e^t L)$ lifts and pushes forward by $\mathcal{O}$ to a Lebesgue measure on the flat twist torus. The Teichmüller geodesic flow g_t satisfies $(g_t \circ \mathcal{O})_* \widehat{\tau}_{\mathcal{P}}(L) = \mathcal{O}_* \widehat{\tau}_{\mathcal{P}}(e^t L)$, and these measures are invariant for the Teichmüller horocycle flow. Seminal work of Eskin, Mirzakhani, and Mohammadi [5, 6] gives that

$$\frac{1}{T} \int_0^T (g_t \circ \mathcal{O})_* \widehat{\tau}_{\mathcal{P}}(L)\ dt \to \nu$$

weak-$*$ as $T \to \infty$, and ν is an measure supported on an *affine invariant submanifold*. In particular, ν gives zero mass to quadratic differentials with a horizontal saddle connection. Thus we can apply Corollary 6 to see that

$$\frac{1}{T} \int_0^T \widehat{\tau}_{\mathcal{P}}(e^t L)\ dt \to \mathcal{O}_*^{-1}\nu.$$

Pushing this down to $\mathcal{M}_g$, we recover the equidistribution result from Theorem 2.

The only thing left to do is identify ν. This is achieved by applying recent work of Apisa–Wright [1] that *high rank* affine invariant submanifolds are large. We estimate the rank of the support of ν using certain *cylinder deformations* [8]. The details are carried out in [4, Section 5].

We do not know if the following statement holds true.

Conjecture 7. As $t \to \infty$, we have that $(g_t \circ \mathcal{O})_* \widehat{\tau}_{\mathcal{P}}(L)$ converges weak-$*$ to the Masur–Veech–Smillie measure on the stratum component into which $\mathcal{O}(\widehat{\mathbb{T}}_{\mathcal{P}}(L))$ lands, so that the averaging step in Theorem 2 can be removed.

References

[1] P. Apisa and A. Wright, *High rank invariant subvarieties*, Ann. of Math. (2) **198** (2023), 657–726.

[2] A. Calderon and J. Farre, *Shear–shape cocycles for measured laminations and ergodic theory of the earthquake flow*, Geom. Top. *(to appear)*, arXiv:2102.13124, (2021).

[3] ———, *Continuity of the orthogeodesic foliation and ergodic theory of the earthquake flow*, Preprint, arXiv:2401.12299, (2024).

[4] A. Calderon and J. Farre, *On Mirzakhani's twist torus conjecture*, (2024).

[5] A. Eskin and M. Mirzakhani, *Invariant and stationary measures for the* $\mathrm{SL}(2,\mathbb{R})$ *action on moduli space*, Publ. Math. Inst. Hautes Études Sci. **127** (2018), 95–324.

[6] A. Eskin, M. Mirzakhani, and A. Mohammadi, *Isolation, equidistribution, and orbit closures for the* $\mathrm{SL}(2,\mathbb{R})$ *action on moduli space*, Ann. of Math. (2) **182** (2015), 673–721.

[7] M. Mirzakhani, *Ergodic theory of the earthquake flow*, Int. Math. Res. Not. IMRN **2008**, (2008).

[8] A. Wright, *Cylinder deformations in orbit closures of translation surfaces*, Geom. Topol. **19** (2015), no. 1, 413–438.

The Diagonal Flow, Topology, and Lyapunov Exponents

Vaibhav Gadre

By the Gauss–Bonnet theorem, an orientable finite type surface with negative Euler characteristic cannot support a flat metric; it does support a flat metric outside a finite set of singular points with the expected "excess angle" concentrated at these singular points. The subclass of singular flat metrics that arise from holomorphic abelian or meromorphic (with simple poles) quadratic differentials are important in many contexts, such as, for example, billiards in rational polygons.

Suppose that S is an oriented surface with finite genus and finitely many marked points. Such a surface carries a conformal structure by charts to $\mathbb{C}$ with holomorphic transitions. The Teichmüller space is the space of marked conformal structures on S. The mapping class group $\mathrm{Mod}(S)$ is the group of orientation-preserving diffeomorphisms of S up to isotopy. $\mathrm{Mod}(S)$ acts on $\mathrm{Teich}(S)$, and the quotient is the moduli space $\mathcal{M}$ of Riemann surfaces. The cotangent bundle of $\mathcal{M}$ is the moduli space of meromorphic quadratic differentials on S with simple poles at marked points and only at marked points. This includes the squares of holomorphic 1-forms (abelian differentials). Contour integration of a square-root of a quadratic differential defines charts from S to $\mathbb{C}$. The transition functions are now of the form $z \to \pm z + c$, that is translations or half-translations. This defines a singular flat metric on S with singularities only at the zeroes and poles of the differential.

The moduli space of such differentials is stratified by the orders (of the differentials) at the singularities and if the orders allow it, whether or not the differentials are squares of abelian differentials. The components of the strata were classified by Kontsevich–Zorich [16], Lanneau [17] and Chen–Möller [7]. Periods of a homology basis relative to the singularities (zeroes and poles of the differentials) defines charts on these moduli spaces. The $SL(2,\mathbb{R})$-action on $\mathbb{C} = \mathbb{R}^2$ preserves the form of the transition functions. Hence it descends to an action on each stratum component. See the survey [13] by Forni–Matheus for background and more details.

There are many viewpoints on these moduli spaces and the interplay between the many viewpoints produces striking results. A prime example is the celebrated result (combining work of Eskin–Mirzakhani [8], Eskin–Mirzakhani–Mohammadi [9] and Filip [11]) that $SL(2,\mathbb{R})$ orbit closures, which a priori are dynamical objects, are also special algebraic sub-varieties given by linear equations in period coordinates.

The diagonal part of the $SL(2,\mathbb{R})$-action is the Teichmüller flow. The Kontsevich–Zorich (KZ) cocycle is the dynamical cocycle that arises from the symplectic monodromy action on the absolute homology of S. Teichmüller flow is ergodic, in fact exponentially mixing, but these properties are subtle as the flow is not uniformly hyperbolic. Therefore, whether the absolute homology cocycle is also rich is a very pertinent question. The famous Kontsevich–Zorich (KZ) conjecture [15] asserts that the absolute cohomology cocycle has a simple Lyapunov spectrum asymptotically. For abelian differentials, this conjecture was proved by Forni in genus two [12] and Avila–Viana [3] for all abelian components. The general case for quadratic strata was open until now.

With Bell–Delecroix–Gutierrez–Romo–Schleimer [4], we resolve the KZ conjecture in the above generality (in technical terms, separately for the plus and minus pieces). Our solution is based on a key new idea that relates the dynamics directly to the topology (specifically, the fundamental group) of a stratum component. As a result, the proof is uniform across stratum components and simplifies Avila–Viana in a crucial way.

While stratum components or linear invariant subvarieties are typically orbifolds, they admit finite manifold covers. The space $\mathcal{C}^{\mathrm{root}}$ of rooted differentials (differentials with a marked horizontal separatrix) over a component $\mathcal{C}$ is a traditionally used finite manifold cover. We arrive at the chain of homomorphisms

$$\pi_1(\mathcal{C}^{\mathrm{root}}) \to \pi_1^{\mathrm{orb}}(\mathcal{C}) \to \pi_1^{\mathrm{orb}}(\mathcal{M}) = \mathrm{Mod}(S) \xrightarrow{\rho} \mathrm{Aut}(H_1(S;\mathbb{Z})) \cong \mathrm{Sp}(2g,\mathbb{Z}).$$

The image in $\mathrm{Mod}(S)$ is the modular monodromy defined using the (flat) Gauss–Manin connection and the whole composition gives the symplectic monodromy. The Avila–Viana criterion for Lyapunov simplicity boils down to “almost Bernoulli” (alternatively called approximate product structure) properties for the flow on $\mathcal{C}^{\mathrm{root}}$ and a certain notion of “largeness” (existence of pinching and twisting elements) for the cocycle. The Teichmüller flow lifted to $\mathcal{C}^{\text{root}}$ can be coded by interval exchange transformations (IET) as a suspension flow over the Rauzy-Veech renormalisation on the IET parameter spaces. The almost-Bernoulli properties for the coding require quantitative estimates for an acceleration of the renormalisation (see Avila–Gouëzel–Yoccoz [1] for abelian strata). Avila–Viana verify the required cocycle largeness directly for abelian strata.

In our work, we prove largeness a lot further up the above sequence. Namely, we consider the flow group $G(U, q_0)$, namely the subgroup of $\pi_1(\mathcal{C}^{\text{root}})$ generated by almost closed Teichmüller geodesic segments starting and ending in a contractible open set U containing the base-point q_0. Going deeper into IET theory, we prove:

Theorem 1. *Suppose that $\mathcal{C}$ is a component of a stratum of abelian/quadratic differentials and $\mathcal{C}^{root}$ a component of rooted differentials over $\mathcal{C}$. Suppose U is a contractible open set in $\mathcal{C}^{root}$ containing a base-point q_0. Then*

$$G(U, q_0) = \pi_1(\mathcal{C}^{root}, q_0).$$

By following along the sequence above, the largeness of the flow group enables us to verify cocycle largeness from the monodromy instead of using loops in Rauzy diagram. By work of Benoist, Zariski density implies the largeness property. By using Filip's work on algebraic hulls [10], we prove Zariski density of the symplectic monodromy of all stratum components (abelian or quadratic). This leads to our proof of the Kontsevich–Zorich conjecture for all strata of abelian/quadratic differentials, namely:

Theorem 2. *Suppose that $\mathcal{C}$ is a component of a stratum of abelian/quadratic differentials. Then the plus and minus Kontsevich–Zorich cocycles over $\mathcal{C}$ have a simple Lyapunov spectrum.*

While we prove Zariski density of the symplectic monodromies, the explicit description of the monodromies (and Rauzy–Veech groups) is still open, specifically the question if they are finite index subgroups of the symplectic group and if so precisely which ones? This information is known for abelian strata (and some quadratic strata) through works of Avila–Matheus–Yoccoz [2] and Gutierrez-Romo [14]. A related question is to describe modular monodromies as explicit subgroups of $\mathrm{Mod}(S)$. For abelian strata, Calderon–Salter show that the modular monodromy is a framed mapping class group (the stabiliser of a framing) giving a spin mapping class group over the closed surface [5, 6]. For quadratic strata, we expect it to be the stabiliser of a line field giving a similar spin description over the closed surface.

References

[1] A. Avila, S. Gouëzel, and J-C. Yoccoz, Exponential mixing for the Teichmüller flow. *Publ. Math. Inst. Hautes Études Sci.* **104** (2006), 143–211.

[2] A. Avila, C. Matheus, and J-P. Yoccoz, $SL(2, \mathbb{R})$-invariant probability measures on the moduli spaces of translation surfaces are regular. *Geom. Funct. Anal.* **23** (2013), 1705–1729.

[3] A. Avila, and M. Viana, Simplicity of Lyapunov spectra. *Acta Math.* **198** (2007), 1–56.

[4] M. Bell, V. Delecroix, V. Gadre, R. Gutierrez-Romo, and S. Schleimer, The flow group of rooted abelian or quadratic differentials. Preprint ArXiv: 2101.12197 (2021).

[5] A. Calderon, and N. Salter, Higher spin mapping class groups and strata of abelian differentials over Teichmüller space. *Adv. Math.* **389** (2021), Paper No. 107926, 56pp.
[6] A. Calderon, and N. Salter, Framed mapping class groups and the monodromy of strata of abelian differentials. *J. Eur. Math. Soc.* **25** (2023), 4719–4790.
[7] D. Chen, and M. Möller, Quadratic differentials in low genus: exceptional and non-varying strata. *Ann. Sci. Éc. Norm. Supér.* **47** (2014), 309–369.
[8] A. Eskin, and M. Mirzakhani, Invariant and stationary measures. *Publ. IHES* **127** (2018), 95–324.
[9] A. Eskin, M. Mirzakhani, and A. Mohammadi, Isolation, equidistribution and orbit closures for the $SL(2,\mathbb{R})$ action on moduli space. *Ann. of Math.* (2), **182** (2015), 673–721.
[10] S. Filip, Zero Lyapunov exponents and monodromy. *Duke Math J.* **166** (2017), 657–706.
[11] S. Filip, Splitting mixed Hodge structures. *Ann. of Math.* (2), **183** (2016), 681–713.
[12] G. Forni, Deviation of ergodic averages for area-preserving flows on surfaces of higher genus. *Ann. of Math.* (2), **155** (2002), 1–103.
[13] G. Forni, and C. Matheus, Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces, and billiards. *J. Mod. Dyn.* **8** (2014), 271–436.
[14] R. Gutierrez-Romo, Classification of RV groups. *Invent. Math.* **215** (2019), 741–778.
[15] M. Kontsevich, and A. Zorich, Lyapunov exponents and Hodge theory. Preprint (1997).
[16] M. Kontsevich, and A. Zorich, Connected components of the moduli space of abelian differentials with prescribed singularities. *Invent. Math.* **153** (2003), 631–678.
[17] E. Lanneau, Connected components of the strata of the moduli spaces of quadratic differentials. *Ann. Sci. Éc. Norm. Supér.* (4) **41**, (2008), 1–56.

Topological recursion and applications

Danilo Lewański

Let us package Witten's ψ-class correlators in a single generating series: let t_d (for $d \geq 0$) be a set of formal variables and set

(1)
$$Z(t_0, t_1, t_2, \dots; \hbar) = \exp\left(\sum_{\substack{g\geq 0,\, n\geq 1 \\ 2g-2+n>0}} \frac{\hbar^{2g-2+n}}{n!} \sum_{d_1,\dots,d_n\geq 0} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{i=1}^{n} t_{d_i} \right).$$

The generating series Z arises as a partition function in topological 2D quantum gravity. The string and dilaton equations may be written as differential operators annihilating Z.

Define the differential operators

$$L_{-1} = \hbar \frac{\partial}{\partial t_0} - \hbar^2 \left(\sum_{k\geq 1} t_k \frac{\partial}{\partial t_{k-1}} + \frac{t_0^2}{2} \right), \tag{2}$$

$$L_0 = \hbar \frac{\partial}{\partial t_1} - \hbar^2 \left(\sum_{k\geq 0} \frac{2k+1}{3} t_k \frac{\partial}{\partial t_k} + \frac{1}{24} \right). \tag{3}$$

For $n \geq 1$, these are given by

$$(4) \quad L_n = \hbar \frac{\partial}{\partial t_{n+1}} - \hbar^2 \Bigg(\sum_{k \geq 0} \frac{(2n+2k+1)!!}{(2n+3)!!(2k-1)!!} t_k \frac{\partial}{\partial t_{k+n}} + \frac{1}{2} \sum_{\substack{a,b \geq 0 \\ a+b=n-1}} \frac{(2a+1)!!(2b+1)!!}{(2n+3)!!} \frac{\partial^2}{\partial t_a \partial t_b} \Bigg).$$

Theorem 1 (Witten's conjecture/Kontsevich's theorem)**.** *The differential operators $(L_n)_{n \geq -1}$ annihilate the partition function Z:*

$$(5) \qquad L_n \; Z = 0 \qquad \forall n \geq -1 \,.$$

Moreover, the above system of equations (known as Virasoro constraints*) uniquely determine all intersection numbers.*

We remark that Witten's original formulation of his conjecture states that Z is the unique tau-function of the Korteweg–de Vries (KdV) hierarchy satisfying the string equation $L_{-1} \; Z = 0$. The KdV hierarchy is an infinite sequence of partial differential equations which extends in a certain sense the KdV equation. The equivalent statement in terms of Virasoro constraints was proved by R. Dijkgraaf, H. Verlinde, E. Verlinde.

The Virasoro constraints are equivalent to the following *topological recursion* for Witten's correlators:

$$(6) \quad \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \sum_{m=2}^{n} \frac{(2d_1+2d_m-1)!!}{(2d_1+1)!!\,(2d_m-1)!!} \langle \tau_{d_1+d_m-1} \tau_{d_2} \cdots \widehat{\tau_{d_m}} \cdots \tau_{d_n} \rangle_g + \frac{1}{2} \sum_{a+b=d_1-2} \frac{(2a+1)!!\,(2b+1)!!}{(2d_1+1)!!} \Bigg(\langle \tau_a \tau_b \tau_{d_2} \cdots \tau_{d_n} \rangle_{g-1} + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{ d_2, \ldots, d_n \}}} \langle \tau_a \tau_{I_1} \rangle_{g_1} \langle \tau_b \tau_{I_2} \rangle_{g_2} \Bigg).$$

Moreover, the above recursion is equivalent to the Eynard–Orantin topological recursion formula on the Airy spectral curve:

$$(\mathbb{P}^1, \; x(z) = \frac{z^2}{2}, \; y(z) = z, \; \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2})$$

producing the correlators

$$(7) \quad \omega_{g,n}(z_1, \ldots, z_n) = (-1)^n \sum_{\substack{d_1, \ldots, d_n \geq 0 \\ d_1 + \cdots + d_n = 3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{i=1}^{n} \frac{(2d_i+1)!!}{z_i^{2d_i+2}} \mathrm{d}z_i \,.$$

Through resurgence techniques one can compute the large genus asymptotic Witten's correlators:

$$(8) \qquad \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{i=1}^{n} (2d_i+1)!! = \frac{2^{n-1}}{2\pi} \frac{\Gamma(2g-2+n)}{(\frac{2}{3})^{2g-2+n}} \big(1 + \mathrm{O}(g^{-1})\big) \,.$$

Another example is given by the Cohomological Field Theory (CohFT) associated with the spectral curve

$$\left(\mathbb{P}^1,\ x(z) = -f\log(z) - \log(1-z),\ y(z) = -\log(z),\ B(z_1,z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}\right). \tag{9}$$

is the triple Hodge $\Lambda(1)\Lambda(f)\Lambda(-f-1)$. This is the CohFT underlying the framed topological vertex and the topological recursion formula for the triple Hodge class is nothing other than the BKMP remodelling conjecture for the vertex. The large framing limit recovers the so-called Lambert curve that computes Hurwitz numbers.

Another example is given by JT gravity. In this model the path integral of the theory is over the space of hyperbolic metrics (rather than the space of complex structures). In other words, the 'correct' moduli space is that of *hyperbolic structures*:

$$\mathcal{M}_{g,n}^{\mathrm{hyp}}(L_1,\ldots,L_n) = \left\{ X \;\middle|\; \begin{array}{c} X \text{ is a hyperbolic surface of genus } g \\ \text{with } n \text{ labelled geodesics boundaries} \\ \text{of lengths } L_1,\ldots,L_n \end{array} \right\} \Big/ \sim \tag{10}$$

where $X \sim X'$ if and only if there exists an isometry from X to X' preserving the labelling of the boundary components.

The space $\mathcal{M}_{g,n}^{\mathrm{hyp}}(L)$ is a smooth real orbifold of dimension $2(3g-3+n)$. Moreover, for all $L \in \mathbb{R}_+^n$, it is homeomorphic (as a smooth real orbifold) to the moduli space of smooth Riemann surfaces:

$$\mathcal{M}_{g,n}^{\mathrm{hyp}}(L) \cong \mathcal{M}_{g,n}\,. \tag{11}$$

For any fixed $L \in \mathbb{R}_+^n$, the moduli space $\mathcal{M}_{g,n}^{\mathrm{hyp}}(L)$ comes equipped with a natural symplectic form, called the Weil–Petersson form and denoted ω_{WP}. In particular, we can define the volumes

$$V_{g,n}^{\mathrm{WP}}(L) = \int_{\mathcal{M}_{g,n}^{\mathrm{hyp}}(L)} \frac{\omega_{\mathrm{WP}}^{3g-3+n}}{(3g-3+n)!}\,. \tag{12}$$

Under the homeomorphism $\mathcal{M}_{g,n}^{\mathrm{hyp}}(L) \cong \mathcal{M}_{g,n}$, the Weil–Petersson form extends as a closed form to $\overline{\mathcal{M}}_{g,n}$ and defines the cohomology class

$$2\pi^2\kappa_1 + \frac{1}{2}\sum_{i=1}^{n} L_i^2\,\psi_i\,. \tag{13}$$

An immediate consequence of the above result is that the Weil–Petersson volumes are finite (this was not obvious because $\mathcal{M}_{g,n}^{\mathrm{hyp}}(L)$ is not compact) and is a symmetric polynomial in boundary lengths squared whose coefficients are intersection numbers involving ψ-classes and $\exp(2\pi^2\kappa_1)$:

$$V_{g,n}^{\mathrm{WP}}(L) = \sum_{\substack{d_1,\ldots,d_n \geq 0 \\ d_1+\cdots+d_n \leq 3g-3+n}} \int_{\overline{\mathcal{M}}_{g,n}} e^{2\pi^2\kappa_1} \prod_{i=1}^{n} \psi_i^{d_i}\,\frac{L_i^{2d_i}}{2^{d_i}\,d_i!}\,. \tag{14}$$

These intersection numbers are precisely in the form of CohFT correlators, and as such can be computed by topological recursion. The spectral curve

$$\left(\mathbb{P}^1,\ x(z) = \frac{z^2}{2},\ y(z) = \frac{\sin(2\pi z)}{2\pi},\ \omega_{0,2}(z_1,z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}\right). \tag{15}$$

produces topological recursion correlators associated with Weil–Petersson volumes:

$$\omega_{g,n}(z_1,\ldots,z_n) = \mathrm{d}_{z_1}\cdots\mathrm{d}_{z_n}\left(\prod_{i=1}^{n}\int_0^\infty \mathrm{d}L_i\, e^{-z_iL_i}\right)V_{g,n}^{\mathrm{WP}}(L_1,\ldots,L_n). \tag{16}$$

Metric maps and hyperbolic surfaces in large genus

Baptiste Louf

(joint work with Svante Janson)

We start with the following two results about counting small curves in large genus:

Theorem 1 (Mirzakhani–Petri [7]). *Let $N_g^{hyp}(x,y)$ be the number of simple closed geodesics of length $\in [x,y]$ in a random*[1] *hyperbolic surface of genus g. As $g \to \infty$, we have*

$$N_g^{hyp}(x,y) \xrightarrow{d} \mathit{Poisson}\left(\int_x^y \frac{\cosh(t)-1}{t}dt\right)$$

Theorem 2 (Janson–L. [6], Barazer–Giacchetto–Liu [2]). *Let $N_g^{comb}(x,y)$ be the number of simple closed geodesics of length $\in [x,y]$ in a random*[2] *metric map (or ribbon graph) surface of genus g and one face of perimeter $12g$. As $g \to \infty$, we have*

$$N_g^{comb}(x,y) \xrightarrow{d} \mathit{Poisson}\left(\int_x^y \frac{\cosh(t)-1}{t}dt\right)$$

Although a Poisson law is often expected, the fact that the parameters match in both cases in a surprising coincidence! It is tempting to search for more links between these two models of large genus surfaces ...

Geometric questions. The strongest possible equivalence would be that in some sense, the two models become "the same" in the large genus limit. This could be made possible if one sees the map as the gluing of a polygon, i.e. a random hyperbolic surface could be constructed as the gluing of a hyperbolic polygon whose law we control well. This would entail a nice asymptotic parametrization of the moduli space $\mathcal{M}_g$.

In order to find a good candidate for the construction, one can look for a "canonical" embedded one face map in a hyperbolic surface. One possible guess is the "spine" of a surface (see [3] for references).

[1] under the Weil-Petersson measure.

[2] under the Lebesgue measure.

Question 3. What is the law of the spine of a random hyperbolic surface as $g \to \infty$? What are other nice ways to construct a one face map on a hyperbolic surface ?

One of the key tools of the proof of [6] is Chapuy's bijection between one faced maps and decorated trees [4].

Question 4. Is there a nice geometric adaptation of Chapuy's bijection to construct random surfaces ?

Enumerative questions. A more modest question would be to find a better explanation of the following enumerative fact, which is key to Theorems 1 and 2:

Theorem 5. *Fix n and $l_1, l_2, \ldots n, l_n$ then as $g \to \infty$, we have the following convergence of these volume ratios:*

$$\frac{V_{g,n}^{hyp}(\ell_1, \ldots, \ell_n)}{V_{g,n}^{hyp}} \sim \prod_{i=1}^{n} \mathcal{S}(\ell_i) \quad and \quad \frac{V_{g-1,n+2}^{hyp}}{V_{g,n}^{hyp}} \sim 1;$$

$$\frac{V_{g,n+1}^{comb}(12g-6, \ell_1, \ldots, \ell_n)}{V_{g,n+1}^{comb}(12g-6)} \sim 2^{-n} \prod_{i=1}^{n} \mathcal{S}(\ell_i) \quad and \quad \frac{V_{g-1,n+2}^{comb}(12g)}{V_{g,n}^{comb}(12g)} \sim 4,$$

with $\mathcal{S}(x) = \frac{\sinh x/2}{x/2}$.

These two results look very similar, however their proofs are independent and the calculations do not involve the same quantities.

Question 6. Find a unified proof of these identities, and more volumes for which this is true.

Interpolating between Theorems 1 and 2. Since metric maps can be seen as hyperbolic surfaces with a very large boundary [5], we make the following conjecture that, in some sense, interpolates betweens Theorems 1 and 2.

Conjecture 7. Let $N_g^{L_g}(x, y)$ be the number of simple closed geodesics of length $\in [x, y]$ in a random hyperbolic surface of genus g with one boundary of size L_g. There exists a continuous, increasing function r satisfying $r(0) = 1$ and $r(\infty) = \infty$ such that, as $g \to \infty$

$$N_g^{L_g}(x, y) \xrightarrow{d} \text{Poisson}\left(\int_x^y \frac{\cosh(t) - 1}{t} dt\right) \qquad \text{if } L_g = o(g)$$

$$N_g^{L_g}(r(c)x, r(c)y) \xrightarrow{d} \text{Poisson}\left(\int_x^y \frac{\cosh(t) - 1}{t} dt\right) \qquad \text{if } \frac{L_g}{g} \to c$$

$$N_g^{L_g}\left(\frac{L_g}{12g}x, \frac{L_g}{12g}y\right) \xrightarrow{d} \text{Poisson}\left(\int_x^y \frac{\cosh(t) - 1}{t} dt\right) \qquad \text{if } L_g >> g$$

References

[1] A. Aggarwal. Large genus asymptotics for intersection numbers and principal strata volumes of quadratic differentials. *Invent. Math.*, 2021.
[2] S. Barazer, A. Giacchetto, and M. Liu. Length spectrum of large genus random metric maps, 2023.
[3] B. H. Bowditch and D. B. A. Epstein. Natural triangulations associated to a surface. *Topology*, 1988.
[4] G. Chapuy. The structure of unicellular maps, and a connection between maps of positive genus and planar labelled trees. *Probab. Theory Related Fields*, 2010.
[5] N. Do. The asymptotic Weil-Petersson form and intersection theory on $M_{g,n}$, 2010.
[6] S. Janson and B. Louf. Unicellular maps vs. hyperbolic surfaces in large genus: simple closed curves. *Ann. Probab.*, 2023.
[7] M. Mirzakhani and B. Petri. Lengths of closed geodesics on random surfaces of large genus. *Comment. Math. Helv.*, 2019.

Counting saddle connections on meromorphic Abelian differentials

Howard Masur

(joint work with David Aulicino, Huiping Pan, Weixu Su)

We let (X,ω) be a translation surface of genus $g \geq 2$. This means X is a Riemann surface and ω is a holomorphic 1-form on X. We assume it lies in a hyperelliptic stratum $\mathcal{H}(\text{hyp})$. This means that there is a hyperelliptic transformation $\tau : X \to X$ so that $\tau(\omega) = -\omega$.

Examples of hyperelliptic strata are $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$ in genus 2. In addition, the stratum $\mathcal{H}(2k-2)$ for $k \geq 3$ has a hyperelliptic component as does the stratum $\mathcal{H}(k-1,k-1)$ for $k \geq 3$. Let d be the complex dimension of a hyperelliptic component.

Next we let β be a saddle connection on (X,ω) which we will assume is invariant under the involution; namely $\tau(\beta) = \beta$.

We then let $\Lambda(L,\beta)$ be the set of saddle connections α that are interiorly disjoint from β of length at most L. The goal is to find bounds on $|\Lambda(L,\beta)|$, the cardinality of $\Lambda(L,\beta)$.

One motivation for this problem as indicated by the title is suppose we have a finite genus but infinite area translation surface with a cusp, or equivalently a meromorphic Abelian differential, and one wants to count the number of saddle connections. Such a translation surface has a cusp neighborhood with boundary that consists of one or more saddle connections. Any geodesic crossing this boundary ends at the cusp so the counting of saddle connections is the same as counting on a closed surface those saddle connections that miss one or more saddle connections.

Let me state the theorems that I will discuss in the talk. For the upper bound we have:

Theorem 1. *For any $(X,\omega) \in \mathcal{H}(hyp)$ and invariant saddle connection β there is a constant C such that*

$$|\Lambda(L,\beta)| \leq C\frac{L}{|\beta|}\left(\log\frac{L}{|\beta|}\right)^{d-2}.$$

Remark 2. In the above theorem, if we only count the number of $\alpha \in \Lambda(L,\beta)$ that are not invariant under τ, then the exponent of the log is $d-3$.

For the lower bound, at this point we only have a result in $g = 2$. First we recall there is a Lebesgue measure μ (called MV measure) on a stratum.

Theorem 3. *Given an invariant saddle connection β on any set E of positive μ measure of $(X,\omega) \in \mathcal{H}(2)$ or $(X,\omega) \in \mathcal{H}(1,1)$, for μ almost every $(X,\omega) \in E$ there is a constant C' such that for L big enough*

$$|\Lambda(L,\beta)| \geq C'\frac{L}{|\beta|}\left(\log\frac{L}{|\beta|}\right)^{d-2}.$$

Remark 4. We remark that the exponents for upper and lower bounds coincide.

There are some obvious questions that remain:

(1) We would like to extend the lower bound to all hyperelliptic strata.
(2) Can one find corresponding upper and lower bounds for all strata?
(3) Can one remove the almost everywhere statement in the lower bound or are there examples where the given lower bound does not hold?

References

[1] J. Athreya and H. Masur. *Translation surfaces*, Graduate Studies in Mathematics, AMS 242.

[2] A. Wright, *From rational billiards to dynamics on moduli spaces*, Bulletin AMS 53 (2016), 41–56.

[3] A. Zorich, *Flat surfaces*, Frontiers in Number Theory, Physics, and Geometry Vol. I, Springer Verlag, (2006).

Random surfaces with large systoles

Bram Petri

The systole of a closed hyperbolic surface, or of any Riemannian manifold that possesses closed geodesics, is the length of its shortest closed geodesic. It is known, due to work by Mumford [17], that the systole, as a function on the moduli space $\mathcal{M}_g$ of closed hyperbolic surfaces of genus $g \geq 2$, admits a maximum. What this maximum is and how it behaves as a function of g is a classical question.

The answer to this question is however only known in genus 2. In that case, the maximum is uniquely realized by the Bolza surface – the unique closed Riemann surface of genus 2 with 48 automorphisms, as proved by Jenni [9]. In higher genus, there several conjectures due to Schmutz [23], that remain open to this day. Multiple local maxima in low genus have been found by Schmutz [22] and

Hamenstädt [8] and more recently Fortier Bourque and Rafi [5] have produced infinite sequences.

The next natural question is what the asymptotic behavior of the maximal systole is as the genus tends to infinity. Using the fact that the area of a disk in the hyperbolic plane grows exponentially as a function of its radius, one obtains that the maximal possible systole is at most logarithmic as a function of the genus. This trivial bound has been improved by Bavard and more recently Fortier Bourque and Petri [4]. Infinite sequences of surfaces whose systole grows logarithmically as a function of their genus are also known. The first such construction is due to Buser–Sarnak [3] and by now, several constructions are known [1, 11, 19, 20, 10]. However, even the question of whether the sequence

$$\frac{\max\{\mathrm{systole}(X);\ X \in \mathcal{M}_g\}}{\log(g)}$$

admits a limit, is currently open. The results above imply that its limit inferior is at least $\frac{19}{120}$ (due to Katz–Sabourau) and at most 2.

Our joint work with Mingkun Liu is about the question whether random constructions of hyperbolic surfaces can be used to attack this problem.

First of all, there are multiple well-studied models of random hyperbolic surfaces [2, 7, 15, 14]. However, unfortunately the systoles of these random surfaces usually don't grow [18, 16, 14, 13, 21]

As such, if one wants random surfaces with large systoles, new constructions are needed. In our work with Mingkun, we present two such constructions, both inspired by ideas from graph theory [6, 12]. We obtain sequences random surfaces of growing area, whose systoles grow logarithmically as a function of their area. This also yields a new deterministic result, it allows us to prove a lower bound on the maximal systole of a closed orientable hyperbolic surface of a given genus, improving Katz–Sabourau's recent lower bound.

References

[1] R. Brooks. Platonic surfaces. *Comment. Math. Helv.*, **74** (1999), 156–170.

[2] R. Brooks and E. Makover. Random construction of Riemann surfaces. *J. Differential Geom.*, **68** (2004), 121–157.

[3] P. Buser and P. Sarnak. On the period matrix of a Riemann surface of large genus. *Invent. Math.*, **117** (1994), 27–56. With an appendix by J. H. Conway and N. J. A. Sloane.

[4] M. Fortier Bourque and B. Petri. Linear programming bounds for hyperbolic surfaces. Preprint, arXiv: 2302.02540, (2023).

[5] M. Fortier Bourque and K. Rafi. Local maxima of the systole function. *J. Eur. Math. Soc. (JEMS)*, **24** (2022), 623–668.

[6] A. Gamburd, S. Hoory, M. Shahshahani, A. Shalev, and B. Virág. On the girth of random Cayley graphs. *Random Structures Algorithms*, **35** (2009), 100–117.

[7] L. Guth, H. Parlier, and R. Young. Pants decompositions of random surfaces. *Geom. Funct. Anal.*, **21** (2011), 1069–1090.

[8] U. Hamenstädt. New examples of maximal surfaces. *Enseign. Math. (2)*, **47** (2001), 65–101.

[9] F. Jenni. Ueber das spektrum des Laplace-operators auf einer schar konmpakter Riemannscher flächen. PhD thesis, University of Basel, (1981).

[10] M. G. Katz and S. Sabourau. Logarithmic systolic growth for hyperbolic surfaces in every genus. Preprint, (2024).

[11] M. G. Katz, M. Schaps, and U. Vishne. Logarithmic growth of systole of arithmetic Riemann surfaces along congruence subgroups. *J. Differential Geom.*, **76** (2007), 399–422.
[12] N. Linial and M. Simkin. A randomized construction of high girth regular graphs. *Random Structures Algorithms*, **58** (2021), 345–369.
[13] M. Magee, F. Naud, and D. Puder. A random cover of a compact hyperbolic surface has relative spectral gap $\frac{3}{16} - \varepsilon$. *Geom. Funct. Anal.*, **32** (2022), 595–661.
[14] M. Magee and D. Puder. The asymptotic statistics of random covering surfaces. *Forum Math. Pi*, **11** (2023), Paper No. e15, 51.
[15] M. Mirzakhani. Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus. *J. Differential Geom.*, **94** (2013), 267–300.
[16] M. Mirzakhani and B. Petri. Lengths of closed geodesics on random surfaces of large genus. *Comment. Math. Helv.*, **94** (2019), 869–889.
[17] D. Mumford. A remark on Mahler's compactness theorem. *Proc. Amer. Math. Soc.*, **28** (1971), 289–294.
[18] B. Petri. Random regular graphs and the systole of a random surface. *J. Topol.*, **10** (2017), 211–267.
[19] B. Petri. Hyperbolic surfaces with long systoles that form a pants decomposition. *Proc. Amer. Math. Soc.*, **146** (2018), 1069–1081.
[20] B. Petri and A. Walker. Graphs of large girth and surfaces of large systole. *Math. Res. Lett.*, **25** (2018), 1937 – 1956.
[21] D. Puder and T. Zimhoni. Local statistics of random permutations from free products. Preprint, arXiv: 2203.12250, (2022).
[22] P. Schmutz. Riemann surfaces with shortest geodesic of maximal length. *Geom. Funct. Anal.*, **3** (1993), 564–631.
[23] P. Schmutz Schaller. Geometry of Riemann surfaces based on closed geodesics. *Bull. Amer. Math. Soc. (N.S.)*, **35** (1998), 193–214.

Benjamini-Schramm limits of high genus translation surfaces

Kasra Rafi
(joint work with Lewis Bowen, Hunter Vallejos)

We prove that the sequence of Masur-Smilie-Veech (MSV) distributed random translation surfaces with area equal to genus, Benjamini-Schramm converges as genus tends to infinity. This means: for any fixed radius $r > 0$, if X_g is an MSV-distributed random translation surface with area g and genus g and o is a uniformly random point in X_g, then the radius-r neighborhood of o in X_g converges in distribution.

Benjamini-Schramm convergence is a notion of convergence for sequences of (random) finite graphs and finite-volume manifolds. A sequence $(X_i)_i$ of random finite-volume manifolds Benjamini-Schramm converges to a random pointed manifold (X_∞, p_∞) if, when p_i is a random point in X_i chosen uniformly, then the law of (X_i, p_i) converges to the law of (X_∞, p_∞) in the space of Borel probability measures on the space of pointed manifolds, where we use the weak topology on the former and the pointed Gromov-Hausdorff-Prokhorov topology on the latter. This convergence notion admits natural generalizations to manifolds endowed with extra structure, such as abelian differentials. Intuitively, Benjamini-Schramm convergence is a characterization of what it is like to *live* on X_i at a typical point, as $i \to \infty$.

We show that the Benjamini-Schramm limit of a random translation surface is the *Poisson translation plane*, with intensity 4. One can construct a Poisson translation plane of intensity $l > 0$ in the following way. Let $\mathbb{C}_o$ be a copy of the complex plane and set o to be $0 \in \mathbb{C}_o$. Sample a Poisson point process Π_o of intensity l in $\mathbb{C}_o$. At each point $x \in \Pi_o$, we make a cut along the ray $[1, \infty] \cdot x$ and take the path-metric completion. We call this a slit-plane. Let $\mathbb{C}_x$ be a copy of the complex plane. Cut $\mathbb{C}_x$ along the ray $[0, \infty] \cdot x$ and take the path-metric completion. Now glue this to the slit-plane by identifying boundary-components in a holonomy-preserving manner. We have now constructed the depth 1 approximation to the Poisson translation plane.

Now, on each $\mathbb{C}_x$, we take another Poisson point process Π_x of intensity l, all jointly independent, and perform the exact same procedure. Continuing this process forever ends with a Poisson translation plane of intensity l. We make this process rigorous with what we call a *holonomy tree*, which more or less accomplishes for translation planes what period coordinates accomplish for finite type translation surfaces.

Along the way, we obtain bounds on local geometric properties, such as the probability that the random point o has injectivity radius at most r, that may be of independent interest.

Large genus asymptotics for lengths of saddle connections

Anja Randecker

(joint work with Howard Masur, Kasra Rafi)

For random hyperbolic surfaces of large genus, a lot is known about their geometric properties since the seminal work of Mirzakhani [1]. The situation is very different for translation surfaces – much fewer results have been shown on the geometric properties of random large-genus surfaces.

In our work, we study the distribution of the lengths of saddle connections, that is, the lengths of geodesic segments between zeros of the abelian differential. Mirzakhani and Petri have previously shown for hyperbolic surfaces that the number of closed geodesics with lengths in a given range converges to a random variable with Poisson distribution [2]. A similar statement is also true for translation surfaces, for a different length scale and with a different mean of the Poisson distribution. More specifically:

Given a translation surface (X, ω) of genus g and an interval $[a, b] \subset \mathbb{R}_+$, let $N_{g,[a,b]}(X, \omega)$ denote the number of saddle connections on (X, ω) with lengths in the interval $\left[\frac{a}{g}, \frac{b}{g}\right]$.

Theorem 1. *Let $[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k] \subset \mathbb{R}_+$ be disjoint intervals. Then, as $g \to \infty$, the vector of random variables*

$$\left(N_{g,[a_1,b_1]}, \ldots, N_{g,[a_k,b_k]}\right) : \mathcal{H}_g(1, \ldots, 1) \to \mathbb{N}_0^k$$

converges jointly in distribution to a vector of random variables with Poisson distributions of means $\lambda_{[a_i,b_i]}$, *where*

$$\lambda_{[a_i,b_i]} = 8\pi(b_i^2 - a_i^2)$$

for $i = 1, \dots, k$. *That is,*

$$\lim_{g\to\infty} \mathbb{P}_g\left(N_{g,[a_1,b_1]} = n_1, \dots, N_{g,[a_k,b_k]} = n_k\right) = \prod_{i=1}^{k} \frac{\lambda_{[a_i,b_i]}^{n_i} e^{-\lambda_{[a_i,b_i]}}}{n_i!}.$$

Note that the theorem is only stated for the principal stratum $\mathcal{H}_g(1,\dots,1)$ of translation surfaces of genus g where all zeros are simple. However, as other strata of translation surfaces of genus g are part of the boundary of the principal stratum, the theorem is also true when considering the probability of having short saddle connections on the whole space of translation surfaces of genus g.

The proof of the theorem follows the same strategy as the one in [2], based on the method of moments. For this, we determine the factorial moments of the random variable $N_{g,[a,b]}$ by describing a simultaneous collapsing procedure for all the saddle connections in the given range, and we argue that the situation where this collapsing is possible is generic.

References

[1] M. Mirzakhani, *Growth of Weil-Petersson volumes and random hyperbolic surface of large genus*, Journal of Differential Geometry **94** (2013), 267–300.

[2] M. Mirzakhani and B. Petri, *Lengths of closed geodesics on random surfaces of large genus*, Commentarii Mathematici Helvetici **94** No. 4 (2019) 869–889.

Enumerative geometry and isomonodromic foliations

Adrien Sauvaget

1. Introduction

Let g and n be two non-negative integers such that $2g - 2 + n \geq 0$. The moduli space of smooth complex curves (*Riemann surfaces*) of genus g with n marked points is denoted $\mathcal{M}_{g,n}$. This space is a smooth complex orbifold (a generalization of manifolds) of complex dimension $3g - 3 + n$

The study of $\mathcal{M}_{g,n}$ can be approached from various perspectives, and connections between these viewpoints have provided valuable tools for understanding the geometry of $\mathcal{M}_{g,n}$. A notable example of such a connection is found in the work of Mirzakhani, who related intersection numbers of so-called tautological classes (complex analytic/algebraic viewpoint) to Weil-Petersson volumes (symplectic/hyperbolic viewpoint). In particular, her work led to a new proof of Witten's conjecture (see the next section), and provided an expression for the number of "long" closed geodesics on random hyperbolic surfaces in terms of intersection numbers [5, 6, 7].

Here we report recent results and problems towards new interplays between enumerative geometry and isomonodromic foliations. We emphasize two problems

- Flat geometry: how to compute Masur-Veech and Veech volumes in presence of isomonodromic foliations?
- Hyperbolic geometry: how to relate the properties of isomonodromic foliations to cohomological computations?

2. Enumerative geometry

The moduli space of curves admits a standard compactification $\overline{\mathcal{M}}_{g,n}$ constructed in the 60's by Deligne and Mumford by adding divisors that classifies singular complex curves [2]. This compactification is smooth, so well-suited to compute integrals, and intersection numbers of cohomology classes.

An important property of $\overline{\mathcal{M}}_{g,n}$ is the existence of a *universal curve* $\pi\colon \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ whose fiber over a point is a complex curve in the class associated with the given point. Besides, there exists sections $\sigma_1, \dots, \sigma_n$ of the fibration π that map a class of marked curves to the i-th marking. The cohomology class $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is Chern class of the line bundle

$$\sigma_i^* \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}$$

where $\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}$ is the relative co-tangent line. Here, enumerative geometry will be understood as the study of rational numbers of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \alpha \cdot \psi_1^{k_1} \dots \psi_n^{k_n} \in \mathbb{Q}$$

where α is a cohomology class that depends of the problem that we study (e.g. Hurwitz numbers, Gromov-Witten invariants, Quasi-maps, Landau-Ginzburg models). In 1991, Witten conjectured that such integrals with $\alpha = 1$ are computed by induction on g and n [10]. This conjecture was soon proved by Kontsevich [4], and several other proofs were given over time. All the proofs of this foundational results begin by expressing these integrals from a different point of view: e.g. Weil-Petersson volumes in the case of Mirzakhani's proof mentioned above.

3. Moduli spaces of cone surfaces

Let $a = (a_1, \dots, a_n)$ be a vector in $\mathbb{R}^n_{>0}$. Here, a surface of type (g, a) is the datum of $(C, x_1, \dots, x_n, \eta)$ where $(C, x_1, \dots, x_n)$ where is a compact real surface of genus g with n distinct points, and η is Riemannian metric with constant curvature with cone singularities of angle $2\pi a_i$ at x_i for all i. Depending on the sign of the curvature, such surfaces can be obtained (up to a scalar) by gluing geodesic polygons in either the disk, the plane or the sphere, along sides of equal lengths. We denote by $\mathcal{M}_{g,n}(a)$ the moduli space of cone surfaces of type (g, a) up to isometries and scaling. These spaces are canonically endowed with structure of real orbifolds. Moreover, Trovyanov showed that

$$\mathcal{M}_{g,n} \simeq_{C^\infty} \mathcal{M}_{g,n}(a)$$

if $|a| \leq 2g-2+n$ (i.e. for flat or hyperbolic surfaces). Besides the moduli space $\mathcal{M}_{g,n}$, one may be interested in the Teichmüller space, i.e. the universal cover $\mathcal{T}_{g,n}(a)\mathcal{M}_{g,n}(a)$ which is defined as a moduli space of cone metrics on a reference surface. Moreover, if $G = PSL(2,\mathbb{R})$ or $\mathbb{C} \times \mathbb{S}^1$, the group of isometries of the hyperbolic disk and the plane respectively, then we define $\Xi_{g,n}(a)$ to be the character variety with relative conditions prescribed by a: the space of morphism from the fundamental group of reference (punctured) surface to G (up to the action of G by adjunction) with boundary conditions given by imposing that the image of loops around the punctures are mapped to rotations of angles prescribed by a. Then the *monodromy morphism* is the C^∞ morphism

$$\mathrm{Mon}\colon \mathcal{T}_{g,n}(a) \ \to \Xi_{g,n}(a).$$

This is a local isomorphism in general, unless a has an integral entry. In this case, Mon is a submersion and the fiber of a point is a complex manifolds. This defines a foliation on $\mathcal{T}_{g,n}(a)$ that descends to $\mathcal{M}_{g,n}(a)$, this is the *isomonodromy foliation.*

4. Flat surfaces

Here we assume that $|a| = 2g-2+n$. Then $\mathcal{M}_{g,n}(a)$ is endowed with with a canonical measure, known as *Veech measure.* If k is a positive integer such that ka is integral, then can also consider the moduli space $\mathcal{M}_{g,n}(a,k) \subset \mathcal{M}_{g,n}$ defined as the set of marked curves $(C, x_1, \ldots, x_n)$ satisfying: there exists a k-differential with singularities of order $ka_i - k$ at x_i for all i. This space is canonically embedded in $\mathcal{M}_{g,n}(a)$ as the sub-space of flat surfaces with rotational part of the monodromy valued in the group of k-th roots of unity. Besides, this space is also endowed with a canonical measure, the *Masur-Veech measure.*

If a is rational and without integral coordinate, then the volumes of $\mathcal{M}_{g,n}(a)$ and $\mathcal{M}_{g,n}(a,k)$ were both computed in [8] and expressed as integrals of cohomology classes over $\overline{\mathcal{M}}_{g,n}$. However, if a has integral coordinates, i.e. in presence of isomonodromic foliations/deformations, then these volumes are unknown in general (i.e. beyond $k=1$ and 2 by different methods). A conjecture was proposed in [1] and generalized in [8] to express these volumes as integrals of cohomology classes. The integrated cohomology class involves one ψ-class for each of the integral coordinates of a, suggesting that ψ-classes could be thought as Chern classes of the tangent line along isomorphic deformations.

5. Hyperbolic surfaces

In the hyperbolic setting, and for small values of a, the space $\mathcal{M}_{g,n}(a)$ shares common features with the moduli space of hyperbolic surfaces with geodesic boundaries: it has a canonical symplectic form (the Weil-Petersson form), with Darboux coordinates (the Frenchel-Nielsen Coordinates), an expression of the volume in terms of intersection numbers, and all surface satisfy a McShane-type identity allowing for the inductive computation of the volumes, based on the results of Mirzakhani for moduli spaces of surfaces with boundaries.

For a general value of a, these properties fail, although $\mathcal{M}_{g,n}(a)$ is still endowed with a canonical symplectic form. In [9], an expression of the volume is conjectured for a general value of a generalizing all previous results for special choices of g and a. At vectors a with integral coordinates, the symplectic form degenerates, thus the volume vanishes. As we expect that the Weil-Petersson form is the pull-back of Goldman's symplectic form on the character variety (this holds for $n = 0$ by [3]), this vanishing is due to the existence of fibers positive dimension in Mon, i.e. of isomonodromic deformations. In return this vanishing may be used to provide a new proof of Witten's conjecture as explained in [9]. From here, two types of problem occur

- Can we exploit such bridge for other enumerative problems/isomonodromic systems?
- Conversely, can we exploit known results about $H^*(\overline{\mathcal{M}}, \mathbb{Q})$ to describe the properties of the isomonodromic foliation?

References

[1] D. Chen, M. Möller, and A. Sauvaget. Masur-Veech volumes and intersection theory: the principal strata of quadratic differentials. *Duke Math. J.*, **172** (2023), 1735–1779.

[2] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, **36** (1969), 75–109.

[3] W. M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. Math.*, **54** (1984), 200–225.

[4] M. Kontsevich. Intersection theory on the moduli space of curves and the matrix Airy function. *Comm. Math. Phys.*, **147** (1992), 1–23.

[5] M. Mirzakhani. Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. *Invent. Math.*, **167** (2007), 179–222.

[6] M. Mirzakhani. Weil-Petersson volumes and intersection theory on the moduli space of curves. *J. Amer. Math. Soc.*, **20** (2007), 1–23.

[7] M. Mirzakhani. Growth of the number of simple closed geodesics on hyperbolic surfaces. *Ann. of Math. (2)*, **168** (2008), 97–125.

[8] A. Sauvaget. Volumes of moduli spaces of flat surfaces. *arXiv:2004.03198*, (2020).

[9] A. Sauvaget. A flat perspective on moduli spaces of hyperbolic surfaces. *arXiv:2405.10869*, (2024).

[10] E. Witten. Two-dimensional gravity and intersection theory on moduli space. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pages 243–310. Lehigh Univ., Bethlehem, PA, 1991.

Enumerative Invariants from Log Intersection Numbers

Johannes Schmitt

Enumerative geometry counts geometric objects satisfying a list of properties. An important method in the area is to obtain these counts as intersection numbers on a suitable moduli space $\overline{\mathcal{M}}$. In the talk I explain how logarithmic intersection theory can be used in different examples to define intersection numbers which are independent of the precise choice of this space $\overline{\mathcal{M}}$. After a discussion of (double) Hurwitz numbers, we'll also see a new class of invariants - called k-leaky double Hurwitz descendants - defined in joint work with Cavalieri and Markwig. I discuss some of their properties, show tropical formulas calculating them and present some questions on their enumerative interpretation.

1. Introduction

The goal of enumerative geometry is to count geometric objects with specified properties. Intersection theory provides a framework for such enumerative problems. Ideally, one constructs a suitable moduli space $\overline{\mathcal{M}}$ parametrizing the objects of interest, and then represents the desired counts as intersection numbers of appropriate cycle classes on $\overline{\mathcal{M}}$. For example, if one wants to count objects satisfying two properties A and B, one can consider the closed subsets S_A and S_B of $\overline{\mathcal{M}}$ parametrizing objects satisfying the respective properties. If $\overline{\mathcal{M}}$ is smooth and compact, one can compute the intersection number

$$N_{A,B} = \int_{\overline{\mathcal{M}}} [S_A] \cdot [S_B],$$

where $[S_A]$ and $[S_B]$ are the Poincaré duals of the fundamental classes of S_A and S_B. If S_A and S_B intersect transversely, then $N_{A,B}$ should be the desired count. However, difficulties arise when the objects to be counted intersect in the boundary of the moduli space, which may parameterize degenerate objects not of primary interest. This motivates the use of blow-ups to resolve these boundary intersections and the development of intersection theory that is agnostic to the specific blow-up chosen.

2. Compactifying Strata of k-Differentials

As a concrete example in the talk, we consider the problem of compactifying the strata of k-differentials. Fix $g, n, k \geq 0$ and a vector $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ with $\sum a_i = k(2g-2+n)$. The stratum $\mathrm{DR}_g^0(A) \subseteq \mathcal{M}_{g,n}$ parametrizes smooth pointed curves $(C, p_1, \dots, p_n)$ such that there exists a meromorphic k-differential η on C with divisor

$$\mathrm{div}(\eta) = \sum_{i=1}^{n} (a_i - k) p_i \,. \tag{1}$$

This condition can be reformulated in terms of line bundles:

$$(\omega_C^{\log})^{\otimes k} \cong \mathcal{O}_C \left(\sum_{i=1}^{n} a_i p_i \right), \tag{2}$$

where $\omega_C^{\log} = \omega_C \otimes \mathcal{O}_C \left(\sum_{i=1}^{n} p_i \right)$ is the log canonical bundle. Several approaches to compactifying these strata have been proposed:

(1) *Naive closure*: Take the closure of $\mathrm{DR}_g^0(A)$ inside $\overline{\mathcal{M}}_{g,n}$. This often leads to highly singular spaces.
(2) *Multi-scale differentials*: Construct a smooth compactification $\mathcal{MS}_g^k(A)$ by hand, as done in [1, 4]. The space $\mathcal{MS}_g^k(A)$ parameterizes stable curves C together with a multi-scale k-differential. The data of this differential combines additional combinatorial data on the stable graph of C (called

an enhanced level graph), with geometric data such as twisted differentials on the components of C, prong matchings at the node, and an equivalence relation defined by rescaling ensembles.

(3) *Closure in a log blow-up*: Take the closure of $\mathrm{DR}_g^0(A)$ inside a suitable log blow-up (i.e. iterated boundary blow-up) of $\overline{\mathcal{M}}_{g,n}$. This approach, developed in joint work [7] with Chen, Grushevsky, Holmes, and Möller, leads to spaces that are closely related to the moduli spaces of multi-scale differentials.

There exists a natural log blow-up $\overline{\mathcal{M}}_{g,A}^{\diamond}$ of $\overline{\mathcal{M}}_{g,n}$ such that the closure of $\mathrm{DR}_g^0(A)$ inside $\overline{\mathcal{M}}_{g,A}^{\diamond}$ recovers the moduli space of multi-scale differentials for $k = 1$:

$$\overline{\mathrm{DR}_g^0(A)}^{\overline{\mathcal{M}}_{g,A}^{\diamond}} \cong \mathcal{MS}_g^1(A). \tag{3}$$

This is expected to hold for higher k as well. Moreover, the moduli space $\mathcal{MS}_g^k(A)$ admits a modular interpretation in the language of log geometry. It is a union of components of the moduli space $\mathrm{DR}_g(A)$ parametrizing log curves $(C, p_1, \ldots, p_n)$ together with an isomorphism of line bundles

$$(\omega_C^{\log})^{\otimes k} \cong \mathcal{O}_C\left(\sum_{i=1}^n a_i p_i\right)(\alpha), \tag{4}$$

where α is a piecewise linear function on the log curve C and $\mathcal{O}_C(\alpha)$ is an associated line bundle. This main component can be cut out by additionally requiring a Global Residue Condition on twisted differentials induced by the isomorphism (4).

3. Logarithmic Chow Rings and the Logarithmic Double Ramification Cycle

To develop an intersection theory that is independent of the specific log blow-up chosen, we introduce the notion of the *logarithmic Chow ring* $\mathrm{logCH}^*(\overline{\mathcal{M}}_{g,n})$. This ring describes the intersection theory of all log blow-ups of $\overline{\mathcal{M}}_{g,n}$ simultaneously. An element of $\mathrm{logCH}^*(\overline{\mathcal{M}}_{g,n})$ is given by a pair $(\widehat{M}, \alpha)$, where $\widehat{M}$ is a log blow-up of $\overline{\mathcal{M}}_{g,n}$ and $\alpha \in \mathrm{CH}^*(\widehat{M})$. Two pairs $(\widehat{M}, \alpha)$ and $(\widehat{M}', \alpha')$ are equivalent if there exists a common log blow-up $\widehat{M}''$ dominating both $\widehat{M}$ and $\widehat{M}'$ such that the pullbacks of α and α' to $\widehat{M}''$ coincide. The *logarithmic double ramification cycle* $\mathrm{logDR}_g(A)$ is defined as the class of the moduli space $\mathrm{DR}_g(A)$ inside the log blow-up $\overline{\mathcal{M}}_{g,A}^{\diamond}$, equipped with its virtual fundamental class:

$$\mathrm{logDR}_g(A) = [(\overline{\mathcal{M}}_{g,A}^{\diamond}, [\mathrm{DR}_g(A)]^{\mathrm{vir}})] \in \mathrm{logCH}^g(\overline{\mathcal{M}}_{g,n}). \tag{5}$$

4. Applications to Hurwitz Numbers

In the case $k = 0$, the stratum $\mathrm{DR}_g^0(A)$ parametrizes covers of $\mathbb{P}^1$ with prescribed ramification profiles over 0 and ∞. Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be a vector with $\sum a_i = 0$. We write $A = A^+ - A^-$, where A^+ and A^- are the positive and negative parts of A. Then $\mathrm{DR}_g^0(A)$ parametrizes covers $f : C \to \mathbb{P}^1$ such that the ramification profile over 0 is given by A^+ and the ramification profile over ∞ is given by A^-. The double Hurwitz number $H_g(A)$ counts such covers with fixed simple branch points elsewhere. Double Hurwitz numbers satisfy several interesting properties:

- They are piecewise polynomial in the entries of A.
- They admit wall-crossing formulas describing how the polynomials change across the walls.
- They can be computed via character formulas, cut-and-join equations, topological recursion, and tropical geometry.

A theorem of Cavalieri, Markwig, and Ranganathan expresses double Hurwitz numbers as intersection numbers involving the logarithmic double ramification cycle and a specific class $\mathrm{br}_g(A) \in \log\mathrm{CH}^{2g-3+n}(\overline{\mathcal{M}}_{g,n})$ encoding the fixed branch points:

Theorem 1. [3] *The double Hurwitz number $H_g(A)$ is given by*

$$H_g(A) = \int_{\overline{\mathcal{M}}^{\diamond}_{g,A}} \log\mathrm{DR}_g(A) \cdot \mathrm{br}_g(A). \tag{6}$$

The proof involves degenerating the target $\mathbb{P}^1$ into a chain of $\mathbb{P}^1$s and analyzing the corresponding degeneration of the covers. This leads to a combinatorial formula in terms of *tropical covers*, which are graphs mapping to the dual graph of the degenerate target, satisfying certain balancing conditions at the vertices and encoding the ramification profiles at the edges. The double Hurwitz number is then expressed as a weighted sum over tropical covers:

$$H_g(A) = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{e \in E(\Gamma)} \kappa_e, \tag{7}$$

where the sum is over tropical covers Γ, $\mathrm{Aut}(\Gamma)$ is the automorphism group of Γ, $E(\Gamma)$ is the set of edges of Γ, and κ_e is the multiplicity associated to the edge e.

5. Generalization to k-Differentials

The formula for double Hurwitz numbers can be generalized to arbitrary k. This leads to the definition of *k-leaky double Hurwitz numbers* $H_g^k(A)$ and their descendants $H_g^k(A; e)$, which involve intersections with ψ-classes. The k-leaky double Hurwitz number $H_g^k(A)$ is defined by

$$H_g^k(A) = \int_{\overline{\mathcal{M}}^{\diamond}_{g,A}} \log\mathrm{DR}_g(A) \cdot \mathrm{br}_g(A), \tag{8}$$

where $\mathrm{br}_g(A)$ is the same class as in the case $k = 0$. The k-leaky double Hurwitz descendants $H_g^k(A; e)$ are defined by

$$H_g^k(A; e) = \int_{\overline{\mathcal{M}}^{\diamond}_{g,A}} \log\mathrm{DR}_g(A) \cdot \mathrm{br}_g^e(A) \cdot \prod_{i=1}^{n} \psi_i^{e_i}, \tag{9}$$

where $e = (e_1, \dots, e_n)$ is a vector of non-negative integers and $\mathrm{br}_g^e(A)$ is a variant of the branch class with codimension $2g - 3 + n - |e|$. These numbers interpolate between double Hurwitz numbers (when $e = 0$) and intersection numbers involving the usual double ramification cycle (when e is maximal). They satisfy piecewise polynomiality properties and admit wall-crossing formulas. Moreover, they can be expressed as sums over tropical covers with "leaks" at the vertices, reflecting the discrepancy between the multiplicities of zeros and poles of k-differentials.

6. Enumerative Meaning and Future Directions

A major open question is to find an enumerative interpretation of the k-leaky double Hurwitz numbers for $k > 0$. One approach is to consider counting k-differentials with prescribed periods or residues. For example, in genus 0, certain residue conditions lead to enumerative problems whose solutions exhibit striking similarities to the k-leaky double Hurwitz numbers.

Specifically, consider the case $g = 0$, $k = 1$, and $A = (d, -b_1, \dots, -b_n)$ with $d = n - 1 + \sum_{i=1}^n b_i$ and $b_i \geq 0$. Let $\mathrm{DR}_0^0(A)$ be the locus of curves $(C, p_1, \dots, p_{n+1})$ equipped with a meromorphic 1-differential η such that

$$\mathrm{div}(\eta) = (d-1)p_1 - \sum_{i=2}^{n+1} (b_i + 1) p_i.$$

Inside $\mathrm{DR}_0^0(A)$, consider the locus $\mathrm{DR}_0^{\vec{r}}(A)$ where the residues of η at the poles $p_2, \dots, p_{n+1}$ are linearly dependent to a fixed vector $\vec{r} = (r_2, \dots, r_{n+1}) \in \mathbb{C}^n$ with $\sum_{i=2}^{n+1} r_i = 0$. For generic $\vec{r}$, the cardinality of $\mathrm{DR}_0^{\vec{r}}(A)$ is given by

$$|\mathrm{DR}_0^r(A)| = (d-1)(d-2)\cdots(d-(n-2)), \tag{10}$$

see [6, 2, 5]. On the other hand, the 1-leaky double Hurwitz number $H_0^1(A)$ is given by

$$H_0^1(A) = (n-1)! \left(d - \frac{1}{2}\right)\left(d - \frac{2}{2}\right)\cdots\left(d - \frac{n-2}{2}\right). \tag{11}$$

The structural similarity between these formulas suggests a potential connection between the enumerative problem involving residue conditions and the k-leaky double Hurwitz numbers.

Question 2. Is there an interesting enumerative problems for k-differentials in arbitrary genus g and profile A of zeros and poles? Does it have some relationship to the number $H_g^k(A)$?

References

[1] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, *The moduli space of multi-scale differentials*, arXiv e-prints, page arXiv:1910.13492, (2019).

[2] A. Buryak and P. Rossi, *Counting meromorphic differentials on* $\mathbb{CP}^1$, Lett. Math. Phys., **114** (2024), 97.

[3] R. Cavalieri, H. Markwig, and D. Ranganathan, *Pluricanonical cycles and tropical covers*, arXiv e-prints, page arXiv:2206.14034, (2022).

[4] M. Costantini, M. Möller, and J. Zachhuber, *The area is a good enough metric*, Ann. Inst. Fourier (Grenoble), **74**(2024), 1017–1059.

[5] D. Chen and M. Prado, *Counting differentials with fixed residues*, arXiv e-prints, page arXiv:2307.04221, (2023).

[6] Q. Gendron and G. Tahar, *Isoresidual fibration and resonance arrangements*, Lett. Math. Phys., **112**(2022), 33.

[7] D. Holmes, S. Molcho, R. Pandharipande, A. Pixton, and J. Schmitt, *Logarithmic double ramification cycles*, arXiv e-prints, page arXiv:2207.06778, (2022).

Isoperiodic forms and invariant subvarieties of moduli space

Karl Winsor

Fix an integer $g \geq 2$, and let $\kappa = (k_1, \ldots, k_n)$ be a partition of $2g-2$ with positive integer parts. Let $\Omega\mathcal{M}_g(\kappa)$ be the associated stratum of holomorphic abelian differentials (X, ω) with zero orders $k_1, \ldots, k_n$. The stratum $\Omega\mathcal{M}_g(\kappa)$ admits local period coordinates modelled on $\mathrm{Hom}(H_1(X, Z(\omega); \mathbb{Z}), \mathbb{C})$, where $Z(\omega)$ is the set of zeros of ω. By varying $(X, \omega) \in \Omega\mathcal{M}_g(\kappa)$ while fixing the integrals of ω along closed loops, one obtains a holomorphic foliation called the *isoperiodic foliation* of $\Omega\mathcal{M}_g(\kappa)$. We presented work in progress establishing a rigidity theorem for leaf closures of the isoperiodic foliation.

In the generic stratum $\Omega\mathcal{M}_g(1^{2g-2})$, work of Calsamiglia-Deroin-Francaviglia [4] provides a complete classification of isoperiodic leaf closures, and in particular shows they are always suborbifolds. Their approach uses degeneration arguments to induct on genus, and builds on an observation of McMullen [8] to address base cases in low genus. Let $\mathrm{Per}(\omega)$ be the additive subgroup of $\mathbb{C}$ formed by the integrals of ω along closed loops. Riemann's bilinear relations imply that $\mathrm{Per}(\omega)$ contains a lattice in $\mathbb{C}$, so its closure Λ is isomorphic to one of $\mathbb{C}$, $\mathbb{R}+i\mathbb{Z}$, or $\mathbb{Z}+i\mathbb{Z}$. Let $\Omega_a^\Lambda\mathcal{M}_g(\kappa)$ be the subset of differentials (Y, η) of area $a > 0$ such that $\mathrm{Per}(\eta)$ is contained in Λ and meets every connected component of Λ. If $\Lambda = \mathbb{R} + i\mathbb{Z}$, this means $\mathrm{Per}(\eta)$ is not contained in $\mathbb{R}+im\mathbb{Z}$ for some integer $m > 1$, and if $\Lambda = \mathbb{Z}+i\mathbb{Z}$, this means $\mathrm{Per}(\omega) = \Lambda$. Calsamiglia-Deroin-Francaviglia's result shows that all isoperiodic leaf closures in the generic stratum are given by connected components of $\Omega_a^\Lambda\mathcal{M}_g(1^{2g-2})$ for some $a > 0$ and some closed subgroup $\Lambda \subset \mathbb{C}$. We generalized this result to all strata, with a couple of caveats.

First, isoperiodic leaves have complex dimension $n-1$. When $n = 1$, leaves are points and are always closed. Second, most strata $\Omega\mathcal{M}_g(\kappa)$ with $1 < n < 2g-2$ contain algebraic subvarieties arising from covering constructions that are unions of isoperiodic leaves. An example in the stratum $\Omega\mathcal{M}_4(5, 1)$ arises from branched double covers of differentials in $\Omega\mathcal{M}_2(2)$ branched over the zero and over a regular point. These subvarieties are also invariant under the action of $\mathrm{GL}^+(2, \mathbb{R})$ on strata. The main result presented in our talk was the following.

Theorem 1. *Fix $g \geq 3$, and let $\Omega\mathcal{M}_g(\kappa)$ be a stratum with $n > 1$ zeros. There is an algebraic subvariety $\mathcal{V} \subset \Omega\mathcal{M}_g(\kappa)$ of positive codimension, such that if L is an isoperiodic leaf that is not contained in $\mathcal{V}$, then the closure of L is a connected component of $\Omega_a^\Lambda\mathcal{M}_g(\kappa)$ for some $a > 0$ and some closed subgroup $\Lambda \subset \mathbb{C}$.*

Our result is obtained by classifying leaf closures in a nonempty open $\mathrm{GL}^+(2,\mathbb{R})$-invariant subset of each component of $\Omega\mathcal{M}_g(\kappa)$, and then applying the rigidity theorems of Eskin-Mirzakhani-Mohammadi [5] and Filip [7] which tell us that closed $\mathrm{GL}^+(2,\mathbb{R})$-invariant subsets of strata are algebraic subvarieties. By restricting attention to an open $\mathrm{GL}^+(2,\mathbb{R})$-invariant subset, we can exploit configurations of saddle connections that are only guaranteed to exist on a typical differential, see [6]. Our approach involves moving along isoperiodic leaves to find many such configurations related by Dehn twists. In special cases, one can obtain more information about $\mathcal{V}$ using classification results for $\mathrm{GL}^+(2,\mathbb{R})$-orbit closures, see for instance [1], [2], [3]. In many strata for which the numerology of the zero orders prevents the existence of the above type of covering construction, one can show that $\mathcal{V}$ can be taken to be empty.

References

[1] D. Aulicino and D-M. Nguyen. Rank 2 affine manifolds in genus 3. *J. Diff. Geom.*, **116** (2020), 205–280.

[2] P. Apisa. $gl_2\mathbb{R}$ orbit closures in hyperelliptic components of strata. *Duke Math. J.*, **167** (2018), 679–742.

[3] P. Apisa and A. Wright. High rank invariant subvarieties. *Ann. Math.*, **198** (2023), 657–726.

[4] G. Calsamiglia, B. Deroin, and S. Francaviglia. A transfer principle: from periods to isoperiodic foliations. *Geom. Funct. Anal.*, **33** (2023), 57–169.

[5] A. Eskin, M. Mirzakhani, and A. Mohammadi. Isolation, equidistribution, and orbit closures for the $\mathrm{sl}(2,\mathbb{R})$ action on moduli space. *Ann. of Math.*, **182** (2015), 673–721.

[6] A. Eskin, H. Masur, and A. Zorich. Moduli spaces of abelian differentials: the principal boundary, counting problems, and the siegel-veech constants. *Publ. Math. IHES*, **97** (2003), 61–179.

[7] S. Filip. Splitting mixed hodge structures over affine invariant manifolds. *Ann. Math.*, **183** (2016), 681–713.

[8] C. McMullen. Moduli spaces of isoperiodic forms on riemann surfaces. *Duke Math. J.*, **163** (2014), 2271–2323.

Reporter: Andrei Bud

Participants

Prof. Dr. Nalini Anantharaman
I R M A
Université de Strasbourg
7, rue René Descartes
67084 Strasbourg Cedex
FRANCE

Dr. Paul Apisa
Department of Mathematics
University of Wisconsin - Madison
480 Lincoln Dr
Madison, WI 53706
UNITED STATES

Dr. David Aulicino
Department of Mathematics
CUNY - Brooklyn College
2900 Bedford Avenue
Brooklyn, NY 11210-2889
UNITED STATES

Dr. Simon Barazer
Université Paris Saclay
Bâtiment 307, rue Michel Magat
91405 Orsay Cedex
FRANCE

Jérémie Bouttier
IMJ-PRG
Sorbonne Université
4 place Jussieu
Boite Courrier 247
75252 Paris Cedex 05
FRANCE

Andrei Bud
Goethe Universität Frankfurt am Main
Robert-Mayer Straße 6-8
60325 Frankfurt am Main
GERMANY

Dr. Timothy Budd
Institute for Mathematics, Astrophysics
and Particle Physics,
Radboud Universiteit Nijmegen
Postbus 9010
6500 GL Nijmegen
NETHERLANDS

Dr. Guillaume Chapuy
IRIF – UMR 8243 CNRS
Université Paris Cité
Case 7014
75205 Paris Cedex 13
FRANCE

Prof. Dr. Dawei Chen
Department of Mathematics
Boston College
Chestnut Hill MA 02467-3806
UNITED STATES

Matteo Costantini
Fakultät für Mathematik
Universität Duisburg-Essen
45117 Essen
GERMANY

Edmond Covanov
Laboratoire de Mathématiques
Université de Grenoble I
Institut Fourier
B. P. 74
38402 Saint-Martin-d'Hères Cedex
FRANCE

Dr. Vincent Delecroix
LaBRI
Université de Bordeaux
351, cours de la Liberation
33405 Talence Cedex
FRANCE

Prof. Dr. Bertrand Deroin
Département de Mathématiques
Université de Cergy-Pontoise
Site Saint-Martin, BP 222
2, avenue Adolphe Chauvin
95302 Cergy-Pontoise Cedex
FRANCE

Dr. Viveka Erlandsson
University of Bristol
School of Mathematics
Fry Building
Woodland Road
Bristol BS81UG
UNITED KINGDOM

Dr. Samantha Fairchild
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22
04103 Leipzig
GERMANY

Prof. Dr. James Farre
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig
GERMANY

Dr. Samuel Freedman
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Kai Fu
Mathématiques et Informatique
Université Bordeaux I
351, cours de la Liberation
33405 Talence Cedex
FRANCE

Dr. Vaibhav Gadre
Department of Mathematics
University of Glasgow
University Gardens
Glasgow G12 8QQ
UNITED KINGDOM

Dr. Nihar Gargava
Institut de Mathématiques
Université de Strasbourg
7, rue René Descartes
67084 Strasbourg Cedex
FRANCE

Quentin Gendron
Instituto de matématicas
UNAM
Área de la Investigación Científica
04510 Ciudad de México D.F.
MEXICO

Riccardo Giannini
Department of Mathematics
University of Glasgow
University Gardens
Glasgow G12 8QW
UNITED KINGDOM

Prof. Dr. Samuel Grushevsky
Department of Mathematics
Stony Brook University
Stony Brook NY 11794-3651
UNITED STATES

Prof. Dr. Ursula Hamenstädt
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Vivian He
Department of Mathematics
University of Toronto
40 St George Street
Toronto ON M5S 2E4
CANADA

Manuel Kany
Fachrichtung Mathematik
Universität des Saarlandes
Postfach 151150
66041 Saarbrücken
GERMANY

Pascal Kattler
Fachrichtung Mathematik
Universität des Saarlandes
Postfach 151150
66041 Saarbrücken
GERMANY

Yanik Kleibrink
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Str. 6-8
60325 Frankfurt am Main
GERMANY

Victor le Guilloux
Institut de Mathématiques
Université de Strasbourg
7, rue René Descartes
67084 Strasbourg Cedex
FRANCE

Prof. Dr. Danilo Lewanski
Dip. di Matematica e Informatica
Universita di Trieste
Via Alfonso Valerio 12/1
34127 Trieste
ITALY

Dr. Mingkun Liu
University of Luxembourg,
Department of Mathematics
1359 Belvaux
LUXEMBOURG

Dr. Baptiste Louf
Mathématiques et Informatique
Université Bordeaux I
351, cours de la Liberation
33405 Talence Cedex
FRANCE

Hugo Manet
IRIF, CNRS UMR 8243
Université Paris Cité
Case 7014
75205 Paris Cedex 13
FRANCE

Prof. Dr. Howard Masur
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Prof. Dr. Carlos Matheus
Centre de Mathématiques Appliquées
UMR 7641 – CNRS
École Polytechnique
91128 Palaiseau Cedex
FRANCE

Prof. Dr. Martin Möller
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Straße 6-8
60325 Frankfurt am Main
GERMANY

Dr. Bram Petri
Institut de Mathématiques de Jussieu
Paris Rive Gauche (IMJ-PRG)
4 place Jussieu
P.O. Box 247
75252 Paris Cedex 05
FRANCE

Dr. Miguel Prado
Institut für Mathematik
Goethe-Universität
Robert-Mayer-Str. 6-8
60325 Frankfurt am Main
GERMANY

Prof. Dr. Kasra Rafi
Department of Mathematics
University of Toronto
40 St George Street
Toronto ON M5S 2E4
CANADA

Dr. Anja Randecker
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Dr. Adrien Sauvaget
Département de Mathématiques
Université de Cergy-Pontoise
Site Saint-Martin, BP 222
2, avenue Adolphe Chauvin
95302 Cergy-Pontoise Cedex
FRANCE

Johannes Schmitt
Institut für Mathematik
ETH Zürich
8092 Zürich
SWITZERLAND

Dr. Alexander Thomas
Université Lyon 1
21 Av. Claude Bernard
69100 Villeurbanne
FRANCE

Dr. Sahana Vasudevan
Institute for Advanced Study
School of Mathematics
1 Einstein Drive
Princeton, NJ 08540
UNITED STATES

Dr. Karl Winsor
Department of Mathematics
Stony Brook University
Math. Tower
Stony Brook, NY 11794-3651
UNITED STATES

Ivan Yakovlev
LaBRI, Université de Bordeaux
351, cours de la Liberation
33405 Talence Cedex
FRANCE

Prof. Dr. Anton Zorich
Institut de Mathématiques de Jussieu
Université de Paris VII
Paris Rive Gauche (IMJ-PRG)
8 Place Aurélie Nemours
75013 Paris Cedex
FRANCE