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Homotopical Algebra and Higher Structures

Organized by
Michael Batanin, Prague
Andrey Lazarev, Lancaster
Muriel Livernet, Paris
Martin Markl, Prague

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Abstract. Homotopical algebra and higher category theory play an increasingly important role in pure mathematics, and higher methods have seen tremendous development in the last couple of decades. The talks delivered at the workshop described some of the latest progress in this area and applications to various problems of algebra, geometry, and combinatorics.

Mathematics Subject Classification (2020): 18N65, 18N50, 18M65, 18M70, 18M30, 18M85, 18G80.

Introduction by the Organizers

The workshop *Homotopical algebra and higher structures*, organized by Michael Batanin (Prague), Andrey Lazarev (Lancaster), Muriel Livernet (Paris) and Martin Markl (Prague) was a workshop with 41 participants attending in person. It represented a geographically broad selection of pure mathematicians based in Europe, Asia, North and South America, and Australia. Particular care was taken to promote an appropriate gender balance among participants and speakers of the workshop.

We were given the opportunity to organize the workshop in a hybrid format but, after careful deliberation, decided not to take it. Most participants seem to appreciate the opportunity to come to Oberwolfach in person and take advantage of the unique and stimulating atmosphere at the Institute. As always, we observed lots of mathematical conversations held in large and small groups during lunch breaks and in the evenings.

On each day of the workshop there were three 55 minutes talks in the morning. The afternoon schedule was more varied; on Monday, Tuesday and Thursday afternoons we had five 55 minutes talks in total and also three shorter half-hour talks. On Wednesday, following a time-honoured tradition, participants went on a hike in Black Forest in the afternoon, and the Friday schedule included only three morning talks as (partially due to the interruptions with trains around the Hausach area) most participants opted to leave shortly after lunch.

In addition to regular talks, a 'gong talk show' was held on Tuesday from 7:45 p.m. to 9 p.m. during which talks of 5 minutes' duration were delivered, with an additional allowance of one minute for questions after each talk. There were no breaks between talks and the schedule was strictly enforced by the sound of a gong at the end of each five minutes or one minute slot. This format proved to be quite popular among the participants, and six out of ten speakers managed to finish their talks on time without being stopped by the gong (an impressive feat). As a result, all participants who were willing to give a talk, had the opportunity to do so (albeit in this unusual format). If we happen to organize another Oberwolfach workshop, this is a feature that we would definitely want to keep.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows".

Workshop: Homotopical Algebra and Higher Structures

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Abstracts

Triangulated categories from triangulated surfaces

Bernhard Keller

In Fock–Goncharov's approach [7] to higher Teichmüller theory, higher Teichmüller spaces are obtained from cluster varieties $V(S,G)$ associated with pairs consisting of a marked surface S and a split semi-simple (real) Lie group G. The varieties $V(S,G)$ were constructed by Fock-Goncharov [7] for type A and by Goncharov-Shen [8] for all Dynkin types. In this expository talk, we report on the ongoing project of categorifying these cluster varieties using triangulated and extriangulated categories [11], respectively their dg enhancements known as exact dg categories [3, 1, 2]. The talk is based on recent work by Merlin Christ [6, 5, 4], by Miantao Liu [10] and by Yilin Wu [12] as well as on by now classical results of Haiden–Katzarkov–Kontsevich [9].

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Centers and homotopy centers of non-symmetric operads

Florian De Leger

(joint work with Maroš Grego)

In the first part of the talk, we will recall the idea of the Baez-Dolan plus construction [1], then explain how hyperoperads, as we defined them in [4], look like. We will then explain how, just like the planar trees version of the dendroidal category Ω_p [6] naturally extend the simplex category Δ, hyperoperads naturally extend non-symmetric operads. Finally we will recall the main result of our paper [4] which is a triple delooping result for multiplicative hyperoperads analogous to the double delooping result of Turchin [7] and Dwyer-Hess [5].

In the second part of the talk, we will define a notion of centers and homotopy centers of non-symmetric operads analogous to the notion of centers and homotopy centers of monoids given by Batanin and Markl in [3]. We will explain how this notion extends the classical notion of centers of a monoid. As we will explain, we believe that there is an action of the little 3-disks operad on the homotopy center of a non-symmetric operad. This conjecture is the analogue of the result from Batanin and Berger about the action of the little 2-disks operad on the homotopy center of a monoid [2]. We will then proceed to give the plan for the proof of our conjecture.

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Equivalences of higher categories

Viktoriya Ozornova

(joint work with Amar Hadzihasanovic, Félix Loubaton, Martina Rovelli)

In my talk, I discussed the relation of three sorts of higher categories for general natural number n: We can consider strict n-categories, weak n-categories or (∞, n)-categories.

One can give an informal description which would fit any of these mathematical objects. Any of them should have some objects, (1-)morphisms between these objects, then 2-morphisms between 1-morphisms, and so on. One would expect to have some source, target and composition operations for morphisms, satisfying some appropriate unitality, associativity and interchange laws. So far, it may sound that the aforementioned three kinds of higher categories are in fact the same.

The magic lies once in the 'and so on' part and once in the little word 'appropriate'. On the one hand, both 'strict' and 'weak' higher categories refer to the fact that 'and so on' stops at the dimension n of the morphisms, while ∞

of (∞, n)-categories refers to the fact that the process of going higher and higher never stops; the dimension n here refers to the fact that all the higher morphisms are assumed to be appropriately invertible.

On the other hand, the 'appropriate' compatibility laws could be just equalities (leading to strict n-categories) or witnessed by higher morphisms, which should in turn satisfy some coherence conditions. While the strict notion is indisputable, the other two allow for some amount of flexibility in encoding the notion.

Maybe surprisingly at the first glance, the weaker notions ought always be considered as some homotopical notion. One variant asks for considering the corresponding $(\infty, 1)$-category; a more structured version which is often more convenient for (∞, n)-categories is formulating statements in the language of model categories. Barwick and Schommer-Pries [BSP21] give an axiomatic description of the $(\infty, 1)$-category of (∞, n)-categories. Many implementation, in particular in the language of model categories, have been exhibited and are known to be equivalent; e.g. featuring the work of Joyal [Joy08b], Lurie [Lur09b], Bergner–Rezk [BR13, BR20], Ara [Ara14], Barwick [Bar05], Haugseng [Hau18], Gepner–Haugseng [GH15], Rezk [Rez01, Rez10], Loubaton [Lou22b], just to name a few.

The case of weak n-categories is in turn subtle. While for $n = 2$, the notion of the bicategory is both quite explicit and (maybe not independently) rather widespread, the notion becomes increasingly subtle as n grows. There are explicit definitions for weak 3-categories (aka tricategories) and some work on explicit definition for weak 4-categories (aka tetracategories) by Trimble, but for $n > 4$, the situation is quite hopeless. Instead, for the homotopy theory of weak n-categories, one can use - as in [GH15] - a localization of (∞, n)-categories, and it was shown by Haugseng [Hau15] that this coincides with the notion of weak n-categories provided by Tamsamani [Tam99]; also related work was done by Paoli [Pao19].

The relationship of strict n-categories, weak n-categories and (∞, n)-categories is sometimes subtle and our understanding depends a lot on the value of n. I will first report on the largest value - namely $n = 2$ - where we have a quite complete understanding, and then speculate on possible further developments for $n > 2$.

There are various homotopical embeddings for the homotopy theory of strict 2-categories as constructed e.g. by Lack [Lac02, Lac04] into the homotopy theory of $(\infty, 2)$-categories, e.g. by Rovelli and the author [OR21a], by Campbell [Cam20], by Gagna–Harpaz–Lanari [GHL22], by Moser [Mos20], and probably others. In a joint work with Moser and Rovelli, we were able to clarify the relationship of these constructions:

Theorem 1. [MOR22] *All* 2*-categorical nerves are equivalent and satisfy the universal property given by Gepner–Haugseng.*

It turns out that the notion of a coherent equivalence within a 2-category is crucial to define and to understand the corresponding nerve.

How do we reach higher dimensions? To construct a (somewhat explicit) model of the nerve, it would be helpful to have a corresponding notion of equivalence. (One could also work with a more abstract notion based on the model structure due to Lafont–Métayer–Worytkiewicz [LMW10] but for explicit calculations, one needs an explicit model.)

Theorem 2. [HLOR24] *There is an explicit model $\omega\mathcal{E}_n$ of fully coherent n-equivalence.*

While 'explicit' is not a strictly mathematical term, one thing which can be said in a more precise fashion is that the resulting n-category is finitely generated in every given degree.

Now recall that there is a model of (∞, n)-categories based on presheaves on a variant $t\Delta$ of the usual simplicial category Δ (some of it discussed in [OR20]). One can define a potential nerve for n-categories as a right adjoint to the left Kan extension of the functor $t\Delta \to n\mathcal{C}at$ given by n-truncated orientals $\mathcal{O}_n[\bullet]$ on Δ and by $[k]_t \mapsto \mathcal{O}_n[k] \coprod_{C_k} \Sigma^{k-1}\omega\mathcal{E}_n$ on the additional objects of $t\Delta$. Work in progress building upon recent work by Henry–Loubaton [HL23] indicates that the obtained adjunction should yield a Quillen pair between corresponding model categories. It is ongoing work to investigate (fully) faithfulness of the resulting ∞-adjunction for various values of n.

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Global Koszul duality

Matt Booth

(joint work with Andrey Lazarev)

The theory of conilpotent Koszul duality has its roots in Quillen's comparison between the commutative and Lie approaches to rational homotopy theory [11]. The modern formulation of conilpotent Koszul duality, due to Positselski and Lefèfvre-Hasegawa, is as a Quillen equivalence between the categories of augmented dg algebras and conilpotent dg coalgebras [7, 9]. The functors in question are the bar construction $B : \mathbf{dgAlg^{aug}} \to \mathbf{dgCog^{conil}}$, which roughly sends A to a twist of the tensor coalgebra on its augmentation ideal $\bar{A}$, and its left adjoint $\Omega : \mathbf{dgCog^{conil}} \to \mathbf{dgAlg^{aug}}$, defined analogously. The model structure on dg algebras here is the usual one - weak equivalences are the quasi-isomorphisms, fibrations are the degreewise surjections - but important here is both that the weak equivalences in $\mathbf{dgCog^{conil}}$ are created by Ω, and that they are strictly stronger than the quasi-isomorphisms.[1]

One should think of this algebra-coalgebra Koszul duality as a noncommutative version of the derived-geometric Lurie–Pridham correspondence between formal moduli problems and dg Lie algebras [10, 8]. Indeed, in characteristic zero, dg Lie algebras are Koszul dual to cocommutative conilpotent dg coalgebras, which - following a philosophy going back to Hinich [5] - one should think of as formal moduli

[1]The cofibrations in $\mathbf{dgCog^{conil}}$ are the degreewise injections.

problems.[2] At a high level, one should think of this as calculus - a formal moduli problem has a 'linearisation' to its tangent complex, which is a dg Lie algebra, and working formally locally ensures that one can always go back via integration. From this perspective, the above Quillen equivalence shows that augmented dg algebras control noncommutative deformation problems via a similar sort of calculus.[3]

Two natural questions arise: firstly, is there a version of 'nonconilpotent Koszul duality', and secondly, what kind of deformation-theoretic interpretation should this have? Conilpotency in our coalgebras corresponds to the fact that our formal moduli problems accept Artinian local dg algebras as input. So if we want to drop conilpotency (and also the (co)augmentations), our resulting notion of deformation problems should accept all finite dimensional algebras as input. In the commutative world, every finite dimensional algebra splits as a product of local algebras, but this is false in noncommutative geometry (think of, for example, matrix algebras), so these moduli problems should contain interesting 'genuinely noncommutative' data that allows separate points to communicate.

Dropping the (co)augmentations corresponds to introducing **curvature** on the other side of the bar-cobar adjunction.[4] Essentially, a curved algebra is like a dg algebra but instead of asking that the differential squares to zero we ask that $d^2(x) = [h, x]$ for some degree two 'curvature element' h (in particular, a curved algebra with zero curvature is the same thing as a dg algebra). Morphisms of curved algebras have two components: an algebra morphism and a change of curvature term.[5] This means, for example, that the natural inclusion $\mathbf{dgAlg} \to \mathbf{cuAlg}$ is not full. Curved coalgebras are defined similarly.

When removing the conilpotency assumption, one needs to replace the bar construction B by the extended bar construction $\check{B}$; loosely this is a completion of the usual bar construction.[6] For dg algebras the properties of the extended bar construction were first worked out in detail by Anel and Joyal [1] and in the curved setting, Guan and Lazarev [4] showed that there is an adjunction

$$\Omega : \mathbf{cuCog} \longleftrightarrow \mathbf{cuAlg} : \check{B}.$$

[2]a.k.a. 'formal stacks' or 'derived deformation functors'.

[3]This works over any base field, essentially since dg algebras always model E_1-algebras. In positive characteristic, dg Lie algebras are not the correct objects to use, and one must instead use Brantner and Mathew's partition Lie algebras [3].

[4]A fact well known to Positselski, who also gives a Quillen equivalence between conilpotent curved coalgebras and all dg algebras [9].

[5]A curved algebra is a curved A_∞-algebra with only three nonzero operations m_0, m_1, m_2, and a morphism is then the same as an A_∞ morphism with only two components f_1, f_2.

[6]Heuristically, $\check{B}$ is like B but where one replaces the 'cofree conilpotent coalgebra' functor - which is the tensor coalgebra functor - with the 'cofree coalgebra' functor, which is much wilder. For example, the cofree coalgebra on a one-dimensional vector space has dimension at least as large as the number of closed points in $\mathbb{A}^1_k$ - a sharp contrast to the tensor coalgebra, which always has dimension $\aleph_0$.

Our main theorem is that the categories **cuAlg** and **cuCog** admit model structures making the above adjunction into a Quillen equivalence.[7] As the notion of quasi-isomorphism does not make sense for curved (co)algebras, we need to formulate a new type of weak equivalence, the **Maurer–Cartan equivalences**. An MC element in a curved algebra is an element x of degree one with $dx+x^2+h=0$. We denote the set of MC elements in E by $\mathrm{MC}(E)$, and we caution that this set may be empty![8] Just as in the dg case, MC elements in the convolution algebra mediate the bar-cobar adjunction: if C is a curved coalgebra and A is a curved algebra, then the graded vector space $\hom(C, A)$ admits the structure of a curved algebra, and there are natural bijections $\mathbf{cuCog}(C, \check{B}A) \cong \mathrm{MC}\hom(C, A) \cong \mathbf{cuAlg}(\Omega C, A)$. If E is any curved algebra, we define a dg category $\mathrm{MC}_{\mathrm{dg}}(E) \subseteq \mathrm{Tw}(E)$ whose objects are the MC elements of E and whose hom-complexes are given by two-sided twists. Abbreviating $\mathrm{MC}_{\mathrm{dg}}(C, A) := \mathrm{MC}_{\mathrm{dg}}\hom(C, A)$, we can thus view $\mathrm{MC}_{\mathrm{dg}}(C, A)$ as a dg category of maps $C \to \check{B}A$ (equivalently, $\Omega C \to A$). We then say that a map f of curved algebras is an **MC equivalence** if for all[9] curved coalgebras C, the induced map $\mathrm{MC}_{\mathrm{dg}}(C, f)$ is a quasi-equivalence (a.k.a. Dwyer–Kan equivalence) of dg categories. MC equivalences for curved coalgebras are defined analogously.

We show that **cuCog** is a model category, where the cofibrations are the injections and the weak equivalences are the MC equivalences. Moreover, we show that **cuAlg** is a model category, where the fibrations are the maps p inducing fibrations $\mathrm{MC}_{\mathrm{dg}}(C, p)$ for all curved coalgebras C, and the weak equivalences are the MC equivalences.[10] Finally, we show that the extended bar-cobar adjunction is a Quillen equivalence.[11]

Following [1], we also show that **cuCog** is a closed symmetric monoidal model category under $\otimes$, and that **cuAlg** is model enriched over **cuCog**. The external homs are given by setting $\underline{\hom}(\Omega C, A) = \check{B}\hom(C, A)$ and then Kan extending in the first variable. We moreover show that our Koszul duality equivalence is compatible with both the curved and uncurved versions of conilpotent Koszul duality, as well as Holstein and Lazarev's categorical Koszul duality [6]; in particular we show that the left adjoint of the $\mathrm{MC}_{\mathrm{dg}}$ functor gives a Quillen coreflection of **dgCat** into **cuAlg**, and hence that the homotopy theory of dg categories fully faithfully embeds into that of dg algebras.

[7] Strictly, **cuAlg** is not cocomplete as it lacks an initial object, so we formally add one; dually we must also finalise **cuCog**.

[8] We have $\mathrm{MC}(E) \cong \mathbf{cuAlg}(k, E)$, and this set is nonempty precisely when E is curved isomorphic to a dg algebra; in fact, this gives an equivalence $\mathbf{cuAlg}_{k/} \simeq \mathbf{dgAlg}$.

[9] It is actually enough to test against all finite dimensional curved coalgebras.

[10] To partly alleviate this apparent asymmetry, a key intermediate step is to show that a morphism i of curved coalgebras is an injection if and only if, for all curved algebras A, the map $\mathrm{MC}_{\mathrm{dg}}(i, A)$ is a fibration. The rough idea of the proof is to reduce to cosquare-zero extensions and finite dimensional cosemisimple coalgebras. Whilst every fibration of algebras is a surjection, the converse is not true, and so some asymmetry remains.

[11] Using the results of [4] it is relatively straightforward to show that the corresponding ∞-categories are equivalent; the difficult part of [2] consists of actually constructing the model structures.

Finally, we study the global analogue of noncommutative formal moduli problems, which we call **Maurer-Cartan stacks**, defined as the left exact ∞-functors from $\mathbf{cuAlg_{fd}}$ to any finitely complete ∞-category. These are geometric objects modelled on (curved) profinite completions, rather than pro-Artinian completions. We give (pro)representability results for MC stacks valued in simplicial sets and in dg categories, and moreover show that these are compatible with Pridham and Lurie's (pro)representability results for formal moduli problems.

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Higher lax functorialities of the Grothendieck construction

Dimitri Ara

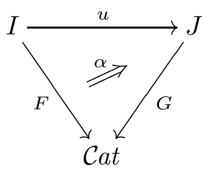
(joint work with Léonard Guetta)

The classical Grothendieck construction defines, for every small category I, a functor

$$\textstyle\int_I : \underline{\mathsf{Hom}}(I, \mathcal{C}at) \to \mathcal{C}at$$

sending a functor $F : I \to \mathcal{C}at$, from I to the category of small categories $\mathcal{C}at$, to the so-called *Grothendieck construction* $\int_I F$ of F. Here $\underline{\mathsf{Hom}}$ denotes the cartesian internal Hom of $\mathcal{C}at$. In particular, its morphisms are (strict) natural transformations. But the functorialities of the Grothendieck construction are more general. First, if $F, G : I \to \mathcal{C}at$ are two such functors and $\alpha : F \Rightarrow G$ is a *lax* transformation (that is, roughly speaking, a transformation in which the naturality squares only commute up to an oriented 2-cell), then one can still integrate α to obtain a functor $\int_I \alpha : \int_I F \to \int_I G$. Second, the construction is also functorial in I.

Putting all these together, we get a functoriality

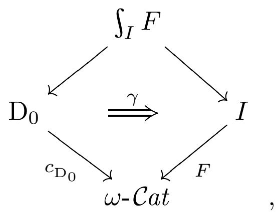


$$\mapsto \qquad \textstyle\int_I F \xrightarrow{\int(u,\alpha)} \int_J G \quad ,$$

where α is a lax transformation.

The purpose of this text, based on our paper [1], is to share some ideas about the higher generalizations of these functorialities, more precisely, in the setting of strict ω-categories. Let's denote by $\omega\text{-}\mathcal{C}at$ the ω-category of strict ω-categories (with the cartesian enrichment). If $F : I \to \omega\text{-}\mathcal{C}at$ is a strict ω-functor, where I is a strict ω-category, then a Grothendieck construction $\int_I F$ was defined by Warren in his work on the model of strict ω-groupoids for dependent type theory [4].

Nevertheless, the definition of Warren is unsatisfactory as it relies on explicit (and complicated) formulas. We propose to *define* the Grothendieck construction of $F : I \to \omega\text{-}\mathcal{C}at$ as the ω-category $\int_I F$ endowed with a universal 2-square



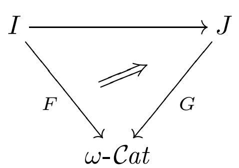
,

where D_0 denotes the terminal ω-category, c_{D_0} the constant ω-functor of value D_0 and γ a lax transformation. This type of universal 2-squares was already studied by myself and Maltsiniotis [2], and is related to the classical comma construction, usually denoted $u \downarrow v$. More precisely, we have

$$\textstyle\int_I F = c_{\mathrm{D}_0} \downarrow F \,,$$

where $\downarrow$ denotes the *lax* comma construction. Although these definitions are abstract, explicit formulas can be extracted and we recover from this abstract point of view the formulas of Warren.

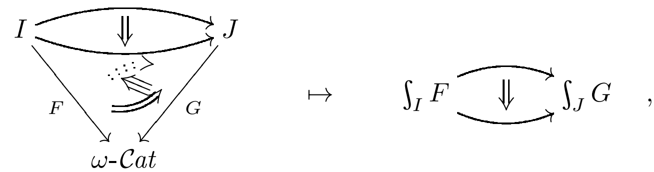
Let's now come back to the functorialities of the Grothendieck construction[1]. The universal property of the Grothendieck construction immediately gives a functoriality



$$\mapsto \qquad \textstyle\int_I F \longrightarrow \int_J G \quad ,$$

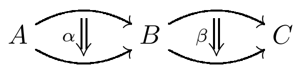
[1]Note that in the paper this text is based on [1] we address the more general question of the functorialities of the comma construction.

where the 2-cell represents a lax transformation. Working a bit harder, one can get a functoriality



where the 2-cells represent lax transformations and the 3-cell represents a lax 2-transformation (also known as a lax modification). And now comes the question: what is the general statement?

The answer to this question uses the language of Gray ω-categories, which we introduced with Maltsiniotis in our work on the join construction and the slices [3]. Indeed, the diagrams above involving 0-cells, 1-cells, 2-cells and 3-cells actually live in $\omega\text{-}\mathcal{C}\mathsf{at}_{\mathrm{lax}}$, in which 0-cells are strict ω-categories, 1-cells are strict ω-functors, 2-cells are lax transformations, 3-cells are lax 2-transformations, and so on. But $\omega\text{-}\mathcal{C}\mathsf{at}_{\mathrm{lax}}$ is not an ω-category! Indeed, if



are two lax transformations, then there are a priori two ways of composing them:

$$(t(\beta) *_0 \alpha)) *_1 (\beta *_0 s(\alpha)) \quad \text{and} \quad (\beta *_0 t(\alpha)) *_1 (s(\beta) *_0 \alpha)\,,$$

where s and t denote the source and the target. In general, these two lax transformations are different! In other words, $\omega\text{-}\mathcal{C}\mathsf{at}_{\mathrm{lax}}$ do not satisfy the exchange rule. What is true is that there is a (non-invertible) canonical lax 2-transformation

$$(t(\beta) *_0 \alpha)) *_1 (\beta *_0 s(\alpha)) \xRightarrow{\beta\circ\alpha} (\beta *_0 t(\alpha)) *_1 (s(\beta) *_0 \alpha)\,.$$

This means that $\omega\text{-}\mathcal{C}\mathsf{at}_{\mathrm{lax}}$ is some kind of lax ω-category. Formally, $\omega\text{-}\mathcal{C}\mathsf{at}_{\mathrm{lax}}$ is what we call a *Gray ω-category*[2], that is, a category enriched in $\omega\text{-}\mathcal{C}at$ endowed with the lax Gray tensor product. Morphisms of Gray ω-categories are called *Gray ω-functors*.

It is now tempting to think that the Grothendieck construction is a Gray ω-functor of target $\omega\text{-}\mathcal{C}\mathsf{at}_{\mathrm{lax}}$. But what would be the source Gray ω-category? Or, in other words, in which Gray ω-category do the triangles and cones we drew on the previous page are 1-cells and 2-cells? Obviously, in some kind of Gray ω-category of strict ω-categories over $\omega\text{-}\mathcal{C}at$. More generally, we prove that if $\mathbb{C}$ is a Gray ω-category and c is an object of $\mathbb{C}$, then there is a natural Gray ω-category $\mathbb{C}_{/c}$ of objects of $\mathbb{C}$ over c. In particular, we can consider the Gray ω-category $\omega\text{-}\mathcal{C}\mathsf{at}_{\mathrm{lax}/\omega\text{-}\mathcal{C}at}$.

[2]Actually, a *skew* Gray ω-category but we will not be precise about that in this text.

We can now finally answer our question:

Theorem. *The Grothendieck construction defines a Gray ω-functor*

$$\textstyle\int : \omega\text{-}\mathcal{C}\mathsf{at}_{\mathrm{lax}}/\omega\text{-}\mathcal{C}at \to \omega\text{-}\mathcal{C}\mathsf{at}_{\mathrm{lax}}\,.$$

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Davydov-Yetter cohomology: some tools and some applications

Christoph Schweigert
(joint work with Matthieu Faitg, Azat Gainutdinov, Jonas Haferkamp)

Davydov-Yetter (DY) cohomology originally assigned to a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between k-linear monoidal categories $\mathcal{C}$ and $\mathcal{D}$ a cochain complex of k vector spaces. 3-cocycles in the DY cochain complex for the identity monoidal functor $\mathrm{id}_{\mathcal{C}}$ describe deformations of associators of the monoidal category $\mathcal{C}$. 2-cocycles in the complex for a monoidal functor F describe deformations of of the monoidal structure on F.

Coefficients for DY cohomology were introduced in [2]; they take their values in a monoidal category $\mathcal{Z}_F(\mathcal{D})$ which for F the identity monoidal functor reduces to the Drinfeld center. DY cohomology with coefficients allows to describe deformations of mixed associators of module categories over monoidal categories. DY cohomology with coefficients also allows to describe deformations of braidings [4]. Moreoever, using coefficients, a conceptual understanding of Ocneanu rigidity can be achieved.

The coefficients also give rise to a pair of adjoint functors $\mathcal{Z}_F(\mathcal{D}) \rightleftarrows \mathcal{D}$ which, for F a right exact functor and $\mathcal{D}$ a finite tensor category, form a resolvent pair, so that DY cohomology can be expressed as relative cohomology [3]. From the general theory of relative cohomology, we obtain long exact sequences which allow to reduce the computation of DY-cohomology to a problem in representation theory. Based on this insight, concrete examples can be computed both by hand and in GAP [1] The project will be continued and more conceptual and computational tools will be developed in [4].

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A directed approach to higher categories

Simona Paoli

Higher categories are motivated by the desire to model structures arising in different areas such as homotopy theory, mathematical physics, logic and computer science. Intuitively, a weak higher category consists of objects (also called 0-cells), and higher cells (also called higher morphisms) in dimensions $1, \cdots, n, \cdots$, which are associative and unital up to invertible cell in the next dimension, in a coherent way. There are several classes of higher categories: weak n-categories have cells in dimensions up to n, and they are called weak n-groupoids when there are weak inverses. Weak ω-categories have cells in all dimensions and are called weak ω-groupoids when there are weak inverses. An important class of weak ω-categories consist of the (∞, n)-categories in which morphisms are weakly invertible after dimension n.

Several different combinatorial machineries (also called models of higher categories) have been developed to make this intuitions into precise mathematical structures; recently, model-independent approaches have also been developed. Several models are based on the simplicial category Δ and its products: for instance Segal n-categories and complete n-fold Segal spaces, based on functors $[\Delta^{n-1^{op}}, Spaces]$ satisfying additional conditions, are models of (∞, n)-categories; Tamsamani n-categories and weakly globular n-fold categories [2], based on functors $[\Delta^{n-1^{op}}, \mathsf{Cat}\,]$ satisfying additional conditions, are models of weak n-categories.

There are some disadvantages in the simplicial approaches: one is that the category $\Delta^{n^{op}}$ is not an inverse category, which makes it difficult to formalize these models into type theory.

Instead of Δ we seek to use a direct category so that morphisms only go in one direction, but which is still capable of encoding compositions and units. The wide subcategory Δ_{mono} of Δ on the injective maps is a direct category. However, this encodes only compositions: $X \in [\Delta^{op}_{mono}, \mathsf{Set}]$ such that the Segal maps are isomorphisms is a semi-category. We want a direct category which is intermediate between Δ_{mono} and Δ. The fat delta $\underline{\Delta}$, introduced by Joachim Kock [1], serves this purpose.

This approach also allows to keep the structure of compositions and units quite separated, and hence to use it to formulate models of higher categories which are minimally weak: such a model, proposed by J.Kock, is called fair n-categories $\mathsf{Fair^n}$, and is based on functors $[\underline{\Delta}^{n-1^{op}}, \mathsf{Cat}\,]$ satisfying additional conditions. These model weak n-categories in which compositions are strictly associative but only weakly unital. The interchange laws are also strict.

It is not known to date if this model satisfies the homotopy hypothesis. This was conjectured by J.Kock [1] as a way to formulate the Simpson's weak units conjecture that every good model of weak n-categories (that is, one that satisfies the homotopy hypothesis) should be suitably equivalent to a minimally weak one.

In this talk I have presented a study of the case $n = 2$ [3] and explained a direct comparison between Fair^2 and the category $\mathsf{Cat}^2_{\mathsf{wg}}$ of weakly globular double categories [4]. This result sheds new light on the way weak units are encoded in $\mathsf{Cat}^2_{\mathsf{wg}}$ in the so called weak globularity condition. It also paves the way to higher dimensions.

I have also presented several directions for future work. One is about the Simpson's weak units conjecture. Weakly globular double categories have been generalised in [2] to the category $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$ of weakly globular n-fold categories which was proved there to satisfy the homotopy hypothesis. Therefore one way to formulate the Simspon's conjecture is that fair n-categories are equivalent to weakly globular n-fold categories.

Another direction for future work is the use of $\underline{\Delta}$ to model $(\infty, 1)$-categories in a way similar to Segal categories and complete Segal spaces.

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Lax additivity, lax matrices, and mutation

Tobias Dyckerhoff

(joint work with Christ-Walde, Kapranov-Schechtman)

Several recent developments (c.f. [7, 6, 5, 3, 1]) suggest the possibility of systematically categorifying certain aspects of homological algebra, replacing abelian groups by stable ∞-categories. In a nutshell, the following table can serve as a basic guide for this type of "stable categorification":

classical	categorified
abelian group A	stable ∞-category $\mathcal{A}$
element $x \in A$	object $X \in \mathcal{A}$
$y - x$	$\mathrm{cone}(X \xrightarrow{f} Y)$
$\sum(-1)^i x_i$	$\mathrm{tot}(\ X_0 \xrightarrow{d} X_1 \xrightarrow{d} \cdots \xrightarrow{d} X_n\ )$
direct sum decomposition $C \cong A \oplus B$	semiorthogonal decomposition $\mathcal{C} \simeq \langle \mathcal{A}, \mathcal{B} \rangle$
$\vdots$	$\vdots$

Additive categories provide a basic axiomatic framework for homological algebra:

Definition 1. *A category $\mathcal{C}$ is called* semiadditive *if*

(1) $\mathcal{C}$ is enriched in abelian monoids,
(2) $\mathcal{C}$ has finite products and coproducts.

It is called additive *if, in addition,*

(3) for every pair $A, B \in \mathcal{C}$ of objects, the monoid $\mathcal{C}(A, B)$ is an abelian group.

The prototypical example of an additive category is, of course, the category of abelian groups. We propose the notion of a lax additive $(\infty, 2)$-category as a counterpart for the categorified context:

Definition 2 (Christ-D.-Walde)**.** *An $(\infty, 2)$-category $\mathbb{C}$ is called* semiadditive *if*

(1) $\mathbb{C}$ is enriched in $(\infty, 1)$-categories with colimits,
(2) $\mathbb{C}$ has lax colimits and limits.

It is called lax additive *if, in addition,*

(3) for every pair $A, B \in \mathbb{C}$ of objects, the $(\infty, 1)$-category $\mathbb{C}(A, B)$ is stable.

The prototypical example of a lax additive $(\infty, 2)$-category is given by presentable stable ∞-categories with colimit-preserving functors.

A basic phenomenon within an additive category is that, for objects A and B, their product and coproduct

$$\begin{array}{ccc} A \times B & \longrightarrow & B \\ \downarrow & & \\ A & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A \amalg B & \longleftarrow & B \\ \uparrow & & \\ A, & & \end{array}$$

are in fact canonically isomorphic (and hence referred to as direct sums). From this comparison isomorphism combined with the universal properties of product and coproduct results a matrix calculus to describe morphisms between direct sums and their compositions.

As analogous categorified universal constructions in a lax additive $(\infty, 2)$-category $\mathbb{C}$, consider a pair of objects $\mathcal{A}$, $\mathcal{B}$, along with a 1-morhpism $F : \mathcal{A} \to \mathcal{B}$. The relevant universal constructions associated to this data are characterized by the universal cones depicted in Figure 1.

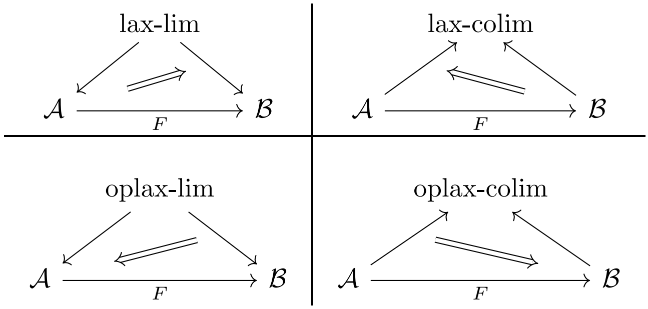

FIGURE 1. The four lax cones with base given by a functor $F\colon \mathcal{A} \to \mathcal{B}$.

Theorem (Lax Additivity, [2]). *In any lax additive $(\infty, 2)$-category, the four lax universal constructions from Figure 1 exist and are canonically equivalent.*

From an axiomatic perspective, the canonical equivalence of these four universal constructions captures the essence of what seems to make lax additive $(\infty, 2)$-categories a suitable context for categorified homological algebra (just like additive categories are used for classical homological algebra).

As in its 1-categorical analog, the equivalence between the various universal constructions yields a "lax matrix calculus" to describe morphisms between them and their compositions. For example, given 1-morphisms $F : \mathcal{A}_0 \to \mathcal{A}_1$ and $G : \mathcal{B}_0 \to \mathcal{B}_1$ in $\mathbb{C}$, a 1-morphism

$$\text{oplax-colim}\, F \longrightarrow \text{lax-colim}\, G$$

can be described by a *lax matrix*

$$\begin{pmatrix} \alpha_{00} & \longrightarrow & \alpha_{01} \\ \downarrow & & \downarrow \\ \alpha_{10} & \longrightarrow & \alpha_{11} \end{pmatrix}$$

with components $\alpha_{ij} \in \mathbb{C}(\mathcal{A}_j, \mathcal{B}_i)$ connected by morphisms a Grothendieck construction of the diagram $\mathbb{C}(\mathcal{A}_\bullet, \mathcal{B}_\bullet)$. The multiplication formula for

$$\begin{pmatrix} \alpha_{00} & \longrightarrow & \alpha_{01} \\ \downarrow & & \downarrow \\ \alpha_{10} & \longrightarrow & \alpha_{11} \end{pmatrix} \begin{pmatrix} \beta_{00} & \longrightarrow & \beta_{01} \\ \downarrow & & \downarrow \\ \beta_{10} & \longrightarrow & \beta_{11} \end{pmatrix}$$

yields the lax matrix

$$\begin{pmatrix} [\alpha_{00}\beta_{00} \to \alpha_{01}\beta_{10}] & \longrightarrow & [\alpha_{00}\beta_{01} \to \alpha_{01}\beta_{11}] \\ \downarrow & & \downarrow \\ [\alpha_{10}\beta_{00} \to \alpha_{11}\beta_{10}] & \longrightarrow & [\alpha_{10}\beta_{01} \to \alpha_{11}\beta_{11}] \end{pmatrix}$$

where a square bracket denotes the cone of the morphism it encloses.

In [4], this lax matrix calculus is used as a computational approach to investigate mutations of semiorthogonal decompositions (SODs). The analysis of the corresponding lax coordinate change matrices leads to criteria for higher periodicity properties of such SODs. The entries in the resulting lax matrix products can be expressed as explicit complexes of functors (higher spherical twists) which, remarkably, turn out to be categorifications of Euler's classical continuants.

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Reflexive homology and involutive Hochschild homology as equivariant Loday constructions

Birgit Richter

(joint work with Ayelet Lindenstrauss)

A non-equivariant Loday construction $\mathcal{L}_X(R)$ combines a finite simplicial set X and a commutative ring R into a simplicial commutative ring. For the circle, its homotopy groups are the Hochschild homology groups of R. Other important cases are higher dimensional spheres and tori.

Equivariantly for a finite group G, the input is a finite simplicial G-set and a G-commutative monoid: For G-spectra these are genuine commutative G ring spectra and for G-Mackey functor they are given by G-Tambara functors. We defined equivariant Loday constructions $\mathcal{L}_X^G(-)$ in these settings in [5]. In the following we will specialize to the group of order 2, C_2, to fixed point Tambara functors $\underline{R}^{\mathrm{fix}}$ of a commutative ring R with C_2-action, and to the one-point compactification of the real sign-representation, S^σ. For well-behaved genuine commutative C_2-ring spectra A we identified $\mathcal{L}_{S^\sigma}^{C_2}(A)$ with the Real topological Hochschild homology of

A, $THR(A)$, in [5]. In the talk, I explained a corresponding result for fixed point Tambara functors [4].

Involutive Hochschild cohomology was defined by Braun [1] and the corresponding homology theory, $\mathsf{iHH}_*^k(A;M)$, for associative k-algebras with anti-involution A and involutive A-bimodules M was developed by Fernàndez-València and Giansiracusa. We identify the latter with the homotopy groups of the C_2/C_2-level of our Loday construction [4]:

Theorem. If 2 is invertible in R and if R is flat as an abelian group, then

$$\pi_*\mathcal{L}_{S^\sigma}^{C_2}(\underline{R}^{\mathrm{fix}})(C_2/C_2) \cong \mathsf{iHH}_*^{\mathbb{Z}}(R;R).$$

Daniel Graves explored reflexive homology in [3]. This is the homology theory for the crossed simplicial group ΔR, where $R_n = C_2$ acts on the simplicial category Δ by reversing the simplicial structure. He showed that for a field of characteristic zero, k, involutive Hochschild homology and reflexive homology, $HR_*^{+,k}(A;M)$ of an associative k-algebra with anti-involution and an involutive A-bimodule M agree. We prove the following comparison result [4]:

Theorem. If 2 is invertible in R and if R is flat as an abelian group, then

$$\pi_*\mathcal{L}_{S^\sigma}^{C_2}(\underline{R}^{\mathrm{fix}})(C_2/C_2) \cong HR_*^{+,\mathbb{Z}}(R;R).$$

In particular, this identifies iHH_* and HR_*^+ in this generality. We also obtain identifications relative to an arbitrary commutative ground ring k under similar flatness conditions if 2 is invertible.

For an arbitrary finite group G there is no meaningful way for G to act on Δ. We propose

$$\pi_*\mathcal{L}_{SG}^{G}(\underline{R}^{\mathrm{fix}})(G/G)$$

as a suitable homology theory for commutative rings R with G action if the order of G is invertible in R and if R is flat. Here, SG is the unreduced suspension of G and G acts on SG by permuting the arcs.

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Kaledin classes and formality criteria

Coline Emprin

The idea of formality originated in the field of rational homotopy theory. In this context, a simply connected topological space is formal if one can recover all the information related to its rational homotopy type from the cohomology ring. More precisely, Sullivan [6] constructs a commutative differential graded algebra (cdga) denoted $\mathcal{A}_{pl}(X)$, whose quasi-isomorphism class faithfully characterizes the rational homotopy type of X. By definition, the space X is then formal if there exists a zig-zag of cdga quasi-isomorphisms relating $\mathcal{A}_{pl}(X)$ to its cohomology $H(X;\mathbb{Q})$. A central formality result is given by Deligne, Griffiths, Morgan, and Sullivan in [1], where they prove that any compact Kälher manifold is formal. This notion of formality generalizes to a wide range of algebraic structures. Let us fix a commutative ground ring R and a chain complex A over R. A differential graded algebraic structure φ on A (e.g. an associative algebra, a Lie algebra, an operad, etc.) is formal if it is related to its homology by a zig-zag

$$(A,\varphi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A),\varphi_*) \ ,$$

of quasi-isomorphisms, i.e. morphisms inducing isomorphisms in cohomology. Kaledin classes were introduced by Kaledin [3] as an obstruction theory fully characterizing the formality of associative algebras over a characteristic zero field. The work of Kaledin was extended by Lunts [4] for homotopy associative algebras over a $\mathbb{Q}$-algebra and by Melani and Rubió [5] for algebras over a binary Koszul operad in characteristic zero. One can ask for the formality of a wide range of other algebraic structures: operads themselves, structures involving operations with several inputs but also several outputs such as dg Frobenius bialgebras, dg involutive Lie bialgebras etc. Such structures are encoded by generalizations of operads: colored operads and properads. This aim of the present talk is to present a generalization of Kaledin classes construction established in [2] to study formality of

- any algebra encoded by a groupoid colored operad or properad;
- over any commutative ground ring R.

On the one hand, this enables us to recover and incorporate previous results into a single theory. On the other hand, this allows us to address new formality problems, such as the formality of algebras over properads and formality results with coefficients in any commutative ring. Thus, we use the resulting Kaledin classes to establish new formality criteria such as formality descent results, an intrinsic formality criterium or formality in families. Finally, this also leads to formality criteria in terms of chain level lifts of certain homology automorphisms. More precisely, we settle conditions on an homology automorphism so that the existence a chain level lift implies formality. This condition is, for example, satisfied by the Frobenius action in the ℓ-adic cohomology of any smooth projective variety thanks to the Weil conjectures and Riemann hypothesis for finite fields. This leads to the following result.

Theorem. *Let $\mathbb{V}$ be a groupoid and let $\mathcal{P}$ be a $\mathbb{V}$-colored operad in sets. Let p be a prime number. Let K be a finite extension of $\mathbb{Q}_p$ and let $K \hookrightarrow \mathbb{C}$ be an embedding. Let X be a $\mathcal{P}$-algebra in the category of smooth and proper schemes over K of good reduction, i.e. for which there exists a smooth and proper model $\mathcal{X}$ over the ring of integers $\mathcal{O}_K$. The dg $\mathcal{P}$-algebra of singular chains $C_*(X_{\mathrm{an}}, \mathbb{Q})$ is formal.*

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THR of Poincaré ∞-categories

Julie Rasmusen

Unimodular symmetric bilinear forms have long been objects of widespread interest, the classification of these however, still remains out of reach, even for simple rings such as the integers. Ideas from K-theory suggests a simplification where one considers for a sufficiently nice ring R, the abelian group $\mathrm{GW}_0(R)$ obtained by group completion of the monoid, consisting of isomorphism classes of finitely generated projective R-modules equipped with unimodular symmetric bilinear forms. This group is commonly known as the Groethendieck-Witt group, and can be decomposed into a K-theoretic and L-theoretic part through the exact sequence

$$K_0(R)_{C_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}_0(R) \to L_0(R) \to 0.$$

Here hyp refers to the map assigning to a projective module P it's hyperbolisation $P \otimes \hom_R(P, R)$ equipped with the evaluation form, and $L_0(R)$ is the cokernel of this map. Similarly to how Quillen extended K_0 to the higher K-groups, both GW_0 and L_0 has been given a more homotopy-theoretic refinement GW, L. By work of Karoubi and Schlichting, these fit into a fiber sequence, extending the above exact sequence into a long exact sequence, but only if 2 is a unit in R.

In the recent paper [1], this theory was moved into the setting of stable ∞-categories, with one of the goals to establish this fiber sequence also in the case where 2 is not a unit in R. Instead of considering a ring R, they instead consider so called Poincaré ∞-categories, generalising the idea that symmetric bilinear forms can be thought of as extra structure on the perfect derived category $D^p(R)$.

Definition 1. *A Poincaré ∞-category $(\mathcal{C}, \Omega\!\!\!\!\!\!\;\text{})$ consists of a small stable ∞-category $\mathcal{C}$ equipped with a reduced functor $Ϙ : \mathcal{C}^{op} \to Sp$, such that the following holds:*

- *The induced functor $B_{Ϙ} : \mathcal{C}^{op} \times \mathcal{C}^{op} \to Sp$ given by $B_{Ϙ}(x, y) = fib(Ϙ(x \oplus y) \to Ϙ(x) \oplus Ϙ(y))$ is exact and representable in each variable.*
- *The unit $ev : id_{\mathcal{C}} \Rightarrow D_{Ϙ}^2$ of the adjunction $D_{Ϙ} \dashv D_{Ϙ} : \mathcal{C}^{op} \to \mathcal{C}$, induced by the representing objects of $B_{Ϙ}$, is an equivalence.*
- *The induced functor $\Lambda_{Ϙ} := cofib((B_{Ϙ} \circ \Delta)_{hC_2} \to Ϙ)$, where Δ denotes the diagonal, is exact.*

A key example of a Poincaré ∞-category is the symmetric forms $(D^p(R), Ϙ^s)$, with $Ϙ^s(P) = \mathrm{map}_{R\otimes R}(P \otimes P, R)^{hC_2}$, where map denotes the mapping spectrum. We can consider it's so called *space of forms*, consisting of pairs (P, q) with $P \in D^p(R)$ and $q \in \Omega^\infty Ϙ^s(P)$. Noting that $\Omega^\infty Ϙ^s(P) \simeq \mathrm{Map}_{R\otimes R}(P \otimes P, R)^{hC_2}$, we see that this recovers the above setting. Using this as input, they continue by constructing the Groethendieck-Witt spectrum $\mathrm{GW}(\mathcal{C}, Ϙ)$ and the L-theory spectrum $L(\mathcal{C}, Ϙ)$, and establishes the desired fiber sequence, generalising the previously know result.

Theorem. *For $(\mathcal{C}, Ϙ)$ a Poincaré ∞-category there exists a fibre sequence in Sp*

$$K(\mathcal{C})_{hC_2} \to GW(\mathcal{C}, Ϙ) \to L(\mathcal{C}, Ϙ).$$

Furthermore, there exists a C_2-spectrum $\mathbb{K}R(\mathcal{C}, Ϙ)$ with $K(\mathcal{C})$ the underlying spectrum, and

$$\mathbb{K}R(\mathcal{C}, Ϙ)^{C_2} \simeq GW(\mathcal{C}, Ϙ),\ \Phi^{C_2}\mathbb{K}R(\mathcal{C}, Ϙ) \simeq L(\mathcal{C}, Ϙ).$$

An important tool for understanding algebraic K-theory is Topological Hochschild Homology, which for a sufficiently nice ring (spectrum) R is defined as $\mathrm{THH}(R) := R \otimes_{R\otimes_R R} R \simeq |\mathrm{N}^{\mathrm{cy}}_{\cdot}(R)|$. Intuitively, this can be thought as tensoring R with itself in a circle, and turning this circle leads to an S^1-action, making this into a genuine S^1-spectrum. Extending this to the case where R is equipped with an anti-involution, one can refine this to a genuine C_2-spectrum by in addition using the reflection of the circle. This leads to Real Topological Hochschild Homology, which is given by $\mathrm{THR}(R) = R \otimes_{NR} R$, where NR is the Hill-Hopkins-Ravenel norm. Using this C_2-action, it was shown in [2] that

$$\Phi^{C_2}\mathrm{THR}(R) \simeq \Phi^{C_2} R \otimes_R \Phi^{C_2} R,$$

A natural question is now, how one could extend this to the setting of Poincaré ∞-categories? One of the immediate problems is that instead of having a spectrum, we have a stable ∞-category. However, a key property of stable ∞-categories is that their mapping space have a canonical refinement to a mapping spectrum, which can be shown to lead to a spectral enrichment. Following the approach in [2], this can be used to extend THR to this new setting:

Theorem (J. Rasmusen)**.** *There exists a functor*

$$\Phi^{C_2}\, THR : Cat^p_\infty \to Sp,$$

which on a Poincaré ∞-category $(\mathcal{C}, Ϙ)$ is given by

$$\left| [n] \mapsto \underset{c_0,\dots,c_n \in ob\mathcal{C}}{colim} \Phi^{C_2}\mathcal{C}(c_0, D_{Ϙ}c_0) \otimes \mathcal{C}(c_1, c_0) \otimes \cdots \otimes \mathcal{C}(c_n, c_{n-1}) \otimes \Phi^{C_2}\mathcal{C}(D_{Ϙ}c_n, c_n) \right|$$

Through an appropriate version of Morita invariance, this can be shown to generalise the known settings:

- $\Phi^{C_2}\mathrm{THR}(D^p(R), \Omega^{gs}) \simeq \Phi^{C_2}R \otimes_R \Phi^{C_2}R$
- $\Phi^{C_2}\mathrm{THR}(D^p(R), \Omega^{s}) \simeq R^{tC_2} \otimes_R \Phi^{C_2}R$.

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New Developments in Feynman Categories: Bar and Koszul

Ralph M. Kaufmann

Summary. We present new results about Feynman categories based on a formalism developed with Michael Monaco. A main set of results is joint with Micheal Monaco and Yang Mo about three different bar resolutions for strong monoidal functor out of Feynman categories, generalizing the work of Muriel Livernet for operads. These are linked to results is the Koszulness for cubical Feynman categories and generalizations which are joint with Ben Ward. The originally announced results on Hopf algebras will be discussed elsewhere.

1. Categories as bimodule monoids and operad like structures. Following the philosophy of [9, 8, 7], which goes back to MacLane [10], a category is constructed in three steps by objects, their isomorphisms and then general morphisms. One can expand and formalize as follows: fix a symmetric monoidal enrichment category $\mathcal{E}$ which is cocomplete. let $\mathcal{B}$ be a category, that of basic morphisms, e.g. isomorphisms, and call a functor $\mathcal{B}^{op} \times \mathcal{B} \to \mathcal{E}$ a $\mathcal{B}$–bimodule (in $\mathcal{E}$). Bi–module morphisms are natural transformations. The standard examples are (i) $Hom_{\mathcal{C}}$, if $\mathcal{C}$ is enriched over $\mathcal{E}$, is a $\mathcal{C}$–bimodule, (ii) for $\mathcal{B} = Iso(\mathcal{C})$, the underlying groupoid, $Hom_{\mathcal{C}} : \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{E}$ (we will call this restriction $\rho_{\mathcal{C}}$) and (iii) for $\mathcal{B} = \mathcal{C}_{disc}$ the underlying discrete category, the restriction $Hom_{\mathcal{C}} : \mathcal{C}_{disc}^{op} \times \mathcal{C}_{disc} \to \mathcal{E}$, we will call these bi–modules $\rho_{\mathcal{C}_{disc}}$. $\mathcal{B}$–bimodules form a monoidal category under the plethysm product $\rho_1 \Box \rho_2(X,Z) = \int^{Y \in \mathcal{B}} \rho_1(Y,Z) \otimes \rho_2(X,Y)$, with unit $u = Hom_{\mathcal{B}}$.

Unital monoids in this monoidal category define category $C(\rho)$ with objects given by those of $\mathcal{B}$ and morphisms $Hom_{C(\rho)}(X,Y) = \rho(X,Y)$ with a left and a right action of the morphisms of $\mathcal{B}$. Composition is given by the monoidal structure and this descends to the coend by associativity. The identity maps are given by the unital structure. This association is an equivalence of categories. The example relevant to operad–like theories being (ii) above. The action is then given by $(\sigma, \sigma')(\phi) = \sigma' \phi \sigma^{op}$. Note that in the groupoid case, we can turn the right action of σ^{op} into a left action of σ^{-1}. This yields an internal category in groupoids, cf. [9].

If $\mathcal{B}$ is a (symmetric) monoidal category, we require that ρ is a lax monoidal functor—the examples (i)–(iii) work in this case as well. As a lax monoidal functor there are morphisms $\mu_\rho : \rho \otimes \rho \to \rho$–the key example is

$$Hom(X,Y) \otimes Hom(X',Y') \to Hom(X \otimes X', Y \otimes Y')$$

A monoidal bimodule monoid (MBM) is then defined by a *monoidal* natural transformation $\gamma : \rho \Box_{\mathcal{B}} \rho \to \rho$.[1] This implies the interchange relation: Writing $\gamma(\phi,\psi) = \phi\psi$ and $\mu(\phi,\psi) = \phi \otimes \psi$ it reads $(\phi_1 \otimes \psi_1)(\phi_2 \otimes \psi_2) = (\phi_1\phi_2) \otimes (\psi_1\psi_2)$. The data of a unital MBM defines a monoidal category $\mathcal{C}(\rho)$. For example, if $\mathcal{B} = \mathbb{S}$ the monoidal groupoid whose objects are natural numbers with addition, and $Hom(n,n) = \mathbb{S}_n$, the symmetric group, with $Hom(n,m) = \emptyset$ if $n \neq m$. The $\mathbb{S}$–bimodules are precisely PROPs as defined by MacLane [10].

An algebra is a functor $\alpha \in [\mathcal{B}, \hat{\mathcal{E}}]$, where $\hat{\mathcal{E}}$ is tensored over $\mathcal{E}$, together with a natural transformation $m : \rho \Box_{\mathcal{B}} \alpha \to \alpha$ which is an action. Note, $\rho_{\mathcal{C}_{disc}}$ algebras are equivalent to a functors from $\mathcal{C}$. In the monoidal case, α is required to be strong monoidal. The free ρ–algebra on a functor α is given by $\rho \Box_{\mathcal{B}} \alpha$. If α is a monoidal functor, $\rho \Box_{\mathcal{B}} \alpha$ is lax monoidal but not necessarily strong monoidal.

Definition. A unital MBM, is called *hereditary* if the following natural monoidal tranformation of functors $\mathcal{B}^{op} \times \mathcal{B} \times \mathcal{B} \to \mathcal{E}$ is an isomorphism.

$$(\rho \otimes \rho) \Box_{\mathcal{B} \times \mathcal{B}} u(id \times \mu_{\mathcal{B}}) \overset{(1)}{\to} \rho(\mu_{\mathcal{B}} \times \mu_{\mathcal{B}}) \Box_{\mathcal{B} \times \mathcal{B}} u(id \times \mu_{\mathcal{B}}) \overset{(2)}{\to} \rho(id \times \mu_{\mathcal{B}}) \Box_{\mathcal{B}} u \simeq \rho_{\mathcal{B}}(id \times \mu_{\mathcal{B}})$$

where (1) is induced by μ_ρ and (2) is defined by the universal property of the coend.

For example, a PROP is hereditary if and only if it is the PROP generated by an operad. Note that the condition above is exactly condition (ii') for a (strict) Feynman category [11] and is closely related to Getzler's pattern condition [12].

Theorem. If ρ is hereditary and α is strong, then $\rho \Box_{\mathcal{B}} \alpha$ is strong.

2. Special Categories. These observations let us rephrase the definition of a *strict Feynman category* $\mathcal{F}$ which is a monoidal category such that (i) the isomorphisms are free monoidal $Iso(\mathcal{F}) \simeq \mathcal{V}^{\otimes}$, (ii') $\rho_{\mathcal{F}}$ is hereditary and (iii) the slice categories are essentially small. These conditions ensure that the forgetful functor G from strong monoidal functors $\mathcal{F} \to \mathcal{E}$ to functors $\mathcal{V} \to \mathcal{E}$ has a left free adjoint F and these former functors are equivalent to algebras over the monad $T = GF$. A standard example is $\mathcal{E}ns$, that is finite sets. Condition (ii) can be weakened to the condition that for a monoidal category $\rho_{\mathcal{M}} = \nu_{\mathcal{M}}^{\otimes}$ is a free monoidal MBM. This condition defines a *unique factorization category* UFC [8]. It is a consequence that $Iso(\mathcal{M}) \simeq \mathcal{V}^{\otimes}$. A pre–hereditary[2] UFC satisfies an additional condition that implies that ν is a two–sided ideal under γ, cf. [8].—examples are cospans [13].

3. Baez–Dolan type plus constructions. Baez–Dolan type plus constructions or opetopes [14] allow one to put algebras and MBMs into relation. The classical example is that given an operad an algebra over an operad is an algebra

[1]The lax–monoidal structure of $\rho \Box_{\mathcal{B}} \rho$ is induced by 2,3 interchange on the domain and monoidal structure of $\mathcal{B}$ via the universal property of colimits

[2]this condition is called hereditary in the first version.

over the $\mathcal{E}ns$ hereditary MBM defined by $\rho_{\mathcal{O}}(n,1) = \mathcal{O}(n)$, while an operad itself is a functor from the Feynman category of operads, which is a plus construction [14, 11]. Formalizing the relationship of $\rho_{\mathcal{O}}$ to $\rho_{\mathcal{E}ns}$ as a natural transformation between bimodules, one arrives at slice categories, which in the unital case correspond to so–called indexed enrichments of categories.

In [8], we constructed plus constructions for categories which corepresent slice categories over $\rho_{\mathcal{C}}$, in the category and the monoidal case. In the latter there is are two constructions classifying lax or strong monoidal transformations. The strong monoidal version $\mathcal{M}^+$ for a monoidal category $\mathcal{M}$ generalizes the constructions of [14, 11, 9]. The upshot is that for any functor $\mathcal{M}^+$ corepresents $\rho_{\mathcal{M}}$–MBMs which are strong over $\rho_{\mathcal{M}}$. There is a unital version called gcp, which corepresents indexed enrichments, that is the slice category over $\mathcal{M}$ with respect to strong monoidal functors. A structural result is

Theorem [8]. $\mathcal{M}^+$ is a Feynman category if and only if $\mathcal{M}$ is a UFC.

This says that $\rho_{\mathcal{F}}$ algebras have a definition as $Iso(\mathcal{M})$ MBMs, the classic example being operads being plethysm monoids in the classical sense as $Iso(\mathcal{E}ns) = \mathbb{S}$[3]

4. Bar constructions. In [15], M. Livernet compared three bar constructions (1) The original Ginzburg-Kapranov bar construction B^{GK} [16], (2) the bar resolution of an operad as an algebra over the monad of trees B^L, and (3) the bar resolution of an operad as a plethysm monoid, cf. [17] B^o. The formulation of (2) is given slightly differently in [15] as a bar construction over an category of trees, which in light of section (1) can be understood as the resolution of an algebra over the monad $T = GF$. The key new observation is that Livernet's categories of trees are equivalent to the slice categories that appear in the left Kan extension. Fresse gave a levelization map $l : B^{GK} \to B^0$ and Livernet defined an inclusion $B^{GK} \to B^L$ which factors as l followed by an explicit quasi–isomorphism. The inclusion uses the identifcation of B^{GK} with a Koszul complex based on the category of trees.

The generalization of the bar construction B^{GK} for operads cyclic and modular operads is given in [5, 6], and was generalized in [11] to $B(\alpha)$ for strong monoidal functors α out of *cubical* Feynman categories. These are special Feynman categories with a non–negative degree function on morphisms, such that (1) isomorphisms are the only degree 0 morphisms, (2) the degree 1 morphisms together with the isomorphisms generate all morphisms under composition and tensor, and (3) every degree n morphism is decomposable in $n!$ ways up to isomorphism into degree 1 morphisms. The Feynman categories for operads, cyclic operads, modular operads are all cubical. Generalizing the results of [15]:

Theorem [3]. Every cubical Feynman category is Koszul.

Theorem [4]. (1) Using the interpretation of Livernet's categories of trees as slice categories, the bar construction B^L for a $\rho_{\mathcal{F}}$ algebra α is possible in any Feynman

[3] To relate this product to that of species, one must induce up from $\mathbb{S}^{op}$–modules to $\mathbb{S}$—bimodule on the left side : $\alpha^{\otimes}\Box_{\mathbb{S}}\alpha$

category as $B^{alg} := B(\rho_{\mathcal{F}}, \rho_{\mathcal{F}}, \alpha)$. This can be identified with the bar resolution of an algebra over the monad $T = GF$. (2) Using the Theorem above for a cubical Feynman category the bar construction of [11] $B(\alpha)$ can identified with the Koszul complex $K(\rho_{\mathcal{F}}, \rho_{\mathcal{F}}, \alpha)$ this maps via a quasi–isomorphism to $B^{alg}(\alpha)$. (3) For a Feynman category $\mathcal{F} = \mathcal{M}^{+,gcp}$ which is the gcp plus construction of a UFC M, there is a bar construction of α as a MBM ρ_α over $\rho_{\mathcal{M}}$ and a bar construction $B^{mon} = B(\rho_\alpha, \rho_\alpha, \rho_\alpha)$. If $\mathcal{F}$ is cubical there is a levelization map $B(\alpha) \to B^{mon}$ as an averaging of insertions of units factors the map to B^{alg}.

5. Generalization of Cubical Koszulness. Together with B. Ward we could extend cubical Koszulness in the following way—rephrased using the new language.

Theorem. [2] If $\rho_{\mathcal{F}} = \rho_1 \Box_{Iso(\mathcal{F})} \rho_2$ where ρ_i are MBMs with a distribute law satisfying the assumptions of the Diamond Lemma, and both are Koszul, so is $\rho_{\mathcal{F}}$.

The motivating example is the Feynman category for Schwarz Modular operads [1], which are the non–connected version of modular operads.

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2-Segal spaces and algebraic K-theory

Julie Bergner
(joint work with Brandon Shapiro, Inna Zakharevich)

The theory of 2-Segal spaces is a generalization of the theory of (1-)Segal spaces. While a Segal space models the data of a category up to homotopy, one can think of a 2-Segal space as a weaker structure, where composition of morphisms need not always exist or be unique, yet is still associative in an appropriate homotopical sense. More precisely, a *2-Segal space* is a simplicial space $X\colon \Delta^{\mathrm{op}} \to \mathcal{SS}et$ such that certain maps

$$X_n \to \underbrace{X_2 \times_{X_1} \cdots \times_{X_1} X_2}_{n-1}$$

associated to triangulations of $(n+1)$-gons for $n \geq 3$ are all weak equivalences of simplicial sets.

As shown by Dyckerhoff and Kapranov [3] and Gálvez-Carrillo, Kock, and Tonks [4], a central example of a 2-Segal space that is not a Segal space is the output of Waldhausen's $S_\bullet$-construction when applied to an exact category, which plays an important role in algebraic K-theory. Given the fact that these two sets of authors had quite different motivations and approaches to studying 2-Segal spaces, the fact that they both came to this particular family of examples suggests that it is central to the subject.

However, the $S_\bullet$-construction does not require the full structure of an exact category but only the following features: the existence of two distinguished classes of morphisms (admissible monomorphisms and admissible epimorphisms), a zero object, and the fact that pushout and pullback squares coincide. In joint work with Osorno, Ozornova, Rovelli, and Scheimbauer, we wanted to know the most general input to the $S_\bullet$-construction that resulted in a 2-Segal space. While Dyckerhoff and Kapranov had already generalized to *proto-exact categories*, which are characterized by the features listed above, our insight was that one could generalize to the setting of double categories, where the two kinds of morphisms need not have a common ambient category. In particular, a *double category* consists of objects, horizontal morphisms, vertical morphisms, and squares, forming categories in various appropriate ways.

Moving to a homotopical setting, the analogue of a double category is a double Segal space, and we ask for those that are pointed (having the analogue of a zero object) and be stable (the double-categorical analogue of pushout and pullback squares agreeing). We proved that the $S_\bullet$-construction gives an equivalence of homotopy theories between pointed stable double Segal spaces and reduced 2-Segal spaces [1]. Thus, pointed stable double Segal spaces can be regarded as a kind of universal input for algebraic K-theory.

From another point of view, Campbell and Zakharevich sought to axiomatize a general input for algebraic K-theory that encompassed examples such as the K-theory of varieties [2]. Their general input is known as a *CGW category*, and also takes the form of a double category satisfying other conditions. They and other authors have sought to understand when, for example, key theorems such as Additivity, hold in this broad context.

A natural question is what the relationship is between CGW categories and pointed stable double Segal spaces. In this project, we prove that every CGW category can be thought of as a pointed stable double Segal space in a straightforward way, using a reformulation of the axiomatization of CGW categories. Since it is not the case that every pointed stable double Segal space is a CGW category, the more difficult question is to characterize those that are.

The key to answering this question uses a generalization of the classifying diagram of Rezk [5]. Given a small category, its *classifying diagram* is the simplicial space $N\mathcal{C}$ whose space of n-simplices is given by the nerve of the groupoid of chains of n composable morphisms in $\mathcal{C}$. Since a CGW category instead has the structure of a double category, we generalize the classifying diagram construction to one that takes a double category $\mathcal{D}$ to a bisimplicial space $N\mathcal{D}$ in an analogous way. Then CGW categories can be characterized as the pointed stable double Segal spaces that look like classifying diagrams of suitable double categories.

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Effectivity in an arbitrary (∞, n)-categories.

Félix Loubaton

It is well known that there is an equivalence

$$\mathrm{Surj}(A) \simeq \mathrm{Cong}(A)$$

where $\mathrm{Surj}(A)$ denotes the $(\infty, 1)$-category of surjections between ∞-groupoids whose domain is A, and where $\mathrm{Cong}(A)$ is the $(\infty, 1)$-category of internal groupoids $\Delta \to \infty\text{-grd}$ whose ∞-groupoid of objects is A.

Moreover, we have a very explicit description of this equivalence. Given a surjection $p : A \to B$, the associated congruence is the simplicial object

$$C_n(p) := A \times_B A \times_B \cdots \times_B A$$

(i.e., a fibered product over B).

We then say that *congruences in* ∞*-grd are effective.*

We can now wonder what is the $(\infty, 1)$-categorical analogue of the effectiveness of ∞-groupoids. By this, we mean finding a notion of surjection and congruence such that the $(\infty, 2)$-category of surjections whose domain is an $(\infty, 1)$-category A is equivalent to the $(\infty, 2)$-category of congruences on A.

The right notion of surjectivity for $(\infty, 1)$-categories corresponds to the idea of essential surjectivity. A few simple examples make it clear that, given a surjection $p : A \to B$, the simplicial object that allows one to reconstruct B is given by

$$C_n^{lax}(p) := A \overset{\rightarrow}{\times}_B A \overset{\rightarrow}{\times}_B \ldots \overset{\rightarrow}{\times}_B A$$

where $\overset{\rightarrow}{\times}_B$ denotes the *lax cartesian product.* This simplicial object should be considered the prototypical example of an $(\infty, 1)$-congruence.

Following this intuition, we define an $(\infty, 1)$-congruence as an internal category $F : \Delta \to (\infty, 1)$-cat such that $F_{[1]} \to F_0 \times F_0$ is a two-sided fibration fibered in groupoids (roughly, a morphism $C \to A \times B$ is a two-sided fibration if its fibers depend contravariantly on A and covariantly on B).

We then recover the equivalence

$$\mathrm{Surj}(A) \simeq \mathrm{Cong}(A)$$

which sends a surjection p to $C_\bullet^{lax}(p)$. We say that $(\infty, 1)$*-congruences in* $(\infty, 1)$*-cat are effective.*

The generalization of the notions of surjection and congruence to the case of (∞, n)-categories is quite natural. We say that a morphism is a *surjection* if it is essentially surjective, and an internal category $F : \Delta \to (\infty, n)$-cat is an (∞, n)-congruence if $F_{[1]} \to F_0 \times F_0$ is a two-sided fibration fibered in $(\infty, n-1)$-categories. Once again, we have an equivalence

$$\mathrm{Surj}(A) \simeq \mathrm{Cong}(A)$$

but this time, the equivalence involves the (∞, n)-categories of Gray cylinders in A. We say that (∞, n)*-congruences in* (∞, n)*-cat are effective.*

Spectral sequences via presheaves

Sarah Whitehouse

(joint work with Muriel Livernet)

This talk presented the work of [6]. We study the category SpSe of spectral sequences and its homotopy theory. Since SpSe is neither complete nor cocomplete, it does not admit model category structures. In [5, Theorem 5.3.1], we established a weaker homotopical framework, that of an *almost Brown category*, and exhibited such a structure, SpSe_r, on SpSe for each $r \geq 0$. The class of weak equivalences is given by maps of spectral sequences which are quasi-isomorphisms on page r. Here, we situate SpSe as a subcategory of the category ESpSe of *extended spectral sequences* and exhibit various model category structures on this category. This setting provides a new perspective on the category of spectral sequences and its homotopy theory and we deduce consequences for the infinity category of spectral sequences.

To this end, we introduce and study a category of linear presheaves closely related to the category of spectral sequences SpSe. This is the category LWB of *linear witness books*, a linear presheaf category built from suitable disc objects. Intermediate between these two categories is the category of extended spectral sequences ESpSe, motivated by the wish to view SpSe as a subcategory of a convenient bicomplete category. There is a choice involved here, as we weaken the requirement in spectral sequences that a page is isomorphic to the homology of the previous one. Here we simply require a map, but not that it be an isomorphism. We have chosen that the maps go from a page to the homology of the previous one. In some sense, this choice effectively gives preference to colimits over limits. It fits well with the notion of *witness* cycles and boundaries appearing in our previous work [3]. Another motivation for our choice is that it is likely to be better behaved in terms of monoidal structure. Although we do not pursue that direction here, we note that Brotherston has shown that, on the category of filtered complexes, model structures which are closely related to spectral sequences are monoidal [2].

It turns out that ESpSe is bicomplete. Colimits are calculated pagewise, but to understand limits is less straightforward. Here the linear presheaf category of linear witness books plays a vital role, together with a pair of adjoint functors $(\mathcal{Q}, \mathcal{N})$. The terminology of linear witness books is chosen because the objects of this category can be viewed as having pages like those of a spectral sequence, with *witness* maps from a page to the previous one, as well as degeneracy maps in the other direction. This means objects have extra data, compared with spectral sequences, witnessing how elements end up on the r-page.

We establish an adjunction $\mathcal{Q} \dashv \mathcal{N}$ of functors between LWB and ESpSe and use its properties to identify subcategories $(\mathsf{LWB})^{\mathsf{e}}$ and $(\mathsf{LWB})^{\mathsf{s}}$ of LWB equivalent to ESpSe and SpSe respectively. As $(\mathsf{LWB})^{\mathsf{e}}$ is a full reflective subcategory of LWB it has all (small) limits and colimits and thus so does ESpSe.

This setting offers insight into *décalage* for spectral sequences. We study truncation functors on the underlying category D on which we take our linear presheaves. The embedding of a suitably truncated version of D into D has both a left and a right adjoint. This triple of adjoint functors gives rise to a chain of five adjoint functors on LWB. Of these, we show that the leftmost three are internal to $(\mathsf{LWB})^{\mathsf{e}}$ and so there is a corresponding triple of adjoints on ESpSe. These also restrict to SpSe. In particular we obtain a shift functor on ESpSe or SpSe with both a left adjoint LDec and a right adjoint Dec, two versions of décalage. The terminology is explained by noting that these functors are suitably compatible with Deligne's functors Dec^* and Dec for filtered complexes [4]. Although décalage in this sense is always closely connected to the study of spectral sequences, we are not aware of other work defining décalage functors directly on the category of spectral sequences, as we do here. Nonetheless, reference is quite often made to décalage of a spectral sequence and there are various important instances of this relationship. For example, Rognes notes in [8] that it is common to call the Whitehead tower spectral sequence the décalage of the Atiyah–Hirzebruch spectral sequence.

The homotopical part of the work extends the study of the homotopy theory of spectral sequences initiated in [5]. For each $r \geq 0$, we study spectral sequences with r-quasi-isomorphisms as a *relative category*, denoted $(\mathsf{SpSe}, \mathcal{E}_r)$. Having situated spectral sequences inside the bicomplete category of extended spectral sequences ESpSe, we establish model category structures there, restricting to the relevant structure on spectral sequences.

For each $r \geq 0$, we show the existence of a model category structure on ESpSe, restricting to recover the corresponding underlying almost Brown category structure on SpSe. Indeed there are two flavours of such model structures; in both the fibrations are those maps f that are surjective on pages 0 to r. There is a model category structure ESpSe_r where the weak equivalences are those maps such that the component on the r-page is a quasi-isomorphism. And there is another ESpSe'_r, where the weak equivalences are those maps such that the component on the r-page is a quasi-isomorphism and the components on all higher pages are isomorphisms. As relative categories, both ESpSe_r and ESpSe'_r restrict to $(\mathsf{SpSe}, \mathcal{E}_r)$.

The methods of proof use the category LWB. For the first family of structures, we obtain a cofibrantly generated model category structure on LWB by transfer of a projective-type model structure on the category of r-bigraded complexes and then modify this in order to produce a version LWB_r which is closely related to the relevant structure on SpSe. This model structure is then transferred to produce ESpSe_r. The second family of model structures is established by directly checking the axioms, making use of the existence of the first family.

These model structures have the following relationships to each other. We show that the model categories ESpSe_r for different r are all Quillen equivalent via shift and décalage functors and indeed, these are all Quillen equivalent to a projective-type model category structure on the category of 0-bigraded complexes. Similarly, the model categories ESpSe'_r for different r are all Quillen equivalent. The identity functor $\mathsf{ESpSe}'_r \to \mathsf{ESpSe}_r$ is a right Bousfield localization which is not a Quillen equivalence. Thus we provide a right *delocalization* of (a model category Quillen equivalent to) the projective model structure on 0-bigraded complexes, a bigraded version of chain complexes.

The model category ESpSe'_0 has the important feature that SpSe_0 is a homotopically full subcategory, in the sense that any object weakly equivalent to a spectral sequence is itself a spectral sequence. In [1] Barwick and Kan provide a model category structure on the category of relative categories, Quillen equivalent to Rezk's complete Segal space model structure on simplicial spaces, thus establishing another model for a homotopy theory of homotopy theories. Results of Meier [7] allow us to conclude that $(\mathsf{SpSe}, \mathcal{E}_0)$ is a fibrant relative category in this model. Our result can therefore be viewed as establishing an infinity-category of spectral sequences. And, via the shift-décalage adjunction, for each r the relative category $(\mathsf{SpSe}, \mathcal{E}_r)$ has $(\mathsf{SpSe}, \mathcal{E}_0)$ as a fibrant replacement.

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Homotopy theory of post-Lie algebras

Yunhe Sheng

(joint work with Andrey Lazarev, Rong Tang)

Homotopy invariant algebraic structures play a prominent role in modern mathematical physics [9]. Historically, the notion of A_∞-algebras, which was introduced by Stasheff in his study of based loop spaces [8], was the first such structure. Relevant later developments include the work of Lada and Stasheff about L_∞-algebras in mathematical physics [6] and the work of Chapoton and Livernet about pre-Lie$_\infty$ algebras [5]. A strong homotopy algebra is typically a Maurer-Cartan element in a certain differential graded (dg) Lie algebra (or possibly an L_∞-algebra) and its Maurer-Cartan twisting is called the cohomology of said strong homotopy algebra; it is known to control its deformation theory.

The notion of a post-Lie algebra has been introduced by Vallette in the course of study of Koszul duality of operads [10]. Munthe-Kaas and his coauthors found that post-Lie algebras also naturally appear in differential geometry and numerical integration on manifolds [7]. Meanwhile, it was found that post-Lie algebras play an essential role in regularity structures in stochastic analysis [3, 4].

A Rota-Baxter operator on a Lie algebra was introduced as the operator form of the classical Yang-Baxter equation. To better understand the classical Yang-Baxter equation and related integrable systems, the more general notion of an $\mathcal{O}$-operator (also called relative Rota-Baxter operator) on a Lie algebra was introduced by Kupershmidt. Relative Rota-Baxter operators naturally give rise to pre-Lie algebras or post-Lie algebras [1, 2].

Guided by Koszul duality theory, we consider the graded Lie algebra of coderivations of the cofree conilpotent graded cocommutative cotrialgebra generated by a graded vector space V. We show that in the case of V being a shift of an ungraded vector space W, Maurer-Cartan elements of this graded Lie algebra are exactly post-Lie algebra structures on W. The cohomology of a post-Lie algebra is then defined using Maurer-Cartan twisting. The second cohomology group of a post-Lie algebra has a familiar interpretation as equivalence classes of infinitesimal deformations. Next we define a post-Lie-infty algebra structure on a graded vector space to be a Maurer-Cartan element of the aforementioned graded Lie algebra. Post-Lie-infty algebras admit a useful characterization in terms of L-infty-actions (or open-closed homotopy Lie algebras). Finally, we introduce the notion of homotopy Rota-Baxter operators on open-closed homotopy Lie algebras and show that certain homotopy Rota-Baxter operators induce post-Lie-infty algebras.

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Higher homotopy categories and their uses

George Raptis

The homotopy category $\mathcal{C}[\mathcal{W}^{-1}]$ of a category with weak equivalences $(\mathcal{C}, \mathcal{W})$ ('homotopy theory') is a fundamental construction in abstract homotopy theory that is characterized by a universal property in the context of ordinary categories. Classically, this used in order to descend to a homotopy invariant context (with respect to $\mathcal{W}$), where categorical properties are invariant under weak equivalence, and it forms a natural domain for the definition of derived functors. Homotopical algebra has long revealed the great limitations of this construction, in connection with problems of homotopy coherence, the study of homotopy invariant algebraic structures, the properties of interesting invariants of homotopy theories, such as algebraic K-theory, and so on. Still, both the properties of the homotopy category $\mathcal{C}[\mathcal{W}^{-1}]$ (leading, for example, to the theory of triangulated categories) and questions about $\mathcal{C}[\mathcal{W}^{-1}]$ potentially determining further structure encoded in $(\mathcal{C}, \mathcal{W})$ (e.g., rigidity theorems, derived tilting theory, approximation theorems, K-theory of triangulated categories or derivators, and so on) have been extensively studied.

Despite its significant drawbacks, the construction of the homotopy category admits a corrected refinement that is essentially based on the same principle but in a higher categorical context: the ∞-localization $\mathcal{C}[\mathcal{W}^{-1}]_\infty$ reveals the homotopy theory encoded in $(\mathcal{C}, \mathcal{W})$ and is characterized by a universal property in the context of ∞-categories (see, for example, [2]). This also suggests an intermediate object: between $\mathcal{C}[\mathcal{W}^{-1}]_\infty$ and $\mathcal{C}[\mathcal{W}^{-1}]$ lies the homotopy n-category (see [5]) – this is characterized similarly by a universal property in the context of n-(that is, $(n, 1)$-)categories. In practice, this is obtained by a suitable truncation of the (derived) mapping spaces in the same way that the classical homotopy category is obtained by passing to the components of these mapping spaces.

The tower of higher homotopy categories associated to a given ∞-category bridges the gap between the ∞-category and its classical homotopy category. Higher homotopy categories define a natural sequence of intermediate refinements for the problem of comparing (naive) homotopy commutativity with (enhanced) homotopy coherence. This talk reviewed recent work concerning the properties and uses of higher homotopy categories that is part of a research project that aims to address similar questions about higher homotopy categories as the ones indicated above for the classical homotopy category. Specifically, it aims to study the properties of homotopy n-categories (with an eye towards notions of "higher triangulated categories"), make a comparative analysis of their relationship with homotopy theories (e.g., via invariants like K-theory), and obtain generalizations of results that link together previously unconnected results for well-behaved ∞-categories and for their homotopy categories (e.g., adjoint functor theorems). In more detail, the following topics were presented and discussed in the talk.

- The construction and main properties of the homotopy n-category were reviewed, especially, in connection with a notion of higher weak (co)limit that was introduced in [9] and further developed in joint work with H. K. Nguyen and C. Schrade [8]. These properties lead to a proposal for a notion of a stable n-category with nice properties, as conjectured by Antieau [1], that contains the homotopy n-categories of stable ∞-categories. Moreover, for $n \geq 2$, the (ordinary) homotopy category of a stable n-category is canonically triangulated – the case $n = \infty$ is well known, see [6]. (The analysis reveals the case $n = 1$ of triangulated categories to be somewhat special in an interesting way.)

- The definition of K-theory for higher homotopy categories (and related n-categories), based on Waldhausen's $\mathcal{S}_\bullet$-construction and using instead higher weak pushouts, was discussed. The main comparison result states that the comparison map from the usual algebraic K-theory to the K-theory of the homotopy n-category is n-connected [9]. This recovers and generalizes the classical fact that the Grothendieck group can be recovered from the triangulated homotopy category.

• (based on joint work with H. K. Nguyen and C. Schrade [8]) n-categorical generalizations of the classical Brown representability theory were presented. In particular, these apply to the homotopy n-category of a presentable ∞-category (for $n \geq 2$, unless it is stable) and link together in a clarifying way classical Brown representability theorems (and the corresponding adjoint functor theorems) for $n = 1$ [3, 4] with adjoint functor theorems (and the associated representability theorems) for $n = \infty$ [5, 7].

• The higher determinacy (or rigidity) of a (presentable) ∞-category in terms of its homotopy n-category was briefly discussed. Interestingly, for $n \geq 2$, the problem of rigidity/determinacy of a homotopy theory can also be formulated in the unstable context, despite the lack of triangulated structures, simply based on the refined categorical properties of higher homotopy categories. A main result (work in preparation) concerning the rigidity of the ∞-category of spaces was presented.

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Operadic 2-categories

Dominik Trnka

The theory of operadic categories was introduced in [BM15] and further developed in [BM23a, BM23b] and [BM24]. The last of the above mentioned works focuses only on 'unary' operadic categories which themselves describe a lot of known structures. On one hand, every discrete decomposition space is a unary operadic category [GKW21]. On the other hand, operads of certain unary operadic categories are monoids, categories, sharades, &c. For simplicity we shall consider only unary operadic categories.

The notion of Grothendieck construction, discrete fibration and categorical fibration is well known. In the operadic context, the operadic Grothendieck construction for Set-valued operads, as well as discrete operadic fibrations were introduced already in the first paper [BM15]. It was shown that for an operadic category $\mathbb{O}$, the operadic Grothendieck construction of an $\mathbb{O}$-operad produces a discrete operadic fibration over $\mathbb{O}$, and this assignment forms a part of an equivalence

$$\mathbb{O}\text{-oper}(\text{Set}) \simeq \text{DoFib}(\mathbb{O})$$

between Set-valued $\mathbb{O}$-operads discrete operadic fibrations over $\mathbb{O}$. Our goal is to establish a correspondence

$$\mathbb{O}\text{-oper}(\text{Cat}) \simeq \text{oFib}(\mathbb{O})$$

between categorical $\mathbb{O}$-operads and operadic functors into $\mathbb{O}$ with certain lifting properties, which we call operadic fibrations. Over time it became clear that the framework of operadic (1-)categories is not sufficient, and that we need to introduce (unary) operadic 2-categories. The definition is motivated by the characterisation of unary operadic categories of [GKW21].

Definition. A unary operadic 2-category is a pair $\mathbb{O} = (X, \mathscr{C})$ of a 2-category $\mathscr{C}$ and a simplicial set X, such that the upper décalage of X is the 2-categorical nerve of $\mathscr{C}$:

$$\text{dec}_\top X \cong \mathscr{N}\mathscr{C}.$$

Morphisms of unary operadic 2-categories are just morphisms of their underlying simplicial sets.

Equivalently, a unary operadic 2-category is a 2-category $\mathscr{C}$ together with a normalised lax functor $\coprod_{x\in\mathscr{C}} \mathscr{C}/x \xrightarrow{\varphi} \mathscr{C}$ from a coproduct of lax slices, together with a set U of chosen local lali-terminal objects u_c in each connected component c of $\mathscr{C}$, such that certain conditions hold. Here, local lali-terminal means that for every object x in the component c of $\mathscr{C}$, the category $\mathscr{C}(x, u_c)$ has a terminal object and require that the terminal object of $\mathscr{C}(u_c, u_c)$ is the identity on u_c. The conditions are essentially 2-categorical extension of the ones of [BM24].

Next, we define categorical operads of unary operadic 2-categories and their operadic Grothendieck construction. For simplicity we shall consider only 1-connected operads, i.e. operads having only the identity operation in every trivial arity. For a monoidal category $\mathscr{V}$, considered as a 1-objects 2-category, the lax slice $\mathscr{V}^\triangle := \mathscr{B}\mathscr{V}/*$ has a natural structure of an operadic 2-category.

Definition. A 1-connected categorical $\mathbb{O}$-operad $\mathcal{P}$ is an operadic functor $\mathcal{P}\colon \mathbb{O} \to \text{Cat}^\triangle$.

A categorical $\mathbb{O}$-operad $\mathcal{P}$ is thus a collection of categories $\mathcal{P}_x$, indexed by objects $x \in \mathscr{C}$, such that $\mathcal{P}_{u_c} = 1$, the terminal category, for every $c \in \pi_0(\mathscr{C})$. We denote the unique object of $\mathcal{P}_{u_c}$ by e_c. The collection is further equipped with functors

$$\mathcal{P}_y \times \mathcal{P}_{\varphi(f)} \xrightarrow{\mathcal{P}_f} \mathcal{P}_x,$$

for every map $f\colon x \to y$ in C which satisfy the following associativity and unit laws. We use the notation $a \cdot_f b$ for $\mathcal{P}_f(a, b)$.

– For any lax triangle α in C,

$$\begin{array}{ccc} x & \xrightarrow{\quad f \quad} & y, \\ & {\scriptstyle h}\searrow \quad \alpha \quad \swarrow{\scriptstyle g} & \\ & z & \end{array}$$

and objects $a \in \mathcal{P}_z, b \in \mathcal{P}_{\varphi(g)}, c \in \mathcal{P}_{\varphi(f)}$,

$$(a \cdot_g b) \cdot_f c = a \cdot_h (b \cdot_{\varphi(\alpha)} c).$$

– For any object $x \in C$ and $a \in \mathcal{P}_x$,

$$a \cdot_{\mathbb{1}_x} e_{\varphi(x)} = a,$$

$$e_{\pi(x)} \cdot_{u_x} a = a.$$

Definition. The operadic Grothendieck construction $\int_{\mathbb{O}} \mathcal{P} = (X_{\mathcal{P}}, \mathscr{C}_{\mathcal{P}})$ is a pullback of $\mathcal{P}$ along $\pi^{\triangle} \colon (*//\mathrm{Cat})^{\triangle} \to \mathrm{Cat}^{\triangle}$, where $*$ denotes the terminal category.

$$\begin{array}{ccc} \int_{\mathbb{O}} \mathcal{P} & \longrightarrow & (*//\mathrm{Cat})^{\triangle} \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle \pi^{\triangle}} \\ \mathbb{O} & \xrightarrow{\quad \mathcal{P} \quad} & \mathrm{Cat}^{\triangle} \end{array}$$

The construction can be extended to non-1-connected operads and the assignment $\int_{\mathbb{O}}$ induces a fully faithful functor $\mathbb{O}$-oper(Cat) $\to$ uOpCat/$\mathbb{O}$.

We present a simple example: Let $\odot$ be the terminal category with one object $*$ and one morphism $\mathbb{1}_*$. It is an operadic 2-category with $\varphi(\mathbb{1}_*) = *$ and $U = \{*\}$. Categorical $\odot$-operad $\mathcal{P}$ is a strict monoidal category and moreover $\int_{\odot} \mathcal{P} \cong \mathcal{P}^{\triangle}$. In machine learning the (1-)category $\mathcal{P}^{\triangle}$ is known as the 'Para' construction of a monoidal category [FST21]. We conclude with the following result.

Theorem. There is a one-to-one correspondence of categorical $\mathbb{O}$-operads and split operadic fibrations over a unary operadic 2-category $\mathbb{O}$, i.e. operadic functors into $\mathbb{O}$, which satisfy certain lifting properties.

The properties are operadic analogues of the lifting properties of classical categorical split fibrations.

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A research program for higher $\mathcal{V}$-topoi

André Joyal
(joint work with Mathieu Anel)

Recall from [2] that a *topos* can be defined to be a left exact localization of the category of presheaves $[\mathcal{C}^{op}, \mathsf{Set}]$ on a small category $\mathcal{C}$. Recall that every locally small category $\mathcal{E}$ admits a locally small free cocompletion (=completion under colimits) $y : \mathcal{E} \to P(\mathcal{E})$, and that a locally presentable category $\mathcal{E}$ is a topos if and only if the colimit functor $P(\mathcal{E}) \to \mathcal{E}$ is left exact. Category theory has a natural extension to ∞-categories. The notion of topos was extended to ∞-categories in [10], [11] and [6]. Let us denote by $\mathcal{S}$ the ∞-category of spaces (= ∞-groupoids). An ∞*-topos* is defined to be left exact (accessible) localization of the ∞-category of ∞-presheaves $[\mathcal{C}^{op}, \mathcal{S}]$ on a small ∞-category $\mathcal{C}$. Recall that every locally small ∞-category $\mathcal{E}$ has a locally small free cocompletion $y : \mathcal{E} \to P(\mathcal{E})$, and that a presentable ∞-category $\mathcal{E}$ [6] is an ∞-topos if and only if the colimit functor $P(\mathcal{E}) \to \mathcal{E}$ is left exact. The theory of enriched categories [4] has a natural extension to enriched ∞-categories [7] and [3]. Our goal is to develop an enriched version of the notion of ∞-topos [1]. Let $\mathcal{V}$ be a presentable smc (=symmetric monoidal closed) ∞-category. A $\mathcal{V}$*-topos* can be defined to be an (accessible) left exact $\mathcal{V}$-localization of the $\mathcal{V}$-category of $\mathcal{V}$-presheaves $[\mathcal{C}^{op}, \mathcal{V}]$ on a small $\mathcal{V}$-category $\mathcal{C}$. This definition is incomplete, since the notion of *finite* $\mathcal{V}$-limit was left undefined. In order to complete the definition, we shall suppose that the smc ∞-category $\mathcal{V}$ is ω*-presentable* in following sense: (i) the underlying ∞-category $\mathcal{V}_o$ is ω-presentable; (ii) the tensor product $A \otimes B$ of ω-compact objects $A, B \in \mathcal{V}$ is ω-compact; (iii) the unit object $I \in \mathcal{V}$ is ω-compact. We then say that a $\mathcal{V}$-limit (=weighted limit) $\{W, F\}$ is *finite* if the weight $W : \mathcal{J} \to \mathcal{V}$ [3] is ω-compact in the ∞-category of weights $\mathsf{Wt}(\mathcal{V})$. Every $\mathcal{V}$-category $\mathcal{E}$ has a free cocompletion under $\mathcal{V}$-colimits $y : \mathcal{E} \to P(\mathcal{E})$, and we have $P(\mathcal{E}) = [\mathcal{E}^{op}, \mathcal{V}]$ when $\mathcal{E}$ is small. We conjecture that a presentable $\mathcal{V}$-category $\mathcal{E}$ is a $\mathcal{V}$-topos if and only if the colimit functor $P(\mathcal{E}) \to \mathcal{E}$ is left exact. This conjecture is connected to the existence of a (pseudo) distributive law $Q_\omega P \to P Q_\omega$ between the free cocompletion monad $P : \mathcal{V}\mathsf{CAT} \to \mathcal{V}\mathsf{CAT}$ and the free finite completion monad $Q_\omega : \mathcal{V}\mathsf{CAT} \to \mathcal{V}\mathsf{CAT}$. More generally, a $\mathcal{V}$*-scale* is defined to be a class of $\mathcal{V}$-weights, or equivalently to be a full subcategory of $\mathsf{Wt}(\mathcal{V})$. There is a notion of α-colimits and a dual notion of α-limits for any $\mathcal{V}$-scale α; a $\mathcal{V}$-category $\mathcal{E}$ is α-cocomplete if and only if the opposite $\mathcal{V}$-category $\mathcal{E}^{op}$ is α-complete. A $\mathcal{V}$-category $\mathcal{E}$ has a free α-cocompletion $y : \mathcal{E} \to P_\alpha(\mathcal{E})$ and a free α-completion $y^o : \mathcal{E} \to Q_\alpha(\mathcal{E}) = P_\alpha(\mathcal{E}^{op})^{op}$. If τ is the largest scale, then $P_\tau(\mathcal{E}) = P(\mathcal{E})$ and $Q_\tau(\mathcal{E}) = Q(\mathcal{E})$. We may say that an α-complete $\mathcal{V}$-category (resp. α-continuous $\mathcal{V}$-functor) is α*-lex*. Let $\mathcal{V}\mathsf{CAT}^{Q_\alpha}$ be the category of α-lex $\mathcal{V}$-categories and α-lex $\mathcal{V}$-functors. We say that the $\mathcal{V}$-scale α is *distributive* if the functor $P : \mathcal{V}\mathsf{CAT} \to \mathcal{V}\mathsf{CAT}$ induces a functor

$P : \mathcal{V}\mathsf{CAT}^{Q_\alpha} \to \mathcal{V}\mathsf{CAT}^{Q_\alpha}$. If α is distributive, we say that a $\mathcal{V}$-category $\mathcal{E}$ is an *α-logos* if $\mathcal{E}$ is α-lex, $\mathcal{V}$-cocomplete and the colimit functor $P(\mathcal{E}) \to \mathcal{E}$ is α-lex. Let us denote the category of α-logoi by $\alpha\mathsf{LOG}$, where a *morphism* is a cocontinuous α-lex $\mathcal{V}$-functor. If the scale α is distributive and $\mathcal{A}$ is an α-lex $\mathcal{V}$-category, then the Yoneda functor $y : \mathcal{A} \to P(\mathcal{A})$ exhibits the α-logos freely generated by $\mathcal{A}$. Hence the left adjoint to the forgetful functor $\alpha\mathsf{LOG} \to \mathcal{V}\mathsf{CAT}^{Q_\alpha}$ is induced by the functor $P : \mathcal{V}\mathsf{CAT} \to \mathcal{V}\mathsf{CAT}^P$, where $\mathcal{V}\mathsf{CAT}^P$ is the category of cocomplete $\mathcal{V}$-categories and cocontinuous $\mathcal{V}$-functors. If $\mathcal{K}$ is a $\mathcal{V}$-category, then the composite $yy^o : \mathcal{K} \to Q_\alpha(\mathcal{K}) \to PQ_\alpha(\mathcal{K})$ exhibits the α-logos freely generated by $\mathcal{K}$. Hence the endofunctor PQ_α has the structure of a monad and we have $\alpha\mathsf{LOG} = \mathcal{V}\mathsf{CAT}^{PQ_\alpha}$. The monad structure of PQ_α can be obtained from a (pseudo) distributive law $\gamma : Q_\alpha P \to PQ_\alpha$ [8]. By construction, the $\mathcal{V}$-functor $\gamma(\mathcal{K}) : Q_\alpha P(\mathcal{K}) \to PQ_\alpha(\mathcal{K})$ is the unique α-lex $\mathcal{V}$-functor extending the $\mathcal{V}$-functor $P(y^o) : P(\mathcal{K}) \to PQ_\alpha(\mathcal{K})$ along the $\mathcal{V}$-functor $y^o : \mathcal{K} \to Q_\alpha(\mathcal{K})$. The largest scale τ is distributive and a τ-logos is a completely distributive ∞-category [9]. We conjecture that if the smc ∞-category $\mathcal{V}$ is ω-presentable, then the scale $\omega(\mathcal{V})$ of ω-compact weights is distributive. Moreover a presentable $\mathcal{V}$-category $\mathcal{E}$ is a $\mathcal{V}$-topos if and only if it is an $\omega(\mathcal{V})$-topos. The ∞-category of small ∞-categories Cat_∞ is presentable and cartesian closed. An $(\infty, 2)$-category can be defined to be ∞-category enriched over Cat_∞. Hence the general theory of $\mathcal{V}$-topoi can be applied to the theory of $(\infty, 2)$-topoi [5]. In a future paper in collaboration with Simon Henry, we shall describe sufficient conditions on a scale α to be distributive.

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Higher Segal spaces and partial groups

Philip Hackney
(joint work with Justin Lynd)

Our purpose is to explain and explore a new connection between the d-Segal spaces of Dyckerhoff and Kapranov and the partial groups of Chermak. The former objects have applications (when $d = 2$) in representation theory, geometry, combinatorics, and elsewhere, and are closely connected to ∞-operads, Span-enriched

A_∞-algebras, and operadic categories. The latter objects played a key role in Chermak's proof of the existence and uniqueness of centric linking systems for saturated fusion systems, a major recent result in p-local finite group theory.

Partial groups

Partial groups [C] are akin to groups, but where the n-fold multiplications $G^{\times n} \to G$ are replaced by partial functions. These may be concisely described as 'reduced spiny symmetric sets' by [HL], as we now explain. Let Υ be the category with the same objects $[n] = \{0, 1, \ldots, n\}$ as the simplicial category Δ, but with arbitrary functions as morphisms. A *symmetric (simplicial) set* is a functor $X\colon \Upsilon^{\mathrm{op}} \to \mathsf{Set}$. Groupoids may be identified with those symmetric sets X such that the Segal maps

$$X_n \to X_1 \times_{X_0} \underset{n}{\cdots} \times_{X_0} X_1$$

are bijections for all $n \geq 2$. A *spiny symmetric set* is a symmetric set X such that the Segal map is an injection for all $n \geq 2$, and a *partial group* is the same thing as a spiny symmetric set with X_0 a point. The partially-defined n-fold multiplication is defined by the span $X_1^{\times n} \hookleftarrow X_n \to X_1$ where the map on the right is given by the endpoint-preserving map $[1] \to [n]$. Every group G can be considered as a partial group, by identifying it with the associated symmetric set BG.

Every nonempty symmetric subset of BG is a partial group, and many important partial groups arise in this way. (Though not every partial group may be embedded into a group.) For example, $B_{\mathrm{com}}G \subseteq BG$ has n-simplices those $[g_1|\cdots|g_n] \in BG_n = G^{\times n}$ where $g_i g_j = g_j g_i$ for all i, j (see [AG]). Let us give another fundamental class of examples:

Example. Suppose G acts on a set V, and U is a subset of V. Then G 'acts partially' on the set U, and we let E be the simplicial set with n-simplices of the form

$$u_0 \xrightarrow{g_1} u_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} u_n$$

with $u_i \in U$ and $g_i \cdot u_{i-1} = u_i$. This E is a groupoid with object set $E_0 = U$, and we let $L \subseteq BG$ be the image of the map $E \to BG$.

For instance, consider the action of G by conjugation on the set V of subgroups of G, and let $U \subseteq V$ be the set of nontrivial subgroups in a fixed Sylow p-subgroup of G. The most important class of partial groups are the *localities*, which are modeled on this situation.

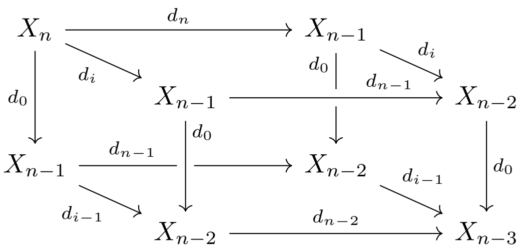
Higher Segal conditions

Higher Segal conditions are certain exactness conditions associated to a simplicial object, generalizing the usual Segal condition which underlies some models for $(\infty, 1)$-categories. The 2-Segal conditions first appeared in [DK], while the d-Segal conditions for $d > 2$ are explored in [P, W].

The d-Segal conditions[1] can be phrased in terms of a simplicial object X having a small number of associated cubes being (homotopy) limit cubes (of dimension $\lceil \frac{d}{2} \rceil + 1$). Write $i \ll j$ to mean $i < j - 1$. The 1-Segal condition is that (1) is a pullback for all $n \geq 2$, the 2-Segal condition is that the squares (2) are pullbacks whenever $0 \ll i \ll n$, and the 3-Segal condition is that the cube (3) is cartesian whenever $0 \ll i \ll n$.

$$(1)\quad \begin{array}{ccc} X_n & \xrightarrow{d_0} & X_{n-1} \\ {\scriptstyle d_n}\downarrow & \lrcorner & \downarrow{\scriptstyle d_{n-1}} \\ X_{n-1} & \xrightarrow[d_0]{} & X_{n-2} \end{array} \qquad (2)\quad \begin{array}{ccc} X_n & \xrightarrow{d_i} & X_{n-1} \\ {\scriptstyle d_0}\downarrow & \lrcorner & \downarrow{\scriptstyle d_0} \\ X_{n-1} & \xrightarrow[d_{i-1}]{} & X_{n-2} \end{array} \quad \begin{array}{ccc} X_n & \xrightarrow{d_i} & X_{n-1} \\ {\scriptstyle d_n}\downarrow & \lrcorner & \downarrow{\scriptstyle d_{n-1}} \\ X_{n-1} & \xrightarrow[d_i]{} & X_{n-2} \end{array}$$

(3)



For the 4-Segal condition, one replaces the cubes (3) associated with integers $0 \ll i \ll n$ by cubes associated with $0 \ll i \ll j (< n)$ and $(0<) i \ll j \ll n$. The 5-Segal condition concerns the four dimensional cubes associated to $0 \ll i \ll j \ll n$, and so on. A d-Segal object is automatically $(d+1)$-Segal, so one could wonder about the minimal d (if any) for a simplicial object to be d-Segal. For partial groups, this will turn out to always be odd. Let us give an indication of why this is true.

Theorem (H–Lynd)**.** *If a symmetric set is 2-Segal, then it is 1-Segal.*

Proof. The symmetric group action implies that for $n \geq 3$, square (1) is isomorphic to any of the squares in (2). The $n = 2$ instance of square (1) is a retract of the $n = 3$ instance of square (1). Thus (1) is a pullback for all $n \geq 2$. □

Definition. The degree of a partial group X, denoted $\deg(X)$, is the least positive integer k such that X is $(2k-1)$-Segal.

Groups are precisely the degree 1 partial groups. One can show that $B_{\mathrm{com}}G$ is 3-Segal, hence has degree 1 or 2. There are rich families of partial groups (arising from the example above) attaining arbitrarily high degree.

A primary method for calculating $\deg(X)$ is to consider sufficiently nice actions of X on various sets U. Such an action can be encoded as a map $\rho\colon E \to X$ satisfying certain properties, where E is a groupoid with $E_0 = U$. (This includes $E \to L$ from our example.) This gives rise to a closure operator $A \mapsto \bar{A}$ on E_0 defined in terms of simplices of X which act on all elements of $A \subseteq E_0$. A collection

[1] For d odd, we only consider the *lower* d-Segal conditions.

Γ of nonempty subsets of E_0 is *independent* if the set

$$\bigcap_{\substack{\Lambda\subset\Gamma \\ |\Gamma\setminus\Lambda|\le 1}} \overline{\bigcup \Lambda}$$

is empty, and $h(\rho)$ is defined to be the size of the largest independent $\Gamma \subseteq 2^{E_0}$.

Theorem (H–Lynd)**.** $\deg(X) \le h(\rho)$.

Corollary. *The degree of a finite partial group is finite.*

Proof. A partial group X is said to be finite just when X_1 is a finite set. Every finite partial group is finite-dimensional as a symmetric set by [HM], and hence has finitely many nondegenerate simplices. The canonical map $E = \coprod_{\mathrm{nd}(X)} \Upsilon^n \xrightarrow{\rho} X$ is a nice action of X on the finite set E_0. It follows that $h(\rho)$ is finite. □

We have now explained the rudiments of the connection between partial groups and higher Segal structures, by realizing partial groups as symmetric simplicial sets. We introduced a new invariant for partial groups – the degree – and a method for producing upper bounds for this invariant. In future work, we will calculate the degree for a number of important classes of examples, providing a source of interesting d-Segal spaces for large d.

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Stable Lie algebra homology and wheeled operads

Vladimir Dotsenko

Following Quillen, algebraic K-theory assigns to a ring R a sequence of groups $K_i(R)$ which are certain homotopy invariants of the group

$$GL(R) = \lim_{n\to\infty} GL_n(R)$$

of infinite invertible matrices. If R contains $\mathbb{Q}$, one can show that these groups are the primitive elements of the Hopf algebra given by the group homology of $GL(R)$. This suggests that, if one replaces the groups $GL_n(R)$ by the Lie algebras $\mathfrak{gl}_n(R)$, the Lie algebra homology of $\mathfrak{gl}(R) = \lim_{n\to\infty} \mathfrak{gl}_n(R)$ gives an infinitesimal version of the K-theory of R. A celebrated result proved independently by Loday–Quillen [4] and Tsygan [3] states that for a unital ring containing $\mathbb{Q}$ the primitive elements of the Hopf algebra given by the Lie algebra homology of $\mathfrak{gl}(R)$ is given, up to a degree shift, by the cyclic homology of R.

Suppose now that $\mathcal{O}$ is an arbitrary operad (the case of a ring R corresponds to an operad supported on elements of arity one). Then we can replace $GL_n(R)$ by $\mathrm{Aut}\mathcal{O}(x_1,\dots,x_n)$, the automorphism group of a finitely generated free $\mathcal{O}$-algebra. There are several reasonable candidates for the infinitesimal approximation of that group (for which we used $\mathfrak{gl}(R)$): for instance, one may look at the Lie algebra $\mathrm{Der}(\mathcal{O}(x_1,\dots,x_n))$ of all derivations, or at the Lie algebra $\mathrm{SDer}(\mathcal{O}(x_1,\dots,x_n))$ of all derivations with zero divergence, for the general notion of divergence for derivations of free algebras recently introduced by Powell [11]. Homotopy invariants of the group

$$\lim_{n\to\infty} \mathrm{Aut}(\mathcal{O}(x_1,\dots,x_n))$$

being extremely hard to compute, one can start with those infinitesimal versions. In fact, to obtain a nontrivial question, it is preferable to consider augmented operads, use the augmentation to "remove" derivations of degree zero (corresponding to linear changes of variables) and thus focus on the Lie algebras $\mathrm{Der}^+(\mathcal{O}(x_1,\dots,x_n))$ and $\mathrm{SDer}^+(\mathcal{O}(x_1,\dots,x_n))$.

To state our main result, we shall need the notion of a wheeled operad. Going back to work of Merkulov [6] (see also [10]), historically wheeled operads first appeared as particular cases of wheeled PROPs. However, we believe that they deserve to be explored as objects in their own merit, with a definition that is more elegant than that of a general wheeled PROP. Specifically, a *wheeled operad* is a two-coloured linear species $\mathcal{U}=\mathcal{U}_{\mathsf{o}}\oplus\mathcal{U}_{\mathsf{w}}$, where:

- $\mathcal{U}_{\mathsf{o}}$ is an operad,
- $\mathcal{U}_{\mathsf{w}}$ is a right $\mathcal{U}_{\mathsf{o}}$-module,
- $\partial(\mathcal{U}_{\mathsf{o}})$ is given a *trace map*

$$\mathrm{tr}\colon \partial(\mathcal{U}_{\mathsf{o}})\to\mathcal{U}_{\mathsf{w}},$$

which is a morphism of right $\mathcal{U}_{\mathsf{o}}$-modules and vanishes on the commutators in the twisted associative algebra $\partial(\mathcal{U}_{\mathsf{o}})$.

Here $\partial(\mathcal{U}_{\mathsf{o}})$ is the "derivative species" which declares one of the inputs of operations "special". Compositions in the special input makes that species a twisted associative algebra (a left module over the associative operad).

Stable limit of the homology of the Lie algebras of derivations of $\mathcal{O}(x_1,\dots,x_n)$ as $\dim(V)\to\infty$ is best understood via the notion of mixed representation stability [12] going back to the work of Brylinski [13], Feigin and Tsygan [8, Chapter 4] and Hanlon [7]. Let us recall the necessary basics that apply in our context. Consider a sequence of $\mathfrak{gl}_n$-modules V_n equipped with maps $V_n\to V_{n+1}$ that are injective for large n and consistent with the embeddings $\mathfrak{gl}_n\hookrightarrow\mathfrak{gl}_{n+1}$. We further assume that the image of V_n in V_{n+1} generates all of V_{n+1} under the action of $\mathfrak{gl}_{n+1}$ for large enough n. This sequence is said to be *mixed representation stable* if for all partitions α,β, the multiplicity of the irreducible $\mathfrak{gl}_n$-module $V(\alpha,\beta)$ with highest weight

$$\sum_i \alpha_i e_i - \sum_j \beta_j e_{n+1-j}$$

in V_n is eventually constant. This sequence is said to be *uniformly mixed representation stable* if all multiplicities become eventually constant simultaneously: there is some N such that for all partitions α, β, the above multiplicity does not depend on n for $n \geq N$. The language of wheeled operads and PROPs allows us to write a very short and compact formula for multiplicities of irreducible modules as follows.

Theorem. *Let $\mathcal{O}$ be an augmented operad. For each $i \geq 0$, the homology groups $H_i(\mathrm{Der}^+(\mathcal{O}(x_1, \ldots, x_n)), \Bbbk)$ and $H_i(\mathrm{SDer}^+(\mathcal{O}(x_1, \ldots, x_n)), \Bbbk)$ are uniformly mixed representation stable. More precisely, for $n \to \infty$, and for any two partitions $\alpha \vdash p, \beta \vdash q$:*

- *the multiplicity of $V(\alpha, \beta)$ in $H_\bullet(\mathrm{Der}^+(\mathcal{O}(x_1, \ldots, x_n)), \Bbbk)$ stabilizes to*
$$\left(\mathcal{P}^c(\mathsf{B}^{\circlearrowleft}(\mathcal{O}))(q, p) \otimes S^\alpha \otimes S^\beta\right)_{S_q \times S_p}$$
- *the multiplicity of $V(\alpha, \beta)$ in $H_\bullet(\mathrm{SDer}^+(\mathcal{O}(x_1, \ldots, x_n)), \Bbbk)$ stabilizes to*
$$\left(\mathcal{P}^c(\mathsf{B}^{\circlearrowleft}(\mathcal{O}^{\circlearrowleft}))(q, p) \otimes S^\alpha \otimes S^\beta\right)_{S_q \times S_p}$$

Here $\mathsf{B}^{\circlearrowleft}(\mathcal{O}))$ is the wheeled bar construction of the operad $\mathcal{O}$ viewed as a wheeled operad (with all trace maps equal to zero), $\mathsf{B}^{\circlearrowleft}(\mathcal{O}))$ is the wheeled bar construction of the wheeled completion of the operad $\mathcal{O}$, and $\mathcal{P}^c$ is the coPROP completion of a wheeled cooperad.

Examining this result in particular cases suggests that (positive parts of) the Lie algebras of divergence zero derivations are much better behaved homologically than the Lie algebras of all derivations, in particular they are stably Koszul for the associative operad and the Lie operad. However, even the more complicated answer for the Lie algebras of all derivations is somewhat tractable, and raises many natural questions about generators and relations of these Lie algebras, which we hope to study in further work.

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Graph complexes and models of chain E_n-operads in odd characteristic

Benoit Fresse

This talk is based on a work in progress.

There are several usual models of E_n-operads. In the setting of the rational homotopy theory, a model of E_n-operads, used by Kontsevich to prove the formality of E_n-operads in characteristic zero, is given by a differential graded (co)operad of graphs (see [5]). The construction of this cooperad of graphs can be formalized by using a twisting procedure, which reflects a fiberwise integration process of semi-algebraic differential forms on Fulton-MacPherson operads. The main purpose of this talk is to explain the definition of an analogue of the graph cooperad model of E_n-operads in characteristic different from 2.

The construction of this odd characteristic analogue of the cooperad of graphs involves particular E_∞-algebra structures governed by the surjection operad. Recall briefly that the surjection operad, denoted by $\mathbb{E}$ in these notes, is a chain operad spanned in arity r and degree d by the sequences $\underline{u} = (u(1), \dots, u(r+d))$ such that $u(t) \in \{1, \dots, r\}$, for $t = 1, \dots, r+d$. In the chain complex $\mathbb{E}(r)$, we also take $\underline{u} \equiv 0$ when the mapping $t \mapsto u(t)$ does not surject over the set $\{1, \dots, r\}$ or when we have a repetition $u(t) = u(t+1)$ in the sequence $\underline{u}$. The differential of the surjection operad is given by the omission of terms in sequences and the operadic composition operations extend the operadic composition of permutations. The normalized cochain complex $N^*(X)$ of any simplicial set X is equipped with a natural $\mathbb{E}$-algebra structure (see e.g. [1]).

We make the following claim:

Theorem.

(1) Let A and B be a pair of $\mathbb{E}$-algebras. Let $A \vee B$ denote the coproduct of A and B in the category of $\mathbb{E}$-algebras. We have a natural deformation

retract in the category of chain complexes

$$A \otimes B \underset{\Delta}{\overset{\nabla}{\rightleftarrows}} A \vee B \circlearrowright H ,$$

where ∇ carries $a \otimes b \in A \otimes B$ to the product of the elements a and b in $A \vee B$, while Δ is induced by an Alexander-Whitney diagonal map on the underlying chain complexes of the surjection operad. The natural transformation Δ is strongly symmetric monoidal (while ∇ is only symmetric up to homotopy).

(2) *In the case of the normalized cochain algebras of simplicial sets $A = N^*(X)$, $B = N^*(Y)$, the map Δ makes the following diagram commute*

$$\begin{array}{ccc} N^*(X) \vee N^*(Y) & \xrightarrow{\mathrm{pr}_X^* + \mathrm{pr}_Y^*} & N^*(X \times Y)\ , \\ & {\scriptstyle\Delta}\searrow \quad \swarrow{\scriptstyle\nabla^*} & \\ & N^*(X) \otimes N^*(Y) & \end{array}$$

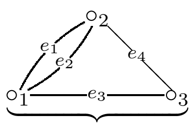
where the right hand side diagonal arrow is yielded by the usual Eilenberg–MacLane morphism, and we consider the morphisms $\mathrm{pr}_X^* : N^*(X) \to N^*(X \times Y)$, $\mathrm{pr}_Y^* : N^*(X) \to N^*(X \times Y)$, induced by the canonical projections $\mathrm{pr}_X : X \times Y \to X$, $\mathrm{pr}_Y : X \times Y \to Y$.

We rely on this result to construct a cooperad in chain complexes $\mathbb{E}\mathrm{Gra}_n^c$ such that:

$$\mathbb{E}\mathrm{Gra}_n^c(r) = \bigvee_{ij} \mathbb{E}(\omega_{ij}),$$

for each arity r, where we take, for each pair $\{i,j\} \subset \{1,\dots,r\}$, a free $\mathbb{E}$-algebra $\mathbb{E}(\omega_{ij})$ on one generator ω_{ij} of degree $n-1$. We can represent the elements of this cooperad as tensors $\underline{u} \otimes \gamma$, where γ is a graph based at vertices $\circ_1,\dots,\circ_r$, with edges $e_k = \circ_{i_k}{-}\circ_{j_k}$ corresponding to factors $\omega_{i_k j_k}$, and where $\underline{u} \in \mathbb{E}(m)$ is an element of the surjection operad whose inputs are in bijection with the edges of our graph e_k, $k = 1,\dots,m$. For instance the tensor

$$\underbrace{(1,3,1,2,4)}_{\underline{u}} \otimes \underbrace{\gamma}_{\gamma} \in \mathbb{E}\mathrm{Gra}_n^c(3)$$



corresponds to the monomial $\underline{u}(\omega_{12},\omega_{12},\omega_{13},\omega_{23})$ with $\underline{u} = (1,3,1,2,4) \in \mathbb{E}(4)$ in the $\mathbb{E}$-algebra $\mathbb{E}\mathrm{Gra}_n^c(3) = \mathbb{E}(\omega_{12}) \vee \mathbb{E}(\omega_{13}) \vee \mathbb{E}(\omega_{23})$.

We have natural morphisms of cooperads such that

$$\mathbb{E}\mathrm{Gra}_n^c(r) = \bigvee_{ij} \mathbb{E}(\omega_{ij}) \xrightarrow{(1)} \bigvee_{ij} N^*(S_{ij}^{n-1}) \xrightarrow{(2)} N^*(\bigtimes_{ij} S_{ij}^{n-1}) \xrightarrow{(3)} N^*(\mathcal{E}_n(r)),$$

for each arity r, where we consider the variant of the above cooperad of graphs defined by taking a coproduct of copies of the cochain algebra of the $n-1$-sphere $N^*(S_{ij}^{n-1}) = N^*(S^{n-1})$ (instead of the free $\mathbb{E}$-algebras $\mathbb{E}(\omega_{ij})$), we take the operad in simplicial sets given by the product $\times_{ij} S^{n-1}_{ij}$ of the copies of the sphere $S^{n-1}_{ij} = S^{n-1}$ and the associated cochain cooperad $N^*(\times_{ij} S^{n-1}_{ij})$, and $\mathcal{E}_n$ is a model of E_n-operad in simplicial sets with $\mathcal{E}_n(0) = *$ (we can take for instance the Barratt-Eccles operad model of [1] for this operad $\mathcal{E}_n$).

We proceed as follows to construct the morphisms of this sequence (1-3). We use the assumption that 2 is invertible in our ring of coefficients to pick a representative of the fundamental class of the sphere $\omega \in N^{n-1}(S^{n-1})$ whose image under the action of the antipode satisfies $\tau^*(\omega) = (-1)^n\omega$. We then form the coproduct of the morphisms of $\mathbb{E}$-algebras $\phi_{ij} : \mathbb{E}(\omega_{ij}) \to N^*(S^{n-1}_{ij})$, which carry the generator ω_{ij} to this element $\omega \in N^{n-1}(S^{n-1}_{ij})$ in our copy of the cochain algebra of the sphere, in order to define the first morphism (1) of our sequence, and we consider the morphisms $\mathrm{pr}^*_{ij} : N^*(S^{n-1}_{ij}) \to N^*(\times_{ij} S^{n-1}_{ij})$ induced by the canonical projections $\mathrm{pr}_{ij} : \times_{ij} S^{n-1}_{ij} \to S^{n-1}_{ij}$ in order to get our second morphism (2). We again rely on the result of the previous theorem to ensure that these morphisms preserve cooperad structures (the invariance assumption on our cochain ω with respect to the action of the antipode ensures that the first morphism preserves the symmetric structures of our cooperads). We eventually use the equivalence $\mathcal{E}_n(2) \sim S^{n-1}$ and the map $\psi_{ij} : \mathcal{E}_n(r) \to \mathcal{E}_n(2)$ such that $\psi_{ij}(\underline{w}) = \underline{w}(*, \dots, \underset{i}{1}, \dots, \underset{j}{1}, \dots, *)$ to get a morphism of operads in simplicial sets such that $\psi : \mathcal{E}_n(r) \to \times_{ij} S^{n-1}_{ij}$. We take the morphism of cochain cooperads induced by this morphism of simplicial operads to get the third morphism (3) of our sequence.

We prolong this sequence of morphisms by using the Koszul duality of E_n-operads in chain complexes (see [3]), which implies the existence of a quasi-isomorphism of cooperads

$$N^*(\mathcal{E}_n) \xrightarrow{\sim} B(\Lambda^n \mathbb{E}_n),$$

where $B(-)$ denotes the operadic bar construction, the notation Λ refers to the operadic suspension, and $\mathbb{E}_n$ is a model of E_n-operads in chain complexes (e.g. we can take the operad of chains on the Barratt-Eccles operad model of E_n-operads $\mathbb{E}_n = N_*(\mathcal{E}_n)$, as in *loc. cit.*) We accordingly have a sequence of morphisms:

$$(*) \qquad \mathbb{E}\mathrm{Gra}^c_n \to N^*(\mathcal{E}_n) \to B(\Lambda^n \mathbb{E}_n) \xrightarrow{\epsilon_*} B(\Lambda^n \mathcal{C}om),$$

where we also adopt the notation $\mathcal{C}om$ for the commutative operad and we consider the morphism induced by the augmentation $\epsilon : \mathbb{E}_n \to \mathcal{C}om$.

For any cooperad $\mathcal{C}$, we have a bijection between the morphisms $\phi_\alpha : \mathcal{C} \to B(\Lambda^n \mathcal{C}om)$ and the set of Maurer-Cartan elements in a differential graded preLie algebra with divided powers $\mathrm{Dfm}(\mathcal{C}, \Lambda^n \mathcal{C}om)$ such that:

$$\mathrm{Dfm}(\mathcal{C}, \Lambda^n \mathcal{C}om) = \mathrm{Hom}(\mathcal{C}, \Lambda^n \mathcal{C}om),$$

where we take the differential graded hom-object of maps of symmetric sequences $\alpha : \mathcal{C} \to \Lambda^n \mathcal{C}om$ (see [6]). We use a cooperad version of the general twisting procedure of [2] to associate a twisted cooperad $\mathrm{Tw}^c\mathcal{C}$ to any such Maurer-Cartan element $\alpha \in \mathrm{MC}(\mathrm{Dfm}(\mathcal{C}, \Lambda^n \mathcal{C}om))$.

We apply this construction to the case $\mathcal{C} = \mathbb{E}\mathrm{Gra}_n^c$. We then get cooperad morphisms:

$$\mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c \to \mathrm{Tw}^c N^*(\mathcal{E}_n) \to \mathrm{Tw}^c B(\Lambda^n\mathbb{E}_n),$$

by functoriality of the twisting construction. We also have a cooperad morphism $\pi_* : \mathrm{Tw}^c B(\Lambda^n\mathbb{E}_n) \to B(\Lambda^n\mathbb{E}_n)$ induced by composites with arity zero operations $* \in \mathbb{E}_n(0)$ at the level of the operad $\mathbb{E}_n$.

We can identify the elements of the cooperad $\mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c$ with tensors $\underline{u} \otimes \gamma$, where γ is a graph in which we split the set of vertices into a subset of internal vertices, usually denoted in black $\bullet_1, \dots, \bullet_k$, and a subset of external vertices $\circ_1, \dots, \circ_r$, which correspond to operadic inputs in $\mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c$.

We have $\mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n\mathcal{C}om) = \prod_r (\mathbb{E}\mathrm{Gra}_n^c(r)_{\Sigma_r})^\vee$, where we take the dual $(-)^\vee$ of the modules of graphs $\mathbb{E}\mathrm{Gra}_n^c(r)$ (moded out by the action of the symmetric group). We let $\mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n\mathcal{C}om)^{conn}$ denote the submodule spanned by the connected graphs inside $\mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n\mathcal{C}om)$. We can check that this submodule is preserved by the preLie algebra structure with divided powers on $\mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n\mathcal{C}om)$. We then make the following crucial observation:

Claim (Theorem in the case $n = 2$. Conjecture in the case $n > 2$)**.**

(1) The Maurer-Cartan element $\alpha \in \mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n\mathcal{C}om)$, which corresponds to the morphism $\phi_\alpha : \mathbb{E}\mathrm{Gra}_n^c \to B(\Lambda^n\mathcal{C}om)$ yielded by our sequence (*), satisfies $\alpha \in \mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n\mathcal{C}om)^{conn}$. This relation implies that we can form a well-defined cooperad in chain complexes by taking the quotients

$$\mathbb{E}\mathrm{Graphs}_n^c(r) = \mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c(r)/\equiv,$$

where we mod out the complex $\mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c(r)$ by relations such that $\underline{u}\otimes\gamma \equiv 0$ when the graph γ contains a connected components of internal vertices.

(2) The morphism $\mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c \to B(\Lambda^n\mathbb{E}_n)$, which we deduce from the map $\pi_* : \mathrm{Tw}^c B(\Lambda^n\mathbb{E}_n) \to B(\Lambda^n\mathbb{E}_n)$ at the level of the bar construction, cancels these elements such that $\underline{u} \otimes \gamma \equiv 0$, and therefore induces a morphism

$$\begin{array}{ccccc} \mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c & \longrightarrow & \mathrm{Tw}^c N^*(\mathcal{E}_n) & \longrightarrow & \mathrm{Tw}^c B(\Lambda^n\mathbb{E}_n) \\ \big\downarrow & & & & \big\downarrow{\scriptstyle \pi_*} \\ \mathbb{E}\mathrm{Graphs}_n^c & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \dashrightarrow & B(\Lambda^n\mathbb{E}_n) \end{array}$$

on our quotient cooperad $\mathbb{E}\mathrm{Graphs}_n^c = \mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c/\equiv$.

We then obtain:

Theorem (depending on the validity of the conjectured property of the claim in the case $n > 2$)**.** *The morphism of cooperads yielded by the above construction defines a quasi-isomorphism:*

$$\mathbb{E}\mathrm{Graphs}_n^c \xrightarrow{\sim} B(\Lambda^n\mathbb{E}_n),$$

and hence the cooperad of graphs $\mathbb{E}\mathrm{Graphs}_n^c$ defines a model of E_n-cooperad in chain complexes through the Koszul duality of E_n-operads.

The cooperad $\mathbb{E}\mathrm{Graphs}_n^c$ is a generalization of the (co)operad of graphs defined in [5]. We also refer to [7] for a thorough study of this characteristic zero version of the (co)operad of graphs, which we usually denote by Graphs_n^c. We have, according to this reference, an action of a Lie algebra of graphs GC_n^2 on the cooperad Graphs_n^c, and one can prove that this differential graded Lie algebra GC_n^2 is quasi-isomorphic to the deformation complex of the object Graphs_n^c in the category of Hopf cooperads. We can use the latter result to compute the homotopy of the space of homotopy automorphisms of the rationalization of E_n-operads (see [4]). When we forget about Hopf structures, we still have a result asserting that the deformation complex of the graph cooperad Graphs_n^c is quasi-isomorphic, as a chain complex, to the symmetric algebra generated by the $n+1$-fold suspension of the differential graded module GC_n^2 (up to an extra degree shift).

In our setting, the role of the differential graded Lie algebra of graphs GC_n^2 is yielded by the preLie algebra with divided powers:

$$\mathbb{E}\mathrm{GC}_n^2 = \mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n \mathcal{C}om)^{conn}$$

which we equip with a twisted differential determined by our Maurer-Cartan element $\alpha \in \mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n \mathcal{C}om)^{conn}$. We can equip this extended graph complex $\mathbb{E}\mathrm{GC}_n^2$ with the structure of an L_∞-algebra with divided powers and we also have an action of this L_∞-algebra with divided powers on the chain cooperad $\mathbb{E}\mathrm{Graphs}_n^c = \mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c/\equiv$. We conjecture that the deformation complex of the object $\mathbb{E}\mathrm{Graphs}_n^c$ (in the category of chain cooperads) is equipped with a natural E_{n+2}-algebra structure and that the action of the L_∞-algebra $\mathbb{E}\mathrm{GC}_n^2$ on $\mathbb{E}\mathrm{Graphs}_n^c$ gives rise to a quasi-isomorphism $U_{E_{n+2}}(\mathbb{E}\mathrm{GC}_n^2) \xrightarrow{\sim} \mathrm{Dfm}(\mathbb{E}\mathrm{Graphs}_n^c, \mathbb{E}\mathrm{Graphs}_n^c)$, where $U_{E_{n+2}}(-)$ denotes the enveloping E_{n+2}-algebra functor from the category of L_∞-algebras (with divided powers) to the category of E_{n+2}-algebras. This conjecture is consistent with the computation of the deformation complex of the graph cooperad Graphs_n^c as a chain complex in characteristic zero.

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Enriched categorical Koszul duality and Calabi-Yau structures

Julian Holstein
(joint work with Andrey Lazarev, Manuel Rivera)

In joint work with A. Lazarev [1] we showed there is a Koszul duality between differential graded (dg) categories and a class of curved coalgebras. This generalizes L. Positselski's non-homogeneous Koszul duality between dg algebras and conilpotent curved coalgebras [4], which itself was a generalization of the classical duality between augmented dg algebras and conilpotent dg coalgebras. We work over a ground field k throughout.

Categorical Koszul duality is given by generalized bar and cobar constructions and takes the form of a Quillen equivalence between suitable model categories of pointed curved coalgebras and dg categories:

$$\Omega : \mathsf{cuCoa}_*^{\mathsf{ptd}} \leftrightarrow \mathsf{dgCat}' : B$$

Here dgCat' is a (Quillen equivalent) modification of dgCat and we may consider the Dwyer-Kan model structure or the Morita model structure on dgCat' and obtain two different model structures on $\mathsf{cuCoa}_*^{\mathsf{ptd}}$. Koszul duality also induces an equivalence between the derived category of a dg category and the coderived category of its cobar construction.

One technical ingredient of independent interest is that we consider dg categories as dg monoids in bicomodules over the cosemisimple coalgebra spanned by the objects. This allows us to treat dg categories as one would treat algebras (and in particular perform a bar construction).

We show furthermore that the normalized chain complex functor transforms the well-known Quillen equivalence between quasi-categories and simplicial categories into our Koszul duality. Thus pointed curved coalgebras are a linearization of quasi-categories. This allows us to give a conceptual interpretation of the dg nerve of a dg category and its adjoint. As an application, we prove that the category of representations of a quasicategory K is equivalent to the coderived category of comodules over $C_*(K)$, the chain coalgebra of K. A corollary of this is a characterization of the category of constructible dg sheaves on a stratified space as the coderived category of a the chain coalgebra of exit paths.

Categorical Koszul duality is compatible with many further structures. In further work with A. Lazarev [2] we construct a monoidal model structure on the category of pointed curved coalgebras $\mathsf{cuCoa}_*^{\mathsf{ptd}}$ and show that the cobar functor to dgCat' is quasi-strong monoidal. We also show that dgCat' is a $\mathsf{cuCoa}_*^{\mathsf{ptd}}$-enriched model category. As a consequence, the homotopy category of dgCat' is closed monoidal and is equivalent as a closed monoidal category to the homotopy category of $\mathsf{cuCoa}_*^{\mathsf{ptd}}$. This remedies the well-known defect that the category of small dg categories dgCat, though it is monoidal, does not form a monoidal model category. In particular, this gives a conceptual construction of a derived internal hom in dgCat, recovering the dg category of A_∞-functors as proposed by Kontsevich

In work in progress with M. Rivera we show that categorical Koszul duality furthermore exchanges smooth and proper Calabi-Yau structures on dg categories and pointed curved coalgebras. (We have learnt that there is independent work in progress by M. Booth, J. Chuang and A. Lazarev that obtains very similar results.)

Concretely let A be a smooth dg category quasi-equivalent to ΩC for $C \in \mathsf{cuCoa}_*^{\mathsf{ptd}}$. Then a smooth n-Calabi-Yau structure on A is a class η in negative cyclic homology $\mathrm{HN}_n(A)$ which induces a weak equivalence $A^! \to A[n]$ of A-bimodules. (A class in $\mathrm{HH}_n(A)$ inducing such a weak equivalence is known as a weak n-Calabi-Yau structure.) Then A has an n-Calabi-Yau structure if and only if there is a classs in negative cocyclic homology in $\mathrm{coHN}_n(C) \cong \mathrm{HN}_n(A)$ that induces a weak equivalence $C^* \to C[n]$ of C-bicomodules. We call this a proper Calabi-Yau structur on C by analogy with the definition of a proper Calabi-Yau structure on a proper dg category.

Applying our result to the coalgebra of chains on an oriented n-manifold this recovers the n-Calabi-Yau structure on the dg algebra of chains on the loop space from Poincaré duality with coefficents (and vice-versa). Similarly Poincaré duality with coefficients for the a unimoduler n-dimensional Lie algebra $\mathfrak{g}$ gives rise to an n-Calabi-Yau structure on the universal envelopping algebra of a finite-dimensional $\mathfrak{g}$. As far as we are aware previously only a weak n-Calabi-Yau structure on $\mathcal{U}(\mathfrak{g})$ was known [3].

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Reporter: Annika Tarnowsky

Participants

Dr. Dimitri Ara
Institut de Mathématiques de Marseille
Université d'Aix-Marseille
13000 Marseille
FRANCE

Prof. Dr. Clemens Berger
Laboratoire J. A. Dieudonné
Université Côte d'Azur
Parc Valrose
06108 Nice Cedex
FRANCE

Prof. Dr. Julie Bergner
Department of Mathematics
University of Virginia
Kerchof Hall
141 Cabell Drive
Charlottesville, VA 22904-4137
UNITED STATES

Dr. Matt Booth
Department of Mathematics
University of Lancaster
Lancaster LA1 4YF
UNITED KINGDOM

Prof. Dr. Joseph Chuang
Department of Mathematics
City, University of London
Northampton Square
London EC1V 0HB
UNITED KINGDOM

Dr. Florian De Leger
Institute of Mathematics of the
Czech Academy of Sciences
Žitná 25
115 67 Praha
CZECH REPUBLIC

Prof. Dr. Vladimir Dotsenko
Institut de Recherche Mathématique
Avancée, Université de Strasbourg
7 rue René Descartes
67084 Strasbourg Cedex
FRANCE

Dr. Tobias Dyckerhoff
Department Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg
GERMANY

Coline Emprin
Ecole Normale Supérieure de Paris
45 Rue d'Ulm
75005 Paris
FRANCE

Sonja Farr
Department of Mathematics
University of Nevada
1664 N. Virginia Street
Reno, NV 89557
UNITED STATES

Prof. Dr. Benoit Fresse
Laboratoire Paul Painlevé
Université de Lille
UMR 8524
Bâtiment M2 003
59655 Villeneuve d'Ascq
FRANCE

Maroš Grego
Mathematical Institute
Charles University
Sokolovska 83
186 75 Praha 8
CZECH REPUBLIC

Dr. Philip Hackney
Department of Mathematics
University of Louisiana at Lafayette
217 Maxim Doucet Hall
P.O. Box 43568
Lafayette, LA 70504-3568
UNITED STATES

Prof. Dr. Rune Haugseng
NTNU Trondheim
7491 Trondheim
NORWAY

Dr. Simon Henry
Department of Mathematics & Statistics
University of Ottawa
585 King Edward Avenue
Ottawa ON K1N 6N5
CANADA

Dr. Julian Holstein
Fachbereich Mathematik
Universität Hamburg
Bundesstraße 55
20146 Hamburg
GERMANY

Prof. Dr. André Joyal
Department of Mathematics
University of Quebec/Montreal
C.P. 8888
Succ. Centre-Ville
Montréal QC H3C 3P8
CANADA

Prof. Dr. Ralph Kaufmann
Department of Mathematics
Purdue University
150 N. University Street
West Lafayette, IN 47907-2067
UNITED STATES

Prof. Dr. Bernhard Keller
U.F.R. de Mathématiques
Université Paris Cité
Bâtiment Sophie Germain, Case 7012
8, place Aurélie Nemours
P.O. Box Case 7012
75013 Paris Cedex 13
FRANCE

Prof. Dr. Andrey Lazarev
Department of Mathematics
University of Lancaster
Lancaster LA1 4YF
UNITED KINGDOM

Prof. Dr. Muriel Livernet
Institut de Mathématiques de Jussieu –
PRG
Université Paris Cité
Bâtiment Sophie Germain
8, place Aurélie Nemours
75013 Paris Cedex
FRANCE

Dr. Félix Loubaton
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Martin Markl
Institute of Mathematics of the
Czech Academy of Sciences
Žitná 25
115 67 Praha
CZECH REPUBLIC

Prof. Dr. Anibal Medina-Mardones
Western University
London N6A 3K7
CANADA

Dr. Lyne Moser
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg
GERMANY

Dr. Viktoriya Ozornova
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Simona Paoli
Dept. of Mathematical Sciences
University of Aberdeen
Fraser Noble Bldg.
Aberdeen AB24 3UE
UNITED KINGDOM

Prof. Dr. Kate Poirier
Department of Mathematics
New York City College of Technology
City University of New York
300 Jay Street
Brooklyn, NY 11201
UNITED STATES

Dr. Georgios Raptis
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

Dr. Nima Rasekh
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Julie Rasmusen
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Birgit Richter
Fachbereich Mathematik
Universität Hamburg
Bundesstraße 55
20146 Hamburg
GERMANY

Dr. Marcy Robertson
Dept. of Mathematics and Statistics
University of Melbourne
Melbourne VIC 3010
AUSTRALIA

Prof. Dr. Chris Rogers
Department of Mathematics
University of Nevada
1664 N. Virginia Street
Reno, NV 89557
UNITED STATES

Prof. Dr. Maria O. Ronco
Instituto de Matemática y Fisica
Universidad de Talca
Campus: Avda. Lircay s/n
Camino Lircay s/n
3460 000 Talca, Maule
CHILE

Prof. Dr. Claudia I. Scheimbauer
Department Mathematik
Technische Universität München
85748 Garching bei München
GERMANY

Prof. Dr. Christoph Schweigert
Universität Hamburg
Fachbereich Mathematik
Bereich Algebra und Zahlentheorie
Bundesstraße 55
20146 Hamburg
GERMANY

Dr. Yunhe Sheng
Department of Mathematics
Jilin University
No. 2699, Qianjin Street
Changchun 130012
CHINA

Annika Tarnowsky
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dominik Trnka
Dept. of Mathematics
Masaryk University
662 95 Brno
CZECH REPUBLIC

Prof. Dr. Sarah Whitehouse
University of Sheffield
School of Mathematics and Statistics
Hicks Building
Hounsfield Road
Sheffield S3 7RH
UNITED KINGDOM