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MATRIX-MFO Tandem Workshop: Invariants in Low-Dimensional Topology: Combinatorics, Geometry, and Computation

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22 September – 27 September 2024

Abstract. The MATRIX-MFO tandem workshop addressed several research questions in low-dimensional topology and related areas. The format of the workshop consisted primarily of discussion sessions focusing on computable invariants of colored spatial graphs, random knots, small clasp number knots, algebraic models for Poincaré duality complexes in dimension 4, and the topological volume of the three-torus.

Mathematics Subject Classification (2020): 57K10, 57K20, 57K30, 57K40.

Introduction by the Organizers

The MATRIX-MFO Tandem Workshop *Invariants and Structures in Low-Dimensional Topology* was organized by Stefan Friedl (Regensburg) and Paula Truöl (MPIM Bonn) on the MFO side and by Joan Licata (Canberra) and Stephan Tillmann (Sydney) on the MATRIX side.

This tandem workshop was modelled on a tandem workshop that took place in 2021. The workshop in September 2021 was rather special: On the positive side for many researchers in Germany it was the first in-person conference in 18 months. On the negative side, the Australian mathematicians were in lockdown. So the tandem workshop in 2021 was an unforgettable event, but not always for the best reasons.

The workshop held in September 2024 covered similar topics and used a similar format, but fortunately the circumstances were back to normal.

This time 24 mathematicians met at MFO and at the same time 20 mathematicians met at MATRIX. To accommodate the hybrid setting and the time difference, the workshop had an unusual format, consisting primarily of discussion sessions and with very few formal talks. Our first goal was to give young mathematicians the opportunity to introduce themselves and their work. The second goal was to encourage conversations among all the participants and to try to make the workshop as interactive as possible. In order to achieve our second goal, we wrote to all participants before the meeting, soliciting questions or topics to be discussed at the meeting. Each participant was also given the opportunity to indicate which problems they would be interested in discussing.

Among the many suggestions from the participants, we picked five problems which had enough traction at MFO as well as on the Australian side:

(1) Computable invariants of colored spatial graphs.
(2) Random knots.
(3) Small clasp number knots.
(4) Algebraic Models for Poincaré Duality Complexes in Dimension 4.
(5) Topological volume of the three-torus.

Each topic had two discussion leaders, one at MFO and one in Australia.

The setup resulted in a rather unusual schedule. Every day we had a common time between 9am and 11am German time. On Monday every participant got one minute to introduce themselves using prepared slides. Afterwards the discussion leaders gave short introductions to the five problems. Finally the various working groups met in different rooms at MFO and they were joined via zoom by the Australian mathematicians to get started. During the day, the groups at MFO had lots of time to work in their respective problem groups. In the evenings, just before dinner, all participants at MFO met again and each working group gave a short summary of what happened and often the summaries would contain ideas, suggestions and requests for the Australian side. These summaries were recorded on Zoom. The participants at MATRIX started their next working day and picked up the threads from the conversations at MFO.

We set up a common Google folder with a subfolder for each working group. These folders were used to share ideas, papers and pictures between the working groups on the two continents.

On Monday and Tuesday we also had a second common time between 1:30pm and 2:15pm. Each of these afternoon time slots was filled with short (5 minute) talks by young researchers. There were three such talks from MATRIX on Monday, and six such talks from MFO on Tuesday (four of them in the afternoon, two of them in the morning tandem time).

On the remaining days of the workshop, the morning tandem time had the same structure. We started with brief summaries from the working groups, and occasionally an indication of the plans for the coming day. The bulk of the shared time was left to the working groups to discuss and collaborate, with more detailed reports from the groups on either continent. Over the course of the week the problems being discussed also evolved – for example the spatial graphs group and the clasp number group met on Thursday for a joint discussion after having observed some similarities of their problems. Participants were encouraged to switch between groups during the week as they wished, but most remained with their original choices.

In summary we feel that the collaboration across time zones and continents was a great success. The technical equipment at MFO and MATRIX worked extremely well and the big screens and the excellent cameras made it easy to share ideas. The eight hour time difference turned out to be a blessing since it gave us enough time to engage with mathematicians across continents, but it also gave each location time to work on the problems separately and in smaller groups.

The extended abstracts are organized as follows. First we provide five summaries of what happened in the working groups. These are followed by the extended abstracts of the six short talks by Ferretti, Galvin, Merz, Santoro, Truöl and Wakelin.

The organizers of this workshop hope that interactive meetings, which are more focused on problem solving instead of a long list of talks, will become more common. The organizers also wish to thank the staff at MFO and MATRIX for making this meeting possible and for providing us with all the technical support which was required.

Acknowledgement: The MFO and the workshop organizers would like to thank the Simons Foundation for supporting Takahiro Kitayama in the "Simons Visiting Professors" program at the MFO.

MATRIX-MFO Tandem Workshop: Invariants in Low-Dimensional Topology: Combinatorics, Geometry, and Computation

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Abstracts

Invariants of spatial graphs

Vitalijs Brejevs, Zsuzsanna Dancso, Livio Ferretti, Stefan Friedl, Tobias Hirsch, Christopher Johnson, Takahiro Kitayama, Justin Lanier, Stephan Tillmann

Let G be an abstract graph, possibly with decorations. Consider an embedding $\phi : G \to S^3$. Its image Γ is called a spatial graph. A basic problem is to classify spatial graphs for a fixed G up to ambient isotopy. This should be viewed as a generalization of the classification of knots and links, and indeed the robust toolkit for analyzing knots and links can be leveraged effectively in studying more general spatial graphs.

A first invariant of a given spatial graph are its constituent knots and links. Any nonempty collection of disjoint cycles in G yields a knot or link in S^3 under the restriction of the embedding ϕ; and the set of all of these, possibly with decorations, is an ambient isotopy invariant, as observed by Kauffman [2].

This invariant is not sufficient to distinguish spatial graphs that have the same underlying abstract graph: Consider the spatial graphs with underlying spatial graph the theta graph, called θ-curves. Two of these, the trivial θ-curve and the Kinoshita θ-curve, are depicted in Figure 1. Although these θ-curves are not equivalent [3, 4, 5, 9], they each have as their collection of constituent knots a set of three unknots. In our collaboration, we have sought more powerful invariants of spatial graphs. We have worked with several algebraic approaches to invariants; we've considered Blanchfield pairings on Alexander modules and have also explored sets of representations of $\pi_1(S^3 \setminus \Gamma)$ into finite groups. We've also worked on extending the notion of constituent knots and links using an "unzipping" construction.

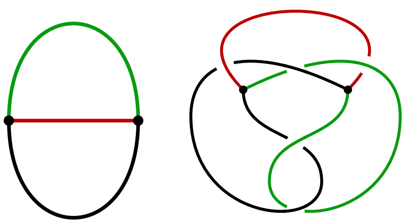

Figure 1. The trivial and Kinoshita θ-curves with edges coloured.

Context. A number of invariants of spatial graphs have been studied. Taking the case of θ-curves as an example, a list of prime θ-curves with up to 7 crossings was put forward by Litherland and later confirmed by Moriuchi [7, 8]. To distinguish θ-curves, these authors used algebraic invariants: Litherland [6] used the Alexander polynomial (first extended to spatial graphs by Kinoshita) while Moriuchi used the

Yamada polynomial. Later work by Heard, Hodgson, Martelli and Petronio [1] distinguished this same catalog by using geometric invariants of hyperbolic 3-orbifolds associated with spatial graphs.

Blanchfield pairing. Let $\Gamma \subset S^3$ be a spatial graph. We write $H := H_1(S^3 \setminus \Gamma)$ and $\Lambda := \mathbb{Z}[H]$. One of the most classical invariants of Γ is the Alexander module $H_1(S^3 \setminus \Gamma; \Lambda)$. As was pointed out in [6], it can also be interesting to consider a relative Alexander module $H_1(S^3 \setminus \Gamma, A; \Lambda)$ where $A \subset \partial(S^3 \setminus \Gamma)$ is a suitable subspace. This approach leads in particular to a suitable generalization of the classical Alexander polynomial of knots.

These Alexander modules can be endowed with extra structure as follows: using Poincaré duality one can define a hermitian form

$$\operatorname{Tor}_\Lambda H_1(S^3 \setminus \Gamma, A; \Lambda) \times \operatorname{Tor}_\Lambda H_1(S^3 \setminus \Gamma, A; \Lambda) \ \to\ \Omega/\Lambda$$

where Ω is the quotient field of $\mathbb{Z}[H]$. For knots the Blanchfield pairing is well-understood, and it turns out that for knots the Blanchfield pairing determines many classical invariants, e.g. the Levine-Tristram signatures.

We discussed how one could define signature invariants of spatial graphs using Blanchfield pairings. But a lot of work remains to be done.

Invariants from representations. The complement of a regular neighborhood of a spatial graph Γ is a compact 3–manifold M with non-empty boundary. The boundary ∂M has a natural boundary pattern $\gamma \subset \partial M$; this is a 1–manifold whose components form a complete set of meridians for the edges in Γ.

Denoting the oriented edges $e_1, \dots, e_m$ we can consider the $(m+1)$-tuple $(\pi_1(S^3 \setminus \Gamma), [\mu_1], \dots, [\mu_m])$ consisting of the fundamental group of the complement of Γ and the conjugacy classes given by the oriented meridians $\mu_1, \dots, \mu_m$ of the oriented edges.

Now given a group G and conjugacy classes $C_1, \dots, C_m \subset G$, one can study the set of representations of $\pi_1(S^3 \setminus \Gamma)$ into G that have the property that μ_i has image in C_i. For instance, a natural choice in [1] is $G = \mathrm{PSL}(2, \mathbb{C})$ and mapping all meridians to parabolics.

If G is finite, one can count the number of epimorphisms $\alpha\colon \pi_1(S^3 \setminus \Gamma) \to G$ with $\alpha([\mu_i]) \in C_i$, $i = 1, \dots, m$. An attractive choice for G is given by the group $G = \mathrm{SL}(2, \mathbb{F}_p)$ for some suitable prime p. For knots these invariants have turned out to be very powerful. We expect that also for spatial graphs these invariants will be very efficient at distinguishing spatial graphs.

The unzipped links invariant. We describe a method of obtaining links from a θ-curve that forgets less information about the graph than looking only at embedded cycles. The same method can also be used for arbitrary underlying graphs. The idea is to produce links by unzipping edges.

Let G denote the standard θ-graph with edges labelled 0,1,2, and let $\varphi : G \to S^3$ denote an embedding with image Γ. One obtains a link from Γ by unzipping (doubling) one of the edges (say, edge i), and connecting to the two adjacent edges at the ends to obtain a two-component link. For an unframed θ-curve, this operation is not well-defined, as the doubled edge supports an undetermined number of twists; however, it can be made well-defined by fixing the linking number of the two components. The linking number is determined by setting the orientations of the two copies of the doubled edge i to coincide, and letting this determine the orientations of the two resulting link components.

For $i = 0, 1, 2$, and $n \in \mathbb{Z}$, denote by $L_i^n(\Gamma)$ the unique two-component link obtained by unzipping edge i with linking number n. The components of L_i^n are labelled by the numbers $\{j, k\} = \{0, 1, 2\} \setminus \{i\}$. Denote by $L_{i,j}^n$ the component labelled by j. By definition, we have $\mathrm{lk}(L_{i,j}^n, L_{i,k}^n) = n$. Notation can be chosen such that the component $L_{i,j}^n$, as a knot, coincides with the constituent knot corresponding to the cycle containing edges i and j of G. In particular, $L_{i,j}^n$ viewed as a knot is the same knot independent of n.

In summary, to the θ-curve Γ we associate an infinite list of knots and links:

$$\Gamma \mapsto \mathcal{L}_\Gamma = (K_i, L_i^n)_{i \in \{0,1,2\}; n \in \mathbb{Z}}$$

Here K_i is the constituent knot obtained by deleting the image of the edge i, and L_i^n are the unzipped links.

Question 1. *Is $\mathcal{L}_\Gamma$ a complete invariant of Γ? If so, what is a minimal sub-list of $\mathcal{L}_\Gamma$ which uniquely determines the isotopy class of Γ?*

Initially we thought that it should be relatively straightforward to give a negative answer to the question. But all of our attempts failed and we now think that there is a good chance that the question might be answered in the affirmative.

Skein module relations. Let $\mathcal{K}$ denote the free $\mathbb{Q}[a, a^{-1}]$–module generated by links in S^3. Factoring out by the Conway skein relation $\times - \times = a \cdot)\,($ we obtain an extended version of the Alexander skein module—extended as we have not set the value of the unknot. Call this module $\mathcal{K}_\nabla$.

We observed an interesting relation between elements of the list $\mathcal{L}_\Gamma$ within $\mathcal{K}_\nabla$, as follows. For any n, the unzipped links L_i^n and L_i^{n+1} differ in a single crossing change. Furthermore, the knot obtained from smoothing this same crossing is isotopic – via some untwisting – to K_i, the knot obtained from Γ by deleting the edge i. Therefore, in $\mathcal{K}_\nabla$, we have equalities $L_i^n - L_i^{n-1} = a \cdot K_i$.

While this suggests, in spirit, that it is enough to record L_i^n for a given n (for example, $n = 0$), we haven't thus far obtained a proof of this.

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Models for random knots

Valentina Bais, Agnese Barbensi, Benjamin Burton, Daniele Celoria, Chun-Sheng Hsueh, Marc Kegel, Joan Licata, Alice Merz, Jonathan Spreer, Annika Thiele, Lucy Tobin

Several fundamental conjectures in low-dimensional topology are verified in many cases but have resisted rigorous proofs. Some examples are the cabling conjecture [1], the Berge conjecture [8], the cosmetic surgery conjecture [9], the L-space conjecture [7], and the slice-ribbon conjecture [16]. An alternative approach that could provide a weaker but still interesting answer is the study of the properties described by such open problems from a probabilistic point of view. More concretely, let P be a property of knots. What is the probability that a random knot K satisfies the property P? The answer will in general depend very heavily on the chosen model and, in particular, it will provide information on the fixed probability measure rather than on the set of knots itself.

Several random models for knots have been defined and developed in the literature. Well studied examples are the braid model and the crossing number model [10, 18, 19], and see [14] for a comprehensive overview over more models from a recent workshop on the topic. In our research proposal, the following two models are taken into account.

The grid model. Every knot K in S^3 can be represented in a *grid diagram*, that is a combinatorial encoding of a knot that behaves more friendly than usual projection diagrams (see *e.g.* monotonic simplification of grid diagrams of the unknot [13]). In particular, this approach is closely related to geometric and algebraic properties of the knot, due to its connections with contact topology [22] and knot Floer homology [23].

For a given integer $n \in \mathbb{N}$ there exist finitely many grid diagrams of size n, where each diagram represents a knot. A simple combinatorial exercise shows that this number is $G(n) = n!(n-1)!$. Among these finitely many diagrams we choose a random one and, by letting n converge to infinity, we get statements about generic knots.

In particular, we have performed computer experiments to study invariants related to the smooth sliceness status of a knot. We used the program GridPyM [3]

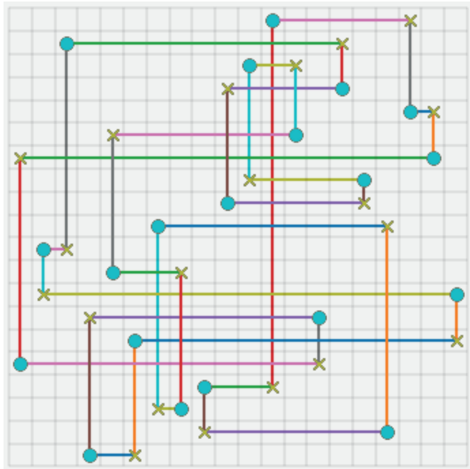

FIGURE 1. Example of a grid diagram

for analyzing and enumerating grid diagrams, the knot Floer homology calculator [24] and the algorithmic search for ribbon knots [17]. In detail:

(1) Let $\tau_0(n)$ be the number of grid diagrams of size n that represent a knot with vanishing τ-invariant. The τ-invariant is a powerful tool from knot Floer homology whose non-vanishing is an obstruction for a knot being slice. We have analyzed the ratio $\tau_0(n)/G(n)$, see Figure 2.
(2) Let $R(n)$ be the number of grid diagrams of size n representing a ribbon knot. Assuming the slice ribbon conjecture is true, certifiably non-ribbon knots can be assumed non-slice. Plotting $R(n)/G(n)$ in the same plot as $\tau_0(n)/G(n)$ leaves a portion of our sample as potentially, but not definitely slice. This portion seems to be a positive fraction of all samples. See Figure 2
(3) Similar computer experiments for the Thurston–Bennequin number (*tb*) suggest the surprising conjecture that the probability of a generic knot in the grid model having positive Thurston–Bennequin invariant is a positive constant, which we experimentally estimate to be $C \sim 0.018$. We plan to analyse this problem combinatorially in more detail and present an actual proof of the above conjecture for some value $C' > 0$.

The 1-vertex triangulation model. Let T be a triangulation of S^3 with a single vertex. Then any edge e in T represents a knot $K \subset S^3$. By drilling out the edge e we get an ideal triangulation of the knot complement $S^3 \setminus K$. Moreover, it is known that any knot in S^3 arises as an edge in a 1-vertex triangulation of the 3-sphere [15].

Since for any natural number t there exist finitely many 1-vertex triangulations of S^3 with t tetrahedra, this yields a random model of knots.

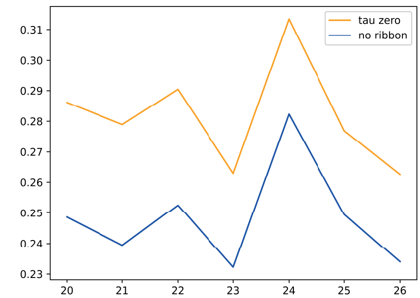

FIGURE 2. Fraction of grids with $\tau = 0$ (orange) and fraction of grids with $\tau = 0$ which are certified **not ribbon** (blue).

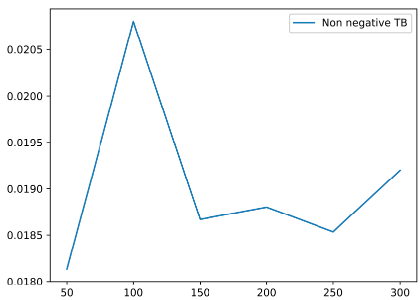

FIGURE 3. Fraction of grid diagrams with non-negative $tb(n)$ per grid size n.

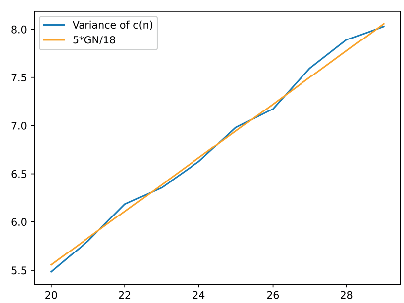

FIGURE 4. Variance of the distribution $c(n)$ of cusps per grid size n.

We performed explicit computer experiments to create random knots in this model, using the same data set of triangulations as used in [2]. In particular, we determined how many of these knots are isotopic to the unknot, and how many have hyperbolic complements. The results are displayed in Figures 5 and 6

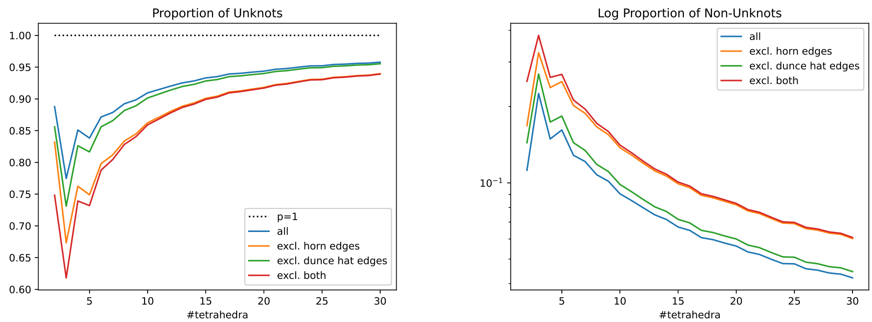

FIGURE 5. Proportion of unknots among sampled edges in S^3.

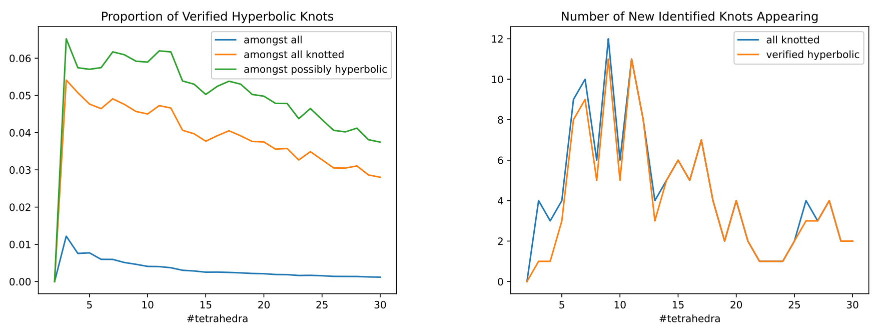

FIGURE 6. Left: Proportion of knots verified to be hyperbolic among sampled edges in S^3. Right: Number of identified knot types among sampled edges in S^3 with n tetrahedra which are not seen for any smaller n.

Since the minimal number of tetrahedra in an ideal triangulation of $S^3 \setminus K$ is at most $t + 2$ (coming from an explicit construction, implemented in *Regina* [20]), we expect (and observe) that many knots in this model appear in the SnapPy [21] census [12]. The knots in the SnapPy census are interesting since approximately half of them are hyperbolic L-space knots [11, 4, 5, 6]. We expect many hyperbolic knots that can be triangulated with 10 or 11 ideal tetrahedra are L-space knots as well.

In further work we plan to confirm this expectation in concrete experiments and check the L-space conjecture on manifolds obtained by small surgeries on these knots [11].

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Poincaré duality 4-complexes

Beatrice Bleile, Diarmuid Crowley, Daniel Galvin, Jayden Hammet, Naageswaran Manikandan, Mark Pencovitch, and Arunima Ray

Our group considered problems concerning 4-dimensional Poincaré duality complexes, i.e. 4-dimensional CW-complexes which are equipped with a top-dimensional homology class which induces Poincaré duality isomorphisms via the cap product. The work of Baues-Bleile [2] showed that 4-dimensional Poincaré complexes (from now, we use the abbreviation PD_4-complexes) are classified by so-called *fundamental triples* (T, w, b), given by the following data: a 2-coconnected CW complex T, an orientation character homomorphism $w\colon \pi_1(T) \to \mathbb{Z}/2$, and a specified homology class $b \in H_4(T;\mathbb{Z})$. For a given PD_4-complex X, we construct the 2-type T by taking the second stage in a Postnikov tower for X, i.e. the space T fits into a fibration sequence

$$K(A, 2) \to T \to K(\pi, 1)$$

where $\pi := \pi_1(X)$ and $A := \pi_2(X)$. In particular, T has the same first and second homotopy groups as X but all of its higher homotopy groups vanish. The space T is defined unqiuely up to homotopy equivalence by π, A and a cohomology class $k \in H^3(\pi; A)$, called the k-invariant, which determines the fibration and is defined as the obstuction to the fibration admitting a section. There is then a canonical inclusion map $c\colon X \to T$ and the class b is defined as $b := c_*[X] \in H_4(T;\mathbb{Z})$.

There is another invariant that is associated to a PD_4-complex: the equivariant intersection form $\lambda\colon H^2(X;\mathbb{Z}\pi) \times H^2(X;\mathbb{Z}\pi) \to \mathbb{Z}\pi$ which is defined using the cap product with the fundamental class $[X]$. In the literature, the pentuple (π, A, w, k, λ) is referred to as the *quadratic 2-type*—but we will refer to it as the *symmetric 2-type* as this more closely follows the terminology of algebraic surgery due to Ranicki—and is an important invariant which can be recovered from the fundamental triple. This is because there is a map

$$H_4(T; Z) \to \mathrm{Herm}(H^2(X;\mathbb{Z}\pi))$$

which recovers the equivariant intersection form from the image of the fundamental class.

We considered the following general problems.

(1) Given a fundamental triple (T, w, b), does there exist a PD_4-complex which realises it? Does there exist a manifold that realises it?
(2) Assuming that a fundamental triple (T, w, b) is realised by a PD_4-complex, for which $b' \in H_4(T;\mathbb{Z})$ is the triple (T, w, b) realised by a PD_4-complex? Which $b \in H_4(T;\mathbb{Z})$ are realised by manifolds?
(3) In general the symmetric 2-type does not determine the homotopy type (see below references). What data should be added to the symmetric 2-type to be able to recover the homotopy type in general? For which π does the symmetric 2-type suffice? Can this enhacement be formulated as a certain quadratic enhancement of the symmetric 2-type, to yield a "quadratic 2-type"?

Table 1. Literature Review.

Authors, and reference	π	w	Result
Hambleton-Kreck [3]	finite	0	$\mathrm{Tors}(\mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2)) = \mathrm{Ker}(\lambda)$
Hambleton-Kreck [3]	finite 4-periodic H^*, e.g. finite cyclic	0	Symm 2-type classifies PD_4
Bauer [1]	finite & Sylow 2-subgp has 4-periodic H^*	0	Symm 2-type classifies PD_4
Hambleton-Kreck-Teichner [13]	geometrically 2-dimensional	0	Symm 2-type does not classify 4-mflds
Kasprowski-Powell-Ruppik [10]	Sylow 2-subgp has ≤ 2 gen	0	Symm 2-type classifies PD_4
Kasprowski-Nicholson-Ruppik [9]	dihedral	0	Symm 2-type classifies PD_4
Kim-Kojima-Raymond [12] and Hambleton-Kreck-Teichner [8]	$\mathbb{Z}/2$	Id	Symm 2-type does not classify 4-mflds
Kasprowski-Teichner [11]	finite	any	$\mathrm{Tors}(\mathbb{Z} \otimes_{\mathbb{Z}\pi} \Gamma(\pi_2)) = \mathrm{Ker}(\lambda)$
Kasprowski-Teichner [11]	$\mathbb{Z}/2$	Id	Symm 2-type classifies top 4-mflds but not PD_4
Hillman [6]	free	any	λ classifies PD_4
Hillman-Kasprowski-Powell-Ray [14, 7]	3-manifold gps, whose finite subgps are cyclic	trivial on finite order elts	Symm 2-type classifies top 4-mflds.

We thought about two particularly special cases, the case of finite π, and the case where π is geometrically 2-dimensional. By geometrically n-dimensional, we mean that $K(\pi, 1)$ has an n-dimensional CW model. We now give a specific example for the second case.

Consider the manifold $X := S^1 \times S^1 \times S^2$. This corresponds to the triple (T, w, b) where $T \simeq S^1 \times S^1 \times \mathbb{CP}^\infty$, w is the zero map, and b is a certain class in $H_4(T;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. The equivariant intersection is the zero form. By the classification of manifolds with fundamental group $\mathbb{Z} \times \mathbb{Z}$ due to Hambleton-Kreck-Teichner [8], there is another manifold with the symmetric 2-type, the manifold $Y := (S^1 \times S^1)\tilde{\times} S^2$, the twisted S^2-bundle over $S^1 \times S^1$. The manifold X is not homotopy equivalent to Y, and this can be detected by the second Stiefel-Whitney class. It is then an interesting problem to try and detect this difference inside $H_4(T;\mathbb{Z})$ and to generalise this procedure to more manifolds with geometrically 2-dimensional fundamental groups. One could also try to find other elements in $H_4(T;\mathbb{Z})$ which are realised by PD_4-complexes. Since we know the manifold classification, such PD_4-complexes (if they exist) are necessarily not homotopy equivalent to any 4-manifold.

In Table 1 we summarise the various known results concerning classification of PD_4-complexes which are in the literature. The table should be reasonably self-explanatory.

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Obstructing ribbon clasp number one

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Ribbon clasp number. A knot $K \subset \mathbb{S}^3$ is null-homotopic and can therefore be seen to bound a smoothly immersed disc $D \looparrowright \mathbb{S}^3$ with transversal self-intersections. The singular set consists of intervals and circles of double points; in fact, we can arrange that only ribbon and clasp singularities occur. In particular, one can always eliminate Whitney umbrellas, circles of double points and triple points (possibly at the cost of increasing the number of clasp and/or ribbon singularities). Thus, the count of clasp intersections seems to be a natural measure of complexity for K.

Definition 1. The *ribbon clasp number* $c_r^*(K)$ of a knot $K \subset \mathbb{S}^3$ is defined as the minimal number of clasp intersections of an immersed disc $D \looparrowright \mathbb{S}^3$ with $\partial D = K$ that has only ribbon and clasp singularities.

If $c_r^*(K) = 0$, then K is a ribbon knot and the interior of any disc D without clasps can be pushed into the four-dimensional ball $\mathbb{B}^4$ to obtain a smoothly embedded slice disc for K. Similar but a priori different invariants can be found in the literature (cf. Shibuya [4]):

- the *three-dimensional clasp number* $c_3(K)$, defined as the minimal number of clasps among all immersed discs $D \looparrowright \mathbb{S}^3$ with only clasp intersections with boundary K;
- the *four-dimensional clasp number* $c_4(K)$, defined as the minimal number of (interior) transverse double points among all smoothly immersed discs $D \looparrowright \mathbb{B}^4$ with boundary K;
- the *smooth four-genus* $g_4(K)$, defined as the minimal genus among all smoothly embedded orientable surfaces in $\mathbb{B}^4$ with boundary K.

The following inequalities are well-known and relatively straightforward to show. For any knot K, we have

$$g_4(K) \leq c_4(K) \leq c_r^*(K) \leq c_3(K).$$

In this project, we studied whether any of these inequalities are strict for certain knots K. Note that a knot K with $g_4(K) = 0$ and $c_r^*(K) > 0$ would be a counterexample to the slice-ribbon conjecture. Instead of looking for such an example, we considered the problem of finding an obstruction to a knot K from having $c_r^*(K) = 1$, hoping to apply such an obstruction to knots of smooth four-genus one.

Our observations are summarised below. Note that stronger results have been proved by Owens–Strle [3] using related techniques. Unfortunately, we have not yet succeeded in applying our methods to examples, so it remains to be seen whether Proposition 2 and Corollary 3 can be used to settle open questions about the ribbon clasp number.

Double branched covers. Let K be a knot. Define the link $L = K \cup \mu$, where μ is a meridian for K. If K has $c_r^*(K) = 1$, then L bounds a ribbon annulus in $\mathbb{S}^3$ and thus an embedded annulus $A \subset \mathbb{B}^4$. Denote by $X_A := \Sigma_2(\mathbb{B}^4, A)$ the double branched cover of $\mathbb{B}^4$ over $A \subset \mathbb{B}^4$. Note that the intersection form of X_A is $Q_{X_A} = (m)$ for some non-zero even integer m. The boundary of X_A is the double branched cover $Y_L = \Sigma_2(\mathbb{S}^3, L)$. Using the long exact sequence of the pair (X_A, Y_L), we obtain the following.

Proposition 2. *Let K be a knot with ribbon clasp number $c_r^*(K) = 1$. Then*

$$\pm 2 \cdot \det(K) = k^2 \cdot \det(Q_{X_A})$$

for some odd integer k.

Proof. Consider the long exact sequence in homology of the pair (X_A, Y_L) with integer coefficients:

$$\begin{tikzcd}
\ldots \arrow[r] & H_2(Y_L) \arrow[r, "\phi"] & H_2(X_A) \arrow[r, "\psi"] & H_2(X_A, Y_L) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztotarget.north) -- (\tikztotarget)}] & \\
 & H_1(Y_L) \arrow[r] & H_1(X_A) \arrow[r] & H_1(X_A, Y_L) \arrow[r] & \ldots
\end{tikzcd}$$

This simplifies to

$$0 \to \mathbb{Z} \oplus T' \to \mathbb{Z} \oplus T \to G \to T \to T' \to 0$$

for some finite groups T, T' and $G = H_1(Y_L)$, which in turn gives

$$0 \to \mathbb{Z}/m\mathbb{Z} \oplus T/T' \to G \to T \to T' \to 0,$$

where $Q_{X_A} = (m)$ is the intersection form of X_A. By comparing the orders of these finite groups, we deduce that $2 \cdot |\det(K)| = |G| = k^2 \cdot m$, where $k \cdot |T'| = |T|$. Since $\det(K)$ is odd, k is also odd with m even. □

Lattice embeddings. Now let K be an alternating knot. Then the link L is also alternating and so L bounds both a positive definite and a negative definite spanning surface with respect to the Gordon-Litherland form. In particular, starting from a definite surface $F_\pm$ for K, we can construct a definite surface $F'_\pm$ for L by attaching a small 1-handle with ± 1 full twist to it. In this way, the Gordon-Litherland form of $F'_\pm$ can be written as

$$GL(F'_\pm) = \begin{pmatrix} GL(F_\pm) & 0 \\ 0 & \pm 2 \end{pmatrix}.$$

Now suppose that $Q_{X_A} = (m)$ with $m > 0$ (the case $m < 0$ is analogous). Let F'_- be a negative definite spanning surface for L and denote by $X_{F'_-} := \Sigma_2(\mathbb{B}^4, F'_-)$ the double branched cover of $\mathbb{B}^4$ over a copy of F'_- pushed into $\mathbb{B}^4$. Recall that the intersection form of $X_{F'_-}$ is isomorphic to the Gordon-Litherland form of F [2], hence X_F is negative definite.

Let $X := X_{F'_-} \cup \overline{X_A}$ be the closed 4-manifold built by gluing these double branched covers along their common boundary Y_L. Then X is a closed, orientable

4-manifold with negative intersection form. Therefore, by Donaldson's diagonalisation theorem [1], the intersection form of X must be isomorphic to the standard lattice $(\mathbb{Z}^n, -I_n)$. Since Y_L is a $\mathbb{Q}HS^3$, we have a lattice embedding

$$\left(H_2(X_{F'_-})/\operatorname{Tors}, Q_{X_{F'_-}}\right) \oplus \left(H_2(\overline{X_A})/\operatorname{Tors}, Q_{\overline{X_A}}\right) \hookrightarrow (\mathbb{Z}^n, -I_n).$$

We can summarise the above discussion in the following corollary.

Corollary 3. *Let K be an alternating knot with $c_r^*(K) = 1$. Let F_+ and F_- be positive and negative spanning surfaces, respectively, for K. Then there exist some $k, m > 0$ such that $2 \cdot \det(K) = k^2 \cdot m$ and, for at least one choice of sign, the lattice*

$$\begin{pmatrix} GL(F_\pm) & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & \pm m \end{pmatrix}$$

embeds in the standard definite lattice of the same rank and signature, where $GL(\cdot)$ denotes the Gordon-Litherland form.

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Computing the topological volume of some three-manifolds

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Every compact orientable three-manifold M with boundary consisting of a (possibly empty) union of tori contains a knot K whose complement $M - K$ is a hyperbolic manifold [12, Corollary 6.3]. This fact prompted Bessières, Besson, Boileau, Maillot, and Porti to consider the set of all possible volumes of hyperbolic manifolds that are obtainable from M by removing a (possibly empty) link [3]. Recall also that the set of hyperbolic volumes $\operatorname{vol}(M)$ of complete hyperbolic 3-manifolds M with finite volume is a well-ordered set by work of Jørgensen and Thurston [13]. Thus, it is natural (as in [3]) to define the following invariant of 3-manifolds:

Definition 1. The **topological volume** of a compact orientable three-manifold with empty or toroidal boundary is

$$\operatorname{volt}(M) = \min\{\operatorname{vol}(M - L) \mid L \text{ is a link in } M \text{ such that } M - L \text{ is hyperbolic}\}.$$

Recent work of Kegel, Ray, Spreer, Thompson, and Tillmann gave general upper and lower bounds for the topological volume and determined the complements of volt-realising links for all but finitely many lens spaces [10].

In this working group we tried to get a better understanding of the topological volume of some manifolds that were not already covered by the work of [3]. In particular, we focused on product three-manifolds of the type $F \times \mathbb{S}^1$, where F is a compact orientable surface.

1. The solid torus (F is a disk)

The easiest case to consider is $D^2 \times \mathbb{S}^1$. In fact, a theorem of Agol states that the Whitehead link complement and the $(-2, 3, 8)$-pretzel link complement are the minimal volume orientable hyperbolic three-manifolds with two cusps [1]. Since hyperbolic volume decreases under Dehn filling, and since there exists a knot K in $D^2 \times \mathbb{S}^1$ whose complement is diffeomorphic to the Whitehead link complement, one deduces the following:

Proposition 1. The topological volume of $D^2 \times \mathbb{S}^1$ is given by

$$\mathrm{volt}(\mathbb{S}^1 \times D^2) = \mathrm{vol}(\mathbb{W}) = \mathrm{v}_8 = 3.66...$$

where $\mathbb{W}$ denotes the Whitehead link complement and v_8 is the volume of a regular ideal octahedron in $\mathbb{H}^3$.

2. The thickened torus (F is an annulus)

One can see the thickened torus $\mathbb{S}^1 \times I \times \mathbb{S}^1$ as the exterior of the Hopf link in $\mathbb{S}^3$, and the three-component link `L6a5` admits the Hopf link as a sublink, thus

$$\mathrm{volt}(\mathbb{S}^1 \times I \times \mathbb{S}^1) \leq \mathrm{vol}(\mathbb{S}^3 - \texttt{L6a5}) = 5.33...$$

The complement of `L6a5` (also called the *Magic Manifold*) is actually conjectured to have minimal volume among hyperbolic 3-manifolds with three cusps [1]. Hence, if this conjecture is true, then the previous upper bound for $\mathrm{volt}(\mathbb{S}^1 \times I \times \mathbb{S}^1)$ would become an equality.

3. When F is a pair of pants

With the same reasoning as before, for F the pair of pants (or three-times punctured sphere), $F \times \mathbb{S}^1$ is the exterior of the connected sum of two Hopf links, and such a connected sum is a sublink of the 4-component link 8^4_2. Since Yoshida proved that 8^4_2 (which is a *minimally twisted chain link* like `L6a5`) has minimal volume among hyperbolic 3-manifolds with four cusps [14], we deduce:

Proposition 2. When F is a pair of pants, we have

$$\mathrm{volt}(F \times \mathbb{S}^1) = \mathrm{vol}(\mathbb{S}^3 - 8^4_2) = 2\mathrm{v}_8 = 7.33...$$

4. When F is the sphere

We then considered the manifold $\mathbb{S}^2 \times \mathbb{S}^1$. We were able to find a hyperbolic knot K in it whose exterior is diffeomorphic to the 0-surgery on one component of the pretzel link $P(-2, 3, 8)$ (called `m043` in the SnapPy census [4]) in Figure 1. The volume of $\mathbb{S}^2 \times \mathbb{S}^1 - K$ provides an upper bound on the topological volume of $\mathbb{S}^2 \times \mathbb{S}^1$, and is $3.25\ldots$. Hence

$$\mathrm{volt}(\mathbb{S}^2 \times \mathbb{S}^1) \leq \mathrm{vol}(\mathtt{m043}) = \mathtt{3.25...} < \mathrm{v}_8$$

This, together with the aforementioned theorem by Agol [1], yields:

Corollary 1. The topological volume of $\mathbb{S}^2 \times \mathbb{S}^1$ is realised by a knot.

At this point, one could prove that $\mathrm{volt}(\mathbb{S}^2 \times \mathbb{S}^1) = \mathrm{vol}(\mathtt{m043}) = 3.25...$ by listing the finite number of hyperbolic 3-manifolds with one cusp and whose volume is between 3.07 and 3.26, and then checking for each one that none of its respective finite number of exceptional surgeries is $\mathbb{S}^2 \times \mathbb{S}^1$. We do not need to check when the volume is smaller than 3.07, thanks to work of Gabai, Haraway, Meyerhoff, Thurston and Yarmola [7, Theorem 1.5]. If such an example with smaller volume exists, it is not in Dunfield's census [6].

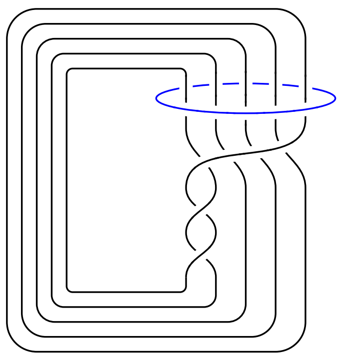

Figure 1. The pretzel link $P(-2, 3, 8)$. The SnapPy census manifold `m043` is obtained by 0-surgery on the unknot component (blue).

Towards a theoretical proof, we tried to exploit the fibration by 2-spheres of $\mathbb{S}^2 \times \mathbb{S}^1$ to study minimisers in $\mathbb{S}^2 \times \mathbb{S}^1$. In fact, an interesting feature of the knot K we found is that it can be isotoped to be transverse to this fibration (in other words, K is the closure of a braid in $\mathbb{S}^2 \times \mathbb{S}^1$). We showed that this is not an accident.

Theorem 2. Suppose that K' is a knot in $\mathbb{S}^2 \times \mathbb{S}^1$ that realises $\mathrm{volt}(\mathbb{S}^2 \times \mathbb{S}^1)$. Then K' can be isotoped to be transverse to the fibration by 2-spheres. Moreover, K' intersects each of these spheres in at least 5 points.

The key idea of the proof is to put K in thin position with respect to the fibration to find an essential punctured sphere in its exterior. We cut along this

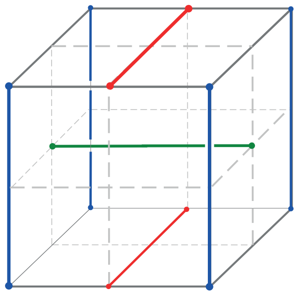

FIGURE 2. The candidate for the link L realising $\mathrm{volt}(\mathbb{T}^3)$ (from [9])

punctured sphere, and then by studying the *guts* of the resulting manifold [2] we argue that, if K' cannot be isotoped to be transverse to the fibration, then the volume of its exterior is at least $\mathrm{v}_8 = 3.66...$ and hence it is not a volume minimiser.

There might be some way to use the fact that the minimiser has to be a braid with at least 5 strands to prove that it is indeed `m043`.

5. The three-torus

We also studied the topological volume of the 3-torus $\mathbb{T}^3$. A starting point of this investigation was the familiar depiction of $\mathbb{T}^3$ as a 3-dimensional cube with opposite faces identified, and the 3-component link L which appears as a set of three disjoint straight line segments, each connecting two opposite faces, and orthogonal to them, as in Figure 2.

It is well-known that $\mathbb{T}^3 - L$ is homeomorphic to the hyperbolic manifold $\mathbb{S}^3 - B$, where B denotes the Borromean rings; for example see [9, Section 2.1]. More precisely, performing 0-framed surgery along B yields $\mathbb{T}^3$. We conjectured that this link L realises $\mathrm{volt}(\mathbb{T}^3)$, and that therefore the inequality

$$\mathrm{volt}(\mathbb{T}^3) \leq \mathrm{vol}(\mathbb{S}^3 - B) = 2\mathrm{v}_8$$

is actually an equality.

We attempted to disprove this conjecture by computing the hyperbolic volumes of various links in $\mathbb{T}^3$. This required representing the link in SnapPy first as a link with the union of the Borromean rings, and then performing $0/1$ Dehn filling on the link components corresponding to the Borromean rings.

Despite much experimentation, we could not find a counterexample to the conjecture. On the contrary, the following observation supports it:

Theorem 3. If L' is a hyperbolic 3-component link in $\mathbb{T}^3$ whose components are geodesics (for the Euclidean metric in $\mathbb{T}^3$), then

$$\mathrm{vol}(\mathbb{T}^3 - L') \geq \mathrm{vol}(\mathbb{T}^3 - L) = 2\mathrm{v}_8.$$

Such links are studied in [9, 8, 5]. In particular, Hui shows that these are hyperbolic if and only if the directions of the components γ_1, γ_2, γ_3 are linearly independent [8]. In that case, we consider the foliation by planes spanned by the directions of (say) γ_1 and γ_2. Then γ_1, γ_2 lie in different leaves F_1, F_2 of this foliation. After removing a small neighbourhood of L', the tori F_i each get cut into an annulus with k punctures F_i', where $k = |\alpha_3 \cap F_i|$. We cut the hyperbolic manifold $\mathbb{T}^3 - L'$ along $F_1' \cup F_2'$, resulting into two components. Each has guts with Euler characteristic at most -1. Thus, their volumes are each bounded below by v_8 due to a result of Agol, Storm, Thurston [2] and Miyamoto [11, Theorem 4.2].

A somewhat promising idea that might allow one to actually show that $\mathrm{volt}(\mathbb{T}^3) = 2\mathrm{v}_8$ would be to try and generalise the guts machinery we used for $\mathbb{S}^2 \times \mathbb{S}^1$ to the case of $\mathbb{T}^3$.

The results presented so far serve as the starting point of an investigation of topological volume for 3-manifolds that fiber over $\mathbb{S}^1$. A natural next example to consider would be the product $F \times \mathbb{S}^1$ with F a once-punctured torus. It would also be interesting to better understand the behaviour of hyperbolic volume under connected sums (although some partial results are known [10, Corollary 2.13 and Section 4.4]), and under gluing along tori.

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A combinatorial formula for the multivariable link signature

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(joint work with David Cimasoni, Jessica Liu)

The Levine-Tristram signature is among the most studied and best understood invariants of links in the 3-sphere. For a link $L \subset S^3$, it is given by a map $\sigma_L : S^1 \setminus \{1\} \to \mathbb{Z}$, $\omega \mapsto \sigma_L(\omega) = \mathrm{sign}(H(\omega))$, where $H(\omega)$ is a certain hermitian matrix constructed from the Seifert form and sign denotes the signature. Slightly less known is the fact that the Levine-Tristram signature admits a multivariable generalization to the setting of colored links, introduced in [3]. Let us recall that, given an integer $\mu > 0$, a *μ-colored link* is an oriented link $L \subset S^3$ each of whose components is endowed with a *color* in $\{1, \dots, \mu\}$ in such a way that all these colors are used. For a μ-colored link L we will use the notation $L = L_1 \cup \cdots \cup L_\mu$, where L_i is the sublink of L consisting of all the components of color i.

The definition of the multivariable signature uses a generalization of Seifert surfaces known as C-complexes.

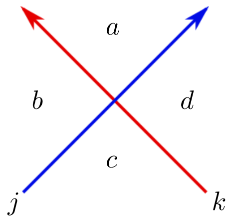
Definition 1. A *C-complex* for a μ-colored link $L = L_1 \cup \cdots \cup L_\mu$ is a union $S = S_1 \cup \cdots \cup S_\mu$ of surfaces embedded in S^3 satisfying the following conditions:

(1) for all i, the surface S_i is a (possibly disconnected) Seifert surface for L_i;
(2) for all $i \neq j$, the surfaces S_i and S_j are either disjoint or intersect in a finite number of *clasps*;
(3) for all i, j, k pairwise distinct, the intersection $S_i \cap S_j \cap S_k$ is empty.

Now, given a C-complex S for a colored link L, choosing a normal direction for every surface S_i allows one to push-off curves on S and define a *generalized Seifert form* (see [3] for a precise definition). In a nutshell, whatever Seifert matrices can do in one variable for oriented links, generalized Seifert matrices can do in μ variables for μ-colored links. For instance, one can construct a multivariable signature $\sigma_L : (S^1 \setminus \{1\})^\mu \to \mathbb{Z}$, defined as the signature of a certain Hermitian matrix given as a linear combination of all the generalized Seifert matrices (for all choices of normal directions).

Much like the Levine-Tristram signature, the multivariable signature is an invariant that has many alternative definitions and remarkable topological properties; for instance, it gives lower bounds on the genus of spanning surfaces in B^4 with boundary the link and is almost everywhere an invariant of topological concordance [4]. However, while the definition using Seifert forms is in principle algorithmic, for a μ-colored link one would need to compute $2^{\mu-1}$ a priori unrelated matrices, which quickly grows impractical for links with many colors. The goal of our work in [2] was to show how the multivariable signature can be defined and computed in a purely combinatorial way, as the signature of a single real symmetric matrix directly constructed from a colored link diagram. This goes as follows:

Definition 2. Given a μ-colored diagram D, let $\tau_D(x)$ be the symmetric matrix with rows and columns indexed by the regions of D and whose coefficients are



	a	b	c	d
a	x_{jk}	x_j	1	x_k
b	x_j	$2x_jx_k - x_{jk}$	x_k	1
c	1	x_k	x_{jk}	x_j
d	x_k	1	x_j	$2x_jx_k - x_{jk}$

FIGURE 1. A crossing v together with the corresponding 4×4 minor of $\tau_v(x)$. The incoming left strand is of color j, the incoming right strand of color k, and the four adjacent regions are a, b, c, and d.

functions of formal variables $x = \{x_j, x_{jk} \mid 1 \leq j, k \leq \mu\}$ indexed by (unordered pairs of) colors, defined by

$$\tau_D(x) = \sum_v \frac{\operatorname{sgn}(v)}{\sqrt{1-x_j^2}\sqrt{1-x_k^2}} \tau_v(x)\,,$$

where the sum is over all crossings of D, the indices $j, k \in \{1, \dots, \mu\}$ are the (possibly identical) colors of the two strands crossing at v, and the only non-vanishing coefficients of the matrix $\tau_v(x)$ are given in Figure 1.

Also, we shall denote by $\widetilde{\tau}_D(x)$ the matrix obtained by removing the two rows and columns corresponding to two adjacent regions of D determined by a marked point on D.

Note that if the regions a, b, c, d around a crossing v are not all distinct, then one should add the corresponding rows and columns of $\tau_v(x)$. We then obtain the following formula.

Theorem 1 ([2])**.** *Let D be an arbitrary μ-colored diagram for a μ-colored link L. For any $\omega = (\omega_1, \dots, \omega_\mu) \in (S^1 \setminus \{1\})^\mu$, the signature of L is given by*

$$\sigma_L(\omega) = \tfrac{1}{2}(\operatorname{sign}(\widetilde{\tau}_D(\omega)) - w_{\mathrm{m}}(D))\,,$$

where $w_{\mathrm{m}}(D)$ is the sum of the signs of all monochromatic crossings of D, and $\tau_D(\omega)$ stands for the evaluation of $\tau_D(x)$ at

$$x_j = \operatorname{Re}(\omega_j^{1/2})\,, \quad x_{jk} = \operatorname{Re}(\omega_j^{1/2}\omega_k^{1/2})\,.$$

To be totally precise, we need to fix one square root of each coordinate $\omega_j \in S^1 \setminus \{1\}$ of ω: our choice is to take $\omega_j = e^{i\theta_j}$ with $\theta_j \in (0, 2\pi)$, and $\omega_j^{1/2} = e^{i\theta_j/2}$. In other words, $\omega_j^{1/2}$ denotes the unique square root such that $\operatorname{Im}(\omega_j^{1/2})$ lies in $(0, 1]$. In particular, we have $\sqrt{1-x_j^2} = \operatorname{Im}(\omega_j^{1/2})$. Note that $x_j^2 \neq 1$, so $\tau_D(\omega)$ is a well-defined symmetric *real* matrix.

Moreover, note that the evaluation of the formal variables satisfies $x_{jj} = 2x_j^2 - 1$ for all j. Therefore, if a crossing v is monochromatic, then the matrix $\tau_v(x)$ can be written in a simple form which only depends on the single variable x_j. In particular, if $\mu = 1$, then $\tau_D(x)$ depends on a single variable. This matrix was first introduced by Kashaev in [5]. Remarkably, Kashaev had discovered this matrix for completely different reasons, with motivations coming from quantum topology, and he only conjectured that this should give a way of computing the Levine-Tristram signature. The conjecture (in the one-variable case), was proved in [1] and [6], so our result can be seen as a multivariable generalization of Kashaev's conjecture.

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Stable smooth isotopy of surfaces in simply-connected 4-manifolds

Daniel Galvin

Let X and X' be homeomorphic, compact, orientable, smooth 4-manifolds. It is the result of Gompf [3] that X and X' are *stably diffeomorphic*, i.e. for some $k \geq 0$ there exists a diffeomorphism

$$X\#(\#_k S^2 \times S^2) \xrightarrow{\cong} X'\#(\#_k S^2 \times S^2).$$

We consider an analogous version for embedded surfaces in a simply-connected 4-manifold. In particular, we present the following theorem.

Theorem 1. *Let X be a compact, orientable, smooth 4-manifold and let Σ_1, $\Sigma_2 \subset X$ be a pair of smoothly, properly embedded surfaces which are topologically isotopic relative to their boundaries. Then Σ_1 and Σ_2 are externally stably smoothly isotopic, i.e. there exists $k \geq 0$ such that Σ_1 and Σ_2 become smoothly isotopic relative to their boundaries in $X\#(\#_k S^2 \times S^2)$, where we perform the connected-sums in the complement of $\Sigma_1 \cup \Sigma_2$.*

We now present a sketch proof of the theorem.

Proof of Theorem (sketch). We will consider the closed case for simplicity, i.e. X, Σ_1 and Σ_2 will all be closed. The topological isotopy between Σ_1 and Σ_2 provides us with a homeomorphism $\widehat{G}\colon X \to X$ which sends Σ_1 to Σ_2. It is a consequence of the uniqueness of normal bundles for codimension two embeddings in 4-manifolds [1, Theorem 9.3A] that we can isotope $\widehat{G}$ relative to Σ_1 to a homeomorphism which sends $\nu(\Sigma_1)$ to $\nu(\Sigma_2)$ via a smooth vector bundle isomorphism, so assume that $\widehat{G}$ already has this property. Restricting $\widehat{G}$ to the exteriors produces a homeomorphism

$$G\colon X \setminus \nu(\Sigma_1) \xrightarrow{\approx} X \setminus \nu(\Sigma_2),$$

i.e. a homeomorphism of the surface exteriors. The idea is now to try to modify G relative to the boundary to a diffeomorphism, allowing taking connected-sums with $S^2 \times S^2$. All we need to ensure is that the resulting stable diffeomorphism of X is smoothly isotopic to the identity (we are allowed to stabilise further to achieve this). A theorem of Quinn [4, Theorem 1.4] says that all this requires is that the resulting stable diffeomorphism induces the identity map on homology.

We return to modifying G. A result of Freedman-Quinn [1, Theorem 8.2] says that there is an obstruction to G being stably pseudo-isotopic to a diffeomorphism, i.e. pseudo-isotopic to a diffeomorphism after taking connected-sums with $S^2 \times S^2$. We use a theorem of the author [2, Theorem 3.2] that says we can modify this obstruction to be zero after taking a connected-sum with a single copy of $S^2 \times S^2$ by modifying the stabilised homeomorphism on a neighbourhood of a curve union the new $S^2 \times S^2$ summand. The result can be then be stably pseudo-isotoped to a diffeomorphism. Our modification that we made to kill the smoothing obstruction can be taken to not change the induced map on homology, and hence the resulting stable diffeomorphism of X is stably isotopic to the identity. It follows that Σ_1 and Σ_2 are externally stably smoothly isotopic. $\square$

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Taut foliations and left orderability

Alice Merz

For a closed, oriented 3-manifold M, a (codimension 1) foliation is, roughly speaking, a partition of M into injectively immersed surfaces, called *leaves*, such that locally it is the partition of $\mathbb{R}^3$ in parallel planes. A foliation is said to be *taut* if every leaf intersects a closed curve transverse to the foliation. It is a well-known theorem, proved independently by Lickorish [7], Novikov [8] and Zieschang, that every 3-manifold admits a coorientable foliation. However, the foliations constructed with their procedure are in general not taut, and in fact not every 3-manifold admits a coorientable taut foliation (e.g. see [8], [10], [11]).

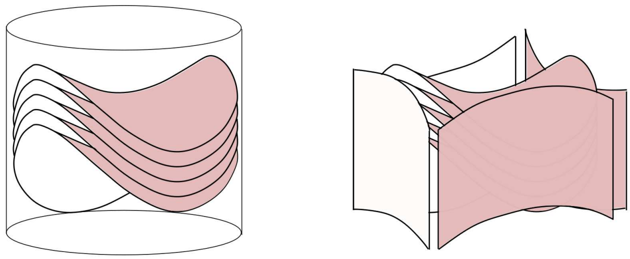

FIGURE 1. Foliating $D^2 \times \mathbb{R}$ by saddles.

In [9] the authors showed that Heegaard Floer homology gives obstructions to the existence of taut foliations. Namely they show that if M is an L-space, then M does not support a coorientable taut foliation, and they ask if the converse also holds.

In [1] the authors conjecture moreover that being an L-space is equivalent to the fundamental group of the manifold being *left-orderable*, meaning that it admits a total order that is invariant by left multiplication. This leads to the formulation of one of the most well-known conjectures in low dimensional topology, called the *L-space conjecture*, and, despite its audacity, it is now known to hold for graph manifolds and on a large class of hyperbolic rational homology spheres.

Conjecture 1 ([1],[6]). *Let M be an irreducible, rational homology 3-sphere. The following are equivalent:*

(1) M is not an L-space;
(2) M admits a coorientable taut foliation;
(3) $\pi_1(M)$ is left-orderable.

In my talk I mostly focused on properties (2) and (3), and more precisely on possible strategies to induce a left order on $\pi_1(M)$ given an explicit coorientable taut foliation $\mathcal{F}$ on M.

Let $\widetilde{M}$ denote the universal cover of M. The foliation $\mathcal{F}$ lifts to a foliation $\widetilde{\mathcal{F}}$ and one can consider the *leaf space* $\mathcal{L} = \widetilde{M}/\widetilde{\mathcal{F}}$. Notice that, since $\pi_1(M)$ acts on $\widetilde{M}$ preserving the foliation $\widetilde{\mathcal{F}}$, it acts on $\mathcal{L}$.

It is a well-known fact (see for example [3]) that if $\mathcal{F}$ is taut, $\mathcal{L}$ is a simply connected 1-manifold, that in general is non-Hausdorff. If the leaf space $\mathcal{L}$ happens to be Hausdorff, it is indeed equal to $\mathbb{R}$, and in this case we say that $\mathcal{F}$ is $\mathbb{R}$-*covered.* However not every taut foliation is $\mathbb{R}$-covered, as shown by the following example.

Example 2. *One can foliate $D^2 \times \mathbb{R}$ as follows: start by foliating its interior by saddles (see left side of Figure 1). To obtain a foliation of $D^2 \times \mathbb{R}$, one can add some limiting leaves, called walls, as in the right-hand side of Figure 1. The leaf space of this foliation is a real line (given by the leaves in the saddles) and some non-Hausdorff points given by the limiting leaves as depicted in Figure 2.*

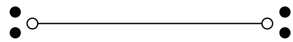

FIGURE 2. The leaf space of $D^2 \times \mathbb{R}$.

It is not too hard to show (see for example [4]) that a countable group G is left-orderable if and only if it acts effectively on $\mathbb{R}$ by orientation preserving homeomorphisms. Thus, if a taut foliation $\mathcal{F}$ on an irreducible 3-manifold is $\mathbb{R}$-covered, we get an action of $\pi_1(M)$ on the real line. In this case it is sufficient that the action is non-trivial (see [2]) to show that $\pi_1(M)$ is left-orderable. Unfortunately, most foliations are not $\mathbb{R}$-covered, but one can hope to improve this technique to some taut foliations that are not $\mathbb{R}$-covered, but such that the leaf space admits a quotient homeomorphic to $\mathbb{R}$, and the action of $\pi_1(M)$ descends to an action on this quotient. This technique was effectively used by Zung [13] to show that the fundamental group of some surgeries on mapping tori of pseudo-Anosov maps are indeed left-orderable, but it is in general not easy to apply since one needs to deeply understand the leaf space of the foliation.

Question 1. *Can we understand the leaf space of some "explicit" taut foliations that have been constructed in the literature such as, for example, the ones constructed in* [12]*, in order to apply this technique?*

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An overview on the L-space conjecture

Diego Santoro

A codimension-1 foliation of a closed 3-manifold M is *taut* if every leaf intersects a closed transversal, i.e. a smooth simple closed curve in M that is everywhere transverse to the foliation. Taut foliations have been a classical and fruitful tool in the study of 3-dimensional manifolds [27, 10]. A foliation is said co-orientable if the line bundle $TM/T\mathcal{F}$ is orientable, where $T\mathcal{F}$ is the plane bundle tangent to $\mathcal{F}$. When the ambient manifold M is orientable, this is equivalent to require that $T\mathcal{F}$ is an orientable plane bundle.

It is a classical theorem by Lickorish [18] and Novikov-Zieschang [20] that every closed orientable 3-manifold M supports a co-orientable (codimension-1) foliation. On the other hand, the existence of a co-orientable taut foliation puts constraints on the topology of M. For example, it is a consequence of the results of [20, 24, 23] that if a closed orientable 3-manifold $M \neq S^2 \times S^1$ contains a co-orientable taut foliation, then M has infinite fundamental group, is irreducible and its universal cover is diffeomorphic to $\mathbb{R}^3$. However, taut foliations are quite abundant: every irreducible closed orientable 3-manifold with positive first Betti number supports a co-orientable taut foliation [10]. A lot of work was required to prove the existence of hyperbolic 3-manifolds not supporting taut foliations. The first examples are due to Roberts-Shareshian-Stein [25].

Some years later, many other examples were found by using techniques coming from Heegaard Floer homology. In fact, Ozsváth-Szabó [21] proved that L-spaces do not admit co-orientable taut foliations. A rational homology sphere M is an *L-space* if it has minimal Heegaard Floer homology, i.e. if it satisfies $\mathrm{rank}\widehat{HF}(M) = |H_1(M,\mathbb{Z})|$. Their theorem was based, among the other things, on an approximation result for C^2-foliations by Eliashberg-Thurston [9], that has been later generalised by Bowden [1] and Kazez-Roberts [14] to less-regular foliations.

Taut foliations and L-spaces are two of the three concepts unified by a recent conjecture proposed by Boyer-Gordon-Watson [4] and Juhász [11]. More specifically, the conjecture is:

L-space conjecture. ([4, 11]) *For an irreducible oriented rational homology* 3-*sphere M, the following are equivalent:*

(1) M supports a co-oriented taut foliation;
(2) M is not an L-space;
(3) M is left-orderable, i.e. $\pi_1(M)$ is left-orderable.

In my talk I gave a partial overwiew of what is know on this conjecture. Perhaps surprisingly, the conjecture is now known to be true for large classes of 3-manifolds, such as graph manifolds [3, 12] and, as already observed, L-spaces do not support co-orientable taut foliations.

Moreover, in [8] Dunfield tested the conjecture on a census of more than $300,000$ hyperbolic rational homology spheres and proved it for more than 60% of these manifolds.

The existence of a co-orientable taut foliation on a rational homology sphere M can be used to produce interesting actions of $\pi_1(M)$ on 1-dimensional manifolds, both on Thurston's universal circle [7] and on the leaf space of the pullback foliation on the universal cover $\widetilde{M}$ of M (this is a simply connected, not necessarily Hausdorff, 1-dimensional manifold). This is particularly interesting in light of a result of Boyer-Rolfsen-Wiest [6] that characterises left-orderability of $\pi_1(M)$ in terms of actions on the real line and in fact, in some cases, both these actions have been succesfully used to prove left-orderability [5, 13, 29, 26].

How one should be able to obtain a taut foliation from a left-order is more misterious. Nonetheless, Li was recently able to do this when the ambient manifold M has Heegaard genus two [16]. On the same lines, left-orders were previously used by Zhao and by Baik-Hensel-Wu to construct respectively taut foliations on M minus a ball [28] and certain types of singular foliations on M [2].

While significant progress has been made, key questions remain open. For example, a concrete and easy to state, yet quite general, question is the following:

Question. Let K be a non-trivial knot in S^3 and suppose that K has no reducible surgeries. Let r be a rational number in $(1-2g, 2g-1)$, where g denotes the genus of K.

- Does the r-surgery on K support a co-orientable taut foliation?
- Is the r-surgery on K left-orderable?

Notice that it follows by [15, 22] that such a surgery on K is never an L-space, and therefore the L-space conjecture predicts an affirmative answer to both questions. The answer to the first question is known to be positive for small enough slopes [17] and recent works of Massoni imply that the set of slopes on K that are strongly realised[1] is open [19].

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Notions of braid positivity and knot concordance

Paula Truöl

This talk gives a brief overview of some of the speaker's past and present research interests.

Knots are closed, connected, oriented, smooth, 1-dimensional submanifolds of the 3-dimensional sphere S^3, which are usually studied up to *(ambient) isotopy.* A natural generalization in dimension 4 of the question whether certain knots are isotopic to the trivial knot, called *unknot*, is the notion of *concordance*, an equivalence relation on the set of all knots. Two knots K and J are called *concordant* if there exists an annulus $A \cong S^1 \times [0, 1]$ smoothly and properly embedded in

$S^3 \times [0,1]$ such that $\partial A = K \times \{0\} \cup J \times \{1\}$ and such that the induced orientation on the boundary of the annulus agrees with the orientation of K, but is the opposite one on J. Knots up to concordance form a group, the *concordance group* $\mathcal{C}$, with the group operation induced by connected sum.

An important open question in the area is the Slice–Ribbon Conjecture, which originated from a question posed by Fox in 1962, asking whether every *slice* knot is a *ribbon* knot [6]. *Slice* knots are those knots that are concordant to the unknot. Equivalently, they bound a smoothly embedded disk in the 4-ball B^4 bounded by S^3, while *ribbon* knots bound an immersed disk in 3-dimensional space with only ribbon singularities. The study of slice knots has been crucial to our understanding of the interactions between 3- and 4-dimensional spaces and is at the center of research in low-dimensional topology.

Isotopic knots are concordant, but the converse is generally not true, as any nontrivial slice knot shows. We are particularly interested in families of knots for which concordance implies isotopy. In [2], Baker showed that for any two strongly quasipositive, fibered knots K_0 and K_1, if $K_0 \# -K_1$ is ribbon (which in particular implies that K_0 and K_1 are concordant), then K_0 is isotopic to K_1. He conjectured the following.

Conjecture 1 (Baker's conjecture, [2]). *If two strongly quasipositive, fibered knots are concordant, then they are isotopic.*

A knot K is *fibered* if its complement in S^3 is the total space of a locally trivial fiber bundle where each fiber is the interior of a Seifert surface for K. A knot is *strongly quasipositive* if it is the closure of a strongly quasipositive braid $\beta \in B_n$ for some $n \geq 1$. Here B_n denotes the braid group on n strands which can be presented by $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| \geq 2 \quad \text{and} \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad [1].$$

See [3] for an overview on braids and their closures. An n-braid is *strongly quasipositive* if it is a product of certain conjugates of the positive Artin generators σ_i of B_n, namely of the *positive band words* $\sigma_{i,j}$, where

$$\sigma_{i,j} = (\sigma_i \cdots \sigma_{j-2})\, \sigma_{j-1}\, (\sigma_i \cdots \sigma_{j-2})^{-1} \quad \text{for } 1 \leq i < j \leq n; \quad \text{see Figure 1.}$$

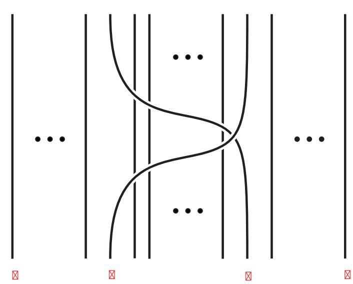

FIGURE 1. The positive band word $\sigma_{i,j}$.

Baker's result described above shows that the Slice–Ribbon Conjecture implies Conjecture 1. In other words, either concordance implies isotopy for the set of strongly quasipositive, fibered knots or the Slice–Ribbon Conjecture is false. In [12], I showed that Baker's conjecture is false in a strong sense if the requirement that the knots are fibered is dropped.

Theorem 2 ([12]). *Every non-trivial strongly quasipositive knot is smoothly concordant to infinitely many pairwise non-isotopic strongly quasipositive knots.*

As far as we know, the following question, which is a weaker version of Conjecture 1, remains open.

Question 3 ([11]). *Are there only finitely many strongly quasipositive, fibered knots in each smooth concordance class in $\mathcal{C}$?*

Braid positive knots are those knots which are closures of *positive braids*, i.e. those braids that can be represented in terms of the positive Artin generators $\sigma_1, \dots, \sigma_{n-1}$ of B_n for some $n \geq 1$. Braid positive knots are fibered [10] and strongly quasipositive, so a special case of Baker's conjecture is the following question.

Question 4. *Are concordant braid positive knots isotopic?*

Focusing on the special case of closures of positive 3-braids (*positive* 3-*braid knots*), in previous work I worked towards understanding the concordance classes of these in [13]. In this case, Question 4 seems to be particularly accessible due to classification results on the conjugacy classes of 3-braids [7, 8]; see also [13, Proposition 3.2]. As a corollary to our main theorem in [13], we provide the following step towards understanding the concordance classes of positive 3-braid knots. Here, $\upsilon(K) = \Upsilon_K(1)$ denotes a (smooth) concordance invariant from knot Floer homology defined by Ozsváth–Stipsicz–Szabó [9] and $g(K)$ denotes the *Seifert genus* of K.

Corollary 1. Let K be a knot that is the closure of a positive 3-braid. Then the minimal r such that K is the closure of $\sigma_1^{p_1}\sigma_2^{q_1}\sigma_1^{p_2}\sigma_2^{q_2}\cdots\sigma_1^{p_r}\sigma_2^{q_r}$ for integers $p_i, q_i \geq 1$, $i \in \{1, \dots, r\}$, is $r = g(K) + \upsilon(K) + 1$. Moreover, if K and J are concordant positive 3-braid knots, then this minimal r is the same for both K and J.

However, there are still many pairs of examples of positive 3-braid knots that we cannot distinguish in concordance with these methods, although we know them to be non-isotopic from the work of Birman–Menasco [4, 5], e.g. the closures of $\sigma_1^3\sigma_2^3\sigma_1^6\sigma_2^6$ and $\sigma_1^3\sigma_2^5\sigma_1^3\sigma_2^7$. In the future, we plan to compute more concordance invariants for positive 3-braid knots to decide whether these pairs differ in concordance. Note that a pair of concordant, but non-isotopic positive 3-braid knots would provide a counterexample to the Slice–Ribbon Conjecture.

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The Dehn surgery characterisation problem

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1. Characterising slopes

Let $\mathbb{S}^3_K(p/q)$ denote the 3-manifold obtained by performing Dehn surgery of slope $p/q \in \mathbb{Q} \cup \{1/0\}$ on a knot $K \subset \mathbb{S}^3$.

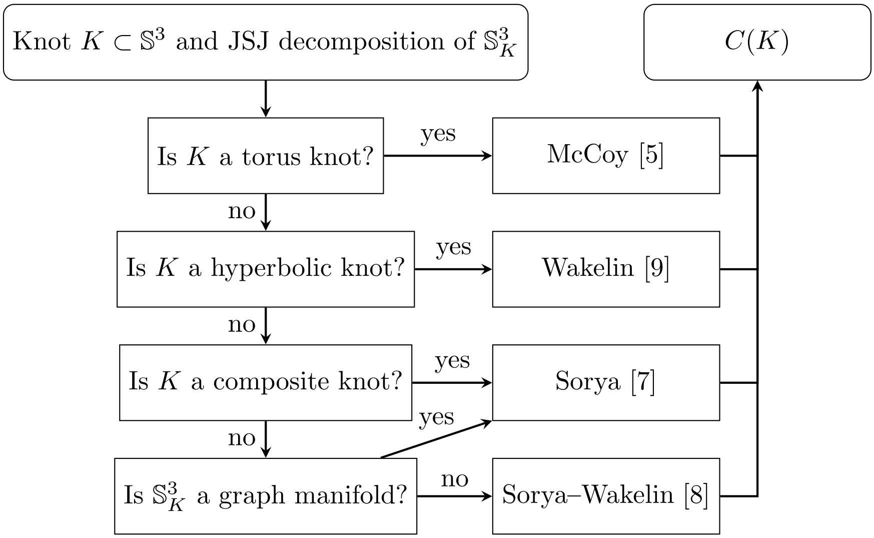
Definition 1. A slope $p/q \in \mathbb{Q}$ is called *characterising* for a knot $K \subset \mathbb{S}^3$ if the existence of a knot $K' \subset \mathbb{S}^3$ and an orientation-preserving homeomorphism $\mathbb{S}^3_K(p/q) \cong \mathbb{S}^3_{K'}(p/q)$ implies that $K = K'$.

Every slope is characterising for the unknot [3], as well as for the trefoils and the figure eight knot [6]. In general, for any given knot, every slope with sufficiently high denominator is characterising.

Theorem 2 (McCoy [5], Lackenby [4], Sorya [7])**.** *For any knot $K \subset \mathbb{S}^3$, there exists a constant $C(K)$ such that every slope $p/q \in \mathbb{Q}$ with $|q| \geq C(K)$ is a characterising slope for K.*

McCoy's proof of the existence of this bound for torus knots is constructive. However, neither Lackenby's proof for hyperbolic knots, nor Sorya's proof for satellite knots in the general case, gives a method for computing $C(K)$ explicitly.

Theorem 3 (McCoy [5], Wakelin [9], Sorya [7], Sorya–Wakelin [8])**.** *For any knot $K \subset \mathbb{S}^3$ whose exterior $\mathbb{S}^3_K$ has a known JSJ decomposition, a value for the constant $C(K)$ can be explicitly computed via the following algorithm.*



2. Non-characterising slopes

Despite the ubiquity of characterising slopes, there are also plenty of examples of non-characterising slopes.

Definition 4. A slope $p/q \in \mathbb{Q}$ is called *non-characterising* for a knot $K \subset \mathbb{S}^3$ if there exists another knot $K' \subset \mathbb{S}^3$ and an orientation-preserving homeomorphism $\mathbb{S}^3_K(p/q) \cong \mathbb{S}^3_{K'}(p/q)$ but $K \neq K'$.

The following result demonstrates that slopes with arbitrarily high denominator can be realised as non-characterising slopes for infinitely many pairs of knots.

Theorem 5 (Brakes [1], Wakelin [9])**.** *Any slope $1/q \in \mathbb{Q}$ can be realised as a non-characterising slope for a pair of multiclasped Whitehead doubles of double twist knots, $K = W^n(T^m_q)$ and $K' = W^m(T^n_q)$, for any non-zero integers $m \neq n$.*

Brakes' construction suggests that this method could also be extended to realise slopes $p/q \in \mathbb{Q}$ with $p \equiv 1 \mod q$ as non-characterising slopes. Joint work in progress proposes a new strategy to generalise this to all slopes.

Theorem 6 (Hayden–Piccirillo–Wakelin [2])**.** *Any slope $p/q \in \mathbb{Q}$ can be realised as a non-characterising slope for a pair of knots which can be constructed explicitly.*

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