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# Prescribing $Q$-curvature on even-dimensional manifolds with conical singularities

Aleks Jevnikar, Yannick Sire and Wen Yang

**Abstract.** On a $2m$-dimensional closed manifold, we investigate the existence of prescribed $Q$-curvature metrics with conical singularities. We present here a general existence and multiplicity result in the supercritical regime. To this end, we first carry out a blow-up analysis of a $2m$th-order PDE associated to the problem, and then apply a variational argument of min-max type. For $m > 1$, this seems to be the first existence result for supercritical conic manifolds different from the sphere.

## 1. Introduction

In conformal geometry, one of the most fundamental problems is understanding the relationship between conformally covariant operators, their associated conformal invariants, and the related PDEs.

As a first example, let us consider the Laplace–Beltrami operator in two dimensions on a closed surface $(M, g)$ and the Gaussian curvature. If we want to prescribe the curvature $K$ through a conformal change of metric $g_v = e^{2v} g$, we have the associated PDE

$$-\Delta_g v + K_g = K e^{2v}, \tag{1.1}$$

where $\Delta_g$ denotes the Laplace–Beltrami operator with respect to the background metric $g$ and $K_g$, $K = K_{g_v}$ are the Gaussian curvatures of the metric $g$ and $g_v$, respectively. Observe that the latter equation yields in particular the conformal invariance of the total Gaussian curvature which is then tight to the topology of the surface via the Gauss-Bonnet formula

$$\int_M K_g \, d\mathrm{vol}_g = \chi(M).$$

Here, $\chi(M)$ is the Euler characteristic of the surface.

A classical issue here is the prescribed Gaussian curvature problem or the Uniformization Theorem about the existence of a conformal metric in the conformal class of $g$ with prescribed (possibly constant) curvature. This amounts to solve the PDE in (1.1) which has been systematically studied since the works of Berger [9], Kazdan–Warner [31] and Chang–Yang [16, 17].

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In higher dimensions, we have the so-called GJMS operators $P_g^{2m}$ and the related $Q$-curvatures $Q_g^{2m}$, which are the higher-order analogues of the Laplace–Beltrami operator and the Gaussian curvature for $2m$-dimensional closed manifolds, see [26, 27]. These are conformally covariant differential operators whose leading term is $(-\Delta_g)^m$. In particular, when $m = 1$, we recover the Laplace–Beltrami operator and the Gaussian curvature. Moreover, for $m = 2$, $P_g^4$ and $Q_g^4$ are related to the Paneitz operator and the standard $Q$-curvature:

$$
\begin{aligned}
P_g^4 f &= P_g f = \Delta_g^2 f + \mathrm{div}_g\Big(\frac{2}{3} R_g g - 2\mathrm{Ric}_g\Big)\, df, \\
Q_g^4 &= 2Q_g = -\frac{1}{6}\,(\Delta_g R_g - R_g^2 + 3\,|\mathrm{Ric}_g|^2),
\end{aligned}
\tag{1.2}
$$

where $\mathrm{Ric}_g$ and $R_g$ stand for the Ricci tensor and the scalar curvature of the manifold $(M, g)$. See the original works of Paneitz [46, 47] and Branson [10] for more details.

The family of GJMS operators and the related $Q$-curvature functions play now an important role in modern differential geometry. As in the lower order case, if we want to prescribe the $Q$-curvature $Q_{g_v}^{2m} = Q$ through a conformal transformation $g_v = e^{2v} g$, then $P_g^{2m}$ and $Q_g^{2m}$ satisfy the following laws:

$$
P_{g_v}^{2m} = e^{-2mv}\, P_g^{2m} \quad \text{and} \quad P_g^{2m} v + Q_g^{2m} = Q\, e^{2mv}. \tag{1.3}
$$

The prescribed $Q$-curvature problem is in thus related to the solvability of (1.3).

One can attack this problem variationally by looking at the critical points of the associated energy functional. A lot of work has been done in this direction, in particular for the four-dimensional case and the Paneitz operator (1.2). In this setting, assuming

$$
P_g \geq 0 \quad \text{and} \quad \mathrm{Ker}\{P_g\} = \{\text{constants}\},
$$

the problem has been first solved by Chang–Yang [18] for

$$
\int_M Q_g^4 \, d\mathrm{vol}_g = 2 \int_M Q_g \, d\mathrm{vol}_g < 16\pi^2 = 2 \int_{\mathbb{S}^4} Q_{g_0} \, d\mathrm{vol}_{g_0}.
$$

Here, $g_0$ is the standard metric of the sphere. See also the related work of Gursky [28]. This is the so-called subcritical case, in which the energy functional is coercive and bounded from below by means of the Adams–Trudinger–Moser inequality [1], and solutions correspond to global minima using the direct methods of the calculus of variations. We refer to the discussion in the sequel for the precise definition of the subcritical, critical and supercritical case. The supercritical case $\int_M Q_g^4 \, d\mathrm{vol}_g > 16\pi^2$, where the energy functional fails to be bounded from below, has been considered by Djadli–Malchiodi [23] via a new min-max method based on improved versions of the Adams–Trudinger–Moser inequality [3, 19], solving the problem provided

$$
\mathrm{Ker}\{P_g\} = \{\text{constants}\} \quad \text{and} \quad \int_M Q_g^4 \, d\mathrm{vol}_g \notin 16\pi^2\mathbb{N}.
$$

Finally, some existence results for the critical case $\int_M Q_g^4 \, d\mathrm{vol}_g \in 16\pi^2\mathbb{N}$ have been derived by Ndiaye [45] by making use of the critical point theory at infinity jointly with a blow-up analysis.

As far as the higher-dimensional case $2m > 4$ is concerned, the subcritical case has been solved in [11] via a geometric flow, while the Djadli–Malchiodi's argument has been generalized by Ndiaye [44] to treat the supercritical case.

In this paper, we are interested in prescribing the $Q$-curvature on a general $2m$-dimensional closed manifold $M$ with conical singularities. Let $g$ be a smooth metric on $M$. We will say that a point $q \in M$ is a conical singularity of order $\alpha \in (-1, +\infty)$ for the new metric $g_v = e^{2v} g$ if

$$g_v(x) = f(x)\,|x|^{2\alpha}\,|dx|^2 \quad \text{locally around } q,$$

for some smooth function $f$. The set of conical singularities $q_j$ of orders $\alpha_j$ is encoded in the formal sum

$$D = \sum_{j=1}^{N} \alpha_j\, q_j\,,$$

while $(M, D)$ will denote the related conical manifold. We define

$$\kappa_g = \int_M Q_g^{2m}\, d\mathrm{vol}_g \quad \text{and} \quad \kappa_{g_v} = \int_M Q_{g_v}^{2m}\, d\mathrm{vol}_{g_v},$$

for which the following relation holds:

$$\kappa_{g_v} = \kappa_g + \frac{\Lambda_m}{2} \sum_{j=1}^{N} \alpha_j\,, \tag{1.4}$$

where $\Lambda_m = (2m-1)!\,|\mathbb{S}^{2m}|$, see for example Theorem 2.3. The critical threshold of a singular manifold is essentially related to the singular Adams–Trudinger–Moser inequality stated in Theorem 2.4. In the spirit of Troyanov [49], we let

$$\tau(M, D) = \Lambda_m \Big(1 + \min_j \{\alpha_j, 0\}\Big)$$

and give the following classification.

**Definition 1.1.** *The singular manifold* $(M, D)$ *is said to be*:

$$\begin{array}{lll} \textit{subcritical} & \textit{if} & \kappa_{g_v} < \tau(M, D), \\ \textit{critical} & \textit{if} & \kappa_{g_v} = \tau(M, D), \\ \textit{supercritical} & \textit{if} & \kappa_{g_v} > \tau(M, D). \end{array}$$

See also the recent work of Fang–Ma [25] for a similar discussion. We point out we have a slightly different notation for $\Lambda_m$ with respect to that paper.

Due to the singular behavior of the conformal factor $v$ around a conical point, prescribing the $Q$-curvature on a manifold with conical singularities at $q_j \in M$ of order $\alpha_j \in (-1, +\infty)$ is related to the solvability of the following singular PDE:

$$P_g^{2m} v + Q_g^{2m} = Q_{g_v}^{2m}\, e^{2mv} - \frac{\Lambda_m}{2} \sum_{j=1}^{N} \alpha_j\, \delta_{q_j}\,, \tag{1.5}$$

where $\delta_{q_j}$ stands for the Dirac measure located at the point $q_j \in M$.

One may desingularize the behavior of $v$ around the conical points by considering

$$u = v + \frac{\Lambda_m}{2} \sum_{j=1}^{N} \alpha_j G(x, q_j),$$

where $G(x, p)$ is the Green function of $P_g^{2m}$, see for example Lemma 2.2. Then $u$ satisfies

$$P_g^{2m} u + Q_g^{2m} + \frac{\Lambda_m}{2|M|} \sum_{j=1}^{N} \alpha_j = \tilde{Q} e^{2mu}, \tag{1.6}$$

where

$$\tilde{Q} = Q_{g_v}^{2m} e^{-m\Lambda_m \sum_{j=1}^{N} \alpha_j G(x,q_j)}, \tag{1.7}$$

which is now singular at the points $q_j$.

The singular equation (1.6) has been studied mainly in the two-dimensional case, that is, in relation to the prescribed Gaussian curvature problem. After the initial work of Troyanov [49], there have been contributions by many authors, as for example [19–21, 34, 40]. This problem has received a lot of attention also in recent years, see [4–6, 15, 36]. See also [24, 39, 41, 42] for further developments in this direction.

In the higher-dimensional case $m > 1$, there are very few results available. The subcritical regime has been just recently solved by Fang–Ma in [25], where the four-dimensional case is considered. The authors point out their method could be applied for higher dimensions too. In any case, the existence here follows by direct methods of the calculus of variations once the singular Adams–Trudinger–Moser inequality in Theorem 2.4 is derived. See also [29] for a related result on the sphere via a fixed point argument. For a blow-up analysis in dimension four, we refer instead to [2]. Concerning the existence problem in the supercritical case, the only result we are aware of is [30], where the authors consider a slightly supercritical problem on the sphere, again with a fixed point argument in the spirit of [29].

The goal of this paper is to give a first general existence result for $2m$-dimensional conic manifolds in the supercritical regime. We define a critical set of values $\Gamma$ as follows:

$$\Gamma = \Big\{ n\Lambda_m + \Lambda_m \sum_{i \in J} (1 + \alpha_i) \mid n \in \mathbb{N} \cup \{0\} \text{ and } J \subset \{1, \ldots, N\} \Big\}. \tag{1.8}$$

Observe that if $\alpha_j \in \mathbb{N}$ for all $j$, then we simply have $\Gamma = \Lambda_m \mathbb{N}$. Recall now the definition of the total curvature $\kappa_g$ given before (1.4). Let $M^{\mathrm{R}} \subset M$ be a closed $n$-dimensional submanifold, $n \in [1, 2m)$, such that the singular points $q_j \notin M^{\mathrm{R}}$ for all $j = 1, \ldots, N$. Then, we have the following.

**Theorem 1.1.** *Let $(M, D)$ be a supercritical singular $2m$-dimensional closed manifold with $\alpha_j > 0$ for $j = 1, \ldots, N$, and let $Q$ be a smooth positive function on $M$. Suppose that there exists a retraction $R\colon M \to M^{\mathrm{R}}$, with $M^{\mathrm{R}} \subset M$ as above. If, moreover,*

$$Ker\{P_g^{2m}\} = \{\text{constants}\} \quad \textit{and} \quad \kappa_g + \frac{\Lambda_m}{2} \sum_{j=1}^{N} \alpha_j \notin \Gamma,$$

*then there exists a conformal metric on $(M, D)$ with $Q^{2m}$-curvature equal to $Q$.*

**Remark 1.1.** We point out that a retraction $R\colon M \to M^{\mathrm{R}}$ as above exists for a wide class of manifolds. For example, we can consider manifolds of the type $M^n \times M^{2m-n}$, where we denote by $M^l$ any $l$-dimensional closed manifold. Indeed, it is easy to see that we can define a retraction $R\colon M^n \times M^{2m-n} \to M^n \times \{p\}$ for some $p \in M^{2m-n}$ with the desired properties. Observe that the torus $\mathbb{T}^{2m}$ belongs to this class of manifolds. One could also consider the connected sum $(M^n \times M^{2m-n})\# N^{2m}$, modifying the above retraction so that it is constant on $N^{2m}$.

We can also deduce the following multiplicity result. Here, $M_k^{\mathrm{R}}$ are the formal barycenters of $M^{\mathrm{R}}$ according to (4.3) and $\widetilde{H}_q(M_k^{\mathrm{R}})$ denotes its reduced $q$-th homology group.

**Theorem 1.2.** *Under the assumptions of Theorem 1.1, let $\kappa_{g_v} \in (k\Lambda_m, (k+1)\Lambda_m)$. Then, if $\mathcal{E}$ in (4.1) is a Morse functional,*

$$\#\{\textit{solutions of } (1.6)\} \geq \sum_{q \geq 0} \dim \widetilde{H}_q(M_k^{\mathrm{R}}).$$

**Remark 1.2.** Consider for example the class of manifolds $M^n \times M^{2m-n}$ in Remark 1.1. We will get an explicit lower bound on the number of solutions as far as we can explicitly estimate the homology groups of $M_k^n$. One can find such computations in [22] for general manifolds $M^n$, focusing on the cases $n = 2$ and $n = 4$. For some simple manifolds, we can easily compute the homology groups. For example, if $M^n$ is a 2-dimensional $G$-torus (connected sum of $G$ tori), then we have at least $\frac{(N+G-1)!}{N!(G-1)!}$ solutions, see [4].

The argument of the proof of the existence result is in the spirit of the celebrated min-max scheme of [23], extended to high dimensions by [44], jointly with some ideas of [4] to treat the singularities. Roughly speaking, the strategy is based on the study of the sublevels of the energy functional, in particular, by showing the low sublevels are non-contractible. This is done by using improved versions of the singular Adams–Trudinger–Moser inequality. We will then overcome the complexity due to the singularities by retracting the manifold onto $M^{\mathrm{R}}$, not containing the singular points. This leads us to study the low sublevels just by looking at functions concentrating on such submanifold, which is enough to gain some non-trivial homology. We refer the interested readers to [8] and [7] for a similar approach applied to surfaces with boundary and Toda systems, respectively.

To conclude the min-max argument, we would need some compactness property as the Palais–Smale conditions are not available in this setting. We thus use Struwe's monotonicity trick [48], which is by now a standard tool in this class of problems, to deduce the existence of a sequence of solutions $u_k$ satisfying (1.6). We will then conclude by showing the following compactness result, which actually holds for any $2m$-dimensional manifold and $\alpha_j > -1$.

**Theorem 1.3.** *Let $u_k$ be a sequence of solutions of (1.6) with $\widetilde{Q} > 0$ and $\alpha_j > -1$ for $j = 1, \ldots, N$. If*

$$Ker\{P_g^{2m}\} = \{\text{constants}\} \quad \textit{and} \quad \kappa_{g_v} \notin \Gamma,$$

*then there exists a constant $C$, independent of $k$, such that*

$$\|u_k\|_{L^\infty(M)} \leq C.$$

This result is a consequence of a quantization phenomenon of blowing-up solutions which is derived via Pohozaev-type inequalities in the spirit of [2, 6, 33].

The above analysis, together with Morse inequalities, allows us to deduce also the multiplicity result of Theorem 1.2.

**Remark 1.3.** We conclude the introduction with the following observations.

(1) The existence result is derived for the case $\alpha_j > 0$ for all $j$. In principle, the same strategy can be carried out for the case $\alpha_j \in (-1, 0)$. However, in this scenario we get a worse Adams–Trudinger–Moser inequality in Theorem 2.4, and this in turn affects the topology of the low sublevels in a non-trivial way, see for instance [15]. We postpone this study to a future paper.

(2) The same analysis should work in the odd-dimensional case with some further technical difficulties, as explained in Section 5 of [44]. We will not discuss this case in the present paper.

This paper is organized as follows. In Section 2, we collect some useful preliminary results. Section 3 is devoted to blow-up analysis and the proof of Theorem 1.3, and in Section 4, we carry out the min-max method to derive the existence and multiplicity results of Theorems 1.1 and 1.2. A Pohozaev-type identity is provided in Appendix A.

**Notations.**

(1) $B_r^M(p)$ is the ball centered at $p \in M$ with geodesic radius $r$ on the manifold $M$.

(2) $B_r(p)$ is the ball centered at $p$ with radius $r$ in $\mathbb{R}^{2m}$.

## 2. Preliminary facts

In this section, we recall briefly some known results which can be easily derived from the existing literature. Let $p$ be a point in $M$ and let $B_r^M(p)$ be the geodesic normal ball such that $B_r^M(p)$ is mapped by $\exp_p^{-1}$ diffeomorphically onto a neighborhood of $0 \in T_p(M)$, where $T_p(M)$ refers to the tangent space of $p$, which can be identified with $\mathbb{R}^{2m}$. The local coordinates defined by the chart $(\exp_p^{-1}, B_r^M(p))$ are called normal coordinates with center $p$. In such coordinates, the Riemannian metric at the point $p$ satisfies

$$g_{ij} = \delta_{ij}, \quad g_{ij,k} = 0, \quad \Gamma_{jk}^i = 0, \quad \text{for all } i, j, k \in \{1, \dots, 2m\}, \tag{2.1}$$

where $\Gamma_{jk}^i$ stands for the Christoffel symbols. With the above preparation, we have:

**Lemma 2.1.** *Let $-\Delta_g$ be the Laplace–Beltrami operator and let $p$ be any point of $M$. In normal coordinates at $p$, we have*

$$(-\Delta_g)^m u = (-\Delta)^m u + \mathcal{D}^{2m} u + \mathcal{D}^{2m-1} u, \tag{2.2}$$

*where $\mathcal{D}^{2m}$ is a linear differential operator, of order $2m$, and whose coefficients are $O(|x-p|^2)$ as $x$ tends to $p$, while $\mathcal{D}^{2m-1}$ is a linear differential operator of order at most $2m-1$, and whose coefficients belong to $C^l_{\text{loc}}(\mathbb{R}^{2m})$ for all $l \geq 0$.*

*Proof.* By the definition of Laplace–Beltrami operator, we have

$$-\Delta_g u = -\frac{1}{\sqrt{\det g}}\,\partial_i(\sqrt{\det g}\, g^{ij}\,\partial_j u), \tag{2.3}$$

where $g^{ij}$ is the inverse of $g_{ij}$. Using (2.1), we can write

$$-\Delta_g u = -g^{ij}\,\partial_{ij}u - \frac{\partial_i(\sqrt{\det g}\, g^{ij})}{\sqrt{\det g}}\,\partial_j u. = -g^{ij}\,\partial_{ij}u - \vartheta_j\,\partial_j u, \tag{2.4}$$

where $\vartheta_j$ is a smooth function. Based on (2.4), it is easy to see that in the final expression of $(-\Delta)^m_g$, the leading differential order is

$$g^{i_1 j_1}\, g^{i_2 j_2} \cdots g^{i_m j_m}\,\partial_{i_1 j_1 i_2 j_2 \cdots i_m j_m}\cdot,$$

where $i_a, j_b \in \{1, \dots, 2m\}$, $\forall a, b \in \{1, \dots, m\}$. While the remaining terms are order at most $2m-1$ and the coefficients are smooth, because the exponential map is differentiable with arbitrary order. Consider the leading term; using (2.1), we see that

$$g^{ij}(x) = \delta_{ij}(x) + O(r^2), \quad \text{if } x \in B^M_r(p).$$

Therefore we can write

$$\prod_{a=1}^{m} g^{i_a j_a}(x) = \prod_{a=1}^{m} \delta_{i_a j_a} + O(r^2), \quad \text{if } x \in B^M_r(p).$$

As a consequence, we can write

$$g^{i_1 j_1}\, g^{i_2 j_2} \cdots g^{i_m j_m}\,\partial_{i_1 j_1 i_2 j_2 \cdots i_m j_m}\cdot = (-\Delta)^m \cdot + \mathcal{D}^{2m}\cdot, \tag{2.5}$$

with $\mathcal{D}^{2m}$ satisfying the property stated in the lemma. This finishes the proof. ■

**Remark 2.1.** Throughout the paper, when performing local computations, we may consider conformal normal coordinates, see [14] or [32], if needed. These are normal coordinates at a point $x_0$ for a metric $g_w = e^{2w} g$ with $\det(g_w) = 1$ in a small neighborhood of $x_0$ and other useful properties, for which we refer the interested reader to [50]. Observe that the differential operator $P^{2m}_g$, after this change of the metric, can be still expanded as the right-hand side of (2.2). Indeed, $w(x) = O(d^2_{g_w}(x, x_0))$, and it is smooth in a small neighborhood of $x_0$. Moreover, by (1.3), we can write (1.6) as

$$P^{2m}_{g_w} u = e^{-2mw} P_g u = e^{-2mw}\Big( -Q^{2m}_g - \frac{\Lambda_m}{2|M|}\sum_{j=1}^{N}\alpha_j + \tilde{Q}\, e^{2mu}\Big),$$

which is equivalent to

$$e^{2mw}\, P^{2m}_{g_w} u = -Q^{2m}_g - \frac{\Lambda_m}{2|M|}\sum_{j=1}^{N}\alpha_j + \tilde{Q}\, e^{2mu}.$$

Concerning the differential operator $e^{2mw} P^{2m}_{g_w}$, it is known that the leading order operator is $e^{2mw}(-\Delta_{g_w})^m$. Then, by using the above Lemma 2.1 and the asymptotic behavior of $w(x)$, we can write

$$e^{2mw}(-\Delta_{g_w})^m u = (-\Delta)^m u + \mathcal{P}^{2m} u + \mathcal{P}^{2m-1} u,$$

with $\mathcal{P}^{2m}$ and $\mathcal{P}^{2m-1}$ satisfying the same properties as $\mathcal{D}^{2m}$ and $\mathcal{D}^{2m-1}$ in Lemma 2.1.

In what follows, recall $\Lambda_m = (2m-1)!|\mathbb{S}^{2m}|$. We will need the following structural result on the Green functional of the operators under consideration, see Lemma 2.1 in [44].

**Lemma 2.2.** *Suppose Ker*$\{P^{2m}_g\} = \{\text{constants}\}$. *Then the Green function* $G(x,y)$ *of* $P^{2m}_g$ *exists and has the following properties*:

(1) *For all* $u \in C^{2m}(M)$ *we have, for* $x \neq y \in M$,

$$u(x) - \bar{u} = \int_M G(x,y) P^{2m}_g u(y)\, dV_g(y), \qquad \int_M G(x,y)\, dV_g(y) = 0,$$

$$P^{2m}_g G(x,p) = \delta_p - \frac{1}{|M|},$$

*where* $\bar{u}$ *is the average of* $u$.

(2) *The function*

$$G(x,y) = H(x,y) + K(x,y)$$

*is smooth on* $M \times M$, *away from the diagonal. The function* $K$ *extends to a* $C^{2,\alpha}$ *function on* $M \times M$ *and* $H$ *satisfies*

$$H(x,y) = \frac{2}{\Lambda_m} \log\left(\frac{1}{r}\right) f(r),$$

*where* $r$ *is the geodesic distance from* $x$ *to* $y$ *and* $f$ *is a smooth positive, decreasing function such that* $f(r) = 1$ *in a neighborhood of* $r = 0$ *and* $f(r) = 0$ *for* $r \geq \mathrm{inj}_g(M)$.

As mentioned in the introduction, the total $Q$-curvature is a conformal invariant for which the following formula holds true.

**Theorem 2.3.** *Consider* $D = \sum_{i=1}^N p_i \alpha_i$, *where* $p_i \in M$ *and* $\alpha_i > -1$. *Let* $g$ *be a smooth metric on* $M$, *and let* $g_v = e^{2v} g$ *be the conical metric representing* $D$ *as explained before* (1.4). *Then, it holds*

$$\int_M Q^{2m}_{g_v}\, \mathrm{dvol}_{g_v} = \int_M Q^{2m}_g\, \mathrm{dvol}_g + \frac{\Lambda_m}{2} \sum_{i=1}^N \alpha_i. \tag{2.6}$$

*Proof.* The proof is a standard argument (see, e.g., [25] for $m = 2$), using Lemmata 2.1 and 2.2. See also [13] for a more general result which implies this statement as a particular case. ■

Finally, we state the general singular Adams–Trudinger–Moser inequality suitable to treat our problem. We focus here for simplicity on the case $P^{2m}_g \geq 0$, and refer to the discussion in [44] for the general case.

**Theorem 2.4.** *Consider $D = \sum_{i=1}^{N} p_i \alpha_i$, where $p_i \in M$ and $\alpha_i > -1$. Let $\widetilde{Q} > 0$ be as in (1.7). Assume $P_g^{2m} \geq 0$ and Ker$\{P_g^{2m}\} = \{\text{constants}\}$. Then, there exists a constant $C = C(\alpha, M)$ such that for any $u \in H^m(M)$, we have*

$$\Lambda_m \Big(1 + \min_j \{\alpha_j, 0\}\Big) \log \int_M \widetilde{Q}\, e^{2m(u-\bar{u})}\, d\mathrm{vol}_g \leq m \int_M u P_g^{2m} u \, d\mathrm{vol}_g + C,$$

*where $\bar{u}$ is the average of $u$.*

*Proof.* The case without singularities is Proposition 2.2 in [44]. The conic case follows by the same approach as in [25]. ∎

## 3. Compactness property

In this section, we shall prove the compactness result of Theorem 1.3. For simplicity of notation, there is no loss of generality to consider a blow-up sequence $u_k$ to

$$P_g^{2m} u_k + Q_g^{2m} = \widetilde{Q} e^{2mu_k}, \tag{3.1}$$

where

$$\widetilde{Q} = Q_{g_v}^{2m}\, e^{-m\Lambda_m \sum_{j=1}^{N} \alpha_j G(x,q_j)} > 0, \quad \text{with} \quad \Lambda_m = (2m-1)!\,|\mathbb{S}^{2m}|.$$

We call $p$ the blow-up point for the blow-up sequence $\{u_k\}$ if $u_k(p) \to +\infty$ as $k \to +\infty$. Collecting all the blow-up points into a set $\mathcal{B}$ and we name it the blow-up set for $\{u_k\}$. Theorem 1.3 will follow by showing a concentration phenomenon:

$$\widetilde{Q} e^{2mu_k} \rightharpoonup \sum_{p \in \mathcal{B}} (1 + \alpha_p) \Lambda_m \delta_p \quad \text{as } k \to +\infty,$$

weakly in the sense of measures,

$$\alpha_p = \begin{cases} 0, & \text{if } p \notin \{q_1, \dots, q_N\}, \\ \alpha_j, & \text{if } p = q_j. \end{cases}$$

It follows that when blow-up occurs, then necessarily

$$\int_M \widetilde{Q}\, e^{2mu_k}\, d\mathrm{vol}_g \to \sigma \in \Gamma \quad \text{as } k \to +\infty,$$

where $\Gamma$ is given in (1.8).

First, we establish the following lemma.

**Lemma 3.1.** *Let $\{u_k\}$ be a sequence of functions on $(M, g)$ satisfying (3.1). Then for $i = 1, \dots, 2m-1$, we have*

$$\int_{B_r^M(x)} |\nabla^i u_k|^l \, dy \leq C(n) r^{2m-il}, \quad 1 \leq l < \frac{2m}{i}, \ \forall x \in M, \ 0 < r < r_{\mathrm{inj}}, \tag{3.2}$$

*where $r_{\mathrm{inj}}$ is the injectivity radius of $(M, g)$.*

*Proof.* Set

$$f_k := \tilde{Q}\, e^{2mu_k} - Q_g^{2m},$$

which is bounded in $L^1(M)$. By Green's representation formula, we have

$$u_k(x) = \fint_M u_k \, d\mathrm{vol}_g + \int_M G(x,y) f_k(y)\, d\mathrm{vol}_g(y). \tag{3.3}$$

For $x, y \in M,\ x \neq y$, we have (see Lemma 2.1 in [44])

$$|\nabla_y^i G(x,y)| \le \frac{C}{d_g(x,y)^i}, \quad 1 \le i \le 2m-1. \tag{3.4}$$

Then differentiating (3.3) and using (3.4) and Jensen's inequality, we get

$$|\nabla^i u_k(x)|^l \le C \Big( \int_M \frac{|f_k(y)|}{d_g(x,y)^i}\, d\mathrm{vol}_g \Big)^l \le C \int_M \Big( \frac{\|f_k\|_{L^1(M)}}{d_g(x,y)^i} \Big)^l \frac{|f_k(y)|}{\|f_k\|_{L^1(M)}}\, d\mathrm{vol}_g .$$

From Fubini's theorem, we conclude that

$$\int_{B_r^M(x)} |\nabla^i u_k(x)|^l \, d\mathrm{vol}_g \le C \sup_{y\in M} \int_{B_r^M(x)} \frac{\|f_k\|^l_{L^1(M)}}{d_g(x,z)^{il}}\, d\mathrm{vol}_g(z) \le C r^{2m-il}. \tag{3.5}$$

This proves the lemma. ∎

Next, we shall give the minimal local mass around a blow-up point.

**Lemma 3.2.** *Let the sequence $u_k$ satisfy (3.1) and blowing-up at $q_j$. Suppose that*

$$\tilde{Q}\, e^{2mu_k} \rightharpoonup \mathfrak{m}, \quad \textit{weakly in the sense of measures in } M.$$

*Then*

$$\mathfrak{m}(q_j) \ge \frac{1}{2} \min\{\Lambda_m(1+\alpha_j), \Lambda_m\}.$$

*Proof.* To show the thesis, it suffices to prove that if the following inequality holds,

$$\int_{B^M(q_j,2r)} \tilde{Q}\, e^{2mu_k}\, d\mathrm{vol}_g < \frac{1}{2} \min\{\Lambda_m(1+\alpha_j), \Lambda_m\}, \quad r < \frac{r_{\mathrm{inj}}}{2}, \tag{3.6}$$

then

$$u_k \le C \quad \text{in } B^M(q_j, r). \tag{3.7}$$

We now study equation (3.1) in terms of the local normal coordinates at $q_j$ (see, for example, Lemma 2.1 and Remark 2.1). By the exponential map, we define the preimage of $B^M(q_j, r)$ by $B_r(0)$ and we use the same notation to denote $x \in M$ and its preimage. We decompose $u_k$ as $u_k = u_{1k} + u_{2k}$, where $u_{1k}$ is the solution of

$$\begin{cases} (-\Delta)^m u_{1k} = \tilde{Q} e^{2mu_k} \Xi_r(x), & \text{in } B_{2r}(0), \\ u_{1k} = \Delta u_{1k} = \cdots = (-\Delta)^{m-1} u_{1k} = 0, & \text{on } \partial B_{2r}(0), \end{cases} \tag{3.8}$$

where

$$\Xi_r(x) = \frac{d\mathrm{vol}_g(x)}{dx} = 1 + O(r^2),$$

due to the metric tensor $g_{ij}(x) = \delta_{ij} + O(r^2)$. By Theorem 7 in [37], we have

$$e^{2m\ell|u_{1k}|} \in L^1(B_{2r}(0)) \quad \text{for } \ell \in \Big(0, \frac{\Lambda_m}{2\|\widetilde{Q}\, e^{2mu_k}\, \Xi_r(x)\|_{L^1(B_{2r}(0))}}\Big) \tag{3.9}$$

and

$$\int_{B_{2r}(0)} e^{2m\ell|u_{1k}|}\, dx \le C(p) r^{2m}. \tag{3.10}$$

Let $G_r(x,y)$ be the Green function of $(-\Delta)^m$ on $B_{2r}(0)$ satisfying the Navier boundary condition, i.e.,

$$\begin{cases} (-\Delta)^m G_r(x,y) = \delta_x(y), & \text{in } B_{2r}(0), \\ G_r(x,y) = \cdots = \Delta^{m-1} G_r(x,y) = 0, & \text{on } \partial B_{2r}(0). \end{cases}$$

the function $G_r(x,y)$ can be decomposed as

$$G_r(x,y) = -\frac{2}{\Lambda_m} \log|x-y| + R_r(x,y),$$

with $R_r(x,y)$ a smooth function for $x, y \in B_{2r}$. By the Green representation formula, we have

$$u_{1k}(x) = -\frac{2}{\Lambda_m} \int_{B_{2r}(0)} \log|x-y|\, \widetilde{Q}\, e^{2mu_k}\, \Xi_r(y)\, dy + O(1), \quad x \in B_{3r/2}(0). \tag{3.11}$$

Observe that

$$\widetilde{Q} = d_g(x,q_j)^{2m\alpha_j}\, \widehat{Q},$$

where

$$\widehat{Q} = Q^{2m}_{g_v}\, e^{-m\Lambda_m \alpha_j R(x,q_j) - m\Lambda_m \sum_{i\neq j}^N \alpha_i G(x,q_i)} \quad \text{is a smooth function in } B_{2r}(0). \tag{3.12}$$

On the other hand, by using $G(x,y)$ and the Green representation formula, we get that

$$\begin{aligned} u_k(x) &= u_{1k}(x) + u_{2k}(x) \\ &= \bar{u}_k + \int_M G(x,y)\widetilde{Q}\, e^{2mu_k}\, d\mathrm{vol}_g - \int_M G(x,y) Q^{2m}_g d\mathrm{vol}_g, \end{aligned} \tag{3.13}$$

where $\bar{u}_k$ is the average of $u_k$. Since it is known that the leading term of $G$ and $G_{2r}$ carry the same singular behavior, we get from Jensen's inequality that

$$u_{2k} = \bar{u}_k + O(1) = \bar{v}_k + O(1) \le \log \fint_M e^{v_k} + O(1) = O(1), \tag{3.14}$$

where we used the average of Green function on $M$ is zero. Therefore, we conclude that $u_{2k}$ is bounded uniformly from above. Next, we shall prove that $u_{1k}$ is bounded, and our discussion is separated into two cases:

*Case* 1. If $\alpha_j > 0$, then (3.6) is equivalent to

$$\int_{B^M_{2r}(q_j)} \tilde{Q}\, e^{2mu_k}\, d\mathrm{vol}_g < \frac{1}{2}\,\Lambda_m.$$

Since $u_{2k}$ is bounded from above in $B_{3r/2}(0)$, we see from (3.10) that there exists some $\ell > 1$ such that $\tilde{Q}\, e^{2mu_k} \in L^\ell(B^M_{2r}(q_1))$. It is easy to see that $u_k \in L^1(M)$. Together with (3.11), we can easily see that $u_{1k}, u_{2k} \in L^1(B_{3r/2}(0))$. By the interior regularity results in Theorem 1 of [12], we get that

$$\|u_{1k}\|_{W^{2m,\ell}(B_r(0))} \le \|\tilde{Q}\, e^{2mu_k}\|_{L^\ell(B_{3r/2}(0))} + \|u_{1k}\|_{L^1(B_r(0))} \le C. \tag{3.15}$$

Thus by the classical Sobolev inequality, we get that $u_{1k} \in L^\infty(B_r(0))$.

*Case* 2. If $\alpha_j \in (-1, 0)$, then (3.6) is equivalent to

$$\int_{B^M_{2r}(q_j)} \tilde{Q}\, e^{2mu_k}\, d\mathrm{vol}_g < \frac{1}{2}\,\Lambda_m(1+\alpha_j).$$

It is not difficult to see $|x|^{2m\alpha_j} \in L^\ell(B_{2r})$ for any $\ell \in [1, -1/\alpha_j)$ and $e^{2mu_{1k}} \in L^p(B_{2r})$ for $p \in [1, 1/(1+\alpha_1) + \varepsilon)$ for some small strictly positive number $\varepsilon$ by (3.9). As a consequence, we get that $|x|^{2m\alpha_1} e^{2mu_{1k}} \in L^\ell(B_{2r}(0))$ for some $\ell > 1$ by Hölder's inequality. Repeating the arguments as in Case 1, we obtain that $u_{1k}$ is bounded uniformly in $B_r(0)$.

After establishing that $u_{1k}$ is bounded in $B_r(0)$, combining with (3.14), we derive that $u_k$ is bounded above in $B^M_r(q_j)$. Then we finish the proof of this lemma. ∎

We shall derive now the quantization result and the concentration property of the bubbling solution.

**Proposition 3.3.** *Let $\{u_k\}$ be a sequence of solutions to (3.1) and let $\mathcal{B}$ be its blow-up set. Then we have the following convergence in the sense of measures:*

$$\tilde{Q} e^{2mu_k} \rightharpoonup \sum_{p\in\mathcal{B}} \Lambda_m(1+\alpha_p)\delta_p, \quad \text{as } k \to +\infty, \tag{3.16}$$

*where*

$$\alpha_p = \begin{cases} \alpha_i, & \text{if } p = q_i \in \{q_1, \dots, q_N\}, \\ 0, & \text{if } p \in \mathcal{B} \setminus \{q_1, \dots, q_N\}. \end{cases}$$

*In particular, $u_k \to -\infty$ uniformly on any compact subset of $M \setminus \mathcal{B}$.*

*Proof.* For any compact set $K \subset M \setminus \mathcal{B}$, we can use the Green representation formula

$$u_k(x) - u_k(y) = \int_M (G(x,z) - G(y,z))\left(\tilde{Q}\, e^{2mu_k} - Q^{2m}_g\right) dz, \tag{3.17}$$

together with the estimate (3.4), to derive that

$$|\nabla^i u_k(x)| \le C(K), \quad \text{for } x \in K,\ 1 \le i \le 2m-1. \tag{3.18}$$

Then from equation (3.1) and classical elliptic estimates, we get that the $2m$ order derivatives of $u_k$ satisfy

$$|\nabla^{2m} u_k(x)| \le C(K), \quad \text{for } x \in K. \tag{3.19}$$

To proceed with our discussion, we introduce the following quantity:

$$\sigma_p = \lim_{r\to 0} \lim_{k\to+\infty} \int_{B^M(p,r)} \widetilde{Q}\, e^{2mu_k}\, d\mathrm{vol}_g. \tag{3.20}$$

It has been shown in Lemma 3.2 that $\sigma_p$ has a positive lower bound at the blow-up point. From the fact that $\int_M Q_g^{2m}\, d\mathrm{vol}_g$ is finite, we conclude that the blow-up points are finitely-many. At a regular blow-up point $p$, it has been already shown in Theorem 2 of [38] that

$$\widetilde{Q}\, e^{2mu_k} \rightharpoonup \Lambda_m \delta_p \quad \text{in } B^M_{r_p}(p),$$

where $r_p$ is chosen such that $B^M(p, r_p) \cap (\mathcal{B} \setminus \{p\}) = \emptyset$. In the following discussion, we will focus on the singular blow-up point. Without loss of generality, we shall consider $u_k$ in $B^M_{2r}(q_1)$, where $r$ is chosen such that $B^M_{2r}(q_1)$ only contains $q_1$ from $\mathcal{B}$. We first claim that

$$u_k \to -\infty, \quad \text{for } x \in B^M_{2r}(q_1) \setminus \{q_1\}. \tag{3.21}$$

We prove it by contradiction. Suppose that $u_k$ is uniformly bounded below at some point away from $q_1$. Then by (3.18), we derive that

$$u_k \to u_0 \quad \text{in } C^{2m-1,\sigma}_{\mathrm{loc}}(B^M_{2r}(q_1) \setminus \{q_1\}),\ \sigma \in (0,1), \tag{3.22}$$

with the limit function satisfying

$$P_g^{2m} u_0 + Q_g^{2m} = d_g(x, q_1)^{2m\alpha_1} \widehat{Q}\, e^{2mu_0} \quad \text{in } B^M_{2r}(q_1) \setminus \{q_1\}, \tag{3.23}$$

where $\widehat{Q}$ is a smooth function around $q_1$ defined analogously as in (3.12) and $d_g(x, q_1)$ denotes the geodesic distance between $x$ and $q_1$ with respect to the metric $g$. According to the definition of $\sigma_p$ (see (3.20)), we see that $u_0$ satisfies

$$P_g^{2m} u_0 + Q_g^{2m} = d_g(x, q_1)^{2m\alpha_1} \widehat{Q}\, e^{2mu_0} + \sigma_{q_1} \delta_{q_1} \quad \text{in } B^M_{2r}(q_1).$$

Using the Green representation formula for $u_0$, we have

$$u_0(x) = \sigma_{q_1} G(x, q_1) + v_0(x), \tag{3.24}$$

where

$$\begin{aligned} v_0(x) = & \int_M \frac{2}{\Lambda_m} \log d_g(x,y) \left(d_g(y, q_1)^{2m\alpha_1} \widehat{Q}\, e^{2mu_0}\right) d\mathrm{vol}_g \\ & + \int_M R(x,y) \left(d_g(y, q_1)^{2m\alpha_1} \widehat{Q}\, e^{2mu_0}\right) d\mathrm{vol}_g \\ & + \bar{u}_0 - \int_M G(x,y)\, Q_g^{2m}\, d\mathrm{vol}_g. \end{aligned} \tag{3.25}$$

Denoting the two terms on the right-hand side by $\hat{v}_1$ and $\hat{v}_2$, respectively, it is not difficult to see that $\hat{v}_2$ is smooth.

In the following, we shall prove that $\hat{v}_1(x)$ is bounded in $x \in B_r^M(q_1)$. In fact, for $x \in B_r^M(q_1)$, we have

$$
\begin{aligned}
v_0(x) &= \int_{B_r^M(q_1)} G(x,y)\, d_g(y,q_1)^{2m\alpha_1}\, \widehat{Q}\, e^{2mu_0}\, d\mathrm{vol}_g + O(1) \\
&\ge \frac{2}{\Lambda_m} \log \frac{1}{r}\, \|\widehat{Q}|x|^{2m\alpha_1} e^{2mu_0}\|_{L^1(B_{2r}^M(q_1))} + O(1).
\end{aligned} \tag{3.26}
$$

This provides a lower bound for $v_0(x)$. On the other hand, we have

$$
d_g(x,q_1)^{2m\alpha_1}\, e^{2mu_0} \ge C d_g(x,q_1)^{2m(\alpha_1 - 2\sigma_{q_1}/\Lambda_m)}. \tag{3.27}
$$

Using the fact that the left-hand side of (3.27) is integrable, we get

$$
\alpha_1 - \frac{2\sigma_{q_1}}{\Lambda_m} > -1. \tag{3.28}
$$

When $\alpha_1 < 0$, we see that the above inequality (3.28) implies that

$$
\sigma_{q_1} < \frac{1}{2}\Lambda_m(1+\alpha_1).
$$

Therefore, $u_k$ cannot blow-up at $q_1$ by Lemma 3.2, and we get a contradiction. This implies (3.21) for $\alpha_1 < 0$. For $\alpha_1 > 0$ we have

$$
\begin{aligned}
C d_g(x,q_1)^{2m\alpha_1 - 4m\sigma_{q_1}/\Lambda_m} e^{v_0(x)} &\ge d_g(x,q_1)^{2m\alpha_1}\, e^{2mu_0} \\
&\ge C d_g(x,q_1)^{2m\alpha_1 - 4m\sigma_{q_1}/\Lambda_m}.
\end{aligned} \tag{3.29}
$$

In order to show that $\hat{v}_1(x)$ is bounded in $B_r^M(q_1)$, we study $\hat{v}_1(x)$ in terms of local coordinates at $q_1$. Then $d_g(x,q_1)$ can be regarded as $|x^p - 0|$, where $x^p$ denotes its preimage of $x$ under the exponential map at $q_1$. By a little abuse of notation, we still denote $x^p$ by $x$. Then we notice that $\hat{v}_1(x)$ satisfies

$$
(-\Delta)^m \hat{v}_1(x) = |x|^{2m\alpha_1}\, \widehat{Q}\, e^{2mu_0}\, \Xi(x) \quad \text{in } B_{2r}(0), \tag{3.30}
$$

where $\Xi(x) = d\mathrm{vol}_g(x)/dx$ is bounded above and below in $B_{2r}(0)$ since the metric tensor is comparable to the standard Euclidean metric. Since we only consider the local behavior of $\hat{v}_1(x)$ in $B_r(0)$, by multiplying a cut-off function $\chi(x)$ with $\chi(x) = 1$ for $|x| \le r$ and $\chi(x) = 0$ for $|x| \ge 2r$, we have that

$$
\tilde{v}_1(x) := \chi(x)\, \hat{v}_1(x)
$$

satisfies

$$
\begin{cases}
(-\Delta)^m \tilde{v}_1(x) = |x|^{2m\alpha_1}\, \widehat{Q}\, e^{2mu_0}\, \Xi(x) + \Xi_0(x) & \text{in } B_{2r}(0), \\
\tilde{v}_1(x) = \Delta \tilde{v}_1(x) = \cdots = \Delta^{m-1} \tilde{v}_1(x) = 0 & \text{on } \partial B_{2r}(0),
\end{cases} \tag{3.31}
$$

where $\Xi_0(x)$ is smooth in $B_{2r}(0)$. It is not difficult to see that

$$|x|^{2m\alpha_1}\widehat{Q}\,e^{2mu_0}\Xi(x)+\Xi_0(x)\in L^1(B_2r(0)).$$

We decompose $|x|^{2m\alpha_1}\widehat{Q}\,e^{2mu_0}\Xi(x)+\Xi_0(x)$ into $P_1(x)$ and $P_2(x)$, with

$$\|P_1\|_{L^1(B_{2r}(0))}\le\varepsilon \quad\text{and}\quad P_2\in L^\infty(B_{2r}(0)). \tag{3.32}$$

Correspondingly, we decompose $\tilde{v}_1$ into $\tilde{v}_{11}$ and $\tilde{v}_{12}$, where $\tilde{v}_{1i}$, $i=1,2$, solves

$$\begin{cases}(-\Delta)^m\tilde{v}_{1i}(x)=P_i(x), & \text{in } B_{2r}(0),\\ \tilde{v}_{1i}(x)=\Delta\tilde{v}_{1i}(x)=\cdots=\Delta^{m-1}\tilde{v}_{1i}(x)=0, & \text{on } \partial B_{2r}(0).\end{cases} \tag{3.33}$$

For $\tilde{v}_{11}$, by Theorem 7 in [37] we get that $e^{\frac{\Lambda_m}{2\varepsilon}\tilde{v}_{11}}\in L^1(B_{2r}(0))$. While for $\tilde{v}_{12}$, using the classical elliptic regularity theory, we derive that $\tilde{v}_{12}\in L^\infty(B_{2r}(0))$. Together with (3.29), we can select $\varepsilon$ sufficiently small such that

$$|x|^{2m\alpha_1}\widehat{Q}\,e^{2mu_0}\Xi(x)\in L^l(B_{2r}(0))\quad\text{for some } l>1.$$

Returning to (3.31), we apply the regularity theory to deduce that $\hat{v}_1\in W^{2m,l}(B_{2r}(0))$. As a consequence, we have $v_0\in W^{2m,l}(B_{2r}(0))$, and this implies that $|v_0|\le C$ for some constant $C$ in $B_r(0)$ by the classical Sobolev inequality. Thus we have proved $v_0$ is bounded in $B_r^M(q_1)$. Together with (3.29), we get that

$$d_g(x,q_1)^{2m\alpha_1}\widehat{Q}\,e^{2mu_0}\sim d_g(x,q_1)^{2m(\alpha_1-2\sigma_{q_1}/\Lambda_m)}\quad\text{if }\alpha_1>0. \tag{3.34}$$

Next we shall derive a contradiction by making use of the Pohzozaev identity. It is known that equation (3.1) can be written as

$$(-\Delta_g)^m u_k(x)+\mathcal{A}u_k+Q_g^{2m}=\widetilde{Q}\,e^{2mu_k}\quad\text{in } B_r^M(q_1), \tag{3.35}$$

where $\mathcal{A}$ is a linear differential operator of order at most $2m-1$; moreover, the coefficients of $A$ belong to $C^l_{\text{loc}}(M)$ for all $l\ge 0$. Using the local normal coordinates, by Lemma 2.1 (see also Remark 2.1), we could write (3.35) as

$$(-\Delta)^m u_k+\mathcal{D}^{2m}u_k+\mathcal{C}u_k+Q_g^{2m}=\widetilde{Q}\,e^{2mu_k}\quad\text{in } B_r(0), \tag{3.36}$$

where $\mathcal{D}^{2m}$ is a linear differential operator of order $2m$ and the coefficients are of order $O(|x|^2)$, with its derivative of arbitrary order smooth, while $\mathcal{C}$ is a linear differential operator of order at most $2m-1$, and the coefficients of $B_k$ belong to $C^l_{\text{loc}}(\mathbb{R}^{2m})$ for all $l\ge 0$. Multiplying by $x\cdot\nabla u_k$ on both sides, concerning the right-hand side, we have

$$\begin{aligned}\text{r.h.s. of (3.36)}&=\frac{1}{2m}\int_{B_r(0)}x\cdot\nabla(|x|^{2m\alpha_1}\widehat{Q}\,e^{2mu_k})\,dx-\alpha_1\int_{B_r(0)}|x|^{2m\alpha_1}\widehat{Q}\,e^{2mu_k}\,dx\\&\quad-\frac{1}{2m}\int_{B_r(0)}(x\cdot\nabla\widehat{Q})|x|^{2m\alpha_1}e^{2mu_k}\,dx\\&=\frac{1}{2m}\int_{\partial B_r(0)}|x|^{2m\alpha_1+1}\widehat{Q}\,e^{2mu_k}\,ds-(\alpha_1+1)\int_{B_r(0)}|x|^{2m\alpha_1}\widehat{Q}\,e^{2mu_k}\,dx\\&\quad-\frac{1}{2m}\int_{B_r(0)}(x\cdot\nabla\widehat{Q})|x|^{2m\alpha_1}e^{2mu_k}\,dx\\&\to-(1+\alpha_1)\sigma_{q_1}+o_r(1)\quad\text{as } k\to+\infty.\end{aligned} \tag{3.37}$$

Next, we consider the left-hand side of (3.36). At first, for the fourth term, we have

$$\Big| \int_{B_r(0)} \langle x, \nabla u_k \rangle \, Q_g^{2m} \, dx \Big| \le r \int_{B_r(0)} |\nabla u_k| \, dx \le C r^{2m}, \tag{3.38}$$

where we used Lemma 3.1. Therefore,

$$\lim_{r\to 0} \lim_{k\to+\infty} \Big| \int_{B_r(0)} \langle x, \nabla u_k \rangle \, Q_g^{2m} \, dx \Big| = 0. \tag{3.39}$$

For the third term, we have

$$\begin{aligned} \Big| \int_{B_r(0)} \langle x, \nabla u_k \rangle \mathscr{C} u_k \Big| &\le C \sum_{i=0}^{2m-1} \int_{B_r(0)} |x| |\nabla^i u_k| |\nabla u_k| \, dx \\ &\le \sum_{i=1}^{2m-1} \Big( \int_{B_r(0)} |x|^{s_i} |\nabla u_k|^{s_i} \, dx \Big)^{1/s_i} \Big( \int_{B_r(0)} |\nabla^i u_k|^{t_i} \, dx \Big)^{1/t_i} \\ &\quad + C \int_{B_r(0)} |\nabla u_k| dx, \end{aligned} \tag{3.40}$$

where we have used that

$$|x| |u_k| \le C \quad \text{in } B_r(0),$$

and where

$$t_i = \frac{2m}{i} - \delta \quad \text{and} \quad s_i = \frac{2m - \delta i}{2m - i - \delta i}, \quad \delta \in \Big(0, \frac{1}{2(2m-1)}\Big).$$

From (3.24) and (3.25), and since $\hat{v}_1$ and $\hat{v}_2$ are bounded from above, we can see that

$$|x \cdot \nabla u_0| \le C \quad \text{in } B_r(0).$$

Together with Lemma 3.1, we have

$$\int_{B_r(0)} |x|^{s_i} |\nabla u_0|^{s_i} \, dx \le C r^{2m} \quad \text{and} \quad \int_{B_r(0)} |\nabla^i u_0|^{t_i} \, dx \le C r^{i\delta}. \tag{3.41}$$

As a consequence of (3.40) and (3.41), we see that

$$\lim_{r\to 0} \lim_{k\to+\infty} \Big| \int_{B_r(0)} \langle x, \nabla u_k \rangle \mathscr{C} u_k \Big| \, dx = 0. \tag{3.42}$$

For the second term, we have already seen that $v_0 \in W^{2m,l}(B_r(0))$ for some $l > 1$. Then, using (3.24), we see that

$$\int_{B_r(0)} x \cdot \nabla u_0 \, \mathscr{D}^{2m} u_0 \, dx \le C \int_{B_r(0)} \frac{1}{|x|^{2m-2}} \, dx + C r^2 \|v_0\|_{W^{2m,1}(B_r(0))} \le C r^2.$$

This leads to

$$\lim_{r\to 0}\lim_{k\to+\infty}\left|\int_{B_r(0)}\langle x,\nabla u_k\rangle\,\mathcal{D}^{2m}u_k\,dx\right|=0. \tag{3.43}$$

Therefore, from (3.39), (3.42) and (3.43), we get that except from the first term on the left-hand side of (3.36), the other terms vanish in the limit. It remains to study the term

$$\int_{B_r(0)}(-\Delta)^m u_k x\cdot\nabla u_k\,dx.$$

We shall only consider the case where $m$ is even; the argument for the case where $m$ is odd goes almost the same. We set $m=2m_0$. Using the Pohozaev identity (A.1), and replacing $f$ by $\frac{2\sigma_{q_1}}{\Lambda_m}\log|x|$ plus a smooth function, after direct computations we get that

$$\begin{aligned}
&\int_{B_r(0)}(-\Delta)^m u_0\langle x,\nabla u_0\rangle\,dx\\
&\quad=\sum_{i=2}^{m_0}\int_{\partial B_r(0)}2^{2m}(m-1)!(m-1)!\left(1-\frac{i-1}{m-i}\right)\frac{\sigma_{q_1}^2}{\Lambda_m^2}\frac{1}{r^{2m-1}}\,ds\\
&\qquad+\sum_{i=1}^{m_0}\int_{\partial B_r(0)}2^{2m}(m-1)!(m-1)!\left(\frac{i-1}{m-i}-1\right)\frac{\sigma_{q_1}^2}{\Lambda_m^2}\frac{1}{r^{2m-1}}\,ds\\
&\qquad-\int_{\partial B_r(0)}2^{2m-1}(m-1)!(m-1)!\frac{\sigma_{q_1}^2}{\Lambda_m^2}\frac{1}{r^{2m-1}}\,ds+o_r(1)\\
&\quad\to-2^{2m-1}(m-1)!(m-1)!\frac{\sigma_{q_1}^2}{\Lambda_m^2}|\mathbb{S}^{2m-1}|\quad\text{as } r\to 0.
\end{aligned}\tag{3.44}$$

It is known that we can write

$$\Lambda_m=(2m-1)!\,|\mathbb{S}^{2m}|=2^{2m-1}(m-1)!(m-1)!\,|\mathbb{S}^{2m-1}|.$$

Together with (3.37) and (3.44), we derive that

$$(1+\alpha_1)\sigma_{q_1}=\frac{\sigma_{q_1}^2}{\Lambda_m}. \tag{3.45}$$

Recalling also Lemma 3.2, we get

$$\sigma_{q_1}=(1+\alpha_1)\Lambda_m. \tag{3.46}$$

Returning to equation (3.34) we see that

$$d_g(x,q_1)^{2m\alpha_1}\widehat{Q}\,e^{2mu_0}\sim d_g(x,q_1)^{-2m-2m(1+\alpha_1)}\ge d_g(x,q_1)^{-2m}, \tag{3.47}$$

which contradicts

$$d_g(x,q_1)^{2m\alpha_1}\widehat{Q}\,e^{2mu_0}\in L^1(B_r(0)).$$

Therefore, $u_k\to-\infty$ uniformly on any compact subset of $B_{2r}^M(q_1)\setminus\{q_1\}$.

It remains to show the quantization $\sigma_{q_1}$ is exactly $\Lambda_m(1+\alpha_1)$. We consider the function

$$\hat{u}_k(x) = u_k(x) - c_k,$$

with

$$c_k = \fint_{\partial B_r(0)} u_k(x) \to -\infty.$$

As before, we can show that $\hat{u}_k(x)$ converges to some function $\hat{u}_0(x)$ in $C^{2m}_{\mathrm{loc}}(B_{2r} \setminus \{0\})$ and we can write

$$\hat{u}_0(x) = -\frac{2\sigma_{q_1}}{\Lambda_m} \log d_g(x, q_1) + \hat{v}(x).$$

Repeating the previous argument, again by the Pohozaev identity we derive that

$$\sigma_{q_1} = \Lambda(1+\alpha_1),$$

and we finish the proof. ■

## 4. Existence result

In this section, we are going to prove the existence and multiplicity results of Theorems 1.1 and 1.2. To make the exposition more transparent, we assume hereafter for simplicity that $P_g^{2m} \geq 0$. The general case can be treated by suitable adaptations, see Remark 4.1.

Solutions of (1.6) are critical points of the functional

$$\begin{aligned} \mathcal{E}(u) = 2m \int_M u P_g^{2m} u \, d\mathrm{vol}_g + 4m \int_M \Big( Q_g^{2m} + \frac{\Lambda_m}{2|M|} \sum_{j=1}^N \alpha_j \Big) u \, d\mathrm{vol}_g \\ - 2\kappa_{g_v} \log \int_M \widetilde{Q}\, e^{2mu} \, d\mathrm{vol}_g \end{aligned} \tag{4.1}$$

with $u \in H^m(M)$, where we recall

$$\widetilde{Q} = Q_{g_v}^{2m} \, e^{-m\Lambda_m \sum_{j=1}^N \alpha_j G(x,q_j)} > 0, \quad \Lambda_m = (2m-1)! \, |\mathbb{S}^{2m}|$$

and

$$\kappa_{g_v} = \int_M \widetilde{Q}\, e^{2mu} \, d\mathrm{vol}_g = \int_M Q_g^{2m} \, d\mathrm{vol}_g + \frac{\Lambda_m}{2} \sum_{j=1}^N \alpha_j .$$

We point out we consider here

$$\alpha_j > 0, \quad \forall j = 1, \dots, N.$$

Then, in particular, the Adams–Trudinger–Moser inequality in Theorem 2.4 implies the functional $\mathcal{E}$ is coercive and bounded from below provided $\kappa_{g_v} < \Lambda_m$. Thus, existence of solutions in this subcritical case follows by direct method of calculus of variations.

In the supercritical case $\kappa_{g_v} > \Lambda_m$, the functional is unbounded from below and we need to apply a min-max method based on the topology of the sublevels of the functional

$$\mathcal{E}^a = \{u \in H^m(M) : \mathcal{E}(u) \leq a\}.$$

The rough idea is that the low sublevels carry some non-trivial topology, while the high sublevels are contractible, and such change of topology jointly with the compactness property of Theorem 1.3 (provided $\kappa_{g_v} \notin \Gamma$) detects a critical point. The main step is the study of low sublevels, which is done by an improved version of the Adams–Trudinger–Moser inequality and suitable test functions.

Let us start by pointing out that once the Adams–Trudinger–Moser inequality is available (Theorem 2.4), then a standard argument yields improved versions of it under a spreading of the conformal volume $\widetilde{Q}\, e^{2mu}$ in, say, $l$ disjoint subsets as expressed in (4.2). Somehow, it is possible to sum up localized versions of the inequality, which are in turn based on cut-off functions, and improve the Adams–Trudinger–Moser constant to roughly $l\Lambda_m$. We refer the interested readers for example to Lemma 4.1 in [44] and references therein.

**Lemma 4.1.** *Let $\delta, \theta > 0$, $l \in \mathbb{N}$ and $\Omega_1, \ldots, \Omega_l \subset M$ be such that $d(\Omega_i, \Omega_j) > \delta$ for any $i \neq j$. Then, for any $\varepsilon > 0$, there exists $C = C(\varepsilon, \delta, \theta, L, M)$ such that for any $u \in H^m(M)$ such that*

$$\int_{\Omega_i} \frac{\widetilde{Q}\, e^{2mu}\, d\mathrm{vol}_g}{\int_M \widetilde{Q}\, e^{2mu}\, d\mathrm{vol}_g} \geq \theta, \quad \forall i \in \{1, \ldots, l\}, \tag{4.2}$$

*it holds*

$$l\Lambda_m \log \int_M \widetilde{Q}\, e^{2m(u-\bar{u})}\, d\mathrm{vol}_g \leq (1+\varepsilon) m \int_M u P_g^{2m} u\, d\mathrm{vol}_g + C,$$

*where $\bar{u}$ is the average of $u$.*

Improved inequalities then yield lower bounds on the functional $\mathcal{E}$. As a consequence, in the low sublevels, $\widetilde{Q}\, e^{2mu}$ has to be concentrated in not too many different subsets, as shown in the following result.

**Lemma 4.2.** *Suppose $\kappa_{g_v} < (k+1)\Lambda_m$ for some $k \in \mathbb{N}$. Then, $\forall \varepsilon, r > 0$, there exists $L = L(\varepsilon, r) \gg 1$ such that $\forall u \in \mathcal{E}^{-L}$ there exist $k$ points $\{p_1, \ldots, p_k\} \subset M$ such that*

$$\int_{\bigcup_{i=1}^k B_r^M(p_i)} \frac{\widetilde{Q}\, e^{2mu}\, d\mathrm{vol}_g}{\int_M \widetilde{Q}\, e^{2mu}\, d\mathrm{vol}_g} \geq 1 - \varepsilon.$$

*Proof.* We sketch here the proof. Suppose the thesis is false. Then, using a standard covering argument as in Lemma 2.3 of [23], we find $k+1$ disjoint subsets $\Omega_1, \ldots, \Omega_{k+1} \subset M$ in which $\widetilde{Q}\, e^{2mu}$ is spread in the sense of (4.2). Therefore, applying the improved Adams–Trudinger–Moser inequality of Lemma 4.1, we would get a lower bound of the functional

$$\mathcal{E}(u) \geq 2m\Big(1 - \frac{\kappa_{g_v}}{(k+1)\Lambda_m}(1+\varepsilon)\Big) \int_M u P_g^{2m} u\, d\mathrm{vol}_g + \text{l.o.t.}$$

By assumption,

$$\kappa_{g_v} < (k+1)\Lambda_m,$$

and hence we can take a sufficiently small $\varepsilon > 0$ such that

$$1 - \frac{\kappa_{g_v}}{(k+1)\Lambda_m}(1+\varepsilon) \geq 0$$

which yields $\mathcal{E}(u) \geq -L$ for some $L \gg 1$. But this is not possible, since we were considering $u \in \mathcal{E}^{-L}$. ■

It is then convenient to describe the low sublevels by means of formal barycenters of $M$ of order $k$, that is, unit measures supported in at most $k$ points on $M$, defined by

$$M_k = \left\{ \sum_{i=1}^{k} t_i \delta_{p_i} \, : \, \sum_{i=1}^{k} t_i = 1, \, t_i \geq 0, \, p_i \in M, \, \forall i = 1, \dots, k \right\}. \tag{4.3}$$

The idea is to use a projection within unit measures such that

$$\frac{\widetilde{Q}\, e^{2mu}\, d\mathrm{vol}_g}{\int_M \widetilde{Q}\, e^{2mu}\, d\mathrm{vol}_g} \mapsto \sigma \in M_k.$$

This is done exactly as in Proposition 3.1 of [23] by using Lemma 4.2 to get the following.

**Proposition 4.3.** *Suppose $\kappa_{g_v} < (k+1)\Lambda_m$ for some $k \in \mathbb{N}$. Then, there exist $L \gg 1$ and a projection $\Psi\colon \mathcal{E}^{-L} \to M_k$.*

Recall now that we are assuming there exists a retraction $R\colon M \to M^{\mathrm{R}}$, with $M^{\mathrm{R}} \subset M$ a closed $n$-dimensional submanifold, $n \in [1, 2m]$, such that $\alpha_j \notin M^{\mathrm{R}}$ for all $j = 1, \dots, N$. Let $M_k^{\mathrm{R}}$ be the set of formal barycenters of $M^{\mathrm{R}}$. We can then define a map $\Psi_R\colon \mathcal{E}^{-L} \to M_k^{\mathrm{R}}$ simply by considering the composition

$$\mathcal{E}^{-L} \xrightarrow{\Psi} M_k \xrightarrow{R_*} M_k^{\mathrm{R}},$$

where $R_*$ is the push-forward of measures induced by the retraction $R$. Therefore, we have the following result.

**Lemma 4.4.** *Suppose that $\kappa_{g_v} < (k+1)\Lambda_m$ for some $k \in \mathbb{N}$. Then, there exist $L \gg 1$ and a continuous map $\Psi_R\colon \mathcal{E}^{-L} \to M_k^{\mathrm{R}}$.*

The low sublevels are thus naturally described (at least partially) by $M_k^{\mathrm{R}}$. As a matter of fact, we are going to construct a reverse map, mapping continuously $M_k^{\mathrm{R}}$ into $\mathcal{E}^{-L}$. This is done by suitable test functions on which the functional attains low values. The idea here is that, since $M^{\mathrm{R}}$ does not contain conical points $q_j$, we may consider a family of *regular* bubbles centered on $M^{\mathrm{R}}$. We thus take a non-decreasing cut-off function $\chi_\delta$ such that

$$\begin{cases} \chi_\delta(t) = t, & t \in [0, \delta], \\ \chi_\delta(t) = 2\delta, & t \geq 2\delta, \end{cases}$$

let $\lambda > 0$ and then define $\Phi\colon M_k^{\mathrm{R}} \to H^m(M)$ by

$$\Phi(\sigma) = \varphi_{\lambda,\sigma}, \quad \sigma = \sum_{i=1}^{k} t_i \delta_{p_i} \in M_k^{\mathrm{R}},$$

where

$$\varphi_{\lambda,\sigma}(y) = \frac{1}{2m} \log \sum_{i=1}^{k} t_i \Big( \frac{2\lambda}{1 + \lambda^2 \chi_\delta^2(d(y, p_i))} \Big)^{2m}.$$

Now, since we are considering bubbles centered on $M^{\mathrm{R}}$, which does not contain conical points, we can neglect the effect of the singularities, and all the following estimates are carried out exactly as in the regular case, see Lemmas 4.5 and 4.6 in [44] and references therein. To avoid technicalities, and with a little abuse of notation, we will write $o(1)$ to denote quantities which do not necessarily tend to zero, but that can be made arbitrarily small.

**Lemma 4.5.** *Let $\varphi_{\lambda,\sigma}$ be as above. Then, for $\lambda \to +\infty$, it holds*

$$\int_M \varphi_{\lambda,\sigma} P_g^{2m} \varphi_{\lambda,\sigma} \, d\mathrm{vol}_g \le 2k\Lambda_m (1 + o(1)) \log \lambda,$$

$$\int_M \Big( Q_g^{2m} + \frac{\Lambda_m}{2|M|} \sum_{j=1}^{N} \alpha_j \Big) \varphi_{\lambda,\sigma} \, d\mathrm{vol}_g = -\kappa_{g_v} (1 + o(1)) \log \lambda,$$

$$\log \int_M \widetilde{Q} \, e^{2m\varphi_{\lambda,\sigma}} \, d\mathrm{vol}_g = O(1).$$

By the latter estimates, we readily get the map we were looking for if we take

$$\kappa_{g_v} > k\Lambda_m.$$

Indeed, it is enough to observe that by Lemma 4.5, we have

$$\mathcal{E}(\Phi(\sigma)) \le 4m(k\Lambda_m - \kappa_{g_v} + o(1)) \log \lambda \to -\infty$$

as $\lambda \to +\infty$. Therefore, we can state the following result.

**Proposition 4.6.** *Suppose $\kappa_{g_v} > k\Lambda_m$ for some $k \in \mathbb{N}$. Then, for any $L > 0$, there exists $\lambda \gg 1$ such that $\Phi\colon M_k^{\mathrm{R}} \to \mathcal{E}^{-L}$.*

We are now in position to prove the existence result.

*Proof of Theorem* 1.1. Suppose $\kappa_{g_v} \in (k\Lambda_m, (k+1)\Lambda_m)$ for some $k \in \mathbb{N}$ and $\kappa_{g_v} \notin \Gamma$, where $\Gamma$ is the critical set given in (1.8). The proof is based on a min-max argument relying on the set $M_k^{\mathrm{R}}$ which will keep track of the topological properties of the low sublveles of the functional $\mathcal{E}$, jointly with the compactness property in Theorem 1.3.

*Step* 1. Recalling Lemma 4.4, let $L \gg 1$ be such that there exists a continuous map $\Psi_R\colon \mathcal{E}^{-L} \to M_k^{\mathrm{R}}$. Then, by Proposition 4.6, we can take $\lambda \gg 1$ such that $\Phi\colon M_k^{\mathrm{R}} \to \mathcal{E}^{-L}$.

Consider now the composition

$$\begin{array}{ccccc} M_k^{\mathrm{R}} & \xrightarrow{\Phi} & \mathcal{E}^{-L} & \xrightarrow{\Psi_R} & M_k^{\mathrm{R}} \\ \sigma & \mapsto & \varphi_{\lambda,\sigma} & \mapsto & \Psi_R\Big(\frac{\widetilde{Q}\, e^{2m\varphi_{\lambda,\sigma}}\, d\mathrm{vol}_g}{\int_M \widetilde{Q}\, e^{2m\varphi_{\lambda,\sigma}}\, d\mathrm{vol}_g}\Big). \end{array}$$

It is not difficult to see that the latter composition is homotopic to the identity map on $M_k^{\mathrm{R}}$. We have just to notice that, as $\lambda \to +\infty$,

$$\frac{\widetilde{Q}\, e^{2m\varphi_{\lambda,\sigma}}\, d\mathrm{vol}_g}{\int_M \widetilde{Q}\, e^{2m\varphi_{\lambda,\sigma}}\, d\mathrm{vol}_g} \rightharpoonup \sigma$$

in the sense of measures, that $\Psi$ is a projection, and that $R$ is a retraction onto $M^{\mathrm{R}}$. The homotopy is thus realized by letting $\lambda \to +\infty$. As a consequence, if we consider the induced maps between homology groups $H_*$ we get that

$$H_*(M_k^{\mathrm{R}}) \hookrightarrow H_*(\mathcal{E}^{-L}) \text{ injectively.} \tag{4.4}$$

Now, since $M^{\mathrm{R}}$ is a closed manifold, it is well known that $M_k^{\mathrm{R}}$ has non-trivial homology groups and hence, in particular, it is non-contractible. We refer the interested readers for example to Lemma 3.7 in [23], where the 4-dimensional case is considered. By the above discussion, this implies

$$\Phi(M_k^{\mathrm{R}}) \subset \mathcal{E}^{-L} \text{ is non-contractible.} \tag{4.5}$$

*Step* 2. We next consider the topological cone over $M_k^{\mathrm{R}}$, which is defined through the equivalence relation

$$\mathcal{C} = \frac{M_k^{\mathrm{R}} \times [0,1]}{M_k^{\mathrm{R}} \times \{0\}},$$

that is, $M_k^{\mathrm{R}} \times \{0\}$ is collapsed to a single point which is the tip of the cone. We then define the min-max value

$$m = \inf_{f \in F} \sup_{\sigma \in \mathcal{C}} \mathcal{E}(f(\sigma)),$$

where

$$F = \big\{ f : \mathcal{C} \to H^m(M) \text{ continuous } : f(\sigma) = \varphi_{\lambda,\sigma},\ \forall \sigma \in \partial\mathcal{C} = M_k^{\mathrm{R}} \big\},$$

which is non-empty since $t\Phi \in F$. Still by Proposition 4.6, we can take $\lambda \gg 1$ sufficiently large such that

$$\sup_{\sigma \in \partial\mathcal{C}} \mathcal{E}(f(\sigma)) = \sup_{\sigma \in M_k^{\mathrm{R}}} \mathcal{E}(\varphi_{\lambda,\sigma}) \le -2L.$$

On the other hand, we claim that

$$m \ge -L.$$

To prove it, we just need to observe that $\partial\mathcal{C} = M_k^{\mathrm{R}}$ is contractible in $\mathcal{C}$ (by construction of the cone), and thus $\Phi(M_k^{\mathrm{R}})$ is contractible in $f(\mathcal{C})$ for any $f \in F$. Hence, we deduce by (4.5) that $f(\mathcal{C})$ cannot be contained in $\mathcal{E}^{-L}$, which proves the claim.

We conclude that the functional $\mathcal{E}$ has a min-max geometry at the level $m$ which in turn implies there exists a Palais–Smale sequence at this level.

*Step* 3. Since the Palais–Smale condition is not available in this framework, we cannot directly pass to the limit to obtain a critical point. To overcome this problem, we use the so-called monotonicity trick jointly with the compactness property in Theorem 1.3. This argument has been first introduced by Struwe in [48], and has been then applied by many authors, see for example [23, 44]. Therefore, we omit the details and just sketch the main ideas.

One considers a small perturbation $\mathcal{E}_\varepsilon$ of the functional so that the above min-max scheme applies uniformly. By using a monotonicity property of the perturbed min-max values $m_\varepsilon$, it is possible to obtain a bounded Palais–Smale sequence which then converges to a solution of the perturbed problem. We then pass to the limit as $\varepsilon \to 0$ by using the compactness property in Theorem 1.3 to eventually recover a solution of the original problem. This concludes the proof. ∎

Finally, we present the proof of the multiplicity result in Theorem 1.2.

*Proof of Theorem* 1.2. Once the above analysis (needed to prove the existence result) is carried out, the multiplicity result is essentially a straightforward application of the Morse inequalities. Thus, we will be sketchy and refer for example to [4, 22] for further details. Recall also that $\mathcal{E}$ is assumed to be a Morse functional. The (weak) Morse inequalities would assert that

$$\begin{aligned}\#\{\text{solutions of (1.6)}\} &\geq \sum_{q\geq 0} \#\{\text{critical points of } \mathcal{E} \text{ in } \{-L \leq \mathcal{E} \leq L\} \text{ with index } q\} \\ &\geq \sum_{q\geq 0} \dim H_q(\mathcal{E}^L, \mathcal{E}^{-L}),\end{aligned}$$

where $H_q(\mathcal{E}^L, \mathcal{E}^{-L})$ stands for the relative homology group of $(\mathcal{E}^L, \mathcal{E}^{-L})$, see, for example, Theorem 2.4 in [22]. Now, it is known that the high sublevels $\mathcal{E}^L$ are contractible. Roughly speaking, one can take $L \gg 1$ sufficiently large so that there are no critical points above the level $L$ which then allows to construct a deformation retract of $\mathcal{E}^L$ onto $H^m(M)$, which is of course contractible, see for example the argument in [35]. Then, by the long exact sequence of the relative homology, it easily follows that

$$H_q(\mathcal{E}^L, \mathcal{E}^{-L}) \cong \widetilde{H}_q(\mathcal{E}^{-L}).$$

But we already now from (4.4) that

$$H_*(M_k^{\mathrm{R}}) \hookrightarrow H_*(\mathcal{E}^{-L}) \text{ injectively,}$$

thus

$$\dim \widetilde{H}_q(\mathcal{E}^{-L}) \geq \dim \widetilde{H}_q(M_k^{\mathrm{R}})$$

and we are done. ∎

**Remark 4.1.** As already pointed out, for simplicity, all the argument has been carried out in the case $P_g^{2m} \geq 0$. In general, one needs to modify the Adams–Trudinger–Moser inequality and its improvements by adding a bound to the component $u_-$ of the function $u$ lying in the direct sum of the negative eigenspaces of $P_g^{2m}$. As a consequence, in the low subleveles, $\mathcal{E}^{-L}$ either the function $u$ concentrates or $u_-$ tends to infinity, or both alternative can hold. To express this alternative, one can use the topological join (see [43]) between $M_k^{\mathrm{R}}$ and a set representing the negative eigenvalues of $P_g^{2m}$. We refer the interested reader to [23].

# A. Appendix: Pohozaev identity

Here we state a Pohozaev-type identity which is used in the blow-up argument.

**Lemma A.1.** *Let $B_r(0)$ be a ball in $\mathbb{R}^{2m}$. We have the following identities:*

(a) *If $m = 2m_0,\ m_0 \geq 1$, then*

$$
\begin{aligned}
&\int_{B_r(0)} x \cdot \nabla f (-\Delta)^{2m_0} f dx \\
&= \sum_{i=2}^{m_0} \int_{\partial B_r(0)} 2(i-1)\Big((-\Delta)^{m-i} f \, \frac{\partial(-\Delta)^{i-1} f}{\partial \nu} - \frac{\partial(-\Delta)^{m-i} f}{\partial \nu} (-\Delta)^{i-1} f\Big)\, ds \\
&+ \sum_{i=1}^{m_0} \int_{\partial B_r(0)} (-\Delta)^{m-i} f \, \partial_\nu \langle x, \nabla(-\Delta)^{i-1} f\rangle \, ds \\
&- \sum_{i=1}^{m_0} \int_{\partial B_r(0)} \langle x, \nabla(-\Delta)^{i-1} f\rangle \, \frac{\partial(-\Delta)^{m-i} f}{\partial \nu}\, ds \\
&+ \int_{\partial B_r(0)} \frac{1}{2}\, |x| \, ((-\Delta)^{m_0} f)^2 \, ds.
\end{aligned} \tag{A.1}
$$

(b) *If $m = 2m_0 + 1,\ m_0 \geq 1$, then*

$$
\begin{aligned}
&\int_{B_r(0)} x \cdot \nabla f (-\Delta)^{2m_0+1} f \\
&= \sum_{i=2}^{m_0} \int_{\partial B_r(0)} 2(i-1)\Big((-\Delta)^{m-i} f \, \frac{\partial(-\Delta)^{i-1} f}{\partial \nu} - \frac{\partial(-\Delta)^{m-i} f}{\partial \nu} (-\Delta)^{i-1} f\Big)\, ds \\
&+ \sum_{i=1}^{m_0} \int_{\partial B_r(0)} (-\Delta)^{m-i} f \, \partial_\nu \langle x, \nabla(-\Delta)^{i-1} f\rangle \, ds \\
&- \sum_{i=1}^{m_0} \int_{\partial B_r(0)} \langle x, \nabla(-\Delta)^{i-1} f\rangle \, \frac{\partial(-\Delta)^{m-i} f}{\partial \nu}\, ds \\
&+ \int_{\partial B_r(0)} \frac{1}{2}\, |x| \, (\nabla(-\Delta)^{m_0} f)^2 \, ds - 2m_0 \int_{\partial B_r(0)} \partial_\nu (-\Delta)^{m_0} f (-\Delta)^{m_0} f \, ds \\
&- \int_{\partial B_r(0)} \partial_\nu (-\Delta)^{m_0} f \, \langle x, \nabla(-\Delta)^{m_0} f\rangle \, ds.
\end{aligned} \tag{A.2}
$$

*Here, $\nu$ is outward normal vector along the boundary $\partial B_r(0)$.*

*Proof.* The proof is based on the following identity:

$$(-\Delta)\langle x, \nabla(-\Delta)^i f\rangle = 2(-\Delta)^{i+1} f + \langle x, \nabla(-\Delta)^{i+1} f\rangle.$$

Using the second Green identity repeatedly, we can get above formula by straightforward computations. We omit the details. ∎

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**Aleks Jevnikar**
Department of Mathematics, Computer Science and Physics, University of Udine
Via delle Scienze 206, 33100 Udine, Italy;
aleks.jevnikar@uniud.it

**Yannick Sire**
Department of Mathematics, Johns Hopkins University
404 Krieger Hall, 3400 N. Charles Street, Baltimore, MD 21218, USA;
ysire1@jhu.edu

**Wen Yang**
Department of Mathematics, Faculty of Science and Technology, University of Macau
Macau, P. R. China;
wenyang@um.edu.mo