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Ahlfors regular conformal dimension and Gromov–Hausdorff convergence

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Abstract. We prove that the Ahlfors regular conformal dimension is upper semicontinuous with respect to Gromov–Hausdorff convergence when restricted to the class of uniformly perfect, uniformly quasi-selfsimilar metric spaces. Moreover, we show the continuity of the Ahlfors regular conformal dimension in case of limit sets of discrete, quasiconvex-cocompact group of isometries of uniformly bounded codiameter of δ-hyperbolic metric spaces under equivariant pointed Gromov–Hausdorff convergence of the spaces.

1. Introduction

The Ahlfors regular conformal gauge of a metric space (X, d) is the set $\mathcal{J}_{\mathrm{AR}}(X, d)$ of all metrics on X that are quasisymmetric equivalent to d and are Ahlfors regular. By definition, a homeomorphism $F\colon (X, d_X) \to (Y, d_Y)$ is a quasisymmetric equivalence if there exists a strictly increasing map $\eta\colon [0, +\infty) \to [0, +\infty)$ with $\eta(0) = 0$ such that

$$\frac{d_Y(F(x), F(x'))}{d_Y(F(x), F(x''))} \leq \eta\Big(\frac{d_X(x, x')}{d_X(x, x'')}\Big)$$

for every $x, x', x'' \in X$ with $d_X(x, x'') > 0$. The notion of quasisymmetric maps was introduced in [25], and it has played an important role in the study of quasiconformal structure on metric spaces. The *Ahlfors regular conformal dimension* of a metric space (X, d) is defined as

$$\mathrm{CD}(X, d) := \inf\{\mathrm{HD}(X, d') \text{ such that } d' \in \mathcal{J}_{\mathrm{AR}}(X, d)\}, \tag{1.1}$$

where HD denotes the Hausdorff dimension. In general, $\mathcal{J}_{\mathrm{AR}}(X, d)$ can be empty, implying $\mathrm{CD}(X, d) = +\infty$. On the other hand, the conformal dimension of every doubling, uniformly perfect metric space is always finite by Corollary 14.5 in [17]. There is a special class of metric spaces that are doubling and uniformly perfect: the class of perfect quasi-selfsimilar metric spaces (see Proposition 2.2).

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Definition 1.1. Let $\rho_0 > 0$ and $L_0 \geq 1$. A compact metric space (X, d) is said to be (L_0, ρ_0)-quasi-selfsimilar (shortly, (L_0, ρ_0)-q.s.s.) if for every open ball $B(x, \rho)$ in X with $0 < \rho \leq \rho_0$, there is a map $\Phi\colon (B(x, \rho), (\rho_0/\rho) \cdot d) \to X$ which is L_0-biLipschitz and is such that $\Phi(B(x, \rho)) \supseteq B(\Phi(x), \rho_0/L_0)$.

The notation $(\rho_0/\rho) \cdot d$ means the metric obtained by multiplying the original metric d by the positive number ρ_0/ρ. In other words, a metric space is (L_0, ρ_0)-quasi-selfsimilar if every ball of radius smaller than ρ_0 is, up to rescaling it to the right size, biLipschitz comparable to a ball of radius exactly ρ_0 of the same space.

This notion arises naturally in the study of limit sets of Gromov-hyperbolic groups and semi-hyperbolic rational fractions, see [1, 3, 16, 24]. Examples of spaces that are quasi-selfsimilar include Lipschitz manifolds with uniform Lipschitz constants and a positive lower bound on the injectivity radius, simplicial complexes with a metric of fixed constant curvature on each simplex and a lower bound on the injectivity radius, self-similar fractals, boundaries of cocompact Gromov-hyperbolic spaces, and Julia sets of semi-hyperbolic rational fractions. If one puts natural geometric constraints to each of the above classes, then it is possible to quantify the quasi-selfsimilarity constants in terms of the constraints.

For perfect quasi-selfsimilar spaces that are connected and locally connected, the Ahlfors regular conformal dimension $\mathrm{CD}(X, d)$ can be equivalently computed as the infimum of the Hausdorff dimension of all metrics d' that are quasisymmetric to d, but not necessarily Ahlfors regular. This is proved in Theorem 1.6 of [15].

The aim of this paper is to study the behaviour of the Ahlfors regular conformal dimension on quasi-selfsimilar metric spaces under Gromov–Hausdorff convergence, whose definition will be recalled in Section 5.

Theorem A. *Let (X_n, d_n) be a sequence of compact, a_0-uniformly perfect, (L_0, ρ_0)-q.s.s. spaces. Suppose it converges in the Gromov–Hausdorff sense to (X_∞, d_∞). Then* $\mathrm{CD}(X_\infty, d_\infty) \geq \limsup_{n\to+\infty} \mathrm{CD}(X_n, d_n)$.

For quasi-selfsimilar spaces, uniform perfectness is quantitatively equivalent to a uniform lower bound of the diameter of the balls $B(\Phi(x), \rho_0/L_0)$ appearing in Definition 1.1, see Proposition 2.2.

Theorem A is false if the spaces are not (L_0, ρ_0)-q.s.s.: the sequence $X_n = [0, 1/n] \subseteq \mathbb{R}$ converges in the Gromov–Hausdorff sense to $X_\infty = \{0\}$, but $\mathrm{CD}(X_n, d_E) = 1$ for every n, while $\mathrm{CD}(X_\infty, d_E) = 0$. Here, d_E is the standard Euclidean metric. Moreover, the upper semicontinuity in Theorem A cannot be improved to continuity in general.

Example 1.2. Let X_n be the set built in this way: we start with $[0, 1]$, and we remove the central segment of length $1/(2n + 1)$. We do the same for each of the two remaining parts. We continue this procedure infinitely many times and we call X_n the resulting metric space endowed with the Euclidean metric d_E. For instance, X_1 is the standard Cantor set. The sequence X_n is made of compact, uniformly perfect, quasi-selfsymilar spaces with uniform constants. It converges to $X_\infty = [0, 1]$ in the Gromov–Hausdorff sense. However, $\mathrm{CD}(X_n, d_E) = 0$ for every n by Proposition 15.11 in [13] (see also Theorem 2.16 in [5]), while $\mathrm{CD}(X_\infty, d_E) = 1$.

On the other hand, we have continuity in a particular setting. In [8], the author studied the class $\mathcal{M}(\delta, D)$ of triples (X, x, Γ), where X is a proper δ-hyperbolic metric space, Γ is a discrete, non-elementary, quasiconvex-cocompact, torsion-free group of isometries of X with codiameter bounded above by D, and x belongs to the quasiconvex hull of the limit set $\Lambda(\Gamma)$. We refer to Section 6 for more details about these terms. One of the main results of [8] is the closure of $\mathcal{M}(\delta, D)$ under equivariant pointed Gromov–Hausdorff convergence. We refer to Section 6 for the precise definition; let us just mention here that, if we denote by $B_\Gamma(x, r)$ the subset of elements $\gamma \in \Gamma$ moving $x \in X$ less than r, saying that a sequence of groups (Γ_j, X_j) converges towards a limit action $(\Gamma_\infty, X_\infty)$ simply means that there exist Gromov–Hausdorff ε-approximations $f_\varepsilon\colon B_{X_j}(x_j, 1/\varepsilon) \to B_{X_\infty}(x_\infty, 1/\varepsilon)$ between larger and larger balls of X_j and X_∞ centered at basepoints x_j and x_∞, which are ε-equivariant with respect to maps $\phi_\varepsilon\colon B_{\Gamma_j}(x_j, 1/\varepsilon) \to B_{\Gamma_\infty}(x_\infty, 1/\varepsilon)$ (that is, with an equivariancy error smaller than ε), for $\varepsilon \to 0$.

Under this convergence, it is possible to prove that the limit sets $\Lambda(\Gamma_n)$ and $\Lambda(\Gamma_\infty)$ are quasisymmetric equivalent for n big enough (see Section 6.1). As a consequence, we get the following.

Theorem B. *Let $(X_n, x_n, \Gamma_n) \subseteq \mathcal{M}(\delta, D)$ be a sequence of triples converging in the equivariant pointed Gromov–Hausdorff sense to $(X_\infty, x_\infty, \Gamma_\infty)$. Then, for suitable metrics, the sequence $\Lambda(\Gamma_n)$ is uniformly perfect and uniformly q.s.s., and converges in the Gromov–Hausdorff sense to $\Lambda(\Gamma_\infty)$. Moreover,* $\mathrm{CD}(\Lambda(\Gamma_\infty)) = \mathrm{CD}(\Lambda(\Gamma_n))$ *for n big enough.*

Here the Ahlfors regular conformal dimension is computed with respect to any visual metric on the limit sets, see Proposition 6.3 and the discussion below. Motivated by Theorem B, we propose the following question.

Question 1.3. *Are there conditions on the metric spaces X_n that ensure continuity of the conformal dimension under Gromov–Hausdorff convergence?*

It is useful to consider the following example (the author thanks M. Murugan for bringing the reference [26] to his attention).

Example 1.4. Let us repeat the construction of Example 1.2 in dimension 2. For every n, we define X_n in the following way: we start with $[0, 1]^2$, we divide it into $1/(2n + 1)^2$ squares and we delete the central one. Now we do the same for every remaining squares. We repeat the procedure infinitely many times. We endow X_n with the Euclidean metric d_E. For instance, the space X_1 is the standard Sierpiński carpet. The sequence (X_n, d_E) converges in the Gromov–Hausdorff sense to $X_\infty = [0, 1]^2$, whose Ahlfors regular conformal dimension is 2. In this case, we have $\lim_{n\to+\infty} \mathrm{CD}(X_n, d_E) = 2$ by a well-known argument (see, for instance, Theorem 3.4 and Example 3.2 in [26]).

We will briefly discuss Question 1.3 and the example above at the end of Section 5.

2. Preliminaries

We denote a metric space by (X, d). The open (respectively, closed) ball of center $x \in X$ and radius $\rho > 0$ is denoted by $B(x, \rho)$ (respectively, $\overline{B}(x, \rho)$). Given $r > 0$ and $Y \subseteq X$, we say that a subset S of Y is r-separated if $d(x, y) > r$ for all $x, y \in S$, while a subset N

of Y is a r-net if for all $y \in Y$ there exists $x \in N$ such that $d(x, y) \leq r$. It is straightforward from the definitions that a maximal r-separated subset of Y is a r-net.

A metric space (X, d) is said D-doubling if the cardinality of any $(\rho/2)$-separated subset inside any ball of radius ρ is at most D.

A metric space (X, d) is perfect if it has no isolated points, while it is said a-uniformly perfect, $0 < a < 1$, if $\overline{B}(x, \rho) \setminus B(x, a \cdot \rho) \neq \emptyset$ for all $x \in X$ and $0 \leq \rho < \mathrm{Diam}(X)$, where Diam denotes the diameter. Clearly, every uniformly perfect metric space is perfect.

Gromov–Hausdorff convergence will be considered in the class of compact metric spaces. A pointed and equivariant version will be defined in Section 6. Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $\varepsilon > 0$. A ε-approximation from X to Y is a function $f: X \to Y$ such that

- $|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| < \varepsilon$ for all $x_1, x_2 \in X$;
- for all $y \in Y$, there exists $x \in X$ such that $d_Y(f(x), y) < \varepsilon$.

A sequence of compact spaces (X_n, d_n) *converges in the Gromov–Hausdorff sense* to the compact space (X_∞, d_∞) if for every $\varepsilon > 0$, there exists $n_\varepsilon \geq 0$ such that if $n \geq n_\varepsilon$, then there is a ε-approximation $f_n: X_n \to X_\infty$. In this case, we use the notation

$$(X_n, d_n) \underset{\mathrm{GH}}{\longrightarrow} (X_\infty, d_\infty).$$

2.1. Ahlfors regular spaces

A metric space (X, d) is said (A, s)-Ahlfors regular, for given $A, s \geq 0$, if there is a measure μ on X satisfying

$$\frac{1}{A} \cdot \rho^s \leq \mu(B(x, \rho)) \leq A \cdot \rho^s$$

for all $x \in X$ and all $0 < \rho \leq \mathrm{Diam}(X)$. The following lemma is classical.

Lemma 2.1. *Let $A_0 \geq 1$ and $s_0 > 0$. Then there exists $0 < a_0 = a_0(A_0, s_0) < 1$ such that every (A, s)-Ahlfors regular metric space (X, d) with $A \leq A_0$ and $s \geq s_0$ is a_0-uniformly perfect.*

Proof. We claim that (X, d) is a-uniformly perfect for all $a < A^{-2/s}$. Indeed, for every such a, for every $x \in X$ and every $0 < \rho \leq \mathrm{Diam}(X)$, we have

$$\mu(\overline{B}(x, \rho)) \geq \frac{1}{A}\rho^s \quad \text{and} \quad \mu(B(x, a \cdot \rho)) \leq A \cdot a^s \rho^s < \frac{1}{A} \cdot \rho^s.$$

Hence $\mu(\overline{B}(x, \rho) \setminus B(x, a \cdot \rho)) > 0$, and in particular, this set is not empty. It is then clear we can choose every $a_0 < A_0^{-2/s_0}$. ■

2.2. Quasi-selfsimilar spaces

We collect now some basic properties of the quasi-selfsimilar metric spaces from Definition 1.1. We say a quasi-selfsimilar metric space (X, d) has diameters bounded below by some $c_0 > 0$ if the ball $B(\Phi(x), \rho_0/L_0)$ that appears in Definition 1.1 has diameter $\geq c_0$ for every $x \in X$ and every $0 < \rho \leq \rho_0$.

Proposition 2.2 (Compare with Lemma 2.2 and Proposition 2.3 in [5]). *Let (X,d) be a quasi-selfsimilar metric space as in Definition 1.1. Then*

(i) *it is doubling*;

(ii) *if it is perfect, then it is uniformly perfect*;

(iii) *it is uniformly perfect if and only if it has diameters bounded below, quantitatively in terms of the relative constants and the diameter of X.*

Proof. If (i) is not true, then for every $n \in \mathbb{N}$, there exist $x_n \in X$ and $\rho_n > 0$ such that there is a $(\rho_n/2)$-separated set inside $B(x_n,\rho_n)$ of cardinality $\geq n$. Up to passing to a subsequence, we can suppose that $\lim_{n\to+\infty}\rho_n = \rho_\infty \in [0,+\infty)$ and that x_n converges to $x_\infty \in X$. If $\rho_\infty > 0$, then X is not totally bounded, a contradiction. If $\rho_\infty = 0$, we can find L_0-biLipschitz maps $\Phi_n\colon (B(x_n,\rho_n),(\rho_0/\rho_n)\cdot d) \to X$. Hence there exists a $(\rho_0/2L_0)$-separated set inside $B(\Phi_n(x_n), L_0\rho_0)$ with cardinality $\geq n$. Once again, this contradicts the compactness of X.

Now we show (ii). Since X is perfect and compact, we have the following property, as in Lemma 2.2 of [5]: for all $\rho > 0$, there exists $d(\rho) > 0$ such that $\mathrm{Diam}(B(x,\rho)) \geq d(\rho)$ for every $x \in X$. Suppose now X is not uniformly perfect: then for all $n \in \mathbb{N}$, there exist $x_n \in X$ and $0 < \rho_n \leq \mathrm{Diam}(X)$ such that $\overline{B}(x_n,\rho_n)\setminus B(x_n,\rho_n/n) = \emptyset$. Up to taking a subsequence, we can suppose that x_n converges to x_∞ and ρ_n converges to ρ_∞. Suppose first $\rho_\infty > 0$. Let y be a point inside $B(x_\infty,\rho_\infty)$. It also belongs to $B(x_n,\rho_n)$ for n big enough, and so it belongs to $B(x_n,\rho_n/n)$. In other words, $d(x_\infty,y) \leq d(x_\infty,x_n) + \rho_n/n$ for every n big enough, i.e., $d(x_\infty,y) = 0$ and x_∞ is an isolated point. This shows that X is not perfect, a contradiction. Suppose now $\rho_\infty = 0$. For all n big enough, we take the map $\Phi_n\colon B(x_n,\rho_n) \to X$ given by Definition 1.1. Therefore

$$B\Big(\Phi_n(x_n),\frac{\rho_0}{L_0}\Big) \subseteq \Phi_n(B(x_n,\rho_n)) = \Phi_n\Big(B\Big(x_n,\frac{\rho_n}{n}\Big)\Big).$$

From one side we have $\mathrm{Diam}(B(\Phi(x_n),\rho_0/L_0)) \geq d(\rho_0/L_0) > 0$ for every n. On the other hand,

$$\mathrm{Diam}\Big(\Phi_n\Big(B\Big(x_n,\frac{\rho_n}{n}\Big)\Big)\Big) \leq L_0\cdot\frac{\rho_0}{\rho_n}\cdot\frac{2\rho_n}{n} \underset{n\to+\infty}{\longrightarrow} 0,$$

which is a contradiction.

Finally, we prove (iii). Let us suppose X has diameters bounded below by $c_0 > 0$. We fix $x \in X$ and $0 < \rho \leq \rho_0$. We take the map $\Phi\colon B(x,\rho) \to X$ given by the definition of quasi-selfsimilarity. Since $\Phi(B(x,\rho))$ contains a set with diameter $\geq c_0$, then there exists $y \in B(x,\rho)$ such that $d(\Phi(x),\Phi(y)) \geq c_0/2$. Therefore

$$d(x,y) \geq \frac{1}{L_0}\cdot\frac{\rho}{\rho_0}\cdot\frac{c_0}{2} = a(L_0,\rho_0,c_0)\cdot\rho,$$

with $0 < a(L_0,\rho_0,c_0) =: a < 1$. If ρ is bigger than ρ_0, then we apply what said above to ρ_0 finding $B(x,\rho_0)\setminus B(x,a\cdot\rho_0) \neq \emptyset$, so $B(x,\rho)\setminus B(x,\frac{a\cdot\rho_0}{\rho}\cdot\rho) \neq \emptyset$. Since X is compact, we have

$$\frac{a\cdot\rho_0}{\rho} \geq \frac{a\cdot\rho_0}{\mathrm{Diam}(X)} =: a_0 > 0,$$

showing that X is a_0-uniformly perfect and that a_0 depends only on L_0, ρ_0, c_0, and the diameter of X. Vice versa, if X is a_0-uniformly perfect, then $\overline{B}(\Phi(x), \rho_0/L_0) \setminus B(\Phi(x), a_0 \cdot \rho_0/L_0) \neq \emptyset$ for all $x \in X$ and Φ as in Definition 1.1. Therefore,

$$\mathrm{Diam}\Big(B\Big(\Phi(x), \frac{\rho_0}{L_0}\Big)\Big) \geq a_0 \cdot \frac{\rho_0}{L_0} =: c_0. \qquad \blacksquare$$

We remark that the diameter bounded below condition is part of the definition of quasi-selfsimilar spaces in [20] and in [5]. However, the definition in [20] differs from the one in Definition 1.1 from the fact that $\Phi(B(x,\rho))$ is required to contain an open set of diameter bounded from below, but which is not necessarily a ball. When an upper bound on the diameter of the metric space is fixed, the diameter bounded below condition is equivalent to bounded uniform perfectness of the metric space by Proposition 2.2. For instance, in the context of Theorem A, there is a uniform upper bound on the diameter of the spaces X_n, since they are converging in the Gromov–Hausdorff sense to the compact space X_∞, so the spaces X_n have all diameter bounded below by some $c_0 > 0$ if and only if they are all a_0-uniformly perfect for some $0 < a_0 < 1$.

3. Combinatorial modulus

It is known that the conformal dimension of a metric space is closely related to the combinatorial modulus, see for instance [3, 6, 22] and the references therein. In this section, we recall the definition of combinatorial modulus and we prove some technical lemmas.

From now on, we fix a D-doubling metric space (X, d). For every $k \in \mathbb{N}$, we choose a finite 10^{-k}-net X_k of X. To simplify notation, given a real number $\lambda > 0$ and $k \in \mathbb{N}$, we will denote by $B_{\lambda,k}(x)$ the open ball of center x and radius $\lambda \cdot 10^{-k}$, namely $B(x, \lambda \cdot 10^{-k})$. The same convention holds for closed balls.

A (λ, k)-path is a finite collection $\gamma = \{q_j\}_{j=0}^M$ of elements of X_k satisfying $\overline{B}_{\lambda,k}(q_j) \cap \overline{B}_{\lambda,k}(q_{j+1}) \neq \emptyset$ for all $j = 0, \dots, M-1$. The points q_0 and q_M are called, respectively, the starting and the ending point of the path.

Given two subsets $E, F \subseteq X$, we denote by $P_{\lambda,k}(E,F)$ the set of (λ,k)-paths with starting point in E and ending point in F. We denote by $\mathcal{A}_{\lambda,k}(E,F)$ the set of admissible functions, i.e., functions $f\colon X_k \to [0,+\infty)$ such that $\sum_{i=0}^M f(q_i) \geq 1$ for every $\{q_i\}_{i=0}^M \in P_{\lambda,k}(E,F)$.

Given a real number $p \geq 0$, we define

$$p\text{-}\mathrm{Mod}_{\lambda,k}(E,F) = \inf_{f \in \mathcal{A}_{\lambda,k}(E,F)} \sum_{q \in X_k} f(q)^p,$$

and we call it the p-modulus of the couple (E,F) at level (λ,k). The infimum is actually realized: any admissible function realizing the minimum is said optimal. If there are no (λ,k)-paths joining E and F, we set $p\text{-}\mathrm{Mod}_{\lambda,k}(E,F) = 0$.

Lemma 3.1. *If $E' \subseteq E$ and $F' \subseteq F$, then $p\text{-}\mathrm{Mod}_{\lambda,k}(E',F') \leq p\text{-}\mathrm{Mod}_{\lambda,k}(E,F)$.*

Proof. If $P_{\lambda,k}(E',F') = \emptyset$, then the result is trivial by definition. Otherwise, every path in $P_{\lambda,k}(E',F')$ belongs to $P_{\lambda,k}(E,F)$. This implies that $\mathcal{A}_{\lambda,k}(E,F) \subseteq \mathcal{A}_{\lambda,k}(E',F')$, and the result follows from the definition. $\blacksquare$

Let $1 \le L_1 < L_2$ be two real numbers. For every $i \in \mathbb{N}$ and every point $y \in X_i$, we set

$$p\text{-Mod}_{\lambda,k}^{L_1,L_2}(y) := p\text{-Mod}_{\lambda,i+k}(\overline{B}_{L_1,i}(y), X \setminus B_{L_2,i}(y)).$$

We remark that this is a modulus at level $(\lambda, i+k)$. Finally, we define

$$p\text{-Mod}_{\lambda,k}^{L_1,L_2}(X) = \sup_{i\in\mathbb{N}} \sup_{y\in X_i} p\text{-Mod}_{\lambda,k}^{L_1,L_2}(y).$$

We want to control how this quantity changes when L_1, L_2 and λ change. We recall that D denotes the doubling constant of X.

Lemma 3.2 (Lemma 4.4 in [5]). *Let k, λ and p be fixed quantities as above. Let $1 \le L_1' \le L_1 < L_2 \le L_2'$. Then there exist $\ell \in \mathbb{N}$ and $C > 0$, depending only on L_1, L_1', L_2, L_2' and D, such that*

$$p\text{-Mod}_{\lambda,k}^{L_1',L_2'}(X) \le p\text{-Mod}_{\lambda,k}^{L_1,L_2}(X)$$

and

$$p\text{-Mod}_{\lambda,k+\ell}^{L_1,L_2}(X) \le C \cdot p\text{-Mod}_{\lambda,k}^{L_1',L_2'}(X).$$

Proof. For every $y \in X_i$, $i \in \mathbb{N}$, we have that $\overline{B}_{L_1',i}(y) \subseteq \overline{B}_{L_1,i}(y)$ and $X \setminus B_{L_2',i}(y) \subseteq X \setminus B_{L_2,i}(y)$, so the first inequality follows by Lemma 3.1.

In order to prove the second inequality, we define ℓ as the minimum integer satisfying $10^{-\ell} \le (L_2 - L_1)/(L_2' + L_1')$. We fix $y \in X_i$ for some $i \in \mathbb{N}$, and we consider the set

$$X_{i+\ell}(y) = \{z \in X_{i+\ell} \text{ such that } B_{L_1',i+\ell}(z) \cap \overline{B}_{L_1,i}(y) \neq \emptyset\}.$$

We fix any $(\lambda, i+\ell+k)$-path $\gamma = \{q_j\}_{j=0}^M$ joining $\overline{B}_{L_1,i}(y)$ and $X \setminus B_{L_2,i}(y)$. This means in particular that $d(y, q_0) \le L_1 \cdot 10^{-i}$ and $d(y, q_M) \ge L_2 \cdot 10^{-i}$. We can find $z \in X_{i+\ell}$ such that $d(z, q_0) \le 10^{-i-\ell}$, so by definition, $z \in X_{i+\ell}(y)$. We claim that the $(\lambda, i+\ell+k)$-path γ joins $\overline{B}_{L_1',i+\ell}(z)$ and $X \setminus B_{L_2',i+\ell}(z)$. Indeed, we know that $d(z, q_0) \le 10^{-i-\ell} \le L_1' \cdot 10^{-i-\ell}$. Moreover, $d(z, y) \le L_1 \cdot 10^{-i} + L_1' \cdot 10^{-i-\ell}$. Therefore,

$$d(z, q_N) \ge L_2 \cdot 10^{-i} - L_1 \cdot 10^{-i} - L_1' \cdot 10^{-i-\ell} \ge L_2' \cdot 10^{-i-\ell}$$

by the choice of ℓ. This means that any path $\gamma \in P_{\lambda,i+\ell+k}(\overline{B}_{L_1,i}(y), X \setminus B_{L_2,i}(y))$ belongs to $P_{\lambda,i+\ell+k}(\overline{B}_{L_1',i+\ell}(z), X \setminus B_{L_2',i+\ell}(z))$ for some $z \in X_{i+\ell}(y)$.

For each $z \in X_{i+\ell}(y)$, we take optimal functions $f_z \in \mathcal{A}_{\lambda,i+\ell+k}(\overline{B}_{L_1',i+\ell}(z), X \setminus B_{L_2',i+\ell}(z))$, and we define the function $f\colon X_{i+\ell+k} \to [0, +\infty)$ as

$$f(q) = \max_{z\in X_{i+\ell}(y)} f_z(q).$$

We claim that $f \in \mathcal{A}_{\lambda,i+\ell+k}(\overline{B}_{L_1,i}(y), X \setminus B_{L_2,i}(y))$. Indeed, every path $\{q_j\}_{j=0}^M \in P_{\lambda,i+\ell+k}(\overline{B}_{L_1,i}(y), X \setminus B_{L_2,i}(y))$ belongs to $P_{\lambda,i+\ell+k}(B_{L_1',i+\ell}(z), X \setminus B_{L_2',i+\ell}(z))$ for some $z \in X_{i+\ell}(y)$, therefore

$$\sum_{j=0}^M f(q_j) \ge \sum_{j=0}^M f_z(q_j) \ge 1.$$

Finally, we have

$$\sum_{q\in X_{i+\ell+k}} f(q)^p = \sum_{q\in X_{i+\ell+k}} \max_{z\in X_{i+\ell}(y)} f_z(q)^p \leq \sum_{z\in X_{i+\ell}(y)} \sum_{q\in X_{i+\ell+k}} f_z(q)^p$$
$$= \sum_{z\in X_{i+\ell}(y)} p\text{-Mod}_{\lambda,k}^{L_1',L_2'}(z) \leq C\cdot p\text{-Mod}_{\lambda,k}^{L_1',L_2'}(X),$$

where C is a constant depending only on the doubling constant D, on ℓ and on L_1'. This shows that

$$p\text{-Mod}_{\lambda,k+\ell}^{L_1,L_2}(y) \leq C\cdot p\text{-Mod}_{\lambda,k}^{L_1',L_2'}(X).$$

Since this is true for every $y\in X_i$ and for every i, we get

$$p\text{-Mod}_{\lambda,k+\ell}^{L_1,L_2}(X) \leq C\cdot p\text{-Mod}_{\lambda,k}^{L_1',L_2'}(X).$$ ∎

Lemma 3.3. *Let $k\in\mathbb{N}$, $p\geq 0$, $1\leq L_1<L_2$ and $2<\lambda\leq\lambda'$. Then there exist $\ell\in\mathbb{N}$ and $C>0$, depending only on λ, λ' and D, such that*

$$p\text{-Mod}_{\lambda,k}^{L_1,L_2}(X) \leq p\text{-Mod}_{\lambda',k}^{L_1,L_2}(X)$$

and

$$p\text{-Mod}_{\lambda',k+\ell}^{L_1,L_2}(X) \leq C\cdot p\text{-Mod}_{\lambda,k}^{L_1,L_2}(X)$$

for all $k>k_0=\log_{10}(\frac{2}{L_2-L_1})$.

Proof. For every $y\in X_i$, $i\in\mathbb{N}$, we have

$$P_{\lambda,k}(\overline{B}_{L_1,i}(y), X\setminus B_{L_2,i}(y)) \subseteq P_{\lambda',k}(\overline{B}_{L_1,i}(y), X\setminus B_{L_2,i}(y)).$$

Therefore, arguing as in the proof of Lemma 3.1, we get

$$p\text{-Mod}_{\lambda,k}^{L_1,L_2}(y) \leq p\text{-Mod}_{\lambda',k}^{L_1,L_2}(y).$$

Taking the supremum on $i\in\mathbb{N}$ and $y\in X_i$, we obtain the first inequality.

In order to show the second inequality, we define ℓ as the smallest integer such that $\lambda'\cdot 10^{-\ell}\leq\lambda/2-1$. It is well defined since $\lambda>2$. We restrict the attention to the integers k bigger than k_0, so that $10^{-k}<(L_2-L_1)/2$. We fix $y\in X_i$, $i\in\mathbb{N}$, and a $(\lambda', i+k+\ell)$-path $\gamma=\{q_j\}_{j=0}^M$ joining $\overline{B}_{L_1,i}(y)$ to $X\setminus B_{L_2,i}(y)$. For every $j=0,\ldots,M$, we take a point $\tilde{q}_j\in X_{i+k}$ such that $d(q_j,\tilde{q}_j)\leq 10^{-i-k}$. We claim $\tilde{\gamma}=\{\tilde{q}_j\}_{j=0}^M$ is a $(\lambda,i+k)$-path joining $\overline{B}_{L_1',i}(y)$ to $X\setminus B_{L_2',i}(y)$, where $L_1'=L_1+10^{-k}$ and $L_2'=L_2-10^{-k}$. Indeed, we have

$$d(y,\tilde{q}_0)\leq d(y,q_0)+d(q_0,\tilde{q}_0)\leq L_1\cdot 10^{-i}+10^{-i-k}=L_1'\cdot 10^{-i},$$
$$d(y,\tilde{q}_M)\geq d(y,q_M)-d(q_M,\tilde{q}_M)\geq L_2\cdot 10^{-i}-10^{-i-k}=L_2'\cdot 10^{-i},$$

and

$$d(\tilde{q}_j,\tilde{q}_{j+1})\leq d(\tilde{q}_j,q_j)+d(q_j,q_{j+1})+d(q_{j+1},\tilde{q}_{j+1})$$
$$\leq 2\cdot 10^{-i-k}+2\lambda'\cdot 10^{-i-k-\ell}\leq 2\cdot 10^{-i-k}+2\Big(\frac{\lambda}{2}-1\Big)\cdot 10^{-i-k}=\lambda\cdot 10^{-i-k}$$

for every $j=0,\ldots,M-1$. Observe that the condition on k implies $L_1'<L_2'$.

We are ready to compare the combinatorial moduli. We take an optimal function $\tilde{f} \in \mathcal{A}_{\lambda,i+k}(\overline{B}_{L'_1,i}(y), X \setminus B_{L'_2,i}(y))$, and we define the function $f\colon X_{i+k+\ell} \to [0,+\infty)$ by

$$f(q) := \max\{\tilde{f}(\tilde{q}) \text{ such that } \tilde{q} \in X_{i+k} \text{ and } d(q,\tilde{q}) \leq 10^{-i-k}\}.$$

First of all, we show that $f \in \mathcal{A}_{\lambda',i+k+\ell}(\overline{B}_{L_1,i}(y), X \setminus B_{L_2,i}(y))$. Indeed, we have seen that given any $(\lambda', i+k+\ell)$-path $\{q_j\}_{j=0}^M$ joining $\overline{B}_{L_1,i}(y)$ to $X \setminus B_{L_2,i}(y)$, there is an associated $(\lambda, i+k)$-path $\{\tilde{q}_j\}_{j=0}^M$ joining $\overline{B}_{L'_1,i}(y)$ to $X \setminus B_{L'_2,i}(y)$ such that $d(q_j,\tilde{q}_j) \leq 10^{-i-k}$ for every $j = 0,\ldots,M$. Therefore, by definition of f, we have

$$\sum_{j=0}^{M} f(q_j) \geq \sum_{j=0}^{M} \tilde{f}(\tilde{q}_j) \geq 1.$$

Finally, we observe that

$$\begin{aligned} p\text{-Mod}_{\lambda',k+\ell}^{L_1,L_2}(y) &\leq \sum_{q \in X_{i+k+\ell}} f^p(q) \leq C' \cdot \sum_{\tilde{q} \in X_{i+k}} \tilde{f}^p(\tilde{q}) \\ &= C' \cdot p\text{-Mod}_{\lambda,k}^{L'_1,L'_2}(y) \leq C' \cdot p\text{-Mod}_{\lambda,k}^{L'_1,L'_2}(X), \end{aligned}$$

where C' is a constant depending only on D and ℓ. Since this is true for every $y \in X_i$ and for every $i \in \mathbb{N}$, we conclude that

$$p\text{-Mod}_{\lambda',k+\ell}^{L_1,L_2}(X) \leq C' \cdot p\text{-Mod}_{\lambda,k}^{L'_1,L'_2}(X).$$

This inequality is true for all $k \geq k_0$. Choosing $L''_1 = L_1 + 10^{-k_0}$ and $L''_2 = L_2 - 10^{-k_0}$, one concludes, using the easy inequality in Lemma 3.2, that

$$p\text{-Mod}_{\lambda',k+\ell}^{L_1,L_2}(X) \leq C' \cdot p\text{-Mod}_{\lambda,k}^{L''_1,L''_2}(X)$$

for all $k \geq k_0$. An application of the non-trivial inequality of Lemma 3.2 concludes the proof. ∎

In order to normalize the notation, from now on we choose, for technical reasons, $\lambda = 10$, $L_1 = 3$, $L_2 = 4$, and we set

$$p\text{-Mod}_k(y) := p\text{-Mod}_{10,i+k}(\overline{B}_{3,i}(y), X \setminus B_{4,i}(y))$$

for every $i \in \mathbb{N}$ and every point $y \in X_i$. In the same way, we put

$$p\text{-Mod}_k(X) = \sup_{i \in \mathbb{N}} \sup_{y \in X_i} p\text{-Mod}_k(y).$$

4. Combinatorial modulus on quasi-selfsimilar spaces

In this section, we consider the class of quasi-selfsimilar metric spaces as given in Definition 1.1. On these spaces, the computation of the combinatorial moduli is easier. Before that, we need an easy result.

Lemma 4.1. *If X is (L_0,ρ_0)-q.s.s., then it is (L_0,ρ_1)-q.s.s. for every* $0 < \rho_1 \leq \rho_0$.

Proof. We fix $x \in X$ and $0 < \rho \le \rho_1$. We apply the definition of (L_0, ρ_0)-quasi-self-similarity to the ball $B(x, \frac{\rho_0}{\rho_1}\rho)$: we can find a L_0-biLipschitz map

$$\Phi : \left(B\left(x, \frac{\rho_0}{\rho_1}\rho\right), \frac{\rho_1}{\rho} \cdot d\right) \to X$$

such that $\Phi(B(x, \frac{\rho_0}{\rho_1}\rho)) \supseteq B(\Phi(x), \rho_0/L_0)$. Then it is straightforward to see that the restriction of Φ to $(B(x, \rho), (\rho_1/\rho) \cdot d)$ is still L_0-biLipschitz. We now take a point $z \in B(\Phi(x), \rho_1/L_0)$. We know there exists a point $y \in B(x, \frac{\rho_0}{\rho_1}\rho)$ such that $z = \Phi(y)$. By the property of Φ, we get $d(x, y) < \rho$. This shows that $\Phi(B(x, \rho)) \supseteq B(\Phi(x), \rho_1/L_0)$. ■

Let X be a (L_0, ρ_0)-q.s.s. space.

We denote by i_0 the smallest integer such that $2(L_0 + 5)^2 \cdot 10^{-i_0} \le \rho_0$. We define $I_0 = \{i \in \mathbb{N}$ such that $(L_0 + 5) \cdot 10^{-i} \ge 10^{-i_0}\}$. Observe that the set I_0 is of the form $\{1, \dots, n_0\}$, where n_0 depends only on L_0 and ρ_0. We fix this value of n_0 for the rest of the section. We set

$$p\text{-Mod}_k(X, n_0) := \sup_{i \le n_0} \sup_{y \in X_i} p\text{-Mod}_k(y).$$

The following is the main result of the section: it allows to use *the fixed sizes up to* n_0 to estimate the combinatorial modulus. Since the explicit doubling constant of our metric space plays an important role, we sometimes add it in the definition: we say a metric space is (L_0, ρ_0, D_0)-q.s.s. if it is (L_0, ρ_0)-q.s.s. and D_0-doubling.

Proposition 4.2. *Let X be (L_0, ρ_0, D_0)-q.s.s. Then there exist a constant $C_0 \ge 1$ and an integer ℓ_0, both depending only on L_0 and D_0, such that*

$$p\text{-Mod}_{k+\ell_0}(X, n_0) \le p\text{-Mod}_{k+\ell_0}(X) \le C_0 \cdot p\text{-Mod}_k(X, n_0)$$

for every integer $k \ge 1$.

Proof. The first inequality is trivial, since we are doing a supremum among less elements. In order to show the second inequality, we fix $i \in \mathbb{N}$ and $y \in X_i$. Clearly, we can suppose that $i > n_0$. By Lemma 4.1, we know that X is also $(L_0, 2(L_0 + 5)^2 \cdot 10^{-i_0})$-q.s.s.. Since $2(L_0 + 5)^3 \cdot 10^{-i} < 2(L_0 + 5)^2 \cdot 10^{-i_0}$, then there is a L_0-biLipschitz map

$$\Phi\colon \left(B(y, 2(L_0 + 5)^3 \cdot 10^{-i}), \frac{10^{-i_0}}{(L_0 + 5) \cdot 10^{-i}} \cdot d\right) \to X.$$

We choose a point $x \in X_{i_0}$ such that $d(x, \Phi(y)) \le 10^{-i_0}$. We consider a $(10, i + k)$-path $\{q_j\}_{j=0}^M$ joining $\overline{B}_{1,i}(y)$ to $X \setminus B_{(L_0+5)^3,i}(y)$. This means that

- $q_0 \in \overline{B}(y, 10^{-i})$ and $q_M \notin B(y, (L_0 + 5)^3 \cdot 10^{-i})$;
- $\overline{B}(q_j, 10 \cdot 10^{-i-k}) \cap \overline{B}(q_{j+1}, 10 \cdot 10^{-i-k}) \neq \emptyset$ for every $j = 0, \dots, M - 1$.

Suppose that $q_j \in B_{2(L_0+5)^3,i}(y)$ for every $j = 0, \dots, M$. Then we can choose a point $\tilde{q}_j \in X_{i_0+k-1}$ such that $d(\tilde{q}_j, \Phi(q_j)) \le 10^{-i_0-k+1}$ for every $j = 0, \dots, M$. By the property of Φ, we get

$$d(\Phi(y), \Phi(q_0)) \le \frac{L_0}{L_0 + 5} \cdot 10^{-i_0} \quad \text{and} \quad d(\Phi(y), \Phi(q_M)) \ge \frac{(L_0 + 5)^2}{L_0} \cdot 10^{-i_0}.$$

Therefore we have

$$d(x,\tilde{q}_0) \le d(x,\Phi(y)) + d(\Phi(y),\Phi(q_0)) + d(\Phi(q_0),\tilde{q}_0) \le 3\cdot 10^{-i_0},$$
$$d(x,\tilde{q}_M) \ge d(\Phi(y),\Phi(q_M)) - d(x,\Phi(y)) - d(\Phi(q_M),\tilde{q}_M) \ge (L_0+3)10^{-i_0} \ge 4\cdot 10^{-i_0}.$$

Moreover, we know that $d(q_j,q_{j+1}) \le 20\cdot 10^{-i-k}$ for every $j=0,\dots,M-1$. Therefore we get

$$\begin{aligned} d(\tilde{q}_j,\tilde{q}_{j+1}) &\le d(\tilde{q}_j,\Phi(q_j)) + d(\Phi(q_j),\Phi(q_{j+1})) + d(\Phi(q_{j+1}),\tilde{q}_{j+1}) \\ &\le 10^{-i_0-k+1} + 20\cdot 10^{-i_0-k} + 10^{-i_0-k+1} \le 10\cdot 10^{-i_0-k+1}. \end{aligned}$$

In other words, $\{\tilde{q}_j\}_{j=0}^M$ is a $(10, i_0+k-1)$-path joining $\overline{B}_{3,i_0}(x)$ to $X\setminus B_{4,i_0}(x)$.

We take an optimal map $\tilde{f} \in \mathcal{A}_{10,i_0+k-1}(\overline{B}_{3,i_0}(x), X\setminus B_{4,i_0}(x))$. We define the map $f\colon X_{i+k}\to[0,+\infty)$ by

$$f(q) := \max\{\tilde{f}(\tilde{q}) : \tilde{q}\in X_{i_0+k-1}\cap \overline{B}(\Phi(q),10^{-i_0-k+1})\}$$

if $q\in\overline{B}(y,2(L_0+5)^3\cdot 10^{-i})$ and 0 otherwise.

We want to show that $f\in\mathcal{A}_{i+k}(\overline{B}_{1,i}(y), X\setminus B_{(L_0+5)^3,i}(y))$. We consider any path $\{q_j\}_{j=0}^M \in P_{10,i+k}(\overline{B}_{1,i}(y), X\setminus B_{(L_0+5)^3,i}(y))$. First of all, we can extract the minimal subpath $\{q_j\}_{j=0}^{M'}$ such that $q_{M'}\notin B_{(L_0+5)^3,i}(y)$. Clearly, if $\sum_{j=0}^{M'} f(q_j)\ge 1$, then also $\sum_{j=0}^{M} f(q_j)\ge 1$, so it is enough to check the admissibility condition on this minimal subpath. For such a minimal subpath, we can construct the path $\{\tilde{q}_j\}_{j=0}^{M'}$ as in the first part of the proof, since $q_j\in B_{2(L_0+5)^3,i}(y)$ for every $j=0,\dots,M'$. By definition, it holds

$$\sum_{j=0}^{M'} f(q_j) \ge \sum_{j=0}^{M'} \tilde{f}(\tilde{q}_j) \ge 1.$$

Moreover, we have

$$\sum_{q\in X_{i+k}} f(q)^p \le C'\cdot \sum_{\tilde{q}\in X_{i_0+k-1}} \tilde{f}(\tilde{q})^p \le C'\cdot p\text{-Mod}_{k-1}(x) \le C'\cdot p\text{-Mod}_{k-1}(X,n_0),$$

since $x\in X_{i_0}$ and $1\le i_0\le n_0$ by definition. Here C' is a constant depending only on D_0. By the arbitrariness of $i\in\mathbb{N}$ and $y\in X_i$, we conclude

$$p\text{-Mod}_{10,k}^{1,(L_0+5)^3}(X) \le C'\cdot p\text{-Mod}_{k-1}(X,n_0)$$

for every $k\ge 1$. Using Lemma 3.2, we obtain the second inequality; indeed,

$$p\text{-Mod}_k(X) = p\text{-Mod}_{10,k}^{3,4}(X) \le C\cdot p\text{-Mod}_{10,k-\ell}^{1,(L_0+5)^3}(X) \le C\cdot C'\cdot p\text{-Mod}_{k-\ell-1}(X,n_0),$$

where C and ℓ are constants depending only on L_0 and D_0. The thesis follows choosing $C_0 = C\cdot C'$ and $\ell_0=\ell+1$. ∎

The Ahlfors regular conformal dimension of a compact, doubling, uniformly perfect metric space (X,d) coincides with the critical exponent of the combinatorial modulus.

Theorem 4.3 (Theorem 4.5 in [5]). *Let (X, d) be a compact, doubling, uniformly perfect metric space. Then*

$$\mathrm{CD}(X, d) = \inf\{p \geq 0 \text{ such that } \liminf_{k\to+\infty} p\text{-Mod}_k(X) = 0\}.$$

By Lemma 3.2 and Lemma 3.3, the right-hand quantity does not depend on our specific choices of $\lambda = 10$, $L_1 = 3$ and $L_2 = 4$ in the definition of p-Mod$_k(X)$: the critical exponent associated to any other admissible choice of λ, L_1 and L_2 equals the Ahlfors regular conformal dimension of (X, d). Moreover, following again [5] and [3], in the quasi-selfsimilar setting it is possible to find a uniform estimate which will be the key ingredient of the proof of Theorem A.

Proposition 4.4. *Let (X, d) be a perfect (L_0, ρ_0, D_0)-q.s.s. metric space and let $p <$ $\mathrm{CD}(X, d)$. Then there exists a constant λ_0, depending only on D_0, L_0 and p, such that*

$$p\text{-Mod}_k(X, n_0) \geq \lambda_0 > 0$$

for every $k > 0$.

Proof. The space (X, d) is uniformly perfect and doubling, by Proposition 2.2, so the Ahlfors regular conformal dimension of (X, d) can be computed as in Theorem 4.3. The result follows by a submultiplicative estimate. Lemma 4.9 in [5] proves

$$p\text{-Mod}^{1,4}_{10,k+h}(X) \leq C \cdot p\text{-Mod}^{11/10,39/10}_{10,k}(X) \cdot p\text{-Mod}^{1,4}_{10,h}(X)$$

for all $k, h \geq 0$. Here C is a constant depending only on p and D_0. Applying Lemma 3.2, we get

$$p\text{-Mod}_{k+h}(X) \leq C' \cdot p\text{-Mod}_{k-\ell}(X) \cdot p\text{-Mod}_h(X)$$

for all $k \geq \ell$ and $h \geq 0$, where C' is a constant depending only on p and D_0, and ℓ is a universal constant. Let us denote by a_k the quantity p-Mod$_k(X)$. The inequality above is $a_{k+h} \leq C' \cdot a_{k-\ell} \cdot a_h$. By Theorem 4.3, $\liminf_{k\to+\infty} a_k > 0$ since $p < \mathrm{CD}(X, d)$. This implies that $a_k \geq 1/C'$ for all $k > 0$. Indeed, if there exists $k > 0$ such that $C' \cdot a_k < (1 - \varepsilon)$ for some $\varepsilon > 0$, then

$$a_{n(k+\ell)} \leq C' \cdot a_k \cdot a_{(n-1)(k+\ell)} \leq \cdots \leq (1 - \varepsilon)^n$$

for all $n \in \mathbb{N}$. Therefore the subsequence $\{a_{n(k+\ell)}\}_{n\in\mathbb{N}}$ would converge to 0, which is a contradiction. Hence we have found a constant $\lambda > 0$, depending only on p and D_0, such that $a_k \geq \lambda$ for all $k > 0$. An application of Proposition 4.2 gives the thesis. ∎

5. Upper semicontinuity of the conformal dimension

Our scope is to study the behaviour of the Ahlfors regular conformal dimension under Gromov–Hausdorff convergence. For technical reasons, it is often useful to study ultralimits instead of Gromov–Hausdorff limits. It essentially avoids to extract converging subsequences. For more detailed notions on ultralimits, we refer to [14] and [10]. A non-principal ultrafilter ω is a finitely additive measure on $\mathbb{N}$ such that $\omega(A) \in \{0, 1\}$ for every $A \subseteq \mathbb{N}$ and $\omega(A) = 0$ for every finite subset of $\mathbb{N}$. Accordingly, we write ω-a.s. and for ω-a.e.(n) in the usual measure theoretic sense.

Given a bounded sequence (a_n) of real numbers and a non-principal ultrafilter ω, there exists a unique $a \in \mathbb{R}$ such that for every $\varepsilon > 0$ the set $\{n \in \mathbb{N}$ such that $|a_n - a| < \varepsilon\}$ has ω-measure 1, see, for instance, Lemma 10.25 in [14]. The real number a is called the ultralimit of the sequence a_n, and it is denoted by ω-$\lim a_n$.

If (X_n, d_n, x_n) is a sequence of pointed metric spaces, we denote by $(X_\omega, d_\omega, x_\omega)$ the ultralimit pointed metric space. It is the set of sequences (y_n), where $y_n \in X_n$ for every n, such that ω-$\lim d(x_n, y_n) < +\infty$, modulo the relation $(y_n) \sim (y'_n)$ if and only if ω-$\lim d(y_n, y'_n) = 0$. The point of X_ω defined by the class of the sequence (y_n) is denoted by $y_\omega = \omega$-$\lim y_n$. The formula $d_\omega(\omega\text{-}\lim y_n, \omega\text{-}\lim y'_n) = \omega\text{-}\lim d(y_n, y'_n)$ defines a metric on X_ω which is called the ultralimit distance on X_ω.

The relation between Gromov–Hausdorff convergence and ultralimits is summarized here.

Proposition 5.1 (Proposition 3.11 in [18] and Proposition 3.13 in [8]). *Let (X_n, d_n, x_n) be a sequence of pointed, compact metric spaces, and let ω be a non-principal ultrafilter.*

(i) *If $(X_n, d_n) \underset{\text{GH}}{\longrightarrow} (X_\infty, d_\infty)$, then (X_ω, d_ω) is isometric to (X_∞, d_∞). In particular, the ultralimit does not depend on the choice of the basepoints.*

(ii) *If $(X_\omega, d_\omega, x_\omega)$ is compact, then $(X_{n_k}, d_{n_k}) \underset{\text{GH}}{\longrightarrow} (X_\omega, d_\omega)$ for some subsequence $\{n_k\}$.*

Let (X_n, d_{X_n}, x_n), (Y_n, d_{Y_n}, y_n) be two sequences of pointed metric spaces, and let ω be a non-principal ultrafilter. A sequence of maps $f_n \colon X_n \to Y_n$ is said admissible if ω-$\lim d_{Y_n}(f_n(x_n), y_n) < +\infty$. A sequence of admissible L-Lipschitz maps f_n defines a L-Lipschitz map $f_\omega = \omega\text{-}\lim f_n \colon (X_\omega, x_\omega) \to (Y_\omega, y_\omega)$ by $f_\omega(\omega\text{-}\lim x_n) = \omega\text{-}\lim f_n(x_n)$.

The class of uniformly perfect (L_0, ρ_0)-quasi-selfsimilar metric spaces is closed under Gromov–Hausdorff convergence.

Proposition 5.2. *Let (X_n, d_n) be a sequence of compact, a_0-uniformly perfect, (L_0, ρ_0)-q.s.s. metric spaces. Suppose it converges in the Gromov–Hausdorff sense to a metric space (X_∞, d_∞). Then (X_∞, d_∞) is a compact, a_0-uniformly perfect, (L_0, ρ_0)-q.s.s. metric space.*

Proof. The set X_∞ is compact by our definition of Gromov–Hausdorff convergence. We fix a non-principal ultrafilter ω and we call X_ω the ultralimit space: it does not depend on the basepoints, and it is isometric to X_∞ by Proposition 5.1. We fix a point $x_\omega = \omega\text{-}\lim x_n \in X_\omega$ and a positive real number $\rho \le \rho_0$. For every n, there exists a L_0-biLipschitz map $\Phi_n \colon (B(x_n, \rho), (\rho_0/\rho) \cdot d_n) \to X_n$ with $\Phi_n(B(x_n, \rho)) \supseteq B(\Phi_n(x_n), \rho_0/L_0)$. The sequence of maps Φ_n is clearly admissible, so it defines a ultralimit L_0-biLipschitz map Φ_ω, which is defined on the ultralimit space of the sequence $(B(x_n, \rho), (\rho_0/\rho) \cdot d_n)$. We observe that this ultralimit space contains $B(x_\omega, \rho)$. Indeed, if $y_\omega = \omega\text{-}\lim y_n \in B(x_\omega, \rho)$, then $d_n(y_n, x_n) < \rho$ ω-a.s. Moreover, the ultralimit metric of the metrics $(\rho_0/\rho) \cdot d_n$ is $(\rho_0/\rho) \cdot d_\omega$. So we can restrict Φ_ω to a L_0-biLipschitz map from $(B(x_\omega, \rho), (\rho_0/\rho) \cdot d_\omega) \to X_\omega$. We need to show that $\Phi_\omega(B(x_\omega, \rho)) \supseteq B(\Phi_\omega(x_\omega), \rho_0/L_0)$. We take $y_\omega = \omega\text{-}\lim y_n$ such that $d_\omega(y_\omega, \Phi_\omega(x_\omega)) \le (1 - 2\varepsilon)\, \rho_0/L_0$, with $\varepsilon > 0$. By definition, we have $d_n(y_n, \Phi_n(x_n)) \le (1 - \varepsilon)\, \rho_0/L_0$ for ω-a.e.(n). By assumption, we can find points

$z_n \in B(x_n, \rho)$ such that $\Phi_n(z_n) = y_n$, ω-a.s. These points satisfy

$$\frac{\rho_0}{\rho} \cdot d_n(z_n, x_n) \leq L_0 \cdot d_n(y_n, \Phi_n(x_n)) \leq (1-\varepsilon) \cdot \rho_0,$$

so $d_n(x_n, z_n) \leq (1-\varepsilon) \cdot \rho$. Clearly, the point $z_\omega = \omega\text{-lim}\, z_n$ belongs to $B(x_\omega, \rho)$ and satisfies $\Phi_\omega(z_\omega) = y_\omega$.

It remains only to prove that X_ω is a_0-uniformly perfect. We fix $x_\omega = \omega\text{-lim}\, x_n \in X_\omega$ and $0 < \rho \leq \operatorname{Diam}(X_\omega)$. For every $\varepsilon > 0$, we have $(1-\varepsilon)\rho \leq \operatorname{Diam}(X_n)$ for ω-a.e.(n), and thus there exists a point $y_n^\varepsilon \in X_n$ with $d(x_n, y_n^\varepsilon) \leq (1-\varepsilon)\rho$ and $d(x_n, y_n^\varepsilon) \geq a_0(1-\varepsilon)\rho$ for ω-a.e.(n). We consider the ultralimit point $y_\omega^\varepsilon = \omega\text{-lim}\, y_n^\varepsilon \in X_\omega$. It satisfies $d(x_\omega, y_\omega^\varepsilon) \leq (1-\varepsilon)\rho$ and $d(x_\omega, y_\omega^\varepsilon) \geq a_0(1-\varepsilon)\rho$. Since this is true for every $\varepsilon > 0$, and since X_ω is compact, we can find a point $y_\omega \in X_\omega$ such that $d(x_\omega, y_\omega) \leq \rho$ and $d(x_\omega, y_\omega) \geq a_0\rho$, showing that X_ω is a_0-uniformly perfect. ■

Remark 5.3. This proposition, together with Proposition 2.2, implies that the Gromov–Hausdorff limit of a sequence of compact (L_0, ρ_0)-q.s.s. metric spaces with diameters bounded below by $c_0 > 0$, as considered by [20] and [5], is still uniformly perfect.

We can now give the:

Proof of Theorem A. We notice that since X_∞ is compact, then the diameters of X_n are uniformly bounded above by some $\Delta_0 \geq 0$. We proceed in several steps.

Step 1. *There exists $D_0 \geq 0$ such that X_n is D_0-doubling for every n.*

Suppose it is not true: then for every $j \in \mathbb{N}$, there exist n_j, $x_{n_j} \in X_{n_j}$ and $\rho_{n_j} > 0$ such that there is a $(\rho_{n_j})/2)$-separated set inside $B(x_{n_j}, \rho_{n_j})$ of cardinality $\geq j$. Up to passing to a subsequence, we can suppose $\lim \rho_{n_j} = \rho_\infty \in [0, +\infty)$. Clearly X_∞ is not totally bounded when $\rho_\infty > 0$, and this is impossible since X_∞ is compact. If $\rho_\infty = 0$, we use the quasi-selfsimilarity to get L_0-biLipschitz maps $\Phi_j \colon (B(x_{n_j}, \rho_{n_j}), (\rho_0/\rho_{n_j}) \cdot d_{n_j}) \to X_{n_j}$ for every j for which $\rho_{n_j} \leq \rho_0$. Hence we can find a $\frac{\rho_0}{2L_0}$-separated set inside $B(\Phi_j(x_{n_j}), L_0\rho_0)$ with cardinality $\geq j$. Once again this contradicts the compactness of X_∞.

In order to simplify the notations, we fix a non-principal ultrafilter ω and we call X_ω the ultralimit space, which is isometric to X_∞ by Proposition 5.1.

Step 2. *Let $k \in \mathbb{N}$. We fix a maximal 10^{-k}-separated subset $X_{k,n}$ of X_n. Then*

(i) *the cardinality of $X_{k,n}$ is uniformly bounded from above, and each $X_{k,n}$ is a 10^{-k}-net of X_n;*

(ii) *the set $X_{k,\omega} := \{\omega\text{-}\lim q_n$ such that $q_n \in X_{k,n}\}$ is a 10^{-k}-net of X_ω;*

(iii) *there exists $A_k \subseteq \mathbb{N}$, with $\omega(A_k) = 1$, such that the function $\pi_n \colon X_{k,\omega} \to X_{k,n}$, $\pi_n(\omega\text{-}\lim q_n) := q_n$, is well defined and bijective for all $n \in A_k$.*

By Step 1, we know that each X_n is D_0-doubling, therefore the cardinality of $X_{k,n}$ is uniformly bounded above in terms of D_0 and k. The second statement of (i) has been explained at the beginning of Section 2.

We take two points $\omega\text{-lim}\, q_n, \omega\text{-lim}\, q_n' \in X_{k,\omega}$. If $d_\omega(\omega\text{-lim}\, q_n, \omega\text{-lim}\, q_n') < 10^{-k}$, then $d_n(q_n, q_n') < 10^{-k}$ ω-a.s., and by definition $q_n = q_n'$ ω-a.s., implying $\omega\text{-lim}\, q_n = \omega\text{-lim}\, q_n'$. Since X_ω is compact, we conclude that the set $X_{k,\omega}$ is finite, being 10^{-k-1}-separated, and that π_n is well defined ω-a.s. Indeed, the proof given above shows that for every $q_\omega = \omega\text{-}\lim q_n \in X_{k,\omega}$, there exists a set $A_{q_\omega} \subseteq \mathbb{N}$ such that $\omega(A_{q_\omega}) = 1$ and such that if $q_\omega = \omega\text{-}\lim q'_n$ with $q'_n \in X_{k,n}$, then $q'_n = q_n$. Therefore, for every $n \in A_k := \bigcap_{q_\omega \in X_{k,\omega}} A_{q_\omega}$, the map π_n is well defined. Since the cardinality of $X_{k,\omega}$ is finite, we have that $\omega(A_k) = 1$.

We suppose $X_{k,\omega}$ is not a 10^{-k}-net of X_ω. Therefore we can find $y_\omega = \omega\text{-}\lim y_n \in X_\omega$ such that $d_\omega(y_\omega, q_\omega) > 10^{-k}$ for all $q_\omega \in X_{k,\omega}$. Since $X_{k,\omega}$ is finite, we know that $d_n(y_n, q_n) > 10^{-k}$ for all $q_n \in X_{k,n}$, ω-a.s. This contradicts the fact that $X_{k,n}$ is a 10^{-k}-net for every n, so also $X_{k,\omega}$ is a 10^{-k}-net of X_ω. Since

$$|d_\omega(q_\omega, q'_\omega) - d_n(q_n, q'_n)| < \frac{10^{-k}}{2}$$

for all $q_\omega = \omega\text{-}\lim q_n$, $q'_\omega = \omega\text{-}\lim q'_n \in X_{k,\omega}$ and for ω-a.e.(n), we conclude that π_n is injective ω-a.s. Finally, suppose π_n is not surjective ω-a.s. Then it is possible to find a set $A \in \omega$ such that for every $n \in A$ there exists $q_n \in X_{k,n}$ which is not in the image of π_n. In this case, we consider the point $\omega\text{-}\lim q_n$ that belongs to $X_{k,\omega}$, finding a contradiction. This ends the proof of (iii).

Since we have fixed 10^{-k}-nets $X_{k,n}$ and $X_{k,\omega}$ of X_n and X_ω, respectively, every path will be intended with respect to these sets.

Step 3. *Let* $k \in \mathbb{N}$. *There exists a subset* $B_k \subseteq \mathbb{N}$ *of* ω*-measure* 1 *such that*

(i) *the map* $\pi_n: X_{k,\omega} \to X_{k,n}$ *from Step* 2 *is well defined and bijective for every* $n \in B_k$;

(ii) *for every* $n \in B_k$ *and for every* $(10,k)$*-path* $\gamma_n = \{q_j^n\}_{j=0}^M$ *of* X_n, *the associated path* $\gamma_\omega = \{\pi_n^{-1}(q_j^n)\}_{j=0}^M$ *is a* $(30,k)$*-path of* X_ω.

Since $X_{k,\omega}$ is finite, we can find a subset B_k of A_k with ω-measure 1 such that

$$|d_\omega(q_\omega, q'_\omega) - d_n(\pi_n(q_\omega), \pi_n(q'_\omega))| \le 10 \cdot 10^{-k}$$

for all $q_\omega, q'_\omega \in X_{k,\omega}$ and for all $n \in B_k$. Let us take a $(10,k)$-path $\gamma_n = \{q_j^n\}_{j=0}^M$ of X_n, for $n \in B_k$. This means $d_n(q_j^n, q_{j+1}^n) \le 20 \cdot 10^{-k}$ for all $j = 0, \dots, M-1$. Therefore $d_\omega(\pi_n^{-1}(q_j^n), \pi_n^{-1}(q_{j+1}^n)) \le 30 \cdot 10^{-k}$, i.e., the thesis.

Step 4. *Let* $i, k \in \mathbb{N}$ *and* $p \ge 0$. *Then* $p\text{-Mod}_{30,k}^{13/4,15/4}(y_\omega) \ge \omega\text{-}\lim p\text{-Mod}_k(y_n)$ *for every* $y_\omega = \omega\text{-}\lim y_n \in X_{i,\omega}$.

We apply Step 3 to the integer $i+k$ finding $B_{i+k} \subseteq \mathbb{N}$, $\omega(B_{i+k}) = 1$, and bijective maps $\pi_n: X_{i+k,\omega} \to X_{i+k,n}$ for all $n \in B_{i+k}$. We take an optimal function $f_\omega \in \mathcal{A}_{30,i+k}(\overline{B}_{13/4,i}(y_\omega), X_\omega \setminus B_{15/4,i}(y_\omega))$. By definition, f_ω maps points of $X_{i+k,\omega}$ to $[0,+\infty)$. For all $n \in B_{i+k}$, we define the functions $f_n: X_{i+k,n} \to [0,+\infty)$ by $f_n(q) = f_\omega(\pi_n^{-1}(q))$. We find another subset $C_{i+k,y_\omega} \subseteq B_{i+k}$ of ω-measure 1 such that

$$|d_\omega(q_\omega, y_\omega) - d_n(\pi_n(q_n), y_n)| \le \frac{1}{4} \cdot 10^{-i}$$

for all $q_\omega \in X_{i+k,\omega}$ and for all $n \in C_{i+k,y_\omega}$.

We want to check that $f_n \in \mathcal{A}_{10,i+k}(\overline{B}_{3,i}(y_n), X_n \setminus B_{4,i}(y_n))$ for all $n \in C_{i+k,y_\omega}$. We fix $n \in C_{i+k,y_\omega}$ and take a $(10, i+k)$-path $\gamma_n = \{q_j^n\}_{j=0}^M$ such that $d_n(y_n, q_0^n) \le 3 \cdot 10^{-i}$

and $d_n(y_n, q_M^n) > 4 \cdot 10^{-i}$. We denote by $\gamma_\omega = \{\pi_n^{-1}(q_j^n)\}$ the $(30, i+k)$-path given by Step 3. We observe that $d_\omega(y_\omega, \pi_n^{-1}(q_0^n)) \leq \frac{13}{4} \cdot 10^{-i}$ and $d_\omega(y_\omega, q_M^\omega) > \frac{15}{4} \cdot 10^{-i}$, i.e., the $(30, i+k)$-path γ_ω joins $\overline{B}_{13/4,i}(y_\omega)$ and $X_\omega \setminus B_{15/4,i}(y_\omega)$. By definition of f_n, we get

$$\sum_{j=0}^{M} f_n(q_j^n) = \sum_{j=0}^{M} f_\omega(\pi_n^{-1}(q_j^n)) \geq 1.$$

Moreover, it holds

$$p\text{-Mod}_k(y_n) \leq \sum_{q \in X_{i+k,n}} f_n^p(q) = \sum_{q \in X_{i+k,\omega}} f_\omega^p(\pi_n^{-1}(q)) = p\text{-Mod}_{30,k}^{13/4,15/4}(y_\omega).$$

Since this is true for all $n \in C_{i+k,y_\omega}$, we get

$$p\text{-Mod}_{30,k}^{13/4,15/4}(y_\omega) \geq \omega\text{-}\lim p\text{-Mod}_k(y_n).$$

Step 5. *Conclusion.*

We fix $k \in \mathbb{N}$ and $0 \leq p < \omega\text{-}\lim \mathrm{CD}(X_n, d_n)$. By Proposition 4.4, we find a constant $\lambda_0 > 0$ depending only on D_0, L_0 and p such that

$$\sup_{i \leq n_0} \sup_{y \in X_{i,n}} p\text{-Mod}_k(y) \geq \lambda_0 \tag{5.1}$$

for ω-a.e.(n). For all these n's, we take a point $y_n \in X_{i_n,n}$, $1 \leq i_n \leq n_0$, realizing the supremum in (5.1). The sequence i_n is ω-a.s. equal to some $i_* \in \{1, \dots, n_0\}$. So the limit point y_ω belongs to $X_{i_*,\omega}$. By Step 4, we get

$$p\text{-Mod}_{30,k}^{13/4,15/4}(X_\omega) \geq p\text{-Mod}_{30,k}^{13/4,15/4}(y_\omega) \geq \omega\text{-}\lim p\text{-Mod}_k(y_n) \geq \lambda_0.$$

Applying the easy inequalities of Lemmas 3.2 and 3.3, we conclude that $p\text{-Mod}_k(X_\omega) \geq \lambda_0$ for every k. This shows $\liminf_{k \to +\infty} p\text{-Mod}_k(X_\omega) > 0$. As X_ω is a compact, doubling and uniformly perfect metric space by Proposition 5.2 and Proposition 2.2, we can apply Theorem 4.3 to get $p \leq \mathrm{CD}(X_\omega, d_\omega)$. In conclusion, we proved

$$\omega\text{-}\lim \mathrm{CD}(X_n, d_n) \leq \mathrm{CD}(X_\omega, d_\omega).$$

This inequality is true for every non-principal ultrafilter ω, so by Lemma 6.3 of [8], we have $\limsup_{n \to +\infty} \mathrm{CD}(X_n, d_n) \leq \mathrm{CD}(X_\infty, d_\infty)$. ■

The main tools used in this proof are: the reduction of the computation of the combinatorial modulus to a *finite* set of scales and the uniform lower bound on the combinatorial modulus, independent of k, given by Proposition 4.4. The approach to the lower semicontinuity problem is more difficult because from one side it is still possible to reduce the computation to a finite set of scales, but from the other side there is no more any control on the behaviour of the p-modulus, independent of k. If we take some $p > \mathrm{CD}(X_n, d_n)$ for every n, then by Theorem 4.3 it holds $\liminf_{k \to +\infty} p\text{-Mod}_k(X_n) = 0$. But a priori it is not possible to conclude that $\liminf_{k \to +\infty} p\text{-Mod}_k(X_\infty) = 0$. Indeed, for given $\varepsilon > 0$, we cannot control the threshold k_ε such that $p\text{-Mod}_k(X_n) < \varepsilon$ for $k \geq k_\varepsilon$. Clearly, if we have this kind of uniform control on the spaces X_n, then the Ahlfors regular conformal dimension of X_∞ is equal to the limit of the Ahlfors regular conformal dimensions of X_n.

Then Question 1.3 can be rephrased in the following way: are there (interesting) geometric conditions on a quasi-selfsimilar space that gives a uniform control on the thresholds k_ε defined above?

This question seems to be related to (uniform) weak super-multiplicative properties of the sequence $p\text{-Mod}_k(X)$, as studied in relation with the combinatorial Lowner property in Sections 4 and 8 of [3], and in the special case of the Sierpiński carpet in Theorem 1.3 of [21]. This weak super-multiplicative property seems to hold true only for spaces in which curves are uniformly distributed in some sense, as suggested by the arguments used again in Lemmas 4.3 and 8.1 of [3]. This observation gives a possible approach to the question presented in the introduction in case of spaces satisfying a uniform combinatorial Lowner property.

6. Gromov-hyperbolic spaces

In this second part of the paper, we prove Theorem B. We briefly recall the definition of Gromov-hyperbolic metric spaces. Good references are for instance [4] and [12]. Let X be a metric space. Given three points $x, y, z \in X$, the *Gromov product* of y and z with respect to x is

$$(y, z)_x = \frac{1}{2}\left(d(x, y) + d(x, z) - d(y, z)\right).$$

The space X is said *δ-hyperbolic*, $\delta \geq 0$, if for every four points $x, y, z, w \in X$, the following 4-*points condition* holds:

$$(x, z)_w \geq \min\{(x, y)_w, (y, z)_w\} - \delta, \tag{6.1}$$

or, equivalently,

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + 2\delta. \tag{6.2}$$

The space X is *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

Let X be a δ-hyperbolic metric space, and let x be a point of X. The *Gromov boundary* of X is defined as the quotient

$$\partial X = \left\{(z_n)_{n\in\mathbb{N}} \subseteq X \mid \lim_{n,m\to+\infty} (z_n, z_m)_x = +\infty\right\} / \approx,$$

where $(z_n)_{n\in\mathbb{N}}$ is a sequence of points in X and $\approx$ is the equivalence relation defined by $(z_n)_{n\in\mathbb{N}} \approx (z'_n)_{n\in\mathbb{N}}$ if and only if $\lim_{n,m\to+\infty}(z_n, z'_m)_x = +\infty$. We will write $z = [(z_n)] \in \partial X$ for short, and we say that (z_n) *converges* to z. This definition does not depend on the basepoint x. There is a natural topology on $X \cup \partial X$ that extends the metric topology of X. The Gromov product can be extended to points $z, z' \in \partial X$ by

$$(z, z')_x = \sup_{(z_n),(z'_n)} \liminf_{n,m\to+\infty} (z_n, z'_m)_x,$$

where the supremum is taken among all sequences such that $(z_n) \in z$ and $(z'_n) \in z'$. For every $z, z', z'' \in \partial X$, it continues to hold

$$(z, z')_x \geq \min\{(z, z'')_x, (z', z'')_x\} - \delta. \tag{6.3}$$

Moreover, for all sequences (z_n) and (z'_n) converging to z and z', respectively, it holds

$$(z, z')_x - \delta \leq \liminf_{n,m \to +\infty} (z_n, z'_m)_x \leq (z, z')_x. \tag{6.4}$$

The Gromov product between a point $y \in X$ and a point $z \in \partial X$ is defined in a similar way, and it satisfies a condition analogue of (6.4).

The boundary of a δ-hyperbolic metric space is metrizable. A metric $D_{x,a}$ on ∂X is called a *visual metric* of center $x \in X$ and parameter $a \in (0, \frac{1}{2\delta \cdot \log_2 e})$ if there exists $V > 0$ such that for all $z, z' \in \partial X$, it holds

$$\frac{1}{V} e^{-a(z,z')_x} \leq D_{x,a}(z, z') \leq V e^{-a(z,z')_x}. \tag{6.5}$$

A visual metric is said *standard* if for all $z, z' \in \partial X$, it holds

$$(3 - 2e^{a\delta}) e^{-a(z,z')_x} \leq D_{x,a}(z, z') \leq e^{-a(z,z')_x}. \tag{6.6}$$

For all a as before and $x \in X$, there exists always a standard visual metric of center x and parameter a (cf. [2,23]). Every two different visual metrics are quasisymmetric equivalent, and the quasisymmetric homeomorphism is the identity (Lemma 6.1 in [2]). This defines a well-defined quasisymmetric gauge on ∂X, that we denote by $\mathcal{J}_{\text{AR}}(\partial X)$. If C is a subset of ∂X, then the restriction of two visual metrics on C define again two quasisymmetric distances, so the quasisymmetric gauge $\mathcal{J}_{\text{AR}}(C)$ is well defined.

We will deal with proper metric spaces, i.e., spaces in which every closed ball is compact. A metric space X is K-almost geodesic if for all $x, y \in X$ and for all $t \in [0, d(x, y)]$, there exists $z \in X$ such that $|d(x, z) - t| \leq K$ and $|d(y, z) - (d(x, y) - t)| \leq K$. If we do not need to specify the value of K, we simply say that X is almost geodesic. A metric space is geodesic if it is 0-almost geodesic. Let X be a proper, geodesic, Gromov-hyperbolic metric space. Every geodesic ray ξ defines a point $\xi^+ = [(\xi(n))_{n\in\mathbb{N}}]$ of the Gromov boundary ∂X. Moreover, for every $z \in \partial X$ and every $x \in X$, it is possible to find a geodesic ray $\xi_{x,z}$ such that $\xi_{x,z}(0) = x$ and $\xi_{x,z}^+ = z$. Analogously, given different points $z = [(z_n)]$ and $z' = [(z'_n)] \in \partial X$, there exists a geodesic line γ joining z to z', i.e., such that $\gamma|_{[0,+\infty)}$ and $\gamma|_{(-\infty,0]}$ join $\gamma(0)$ to z and z', respectively. We call z and z' the *positive* and *negative endpoints* of γ, respectively, denoted $\gamma^\pm$.

The *quasiconvex hull* of a subset C of ∂X is the union of all the geodesic lines joining two points of C, and it is denoted by QC-Hull(C). If X is proper and geodesic and C has more than one point, then QC-Hull(C) is non-empty by the discussion above. We can say more, see also Lemma 3.6 in [19].

Proposition 6.1. *Let X be a proper, geodesic, δ-hyperbolic metric space and let $C \subseteq \partial X$ be a closed subset with at least two points. Then* QC-Hull(C) *is proper,* 36δ*-almost geodesic and δ-hyperbolic. Moreover, $\mathcal{J}_{\text{AR}}(\partial\text{QC-Hull}(C)) \cong \mathcal{J}_{\text{AR}}(C)$, in the sense that there exists a homeomorphism $F : C \to \partial\text{QC-Hull}(C)$ which is a quasisymmetric equivalence when we equip C and ∂QC-Hull(C) with every metrics in the gauges $\mathcal{J}_{\text{AR}}(C)$ and $\mathcal{J}_{\text{AR}}(\partial\text{QC-Hull}(C))$, respectively.*

We need the following approximation result.

Lemma 6.2 (Lemma 4.6 in [8]). *Let X be a proper, geodesic, δ-hyperbolic metric space. Let $C \subseteq \partial X$ be a subset with at least two points and let $x \in$ QC-Hull(C). Then*

$$d(\xi_{x,z}(t), \text{QC-Hull}(C)) \leq 14\delta$$

for all $z \in C$, every geodesic ray $\xi_{x,z}$ with $\xi_{x,z}(0) = x$ and $\xi_{x,z}^+ = z$ and every $t \in [0, +\infty)$.

Proof of Proposition 6.1. We have that QC-Hull(C) is closed and is 36δ-quasiconvex (Lemma 4.5 in [8]), i.e., every point of every geodesic segment joining every two points y and y' of QC-Hull(C) is at distance at most 36δ from QC-Hull(C). This implies that QC-Hull(C) is 36δ-almost geodesic and proper. Condition (6.1) involves only the distance function, so QC-Hull(C) is δ-hyperbolic. We define a map $F \colon C \to \partial$QC-Hull(C) in the following way. We fix $x \in$ QC-Hull(C). For every $z \in C$, we take a sequence $(z_n) \in$ QC-Hull(C) such that $d(\xi_{x,z}(n), z_n) \leq 14\delta$, as provided by Lemma 6.2. The sequence (z_n) defines a point $\hat{z} \in \partial$QC-Hull(C), since $\lim_{n,m\to+\infty}(z_n, z_m)_x = +\infty$. We set $F(z) := \hat{z}$. It is straightforward to check that F is well defined, i.e., it does not depend on the choice of the sequence (z_n). The Gromov products on C and ∂QC-Hull(C) are comparable by (6.4), namely

$$(z, z')_x - \delta \leq (F(z), F(z'))_x \leq (z, z')_x. \tag{6.7}$$

Fix visual distances $D_{x,a}$ and $\hat{D}_{x,a}$ on ∂X and ∂QC-Hull(C). By (6.7) and (6.5), we have that F is injective. If moreover it is surjective, then it is a quasisymmetric homeomorphism from $(C, D_{x,a})$ to (QC-Hull(C), $\hat{D}_{x,a}$), which is the thesis. So fix a point $\hat{z} \in \partial$QC-Hull(C). By definition, it is represented by a sequence $(z_n) \in$ QC-Hull(C) such that $\lim_{n,m\to+\infty}(z_n, z_m)_x = +\infty$. Let γ_n be a geodesic line of X such that $\gamma_n^{\pm} \in C$ and $z_n \in \gamma_n$. We claim that, up to changing the orientation of γ_n, it holds $\lim_{n\to+\infty}(\gamma_n^+, z_n)_x = +\infty$, so that γ_n^+ converges to $\hat{z} \in \partial X$ as n goes to $+\infty$. Since C is closed, we deduce that $\hat{z} \in C$. Let us suppose the claim is false, so both $(z_n, \gamma_n^{\pm})_x \leq M$ for every n, for some M. By Lemma 3.2 in [11] applied to both the segments $[z_n, \gamma_n^{\pm}]$, we get $d(x, [z_n, \gamma_n^{\pm}]) \leq M + 4\delta$. Let us call $p_n^{\pm}$ points on the rays $[z_n, \gamma_n^{\pm}]$ realizing the distance from x. The 4-point condition (6.2) gives

$$d(x, z_n) + d(p_n^+, p_n^-) \leq \max\{d(x, p_n^+) + d(z_n, p_n^-), d(x, p_n^-) + d(z_n, p_n^+)\} + 2\delta.$$

Since $d(p_n^+, p_n^-) = d(z_n, p_n^-) + d(z_n, p_n^+)$, the inequality above implies

$$d(x, z_n) \leq \max\{d(x, p_n^+), d(x, p_n^-)\} + 2\delta \leq 2M + 10\delta.$$

But this is impossible since $\lim_{n\to+\infty} d(x, z_n) = +\infty$. ∎

The quasisymmetric gauge of an almost geodesic Gromov-hyperbolic space is preserved by quasi-isometries. Recall that a quasi-isometry is a map $f \colon X \to Y$ between metric spaces for which there exist $K \geq 0$ and $\lambda \geq 1$ such that

(i) $f(X)$ is K-dense in Y;

(ii) $\frac{1}{\lambda} d(x, x') - K \leq d(f(x), f(x')) \leq \lambda d(x, x') + K$ for all $x, x' \in X$.

Proposition 6.3 (Theorem 6.5 in [2]). *Let X and Y be two almost geodesic, Gromov-hyperbolic metric spaces, and let $f: X \to Y$ be a quasi-isometry. Then f induces a quasisymmetric homeomorphism $\partial f: \partial X \to \partial Y$.*

This statement means that for one (hence every) choice of metrics on $\mathcal{J}_{\mathrm{AR}}(\partial X)$ and on $\mathcal{J}_{\mathrm{AR}}(\partial Y)$, the map ∂f is a quasisymmetric homeomorphism. In this case, we write $\mathcal{J}_{\mathrm{AR}}(\partial X) \cong \mathcal{J}_{\mathrm{AR}}(\partial Y)$ as in Proposition 6.1.

6.1. The proof of Theorem B

We recall the definition of the class $\mathcal{M}(\delta, D)$ appearing in Theorem B. Let X be a proper, geodesic, δ-hyperbolic metric space. Every isometry of X acts naturally on ∂X, and the resulting map on $X \cup \partial X$ is a homeomorphism. A group of isometries Γ of X is said discrete if it is discrete in the compact-open topology. The *limit set* $\Lambda(\Gamma)$ of a discrete group of isometries Γ is the set of accumulation points of the orbit Γx on ∂X, where x is any point of X. The group Γ is called *elementary* if $\#\Lambda(\Gamma) \leq 2$. The set $\Lambda(\Gamma)$ is closed and Γ-invariant, so it is its quasiconvex hull. A discrete group of isometries Γ is *quasiconvex-cocompact* if its action on QC-Hull$(\Lambda(\Gamma))$ is cocompact, i.e., if there exists $D \geq 0$ such that for all $x, y \in$ QC-Hull$(\Lambda(\Gamma))$, it holds $d(gx, y) \leq D$ for some $g \in \Gamma$. The smallest D satisfying this property is called the *codiameter* of Γ.

Given two real numbers $\delta \geq 0$ and $D > 0$, we define $\mathcal{M}(\delta, D)$ to be the class of triples (X, x, Γ), where X is a proper, geodesic, δ-hyperbolic metric space, Γ is a discrete, non-elementary, torsion-free, quasiconvex-cocompact group of isometries with codiameter $\leq D$, and $x \in$ QC-Hull$(\Lambda(\Gamma))$.

Let Γ be a finitely generated. Given a finite generating set Σ of Γ, one can construct the Cayley graph Cay(Γ, Σ) of Γ relative to Σ. Any two Cayley graphs, made with respect to different generating sets, are quasi-isometric. The group Γ is said to be Gromov-hyperbolic if one (and hence all) of its Cayley graphs is Gromov-hyperbolic. If it is the case, the Gromov boundaries of every two Cayley graphs are quasisymmetric equivalent, by Proposition 6.3. We denote the corresponding quasisymmetric gauge by $\mathcal{J}_{\mathrm{AR}}(\partial\Gamma)$. A straightforward modification of the classical proof of the Svarc–Milnor lemma (along the same lines of Lemma 5.1 in [8]) says that every Cayley graph of Γ is quasi-isometric to QC-Hull$(\Lambda(\Gamma))$, if $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$. So both these spaces are Gromov-hyperbolic and almost geodesic. By Proposition 6.3, the gauges $\mathcal{J}_{\mathrm{AR}}(\partial\Gamma)$ and $\mathcal{J}_{\mathrm{AR}}(\partial\text{QC-Hull}(\Lambda(\Gamma))) \cong \mathcal{J}_{\mathrm{AR}}(\Lambda(\Gamma))$ are quasisymmetric equivalent. The last equality is Proposition 6.1. This is enough, together with the results of [8], to prove the last part of Theorem B. Before that, we recall the definition of equivariant pointed Gromov–Hausdorff convergence.

A triple is (X, x, Γ), where X is a proper metric space, $x \in X$ is a basepoint, and Γ is a group of isometries of X. Given $R > 0$, we define $\Sigma_R(\Gamma, x) := \{g \in \Gamma : d(x, gx) \leq R\}$. Let (X, x, Γ), (Y, y, Λ) be two triples and $\varepsilon > 0$. An *equivariant ε-approximation* from (X, x, Γ) to (Y, y, Λ) is a triple of functions (f, ϕ, ψ), where

- $f: B(x, 1/\varepsilon) \to B(y, 1/\varepsilon)$ is a map such that $f(x) = y$ and satisfying
  - $|d(f(x_1), f(x_2)) - d(x_1, x_2)| < \varepsilon$, for all $x_1, x_2 \in B(x, 1/\varepsilon)$;
  - for all $y_1 \in B(y, 1/\varepsilon)$, there exists $x_1 \in B(x, 1/\varepsilon)$ such that $d(f(x_1), y_1) < \varepsilon$;

- $\phi\colon \Sigma_{1/\varepsilon}(\Gamma, x) \to \Sigma_{1/\varepsilon}(\Lambda, y)$ is a map satisfying $d(f(gx_1), \phi(g)f(x_1)) < \varepsilon$ for all $g \in \Sigma_{1/\varepsilon}(\Gamma, x)$ and for all $x_1 \in B(x, 1/\varepsilon)$ such that $gx_1 \in B(x, 1/\varepsilon)$;
- $\psi\colon \Sigma_{1/\varepsilon}(\Lambda, y) \to \Sigma_{1/\varepsilon}(\Gamma, x)$ is a map satisfying $d(f(\psi(g)x_1), gf(x_1)) < \varepsilon$ for all $g \in \Sigma_{1/\varepsilon}(\Lambda, y)$ and for all $x_1 \in B(x, 1/\varepsilon)$ such that $\psi(g)x_1 \in B(x, 1/\varepsilon)$.

A sequence of triples (X_n, x_n, Γ_n) *converges in the equivariant, pointed Gromov–Hausdorff sense* to $(X_\infty, x_\infty, \Gamma_\infty)$ if for every $\varepsilon > 0$, there exists $n_\varepsilon \geq 0$ such that, if $n \geq n_\varepsilon$, there exists an equivariant ε-approximation from (X_n, x_n, Γ_n) to $(X_\infty, x_\infty, \Gamma_\infty)$. In this case, we write

$$(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X_\infty, x_\infty, \Gamma_\infty).$$

This convergence can be also expressed via ultralimits, similarly to Proposition 5.1. Namely, given a sequence of triples (X_n, x_n, Γ_n) and a non-principal ultrafilter ω, we define the ultralimit group

$$\Gamma_\omega = \{\omega\text{-}\lim g_n : g_n \in \Gamma_n \text{ and } (g_n) \text{ is admissible}\},$$

where we recall that ω-lim g_n acts by isometries on X_ω via

$$\omega\text{-}\lim g_n(\omega\text{-}\lim y_n) := \omega\text{-}\lim(g_n y_n),$$

and that (g_n) is admissible if $\omega\text{-}\lim d(x_n, g_n x_n) < +\infty$. Then the following holds.

Proposition 6.4 (Proposition 3.13 in [8]). *Let (X_n, x_n, Γ_n) be a sequence of triples, and let ω be a non-principal ultrafilter.*

(i) *If $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X_\infty, x_\infty, \Gamma_\infty)$, then $(X_\omega, x_\omega, \Gamma_\omega)$ is isometric to $(X_\infty, x_\infty, \Gamma_\infty)$.*

(ii) *If $(X_\omega, x_\omega, \Gamma_\omega)$ is proper, then $(X_{n_k}, x_{n_k}, \Gamma_{n_k}) \underset{\text{eq-pGH}}{\longrightarrow} (X_\omega, x_\omega, \Gamma_\omega)$ for some subsequence $\{n_k\}$.*

If the triples (X_n, x_n, Γ_n) belong to $\mathcal{M}(\delta, D)$ and $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X_\infty, x_\infty, \Gamma_\infty)$, then also $(X_\infty, x_\infty, \Gamma_\infty) \in \mathcal{M}(\delta, D)$, by Theorem A in [8]. In particular, it is meaningful to talk about $\Lambda(\Gamma_\infty)$. Recall that the conformal dimensions are the conformal dimensions of the quasisymmetric gauges $\mathcal{J}_{\mathrm{AR}}(\Lambda(\Gamma_n))$, $n \in \mathbb{N} \cup \{\infty\}$.

Proposition 6.5. *If $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X_\infty, x_\infty, \Gamma_\infty)$, with $(X_n, x_n, \Gamma_n) \in \mathcal{M}(\delta, D)$, then* $\lim_{n\to+\infty} \mathrm{CD}(\Lambda(\Gamma_n)) = \mathrm{CD}(\Lambda(\Gamma_\infty))$.

Proof. By Theorem A in [8], the triple $(X_\infty, x_\infty, \Gamma_\infty)$ belongs to $\mathcal{M}(\delta, D)$. Moreover, Corollary 7.7 in [8] implies that Γ_n is isomorphic to Γ_∞ for n big enough. Using Proposition 6.3, we conclude that $\mathcal{J}_{\mathrm{AR}}(\partial\Gamma_n) \cong \mathcal{J}_{\mathrm{AR}}(\partial\Gamma_\infty)$. The discussion above about the definition of equivariant pointed Gromov–Hausdorff convergence says that $\mathcal{J}_{\mathrm{AR}}(\Lambda(\Gamma_n)) \cong \mathcal{J}_{\mathrm{AR}}(\Lambda(\Gamma_\infty))$. Therefore, by definition, $\mathrm{CD}(\Lambda(\Gamma_n)) = \mathrm{CD}(\Lambda(\Gamma_\infty))$ for n big enough. ∎

The next step is to show that, under the assumptions of Theorem B, the spaces $\Lambda(\Gamma_n)$ are uniformly perfect and uniformly quasi-selfsimilar, when equipped with suitable visual metrics. Let $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$. We always consider a standard visual metric D_x centered at x and with parameter $a_\delta = \frac{1}{4\delta \log_2 e}$. All the estimates will be done with respect to this metric D_x.

The following three results are essentially known, see for instance [20]. We provide quantified version of them. The critical exponent of Γ is

$$h_\Gamma := \lim_{T\to+\infty} \frac{1}{T} \log \#\Gamma x \cap \overline{B}(x, T).$$

For more details on its geometric meaning, see for instance [7, 9].

Proposition 6.6. *Let $\delta, D, H \geq 0$. There exists $A = A(\delta, D, H) > 0$ such that, for all $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$ with $h_\Gamma \leq H$, the limit set $\Lambda(\Gamma)$ is $(A, h_\Gamma/a_\delta)$-Ahlfors regular.*

Proof. It follows from Theorem 6.1 and Lemma 4.9 in [8]. ∎

Corollary 6.7. *Let $\delta, D, H \geq 0$. There exists $a_0 = a_0(\delta, D, H)$ such that, for all $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$ with $h_\Gamma \leq H$, the limit set $\Lambda(\Gamma)$ is a_0-uniformly perfect.*

Proof. Proposition 5.2 in [8] says that $h_\Gamma \geq \frac{\log 2}{99\delta+10D}$. The conclusion follows by Proposition 6.6 and Lemma 2.1. ∎

Proposition 6.8. *Let $\delta, D \geq 0$. There are $L_0 = L_0(\delta, D)$ and $\rho_0 = \rho_0(\delta, D)$ such that for all $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$, the set $\Lambda(\Gamma)$ is (L_0, ρ_0)-q.s.s.*

Before the proof of this last property, we need a bit of preparation.

Lemma 6.9 (Lemma 4.2 in [8]). *Let X be a proper, geodesic, δ-hyperbolic metric space, and let $z, z' \in \partial X$ and $x \in X$.*

(i) *If $(z, z')_x \geq T$, then $d(\xi_{x,z}(T-\delta), \xi_{x,z'}(T-\delta)) \leq 4\delta$.*

(ii) *If $d(\xi_{x,z}(T), \xi_{x,z'}(T)) < 2b$, then $(z, z')_x > T - b$, for all $b > 0$.*

Lemma 6.10 (Lemma 4.4 in [8]). *Let X be a proper, geodesic, δ-hyperbolic metric space. Then every two geodesic rays ξ and ξ' with the same endpoints at infinity are at distance at most 8δ. More precisely, there exist $t_1, t_2 \geq 0$ such that $t_1 + t_2 = d(\xi(0), \xi'(0))$ and $d(\xi(t+t_1), \xi'(t+t_2)) \leq 8\delta$ for all $t \geq 0$.*

Recall that, on ∂X, we always consider a visual metrics of parameter a_δ.

Corollary 6.11. *Let $(X, x, \Gamma) \in \mathcal{M}(\delta, D)$, and let $z, z' \in \partial X$. Let $\rho > 0$ and R be such that $e^{-a_\delta R} = \rho$. If $D_x(z, z') \leq \rho$, then $d(\xi_{x,z}(R), \xi_{x,z'}(R)) \leq 14\delta$.*

Proof. With this choice of a_δ, we have $\frac{1}{2}e^{-a_\delta (z,z')_x} \leq D_x(z, z') \leq e^{-a_\delta (z,z')_x}$. So, if $D_x(z, z') \leq \rho$, then $(z, z')_x \geq R - \log(2)/a_\delta = R - 4\delta$. By Lemma 6.9, we get $d(\xi_{x,z}(R - 5\delta), \xi_{x,z'}(R - 5\delta)) \leq 4\delta$, so by the triangle inequality, $d(\xi_{x,z}(R), \xi_{x,z'}(R)) \leq 14\delta$. ∎

We can finally give the:

Proof of Proposition 6.8. We claim that $\rho_0 = e^{-a_\delta \cdot D}$ works. We fix $0 < \rho \leq \rho_0$, and we call $R \geq 0$ the real number such that $\rho = e^{-a_\delta \cdot R}$. Let $z \in \Lambda(\Gamma)$ and let $\xi_{x,z}$ be a geodesic ray joining x to z. By Lemma 6.2, there is a point $y \in$ QC-Hull$(\Lambda(\Gamma))$ such that $d(\xi_{x,z}(R), y) \leq 14\delta$. Moreover, by definition of quasiconvex-cocompactness, there exists $g \in \Gamma$ such that $d(y, gx) \leq D$, so $d(\xi_{x,z}(R), gx) \leq 14\delta + D$. Observe that $d(x, gx) \leq R + 14\delta + D$. We call Φ the map induced by g^{-1} on $\Lambda(\Gamma)$, which is well defined since $\Lambda(\Gamma)$ is Γ-invariant. We claim it satisfies the properties required by Definition 1.1.

Let w and w' be two points of $B(z,\rho) \cap \Lambda(\Gamma)$, so $D_x(w,z), D_x(w',z) < \rho$. Let $T \geq 0$ be the real number such that $e^{-a_\delta \cdot T} = D_x(w,w')$. Since $D_x(w,w') < 2\rho$, we get $e^{-a_\delta \cdot T} < 2e^{-a_\delta \cdot R}$ and, by definition of a_δ, $T > R - 4\delta$. We apply three times Corollary 6.11 to get

$$d(\xi_{x,z}(R), \xi_{x,w}(R)) \leq 14\delta, \quad d(\xi_{x,z}(R), \xi_{x,w'}(R)) \leq 14\delta, \quad d(\xi_{x,w}(T), \xi_{x,w'}(T)) \leq 14\delta.$$

By the triangle inequality, we have

$$|T-R| - D - 28\delta \leq d(x, g^{-1}\xi_{x,w}(T)) \leq |T-R| + D + 28\delta.$$

Similar estimates hold for $d(x, g^{-1}\xi_{x,w'}(T))$.

We want to estimate $d(\xi_{x,g^{-1}w}(|T-R|), g^{-1}\xi_{x,w}(T))$. The two rays $\xi_{x,g^{-1}w}$ and $g^{-1}\xi_{x,w}$ define the same point $g^{-1}w$ of ∂X, and $d(x,gx) \leq R + 14\delta + D$. Thus, by Lemma 6.10, there exist $t_1, t_2 \geq 0$ with $t_1 + t_2 \leq R + 14\delta + D$ such that $d(\xi_{x,g^{-1}w}(t+t_1), g^{-1}\xi_{x,w}(t+t_2)) \leq 8\delta$ for all $t \geq 0$. We apply this property to $t = T - t_2 + 18\delta + D$, which is non-negative since $T \geq R - 4\delta$, finding

$$d\big(\xi_{x,g^{-1}w}(T - t_2 + t_1 + 18\delta + D), g^{-1}\xi_{x,w}(T + 18\delta + D)\big) \leq 8\delta.$$

By this inequality and the estimates on $d(x, g^{-1}\xi_{x,w}(T))$, we get

$$|T-R| - 2D - 54\delta \leq d\big(x, \xi_{x,g^{-1}w}(T - t_2 + t_1 + 18\delta + D)\big) \leq |T-R| + 2D + 54\delta,$$

so

$$|T-R| - 2D - 54\delta \leq T - t_2 + t_1 + 18\delta + D \leq |T-R| + 2D + 54\delta.$$

Therefore, by the triangle inequality,

$$d(\xi_{x,g^{-1}w}(|T-R|), g^{-1}\xi_{x,w}(T)) \leq 80\delta + 3D.$$

Analogously, we get $d(\xi_{x,g^{-1}w'}(|T-R|), g^{-1}\xi_{x,w'}(T)) \leq 80\delta + 3D$. Combining these two estimates, we conclude $d(\xi_{x,g^{-1}w}(|T-R|), \xi_{x,g^{-1}w'}(|T-R|)) \leq 174\delta + 6D$. By Lemma 6.9, we have $(g^{-1}w, g^{-1}w')_x > |T-R| - 87\delta - 3D$, so

$$D_x(g^{-1}w, g^{-1}w') \leq e^{-a_\delta(|T-R| - 87\delta - 3D)} \leq e^{a_\delta(87\delta + 4D)} \cdot \frac{e^{-a_\delta D}}{e^{-a_\delta R}} \cdot e^{-a_\delta T}$$
$$= e^{a_\delta(87\delta+4D)} \cdot \frac{\rho_0}{\rho} \cdot D_x(w,w'),$$

where we used the definition of standard visual metric, $|T-R| \geq T - R$ and $\rho_0 = e^{-a_\delta D}$.

We prove now the other inequality. We have $D_x(w,w') = e^{-a_\delta T} \leq e^{-a_\delta (w,w')_x}$, so $(w,w')_x \leq T$. We set $b = 47\delta + D$ and $T' = T + b$. By Lemma 6.9(ii), we know that $d(\xi_{x,w}(T'), \xi_{x,w'}(T')) \geq 2b$. We can argue in the same way as before with $t = T' - t_2$, finding

$$d(\xi_{x,g^{-1}w}(T' - R), g^{-1}\xi_{x,w}(T')) \leq 44\delta + D,$$

and the analogous estimate for w'. Therefore,

$$d(\xi_{x,g^{-1}w}(T'-R), \xi_{x,g^{-1}w'}(T'-R)) \geq 2b - 88\delta - 2D > 4\delta.$$

By Lemma 6.9(i), we have $(g^{-1}w, g^{-1}w')_x < T' - R + \delta = T - R + 48\delta + D$. Thus,

$$D_x(g^{-1}w, g^{-1}w') \geq \frac{1}{2}\, e^{-a_\delta(T - R + 48\delta + D)} = \frac{1}{2}\, e^{-48 \cdot a_\delta \cdot \delta} \cdot \frac{\rho_0}{\rho} \cdot D_x(w,w').$$

Thus, Φ is L_0-biLipschitz, with $L_0 = L_0(\delta, D) = \max\{e^{a_\delta(87\delta+4D)}, 2e^{48\cdot a_\delta\cdot\delta}\}$, from $(B(z,\rho)\cap\Lambda(\Gamma), (\rho_0/\rho)\cdot D_x) \to \Lambda(\Gamma)$.

It remains to show that $\Phi(B(z,\rho)\cap\Lambda(\Gamma)) \supseteq B(\Phi(z), \rho_0/L_0)\cap\Lambda(\Gamma)$. The map Φ is a well-defined self-homeomorphism of $\Lambda(\Gamma)$, so every $w \in B(\Phi(z), \rho_0/L_0)\cap\Lambda(\Gamma)$ is of the form $\Phi(w')$ for some $w' \in \Lambda(\Gamma)$. Moreover, the same proof as above implies that the map Φ^{-1} induced by g is L_0-biLipschitz from $(B(z,\rho_0)\cap\Lambda(\Gamma), (\rho/\rho_0)\cdot D_x) \to \Lambda(\Gamma)$. We know that $D_x(\Phi(w'), \Phi(z)) \le \rho_0/L_0$, then

$$D_x(w', z) = D_x(\Phi^{-1}(\Phi(w')), \Phi^{-1}(\Phi(z))) \le L_0 \cdot \frac{\rho}{\rho_0} \cdot D_x(\Phi(w'), \Phi(z)) \le \rho,$$

i.e., $w' \in B(z,\rho)\cap\Lambda(\Gamma)$. This concludes the proof. ∎

Corollary 6.12. *Let* $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X_\infty, x_\infty, \Gamma_\infty)$, *with* $(X_n, x_n, \Gamma_n) \in \mathcal{M}(\delta, D)$. *Then the spaces* $\Lambda(\Gamma_n)$ *are uniformly q.s.s. and uniformly perfect.*

Proof. By Proposition 6.8, all the spaces $\Lambda(\Gamma_n)$ are compact and (L_0, ρ_0)-q.s.s. By Corollary 5.9 in [8], there exists $H \ge 0$ such that $h_{\Gamma_n} \le H$ for all $n \in \mathbb{N}$. Then $\Lambda(\Gamma_n)$ is a_0-uniformly perfect for the same $0 < a_0 < 1$, by Corollary 6.7. ∎

The last step we need is the following.

Proposition 6.13. *If* $(X_n, x_n, \Gamma_n) \underset{\text{eq-pGH}}{\longrightarrow} (X_\infty, x_\infty, \Gamma_\infty)$, *with* $(X_n, x_n, \Gamma_n) \in \mathcal{M}(\delta, D)$, *there exist visual metrics* $D_n \in \mathcal{J}_{\text{AR}}(\Lambda(\Gamma_n))$ *for* $n \in \mathbb{N}\cup\{\infty\}$ *such that* $(\Lambda(\Gamma_n), D_n) \underset{\text{GH}}{\longrightarrow} (\Lambda(\Gamma_\infty), D_\infty)$, *up to a subsequence.*

Proof. We fix a non-principal ultrafilter ω. We denote by $(X_\omega, x_\omega, \Gamma_\omega)$ the ultralimit triple of the sequence (X_n, x_n, Γ_n): it is equivariantly isometric to $(X_\infty, x_\infty, \Gamma_\infty)$, by Proposition 6.4. We equip each $\Lambda(\Gamma_n)$ with a standard visual metric D_n of center x_n and parameter a_δ. Every point of the space ω-$\lim(\Lambda(\Gamma_n), D_n)$ is an equivalence class of sequences (z_n) with $z_n \in \Lambda(\Gamma_n)$. Associated to this sequence, there is a sequence of geodesic rays ξ_{x_n,z_n} of X_n. It is classical (cf. [10], Lemma A.7) that this sequence of geodesic rays define a limit geodesic ray ξ_{x_ω,z_ω}, with $z_\omega \in \partial X_\omega$. The map $\Psi\colon \omega\text{-}\lim(\Lambda(\Gamma_n), D_n) \to \partial X_\omega$ defined by $\Psi((z_n)) = z_\omega$ is a well-defined homeomorphism, by Proposition 5.11 in [8]. Moreover, the proof of Theorem A(i) in [8] shows that the image of Ψ is exactly $\Lambda(\Gamma_\omega)$. We denote by D_ω the distance induced by Ψ on $\Lambda(\Gamma_\omega)$. By definition, the two spaces ω-$\lim(\Lambda(\Gamma_n), D_n)$ and $(\Lambda(\Gamma_\omega), D_\omega)$ are isometric. Proposition 5.1 implies that, up to a subsequence, $(\Lambda(\Gamma_n), D_n) \underset{\text{GH}}{\longrightarrow} (\Lambda(\Gamma_\omega), D_\omega)$. We claim that D_ω is a visual metric on $\Lambda(\Gamma_\omega)$. This would imply, since $(X_\infty, x_\infty, \Gamma_\infty)$ and $(X_\omega, x_\omega, \Gamma_\omega)$ are equivariantly isometric, that $(\Lambda(\Gamma_n), D_n) \underset{\text{GH}}{\longrightarrow} (\Lambda(\Gamma_\infty), D_\infty)$ for a visual metric D_∞ on $\Lambda(\Gamma_\infty)$.

Let us prove the claim. We take two points $\Psi(z), \Psi(z') \in \Lambda(\Gamma_\omega)$, with $z = \omega\text{-}\lim z_n$, $z' = \omega\text{-}\lim z'_n \in \omega\text{-}\lim(\Lambda(\Gamma_n), D_n)$. By definition, $D(\Psi(z), \Psi(z')) = \omega\text{-}\lim D_n(z_n, z'_n) = e^{-a_\delta R}$ for some $R \ge 0$. From (6.6), we have that $\omega\text{-}\lim(z_n, z'_n)_{x_n} \ge R - 4\delta$, and then $\omega\text{-}\lim d(\xi_{x_n,z_n}(R-5\delta), \xi_{x_n,z'_n}(R-5\delta)) \le 4\delta$, by Lemma 6.9(i). This implies that

$$d(\xi_{x_\omega,z_\omega}(R-5\delta), \xi_{x_\omega,z'_\omega}(R-5\delta)) \le 4\delta,$$

and so $(\Psi(z), \Psi(z'))_{x_\omega} > R - 8\delta$ by Lemma 6.9(ii). This means

$$D_\omega(\Psi(z), \Psi(z')) = e^{-a_\delta R} \geq e^{-a_\delta \cdot 8\delta} \cdot e^{-a_\delta(\Psi(z),\Psi(z'))_{x_\omega}} = \frac{1}{4} \cdot e^{-a_\delta(\Psi(z),\Psi(z'))_{x_\omega}}.$$

Analogously, with the same notation as above, we have ω-$\lim(z_n, z'_n)_{x_n} \leq R$ and so

$$\omega - \lim d(\xi_{x_n,z_n}(R + 3\delta), \xi_{x_n,z'_n}(R + 3\delta)) \geq 6\delta$$

by Lemma 6.9.(ii).

By definition, $d(\xi_{x_\omega,z_\omega}(R + 3\delta), \xi_{x_\omega,z'_\omega}(R + 3\delta)) \geq 6\delta$, so $(\Phi(z), \Phi(z'))_{x_\omega} \leq R + 4\delta$ by Lemma 6.9.(i). This means

$$D_\omega(\Psi(z), \Psi(z')) = e^{-a_\delta R} \leq e^{a_\delta \cdot 4\delta} \cdot e^{-a_\delta(\Psi(z),\Psi(z'))_{x_\omega}} = 2 \cdot e^{-a_\delta(\Psi(z),\Psi(z'))_{x_\omega}}.$$

This shows that D_ω is a visual metric on $\Lambda(\Gamma_\omega)$ and concludes the proof. ∎

Theorem B follows from Corollary 6.12, Proposition 6.13 and Proposition 6.5.

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